LINEAR ALGEBRA:
CLASS #03: SUMMARY.

1. Generalized Pythagoras theorem

Theorem. *If vectors* A *and* B *are orthogonal, then*

$$\|A+B\|^2 = \|A\|^2 + \|B\|^2.$$

2. Useful result

If A is orthogonal to B, and x is any number, then A is also orthogonal to xB because:

$$A \cdot xB = xA \cdot B = x(A \cdot B) = 0.$$

3. Projection

Definition. The *component* of A along B is the number

$$c = \frac{A \cdot B}{B \cdot B}.$$

The *projection* of A along B is the vector cB.

Notice that $A - cB$ is orthogonal to B, because:

$$(A - cB) \cdot B = A \cdot B - c(B \cdot B) = A \cdot B - \frac{A \cdot B}{B \cdot B}(B \cdot B) = 0$$

Thus A can be written as the sum of two orthogonal vectors cB and $A - cB$.

4. Scalar product: Geometric interpretation

If θ is the angle between A and B, then

$$A \cdot B = \|A\|\|B\| \cos\theta$$

5. Schwarz inequality

Theorem (Schwarz[1] inequality). *If* A *and* B *are vectors, then*

$$|A \cdot B| \leqslant \|A\|\|B\|$$

[1]Hermann Schwarz (1843–1921), was a German mathematician

Proof. If $B = 0$, then both sides of the inequality are equal.

If $B \neq 0$, write

$$A = A - cB + cB.$$

By Pythagoras theorem:

$$\|A\|^2 = \|A - cB\|^2 + \|cB\|^2 = \|A - cB\|^2 + c^2\|B\|^2.$$

Thus, $c^2\|B\|^2 \leqslant \|A\|^2$ and since

$$c^2\|B\|^2 = \frac{(A \cdot B)^2}{(B \cdot B)^2}\|B\|^2 = \frac{|A \cdot B|^2}{\|B\|^2},$$

it follows that

$$\frac{|A \cdot B|^2}{\|B\|^2} \leqslant \|A\|^2.$$

Multiplying both sides by $\|B\|^2$ and taking the square root concludes the proof. □

6. Triangle inequality

Theorem (Triangle inequality). *If* A *and* B *are vectors, then*

$$\|A + B\| \leqslant \|A\| + \|B\|$$

Proof. Consider

$$\|A + B\|^2 = (A + B) \cdot (A + B) = A \cdot A + 2\ A \cdot B + B \cdot B.$$

By Schwarz inequality:

$$A \cdot A + 2\ A \cdot B + B \cdot B \leqslant \|A\| + 2\ \|A\|\|B\| + \|B\|,$$

and the right-hand side is just

$$(\|A\| + \|B\|)^2.$$

Thus,

$$\|A + B\|^2 \leqslant (\|A\| + \|B\|)^2$$

Taking the square root concludes the proof. □