Analytical Exercises

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Exercise 1

The constrained optimization problem here is to maximize $||Ax||$ subject to the constraint that $||x|| = 1$. The corresponding Lagrangian for this problem is $\mathscr{L} = ||Ax|| - \lambda(||x|| - 1)$.

Exercise 2

Pick $r = p(A)$, and $C = \lceil \frac{||\mathbf{A}||}{r} \rceil$. Note that the ceiling function's return will never be 0, as the norm and radii are nonnegative, and further that $||\mathbf{A}|| \leq rC$, as the ceiling function returns something greater than or equal to its argument. Now, using Gelfand's Formula, we can perform a series of manipulations to show a proof by contraposition:

$$\begin{aligned}
\text{Negation: } \lim_{k\to\infty} ||\mathbf{A}^k|| &> \lim_{k\to\infty} r^k C & (1) \\
\text{Using } k \in \mathbb{N}, \text{ take roots: } \lim_{k\to\infty} ||\mathbf{A}^k||^{1/k} &> r \lim_{k\to\infty} \sqrt[k]{C} & (2) \\
\text{Apply Gelfand: } p(A) &> r \lim_{k\to\infty} \sqrt[k]{C} & (3) \\
\text{The unbounded } k\text{-th root of a constant} \geq 1 \text{ is 1: } p(A) &> r & (4)
\end{aligned}$$

But, by assumption $p(A) = r$. Therefore, the condition cannot be false under our assumption, so by the law of the excluded middle it must be true.

Exercise 3

Instead of using Gelfand's Formula, we argue by induction, and submultiplicativity.

- Pick r, such that $\frac{1}{r}\lceil ||\mathbf{A}|| \rceil$ is an integer, and where $\lceil \cdot \rceil$ is the ceiling function. For example, $r = \frac{1}{\lceil ||\mathbf{A}|| \rceil}$ would serve. We know that $||A^1|| \leq r \cdot \frac{1}{r}\lceil ||\mathbf{A}|| \rceil$.

- Now, assume the proposition holds for $k-1$. So,

$$||\mathbf{A}^{k-1}|| \leq r^{k-1} \cdot (\frac{1}{r}\lceil||\mathbf{A}||\rceil) \tag{5}$$

Multiply both sides by $||\mathbf{A}||$.

$$||\mathbf{A}^{k-1}|| \cdot ||\mathbf{A}|| \leq r^{k-1} \cdot (\frac{1}{r}\lceil||\mathbf{A}||\rceil)||\mathbf{A}|| \tag{6}$$

Since the norm is submultiplicative, we can reduce to:

$$||\mathbf{A}^{k}|| \leq r^{k-1} \cdot (\frac{1}{r}\lceil||\mathbf{A}||\rceil)||\mathbf{A}|| \tag{7}$$

But, we have an inequality on $||\mathbf{A}||$ that we can use (we can substitute since the norms are nonnegative always).

$$||\mathbf{A}^{k}|| \leq r^{k-1}(\frac{1}{r}\lceil||\mathbf{A}||\rceil)(r \cdot \frac{1}{r}\lceil||\mathbf{A}||\rceil) \tag{8}$$

Some algebraic manipulation yields:

$$||\mathbf{A}^{k}|| \leq r^{k}(\frac{1}{r}\lceil||\mathbf{A}||\rceil)(\frac{1}{r}\lceil||\mathbf{A}||\rceil) \tag{9}$$

But, we defined r to be $\frac{1}{\lceil||\mathbf{A}||\rceil}$. So, since the factor $(\frac{1}{r}\lceil||\mathbf{A}||\rceil)$ is 1, we can write:

$$||\mathbf{A}^{k}|| \leq r^{k}(\frac{1}{r}\lceil||\mathbf{A}||\rceil) \tag{10}$$

Therefore, the proposition in Exercise 2 is true.

Exercise 4

Before starting this problem set, we show that pointwise convergence of the elements of a matrix yields the (spectral) norm sense of convergence. First, define a polynomial norm with pointwise absolute differences between terms, and take convergence of polynomials to be in the sense of that norm (formally, this is creating a Banach space out of a polynomial ring — if I'm thinking correctly). Clearly, pointwise convergence of matrices leads to convergence of their characteristic polynomials in the sense of the above norm. This, in turn, gives us convergence of the characteristic polynomial of the differenes to the polynomial $\lambda^n = 0$, whose only root is 0 with multiplicity n. The spectral radius of the limiting matrix difference between terms is therefore 0, which implies that the spectral norm is 0, which gives us the sense of convergence we're looking for.

We know that $S = \big(M(n \times n), ||\cdot||\big)$ is a Banach space. We also know that, in a complete metric space, a subset is closed iff it is complete, so the problem reduces to showing that the nonnegative-definite matrices are a complete subset $M \subset S$.

We know that a matrix is nonnegative-definite (i.e., positive-semidefinite) if its eigenvalues are all nonnegative. We can write the eigenvalues of a matrix $m \in M$ as a sequence $(e_1, e_2, ..., e_n)$, where $\{e_i\}$ is a family of nonnegative real numbers (in general, roots of the characteristic polynomial will not be strictly real, but nonnegative-definiteness gives us that property). We also know that there *are* n roots, maybe repeated, because the characteristic polynomial is by construction of degree n (i.e., $p(x) = \prod_{i \in \{1,...,n\}} (x - \lambda_i)$

We know that, given a matrix M, a sequence of pointwise convergent matrices whose limit is M will give us a sequence of norm-convergent matrices whose limit is M. So, our first step is to replace each norm-convergent sequence of matrices with a pointwise-convergent sequence, and then show that the limit of the latter is nonnegative-definite.

We know, as argued above, that we have a characteristic polynomial whose coefficients are converging. This implies that the roots of the characteristic polynomial are converging to a sequence $\lambda_1, \lambda_2, ..., \lambda_n$. Writing the families of roots at each term in the sequence as I_t, we know that each I_i will be a family of nonnegative reals, as each matrix in our sequence is nonnegative-definite.

We know that a converging family of nonnegative reals will always converge to a family of nonnegative reals, as the nonnegative reals are a closed subset of the reals. Therefore, we have that the limiting characteristic polynomial has roots all nonnegative, which is to say that the limiting matrix has eigenvalues all nonnegative, which is to say that the limiting matrix is nonnegative-definite.

Exercise 5

For the first part, observe that $(X^*)' = (AX^*A' + M)' = (AX^*A')' + M' = A(X^*)'A' + M'$, by the rules for taking the transposes of sums and products (i.e., $(AX^*A')' = (A(X^*A'))' = (X^*A')'A' = A(X^*)'A'$. We know because M is symmetric that $M' = M$, which is to say that $(X^*)' = A(X^*)'A' + M$, or that $(X^*)'$ is a fixed-point of the Lyapunov equation. But, such fixed points are unique for $p(A) < 1$, which is to say that $(X^*)' = X^*$, or that X^* is symmetric.

For the second and third parts, start with a positive-definite, symmetric guess for X. Assume that A is invertible. This is then the definition of matrix congruence[1], and Sylvester's Law of Inerita says that AXA' will also have all positive eigenvalues. Since the sum of two positive-definite matrices is also positive-definite, we know that iterating will never give us something not positive-definite. The only case to guard against is the case where we have a Cauchy sequence of positive-definite matrices approximating some fixed point that's not, which I'm not precisely sure how to do.

The semidefinite case is easier, however, as we showed that positive-semidefinite matrices form a closed subset of $M(n \times n)$, which is to say that that pathological Cauchy sequence is no longer something we have to worry about.

[1] This fact, and those that follow, are heavy on reading.

There's probably a way to argue about the definition of congruence preserving the associated bilinear form with the matrix, up to the change of basis encoded by the congruency, but I don't know enough to do that.

Exercise 6

Computational.

References

Sometimes when I wanted to check if a fact was true, or look for useful facts that could help/etc., I would look for a relevant theorem, i.e. on StackExchange, etc. Can be furnished if desired.