Computational Economics - Homework Set 9

James Graham

April 13, 2016

1

1.1

Note that we can write the Euclidean norm of Ax as a dot product:

$$\|Ax\|^2 = Ax \cdot Ax = x'A'Ax = x'Bx$$

where $B = A'A$ is a symmetric, positive definite matrix. This means that B has an eigenvalue decomposition with eigenvectors that form an orthonormal basis in $\mathbb{R}^n$ and eigenvalues that are all non-negative. Let λ_i and b_i be the eigenvalues and associated eigenvectors so that we can write $Bb_i = \lambda_i b_i$ for $i = 1, \cdots, n$.

Notice that for any i we can write

$$b_i' B b_i = b_i' \lambda_i b_i = \lambda_i (b_i' b_i) = \lambda_i \tag{1.1}$$

where the final equality follows from the fact that b_i are orthonormal vectors. Now, since b_i are an orthonormal basis for $\mathbb{R}^n$, we can write any vector $x \in \mathbb{R}^n$ as $x = \sum_{i=1}^n y_i b_i$, for scalars $y_i \in \mathbb{R}$.

To find a vector x such that $\|x\| = 1$, then, we need $\sum_{i=1}^n y_i^2 = 1$, since the b_i are orthonormal. Now we can write

$$\|Ax\|^2 = x'Bx = \left(\sum_{i=1}^n y_i b_i\right)' B \left(\sum_{i=1}^n y_i b_i\right) = \left(y_1 b_1' + \cdots y_n b_n'\right) B \left(y_1 b_1 + \cdots y_n b_n\right) = \sum_{i=1}^n y_i^2 b_i' B b_i = \sum_{i=1}^n y_i^2 \lambda_i$$

where the fourth equality is dues to all cross product terms being zero since b_i are orthogonal, and the final equality follows from result (1.1).

Now, using the matrix norm – $\|A\| = \max\limits_{s.t.\ \|x\|=1} \|Ax\|$ – we can write a Lagrangian (constrained optimization) problem:

$$\mathcal{L} = \max_{\{y_i\}_{i=1,\ldots,n}} \sum_{i=1}^n y_i^2 \lambda_i + \mu \left(1 - \sum_{i=1}^n y_i^2\right)$$

The first order condition for a given y_i yields:

$$\begin{aligned} &2y_i\lambda_i - 2\mu y_i = 0 \\ \Rightarrow &\lambda_i = \mu \end{aligned}$$

Since $\lambda_i = \mu$ holds for all i, we can see that the maximum is given by $\mu \sum_{i=1}^n y_i^2 = \mu$. This means that the norm is maximized when μ is the largest eigenvalue, λ_{max}. This we have that

$$\|A\|^2 = \max_i(\lambda_i) = \max_i(|\lambda_i|) = \rho(B) = \rho(A'A)$$

which means that

$$\|A\| = \sqrt{\rho(A'A)}$$

as we wanted to show.

1.2

Gelfand's formula states that $\forall A \in \mathcal{M}(n \times n)$ we have that $\|A^k\|^{1/k} \to \rho(A)$ as $k \to \infty$. This means that for any $\varepsilon > 0$ there exists a $K \in \mathbb{N}$ such that $\forall k \geq K$ we have $|\|A^k\|^{1/k} - \rho(A)| < \varepsilon$. This means that $\forall k \geq K$

$$\begin{aligned}\|A^k\|^{1/k} - \rho(A) &< \varepsilon \\ \|A^k\|^{1/k} &< \rho(A) + \varepsilon \\ \|A^k\| &< (\rho(A) + \varepsilon)^k\end{aligned}$$

Since $\rho(A) < 1$ and we can set ε arbitrarily small, let $r = \rho(A) + \varepsilon < 1$. Then we have that $\|A^k\| < r^k C$, where in this case $C = 1$.

Now, we have

$$\sum_{k=0}^{\infty}\|A^k\| = \sum_{k=0}^{K}\|A^k\| + \sum_{k=K+1}^{\infty}\|A^k\| < \sum_{k=0}^{K}\|A^k\| + \sum_{k=K+1}^{\infty} r^k = \sum_{k=0}^{K}\|A^k\| + \frac{r^{K+1}}{1-r} < \infty$$

where the result of the second summation in the final equality follows from the properties of partial geometric sums. The final inequality follows from the facts that: $\|A^k\|$ is well defined for any matrix, and so is finite for any k and a finite sum of finite values is itself finite; $\frac{r^{K+1}}{1-r} < \infty$ as long as $r \neq 1$.

1.3

If A is diagonalizable, we can express it as $A = PVP^{-1}$, where P is matrix whose columns are the eigenvectors of A and form an orthogonal basis of $\mathbb{R}^n$, and where V is a diagonal matrix with diagonal elements equal to the eigenvalues of A. We can thus write

$$\|A^k\| = \|(PVP^{-1})^k\| = \|PV^kP^{-1}\| \leq \|P\|\|V^k\|\|P^{-1}\| \leq \|P\|\|P^{-1}\|\|V\|^k$$

Consider the matrix norm over V

$$\|V\| = \max_{s.t.\ \|x\|=1}\|Vx\| = \max_{s.t.\ \|x\|=1}\sqrt{(\lambda_1 x_1)^2 + \cdots + (\lambda_n x_n)^2} = \max_i\sqrt{\lambda_i^2} = \max_i|\lambda_i| = \rho(A)$$

where the final equality is simply via the definition of the spectral radius. Now observe that

$$\|A^k\| \leq \|P\|\|P^{-1}\|\|V\|^k = \|P\|\|P^{-1}\|\rho(A)^k$$

and since $\rho(A) < 1$, we have that $\|A^k\| \to 0$ as $k \to \infty$.

1.4

We want to show that the set of nonnegative definite matrices is a closed subset of $(\mathcal{M}(n \times n), \|\cdot\|)$. To do this, we will show that for any sequence of nonnegative definite matrices $\{M_k\}$, its limit point, M, is also a nonnegative definite matrix.

First, note that for any matrix, A, nonnegative definiteness means that $z'Az \geq 0$ for all $z \in \mathbb{R}^n/\{0\}$. Notice, also, that vector and matrix multiplication is a continuous function, i.e. $f(A) := z'Az$ for some z.

Now, let M_k be a sequence of nonnegative definite matrices such that $M_k \to M$ as $k \to \infty$ under the matrix norm, $\|\cdot\|$. We will work towards a contradiction. Suppose the limit point M is such that $z'Mz$ for some $z \in \mathbb{R}^n/\{0\}$ (i.e. M is not nonnegative definite).

$M_k \to M$ means that $\|M_k - M\| < \varepsilon \forall k \geq K \in \mathbb{N}$. Because $f(\cdot)$ is a continuous function we know that for all $\varepsilon > 0$ and all $k \geq K$, there exists a $\delta > 0$ such that $\|M_k - M\| < \delta$ implies that $|f(M_k) - f(M)| < \varepsilon$.

Now, since ε is arbitrary, let $\varepsilon = \frac{z'Mz}{2}$. Then $|f(M_k) - f(M)| = |z'M_kz - z'Mz| < \frac{z'Mz}{2}$ implies that $-\frac{z'Mz}{2} < z'M_kz - z'Mz < \frac{z'Mz}{2}$. Rearranging, we find that $\frac{z'Mz}{2} < z'M_kz < \frac{3z'Mz}{2}$. Since we assumed that $z'Mz < 0$, this implies that $z'M_kz < 0$, which is a contradiction.

Therefore, it must be that the limit point, M, is also a nonnegative definite matrix. Thus, all sequences in the space of nonnegative definite matrices have limit points that also fall inside the space. Therefore, the space of nonnegative definite matrices is a closed subset of $(\mathcal{M}(n \times n), \|\cdot\|)$.

1.5

For this question, we will make use of the following theorem:

Theorem 1.1 (Corollary to the Contraction Mapping Theorem)**.** *If T is a uniform contraction on a Banach space $(S, d(\cdot))$, $T : S' \to S'$, and $S' \underset{closed}{\subset} S$, then the fixed point $Tv = v$ is such that $v \in S'$. Further, if $TS' \subseteq S'' \subset S$, then $v \in S''$.*

Now, let L be the Lyapunov operator on $(\mathcal{M}(n \times n), \|\cdot\|)$, where

$$LX = AXA' + M \tag{1.2}$$

From the lecture notes we know that L is a uniform contraction on $(\mathcal{M}(n \times n), \|\cdot\|)$, and so has a unique fixed point X^*.

Part 1: M symmetric $\Rightarrow$ X^* symmetric

Consider the solution to the Lyapunov equation, X^*. We have $X^* = AX^*A' + M$. We know that M is symmetric. We know that for any $X^* \in \mathcal{M}(n \times n)$, it will be the case that

$$(X^*)' = (AX^*A' + M)' = (AX^*A')' + M' = A(X^*)'A' + M$$

This means that there is another solution to the Lyapunov equation, $(X^*)'$, but we know that there is only a unique solution, therefore $X^* = (X^*)'$. And so we have showed that if M is symmetric, then so is X^*.

Part 2: M nonnegative definite $\Rightarrow$ X^* nonnegative definite

Denote the space of nonnegative definite matrices as $\mathcal{ND}(n \times n)$. As we saw in the previous problem, a nonnegative definite matrix B has the property that $z'Bz \geq 0$ for all $z \in \mathbb{R}^n/\{0\}$. Let M be a nonnegative definite matrix. Consider our Lyapunov operator in (1.2). Note that for any $X \in \mathcal{M}(n \times n)$,

AXA' is nonnegative definite because AXA' is a quadratic form, and so is automatically nonnegative definite. Because the space of nonnegative definite matrices is a linear space, it is closed under addition, and so $AXA' + M$ is also nonnegative definite. Thus, we can see that if X is nonnegative definite, the Lyapunov operator is such that: $L : \mathcal{ND}(n \times n) \to \mathcal{ND}(n \times n)$.

As I proved in exercise 4, $\mathcal{ND}(n \times n) \underset{closed}{\subset} \mathcal{M}(n \times n)$. So we can apply the Corollary to the Contraction Mapping Theorem again, and we immediately see that the fixed point of the Lyapunov operator, X^*, is contained in $\mathcal{ND}(n \times n)$. Thus, M nonnegative definite $\Rightarrow X^*$ nonnegative definite, as we wanted to show.

Part 3: M positive definite $\Rightarrow X^*$ positive definite

Denote the space of positive definite matrices as $\mathcal{P}(n \times n)$. A positive definite matrix B has the property that $z'Bz > 0$ for all $z \in \mathbb{R}^n/\{0\}$. Let M be a positive definite matrix. Consider our Lyapunov operator in (1.2). Note that for any $X \in \mathcal{M}(n \times n)$, AXA' is nonnegative definite because AXA' is a quadratic form, and so is automatically nonnegative definite.

Now, note that $z'(AXA' + M)z = z'AXA'z + z'Mz$. We argued that AXA' is nonnegative definite, and since $z'AXA'z$ is another quadratic form and must be greater than or equal to zero. So we have $z'AXA'z \geq 0$ and $z'Mz > 0$ and so $z'AXA'z + z'Mz > 0$. Thus, we know that the Lyapunov operator takes the space of nonnegative definite matrices to the space of positive definite matrices $L : \mathcal{ND}(n \times n) \to \mathcal{P}(n \times n)$.

Again from above, we know that $\mathcal{ND}(n \times n) \underset{closed}{\subset} \mathcal{M}(n \times n)$. Note, too, that that $\mathcal{P}(n \times n) \subset \mathcal{ND}(n \times n)$. This is because the set of positive definite matrices is just the set of nonnegative definite matrices less all of the matrices that are indefinite i.e. $z'Mz = 0$ for some $z \in \mathcal{R}^n/\{0\}$.

This time we can apply the second part of the Corollary to the Contraction Mapping Theorem: $L(\mathcal{ND}(n \times n)) \subseteq \mathcal{P}(n \times n) \subset \mathcal{ND}(n \times n)$ implies that the fixed point, X^*, is contained in $\mathcal{P}(n \times n)$. Thus, M positive definite $\Rightarrow X^*$ positive definite, as we wanted to show.

1.6