1: Prove that for every A $\in \mathcal{M}(n \times n)$ we have $\| A \| = \sqrt{\rho(A'A)}$. Hint: Use the Lagrange method for constrained optimization.

Proof. Recall that we can normalize the norm of x such that $||x|| = 1$. Then by Lagrangian,

$$L(x, \lambda) = ||Ax|| - \lambda(||x|| - 1)$$

FOC:

$$2A'Ax^* - 2\lambda x^* = 0$$

Notice that λ is a eigenvalue of matrix $A'A$ by definition of eigenvalue. Therefore

$$||Ax^*|| =< Ax^*, Ax^* >= x'^*\lambda x^* = \sqrt{\lambda}||x^*||$$

Notice that here x^* is the solution of $\max ||Ax|| s.t. ||x|| = 1$

$$||Ax^*|| = \| A \| \| x^* \|$$

So $\sqrt{\lambda} = ||A||$. $\sqrt{\rho(A'A)} = \sqrt{max|\lambda|} = ||A||$

□

2: Show that if $\rho(A) < 1$, then $\sum_{k=0}^{\infty} \| A^k \| < \infty$. In particular, show that under the stated condition there exists an $r < 1$ and a $C \in \mathbb{N}$ such that $\| A^k \| \leq r^k C$ for all $k \in \mathbb{N}$

Proof. By Gelfand's formula, we have

$$\| A^k \|^{1/k} \to \rho(A), \qquad \text{as } k \to \infty$$

Given any $\epsilon > 0$ such that $\epsilon < 1 - \rho(A)$, there exists $M_1 \in \mathbb{N}$ such that if $k \geq M_1$, then

$$\left| \| A^k \|^{1/k} - \rho(A) \right| < \epsilon/2 \Rightarrow \| A^k \|^{1/k} < \rho(A) + \epsilon/2$$

Let $r = \epsilon/2 + \rho(A)$. Then $\| A^k \| < r^k, \forall k \geq M_1$. Let
$C' = \max\{\| A^1 \| / r^{M_1}, \| A^2 \|^{1/2} / r^{M_1}, \ldots, \| A_1^M \| / r^{M_1}, 1\}$. And $C = \max\{C', C'^2, \ldots, C'^{M_1}, 1\}$.
Apparently, $\| A^k \| \leq r^k C$, where $r < 1$.

By the convergence of geometric series $\sum_{k=0}^{\infty} r^k$, we can conclude that $\sum_{k=0}^{\infty} \| A^k \| < \infty$ □

3: Show that if $\rho(A) < 1$, then $\| A^k \| \to 0$ as $k \to \infty$ without using Gelfand's formula. Assume that A is diagonalizable.

Proof. Since A is diagonalizable, then there exits an matrix P such that $A = PVP^{-1}$. The diagonal values of matrix V are eigenvalues of A.

Thus $||A^k|| = ||(PVP^{-1})^k|| = ||PV^kP^{-1} \| \leq \| \| P \| \| \| V^k \| \| \| P^{-1} \|$.

Since the maximum eigenvalue $\rho(A) < 1$, then each diagonal element in matrix $V^k \to 0$. So $V^k \to 0$ as $k \to 0$

Therefore $||A^k|| \to 0$ □

4: Show that the set of nonnegative definite matrices is a closed subset of $\mathcal{M}(n \times n, \| \cdot \|)$

Proof. There exists a matrix $B \in \mathcal{M}(n \times n, \| \cdot \|)$ such that a sequence of nonnegative definite matrix$(A_m)_{m \geq 1} \in \mathcal{M}$ converges to B.

Given $\epsilon > 0$, there exists $M \in \mathbb{N}$ such that if $m \geq M$, then $\| A_m - B \| < \epsilon$

Suppose that B is not nonnegative definite. Then there exists $x \in \mathbb{R}^n$ such that $x'Bx < 0$

Let $f(matrix) = x'$matrix x. Clearly, this is a continuous function on $\mathcal{M}$. By property of continuity, there exists $\delta > 0$, for some $A_m \in N_\delta(B)$, we have $x'A_mx < 0$. It contradicts to $A_m, \forall m$ are nonnegative. □

5: Let M, A be in $\mathcal{M}(n \times n)$ with $\rho(A) < 1$. Let X^* be the unique solution to the Lyapunov equation.

$$X = AXA' + M$$

Show that

1.**M symmetric** $\Rightarrow X^*$ **symmetric**

2.**M nonnegative definite** $\Rightarrow X^*$ **nonnegative defininte**

3.M **positive definite** $\Rightarrow X^*$ **positive definite.**

Part 1

Proof. $X'^* = (AXA' + M)' = AX'^*A' + M' = AX'^*A + M$, since $M = M'$.

Apparently, $X^* = X'^*$, since they maintain the same formula, and X^* is the unique solution. □

Part 2

Proof. We have proved in Exercise 4 that that the space of nonnegative definite matrices is closed. Let L be Lyapunov operator. X^* is the unique solution, so we can apply contraction mapping theorem, then X^* should be in the space of nonnegative definite matrices. Therefore the proof is complete. □

Part 3

Proof. By part 2, LX is at least nonnegative definite, since M is positive definite. Therefore $a'AX^*A'a + a'Ma \geq 0, \forall a \in \mathbb{R}^n \backslash \{0\}$. Since $a'Ma > 0$, we have $a'(LX^*)a > 0$. Therefore X^* is positive definite. □