Financial Economics 2 Homework 1

Analytical and Computational Solution of Long-run Risk Model

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1 Question 1: Analytical Solution of Risk Premium

1.1 Model Setup

In Long-run Risk model, the Epstein-Zin preference gives the following asset pricing function for any $R_{i,t+1}$ (including risk free rate $R_{f,t+1}$ and return to wealth $R_{a,t+1}$):

$$E_t\left[\delta^{\theta} G_{t+1}^{-\frac{\theta}{\psi}} R_{a,t+1}^{-(1-\theta)} R_{i,t+1}\right] = 1 \tag{1.1}$$

in which $\theta \equiv \frac{1-\gamma}{1-\frac{1}{\psi}}$ is a parameter, and $G_{t+1} =\equiv \frac{C_{t+1}}{C_t}$ is defined as the consumption growth.

The stochastic discount factor (SDF) and the logarithm SDF are:

$$M_t = \delta^{\theta} G_{t+1}^{-\frac{\theta}{\psi}} R_{a,t+1}^{-(1-\theta)} \tag{1.2}$$

$$m_{t+1} = \theta \log \delta - \frac{\theta}{\psi} g_{t+1} + (\theta - 1) r_{a,t+1} \tag{1.3}$$

The model further assume that the dividend growth rate and volatility is given by the following process:

$$\begin{aligned} g_{t+1} =& \mu + x_t + \sigma \eta_{t+1} \\ x_{t+1} =& \rho x_t + \varphi_e \sigma e_{t+1} \\ g_{d,t+1} =& \mu_d + \phi x_t + \varphi_d \sigma u_{t+1} \\ & w_{t+1}, e_{t+1}, \eta_{t+1} \sim N.i.i.d.(0,1) \end{aligned} \tag{1.4}$$

1.2 Campbell-Shiller Decomposition

In order to get analytical solution, we need to use the Campbell-Shiller Decomposition:

$$r_{i,t+1} = \kappa_{i,0} + \kappa_{i,1} z_{i,t+1} - z_{i,t} + g_{i,t+1} \tag{1.5}$$

in which $g_{i,t+1} = d_{t+1} - d_t = \log \frac{D_{t+1}}{D_t}$ stands for the logarithm dividend growth rate, and $z_{i,t} = p_t - d_t = \log \frac{P_t}{D_t}$ is the logarithm price to dividend ratio. The two parameter is given by

$$\begin{aligned}\kappa_{i,1} &= \frac{e^{\bar{z}_i}}{1 + e^{\bar{z}_i}} \\ \kappa_{i,0} &= -\log\left((1 - \kappa_{i,1})^{1-\kappa_{i,1}} \kappa_{i,1}^{\kappa_{i,1}}\right)\end{aligned} \tag{1.6}$$

One special case is the Campbell-Shiller Decomposition for the return of wealth:

$$r_{a,t+1} = \kappa_0 + \kappa_1 z_{t+1} - z_t + g_{t+1} \tag{1.7}$$

As we have shown during our class, $g_{t+1} = c_{t+1} - c_t = \log \frac{C_{t+1}}{C_t}$ is the logarithm consumption growth rate, and $z_t = \log \frac{P_t}{C_t}$ is the logarithm price to consumption ratio, and g_{t+1} is the consumption growth rate.

From the equation (1.1) we can get equation:

$$E_t \left\{ \exp \left[\theta \log(\delta) - \frac{\theta}{\psi} g_{t+1} + (\theta - 1) r_{a,t+1} + r_{i,t+1} \right] \right\} = 1 \tag{1.8}$$

1.3 Logarithm Consumption to Price Ratio z_t

A special case is take $r_{i,t+1}$ as $r_{a,t+1}$ we get:

$$E_t \left\{ \exp \left[\theta \log(\delta) - \frac{\theta}{\psi} g_{t+1} + \theta r_{a,t+1} \right] \right\} = 1 \tag{1.9}$$

Because we assume i.i.d. log-normal distribution of innovation, then naturally all the random variables in the above equation is jointly-normal distribution. We can rewrite the above equation using the feature of the lognormal distribution:

$$\exp \left\{ E_t \left[\theta \log(\delta) - \frac{\theta}{\psi} g_{t+1} + \theta r_{a,t+1} \right] + \frac{1}{2} var_t \left[\theta \log(\delta) - \frac{\theta}{\psi} g_{t+1} + \theta r_{a,t+1} \right] \right\} = 1 \tag{1.10}$$

Then we can spread out all the terms, and take log:

$$\begin{aligned}&\theta \log(\delta) - \frac{\theta}{\psi} E_t(g_{t+1}) + \theta E_t(r_{a,t+1}) \\ &+ \frac{\theta^2}{2\psi^2} var_t(g_{t+1}) + \frac{\theta^2}{2} var_t(r_{a,t+1}) - \frac{\theta^2}{\psi} cov_t(g_{t+1}, r_{a,t+1}) = 0\end{aligned} \tag{1.11}$$

In order to write the above conditional expectation, variance and covariance as a function of the state variable. We conjecture that the z_t (logarithm price to consumption ratio) take linear form in two state variable: the long-run risk shock x_t:

$$z_t = A_0 + A_1 x_t \tag{1.12}$$

Using this equation and equation 1.4 to write all the term expectation, variance and covariance in state variables $\{x_t, \sigma_t^2\}$:

$$E_t(g_{t+1}) = E_t[\mu + x_t + \sigma\eta_{t+1}] = \mu + x_t \tag{1.13}$$

$$var_t(g_{t+1}) = var_t(\sigma\eta_{t+1}) = \sigma^2 \tag{1.14}$$

$$\begin{aligned} E_t(r_{a,t+1}) =& E_t(K_0 + K_1 z_{t+1} - z_t + g_{t+1}) \\ =& K_0 + K_1 E_t(A_0 + A_1 x_{t+1}) - (A_0 + A_1 x_t) + (\mu + x_t) \\ =& K_0 + K_1 A_0 + K_1 A_1 \rho x_t - A_0 - A_1 x_t + (\mu + x_t) \end{aligned} \tag{1.15}$$

$$\begin{aligned} var_t(r_{a,t+1}) =& var_t(K_1 z_{t+1} + g_{t+1}) \\ =& var_t[K_1(A_0 + A_1 x_{t+1}) + (\mu + x_t + \sigma\eta_{t+1})] \\ =& var_t[K_1 A_1 \varphi_e \sigma e_{t+1} + \sigma\eta_{t+1}] \\ =& (K_1 A_1 \varphi_e)^2 \sigma^2 + \sigma^2 \end{aligned} \tag{1.16}$$

$$\begin{aligned} cov_t(g_{t+1}, r_{a,t+1}) =& cov_t(g_{t+1}, K_1 z_{t+1} + g_{t+1}) \\ =& var_t(g_{t+1}) + K_1 cov_t(g_{t+1}, z_{t+1}) \\ =& var_t(g_{t+1}) + K_1 cov_t(\sigma\eta_{t+1}, A_1 x_{t+1}) \\ =& var_t(g_{t+1}) + K_1 cov_t(\sigma\eta_{t+1}, A_1 \varphi\sigma e_{t+1}) \\ =& var_t(g_{t+1}) \\ =& \sigma^2 \end{aligned} \tag{1.17}$$

Take equation (1.13) - (1.17) to (1.11):

$$\begin{aligned} &\theta \log(\delta) - \frac{\theta}{\psi}(\mu + x_t) + \theta\{K_0 + K_1 A_0 + K_1 A_1 \rho x_t - A_0 - A_1 x_t + \mu + x_t\} \\ &+ \frac{\theta^2}{2\psi^2}\sigma^2 + \frac{\theta^2}{2}[(K_1 A_1 \varphi_e)^2 \sigma^2 + \sigma^2] - \frac{\theta^2}{\psi}\sigma^2 = 0 \end{aligned} \tag{1.18}$$

Because this equation have to hold true for all the state variable $\{x_t, \sigma_t^2\}$, we know that the coefficient before these two state variable must be zero. This fact gives rise to (1.19), and (1.20):

$$-\frac{\theta}{\psi}x_t + \theta\left[K_1 A_1 \rho x_t - A_1 x_t + x_t\right] = 0 \tag{1.19}$$

$$\Leftrightarrow \quad A_1 = \frac{1-\frac{1}{\psi}}{1-K_1\rho}$$

$$\theta(K_1 A_2 \mu_1 \sigma_t^2 - A_2 \sigma_t^2) + \frac{\theta^2}{2\psi}\sigma_t^2 + \frac{\theta^2}{2}[(K_1 A_1 \varphi_e)^2 \sigma_t^2 + \sigma_t^2] - \frac{\theta^2}{\psi}\sigma_t^2 = 0$$

$$\begin{aligned} &\theta\log(\delta) - \frac{\theta}{\psi}\mu + \theta\{K_0 + K_1 A_0 - A_0 + \mu\} \\ &+ \frac{\theta^2}{2\psi^2}\sigma^2 + \frac{\theta^2}{2}[(K_1 A_1 \varphi_e)^2 \sigma^2 + \sigma^2] - \frac{\theta^2}{\psi}\sigma^2 = 0 \end{aligned} \tag{1.20}$$

$$\log(\delta) - \frac{1}{\psi}\mu + K_0 + \mu + \frac{\theta}{2\psi^2}\sigma^2 + \frac{\theta}{2}[(K_1 A_1 \varphi_e)^2 \sigma^2 + \sigma^2] - \frac{\theta}{\psi}\sigma^2 = (1-K_1)A_0$$

$$A_0 = \frac{\log(\delta) + (1-\frac{1}{\psi})\mu + K_0 + \frac{\theta}{2}[(K_1 A_1 \varphi_e)^2 + (1-\frac{1}{\psi})^2]\sigma^2}{1-K_1}$$

Therefore, the log linearization of z_t is:

$$z_t = A_0 + A_1 x_t \tag{1.21}$$

where

$$A_1 = \frac{1-\frac{1}{\psi}}{1-K_1\rho} \tag{1.22}$$

$$A_0 = \frac{\log(\delta) + (1-\frac{1}{\psi})\mu + K_0 + \frac{\theta}{2}[(K_1 A_1 \varphi_e)^2 + (1-\frac{1}{\psi})^2]\sigma^2}{1-K_1} \tag{1.23}$$

1.4 Logarithm Risk Premium of Return of Wealth $E_t\left[r_{a,t+1} - r_{f,t}\right]$

Firstly, we can solve out the **Unexpected Shock to Return to Asset** $r_{a,t+1} - E_t(r_{a,t+1})$ as follow:

Use Campbell-Shiller Decomposition and take expectation:

$$r_{a,t+1} = K_0 + K_1 z_{t+1} - z_t + g_{t+1} \tag{1.24}$$

Then take conditional expectation:

$$E_t(r_{a,t+1}) = K_0 + K_1 E_t(z_{t+1}) - z_t + E_t(g_{t+1}) \tag{1.25}$$

Therefore,

$$\begin{aligned} & r_{a,t+1} - E_t(r_{a,t+1}) \\ & = K_1[z_{t+1} - E_t(z_{t+1})] + [g_t - E_{t(}g_{t+1})] \\ & = K_1 A_1 [x_{t+1} - E_t(x_{t+1})] + [g_t - E_{t(}g_{t+1})] \\ & = K_1 A_1 \varphi_e \sigma e_{t+1} + \sigma_t \eta_{t+1} \\ & = B\sigma e_{t+1} + \sigma \eta_{t+1} \end{aligned} \tag{1.26}$$

where B is a parameter defined as:

$$B = \kappa_1 A_1 \varphi_e = \kappa_1 \frac{\varphi_e}{1 - \kappa_1 \rho} \left(1 - \frac{1}{\psi}\right) \tag{1.27}$$

Then **Conditional Variance of the Return to Asset** is:

$$\begin{aligned} var_t(r_{a,t+1}) &= var_t[B\sigma e_{t+1} + \sigma_t \eta_{t+1}] \\ &= \left(1 + B^2\right) \sigma^2 \end{aligned} \tag{1.28}$$

Secondly, we can solve out the **Unexpected Shock to Logarithm SDF** as follow:

By definition

$$m_{t+1} \equiv \ln M_{t+1} = \theta \log \delta - \frac{\theta}{\psi} g_{t+1} + (\theta - 1) r_{a,t+1} \tag{1.29}$$

Take expectation and use equation (1.15) to simplify the expression:

$$\begin{aligned} E_t(m_{t+1}) =& \theta \log \delta - \frac{\theta}{\psi} E_t(g_{t+1}) + (\theta - 1) E_t(r_{a,t+1}) \\ =& \theta \log \delta - \frac{\theta}{\psi}(\mu + x_t) + (\theta - 1) E_t(r_{a,t+1}) \\ =& \theta \log \delta - \frac{\theta}{\psi}(\mu + x_t) + (\theta - 1)[K_0 + K_1 A_0 + K_1 A_1 \rho x_t - A_0 - A_1 x_t + (\mu + x_t)] \\ =& \theta \log \delta - \frac{\theta}{\psi}\mu + (\theta - 1)[K_0 + K_1 A_0 - A_0 + \mu] \\ & - \frac{\theta}{\psi} x_t + (\theta - 1)[A_1(K_1 \rho - 1) + 1] x_t \end{aligned}$$

For simplification, the we define the red part as m_0:

$$m_0 \equiv \theta \log \delta - \frac{\theta}{\psi}\mu + (\theta - 1)[K_0 + K_1 A_0 - A_0 + \mu]$$

For the blue part, we take into the expression of A_1 and get:

$$-\frac{\theta}{\psi} x_t + (\theta - 1)[A_1(K_1 \rho - 1) + 1] x_t = -\frac{x_t}{\psi} \tag{1.30}$$

Therefore the equation can be simplified as:

$$E_t\left[m_{t+1}\right] = m_0 - \frac{x_t}{\psi} \tag{1.31}$$

where

$$m_0 \equiv \theta \log \delta - \frac{\theta}{\psi}\mu + (\theta - 1)[K_0 + K_1 A_0 - A_0 + \mu] \tag{1.32}$$

Well, the m_{t+1} can be write as the following:

$$\begin{aligned}
m_{t+1} =& \theta \log \delta - \frac{\theta}{\psi} g_{t+1} + (\theta - 1) r_{a,t+1} \\
=& \theta \log \delta - \frac{\theta}{\psi}(\mu + x_t + \sigma \eta_{t+1}) + (\theta - 1)(K_0 + K_1 z_{t+1} - z_t + g_{t+1}) \\
=& \theta \log \delta - \frac{\theta}{\psi}(\mu + x_t + \sigma_t \eta_{t+1}) + (\theta - 1)\{K_0 \\
&+ K_1[A_0 + A_1 x_{t+1}] - (A_0 + A_1 x_t) + (\mu + x_t + \sigma \eta_{t+1})\} \\
=& \theta \log \delta - \frac{\theta}{\psi}(\mu + x_t + \sigma \eta_{t+1}) + (\theta - 1)\{K_0 \\
&+ K_1[A_0 + A_1(\rho x_t + \varphi_e \sigma e_{t+1})] - (A_0 + A_1 x_t) + (\mu + x_t + \sigma \eta_{t+1})\}
\end{aligned}$$

In order to simplify this equation, we notice that all the term except the innovation should be captured by $E_t m_{t+1}$, so we only need to write down the innovation term:

$$m_{t+1} = E_t m_{t+1} + \left(-\frac{\theta}{\psi} + \theta - 1\right)\sigma_t \eta_{t+1} + (\theta - 1)\left(A_1 K_1 \varphi_e\right)\sigma_t e_{t+1}$$

Therefore the **Unexpected Shock to Logarithm SDF** is:

$$\begin{aligned}
m_{t+1} - E_t\left(m_{t+1}\right) =& \left(-\frac{\theta}{\psi} + \theta - 1\right)\sigma_t \eta_{t+1} + (\theta - 1)\left(A_1 K_1 \varphi_e\right)\sigma_t e_{t+1} \\
=& \lambda_{m,\eta}\sigma_t \eta_{t+1} - \lambda_{m,e}\sigma_t e_{t+1}
\end{aligned} \tag{1.33}$$

where,

$$\lambda_{m,\eta} \equiv \left[-\frac{\theta}{\psi} + (\theta - 1)\right] = -\gamma \tag{1.34}$$

$$\lambda_{m,e} \equiv (1 - \theta) B \tag{1.35}$$

From the fundamental asset pricing function $E_t R_{i,t+1} M_{t+1} = 1$, we know that:

$$E_t[\exp r_{i,t+1} + m_{t+1}] = 1 \tag{1.36}$$

$$E_t[\exp r_{i,t+1} + m_{t+1}] = 1 \tag{1.37}$$

Take using the feature of jointly normal distribution and then take logarithm:

$$E_t(r_{i,t+1} + m_{t+1}) + 0.5var_t(r_{i,t+1} + m_{t+1}) = 0 \tag{1.38}$$

If r_i is the logarithm return to wealth r_a then:

$$E_t(r_{a,t+1} + m_{t+1}) + 0.5var_t(r_{a,t+1} + m_{t+1}) = 0$$

$$E_t(r_{a,t+1} + m_{t+1}) + 0.5var_t(r_{a,t+1}) + 0.5var_t(m_{t+1}) + cov_t(r_{a,t+1}, m_{t+1}) = 0 \tag{1.39}$$

If r_i is the logarithm risk free return r_f then:

$$E_t(r_{f,t+1} + m_{t+1}) + 0.5var_t(m_{t+1}) = 0 \tag{1.40}$$

By (1.39)-(1.40), we can write the **Risk Premium of Return to Wealth**:

$$\begin{aligned} E_t\left(r_{a,t+1} - r_{f,t}\right) &= -cov_t\left[m_{t+1}, r_{a,t+1}\right] - 0.5var_t\left(r_{a,t+1}\right) \\ &= -cov_t\left[m_{t+1} - E_t\left(m_{t+1}\right), r_{a,t+1} - E_t\left(r_{a,t+1}\right)\right] - 0.5var_t\left(r_{a,t+1}\right) \end{aligned} \tag{1.41}$$

Then by using equation (1.26), (1.28) and (1.33), we can further simplify the above equation:

$$E_t\left[r_{a,t+1} - r_{f,t}\right] = -\lambda_{m,\eta}\sigma_t^2 + \lambda_{m,e}B\sigma_t^2 - 0.5var_t\left(r_{a,t+1}\right)$$

$$E_t\left[r_{a,t+1} - r_{f,t}\right] = -\lambda_{m,\eta}\sigma_t^2 + \lambda_{m,e}B\sigma_t^2 - 0.5\left(1 + B^2\right)\sigma^2 \tag{1.42}$$

1.5 Logarithm Price to Dividend Ratio: z_m

Another special case of equation (1.8) is take r_i as r_m:

$$E_t\left\{\exp\left[\theta\log(\delta) - \frac{\theta}{\psi}g_{t+1} + (\theta-1)r_{a,t+1} + r_{m,t+1}\right]\right\} = 1 \tag{1.43}$$

therefore,

$$\begin{aligned} &E_t\left[\theta\log(\delta) - \frac{\theta}{\psi}g_{t+1} + +(\theta-1)r_{a,t+1} + r_{m,t+1}\right] + \\ &0.5var_t\left[\theta\log(\delta) - \frac{\theta}{\psi}g_{t+1} + (\theta-1)r_{a,t+1} + r_{m,t+1}\right] = 0 \end{aligned} \tag{1.44}$$

further spread out the expression,

$$\begin{aligned}
&\theta\log(\delta) - \frac{\theta}{\psi}E_t(g_{t+1}) + (\theta-1)E_t(r_{a,t+1}) + E_t(r_{m,t+1}) \\
&+ \frac{\theta}{2\psi^2}var_t(g_{t+1}) + \frac{(\theta-1)^2}{2}var_t(r_{a,t+1}) + \frac{1}{2}var_t(r_{m,t+1}) \\
&- \frac{\theta(\theta-1)}{\psi}cov_t(g_{t+1}, r_{a,t+1}) - \frac{\theta}{\psi}cov_t(g_{t+1}, r_{m,t+1}) + (\theta-1)cov_t(r_{a,t+1}, r_{m,t+1}) = 0
\end{aligned} \tag{1.45}$$

Using the expression (1.15)-(1.17) and the following relationships:

$$\begin{aligned}
E_t(r_{m,t+1}) =& E_t(K_{0,m} + K_{1,m}z_{t+1,m} - z_{t,m} + g_{t+1,d}) \\
=& K_{0,m} + K_{1,m}E_t(A_{0,m} + A_{1,m}x_{t+1}) - (A_{0,m} + A_{1,m}x_t) + (\mu_d + \phi x_t) \\
=& K_{0,m} + K_{1,m}A_{0,m} + K_{1,m}A_{1,m}\rho x_t - A_{0,m} - A_{1,m}x_t + \mu_d + \phi x_t
\end{aligned} \tag{1.46}$$

$$\begin{aligned}
var_t(r_{m,t+1}) =& var_t(K_{1,m}z_{t+1,m} + g_{t+1,d}) \\
=& var_t[K_{1,m}(A_{0,m} + A_{1,m}x_{t+1}) + (\mu_d + \phi x_t + \varphi_d\sigma u_{t+1})] \\
=& var_t[K_{1,m}A_{1,m}\varphi_e\sigma e_{t+1} + \varphi_d\sigma u_{t+1}] \\
=& (K_{1,m}A_{1,m}\varphi_e)^2\sigma^2 + \varphi_d^2\sigma^2
\end{aligned} \tag{1.47}$$

$$\begin{aligned}
cov_t(g_{t+1}, r_{m,t+1}) =& cov_t(g_{t+1}, K_{1,m}z_{t+1,m} + g_{t+1,d}) \\
=& K_{1,m}cov_t(g_{t+1}, z_{t+1,m}) + cov_t(g_{t+1}, g_{t+1,d}) \\
=& K_{1,m}cov_t(g_{t+1}, A_{0,m} + A_{1,m}x_{t+1}) \\
=& K_{1,m}A_{1,m}cov_t(g_{t+1}, x_{t+1}) \\
=& 0
\end{aligned} \tag{1.48}$$

$$\begin{aligned}
cov_t(r_{a,t+1}, r_{m,t+1}) =& cov_t(K_1z_{t+1} + g_{t+1}, K_{1,m}z_{t+1,m} + g_{t+1,d}) \\
=& K_1K_{1,m}cov_t(z_{t+1}, z_{t+1,m}) + K_1cov_t(z_{t+1}, g_{t+1,d}) \\
&+ K_{1,m}cov_t(g_{t+1}, z_{t+1,m}) + cov_t(g_{t+1}, g_{t+1,d}) \\
=& K_1K_{1,m}A_1A_{1,m}var_t(x_{t+1}) + 0 + 0 + 0 \\
=& K_1K_{1,m}A_1A_{1,m}\varphi_e^2\sigma^2
\end{aligned} \tag{1.49}$$

Using the expression (1.15)-(1.17) and (1.46)-(1.49) to simplify (1.50):

further spread out the expression,

$$\begin{aligned}
&\theta\log(\delta) - \frac{\theta}{\psi}(\mu + x_t) \\
&+ (\theta-1)[K_0 + K_1A_0 + K_1A_1\rho x_t - A_0 - A_1x_t + \mu + x_t] \\
&+ [K_{0,m} + K_{1,m}A_{0,m} + K_{1,m}A_{1,m}\rho x_t - A_{0,m} - A_{1,m}x_t + \mu_d + \phi x_t] \\
&+ \frac{\theta}{2\psi^2}\sigma^2 + \frac{(\theta-1)^2}{2}[(K_1A_1\varphi_e)^2\sigma^2 + \sigma^2] + \frac{1}{2}[(K_{1,m}A_{1,m}\varphi_e)^2\sigma^2 + \varphi_d^2\sigma^2] \\
&- \frac{\theta(\theta-1)}{\psi}\sigma^2 + (\theta-1)K_1K_{1,m}A_1A_{1,m}\varphi_e^2\sigma^2 = 0
\end{aligned} \tag{1.50}$$

Therefore, we can collect all the term and get the following equation:

$$\begin{aligned}
&\theta\log(\delta) - \frac{\theta}{\psi}\mu \\
&+(\theta-1)[K_0 + K_1A_0 - A_0 + \mu] \\
&+ [K_{0,m} + K_{1,m}A_{0,m} - A_{0,m} + \mu_d] \\
&+\frac{\theta}{2\psi^2}\sigma^2 + \frac{(\theta-1)^2}{2}[(K_1A_1\varphi_e)^2\sigma^2 + \sigma^2] + \frac{1}{2}[(K_{1,m}A_{1,m}\varphi_e)^2\sigma^2 + \varphi_d^2\sigma^2] \\
&-\frac{\theta(\theta-1)}{\psi}\sigma^2 + (\theta-1)K_1K_{1,m}A_1A_{1,m}\varphi_e^2\sigma^2 = 0
\end{aligned} \tag{1.51}$$

In order to simplify the expression, we define the red part as A, and blue part as C:

$$A_{0,m} = \frac{A + C + K_{0,m} + \mu_d}{1 - K_{1,m}}$$

$$A_{1,m} = \frac{\phi - \frac{1}{\psi}}{1 - K_{1,m}\rho}$$

Therefore, the expression for Logarithm Market Price to Dividend ratio z_m:

$$z_{t,m} = A_{0,m} + A_{1,m}x_t \tag{1.52}$$

where

$$A_{0,m} = \frac{A + C + K_{0,m} + \mu_d}{1 - K_{1,m}} \tag{1.53}$$

$$A_{1,m} = \frac{\phi - \frac{1}{\psi}}{1 - K_{1,m}\rho} \tag{1.54}$$

$$A \equiv \theta\log(\delta) - \frac{\theta}{\psi}\mu + (\theta-1)[K_0 + K_1A_0 - A_0 + \mu] \tag{1.55}$$

$$
\begin{aligned}
C \equiv & \frac{\theta}{2\psi^2}\sigma^2 + \frac{(\theta-1)^2}{2}[(K_1 A_1 \varphi_e)^2\sigma^2 + \sigma^2] + \frac{1}{2}[(K_{1,m}A_{1,m}\varphi_e)^2\sigma^2 + \varphi_d^2\sigma^2] \\
& - \frac{\theta(\theta-1)}{\psi}\sigma^2 + (\theta-1)K_1 K_{1,m} A_1 A_{1,m}\varphi_e^2\sigma^2
\end{aligned} \tag{1.56}
$$

1.6 Logarithm Risk Premium for Market Return $E_t\left(r_{m,t+1} - r_{f,t}\right)$

By using the same method we used to calculate the logarithm risk premium for return to wealth, we can calculate the logarithm risk premium for market return:

$$
var_t\left(r_{m,t+1}\right) = \left(\beta_{m,e}^2 + \varphi_d^2\right)\sigma_t^2 \tag{1.57}
$$

$$
\begin{aligned}
E_t\left(r_{m,t+1} - r_{f,t}\right) &= -cov_t\left[m_{t+1}, r_{m,t+1}\right] - 0.5var_t\left(r_{m,t+1}\right) \\
&= -cov_t\left[m_{t+1} - E_t\left(m_{t+1}\right), r_{m,t+1} - E_t\left(r_{m,t+1}\right)\right] - 0.5var_t\left(r_{m,t+1}\right) \\
&= \beta_{m,e}\lambda_{m,e}\sigma_t^2 + \beta_{m,w}\lambda_{m,w}\sigma_w^2 - 0.5var_t\left(r_{m,t+1}\right) \\
&= \beta_{m,e}\lambda_{m,e}\sigma_t^2 + \beta_{m,w}\lambda_{m,w}\sigma_w^2 - 0.5\left(\beta_{m,e}^2 + \varphi_d^2\right)\sigma_t^2
\end{aligned} \tag{1.58}
$$

1.7 Logarithm Risk Free Return r_f

For risk free rate:

$$
\begin{aligned}
r_{f,t} = & - E_t[m_{t+1}] - 0.5var_t(m_{t+1}) \\
= & - A - \left[(\theta-1)(K_1 A_1 \rho - A_1 + 1) - \frac{\theta}{\psi}\right] x_t \\
& - \frac{\sigma^2}{2}\left\{[(\theta-1) - \frac{\theta}{\psi}]^2 + [(\theta-1)K_1 A_1 \varphi_e]^2\right\}
\end{aligned} \tag{1.59}
$$

2 Question 2: Numerically Solving out z_a and z_m

2.1 Solving out $z_{a,t} = f(x_t)$ Numerically

2.1.1 Policy Function Iteration

The key algorithm we need to use in this question is policy function iteration, we first need to arrive at the iteration function.

$$\begin{aligned} R_{a,t+1} &= \frac{W_{t+1}}{W_t - C_t} = \frac{W_{t+1}}{W_t} \cdot \frac{W_t}{W_t - C_t} \\ &= \frac{W_{t+1}}{C_{t+1}} \cdot \frac{C_{t+1}}{C_t} \cdot \frac{1}{\frac{W_t}{C_t} - 1} \end{aligned} \tag{2.1}$$

Then we can take this equation into the fundamental asset pricing equation:

$$\begin{aligned} 1 &= E_t \left[\delta^\theta \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{\theta}{\psi}} (R_{a,t+1})^{\theta-1} R_{i,t+1} \right] \\ &= E_t \left[\delta^\theta \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{\theta}{\psi}} R_{a,t+1}^\theta \right] \\ &= E_t \left[\delta^\theta \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{\theta}{\psi}} \left(\frac{W_{t+1}}{C_{t+1}} \right)^\theta \left(\frac{C_{t+1}}{C_t} \right)^\theta \cdot \left(\frac{1}{\frac{W_t}{C_t} - 1} \right)^\theta \right] \end{aligned} \tag{2.2}$$

Then we can spread out (2.2) using (2.1), we get:

$$\frac{W_t}{C_t} = \delta \left\{ E_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{\theta(1-\frac{1}{\psi})} \left(\frac{W_{t+1}}{C_{t+1}} \right)^\theta \right] \right\}^{\frac{1}{\theta}} + 1 \tag{2.3}$$

Furthermore, because the the growth rate of C is independent from other parts in the expectation, we can take it outside the equation:

$$\underbrace{\frac{W_t}{C_t}}_{f_n(x_t)} = \underbrace{\delta e^{\left(1-\frac{1}{\psi}\right)(\mu+x_t)+\frac{1}{2}\theta\left(1-\frac{1}{\psi}\right)^2\sigma^2}}_{constant} \left\{ E_t \left[\left(\underbrace{\frac{W_{t+1}}{C_{t+1}}}_{f_{n-1}(x_{t+1})} \right)^\theta \right] \right\}^{\frac{1}{\theta}} + 1 \tag{2.4}$$

2.1.2 Program and Algorithm

We can use the above equation to iterate, where $\frac{W}{C}$ is wealth to consumption ratio. According to fixed point theorem, if we keep on calculate the left hand side of the equation and take back to the right hand side infinity times, the function will finally converge to the solution.

The program contain the following pieces of codes:

- *Run_Question2.m*: This is the main program that calculate the result using policy function iteration method. It also contain code to plot the results;

- *rouwen.m*: This is a function to approximate the AR(1) process using the Rouwenhorst (1995). The function generate grid point of long-run risk x and the transition matrix;
- *loglinearLogPcRatio.m*: This is a function calculate the analytical result using log-linearization method proposed in Bensal and Yaron (2004).

2.1.3 Result

The first plot is the result if we take Number of Grid = 500; Tolerance = 10^{-5}. It takes the MATLAB 5 seconds to calculate the result, but the approximation is not very good.

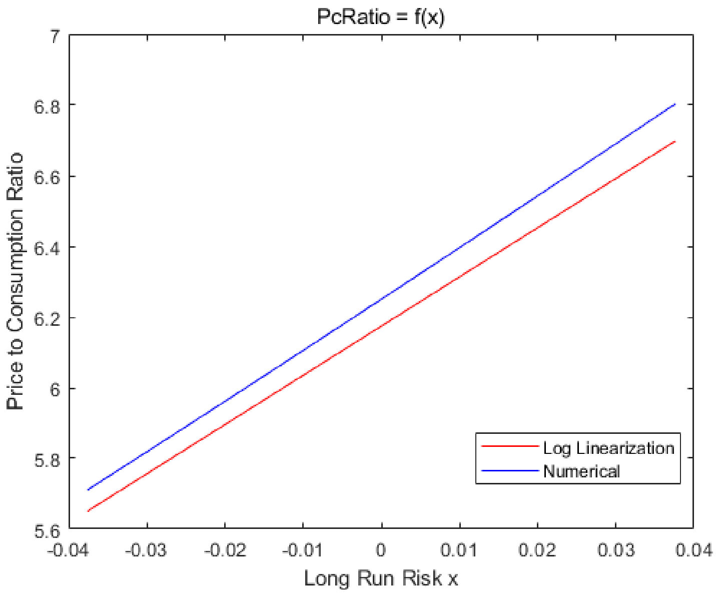

Figure 2.1: Logarithm PC Ratio: Number of Grid = 500; Tolerance = 10^{-5}

Then, we take Number of Grid = 5000; Tolerance = 10^{-10}. It takes the MATLAB about 1.5 hour to calculate the result, and we can see these two lines are much closer than before.

This is probability because the Rouwenhorst (1995) requires a large number of grid point, but it also makes the computation much slower. We can conjecture that two line will finally converge and cut each other at $x = 0$.

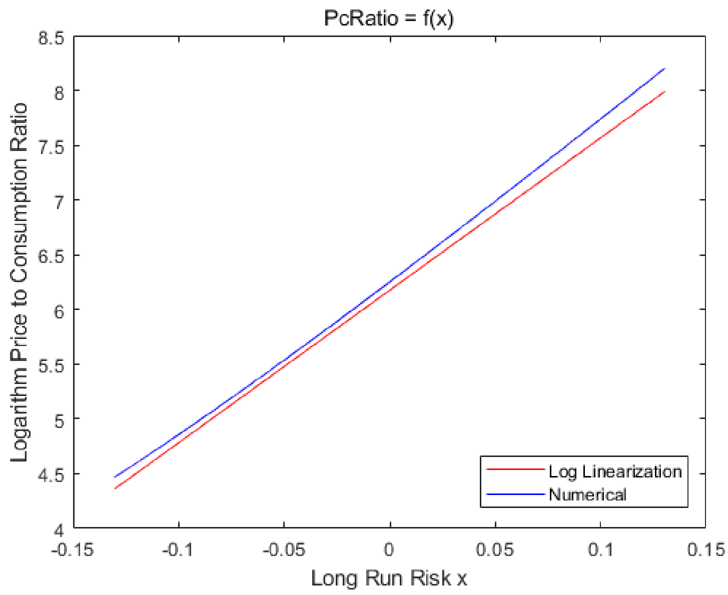

Figure 2.2: Logarithm PC Ratio: Number of Grid = 5000; Tolerance = 10^{-10}

2.2 Solving out $z_{m,t} = g(x_t)$ Numerically

Similarly we can decompose $R_{m,t+1}$

$$R_{m,t+1} = \frac{D_{t+1} + P_{t+1}}{P_t} = \frac{D_{t+1} + P_{t+1}}{D_{t+1}} \cdot \frac{D_t}{P_t} \cdot \frac{D_{t+1}}{D_t} = \left(1 + \frac{P_{t+1}}{D_{t+1}}\right) \cdot \frac{D_t}{P_t} \cdot \frac{D_{t+1}}{D_t} \tag{2.5}$$

Using the fundamental asset pricing equation we can get:

$$1 = E_t\left[\delta^\theta \left(\frac{C_{t+1}}{C_t}\right)^{-\frac{\theta}{\psi}} (R_{a,t+1})^{\theta-1} R_{i,t+1}\right] \tag{2.6}$$

$$\frac{P_t}{D_t} = E_t\left[\delta^\theta \left(\frac{C_{t+1}}{C_t}\right)^{-\frac{\theta}{\psi}} (R_{a,t+1})^{\theta-1} \left(1 + \frac{P_{t+1}}{D_{t+1}}\right) \cdot \frac{D_{t+1}}{D_t}\right] \tag{2.7}$$

Then take in (2.1):

$$\frac{P_t}{D_t} = E_t\left[\delta^\theta \left(\frac{C_{t+1}}{C_t}\right)^{-\frac{\theta}{\psi}} \left(\frac{W_{t+1}}{C_{t+1}} \cdot \frac{C_{t+1}}{C_t} \cdot \frac{1}{\frac{W_t}{C_t} - 1}\right)^{\theta-1} \left(1 + \frac{P_{t+1}}{D_{t+1}}\right) \cdot \frac{D_{t+1}}{D_t}\right] \tag{2.8}$$

$$\frac{P_t}{D_t} = \frac{1}{\left(\frac{W_t}{C_t} - 1\right)^{\theta-1}} E_t\left[\delta^\theta \left(\frac{C_{t+1}}{C_t}\right)^{-\frac{\theta}{\psi}} \left(\frac{W_{t+1}}{C_{t+1}} \cdot \frac{C_{t+1}}{C_t}\right)^{\theta-1} \left(1 + \frac{P_{t+1}}{D_{t+1}}\right) \cdot \frac{D_{t+1}}{D_t}\right] \tag{2.9}$$

$$\frac{P_t}{D_t} = \frac{1}{\left(\frac{W_t}{C_t} - 1\right)^{\theta-1}} E_t\left[\delta^\theta \left(\frac{C_{t+1}}{C_t}\right)^{\theta-1-\frac{\theta}{\psi}} \left(\frac{W_{t+1}}{C_{t+1}}\right)^{\theta-1} \left(1 + \frac{P_{t+1}}{D_{t+1}}\right) \cdot \frac{D_{t+1}}{D_t}\right] \tag{2.10}$$

$$\frac{P_t}{D_t} = \frac{\delta^\theta e^{\left(\left(\theta-1-\frac{\theta}{\psi}\right)(\mu+x_t)+\frac{1}{2}\left(\theta-1-\frac{\theta}{\psi}\right)^2\sigma^2\right)} e^{\left(\mu_d+\phi x_t+\frac{1}{2}\varphi_d^2\sigma^2\right)}}{\left(\frac{W_t}{C_t}-1\right)^{\theta-1}} E_t\left[\left(\frac{W_{t+1}}{C_{t+1}}\right)^{\theta-1}\left(1+\frac{P_{t+1}}{D_{t+1}}\right)\right] \tag{2.11}$$

$$\underbrace{\frac{P_t}{D_t}}_{g_n(x_t)} = \underbrace{\frac{\delta^\theta e^{\left(\left(\theta-1-\frac{\theta}{\psi}\right)(\mu+x_t)+\frac{1}{2}\left(\theta-1-\frac{\theta}{\psi}\right)^2\sigma^2\right)} e^{\left(\mu_d+\phi x_t+\frac{1}{2}\varphi_d^2\sigma^2\right)}}{\left(\frac{W_t}{C_t}-1\right)^{\theta-1}}}_{Constant} E_t\left[\left(\frac{W_{t+1}}{C_{t+1}}\right)^{\theta-1}\left(1+\underbrace{\frac{P_{t+1}}{D_{t+1}}}_{g_{n-1}(x_{t+1})}\right)\right] \tag{2.12}$$

2.2.1 Program and Algorithm

We can use the above equation to iterate, where $\frac{P}{D}$ is price to dividend ratio, and $\frac{W}{C}$ is wealth to consumption ratio we have already solved out.

The program contain the following pieces of codes:

- *Run_Question2.m*: This is the main program that calculate the result using policy function iteration method. It also contain code to plot the results;
- *rouwen.m*: This is a function to approximate the AR(1) process using the Rouwenhorst (1995). The function generate grid point of long-run risk x and the transition matrix;
- *loglinearLogPdRatio.m*: This is a function calculate the analytical result using log-linearization method proposed in Bensal and Yaron (2004).

2.2.2 Result

As we can see from the result, the approximation for logarithm price to dividend ratio is good near $x = 0$.

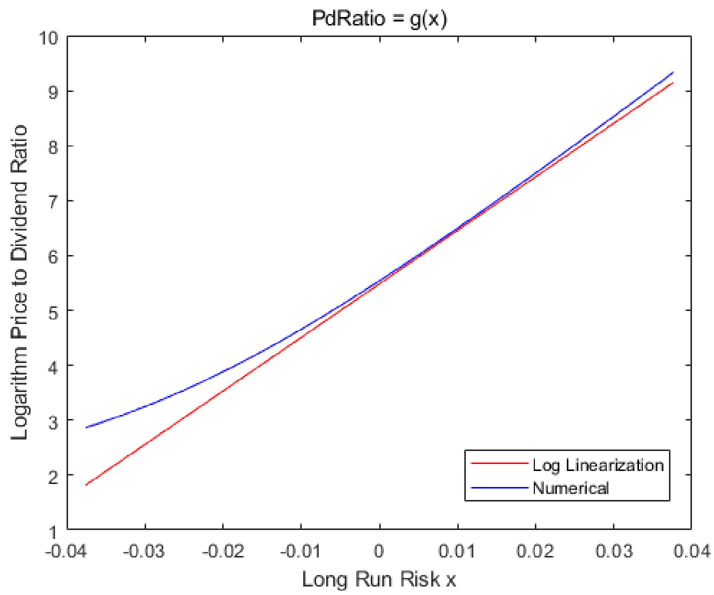

Figure 2.3: Logarithm PD Ratio (Number of Grid = 500; Tolerance = 10^{-5})

3 Question 3: Moments by Simulation and Log-Linearization

For risk free rate:

$$\begin{aligned} r_{f,t} =& - E_t[m_{t+1}] - 0.5var_t(m_{t+1}) \\ =& - A - \left[(\theta - 1)(K_1 A_1 \rho - A_1 + 1) - \frac{\theta}{\psi}\right] x_t \\ & - \frac{\sigma^2}{2}\left\{[(\theta - 1) - \frac{\theta}{\psi}]^2 + [(\theta - 1)K_1 A_1 \varphi_e]^2\right\} \end{aligned} \tag{3.1}$$

For maket return:

$$r_{m,t+1} = g_{d,t+1} + K_1 A_{1,m} x_{t+1} - A_{1,m} x_t \tag{3.2}$$

```
>> Run_Question3
Simulated Risk Free Rate Mean = 2.637189
Simulated Risk Free Rate Std= 0.475787
Simulated Market Rate Mean= 6.598017
Simulated Market Rate Std = 17.066688
Simulated Market Risk Premium= 3.960827
Simulated Market Risk Premium= 0.100255
```

```
>> Run_Question3
Simulated Risk Free Rate Mean = 2.637202
Simulated Risk Free Rate Std= 0.475495
Simulated Market Rate Mean= 7.407580
Simulated Market Rate Std = 18.877049
Simulated Market Risk Premium= 4.770378
Simulated Market Risk Premium= 0.121693
```

```
>> Run_Question3
Simulated Risk Free Rate Mean = 3.018339
Simulated Risk Free Rate Std= 0.000000
Simulated Market Rate Mean= 2.265580
Simulated Market Rate Std = 12.152555
Simulated Market Risk Premium= -0.752759
Simulated Market Risk Premium= 0.000000
```