Interest Rate Liberalization and Capital Misallocations (Liu et al., 2021) Numerical Replication

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1 Abstract

This project aims to replicate the model simulation result of Liu et al. (2021) *Interest Rate Liberalization and Capital Misallocations.* In this paper, the authors build a tractable heterogeneous-agent model to analysis the consequence of interest rate liberalization, a major reform of Chinese financial system in 2010s. Astonishingly, the model suggests that interest rate liberalization decrease social welfare by worsening the capital misallocations toward non-productive state owned enterprises (SOEs). Therefore, authors point out that SOE reform should be conducted before the liberalization of interest rate.

In addition to its seminal policy implication, this paper is a good example of heterogeneous agent model with clear analytical solution, and the aggregation techniques might shed light on future research on macroeconomics.

2 Static Model: Setup and Analytical Result

2.1 Model Setups:

- Firms: SOE sector (denote as s), and POE sector (denote as p);
  - There are μ measure of SOEs, with sector specific average productivity z^s, but they get τz^s due to subsidy ($\tau > 1$);
  - There are $1 - \mu$ measure of POEs, with sector specific average productivity z^p;
  - Assume $z^p > z^s$: the private firms are more productive;
  - Firms are heterogeneous in sector s, p, and a distribution of firm specific productivity ε which follow CDF $\varepsilon \sim \mathbf{F}^p(\varepsilon)$ in POEs and $\varepsilon \sim \mathbf{F}^s(\varepsilon)$ in SOEs;
  - Each firm born with initial capital (wealth) h.
- Financial Market:
  - Firms can borrow b at r_l and save s at r_d;
  - $r_l = r_d + \phi$, where $\phi \geq 0$ is rate difference exogenously set by central bank.
- Credit Constrain:
  - Borrowing cannot exceed θ fraction of initial capital, and $\theta^p < \theta^s$: SOEs are favored by the financial system so that POEs are more credit constrained;
  - Saving cannot exceed the initial capital amount.

The POE problem:

$$\max_{k^p(\varepsilon), l^p(\varepsilon), s^p(\varepsilon)} z^p \varepsilon k^p(\varepsilon) - r_l l^p(\varepsilon) + r_d s^p(\varepsilon) \tag{2.1}$$

Subject to flow of fund constrain:

$$k^p(\varepsilon) = h + l^p(\varepsilon) - s^p(\varepsilon) \tag{2.2}$$

and the credit constrains:

$$0 \leq l^p(\varepsilon) \leq \theta^p h \tag{2.3}$$

$$0 \leq s^p(\varepsilon) \leq h \tag{2.4}$$

The SOE problem:

$$\max_{k^s(\varepsilon), l^s(\varepsilon), s^s(\varepsilon)} \tau z^s \varepsilon k^s(\varepsilon) - r_l l^s(\varepsilon) + r_d s^s(\varepsilon) \tag{2.5}$$

Subject to flow of fund constrain:

$$k^s(\varepsilon) = h + l^s(\varepsilon) - s^s(\varepsilon) \tag{2.6}$$

and the credit constrains:

$$0 \leq l^s(\varepsilon) \leq \theta^s h \tag{2.7}$$

$$0 \leq s^s(\varepsilon) \leq h \tag{2.8}$$

Definition of Equilibrium:
Equilibirum consists of:

- Interest rate $\{r_d, r_l\}$;
- Resources allocation for SOEs $\{k^s(\varepsilon), l^s(\varepsilon), s^s(\varepsilon)\}$ and for POEs $\{k^p(\varepsilon), l^p(\varepsilon), s^p(\varepsilon)\}$;

such that:

- SOEs and POEs solve their maximization problem ;
- Capital market clear:

$$\mu \int k^s(\varepsilon) d\mathbf{F}^s(\varepsilon) + (1-\mu) \int k^p(\varepsilon) d\mathbf{F}^p(\varepsilon) = h \tag{2.9}$$

Aggregate output:

$$Y = \mu \int z^s \varepsilon k^s(\varepsilon) d\mathbf{F}^s(\varepsilon) + (1-\mu) \int z^p \varepsilon k^p(\varepsilon) d\mathbf{F}^p(\varepsilon) \tag{2.10}$$

2.2 Analytical Result with Homogeneous Firms

Proposition 1: Homothetic firms, with interest rate wedges:
Assume that $\tau z^s > z^p > z^s$ and $\phi > \tau z^s - z^p$,. The only equilibrium is in the economy is one with **financial autarky** (no firms save and no firms borrow), with $r_d \in [\tau z^s - \phi, z^p]$. Equilibrium capital is given by:

$$k^p = k^s = h \tag{2.11}$$

Equilibrium output is given by:

$$Y = (1-\mu)z^p h + \mu z^s h \tag{2.12}$$

Proposition 2: Homothetic firms, no interest rate wedges:
In the liberalized economy with $\phi = 0$, the deposit rate and lending rate are identical and the equilibrium interest rate lies in the interval $r^d = r^s \in [z^p, \tau z^s]$. Equilibrium capital is given by:

$$k^p = \frac{1-\mu\left(1+\theta^s\right)}{1-\mu} h < h \tag{2.13}$$

$$k^s = \left(1+\theta^s\right) h > h \tag{2.14}$$

Equilibrium output is given by:

$$Y = \begin{cases} z^p h - \left(z^p - z^s\right)\mu\left(1+\theta^s\right) h & \text{if } \theta^s < 1/\mu - 1 \\ z^s h & \text{if } \theta^s \geq 1/\mu - 1 \end{cases} \tag{2.15}$$

which is lower than the level givn by proposition 1.

Intuition of Proposition 1 & 2:

Proposition 1 and 2 directly point out the possibility that **liberalization of the interest rate may decrease output and welfare**, which is counter-intuitive. This is because there is a distortion due to subsidy of SOE: although the return of capital of SOEs z^s are lower than the POEs z^p, they can get more $\tau z^s > z^p$. This distortion makes the SOEs tend to borrow in the financial market and POEs tend to save in the market.

When there is no interest rate wedge ϕ, the SOEs are borrower and POEs are lender. If the credit constrain is tight $\theta^s < 1/\mu - 1$, then SOEs will expand their production by borrowing the money from the POEs until they the borrowing hit the credit constrain. On the other hand, if the credit constrain is loose $\theta^s \geq 1/\mu - 1$ then SOEs will borrow all the money from the POEs (and they are not credit constrained): Therefore, the only production firms are SOEs.

However, when the interest rate wedge is large (as before the Chinese interest rate liberalization), given that $\phi > \tau z^s - z^p$, the borrowing and saving rate wedge is even larger than the return difference of SOEs and POEs. Therefore, SOEs will not borrowing from the POEs, because the return difference $\tau z^s - z^p$ is not enough to coverage the interest rate wedge ϕ. Thus, SOEs will not borrowing and POEs will not lending, with the economy ending up with financial autarky.

Because the productivity of POEs are higher than SOEs, any lending from POEs to SOEs will decrease the total output. Therefore, the interest rate liberalization is bad for the production.

2.3 Analytical Result with Heterogeneous Firms

Proposition 3: Heterogeneous Firms In the economy with heterogeneous firms, there exist two threshold level of idiosyncratic productivity, denoted by $\underline{\varepsilon}^J$ and $\bar{\varepsilon}^J$ for each sector $j \in \{s, p\}$, such that:

$$s^j(\varepsilon) = \begin{cases} h & \text{if } \varepsilon_{\min} \leq \varepsilon < \underline{\varepsilon}^j \\ 0 & \text{if } \underline{\varepsilon}^j \leq \varepsilon \end{cases} \tag{2.16}$$

$$l^j(\varepsilon) = \begin{cases} 0 & \text{if } \varepsilon_{\min} \leq \varepsilon < \bar{\varepsilon}^j \\ \theta^j h & \text{if } \bar{\varepsilon}^j \leq \varepsilon \end{cases} \tag{2.17}$$

$$k^j(\varepsilon) = \begin{cases} 0 & \text{if } \varepsilon_{\min} \leq \varepsilon < \underline{\varepsilon}^j \\ h & \text{if } \underline{\varepsilon}^j \leq \varepsilon < \bar{\varepsilon}^j \\ \left(1+\theta^j\right) h & \text{if } \bar{\varepsilon}^j \leq \varepsilon \end{cases} \tag{2.18}$$

The thresholds $\underline{\varepsilon}^J$ and $\bar{\varepsilon}^J$ are defined as:

$$\underline{\varepsilon}^j = \frac{r_d}{z^j \tau^j} \tag{2.19}$$

$$\bar{\varepsilon}^j = \frac{r_d + \phi}{z^j \tau^j} \tag{2.20}$$

Where τ^j denotes the output subsidy rate for sector j, and τ^p is normalized to 1.

Intuition of Proposition 3:

In order to characterized the equilibrium, we need to know two set of thresholds $\{\underline{\varepsilon}^s, \bar{\varepsilon}^s\}$ for SOEs and the $\{\underline{\varepsilon}^p, \bar{\varepsilon}^p\}$ for POEs.

In this model, capital is the only factor of production and the model assume constant return to scale: this the the feature of $Y = AK$ model. Therefore, the firms with productivity higher than the upper threshold will borrow until the credit constrain is tight, because their MPK is always larger than the borrowing rate. The firms with productivity lower than the lower threshold will save out all the initial wealth, because their MPK is always lower than the saving rate. Due to the interest rate wedge $\phi > 0$, firms in the middle will not save, because their MPK is higher than saving rate, but they will also not borrowing because their MPK is lower than the lending rate. Therefore, they will use their own wealth to produce. These middle firms seems like the 'individual workshops' in real economic world, which can be potentially explained by this theory.

Notice that:

- Both threshold is lower in SOEs sectors, which imply that SOEs are more likely to produce and borrow, compare to POE with same productivity level;
- Given ϕ, a increase in r_d will leads to more saving and less borrowing, so that there exist unique endogenous r_d that solve the (**??**) $\Rightarrow$ the equilibrium is unique;
- A increase in the ϕ will increases the upper threshold and decreases the lower threshold for both SOEs and POEs.

Simplified market clearing condition

The market clearing condition (2.9) can be written as following:

$$K = (1-\mu)K^p + \mu K^s = h \tag{2.21}$$

where K^p and K^s are

$$K^j = \left[\int_{\underline{\varepsilon}^j}^{\bar{\varepsilon}^j} d\mathbf{F}(\varepsilon) + \left(1+\theta^j\right)\int_{\bar{\varepsilon}^j}^{\infty} d\mathbf{F}(\varepsilon)\right] h, \quad j \in \{s,p\} \tag{2.22}$$

Where $\underline{\varepsilon}^j$ and $\bar{\varepsilon}^j$ are functions of r_d and ϕ. The equilibrium r_d solve the above two equations given the exogenous variable $\phi > 0$ set by the central bank.

Aggregate TFP of the Model:

Because the model have feature of $Y = AK$ feature, we can use aggregate variable to rewrite the sector specific aggregate output:

$$Y^j = A^j K^j \tag{2.23}$$

where the sector specific average productivity

$$A^j = E(z^j\varepsilon|\text{Production}) = z^j \frac{\int_{\underline{\varepsilon}}^{\bar{\varepsilon}}{}^j \varepsilon d\mathbf{F}(\varepsilon) + \left(1+\theta^j\right)\int_{\bar{\varepsilon}^j}^{\infty} \varepsilon d\mathbf{F}(\varepsilon)}{\int_{\underline{\varepsilon}^j}^{\bar{\varepsilon}^j} d\mathbf{F}(\varepsilon) + (1+\theta^j)\int_{\bar{\varepsilon}^j}^{\infty} d\mathbf{F}(\varepsilon)} \tag{2.24}$$

The total output is given by :

$$Y = A^s K + (A^p - A^s)(1-\mu)K^p \tag{2.25}$$

Aggregate TFP is given by:

$$\text{TFP} = \frac{Y}{K} = A^s + (A^p - A^s)(1-\mu)\frac{K^p}{h} \tag{2.26}$$

Intuition of 'Dual Effects of Interest Rate Liberalization:

We can clearly see from equation (2.24) and (2.26) that:

- **Within sector reallocation effect:** From equation (2.24), we can see that the liberalization (decrease of ϕ) might reallocate the wealth from the less productive firms to the more productive firms within the sector, and boost the sector specific TFP.

- **Cross sector reallocation effect:** From equation (2.26), we can see that the liberalization (decrease of ϕ) might reallocate the wealth from the more productive SOE sector to the less productive POE sector (captured by the second term in the equation 2.26), and this effect might decrease the aggregate TFP.

The authors provides a sufficient condition that cross sector reallocation effect happens, and under this condition, the within sector reallocation effect boost the sector specific TFP in the POE sector. However, the sector specific TFP changes in SOE sectors is ambiguous. This result is concluded by the Proposition 5:

Proposition 5: Dual Effects of Interest Rate Liberalization

Assume that the idiosyncratic shocks in the two sectors are drawn from the same distribution, with the probability density function $f(\varepsilon)$, and Assume further that the density function satisfies the condition that $g(\varepsilon) \equiv f'(\varepsilon)\varepsilon/f(\varepsilon)$ decreases with ε. Under these conditions, we obtain

$$\frac{\partial K^s}{\partial \phi} < 0, \quad \frac{\partial K^p}{\partial \phi} > 0 \tag{2.27}$$

We also obtain that $\partial A^p/\partial\phi < 0$ and $\partial A^s/\partial\phi$ has an ambiguous sign. Furthermore, the relation between aggregate output and the interest rate wedge is also ambiguous (i.e., $\partial Y/\partial\phi$ has an ambiguous sign). The same is true for aggregate TFP.

3 Static Model: Numerical Computation

3.1 Computation Target

As we have seen by proposition 5, the relationship between aggregate output Y and the interest rate wedge ϕ controlled by the central bank is of ambiguous sign and can only be solved out numerically (this is figure 2) provided by the paper.

3.2 Algorithm

The algorithm of the static model is quite simple:

- 1. Set up an initial value of ϕ;
  - 2a. Guess the value of interest rate r_d, which also determine the $r_l = r_d + \phi$;
  - 2b. Check the equilibrium condition (2.9), if the condition holds (numerically lies in a small threshold near 0), we know this must be the equilibrium interest rate. If not, redo 2a.
- 3. Reset another ϕ and calculate the equilibrium interest rate r_d. We get a function of $r_d = r_d(\phi)$
- 4. Take interest rate to analytical result derived above, calculated the output given by 2.25.

3.3 Numerical Tools and Personal Contribution

3.3.1 Solve a Single Variable Nonlinear Equation: *fminsearchbnd.m* or *fminbnd.m*

One problem lies in algorithm above is **how to solve a single implicit equation**? There are plenty of choices like Binary SearchNewton Integration Search, Inverse Interpolation Method, etc. MATLAB has already prepare plenty of function so that we do not need to do ourselves:

I notice that the original code use *fminsearchbnd.m* provided by John D'Errico as a optimization function. This function gives the bounded minimized value on an interval $[x1, x2]$, with initial guess $x0$. Therefore, if we want to solve a function, we can define $f(x) = |RHS(x) - LHS(x)|$ (where RHS is Right Hand Side) and use the function to find the minimized value on the interval.

This method is wise, but we do not have to use the *fminsearchbnd.m* provided. In reality, MATLAB itself have function *fminbnd.m* and *fminsearch.m* which can complete the same goal.

In this replication I used the *fminbnd.m* provided by MATLAB.

3.3.2 The Partial Expectation of Log Normal Distribution

Another important numerical method we need to deal with is how to compute the **Partial Expectation of Log Normal Distribution**, defined as below:

$$g(\theta) = E(x|x \geq \theta) = \int_{\theta}^{+\infty} xf(x)dx \tag{3.1}$$

Where $f(x)$ is the PDF of the log normal distribution:

$$f(x) = \begin{cases} \frac{1}{\sigma x\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln x-\mu}{\sigma}\right)^2\right] & x > 0 \\ 0 & x \leq 0 \end{cases} \tag{3.2}$$

We can be solved out analytically ($\Phi()$ is CDF of normal distribution):

$$g(\theta) = \exp\left(\mu + \sigma^2/2\right)\Phi\left(\frac{\mu + \sigma^2 - \log(\theta)}{\sigma}\right) \tag{3.3}$$

We can also define the **k Order Partial Expectation of Log Normal Distribution** as below:

$$g^k(\theta) = E(x^k|x \geq \theta) = \int_{\theta}^{+\infty} x^k f(x)dx \tag{3.4}$$

Which can be solved out analytically:

$$g^k(\theta) = E(x^k|x > \theta) = \exp(k\mu + k^2\sigma^2/2)\Phi\left(\frac{\mu + k\sigma^2 - \log(\theta)}{\sigma}\right) \tag{3.5}$$

The reason why we need to compute the partial expectation is that in equations (2.9), (2.24), and so on, the integral take form of partial expectation. Definitely, we can use the analytical result above to compute the k order partial expectation of log normal distribution. However, not all the distribution have analytical result, so that it is also a better to provide a solution numerically:

In this replication, in addition to *RepFun_g.m* which compute the k order partial expectation analytically, I provided an alternative MATLAB function *RepFun_g_through_int.m* solve out the integral numerically using the *int* function provided by MATLAB, and it can be used to solve out **other kinds of distribution that has no analytical solution**. Although the function is definitely slower than the original code (if there is analytical solution, using analytical solution is always faster than take integral numerically).

Caveat: Both functions' inputs are *(k,thet,mu,sig)* where *mu* is the mean $E(X)$ of the lognormal distribution and *sig* is the standard distribution $D(X)$ of the lognormal distribution:

$$E(x) = e^{\mu+\frac{\sigma^2}{2}} \tag{3.6}$$

$$D(x) = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right) \tag{3.7}$$

Notice that it is different from the μ and σ parameter which denoted as *mu_bar* and *sig_bar* in the MATLAB code.

3.4 Replication

The replication of static model consists of three different components:

- **Baseline Replication:** replicate Figure 2 using the same algorithm as the authors using analytical result of partial expectation of log normal distribution

- **Numerical-integral Replication:** replicate Figure 2 using the same algorithm using numerical computation of partial expectation of log normal distribution
- **Play-with-parameter Replication:** change θ and see how the output changes with the changes.

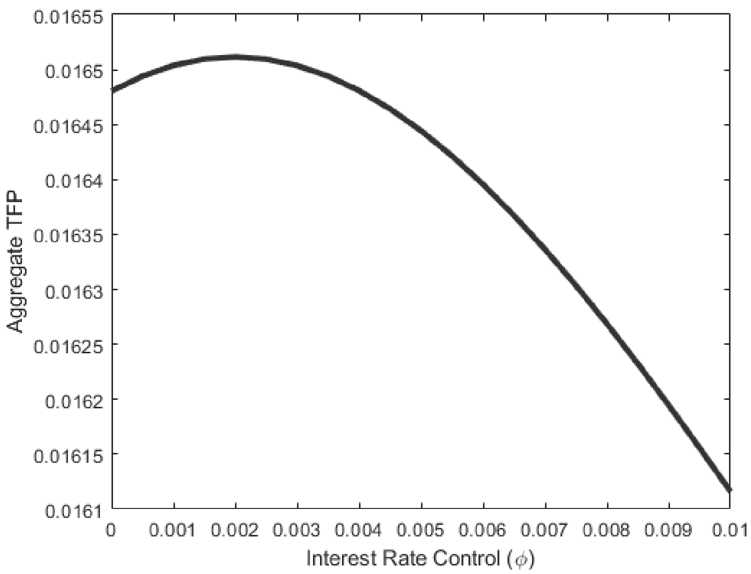

Figure 3.1: Replication of Figure 2 in the Paper

3.4.1 Baseline Replication

The replication file include the following components:

- *RepRunOptimalPhi.m:* Main Function that replicate Figure 2 using analytical solution of partial expectation, which used the above functions except *fminsearchbnd.m*.
  - 1: Define parameters and a vector of ϕ;
  - 2: For each ϕ calculate the market clearing r_d that solves the equation 2.9;
  - 3: For each set of $\{\phi, r_d\}$ calculate the output Y
  - 4: Plot the Y on y axis and ϕ on x-axis.
- *Repfun_g.m:* Function that compute partial expectation of log normal distribution analytically;
- *RepFun_SolveR.m:* Function that gives the $f(x) = |RHS(x) - LHS(x)|$ of market clearing condition equation (2.9), which should equals to 0 when r_d clear the financial market. This function use *Repfun_g.m:* because market clearing equation (2.9) use the partial expectation of lognormal distribution;
  - 1: calculate cutoff value of productivity;

- 2: calculate the integral in equation (2.9);
- 3, calculate the $f(x) = |RHS(x) - LHS(x)|$ of market clearing condition equation (2.9).

- *RepFun_SolveY.m:* Function that compute the output, capital, TFP, and productivity cutoff given r_d clear the financial market;
  - 1: Calculate cutoff value of productivity;
  - 2: Calculate the integral in equation (2.26);
  - 3, Calculate the aggregate output in each sector;
  - 4, Calculate the TFP using equation (2.26).
- *fminsearchbnd.m*: Function provided by John D'Errico as a optimization function that gives the bounded minimization value on an interval. I do not use this function in the replication.

3.4.2 Numerical-integral Replication

The other functions are the same, but I use the numerical integration rather than analytical result to compute the partial expectation of log normal distribution:

- *RepFun_g_through_int.m:* Function that compute partial expectation of log normal distribution numerically.

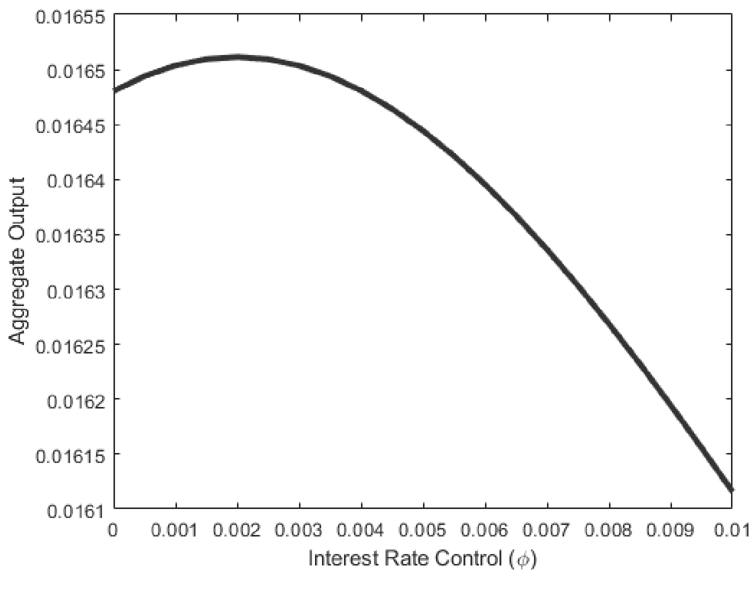

Figure 3.2: Replication of Figure 2 in the Paper Using Numerical Integration

As we can see from the graph, the result is the same as the baseline replication, but the program run much slower: It takes 3 minutes rather than few second if we use numerical result of integration.

3.4.3 Play-with-parameter Replication

In this figure, I plot the value with different θ: $\theta = 0.2, 0.3$ and 0.4:

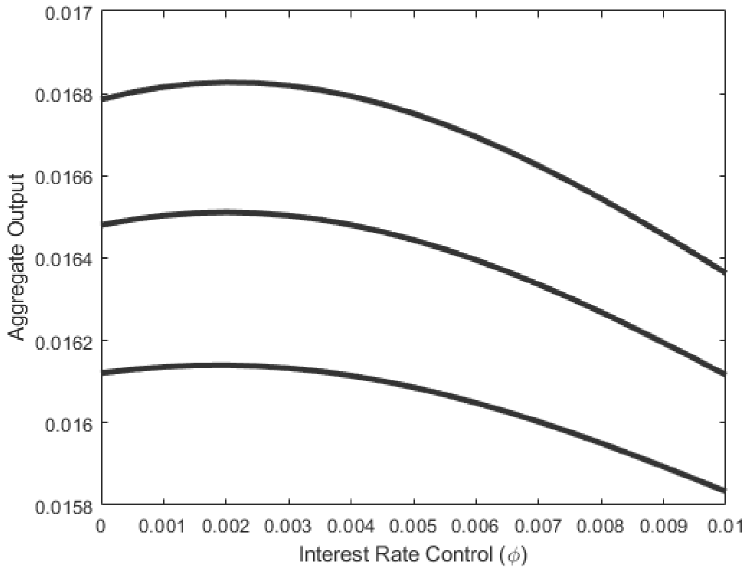

Figure 3.3: Replication of Figure 2 with Different θ

3.5 Result

The simulation of the static model shows that there is a **non-monotonic relationship between the output (welfare) changes and the interest rate wedge**, when the wedge is large, an liberalization of interest will increase the welfare (intuitive). However, it is also true that small interest rate wedge can **increase output and TFP**, because it lets the less efficient SOE sector invest less, so that it correct the over investment problem due to subsidy to the SOE.

4 Dynamic Model: Model Setup and Analytical Result

4.1 Model Setup

4.1.1 The Households

There is a continuum of identical and infinitely lived households with measure one. The representative household supplies inelastically one unit of labor to firms.

$$\max_{C_t, B_{t+1}, x^j_{i,t+1}} \sum_{t=0}^{\infty} \beta^t \log C_t \tag{4.1}$$

subject to the **flow of fund constrain**:

$$\begin{aligned} C_t + & \frac{B_{t+1}}{1+r_{dt}} + \sum_{j=\{s,p\}} \int x^j_{i,t+1} \left(P^j_{it} - d^j_{it} \right) di \\ \leq & W_t N_t + B_t + \sum_{j=\{s,p\}} \int x^j_{it} P^j_{it} di - T_t \end{aligned} \tag{4.2}$$

As we can see from the equations:

- Households can trade risk free bond B_{t+1} with interest rate r_{dt};
- Households can trade stock $x^j_{i,t+1}$ with dividend d^j_{it} and price P^j_{it}, where $j \in s, p$ represent the investment sector and i represents the firms.
- T_i is the lump-sum tax that helps to subsidize the SOE;
- We normalize $N_t = 1$, which means that we assume the labor supply is inelastic.

Furthermore, we assume that the household also subject to the **borrowing constrain**, which does not allow the consumers to borrow from the financial market

$$\frac{B_{t+1}}{1+r_{dt}} \geq 0 \tag{4.3}$$

Naturally, flow of fund constrain must hold with equality because consumer can always rise C_t. This is a standard perfectly forecast household problem and we can setup Lagrange function using Λ_t and ξ_t as multiplier for flow of fund constrain and borrowing constrain:

$$\begin{aligned} \mathcal{L} = & \sum_{t=0}^{\infty} \beta^t \log C_t + \sum_{t=0}^{\infty} \Lambda_t \Bigg[- C_t - \frac{B_{t+1}}{1+r_{dt}} - \sum_{j=\{s,p\}} \int x^j_{i,t+1} \left(P^j_{it} - d^j_{it} \right) di \\ & + W_t N_t + B_t + \sum_{j=\{s,p\}} \int x^j_{it} P^j_{it} di - T_t \Bigg] + \sum_{t=0}^{\infty} \xi_t \frac{B_{t+1}}{1+r_{dt}} \end{aligned} \tag{4.4}$$

Take first order condition we get:

$$\Lambda_t = \beta \left(1 + r_{dt} \right) \Lambda_{t+1} + \xi_t \tag{4.5}$$

$$P^j_{it} = d^j_{it} + \beta \frac{\Lambda_{t+1}}{\Lambda_t} P^j_{it+1} \tag{4.6}$$

$$\Lambda_t = 1/C_t \tag{4.7}$$

where $\xi_t > 0$ only when the borrowing constrain is binding (the consumer deplete all his saving).

Suppose that it is a symmetric equilibrium with homogeneous households. The net supply of bond is 0 and each firm stock supply is normalized to 1. We know that like the economy describe as the CAPM model, all the shares are equally held by all the consumers, and each household do not borrow and lend in the monetray market:

$$x^j_{i,t+1} = 1 \text{ for all } i \text{ and } j \tag{4.8}$$

$$B_{t+1} = 0 \text{ and } \xi_t > 0 \tag{4.9}$$

4.1.2 The Firms

A key difference in the dynamic model is that the authors supposed that the production technology is decreasing return to scale (which is a standard assumption in the firm dynamics theory). The parameter $\eta \in (0,1)$ capture the nature of decreasing return to scale:

$$y^j_t = \left[\left(z^j \varepsilon^j_t k^j_t\right)^{\alpha} \left(n^j_t\right)^{1-\alpha}\right]^{\eta} \qquad j \in \{s,p\} \tag{4.10}$$

In each period after observe the idiosyncratic productivity shock, the firms will draw a idiosyncratic shock term ε^j_t from CDF $\mathbf{F}^j(\cdot)$ the firms solve the **separable instant profit maximization problem** by choosing the optimal level of labor used in period t:

$$\pi^j_t \equiv \max_{n^j_t} \tilde{\tau}^j \left(z^j \varepsilon^j_t k^j_t\right)^{\alpha\eta} \left(n^j_t\right)^{(1-\alpha)\eta} - W_t n^j_t \tag{4.11}$$

Same as the static model, the $\tilde{\tau}^j$ is the subsidy rate in sector j, and we also assume the subsidy rate in the POE sector is 0. Solve out the equation by take first order condition we get:

$$n^j_t = \left(\frac{\eta(1-\alpha)\tilde{\tau}^j}{W_t}\right)^{\frac{1}{1-(1-\alpha)\eta}} \left(z^j \varepsilon^j_t k^j_t\right)^{\frac{\alpha\eta}{1-(1-\alpha)\eta}} \tag{4.12}$$

$$\pi^j_t = \left(\tilde{\tau}^j\right)^{\frac{1}{1-\gamma}} (1-\gamma) \left(\frac{\gamma}{W_t}\right)^{\frac{\gamma}{1-\gamma}} \left(z^j \varepsilon^j_t k^j_t\right)^{\frac{\alpha\eta}{1-\gamma}} \equiv \tau^j R_t \left(z^j \varepsilon^j_t k^j_t\right)^{\tilde{\alpha}} \tag{4.13}$$

For simplicity, we define:

- $\gamma \equiv (1-\alpha)\eta$, $\tilde{\alpha} \equiv \alpha\eta/(1-\gamma)$;
- $T^j \equiv \left(\tilde{\tau}^j\right)^{1/1-\gamma}$ denotes the effective capital subsidy rate;
- $R_t \equiv (1-\gamma)\left(\gamma/W_t\right)^{\gamma/1-\gamma}$ is the presubsidy rate of return on capital;

In each period, the firms will also solve the **current value maximization problem** by choosing how much to borrow l_t^j, save s_t^j, how much capital k_t^j to hold, and how much retain earning h_{t+1}^j to hold as cash for next period:

$$V_t^j\left(h_t^j,\varepsilon_t^j\right)=\max_{l_t^j,s_t^j,k_t^j,h_{t+1}^j}\left\{d_t^j\left(h_t^j,\varepsilon_t^j\right)+\beta\frac{\Lambda_{t+1}}{\Lambda_t}\int V_{t+1}^j\left(h_{t+1}^j,\varepsilon_{t+1}^j\right)d\mathbf{F}^j\left(\varepsilon_{t+1}^j\right)\right\} \tag{4.14}$$

where the dividends are given by:

$$d_t^j\left(h_t^j,\varepsilon_t^j\right)\equiv\tau^jR_t\left(z^j\varepsilon_t^jk_t^j\right)^{\tilde{\alpha}}+(1-\delta)k_t^j-(1+r_{lt})\,l_t^j+(1+r_{dt})\,s_t^j-h_{t+1}^j \tag{4.15}$$

subject to **flow of fund constrain**:

$$k_t^j=l_t^j+h_t^j-s_t^j \tag{4.16}$$

credit constrains:

$$0\le l_t^j\le\theta^jh_t^j \tag{4.17}$$

$$0\le s_t^j\le h_t^j \tag{4.18}$$

4.2 Firms' Behavior

Firm's optimal behavior in the dynamic model is concluded in the proposition 6 of the paper:

Proposition 6: Firms' Behavior in Dynamic Model

Assume that the idiosyncratic shock is drawn from the interval $\left(\varepsilon_{\min}^j,\varepsilon_{\max}^j\right)$ according to the distribution function $\mathbf{F}^j(\varepsilon)$, Then, given h_t^j, the optimal allocations of saving s_t^j, borrowing l_t^j and capital input k_t^j are determined by

$$s_t^j=\begin{cases}\left[1-\left(\varepsilon_t^j/\underline{\varepsilon}_t^j\right)^{\frac{\tilde{\alpha}}{1-\tilde{\alpha}}}\right]h_t^j & \text{if } \varepsilon_{\min}^j\le\varepsilon_t^j<\underline{\varepsilon}_t^j\\ 0 & \text{if } \underline{\varepsilon}_t^j\le\varepsilon_t^j<\varepsilon_{\max}^j\end{cases} \tag{4.19}$$

$$l_t^j=\begin{cases}0 & \text{if } \varepsilon_{\min}^j\le\varepsilon_t^j<\hat{\varepsilon}_t^j\\ \left[\left(\varepsilon_t^j/\hat{\varepsilon}_t^j\right)^{\frac{\tilde{\alpha}}{1-\tilde{\alpha}}}-1\right]h_t^j & \text{if } \hat{\varepsilon}_t^j\le\varepsilon_t^j<\bar{\varepsilon}_t^j\\ \theta^jh_t^j & \text{if } \bar{\varepsilon}_t^j\le\varepsilon_t^j<\varepsilon_{\max}^j\end{cases} \tag{4.20}$$

$$k_t^j=\begin{cases}\left(\varepsilon_t^j/\underline{\varepsilon}_t^j\right)^{\frac{\tilde{\alpha}}{1-\tilde{\alpha}}}h_t^j & \text{if } \varepsilon_{\min}^j\le\varepsilon_t^j<\underline{\varepsilon}_t^j\\ h_t^j & \text{if } \underline{\varepsilon}_t^j\le\varepsilon_t^j<\hat{\varepsilon}_t^j\\ \left(\varepsilon_t^j/\hat{\varepsilon}_t^j\right)^{\frac{\tilde{\alpha}}{1-\tilde{\alpha}}}h_t^j & \text{if } \hat{\varepsilon}_t^j\le\varepsilon_t^j<\bar{\varepsilon}_t^j\\ \left(1+\theta^j\right)h_t^j & \text{if } \bar{\varepsilon}_t^j\le\varepsilon_t^j<\varepsilon_{\max}^j.\end{cases} \tag{4.21}$$

where the three cutoff value $\{\underline{\varepsilon}_t^j,\hat{\varepsilon}_t^j,\bar{\varepsilon}_t^j\}$ are given as:

$$\underline{\varepsilon}_t^j=\left(h_t^j\right)^{\frac{1-\tilde{\alpha}}{\tilde{\alpha}}}\left[\frac{r_{dt}+\delta}{\tilde{\alpha}\tau^j\left(z^j\right)^{\tilde{\alpha}}R_t}\right]^{\frac{1}{\tilde{\alpha}}} \tag{4.22}$$

$$\hat{\varepsilon}_t^j = \left(h_t^j\right)^{\frac{1-\tilde{\alpha}}{\tilde{\alpha}}} \left[\frac{r_{lt}+\delta}{\tilde{\alpha}\tau^j \left(z^j\right)^{\tilde{\alpha}} R_t}\right]^{\frac{1}{\tilde{\alpha}}} \tag{4.23}$$

$$\bar{\varepsilon}_t^j = \left[\left(1+\theta^j\right) h_t^j\right]^{\frac{1-\tilde{\alpha}}{\tilde{\alpha}}} \left[\frac{r_{lt}+\delta}{\tilde{\alpha}\tau^j \left(z^j\right)^{\tilde{\alpha}} R_t}\right]^{\frac{1}{\tilde{\alpha}}} \tag{4.24}$$

The net worth h_{t+1}^j can be solved from the envelope condition:

$$1 = \beta \frac{\Lambda_{t+1}}{\Lambda_t} \int \frac{\partial V_{t+1}^j \left(h_{t+1}^j, \varepsilon_{t+1}^j\right)}{\partial h_{t+1}^j} d\mathbf{F}^j \left(\varepsilon_{t+1}^j\right) \tag{4.25}$$

Intuition of Proposition 6: According to the proposition 6, there are **four kinds** of firms in each sector:

- If the productivity is very low in interval $[\varepsilon_{\min}^j, \underline{\varepsilon}_t^j)$, the firm will not produce and lend all the cash available to the other firms, earning interest rate r_{dt};
- If the productivity is slightly low, in the interval $[\underline{\varepsilon}_t^j, \hat{\varepsilon}_t^j)$, the firm will produce but neither borrow or lend in the financial market (like the backyard firms with self-employed entrepreneur);
- If the productivity is slightly high in the interval $[\hat{\varepsilon}_t^j, \bar{\varepsilon}_t^j)$, the firm will produce and borrow from the financial market paying interest rate r_{lt}, but they will not borrow up to the upper limit;
- If the productivity is very high in the interval $[\bar{\varepsilon}_t^j, \varepsilon_{\max}^j]$, the firm will produce and borrow from the financial market up to the upper limit given by the credit constrain.

The reason why there are two types of firms in the $[\underline{\varepsilon}_t^j, \bar{\varepsilon}_t^j)$ rather than only one types of firms as in the static model is that there are two source of finance in the dynamic model: **internal finance through holding profit as cash** and **external finance through borrowing from the market**. The static model on the other hand, only have the external finance because the firm liquidate after one period.

4.3 Aggregation

4.3.1 Sectoral and Aggregate Effective Units of Capital

Define the sectoral effective units of capital as $\tilde{K}_t^j$ (where net worth is independent of the idiosyncratic shocks $h_t^j = H_t^j$):

$$\tilde{K}_t^j = \int \left[z^j \varepsilon_t^j k_t^j \left(H_t^j, \varepsilon_t^j\right)\right]^{\tilde{\alpha}} d\mathbf{F}^j \left(\varepsilon_t^j\right) \tag{4.26}$$

Given Proposition 6, we can further express the sectoral effective capital as below:

$$
\begin{aligned}
\tilde{K}_t^j = \left(H_t^j\right)^{\tilde{\alpha}} \Bigg\{ \int_{\varepsilon_{\min}^j}^{\varepsilon_t} \left[z^j \varepsilon_t^j \left(\varepsilon_t^j / \underline{\varepsilon}_t^j\right)^{\frac{\tilde{\alpha}}{1-\tilde{\alpha}}} \right]^{\tilde{\alpha}} d\mathbf{F}^j\left(\varepsilon_t^j\right) + \int_{\underline{\varepsilon}_t^j}^{\hat{\varepsilon}_t^j} \left(z^j \varepsilon_t^j\right)^{\tilde{\alpha}} d\mathbf{F}^j\left(\varepsilon_t^j\right) \\
+ \int_{\hat{\varepsilon}_t^j}^{\bar{\varepsilon}_t^j} \left[z^j \varepsilon_t^j \left(\varepsilon_t^j / \hat{\varepsilon}_t^j\right)^{\frac{\tilde{\alpha}}{1-\tilde{\alpha}}} \right]^{\tilde{\alpha}} d\mathbf{F}^j\left(\varepsilon_t^j\right) + \left(1+\theta^j\right)^{\tilde{\alpha}} \int_{\bar{\varepsilon}_t^j}^{\varepsilon_{\max}^j} \left(z^j \varepsilon_t^j\right)^{\tilde{\alpha}} d\mathbf{F}^j\left(\varepsilon_t^j\right) \Bigg\}
\end{aligned}
\tag{4.27}
$$

Therefore, we can compute the aggregate effective units of capital as:

$$
\tilde{K}_t = \sum_{j=\{s,p\}} \tilde{K}_t^j \tag{4.28}
$$

4.3.2 Sectoral and Aggregate Capital Input

Define the sectoral capital input K_t^j used in sector $j \in \{s, p\}$:

$$
K_t^j = \int k_t^j\left(H_t^j, \varepsilon_t^j\right) d\mathbf{F}^j\left(\varepsilon_t^j\right) \tag{4.29}
$$

Given Proposition 6, we can further express the sectoral capital input as below:

$$
\begin{aligned}
K_t^j = & H_t^j \Bigg[\int_{\varepsilon_{\min}^j}^{j} \left(\varepsilon_t^j / \underline{\varepsilon}_t^j\right)^{\frac{\tilde{\alpha}}{1-\tilde{\alpha}}} d\mathbf{F}^j\left(\varepsilon_t^j\right) + \int_{\underline{\varepsilon}_t^j}^{\hat{\varepsilon}_t^j} d\mathbf{F}^j\left(\varepsilon_t^j\right) \\
& + \int_{\hat{\varepsilon}_t^j}^{\bar{\varepsilon}_t^j} \left(\varepsilon_t^j / \hat{\varepsilon}_t^j\right)^{\frac{\tilde{\alpha}}{1-\tilde{\alpha}}} d\mathbf{F}^j\left(\varepsilon_t^j\right) + \left(1+\theta^j\right) \int_{\bar{\varepsilon}_t^j}^{\varepsilon_{\max}} d\mathbf{F}^j\left(\varepsilon_t^j\right) \Bigg]
\end{aligned}
\tag{4.30}
$$

Therefore, we can compute the aggregate capital input as

$$
K_t = \sum_{j=\{s,p\}} K_t^j \tag{4.31}
$$

4.3.3 Sectoral and Aggregate Loan

Define the sectoral loan L_t^j in sector $j \in \{s, p\}$:

$$
L_t^j = \int l_t^j\left(h_t^j, \varepsilon_t^j\right) d\mathbf{F}^j\left(\varepsilon_t^j\right) \tag{4.32}
$$

Given Proposition 6, we can further express the sectoral loan as below:

$$
L_t^j = H_t^j \int_{\hat{\varepsilon}_t^j}^{\bar{\varepsilon}_t^j} \left[\left(\varepsilon_t^j / \hat{\varepsilon}_t^j\right)^{\frac{\tilde{\alpha}}{1-\tilde{\alpha}}} - 1 \right] d\mathbf{F}^j\left(\varepsilon_t^j\right) + \theta_t^j H_t^j \int_{\bar{\varepsilon}_t^j}^{\delta_{\max}} {}^j d\mathbf{F}^j\left(\varepsilon_t^j\right) \tag{4.33}
$$

Therefore, we can compute the aggregate loan as

$$
L_t = \sum_{j=\{s,p\}} L_t^j \tag{4.34}
$$

4.3.4 Sectoral and Aggregate Saving

Define the sectoral saving S_t^j in sector $j \in \{s, p\}$:

$$S_t^j = \int s_t^j \left(h_t^j, \varepsilon_t^j\right) d\mathbf{F}^j \left(\varepsilon_t^j\right) \tag{4.35}$$

Given Proposition 6, we can further express the sectoral saving as below:

$$S_t^j = H_t^j \int_{\varepsilon_{\min}^j}^{\varepsilon_t^j} \left[1 - \left(\underline{\varepsilon}_t^j\right)^{-\frac{\tilde{\alpha}}{1-\tilde{\alpha}}} \left(\varepsilon_t^j\right)^{\frac{\tilde{\alpha}}{1-\tilde{\alpha}}}\right] d\mathbf{F}^j \left(\varepsilon_t^j\right) \tag{4.36}$$

Therefore, we can compute the aggregate saving as

$$S_t = \sum_{j=\{s,p\}} S_t^j \tag{4.37}$$

4.3.5 Sectoral and Aggregate Net Worth

Sectoral aggregate net worth $H_t^j = h_t^j$ is given by solving the following implicit function: (net worth is independent of the idiosyncratic shocks so that $h_t^j = H_t^j$)

$$1 = \beta \frac{\Lambda_{t+1}}{\Lambda_t} \int \frac{\partial V_{t+1}^j \left(h_{t+1}^j, \varepsilon_{t+1}^j\right)}{\partial h_{t+1}^j} d\mathbf{F}^j \left(\varepsilon_{t+1}^j\right) \tag{4.38}$$

where

$$\frac{\partial V_t^j \left(h_t^j, \varepsilon_t^j\right)}{\partial h_t^j} = \begin{cases} \Gamma_t^j \left(\varepsilon_t^j, h_t^j\right) \left(1+\theta^j\right)^{\tilde{\alpha}} + (1-\delta) - (\delta + r_{lt})\, \theta^j & \text{if } \varepsilon_t^j \geq \bar{\varepsilon}_t^j \\ \Gamma_t^j \left(\varepsilon_t^j, h_t^j\right) \left[\left(\varepsilon_t^j / \hat{\varepsilon}_t^j\right)^{\frac{\tilde{\alpha}}{1-\tilde{\alpha}}}\right]^{\tilde{\alpha}} + (1-\delta) - (\delta + r_{lt}) \left[\left(\varepsilon_t^j / \hat{\varepsilon}_t^j\right)^{\frac{\tilde{\alpha}}{1-\tilde{\alpha}}} - 1\right] & \text{if } \hat{\varepsilon}_t^j \leq \varepsilon_t^j < \bar{\varepsilon}_t^j \\ \Gamma_t^j \left(\varepsilon_t^j, h_t^j\right) + (1-\delta) & \text{if } \underline{\varepsilon}_t^j \leq \varepsilon_t^j < \hat{\varepsilon}_t^j \\ \Gamma_t^j \left(\varepsilon_t^j, h_t^j\right) \left[\left(\varepsilon_t^j / \underline{\varepsilon}_t^j\right)^{\frac{\tilde{\alpha}}{1-\tilde{\alpha}}}\right]^{\tilde{\alpha}} + (1-\delta) + (\delta + r_{dt}) \left[1 - \left(\varepsilon_t^j / \underline{\varepsilon}_t^j\right)^{\frac{\tilde{\alpha}}{1-\tilde{\alpha}}}\right] & \text{if } \varepsilon_t^j < \underline{\varepsilon}_t^j, \end{cases} \tag{4.39}$$

and

$$\Gamma_t^j \left(\varepsilon_t^j, h_t^j\right) = \tau^j R_t \left(z^j \varepsilon_t^j\right)^{\tilde{\alpha}} \tilde{\alpha} \left(h_t^j\right)^{\tilde{\alpha}-1} \tag{4.40}$$

Therefore, we can compute the aggregate labor demand as

$$H_t = \sum_{j=\{s,p\}} H_t^j \tag{4.41}$$

4.3.6 Sectoral and Aggregate Labor Demand

Sectoral labor demand can be given as:

$$\begin{aligned} N_t^j &= \tau^j \left(\frac{\gamma}{W_t}\right)^{\frac{1}{1-\gamma}} \int \left(z^j \varepsilon_t^j k_t^j\right)^{\tilde{\alpha}} d\mathbf{F}^j \left(\varepsilon_t^j\right) \\ &= \tau^j \left(\frac{\gamma}{W_t}\right)^{\frac{1}{1-\gamma}} \tilde{K}_t^j, \quad j \in \{s,p\} \end{aligned} \tag{4.42}$$

Therefore, we can compute the aggregate labor demand as

$$N_t = \sum_{j=\{s,p\}} N_t^j \tag{4.43}$$

4.3.7 Sectoral and Aggregate Output

We can compute the sectoral aggregate output as:

$$Y_t^j = \left(\tilde{K}_t^j\right)^{1-(1-\alpha)\eta} \left(N_t^j\right)^{(1-\alpha)\eta} \tag{4.44}$$

Therefore, we can compute the aggregate output as

$$Y_t = \sum_{j=\{s,p\}} Y_t^j \tag{4.45}$$

4.3.8 Sectoral and Aggregate TFP

$$\begin{aligned} A_t^j &= \frac{Y_t^j}{\left[\left(K_t^j\right)^{\alpha} \left(N_t^j\right)^{1-\alpha}\right]^{\eta}} \\ &= \left(z^j\right)^{\alpha} \left\{ \int \left[\frac{\varepsilon_t^j k_t^j \left(H_t^j, \varepsilon_t^j\right)}{K_t^j}\right]^{\tilde{\alpha}} d\mathbf{F}^j \left(\varepsilon_t^j\right) \right\}^{1-\gamma} \end{aligned} \tag{4.46}$$

We can compute the aggregate TFP as

$$A_t = \frac{Y_t}{\left(K_t^{\alpha} N_t^{1-\alpha}\right)^{\eta}} \tag{4.47}$$

4.4 Equilibrium Conditions

In the equilibrium, there are four market clearing condition:

A. Loanable funds market clear:

$$\sum_{j=\{s,p\}} L_t^j = \sum_{j=\{s,p\}} S_t^j + B_t \tag{4.48}$$

where

$$L_t^j = \int l_t^j \left(H_t^j, \varepsilon_t^j\right) d\mathbf{F}^j \left(\varepsilon_t^j\right) \qquad S_t^j = \int s_t^j \left(H_t^j, \varepsilon_t^j\right) d\mathbf{F}^j \left(\varepsilon_t^j\right) \tag{4.49}$$

B. Capital market clear:

$$K_t = \sum_{j=\{s,p\}} H_t^j \tag{4.50}$$

where

$$K_t \equiv \sum_{j=\{s,p\}} K_t^j \tag{4.51}$$

C. Labor market clear:

$$N_t^s + N_t^p = 1 \tag{4.52}$$

D. Final goods market clear:

$$C_t = Y_t - K_{t+1} + (1-\delta)K_t \tag{4.53}$$

5 Dynamic Model: Numerical Computation

5.1 Calibration: Table 1

The first step is model calibration: because one period is correspond to one year, the discount factor $\beta = 0.96$ and depreciation rate $\theta = 0.1$ following existing literature.

The labor share $\alpha = 0.5$ because the existing literature shows that the labor share of income is around 50%. Decreasing return to scale parameter $\eta = 0.85$ following existing literature. $\phi = 0.032 = 3.2\%$ is the average interest rate wedge.

The author normalized the subsidy rate $\tilde{\tau}^p = 1$ and $\tilde{\tau}^s = 1.44$ is computed using the following equation which is from equation (3.12).

$$\tilde{\tau}^j = \frac{W_t n^j_{mit}}{(1-\alpha)\eta y^j_{mit}} \tag{5.1}$$

where m stand for industry, i stand for firm and t stand for year.

The credit constrain parameter is calculated as $\theta^s = 0.504$, $\theta^p = 0.279$ using the average loan to value ratio from the China's Annual Survey of Industrial Firms data.

Also using China's Annual Survey of Industrial Firms data, the the authors show that POEs are on average 92% more productive than SOEs. By normalized the average productivity of SOEs $z^s = 1$, the author calibrated the $z^p = 1.92$.

Finally, the most tricky one is the standard deviation of the productivity distribution, the authors follow a two step calibration:

- 1. According to the micro data $\sigma^p/\sigma^s = 1.23$;
- 2. In order to calibrate the σ^s the author compute the equilibrium level of borrowing, and take the σ^p that generate 39% of SOEs are borrowers in the financial market.

RepCalibration.m and *RepRunCalibration.m* are two files that gives the final step of the calibration. Run the program *RepRunCalibration.m* and we get the calibrated value of parameters as Table 1.

5.2 Benchmark Model Transition Path: Figure 3

5.2.1 Algorithm

I notice that this model behave like a standard Ramsey infinite living agent model, so like the Ramsey model, the computation can be divided into two parts: The first part is **Computation of steady-state** which involves saving the static firm profit maximization problem and present value maximization problem. The second part is **deterministic simulation** using Dynare which involve solving the dynamic investment, consumption allocation problem.

β	Discounting factor	0.96
α	Capital share	0.5
η	Returns to scale	0.85
δ	Capital depreciation rate	0.1
ϕ	Interest rate wedge	0.032
θ^s	SOE borrowing constraint	0.504
θ^p	POE borrowing constraint	0.279
z^s	SOE-specific TFP (normalized)	1
z^p	POE-specific TFP	1.92
σ^s	SOE productivity dispersion	0.375
σ^p	POE productivity dispersion	0.461
τ^s	SOE output wedge	1.44
τ^p	POE output wedge (normalized)	1

Figure 5.1: Table 1: Calibrated Parameters

Because the second step is a standard Dynare deterministic simulation, the key to the computation is the first step, computation of the steady state. The key algorith is as follow:

- 1. Compute the before liberalization steady state with $\phi > 0$:
  - 2a. Solve the steady-state set of price vector $X = \{R, r_d, h_s, h_p\}$, in which R is the return to capital (after subsidy), r_d is the deposit (saving) interest rate, h_s is the SOE firm value, and h_p is the POE firm value. These four variables can be solved out using the following system of linear equations:
    * Equation (3.48): Loanable fund market clear where saving and borrowing is given by (3.33), (3.36);
    * Equation (3.50): Capital market clear where capital and firm value is given by (3.30), and (3.38);
    * Equation (3.38) for SOE: SOE net worth is correctly priced;
    * Equation (3.39) for POE: POE net worth is correctly priced.
  - 2b. If $X = \{R, r_d, h_s, h_p\}$ solve the equations, we know that it is the steady state values, and we can compute out all other steady state real variables: capital, effective capital, labor, output, consumption, utility (welfare), and TFP.
- 3. Compute the after liberalization steady state with $\phi = 0$: Redo step 2a-2c.

Notice that step 2a can be solved using the MATLAB function *fslove.m* which solve the system of nonlinear system of equations.

5.2.2 Code

The computation includes the following files:

- *RepRunTransition_Bench.m:* Main file that replicate Figure 3 using function *RepSolveSteadyState.m*, *RepSolveAllSteadyState.m*, ploting file *RunPlotFig.m* and Dynare file *RepTransPath_DRS_Bench.mod*

- 1. Define parameters before and after liberalization;
- 2a. Solve the before liberalization steady-state price vector $X = \{R, r_d, h_s, h_p\}$ using *RepSolveSteadyState.m* and MATLAB internal function *fsolve.m*;
- 2b. Compute all other steady-state variables before liberalization using *RepSolveAllSteadyState.m*;
- 3. Redo 2a-2b using the parameters after liberalization;
- 4. Run Dynare code *RepTransPath_DRS_Bench.mod* to solve the dynamic path;
- 5. Run *RunPlotFig.m* to plot out the dynamic path.

- *RepSolveSteadyState.m*: Function that gives the $f(x) = |RHS(x) - LHS(x)|$ of equation (3.48), (3.50), (3.38) for SOE and (3.38) for POE, given price vector $X = \{R, r_d, h_s, h_p\}$:
  - 1. Compute the cutoff using (3.22), (3.23), (3.24);
  - 2. Compute the integral parts of (3.27), (3.30), (3.33), (3.36);
  - 3. Compute the sectoral capital and effective capital using equation (3.27), (3.30);
  - 4, Compute the sectoral loan and saving using equation (3.33), (3.36);
  - 5, Compute the firm net worth using equation (3.38);
  - 6. $f(x) = |RHS(x) - LHS(x)|$ of equation (3.48), (3.50), (3.38) for SOE and (3.38) for POE, which should take 0 at steady state equilibrium.
- *RepSolveAllSteadyState.m*: Function that compute all other steady-state variables before liberalization using the equilibrium steady state price vector $X = \{R, r_d, h_s, h_p\}$ that has already been solved out.
- *RepTransPath_DRS_Bench.mod*: Dynare file that finished the second step **deterministic simulation**.
- *RunPlotFig.m*: File that plot the dynamic transition path.
- Fun_g.m: Function that compute partial expectation of log normal distribution analytically.

5.2.3 Result

Run *RepRunTransition_Bench.m:* and we get the replication of figure 3:

As we can see from the figure, we assume that the interest rate liberalization take place in the period 1, so that the deposit and loan rate converge to each other in period 2, the economy will converge to a new steady state that:

- The SOE sector capital share is higher in the new steady state;
- Both the SOEs and POEs have a higher TFP in the new steady state because the **within sector reallocation effect** mentioned before;

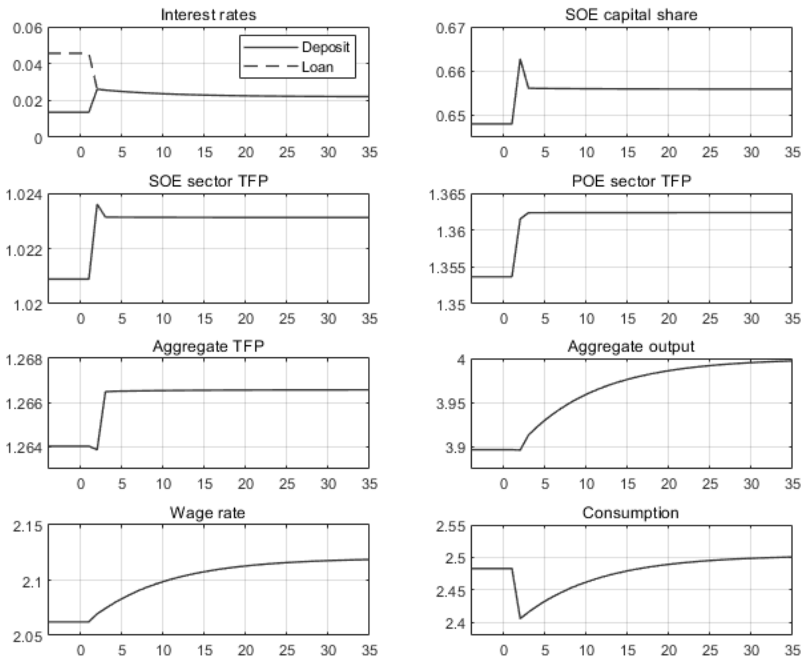

Figure 5.2: Replication of Figure 3 in the Paper

- Aggregate TFP, output and consumption is higher in the new steady state ⇒ **In long run, the liberalization will increase the productivity, output, and consumption of China;**
- Aggregate wage rise in the new steady state because the rise of labor demand and the inelastic supply of labor.

However, although **in long run, the liberalization will increase the productivity, output, and consumption of China, the short run effect of TFP and consumption is not:**

- In period 2 (immediately after the liberalization) the TFP decreases because there is a positive shock on SOE investment. This leads to reallocate resources from the POE to SOEs and this **cross sector reallocation effect** decreases TFP (although it is soon dominated by the within sector reallocation effect that rise TFP).
- More importantly, there is a **decrease in consumption (and therefore welfare) after period 2**, because the liberalization cause **over investment in SOE sector** which takes out large part of the output. The consumption recovers slowly and it takes years to converge to a higher level.

The **long run welfare gain** and **short run welfare loss** are the main result of this paper.

5.3 Benchmark Model Welfare Analysis: Figure 4

In this section, we want to analysis the relationship between **short run welfare loss** and the initial interest rate wedge ϕ. The short run welfare loss is defined as the welfare in the period 2 (immediately after the liberalization) and the welfare in period 1.

5.3.1 Code

The functions are the same with the above the benchmark computation, and the only additional file is:

- *RepRunComputeWelfare.m* Main file that replicate Figure 4 which is a modified version of *RepRun-Transition_Bench.m*, the only difference is that it set different ϕ and each time it compute the short run welfare loss.

5.3.2 Result

Run *RepRunComputeWelfare.m* and we get the replication of figure 4:

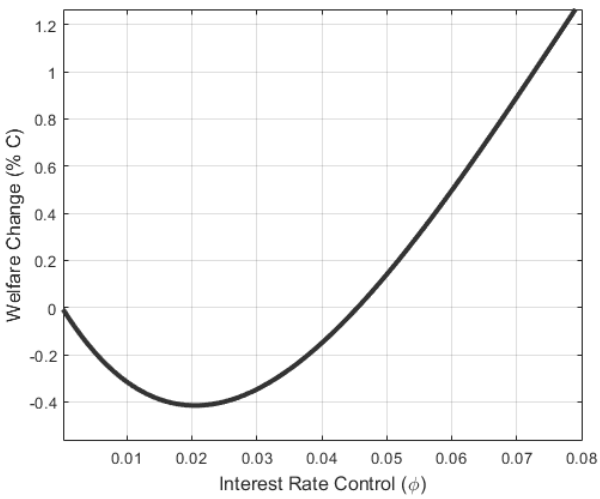

Figure 5.3: Replication of Figure 4 in the Paper

As we can see from the figure, the short run welfare loss is not a monotone function of the interest rate wedge ϕ. The short run loss is the highest at around $\phi = 0.02 = 2\%$. The calibrated value shows that the current wedge is around 3.2% so that there exist a short run loss due to the liberalization.

5.4 Counterfactual Dynamic Transition Analysis: Figure 5

In the original paper, the author also conduct two counterfactual analysis: First, assuming that there is no subsidy to the SOEs; Second, assuming SOEs and POEs have equal access to the credit.

The computation method is the same as the benchmark simulation (only change the related parameters), so I directly use the original code provided by the authors to finished this part. The result is as follow:

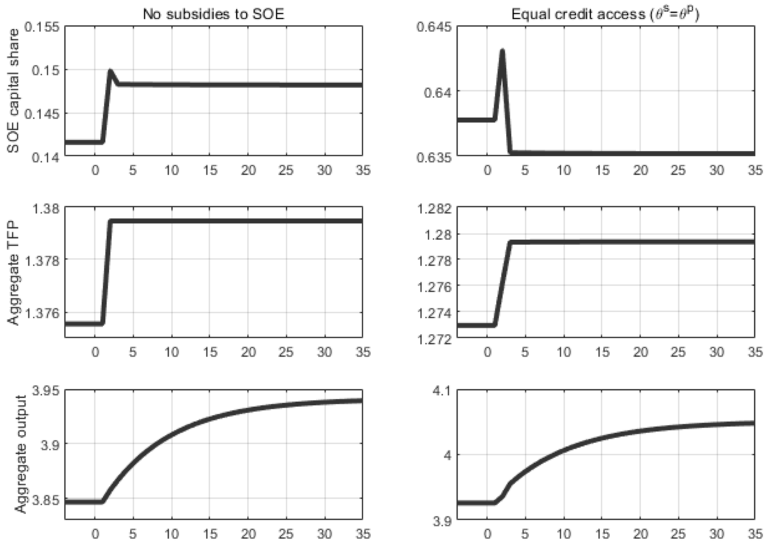

Figure 5.4: Replication of Figure 5 in the Paper

5.5 Counterfactual Dynamic Welfare Analysis: Figure 6

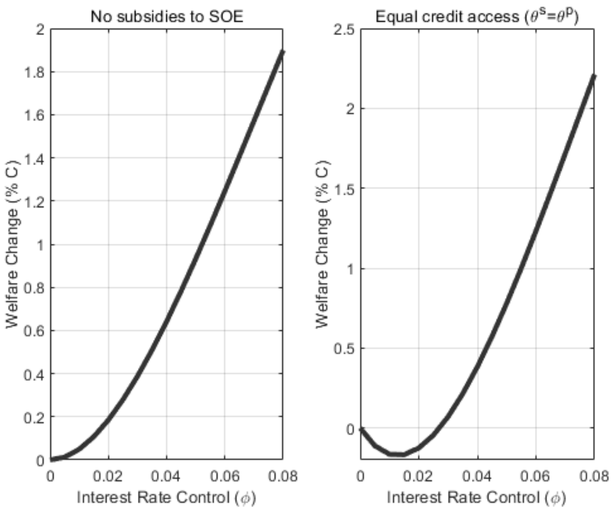

Figure 5.5: Replication of Figure 6 in the Paper

6 Conclusion

In this replication project main focus is the static model, the calibration and the benchmark transition path of the dynamic model. I tried some extensions on solving the static model, including changing parameters and using numerical integration rather than analytical integration. In the calibration and the benchmark transition path of the dynamic model, I recoded all the files and provided detailed annotation on each files which explicitly explain what the authors are doing during the code. I try to understand all the technical details of code and I have learned the following tricks that can greatly help my further research:

- The basic numerical framework to solve out a Ramsey type of infinite-living agent model with exogenous policy shocks: Solve price vector that clear the market in steady state $\Rightarrow$ Solve other real variable in the old and new steady state $\Rightarrow$ Solve the optimal dynamic allocation problem using Dynare;
- How to solve the single price that clear the market: Define a function of $|RHS(P) - LHS(p)|$ of the market clearing condition, and then search the minimized value the *fminsearchbnd.m* or *fminbnd.m* to solve the minimized value. Finally, check whether the function take 0 (if yes, market clearing condition holds; if not, the market clearing condition does not hold)
- How to solve the price vector that clear the market: Similar as before, define a system functions of $|RHS(P) - LHS(p)|$ of the market clearing conditions, and then solve out using **fslove.m**. Cleaver guess of the initial value is required and we also need to check whether the market clearing conditions holds.
- Dynare deterministic simulation.

In addition, I learn many technical details that is very useful:

- The thought of *functional programming*: It is usually better to define functions and recall in functions, rather than finish the code in each program. In my first trail, I tried to do integral separately which is very time consuming.
- The use of *truncation value*: If a simulation is done on a range of parameters, truncation value such as $\min\{xxx, 1e5\}$ is useful to avoid causing errors due to divergence of value.
- The use of analytical result rather than numerical result: the basic the step is, the more important to have a analytical result. As I have tried, using numerical integration takes the computer 5 minutes to replicate figure 2, but it only takes 3 seconds to replicate figure 2 if we use the analytical result: therefore, in building up the model, we need to avoid setting that may result in no close from solution of equations.

Definitely, there are plenty of shortcoming with my replication: due to time constrain, I just redo the benchmark part of the model and I used the original code directly for the counterfactual analysis although it is quite similar to the benchmark part: actually, I spend half an week to derive the analytical result of the model, half an week to refresh myself of the MATLAB. Learning Dynare from 0 to 1 is also time consuming, while it also takes me days to understand the code. Therefore, rather than do an comprehensive replication, I hope my program captures the *core* of the numerical algorithm of this model.

Finally, I would like to sincerely say thank to Prof. Dr. Pengfei Wang for this excellent course. Of all the courses I have taken in my life, I've learned the most from you. This class also makes me full of interest in the field of financial friction: I hope to take this field as my future research direction. In addition, I would also like to sincerely thank Hengxu Song, the assistant teacher of this class. In the past three months, I have received a lot of guidance from teaching assistants. These instructions have taught me a lot. Finally, I would like to thank the three authors of this paper. Your research has profound policy insights and provides a good starting point for my future research.

References

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