Suppose we have a Hilbert space with elements labelled $\{|s\rangle\}$. Suppose G is a group with action $g \mapsto \chi(g)U_g$ for $\chi(g) \in \mathbb{C}, U_g$ an operator. Here $\chi(g)$ depends on the representation. For instance, for translation on n sites, $G = \mathbb{Z}_N$, and $r \mapsto e^{-ikr}T_r$.

Let's understand how we can restrict to a certain symmetry class. Suppose that we have a Hamiltonian H with $[H, U_g] = 0$ for all $g \in G$. Then we should restrict to equivalence classes of states which are eigenvalues of the U_g's. Define

$$|[s]\rangle = \frac{1}{\sqrt{N_s \|G\|}} \sum_{g \in G} \chi(g) U_g |s\rangle \tag{1}$$

where

$$N_s = \sum_{g \in G, g(s)=s} \chi(g). \tag{2}$$

N_s is almost the stabilizer of s, but due to the factors it can take on other values — or be zero, in which case there may not be any equivalence class for that symmetry and eigenvalue. For instance, $|a\rangle = |\uparrow\uparrow\uparrow\uparrow\rangle$ has $k = 1$ equivalence class $|a\rangle - i|a\rangle - |a\rangle + i|a\rangle = 0$, so we should not consider it.

Now let's think about how to construct the Hamiltonian. Consider $H = \sum_j \mathcal{O}_j$. WLOG, we can consider each $\mathcal{O}_j$ and then sum the results. We therefore want to efficiently compute $\langle [r] \,|\, \mathcal{O} \,|\, [s] \rangle$. The action of $\mathcal{O}$ is

$$\mathcal{O}|[s]\rangle = \frac{1}{\sqrt{N_s\|G\|}} \sum_{g \in G} \chi(g) U_g \mathcal{O} |s\rangle \qquad = \frac{1}{\sqrt{N_s\|G\|}} \sum_{g \in G} \chi(g) U_g \mathcal{O}_{sr} |r'\rangle$$

so if $r' \in [r]$ with $h \cdot r = r'$, then $U(h)|r\rangle = |r'\rangle$

$$\begin{aligned}
\langle [r] \,|\, \mathcal{O} \,|\, [s] \rangle &= \langle [r]| \, \mathcal{O}_{sr} \frac{1}{\sqrt{N_s\|G\|}} \sum_{g \in G} \chi(g) U_g |r'\rangle \\
&= \langle [r]| \, \mathcal{O}_{sr} \frac{1}{\sqrt{N_s\|G\|}} \sum_{g \in G} \chi(g) U_g U_h |r\rangle \\
&= \mathcal{O}_{sr} \sqrt{\frac{N_r}{N_s}} \frac{1}{\chi(h)} \langle [r]| \, \frac{1}{\sqrt{N_r\|G\|}} \sum_{gh \in G} \chi(g)\chi(h) U_{gh} |r\rangle
\end{aligned}$$

In other words,

$$\langle [r] \,|\, \mathcal{O} \,|\, [s] \rangle = \mathcal{O}_{sr} \sqrt{\frac{N_r}{N_s}} \chi(h^{-1}). \tag{3}$$