Machine Learning Exercise 3 - 2021

This exercise is composed of 2 parts:

1. probability theory part

2. coding part

Probability Theory Questions

1. Given a random sample $\{x_1, x_2, \dots, x_n\}$, derive the maximum likelihood estimator p of the Binomial distribution.

$$B(x,p) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$L\left(\prod_{i=1}^{n} B(x,p)\right) = L\left(\prod_{i=1}^{n} \binom{n}{x_i} p^{x_i} (1-p)^{n-x_i}\right)$$
$$= \sum_{i=1}^{n} \left(ln\binom{n}{x_i} + ln(p^{x_i}) + ln(1-p)^{n-x_i} \right)$$
$$= \sum_{i=1}^{n} \left(ln\binom{n}{x_i} + \mathrm{x_i}\, ln(p) + (n-x_i) ln(1-p) \right)$$

כיוון שאחרת P≠ 0,1 $B(x,p) = 0$ ונקבל likelihood מינימלי וברצוננו למצוא מקסימלי. לכן למציאת מקסימום צריך לגזור ולהשוות לאפס.

$$\frac{d}{dp} = \sum_{i=1}^{n} \left(\frac{\mathrm{x_i}}{p} - \frac{n-x_i}{1-p} \right) = 0 \rightarrow n^2 p = \sum_{i=1}^{n} x_i \rightarrow p = \frac{1}{n^2} \sum_{i=1}^{n} x_i$$

2. A student wants to know her chances to pass and fail an exam if she studies and if she doesn't study. From last year's results, she sees that $P(pass) = 60\%$. She also found out that $P(studied|pass) = 95\%$, $P(studied|failed) = 60\%$. You can assume that every student either studied or didn't study, and either passed or failed.
   a. What is her probability of passing the exam if she studies?
   b. What is her probability of passing if doesn't study?

Solution:

a- $$P(pass \mid studied) = \frac{P(pass \cap studied)}{P(studied)} =$$
$$= \frac{P(studied|pass) * P(pass)}{P(studied)} =^{*} \frac{0.95 * 0.6}{0.81} = 0.7$$

$$*P(studied) = P(studied|pass) * P(pass) + P(studied|failed) * P(failed) =$$
$$0.95 * 0.6 + 0.6 * 0.4 = 0.81$$

b- $$P(pass|not - studied) = \frac{P(pass \cap not-studied)}{P(not-studied)} =$$
$$= \frac{P(not - studied|pass) * P(pass)}{P(not - studied)} =^{**} \frac{0.05 * 0.6}{0.19} = 0.158$$

$$**P(not - studied) = 1 - P(studied) = 0.19,$$
$$P(not - studied|pass) = 1 - P(studied|pass) = 0.05.$$

3. Find 3 random variables X, Y, C such that:
   a. $X \perp Y \mid C$ (X and Y are independent given C).
   b. X and Y are not independent.
   c. X, Y are integers such that $3 \le X, Y \le 9$ and C is binary.
   d. The following conditions hold:
      i. $P(1 \le X \le 5) = 0.4$
      ii. $P(1 \le Y \le 5) = 0.4$
      iii. $P(C = 0) = 0.3$

   You need to specify the value of $P(X = x, Y = y, C = c)$. How many relevant values exist?

נגדיר באופן הבא:

X|C	X=[3,5]	X=[6,9]	
C=0	0.15	0.15	0.3
C=1	0.25	0.45	0.7
	0.4	0.6	1

Y|C	Y=[3,5]	Y=[6,9]	
C=0	0.15	0.15	0.3
C=1	0.25	0.45	0.7
	0.4	0.6	1

כעת נראה שההתפלגות הזו מקיימת את התנאים הנתונים:

1. ראשית ניתן לראות שההסתברות של X ושל Y אינה תלויה בערך של C לכן $X \perp Y \mid C$
2. ניתן לראות כי X וY אינם בלתי תלויים. נוכיח:

$P(3 \le X \le 5, 3 \le Y \le 5) =$

$P(3 \le X \le 5, 3 \le Y \le 5 | C = 0) * P(C = 0) + P(3 \le X \le 5, 3 \le Y \le 5 | C = 1) * P(C = 1) =$

$P(3 \le X \le 5) * P(3 \le Y \le 5 | C = 0) P(C = 0) + P(3 \le X \le 5) P(3 \le Y \le 5 | C = 1) *$
$P(C = 1) = 0.0505$

כמובן בשונה מ 0.4*0.4=0.16

3. ניתן לראות כי X ו Y הם אכן משתנים בטווח הנתון בין 3 ל 9.
4. ניתן לראות זאת אכן בטבלה , בהתאם לאיך שבנינו את ההתפלגות, אכן מתקיימות ההסתברויות הללו עבור X וY.

4. The probability of Wolt arriving on time is 0.75.
    a. What is the probability of having 2 on-time meals in a week (7 days)?
    b. What is the probability of having at least 4 on-time meals in a week?
    c. A company of 100 employees recorded the number of on-time meals they had during a particular week and averaged their results. What do you expect the value of that average to be?

Solution:

a. X- number of times that Wolt arriving on time in a week.

$$X \sim B(0.75,7)$$

$$P(X=2) = \binom{7}{2} * 0.75^2 * 0.25^5 = 0.01$$

b. We need to compute $P(X \geq 4)$

$$P(X \geq 4) = 1 - P(X=3) - P(X=2) - P(X=1) - P(X=0) =$$

$$= 1 - \binom{7}{3} 0.75^3 * 0.25^4 - \binom{7}{2} * 0.75^2 * 0.25^5 - \binom{7}{1} * 0.75 * 0.25^6 - \binom{7}{0} * 0.25^7 =$$

$$= 1 - 0.07 = 0.93$$

c. Define Y the number of times that Wolt arriving on time for 100 employees in a week.

$$Y = 100X$$

$$E(Y) = E(100X) = 100E(X) = 100 * 0.75 * 7 = 525_{meals}$$

Coding exercise

Follow the instructions supplied for you in the MAP classifier Jupyter notebook.