Waseda University
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Homework 1

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Table of contents

Exercise 1

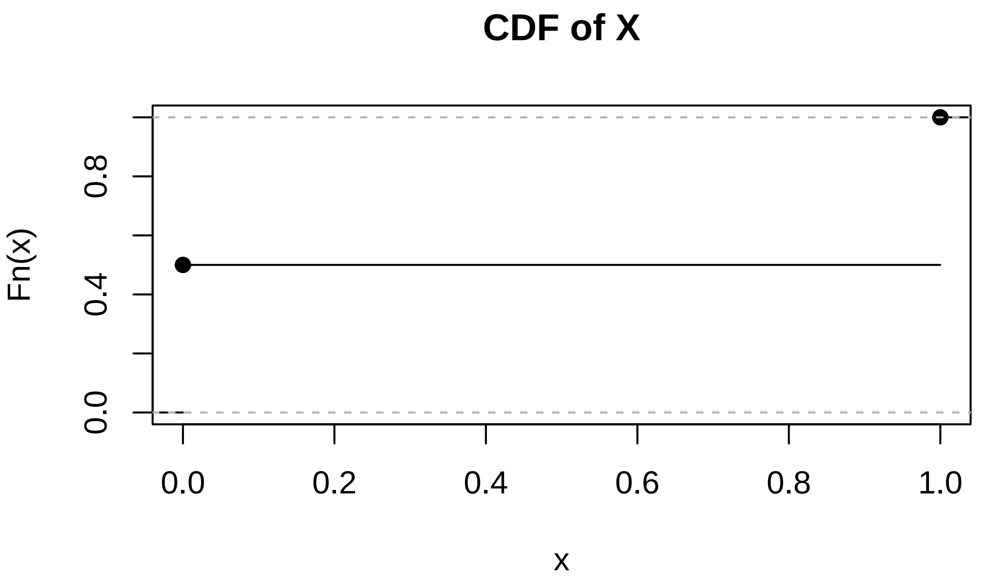
Let X denote the result of flipping a fair coin: X = 1 for head and X = 0 for tail.

1. Draw the CDF of X.

```
# Random variable X  between 0 and 1 where X = 1 is head and X = 0 is tail.
X <- c(1, 0)
X
```

```
[1] 1 0
```

```
# 1. CDF of X
F <- ecdf(X)
plot(F, xlim = c(0,1), main = "CDF of X")
```



2. Calculate E(X).

The expected value of X is given by:

$$E(X) = \sum_{i=1}^{n} x_i p_i$$

where x_i is the value of the random variable and p_i is the probability of the random variable taking the value x_i.

Calculating it will look like this:

$$E(X) = 1 \cdot 0.5 + 0 \cdot 0.5 = 0.5$$

```r
# 2. E(X)
mean(X)
```

```
[1] 0.5
```

3. Calculate Var(X).

The variance of X is given by:

$$Var(X) = E(X^2) - E(X)^2$$

where $E(X^2)$ is the expected value of the random variable squared.

Calculating it will look like this:

$$Var(X) = E(X^2) - E(X)^2 = 1^2 \cdot 0.5 + 0^2 \cdot 0.5 - 0.5^2 = 0.25$$

```r
# 3. Var(X)
# var in R is the sample variance
# so we need to calculate the population variance

var(X) * (1-1/length(X))
```

```
[1] 0.25
```

Exercise 7

Let X be a random variable whose PDF is given by:

$$f(x) = \begin{cases} \frac{1}{2} + \frac{1}{6}x & \text{for } x \in [-3, 0) \\ \frac{1}{2} - \frac{1}{2}x & \text{for } x \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

1. Derive the CDF of X and draw a graph of it.

The CDF of X is given by:

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

where $f(t)$ is the PDF of X.

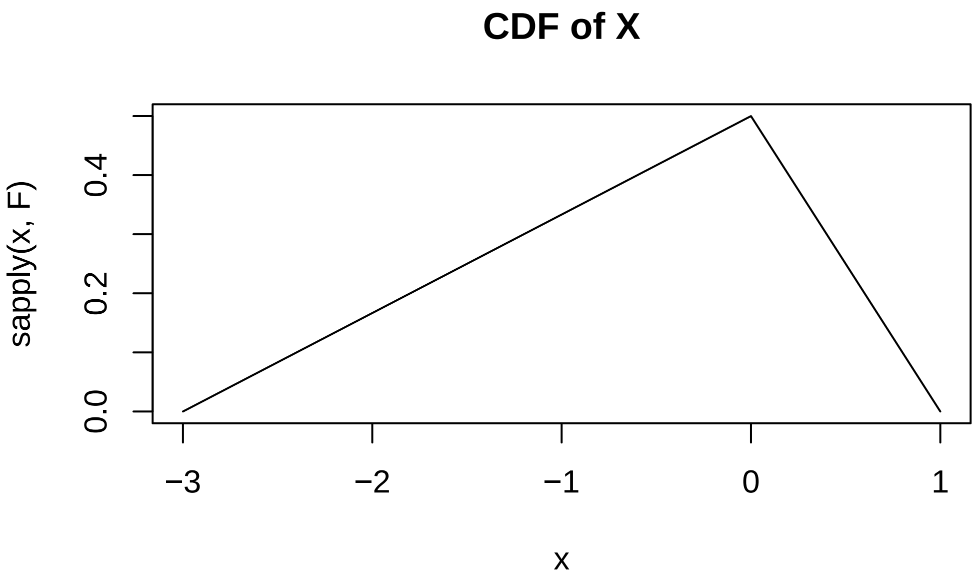
Calculating it will look like this:

$$F(x) = \int_{-3}^{0} \left(\frac{1}{2} + \frac{1}{6}t\right) dt + \int_{0}^{x} \left(\frac{1}{2} - \frac{1}{2}t\right) dt$$

```
# 1. CDF of X
F <- function(x) {
  if (x >= -3 && x < 0) {
    return(1/2 + 1/6 * x)
  } else if (x >= 0 && x <= 1) {
    return(1/2 - 1/2 * x)
  } else {
    return(0)
  }
}

x <- seq(-3, 1, 0.01)

plot(x, sapply(x, F), type = "l", main = "CDF of X")
```



2. Calculate Pr(-1   X).

The probability that X is between -1 and 0 is given by:

$$Pr(-1 \leq X) = F(0) - F(-1)$$

Calculating it will look like this:

$$Pr(-1 \leq X) = F(0) - F(-1) = \left(\frac{1}{2} - \frac{1}{2} \cdot 0\right) - \left(\frac{1}{2} + \frac{1}{6} \cdot -1\right)$$

```
# 2. Pr(-1   X)
F(0) - F(-1)
```

```
[1] 0.1666667
```

$$Pr(-1 \leq X) = F(0) - F(-1) = \left(\frac{1}{2} - \frac{1}{2} \cdot 0\right) - \left(\frac{1}{2} + \frac{1}{6} \cdot -1\right) = \frac{1}{2} - \frac{1}{3} = 0.1666667$$

```
# or
(1/2 - 1/2 * 0) - (1/2 + 1/6 * -1)
```

```
[1] 0.1666667
```

3. Calculate E[X].

The expected value of X is given by:

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

where $f(x)$ is the probability density function of X.

Calculating it will look like this:

$$E(X) = \int_{-3}^{0} x \left(\frac{1}{2} + \frac{1}{6}x\right) dx + \int_{0}^{1} x \left(\frac{1}{2} - \frac{1}{2}x\right) dx = -\frac{3}{4} + \frac{1}{12} = -0.6666667$$

```
# 3. E[X]

# Defining the PDF for each interval
f1 <- function(x) x * (1/2 + 1/6 * x)   # For interval [-3, 0)
f2 <- function(x) x * (1/2 - 1/2 * x)   # For interval [0, 1]

# Integrate over each interval
expectation1 <- integrate(f1, lower = -3, upper = 0)$value
expectation2 <- integrate(f2, lower = 0, upper = 1)$value

# Sum the expectations from both intervals
sum(expectation1, expectation2)
```

```
[1] -0.6666667
```

Exercise 18

Let X be a random variable with E[X] = 1 and V[X] = 5. Calculate the following.

1. E$[(3+X)^2]$

The expected value of $(3+X)^2$ is given by:

$$E[(3+X)^2] = E[9+6X+X^2] = 9+6E[X]+E[X^2]$$

Calculating it will look like this:

$$E[(3+X)^2] = 9+6+1+5 = 21$$

```
# 1. E[(3+X)^2]

9 + 6 + 1 + 5
```

```
[1] 21
```

2. V$[5+5X]$

To find the variance of 5 + 5X we can use the property of variance that says:

$$V[a+bX] = b^2V[X]$$

$$V[5+5X] = V[5]+5^2V[X] = 0+25\cdot 5 = 125$$

Exercise 19

Let D be a binary random variable with Pr(D = 1) = 0.8 and Pr(D = -1) = 0.2, and X and Y be normally distributed as N(0,3) and N(0,5), respectively. Further, let Z = DX + (1-D)Y.

Assume X,Y and D are independent.

Calculate the following:

1. E[(XY)^2]

The expected value of $(XY)^2$ is given by:

$$E[(XY)^2] = E[X^2Y^2] = E[X^2]E[Y^2]$$

D does not affect the expected value of X and Y, so we can calculate the expected value of X and Y separately and then multiply them.

Due to the independence of X and Y, the expected value of their product is the product of their expected values.

$$E[X^2] = V[X] + E[X]^2 = 3 + 0 = 3$$

$$E[Y^2] = V[Y] + E[Y]^2 = 5 + 0 = 5$$

Calculating $E[(XY)^2]$ using independence will look like this:

$$E[(XY)^2] = 3 \cdot 5 = 15$$

Exercise 26

Let X_1 and X_2 be the results of two fair dices. We define $Y = X_1 + X_2$.

Compute the following:

1. E[Y]

The expected value of Y is given by:

$$E[Y] = E[X_1 + X_2] = E[X_1] + E[X_2]$$

Since the dice are fair, the expected value of each die is 3.5.

Calculating E[Y] will look like this:

$$E[Y] = 3.5 + 3.5 = 7$$

```
# 1. E[Y]
Dice_1 <- 1:6
Dice_2 <- 1:6

E_Y <-mean(Dice_1) + mean(Dice_2)
E_Y
```

```
[1] 7
```

2. E[Y|X_1 and X_2 are both even]

The expected value of Y given that X_1 and X_2 are both even restrics the possible values to:

$$X_1, X_2 \in \{2, 4, 6\}$$

Now we need to compute all the different outcomes under this condition:

```
# Define the possible values for dice 1 and 2
values <- c(2, 4, 6)

# Generate all possible combinations of dice 1 and 2
combinations <- expand.grid(Dice_1_even = values, Dice_2_even = values)

# Calculate Y = dice 1 + dice 2 for each combo
Y <- combinations$Dice_1_even + combinations$Dice_2_even

# Store results in a vector
```

```
Y_vector <- as.vector(Y)

# Display the outcomes of Y
Y_vector
```

```
[1]  4  6  8  6  8 10  8 10 12
```

The probability distribution for each pair is equally likely, giving 9 possible outcomes for Y since:

$$X_1, X_2 \in \{2, 4, 6\} \ \cdot \{2, 4, 6\}$$

with probability $\frac{1}{9}$ for each pair.

```
# Count the occurrences of each value of Y
Y_counts <- table(Y_vector)
# Extract the values and probabilities as numeric vector
Y_values <- as.numeric(names(Y_counts))
# Calculate the probabilities
Y_probs <- Y_counts / length(Y_vector)

# Calculate the expected value of Y by summing the products of the
#values and probabilities
expected_Y <- sum(Y_values * Y_probs)

expected_Y
```

```
[1] 8
```

```
# This is the easy way to calculate the expected value of Y given that X1 and X2
# are both even
Dice_1_even <- c(2,4,6)
Dice_2_even <- c(2,4,6)

E_Y_even <- mean(Dice_1_even) + mean(Dice_2_even)
E_Y_even
```

```
[1] 8
```

The expected value of Y given that X_1 and X_2 are both even is given by:

$$E[Y|X_1, X_2 \text{ are both even}] = 4 + 4 = 8$$

3. E[Y|$X_1 \neq X_2$]

The expected value of Y given that $X_1 \neq X_2$ restricts the possible values to:

$$X_1, X_2 \in \{1, 2, 3, 4, 5, 6\} \text{ and } X_1 \neq X_2$$

Now we need to compute all the different outcomes under this condition:

```
# Step 1: Define the values for each die
values <- 1:6

# Step 2: Generate all possible combinations of X1 and X2
combinations <- expand.grid(X1 = values, X2 = values)

# Step 3: Filter for combinations where X1 != X2
combinations_diff <- subset(combinations, X1 != X2)

# Step 4: Calculate Y = X1 + X2 for each combination
Y_vector_diff <- combinations_diff$X1 + combinations_diff$X2

# Step 5: Count occurrences of each unique Y value and calculate probabilities
Y_counts_diff <- table(Y_vector_diff)                     # Frequency of each Y
Y_values_diff <- as.numeric(names(Y_counts_diff))         # Unique Y values
Y_probs_diff <- Y_counts_diff / length(Y_vector_diff)     # Probability of each Y

# Step 6: Calculate the expected value E[Y | X1 != X2]
expected_Y_diff <- sum(Y_values_diff * Y_probs_diff)

# Display the expected value
expected_Y_diff
```

```
[1] 7
```

The expected value of Y given that $X_1 \neq X_2$ is given by:

$$E[Y|X_1 \neq X_2] = 7$$

```
# This is the easy way to calculate the expected value of Y given that
#X1 and X2 are different (code works but is technically not correct since I dont remove
# It works this way since we just remove the case where X1 = X2
#from the possible values of X1 and X2
# and the remaining values cover the same range as the original dice
Dice_1_diff <- 1:6
Dice_2_diff <- 1:6

E_Y_diff <- mean(Dice_1_diff) + mean(Dice_2_diff)
E_Y_diff
```

```
[1] 7
```