LINEAR ALGEBRA: SUMMARY #04

1. Matrices

Let n and m be two positive integers. An array of $n \times m$ numbers

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

is called an m by n *matrix*. It has m *rows* and n *columns*. The number a_{ij} is called the ij–*entry* or ij–*component* of the matrix.

The rows of the matrix may be viewed as n–tuples and the columns as m–tuples. A (row) *vector*

$$(x_1, \dots, x_n)$$

is a $1 \times n$ matrix, and a *column vector*

$$\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$$

is a $m \times 1$ matrix.

Two matrices have the same *size* if they have the same number of columns and the same number of rows. Two matrices A and B are *equal* if they have the same size and their components are equal; $a_{ij} = b_{ij}$ for every i, $1 \leqslant i \leqslant m$, and every j, $1 \leqslant j \leqslant n$.

The *zero matrix* is the matrix with all its components equal to zero,

$$\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

2. Matrix operations

Let A and B be two matrices of the same size; their *sum*, $A+B$, is the matrix whose ij–component is $a_{ij}+b_{ij}$. In other words, matrices of the same size can be added componentwise.

Notice that when A and B are two $1\times n$ matricides, the matrix addition considers with the vector addition defined earlier.

Let O be the zero matrix. Then for any matrix A (of the same size as O)

$$A+O=O+A=A.$$

Let c be a number and A be a matrix, their product, cA, is the matrix whose ij–component is ca_{ij}.

For any matrix A, we define $-A=(-1)A$ to be the matrix whose ij–component is $-a_{ij}$. It is straightforward to see that

$$A+(-A)=O.$$

For that reason, the matrix $-A$ is called the *additive inverse* of A.

3. Transposed, square, and symmetric matrices

Let A be an $m\times n$ matrix. The $n\times m$ matrix B, such that $b_{ji}=a_{ij}$, is the *transpose* of A, and is denoted ny A^T. Taking the transpose of a matrix amounts to turning rows into columns and columns into rows.

A *square matrix* has the same number of rows and columns, $m=n$. The components a_{ii} of the square matrix A are called the *diagonal* components of A.

A *symmetric matrix* is a square matrix A that is equal to its transpose, $A=A^T$ or, in terms of the components, $a_{ij}=a_{ji}$. A *skew-symmetric matrix* is a square matrix A such that $A=-A^T$ or, in terms of the components, $a_{ij}=-a_{ji}$.

4. Numerical example

Let
$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 5 & 1 & -1 \\ 2 & 1 & -1 \end{pmatrix}.$$

Both A and B are 2×3 matrices, each has two rows and three columns. Thus, the matrices A and B are of the same size. The rows of the matrix A are
$$(1, 1, -2) \quad \text{and} \quad (-1, 4, -5)$$
and the columns of the matrix A are
$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -2 \\ -5 \end{pmatrix}.$$

The sum of A and B is
$$A + B = \begin{pmatrix} 6 & 0 & -1 \\ 4 & 4 & 3 \end{pmatrix}.$$

Let $c = 2$, then
$$2A = \begin{pmatrix} 2 & -2 & 0 \\ 4 & 6 & 8 \end{pmatrix} \quad \text{and} \quad 2B = \begin{pmatrix} 10 & 2 & -2 \\ 4 & 2 & -2 \end{pmatrix}.$$

The additive inverses of the matrices A and B are
$$-A = \begin{pmatrix} -1 & 1 & 0 \\ -2 & -3 & -4 \end{pmatrix} \quad \text{and} \quad -B = \begin{pmatrix} -5 & -1 & 1 \\ -2 & -1 & 1 \end{pmatrix}.$$

The transposes of the matrices A and B are
$$A^T = \begin{pmatrix} 1 & 2 \\ -1 & 3 \\ 0 & 4 \end{pmatrix} \quad \text{and} \quad B^T = \begin{pmatrix} 5 & 2 \\ 1 & 1 \\ -1 & -1 \end{pmatrix}.$$