Analytical Exercises

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Exercise 1

Choose x an eigenvector of $A'A$ that yields the eigenvalue with the greatest absolute value.

Exercise 2

Pick $r = p(A)$, and $C = \lceil \frac{||\mathbf{A}||}{r} \rceil$. Note that the ceiling function's return will never be 0, as the norm and radii are nonnegative, and further that $||\mathbf{A}|| \leq rC$, as the ceiling function returns something greater than or equal to its argument. Now, using Gelfand's Formula, we can perform a series of manipulations to show a proof by contraposition:

$$
\begin{aligned}
\text{Negation: } \lim_{k\to\infty} ||\mathbf{A}^k|| &> \lim_{k\to\infty} r^k C && (1)\\
\text{Using } k \in \mathbb{N}\text{, take roots: } \lim_{k\to\infty} ||\mathbf{A}^k||^{1/k} &> r \lim_{k\to\infty} \sqrt[k]{C} && (2)\\
\text{Apply Gelfand: } p(A) &> r \lim_{k\to\infty} \sqrt[k]{C} && (3)\\
\text{The unbounded } k\text{-th root of a constant} \geq 1 \text{ is 1: } p(A) &> r && (4)
\end{aligned}
$$

But, by assumption $p(A) = r$. Therefore, the condition cannot be false under our assumption, so by the law of the excluded middle it must be true.

Exercise 3

Instead of using Gelfand's Formula, we argue by induction, and submultiplicativity.

- Pick r, such that $\frac{1}{r}\lceil ||\mathbf{A}|| \rceil$ is an integer, and where $\lceil \cdot \rceil$ is the ceiling function. For example, $r = \frac{1}{\lceil ||\mathbf{A}|| \rceil}$ would serve. We know that $||A^1|| \leq r \cdot \frac{1}{r}\lceil ||\mathbf{A}|| \rceil$.
- Now, assume the proposition holds for $k - 1$. So,

$$||\mathbf{A}^{k-1}|| \leq r^{k-1} \cdot (\frac{1}{r}\lceil ||\mathbf{A}|| \rceil) \tag{5}$$

Multiply both sides by $||\mathbf{A}||$.

$$||\mathbf{A}^{k-1}|| \cdot ||\mathbf{A}|| \leq r^{k-1} \cdot (\frac{1}{r}\lceil ||\mathbf{A}|| \rceil)||\mathbf{A}|| \tag{6}$$

Since the norm is submultiplicative, we can reduce to:

$$||\mathbf{A}^{k}|| \leq r^{k-1} \cdot (\frac{1}{r}\lceil ||\mathbf{A}|| \rceil)||\mathbf{A}|| \tag{7}$$

But, we have an inequality on $||\mathbf{A}||$ that we can use (we can substitute since the norms are nonnegative always).

$$||\mathbf{A}^{k}|| \leq r^{k-1}(\frac{1}{r}\lceil ||\mathbf{A}|| \rceil)(r \cdot \frac{1}{r}\lceil ||\mathbf{A}|| \rceil) \tag{8}$$

Some algebraic manipulation yields:

$$||\mathbf{A}^{k}|| \leq r^{k}(\frac{1}{r}\lceil ||\mathbf{A}|| \rceil)(\frac{1}{r}\lceil ||\mathbf{A}|| \rceil) \tag{9}$$

But, we defined r to be $\frac{1}{\lceil ||\mathbf{A}|| \rceil}$. So, since the factor $(\frac{1}{r}\lceil ||\mathbf{A}|| \rceil)$ is 1, we can write:

$$||\mathbf{A}^{k}|| \leq r^{k}(\frac{1}{r}\lceil ||\mathbf{A}|| \rceil) \tag{10}$$

Therefore, the proposition in Exercise 2 is true.

Exercise 4

Before starting this problem set, we show that pointwise convergence of the elements of a matrix yields the (spectral) norm sense of convergence. First, define a polynomial norm with pointwise absolute differences between terms, and take convergence of polynomials to be in the sense of that norm (formally, I think this is creating a Banach space out of a polynomial ring out of its underlying field — the field of real numbers — if I'm thinking correctly). Clearly, pointwise convergence of matrices leads to convergence of their characteristic polynomials in the sense of the above norm. This, in turn, gives us convergence to the polynomial $\lambda^n = 0$, whose only root is 0 with multiplicity n. The spectral radius of the limiting matrix is therefore 0, which implies that the spectral norm is 0, which gives us the sense of convergence we're looking for.

We know that $S = \big(M(n \times n), || \cdot ||\big)$ is a Banach space, because it is isometrically isomorphic to $\mathbb{R}^{n^2}$, which is a complete normed vector space. We also know that, in a complete metric space, a subset is closed iff it is complete, so the problem reduces to showing that the nonnegative-definite matrices are a complete subset $M \subset S$.

We know that a matrix is nonnegative-definite (i.e., positive-semidefinite) if its eigenvalues are all nonnegative. We can write the eigenvalues of a matrix $m \in M$ as a sequence $(e_1, e_2, ..., e_n)$, where $\{e_i\}$ is a family of nonnegative real numbers (in general, roots of the characteristic polynomial will not be strictly real, but nonnegative-definiteness gives us that property). We also know that there *are* n roots, maybe repeated, because the characteristic polynomial is by construction of degree n (i.e., $p(x) = \prod_{i \in \{1,...,n\}} (x - \lambda_i)$

We know that if we have a convergent sequence of matrices, we can interpret that convergence pointwise, i.e., we know that each element $a_{ij} \to a^*_{ij}$ for some limiting value. This implies that we have a characteristic polynomial whose coefficients are converging (as the polynomial is $\det A - \lambda I$, which is to say that we have convergent sequences (or families) of roots.

We know that a convergent family of nonnegative reals will always converge to a family of nonnegative reals, as such sequences are bounded by below. So, convergence in matrices implies convergence in the eigenvalues such that the eigenvalues of the limiting matrix are all real and nonnegative, which is to say that the limiting matrix is nonnegative-definite.

Exercise 5

For the first part, observe that $X' = (AXA' + M)' = (AXA')' + M' = AX'A' + M'$, by the rules for taking the transposes of sums and products (i.e., $(AXA')' = (A(XA')' = (XA')'A' = AX'A'$. We know because M is symmetric that $M' = M$, which is to say that $X' = AX'A' + M$, or that X' is a fixed-point of the Lyapunov equation. But, such fixed points are unique for $p(A) < q$, which is to say that $X' = X$, or that X is symmetric.

For the second part, we can use the fact that XX' is always positive definite. See the following set of manipulations:

$$
\begin{array}{rcll}
X & = & AXA' + M/notag//XX' & = \\
(AXA' + M)(AXA' + M)'/notag// & = & (AXA' + M)(AX'A' + M')/notag & (11)
\end{array}
$$

Exercise 6

Computational.

References

Sometimes when I wanted to check if a fact was true, I would look for a relevant theorem, i.e. on StackExchange, etc. Can be furnished if desired.