Problem Set 1 (ver. 2024)

Econometrics 1

Fall semester 2024

- This is a problem set made by Prof. Tadao Hoshino. You can work on the problems to understand the basics of probability better.
- Some of the problems are advanced and more difficult than those given in exams.

Exercise 1 Let X denote the result of flipping a fair coin: $X = 1$ for head and $X = 0$ for tail.

1. Draw the CDF of X.
2. Calculate $\mathbb{E}[X]$.
3. Calculate $\mathbb{V}[X]$.

Exercise 2 Let X_1 and X_2 be drawn independently from Uniform$[0, 1]$, and define $Y = X_1 + X_2$.

1. Derive the CDF and PDF of Y and draw their graphs.
2. Calculate $\mathbb{E}[Y]$.
3. Calculate $\mathbb{V}[Y]$.

Exercise 3 Let X_1 and X_2 be the results of rolling two fair dices. We define

$$Y = \begin{cases} 0 & \text{if } X_1 = X_2 \\ X_1 + X_2 & \text{otherwise} \end{cases}$$

1. Draw the CDF of Y.
2. Calculate $\mathbb{E}[Y]$.

Exercise 4 Let X_1, X_2, and X_3 be the lengths of three segments independently drawn from Uniform$[0, 1]$. What is the probability that a triangle can be formed from the three segments?

Exercise 5 Let X be a random variable whose PDF is given by

$$f(x) = \begin{cases} \frac{1}{2} & \text{for } x \in [0, 1) \\ \frac{3}{4} - \frac{1}{4}x & \text{for } x \in [1, 3] \\ 0 & \text{otherwise.} \end{cases}$$

1. Derive the CDF of X and draw the graph of it.
2. Calculate $\Pr(X > 8/3)$.

3. Calculate $\Pr(1/2 \leq X \leq 4/3)$.

4. Calculate $\mathbb{E}[X]$.

Exercise 6 Let X be a random variable whose PDF is given by

$$f(x) = \begin{cases} \frac{2}{3} - \frac{2}{9}x & \text{if } x \in [0,3] \\ 0 & \text{otherwise} \end{cases}$$

1. Derive the CDF of X and draw the graph of it.

2. Calculate $\Pr(X \leq 1)$.

3. Calculate $\Pr(X \geq 2)$.

Exercise 7 Let X be a random variable whose PDF is given by

$$f(x) = \begin{cases} \frac{1}{2} + \frac{1}{6}x & \text{for } x \in [-3,0) \\ \frac{1}{2} - \frac{1}{2}x & \text{for } x \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

1. Derive the CDF of X and draw the graph of it.

2. Calculate $\Pr(-1 \leq X)$.

3. Calculate $\mathbb{E}[X]$.

Exercise 8 Let X be a random variable whose PDF is given by

$$f(x) = \begin{cases} a + bx^2 & \text{for } x \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

If $\mathbb{E}[X] = 3/5$, find a and b.

Exercise 9 Let X be a continuous random variable and F be its CDF.

1. Derive the CDF of $aX + b$, where a and b are nonzero constants.

2. Derive the CDF of $|X|$.

3. Derive the CDF of X^2.

Exercise 10 Let X be a continuous random variable and F be its CDF. Prove $\Pr(F(X) \leq \alpha) = \alpha$ for any $\alpha \in [0,1]$ (that is, $F(X) \sim \text{Uniform}[0,1]$).

Exercise 11 Let X be distributed as Uniform$[0,1]$. Derive the CDF and PDF of X^n, where n is some constant.

Exercise 12 Let A be any random event. Prove $\mathbb{E}[\mathbf{1}\{A\}] = \Pr(A)$. Here, $\mathbf{1}\{A\}$ denotes the indicator function which takes one (zero) if A is true (false).

Exercise 13 Prove that if $\mathbb{E}[X^2] = (\mathbb{E}[X])^2$, X is a constant (that is, $\Pr(X = a) = 1$ for some a).

Exercise 14 (Jensen's inequality) Prove $\mathbb{E}[X^2] \geq (\mathbb{E}[X])^2$ for any random variable X.

Exercise 15 (Generalized Markov inequality) Let X be a non-negative random variable, and $M(\cdot)$ be a non-decreasing function. Prove $\Pr(X \geq \varepsilon) \leq \frac{\mathbb{E}[M(X)]}{M(\varepsilon)}$ for any $\varepsilon > 0$.

Exercise 16 Let A and B be random events. Prove that if A and B are independent, A and $\{\text{not}B\}$ are also independent.

Exercise 17 Give an example which demonstrates that zero correlation does not imply independence (other than those introduced during the lecture).

Exercise 18 Let X be a random variable with $\mathbb{E}[X] = 1$ and $\mathbb{V}[X] = 5$. Calculate the following.

1. $\mathbb{E}[(3 + X)^2]$
2. $\mathbb{V}[5 + 5X]$

Exercise 19 Let D be a binary random variable with $\Pr(D = 1) = 0.8$ and $\Pr(D = -1) = 0.2$, and X and Y be normally distributed as $N(0, 3)$ and $N(0, 5)$, respectively. Further, let $Z = DX + (1 - D)Y$. Assume X, Y, and D are independent. Calculate the following:

1. $\mathbb{E}[(XY)^2]$
2. $\mathbb{E}[Z]$
3. $\mathbb{V}[Z]$

Exercise 20 Let X and Y be independent random variables. Suppose that X is continuously distributed as Uniform$[-1, 1]$, $\mathbb{E}[Y] = -2$, and $\mathbb{E}[Y^2] = 6$.

1. Calculate $\mathbb{V}[XY]$.
2. Let D be a dummy variable defined by $D = \mathbf{1}\{X > 1/2\}$. Calculate $\mathbb{E}[DY]$.

Exercise 21 Let X and U be random variables, and $Y = U + 0.5X$. Suppose that X is distributed as $N(0, 3)$, U is independent of X, and $\mathbb{E}[U] = 1$. Calculate the following:

1. $\mathbb{E}[Y]$
2. $\mathbb{E}[Y|X]$
3. $\mathbb{E}[4XY]$. Use the law of iterated expectations.

Exercise 22 Suppose that X and Y are independent. Prove $\mathbb{V}(X+Y) = \mathbb{V}(X) + \mathbb{V}(Y)$.

Exercise 23 Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ be random variables. Prove $\mathbb{C}\left(\sum_{i=1}^n X_i, \sum_{j=1}^m Y_j\right) = \sum_{i=1}^n \sum_{j=1}^m \mathbb{C}(X_i, Y_j)$.

Exercise 24 Suppose that $X_1, \ldots, X_n$ are all independent. Let $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ denote the sample average of X. Compute the variance of $\overline{X}_n$.

Exercise 25 Let X be a random variable distributed as the standard normal $N(0,1)$. Further, define $Y = X^2$. Show $\mathbb{C}(Y, X) = 0$. (Fact: $\mathbb{E}(X^3) = 0$ holds for a standard normal X.)

Exercise 26 Let X_1 and X_2 be the results of rolling two fair dices. We define $Y = X_1 + X_2$. Compute the following.

1. $\mathbb{E}[Y]$
2. $\mathbb{E}[Y|X_1 \text{ and } X_2 \text{ are both even}]$
3. $\mathbb{E}[Y|X_1 \neq X_2]$

Exercise 27 (Law of total variance) Prove $\mathbb{V}(Y) = \mathbb{E}[\mathbb{V}(Y|X)] + \mathbb{V}(\mathbb{E}[Y|X])$, where $\mathbb{V}(Y|X)$ is the conditional variance defined as

$$\mathbb{V}(Y|X) = \mathbb{E}\left[(Y - \mathbb{E}[Y|X])^2 \mid X\right].$$

Exercise 28 Let X and Y be random variables. Prove $\mathbb{E}[Y|X = a] = \mathbb{E}[Y|M(X) = M(a)]$ for any strictly increasing function $M(\cdot)$.

Exercise 29 Show that if X and Y are identically distributed, $\mathbb{C}(X+Y, X-Y) = 0$.

Exercise 30 (Convolution) Let X and Y be independent continuous random variables with marginal CDFs F_X and F_Y and marginal PDFs f_X and f_Y, respectively. Derive the CDF and PDF of $X+Y$.