Comp Econ: Problem Set 9
Analytical Part

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Exercise 1

I will first show that $||\mathbf{A}|| \geq \rho(\mathbf{A}'\mathbf{A})$. There exists an $x \in \mathbb{R}^N$ such that $\mathbf{A}'\mathbf{A}x = \rho(\mathbf{A}'\mathbf{A})x$. Then:

$$\begin{aligned}
\rho(\mathbf{A}'\mathbf{A})||x|| &= \langle \rho(\mathbf{A}'\mathbf{A}x, x) \rangle \\
&= \langle \mathbf{A}'\mathbf{A}x, x \rangle \\
&= \langle \mathbf{A}x, \mathbf{A}x \rangle \\
&= ||\mathbf{A}x|| \\
&\leq ||\mathbf{A}||||x||
\end{aligned}$$

Now I will show the other direction, $||\mathbf{A}|| \leq \rho(\mathbf{A}'\mathbf{A})$:

$$\begin{aligned}
||\mathbf{A}||||x|| &= ||\mathbf{A}x|| = \langle \mathbf{A}x, \mathbf{A}x \rangle = \langle \mathbf{A}'\mathbf{A}x, x \rangle \\
&= \langle \sum_i \lambda_i P_i x, x \rangle = \sum_i \lambda_i \langle P_i x, x \rangle = \sum_i \lambda_i \langle P_i' P_i x, x \rangle \\
&= \sum_i \lambda_i \langle P_i x, P_i x \rangle = \sum_i \lambda_i ||P_i x|| \\
&\leq \sum_i \rho(\mathbf{A}'\mathbf{A})||P_i x|| \\
&= \rho(\mathbf{A}'\mathbf{A}) \sum_i ||P_i x|| \\
&= \rho(\mathbf{A}'\mathbf{A}) \left(\sum_i P_i x x \right) \\
&= \rho(\mathbf{A}'\mathbf{A})||x||
\end{aligned}$$

Exercise 2

From Gelfand's formula, we can say that for any $\epsilon > 0$, there exists a $K \in \mathbb{N}$ such that for any $k \geq K$, we have $|||A^k||^{\frac{1}{k}} - \rho(A)| < \epsilon$. Then, for any $k \geq K$:

$$\begin{aligned} \|A^k\|^{1/k} - \rho(A) &< \epsilon \\ \|A^k\|^{1/k} &< \rho(A) + \epsilon \\ \|A^k\| &< (\rho(A) + \epsilon)^k \end{aligned}$$

Now, we can let $r = \rho(A) + \epsilon < 1$. Then, $||A^k|| < r^k C$ where $C = 1$. Observe, using the properties of geometric sums and the fact that $||A^k||$ is finite:

$$\sum_{k=0}^{\infty} \|A^k\| = \sum_{k=0}^{K} \|A^k\| + \sum_{k=K+1}^{\infty} \|A^k\| < \sum_{k=0}^{K} \|A^k\| + \sum_{k=K+1}^{\infty} r^k = \sum_{k=0}^{K} \|A^k\| + \frac{r^{K+1}}{1-r} < \infty$$

Exercise 3

If A is diagonalizable, then it can be written as $A = PVP^{-1}$ where P is a matrix whose columns are the eigenvectors of A and form an orthonormal basis of $\mathbb{R}^N$, and V is a diagonalizable matrix whose diagonal elements are equal to the eigenvalues of A. Then:

$$\|A^k\| = \|(PVP^{-1})^k\| = \|PV^kP^{-1}\| \leq \|P\|\|V^k\|\|P^{-1}\| \leq \|P\|\|P^{-1}\|\|V\|^k$$

We can also see that:

$$\|V\| = \max_{s.t.\ \|x\|=1} \|Vx\| = \max_{s.t.\ \|x\|=1} \sqrt{(\lambda_1 x_1)^2 + \cdots + (\lambda_n x_n)^2} = \max_i \sqrt{\lambda_i^2} = \max_i |\lambda_i| = \rho(A)$$

Together, these imply:

$$\|A^k\| \leq \|P\|\|P^{-1}\|\|V\|^k = \|P\|\|P^{-1}\|\rho(A)^k$$

Since $\rho(A) < 1$, we can see that $\|A^k\| \to 0$ as $k \to \infty$.

Exercise 4

I will show that for any set of non-negative definite matrices, the limit point is also a non-negative definite matrix.

- Let M_k be a sequence of non-negative definite matrixes such that $M_k \to M$ and $k \to \infty$ using the matrix norm, $\|\cdot\|$, and for a contradiction, suppose that the limit point M is non-negative definite.
- We can say that $\|M_k - M\| < \varepsilon\ \forall k \geq K \in \mathbb{N}$. Moreover, we can say that for all $\varepsilon > 0$ and all $k \geq K$, there exists a $\delta > 0$ such that $\|M_k - M\| < \delta$ implies that $|z'M_k z - z'Mz| < \varepsilon$.

- Let $\varepsilon = \frac{z'Mz}{2}$. Then $|z'M_kz - z'Mz| < \frac{z'Mz}{2}$, which implies:
$$-\frac{z'Mz}{2} < z'M_kz - z'Mz < \frac{z'Mz}{2}$$
$$\frac{z'Mz}{2} < z'M_kz < \frac{3z'Mz}{2}$$
- But since we assumed $z'Mz < 0$, then $z'M_kz < 0$, which is a contradiction.
- Then the limit point must also be a non-negative definite matrix. This means that all the sequences in the space of non-negative definite matrices must have limit points that fall inside the space. So the space of nonnegative definite matrices is a closed subset of $(\mathcal{M}(n \times n), \|\cdot\|)$.

Exercise 5

Let L be the Lyapunov operator where $LX = AXA' + M$. L is a uniform contraction on $(\mathcal{M}(n \times n), \|\cdot\|)$, therefore it has a unique fixed point, X^*.

1. We know that $X^* = AX^*A' + M$ and we know that M is symmetric. We also know that for any $X^* \in \mathcal{M}(n \times n)$:
$$(X^*)' = (AX^*A' + M)' = (AX^*A')' + M' = A(X^*)'A' + M$$
So we have found another solution to the Lyapunov equation, $(X^*)'$, but we know that there is only a unique solution, therefore $X^* = (X^*)'$. Therefore, if M is symmetric, then X^* is also symmetric.

2. Let the space of nonnegative definite matrices be $\mathcal{ND}(n \times n)$. We proved in Exercise 4 that a nonnegative definite matrix B has the property that $z'Bz \geq 0$ for all $z \in \mathbb{R}^n/\{0\}$. Let M be a nonnegative definite matrix and consider the Lyapunov operator. For any $X \in \mathcal{M}(n \times n)$, AXA' is nonnegative definite. We can also say that $AXA' + M$ is nonnegative definite. Thus, if X is nonnegative definite, the Lyapunov operator is such that: $L : \mathcal{ND}(n \times n) \to \mathcal{ND}(n \times n)$. From Exercise 4, $\mathcal{ND}(n \times n)$ is a closed subset of $\mathcal{M}(n \times n)$. Applying the corollary to the contraction mapping theorem, we see that X^*, is contained in $\mathcal{ND}(n \times n)$. Therefore, if M is nonnegative definite then X^* is nonnegative definite.

3. Let the space of positive definite matrices br $\mathcal{P}(n \times n)$ and let M be a positive definite matrix. Consider the Lyapunov operator. Note that for any $X \in \mathcal{M}(n \times n)$, AXA' is nonnegative definite.

   Now, notice that $z'(AXA' + M)z = z'AXA'z + z'Mz$. We know that AXA' is nonnegative definite, so $z'AXA'z \geq 0 \implies z'AXA'z + z'Mz > 0$. Thus, we know that the Lyapunov operator takes the space of nonnegative definite matrices to the space of positive definite matrices $L : \mathcal{ND}(n \times n) \to \mathcal{P}(n \times n)$.

   From above, we know that $\mathcal{ND}(n \times n) \underset{closed}{\subset} \mathcal{M}(n \times n)$ and $\mathcal{P}(n \times n) \subset \mathcal{ND}(n \times n)$. This is because the set of positive definite matrices is just the set of nonnegative definite matrices minus all of the matrices that are indefinite i.e. $z'Mz = 0$ for some $z \in \mathcal{R}^n/\{0\}$.

   Applying the corollary to the contraction mapping theorem, $L(\mathcal{ND}(n \times n)) \subseteq \mathcal{P}(n \times n) \subset \mathcal{ND}(n \times n)$ implies that the fixed point, X^*, is contained in $\mathcal{P}(n \times n)$. Thus, M positive definite $\Rightarrow X^*$ positive definite.

Exercise 6

See my Jupyter notebook. In case the plot doesn't show up in the automatically generated .pdf, here it is.