Asset-Pricing Moments in Benchmark RBC Model

Xiangyu Ding [dingdxy@connect.hku.hk]; Pantalfini Matteo [panta@connect.hku.hk]

January 27, 2023

1 Model Setup

- Representative Household:
  - Household provides labor H_t, own and lend capital K_t to firm, and trade consumption claim S_t with other household to maximize lifetime expected utility;
  - Consumption claim is traded at price P_t per unit at period t and gives consumption C_{t+1} next period. The simple return of consumption claim is $R^c_{t+1} = \frac{C_{t+1}+P_{t+1}}{P_t} - 1$:
  - Zero coupon Risk free bond is traded at price P^{rf}_t and gives 1 consumption for sure next period. The simple return of risk-free bond is $R^f_t = \frac{1}{P^{rf}_t} - 1$:

$$\max_{\{C_t,H_t,I_t,K_t,S_t,B_t\}} E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} \left(\log(C_\tau) - \theta \frac{H_\tau^{1+\psi}}{1+\psi} \right) \tag{1.1}$$

$$\text{s.t. } C_t + I_t + P_t S_t + P^{rf}_t B_t \leq W_t H_t + R_t K_{t-1} + \Pi_t + (P_t + C_t) S_{t-1} + B_{t-1} \tag{1.2}$$

$$K_t = \exp(b_t) I_t + (1-\delta) K_{t-1} \tag{1.3}$$

$$\lim_{T\to+\infty} \beta^T MU_T K_T = 0 \tag{1.4}$$

- Representative Firm:
  - Firm hires labor and rents capital from household to produce perfectly substitutable capital and consumption goods to maximize firm value:

$$\max_{\{Y_t,\Pi_t,H_t,K_{t-1}\}} E_t \sum_{\tau=t}^{\infty} M_{t,\tau} \Pi_\tau \tag{1.5}$$

$$\text{s.t. } \Pi_t = Y_t - W_t H_t - R_t K_{t-1} \tag{1.6}$$

$$Y_t = \exp(a_t) K_{t-1}^{\alpha} H_t^{1-\alpha} \tag{1.7}$$

- Model Closure:

– Consumption claim market clear:

$$S_t = 0 \tag{1.8}$$

– Risk-free bond market clear:

$$B_t = 0 \tag{1.9}$$

– Goods market clear:

$$C_t + I_t = Y_t \tag{1.10}$$

– Definition of return on consumption claim:

$$R^c_{t+1} = \frac{C_{t+1} + P_{t+1}}{P_t} - 1 \tag{1.11}$$

– Definition of return on risk-free bond:

$$R^f_t = \frac{1}{P^{rf}_t} - 1 \tag{1.12}$$

– Definition of SDF:

$$M_{t+1} = \beta \frac{C_t}{C_{t+1}} \tag{1.13}$$

– Exogenous shocks:

$$\begin{pmatrix} a_t \\ b_t \end{pmatrix} = \begin{pmatrix} \rho & \tau \\ \tau & \rho \end{pmatrix} \begin{pmatrix} a_{t-1} \\ b_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_t \\ \nu_t \end{pmatrix} \tag{1.14}$$

2 Dynamic System

The dynamic system contains 11 endogenous variables $(H, C, I, K, Y, M, W, PC, R, R^c, R^f)$ and 2 exogenous variables (a, b). We need to find 13 equations linking endogenous variables and exogenous variables:

- Consumer optimization condition:

  – Consumption-labor trade-off:

$$\theta H_t^{\psi} = \frac{1}{C_t} W_t \tag{2.1}$$

  – Capital Euler equation:

$$E_t \left[M_{t+1} \frac{\exp(b_t)}{\exp(b_{t+1})} \left(\exp(b_{t+1}) R_{t+1} + 1 - \delta \right) \right] = 1 \tag{2.2}$$

  – Consumption claim investment Euler equation:

$$E_t \left[M_{t+1} \frac{P_{t+1} + C_{t+1}}{P_t} \right] = E_t \left[M_{t+1} \frac{1 + PC_{t+1}}{PC_t} \frac{C_{t+1}}{C_t} \right] = 1 \tag{2.3}$$

- Risk-free bond investment Euler equation:

$$E_t\left[M_{t+1}\left(1+R_t^f\right)\right]=1 \tag{2.4}$$

- Definition of consumption claim return:

$$R_t^c=\frac{1+PC_t}{PC_{t-1}}\frac{C_t}{C_{t-1}}-1 \tag{2.5}$$

- Definition of SDF:

$$M_t=\beta\frac{C_{t-1}}{C_t} \tag{2.6}$$

- Capital dynamic:

$$K_t=\exp\left(b_t\right)I_t+(1-\delta)K_{t-1} \tag{2.7}$$

- Budget constraint:

$$C_t+I_t=W_tH_t+R_tK_{t-1} \tag{2.8}$$

- Firm optimization:
  - Production function:

$$Y_t=\exp\left(a_t\right)K_{t-1}^{\alpha}H_t^{1-\alpha} \tag{2.9}$$

  - Optimal capital:

$$R_t=\alpha\frac{Y_t}{K_{t-1}} \tag{2.10}$$

  - Optimal labor:

$$W_t=(1-\alpha)\frac{Y_t}{H_t} \tag{2.11}$$

- Exogenous process:
  - Productivity:

$$a_t=\rho a_{t-1}+\tau b_{t-1}+\varepsilon_t \tag{2.12}$$

  - Investment efficiency:

$$b_t=\tau a_{t-1}+\rho b_{t-1}+\nu_t \tag{2.13}$$

  - shocks:

$$E\left(\varepsilon_t\right)=0 \tag{2.14}$$

$$E\left(\nu_t\right)=0 \tag{2.15}$$

$$E\left(\varepsilon_t\varepsilon_s\right)=\begin{cases}\sigma_\varepsilon^2 & \text{if} \quad t=s\\ 0 & \text{if} \quad t\neq s\end{cases} \tag{2.16}$$

$$E\left(\nu_t \nu_s\right) = \begin{cases} \sigma_\nu^2 & \text{if} \quad t = s \\ 0 & \text{if} \quad t \neq s \end{cases} \tag{2.17}$$

$$E\left(\varepsilon_t \nu_s\right) = \begin{cases} \varphi \sigma_\varepsilon \sigma_\nu & \text{if} \quad t = s \\ 0 & \text{if} \quad t \neq s \end{cases} \tag{2.18}$$

3 Deterministic Steady State

- Assume that $\varepsilon_t = \nu_t = 0, \forall t$:
  - From investment shocks and productivity shocks:

$$a_{ss} = 0 \tag{3.1}$$

$$b_{ss} = 0 \tag{3.2}$$

  - From definition of SDF:

$$M_{ss} = \beta \tag{3.3}$$

  - From consumption claim investment Euler equation:

$$PC_{ss} = \frac{\beta}{1-\beta} \tag{3.4}$$

$$R_{ss}^c = \frac{1}{\beta} - 1 \tag{3.5}$$

  - From risk-free bond investment Euler equation:

$$R_{ss}^f = \frac{1}{\beta} - 1 \tag{3.6}$$

  - From capital Euler equation:

$$R_{ss} = \frac{1}{\beta} - (1-\delta) \tag{3.7}$$

  - From capital dynamic:

$$I_{ss} = \delta K_{ss}$$

  - Calculate labor capital ratio from optimal capital and production function:

$$\frac{1}{\beta} - (1-\delta) = \alpha \left(\frac{H_{ss}}{K_{ss}}\right)^{1-\alpha}$$

$$HK_{ss} \equiv \frac{H_{ss}}{K_{ss}} = \left(\frac{1}{\alpha\beta} - \frac{1-\delta}{\alpha}\right)^{\frac{1}{1-\alpha}} \tag{3.8}$$

– Calculate consumption capital ratio from budget constraint:

$$\frac{C_{ss}}{K_{ss}} + \delta = \left(\frac{H_{ss}}{K_{ss}}\right)^{1-\alpha}$$

$$CK_{ss} \equiv \frac{C_{ss}}{K_{ss}} = \frac{1}{\alpha\beta} - \frac{1-\delta}{\alpha} - \delta \tag{3.9}$$

– Combine Consumption-labor trade-off and optimal labor and production function:

$$\theta H_{ss}^{\psi} = \frac{1}{C_{ss}}(1-\alpha)\left(HK_{ss}\right)^{-\alpha} = \frac{1}{K_{ss}CK_{ss}}(1-\alpha)\left(HK_{ss}\right)^{-\alpha}$$

$$\theta H_{ss}^{\psi+1} = \frac{HK_{ss}}{CK_{ss}}(1-\alpha)\left(HK_{ss}\right)^{-\alpha}$$

$$H_{ss} = \left[\frac{1}{\theta}\frac{HK_{ss}}{CK_{ss}}(1-\alpha)\left(HK_{ss}\right)^{-\alpha}\right]^{\frac{1}{\psi+1}} \tag{3.10}$$

– Finally equilibrium capital, consumption, investment, and production:

$$K_{ss} = \frac{H_{ss}}{HK_{ss}} \tag{3.11}$$

$$C_{ss} = CK_{ss}K_{ss} \tag{3.12}$$

$$I_{ss} = \delta K_{ss} \tag{3.13}$$

$$Y_{ss} = K_{ss}^{\alpha}H_{ss}^{1-\alpha} \tag{3.14}$$

4 Dynare Code

```
%% Macro Finance Homework 1: Solve asset prices in RBC Model
% Author: Xiangyu DING and Pantalfini Matteo
% Xiangyu DING <dingdxy@connect.hku.hk>; Pantalfini Matteo <panta@connect.hku.hk>

% The model is a modified version of https://www.dynare.org/assets/tutorial/guide.pdf

%% 1. Variables
var
h       $H$                 (long_name = 'Labor')
c       $C$                 (long_name = 'Consumption')
ii      $I$                 (long_name = 'Investment')
k       $K$                 (long_name = 'Capital')
y       $Y$                 (long_name = 'Production')
m       $M$                 (long_name = 'SDF')
w       $W$                 (long_name = 'Wage')
a       $a$                 (long_name = 'Log Productivity')
b       $b$                 (long_name = 'Log Investment Efficiency')
pc      $PC$                (long_name = 'Price to Consumption Ratio')
r       $R$                 (long_name = 'Return on Capital')
rc      $R^c$               (long_name = 'Return on Consumption Claim')
rf      $R^f$               (long_name = 'Risk Free Return');
```

```

varexo
e       $\varepsilon_t$     (long_name = 'Productivity Shocks')
u       $\nu_t$             (long_name = 'Investment Efficiency Shocks');

%% 2. Parameters
parameters
BETA    $\beta$             (long_name = 'Discount Factor')
ALPHA   $\alpha$            (long_name = 'Capital Share')
DELTA   $\delta$            (long_name = 'Depreciation Rate')
THETA   $\theta$            (long_name = 'Labor Disutility Level')
PSI     $\psi$              (long_name = 'Fisher Elasticity of Labor')
RHO     $\rho$              (long_name = 'VAR Shock Self-Autogression')
TAU     $\tau$              (long_name = 'VAR Shock Spillover');

%% 3. Calibration
% We use the value provided from dynare tutorial, see following link
% https://www.dynare.org/assets/tutorial/guide.pdf
% Quarter (4 month) value since BETA  = 0.99
ALPHA = 0.36;
RHO   = 0.95;
TAU   = 0.025;
BETA  = 0.99;
DELTA = 0.025;
PSI   = 0;
THETA = 2.95;

PHI   = 0.1;

%% 4. Steady State Values
a_ss = 0;
b_ss = 0;
m_ss = BETA;
pc_ss = BETA/(1-BETA);
rc_ss = 1/BETA - 1 ;
rf_ss = 1/BETA - 1 ;
r_ss = 1/BETA - (1-DELTA);

% hk for h/k: labor to capital ratio
hk_ss = (1/(ALPHA*BETA) - (1-DELTA)/ALPHA)^(1/(1-ALPHA));

% ck for c/k: consumption to capital ratio
ck_ss =  1/(ALPHA*BETA) - (1-DELTA)/ALPHA - DELTA;
h_ss = ((1/THETA)*(hk_ss/ck_ss)*(1-ALPHA)*(hk_ss^(-ALPHA)))^(1/(1+PSI));
k_ss = h_ss/hk_ss;
c_ss = ck_ss*k_ss;
ii_ss = DELTA*k_ss;
y_ss = (k_ss^ALPHA)*(h_ss^(1-ALPHA));
w_ss = (1-ALPHA)*y_ss/h_ss;


%% 5. Model
model;
[name = '1. Consumption-labor trade-off']
THETA*(h^PSI) = w/c;
```

```
[name = '2. Capital Euler equation']
m(+1)*(exp(b)/exp(b(+1)))*(exp(b(+1))*r(+1)+1-DELTA)=1;

[name = '3. Consumption claim investment Euler equation']
m(+1)*((pc(+1)+1)/pc)*(c(+1)/c)=1;

[name = '4. Risk-free bond investment Euler equation']
m(+1)*(rf+1) = 1;

[name = '5. Definition of consumption claim return']
rc = (pc+1)*(c/c(-1))/pc(-1) - 1;

[name = '6. Definition of SDF']
m = BETA*c(-1)/c;

[name = '7. Capital dynamic']
k = exp(b)*ii+(1-DELTA)*k(-1);

[name = '8. Budget constraint']
c+ii = w*h+r*k(-1);

[name = '9. Production function']
y = exp(a)*(k(-1)^ALPHA)*(h^(1-ALPHA));

[name = '10. Optimal capital']
r = ALPHA*y/k(-1);

[name = '11. Optimal labor']
w = (1-ALPHA)*y/h;

[name = '12. Productivity shocks']
a = RHO*a(-1)+TAU*b(-1) + e;

[name = '13. Investment shocks']
b = TAU*a(-1)+RHO*b(-1) + u;
end;

%% 6. Initative Values
initval;
h = h_ss;
c = c_ss;
ii = ii_ss;
k = k_ss;
y = y_ss;
m = m_ss;
r = r_ss;
w = w_ss;
a = a_ss;
b = b_ss;
pc = pc_ss;
rc = rc_ss;
rf = rf_ss;
end;

```

```
%% 7. Check steady-state
steady;
check;

%% 8. Exogenous shocks
shocks;
var e; stderr 0.009;
var u; stderr 0.009;
var e, u = PHI*0.009*0.009;
end;

%% 9. Simulation
stoch_simul(periods=200000, order = 2,pruning ,nograph);

%% 10. Write LaTex File

write_latex_definitions;
write_latex_parameter_table;
write_latex_original_model;
write_latex_dynamic_model;
write_latex_static_model;
collect_latex_files;

if system(['pdflatex -halt-on-error -interaction=batchmode ' M_.fname '_TeX_binder.tex'])
    warning('TeX-File did not compile; you need to compile it manually')
end

%% 11. Calculate Moment
a_sd = diag(sqrt(oo_.var))*sqrt(4)*100;
a_moment = [
    oo_.mean(11)*400 ...
    (oo_.mean(11)-DELTA)*400 ...
    a_sd(11) ...
    oo_.mean(12)*400 ...
    a_sd(12)...
    oo_.mean(13)*400 ...
    a_sd(13)]';

Labels =['E[r]              ';
         'E[r]-\delta       ';
         'Sd[r]             ';
         'E[rc]             ';
         'sd[rc]            ';
         'E[rf]             ';
         'sd[rf]            '];
[Labels num2str(a_moment,'%2.3f')]
```

5 Result

Table 1: Quarter Calibration (as in Dynare Tutorial)

Parameter	**Value**	**Description**
β	0.990	Discount Factor
α	0.360	Capital Share
δ	0.025	Depreciation Rate
θ	2.950	Labor Disutility Level
ψ	0.000	Fisher Elasticity of Labor
ρ	0.950	VAR Shock Autoregression
τ	0.025	VAR Shock Spillover
σ_ε	0.009	Productivity Shock Std
σ_ν	0.009	Investment Efficiency Shock Std
φ	0.100	Shocks Correlation

Table 2: Asset Pricing Moments

Parameter	**Value**	**Description**
$E[R]$	14.036%	Mean Capital Return Before Depreciation
$E[R]-\delta$	4.036%	Mean Capital Return After Depreciation
$Std[R]$	0.446%	Std Capital Return
$E[R^c]$	4.049%	Mean Consumption Claim Return
$Std[R^c]$	1.337%	Std Consumption Claim Return
$E[R^f]$	4.038%	Mean Risk-free Return
$Std[R^f]$	0.518%	Std Risk-free Return

As documented by Jermann (1998), Table 2 illustrates that benchmark RBC model with no habit formation and no adjustment cost cannot match asset-pricing moments. Because firms can perfectly adjust capital facing productivity shocks, capital (and consumption) are nearly risk free in this model.

References

Jermann, U. J. (1998). Asset pricing in production economies. *Journal of monetary Economics*, *41*(2), 257–275.