Problem Set 9: Analysis

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April 14, 2016

Exercise 1

Recall that λ is an eigenvalue of A iff $Av = \lambda v$ for some eigenvector $v \in \mathbb{R}^n \neq 0$.
The definition of a norm for a matrix is:

$$\|A\| = \sup \left\{ \frac{\|Ax\|}{\|x\|} : x \in \mathbb{R}^k, x \neq 0 \right\}$$

By fact seen in class, let's focus on $\|x\| = 1$ and note that by monotonicity of the norm, we can take the supremum of $\|Ax\|^2$ instead.
Furthermore, we can use Weierstrass theorem to show that the supremum is attainable (i.e. the maximum exists). This is because Ax is a linear operator (thus continuous) over a finite vector space. The same can be said when we take the square of the norm. Then we can apply Weierstrass.

Using the euclidean norm, the Lagrangean would be:

$$\mathcal{L} = X'A'AX + \lambda X'X$$

where λ is the multiplier. Then the FOC is:

$$2X'A'A = 2\lambda X'$$

$$A'AX = \lambda X$$

Any pair of eigenvectors and eigenvalues can satisfy this FOC. If we use the solution in the objective function:

$$\mathcal{L} = 2\lambda X'X$$

But then this function is maximized for the maximum eigenvalue (which has an associated eigenvector).
So $\rho(A'A) = \|A\|^2$ and again by monotonicity,

$$\|A\| = \sqrt{\rho(A'A)}$$

Exercise 2

Let's first show that if $\rho(A) < 1 \Rightarrow \exists r < 1$ and $C \in \mathbb{R}$ s.t. $\left\|A^k\right\| \leq r^k C$.

Fact: $\left\|A^k\right\|^{1/k} \to \rho(A)$

By definition, $\forall \epsilon > 0,\ \exists M > 0$ s.t. $\left|\left\|A^k\right\|^{1/k} - \rho(A)\right| < \epsilon\ \forall k \geq M$.

But then this means that I can find an $\epsilon > 0$ small enough so that $\rho(A) + \epsilon \equiv r \in (0,1)$.

$\left\|A^k\right\|^{1/k} \leq r \Rightarrow \left\|A^k\right\| \leq r^k$ by monotonicity.

We already have an upper bound for $k \geq M$. What about $k < M$?

For $k < M$, $\left\|A^k\right\| \leq \max_{i \leq M} \left\|A^i\right\| \Rightarrow \left\|A^k\right\| \leq \max_{i \leq M} \left\|A^i\right\| \frac{r^k}{r^M} = Cr^k$.

Hence, $\left\|A^k\right\| \leq Cr^k$ for any $k \geq 0$.

Finally, given the previous proof,

$$\sum_{k=0}^{\infty} \left\|A^k\right\| \leq C \sum_{k=0}^{\infty} r^k = \frac{C}{1-r} < \infty$$

Exercise 3

A square matrix is diagonalizable if there exists an invertible matrix P such that $A = PDP^{-1}$ with D diagonal.

In that case, it is clear that $A^k = PD^kP^{-1}$ (just write it down and it is evident).

Furthermore, one of the possible diagonalizations is by taking D to be a diagonal matrix formed by A's eigenvalues, and P the eigenvectors.

Since $\rho(A) < 1$, then all of the eigenvalues are within the unit circle. This implies that $D^k \to 0$ as $k \to \infty$, so $A^k \to 0$.

Finally, by continuity of the norm, if $A^k \to 0 \Rightarrow \left\|A^k\right\| \to 0$.

Exercise 4

A symmetric non-negative definite matrix (or positive semi-definite) is one in which for any vector $z \neq 0$, the scalar $z^T M z \geq 0$.

Let $\mathcal{M}^*$ be the set of such matrices. To show that it is a closed subset of $\mathcal{M}$, we need to show that every sequence in $\mathcal{M}^*$ that converges, it does so to an element of $\mathcal{M}^*$.

Assume by contra that $\|A - A_j\| \to 0$ but $A \notin \mathcal{M}^*$. That means that $\exists z \neq 0$ s.t. $z'Az < 0$ while $z'A_jz \geq 0$ for all $j \in \mathbb{N}$.

For the contradiction to work, I need to show that then $\|A - A_j\| \nrightarrow 0$.

Note that since $z'Az < 0$ while $z'A_jz \geq 0$, it must be that $\|z'(A - A_j)z\| \nrightarrow 0$.

$\|z'(A-A_j)z\| = \|z'(A-A_j)\|\,\|z\| = \left\|(A-A_j)'z\right\|\|z\| = \|(A-A_j)\|\,\|z\|^2$
Since $\|z\|^2 \neq 0$, $\|(A-A_j)\| \nrightarrow 0$. CONTRADICTION.

Exercise 5

Part 1:

X is symmetric iff $X = X'$.
$X^* = AX^*A' + M$ with M symmetric.

$$M = M'$$

$$X - AXA' = X' - AX'A'$$

$$X - X' = A(X - X')A'$$

$$X = X'$$

Part 2:

Let $z_{(j)} \equiv Az_{(j-1)}$ for $j \geq 1$, with $z_{(0)} = z$. Then:

$$z'_{(j)}Xz_{(j)} = z'A^jX\left(A'\right)^jz + z'_{(j)}Mz_{(j)}$$

Since $\rho(A) < 1$, $A^j \to \mathbf{0}$ as $j \to \infty$. Then $z'_{(j)}Xz_{(j)} \to 0$ as $j \to \infty$.

Now note the following iteration:
$z'Xz = z'Mz + z'AXA'z = z'Mz + z'_1Xz'_1$
$z'_1Xz_1 = z'_1Mz_1 + z'_2Xz'_2$
So that $z'Xz = z'Mz + z'_1Mz_1 + z'_2Xz_2$.
If we repeat this until infinity, we get:

$$z'Xz = \sum_{k=0}^{\infty} z'_kMz_k + \lim_{k\to\infty} z'_kXz_k$$

I already showed that $\lim_{k\to\infty} z'_kXz_k = 0$.
Since M is P.S.D., for any $z \neq 0$, it must be that $z'_kMz_k \geq 0$ for any k.
The point now is to show that $\sum_{k=0}^{\infty} z'_kMz_k < \infty$. This should be the case since $\rho(A) < 1$, so $z'_kMz_k \to 0$.
But note that showing $z'_kMz_k < 1$ for each k is not enough. I would have to show that it converges at a rate fast enough.

A way to show this would be to find an upper bound $z_k' M z_k < a_k$ such that $\sum\limits_{k=0}^{\infty} a_k < \infty$.

Part 3: