# HW # 9

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## 1 Solutions

1. First, I show that the spectral norm is equivalent to finding $\sup_{\|x\|=1} \|Ax\|$. This is fairly obvious since

$$\sup_{x\in\mathbb{R}^n} \frac{\|Ax\|}{\|x\|} = \sup_{x\in\mathbb{R}^n} \frac{\|\frac{|Ax}{\|x\|}\|x\|\|}{\|x\|} = \sup_{x\in\mathbb{R}^n} \frac{\|x\|\|\frac{Ax}{\|x\|}\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\|.$$

From now on, any vector x will be assumed to have norm one. Next, I maximize $\|Ax\|^2$ by the Lagrangian method. $\mathcal{L} = x'A'Ax + \lambda(1 - x'x)$. Using basic matrix calculus, I differentiate with respect to x. Because $A'A$ is symmetric, the the derivative of $x'A'Ax$ with respect to x will be $2A'Ax$. Then we obtain $2A'Ax = 2\lambda x$. It is clear the x is an eigenvector of $A'A$ and $\lambda$ is the corresponding eigenvalue. But then $\|Ax\|^2 = \langle Ax, Ax\rangle = \langle x, A'Ax\rangle = \langle x, \lambda x\rangle = |\lambda|$. The global maximum is achieved with the eigenvector corresponding to the eigenvalue whose absolute value is the spectral radius. Thus $\|Ax\| = \sqrt{\rho(A'A)}$ $\square$

2. Consider the power series $\sum_{k=0}^{\infty} \|A\| x^k$. By Gelfand's formula, $\lim_{k\to\infty} \left\|A^k\right\|^{\frac{1}{k}} = \limsup_{k\to\infty} \left\|A^k\right\|^{\frac{1}{k}} = \rho(A) < 1$. From the convergence theorems for power series, the power series is absolutely convergence if $|x| < \frac{1}{\rho(A)}$. Since $\frac{1}{\rho(A)>1}$, the power series converges when $x = 1$, which is $\sum_{k=0}^{\infty} \|A\|$. Since the series is summable, it is also Abel summable, so $\lim_{x\to 1^-} \sum_{k=0}^{\infty} \|A\| x^k = \sum_{k=0}^{\infty} \|A\|$. We could use the relationship to find $C$ and $r \in (0,1)$ such that $\left\|A^K\right\| \leq Cr^k \ \forall k$, but the details are fuzzy $\square$.

3. Since $A$ is a d diagonalizable square matrix, $A = PDP^{-1}$. Where $D$ is a diagonal matrix of the eigenvalues of $A$. Then $|\lambda_i| < |\rho(A)| < 1$. We use the famous result that $A^k = PD^kP^{-1}$. But the $D^k$ is a diagonal matrix of all the eigenvalues raised to the power k. Then since $\lim_{k\to\infty} |\lambda_i^k| = \lim_{k\to\infty} |\rho(A)^k| = 0$, we conclude that $\lim_{k\to\infty} A^k = \lim_{k\to\infty} PDP^{-1^k} = \lim_{k\to\infty} PD^kP^{-1} = 0\square$.

4. Suppose $\{M_n\}_{n\in\mathbb{N}}$ is a sequence of square non-negative definite matrices that converges to some matrix $M$ which is not. Then there exists some non-zero vector $x$ such that $x'Mx < 0$. Consider the sequence $\{x'M_nx\}_{n\in\mathbb{N}}$. Then it is clear that this sequence convergence to $x'Mx$ , which means that $\forall\epsilon > 0\ \exists N \in \mathbb{N}$ such that $n \geq N$ implies $|x'M_nx - x'Mx| < \epsilon$. Suppose $xMx = -c$, where $c \in \mathbb{R}^+$. I claim there must be some $N$ such that for $n \geq N$, $x'M_nx < 0$. Suppose this were not true. Then $\forall\ n$ we would have $|x'M_nx - x'Mx| = |x'M_nx + c| \geq |0 + c| = c$. This not possible because then the limit definition would not hold if we pick $\epsilon = c/2$. On the other hand, since $M_n$ is positive definite for each $n$, $x'M_nx \geq 0\ \forall n$. Thus we have obtained a contradiction and so $M$ must be non-negative definite also, allowing us to conclude that the set of non-negative definite matrices in $\mathcal{M}(n \times n)$ is closed $\square$.

5. a)We know that $X^* = AX^*A' + M$ and $*$ is unique. Suppose M is symmetric. Then $X^{*\prime} = (AX^*A' + M)' = A(X^*)'A' + M' = A(X^*)'A' + M$ . Then $(X^*)'$ is a solution to the Lyapunov equation, but since the solution is unique $X^* = (X^*)'$ and so $X^*$ is symmetric $\square$

b)Suppose $M$ is non-negative definite. Take non-zero vector $z$. Then $z'X^*z = z'(AX^*A' + M)z = z'AX^*A'z + z'Mz =$.Suppose we repeatedly substitute the expression for $X^*$. We then obtain $z'X^*z = \lim\limits_{k\to\infty} z'A^kX^*A^{k\prime} + \sum_{k=1}^{\infty} z'A^kMA^{k\prime}z + z'Mz$. Since the absolute value of the maximum eigenvalue of A is strictly less the 1, A goes to zero in the power. Let $y_k = A^{k\prime}z$. Each $y_k$ is an nonzero vector and M is non-negative definite so, $\sum_{k=1}^{\infty} Y_k'My + z'Mz \geq 0$ Thus $X^*$ is non-negative definite. $\square$

c)The argument is the same as part (b) except the last step gives strict inequality since M is positive definite.