A CIRCLE METHOD APPROACH TO K-MULTIMAGIC SQUARES

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ABSTRACT. Here we investigate K-multimagic squares of order N, these are $N \times N$ magic squares with remain magic after raising each element to the kth power for all $1 \leqslant k \leqslant K$. Given $K \geqslant 2$, we consider the problem of establishing the smallest $N_2(K)$ for which there exists *non-trivial* K-multimagic squares of for all $N \geqslant N_2(K)$. Previous results on multimagic squares show that $N_2(K) \ll K^K$ for large K, here we utilize the Hardy-Littlewood circle method and establish the bound $N_2(K) \leqslant 2K(K+1)$.

We additionally utilize the Green-Tao theorem to show that we there also exists infinitely many *non-trivial* prime valued K-multimagic squares of order $N_2(K)$.

1. INTRODUCTION

A $N \times N$ matrix $\mathbf{Z} = (z_{i,j})_{1 \leqslant i,j \leqslant N}$ is a magic square of order N if the sum of the entries in each of its rows, columns, and two main diagonals are equal. The concept of magic squares has fascinated mathematicians and laymen for thousands of years. Although the study of these objects dates back millennia, there are still many unresolved problems concerning magic squares. One of the most famous problems in this area concerns the existence of a 3×3 magic square where each element is a distinct square number. This problem became popularized by Martin Gardner in 1996, and was listed in Richard Guy's book, *Unsolved Problems in Number Theory.*

One may also investigate problems related to so called *multimagic* squares. Given $K \geqslant 2$ we say a matrix $\mathbf{Z} \in \mathbb{Z}^{N \times N}$ is a K-multimagic square of order N or $\mathbf{MMS}(K, N)$ for short if the matrices

$$\mathbf{Z}^{\circ k} = (z_{i,j}^k)_{1 \leqslant i,j \leqslant N},$$

remain magic squares for $1 \leqslant k \leqslant K$. Before we can state our problem of interest we must first discuss *trivial* multimagic squares.

It is clear any multiple of the $N \times N$ matrix of all ones is trivially a $\mathbf{MMS}(K, N)$ for every $K \geqslant 2$. However, this is not the only family of trivial multimagic squares that shows up. Suppose $\mathbf{Z}$ is an $N \times N$ matrix in which every row, column, and both main diagonals contain precisely N distinct symbols. Such matrices are known as doubly diagonalized Latin squares of order

N, or **DDLS**(N) for short. **DDLS**(N) are known to exist for all $N \geqslant 4$, see [7]. Then for $N \geqslant 4$ any mapping of these N symbols to the integers yields a **MMS**(K, N) for every $K \geqslant 2$. These trivial cases motivate the following definition.

Definition 1.1. For $K \geqslant 2$ and $N \in \mathbb{N}$ a **MMS**(K, N) is called *trivial* if it utilizes N or less distinct integers.

We begin by observing that for $N = 1$ or 2 the only **MMS**(K, N) are those with every element being equal. Additionally it is known that every 3×3 magic square may be parametrized by three variables a, b, c as follows

$$\mathbf{Z}(a, b, c) = \begin{bmatrix} c - b & c + (a + b) & c - a \\ c - (a - b) & c & c + (a - b) \\ c + a & c - (a + b) & c + b \end{bmatrix}.$$

Thus, solving for a, b, c such that $\mathbf{Z}(a, b, c)^{\circ 2}$ is a magic square one sees that the only solutions are those with $a = b = 0$. Thus for $K \geqslant 2$ and $1 \leqslant N \leqslant 3$ we conclude that the only **MMS**(K, N) are trivial.

In this paper we investigate the minimal value $N_2(K)$ for which there exists a non-trivial **MMS**$(K, N_2(K))$, this type of question has been considered in the past by several authors (See [2],[6],[11],[12],[15]) via constructions of *normal* multimagic squares. A multimagic square is said to be *normal* if its elements consists of the integers $1, 2, \ldots, N^2$. We give a brief overview of the best known results for non-trivial **MMS**(K, N) below.

K	Upper bound on $N_2(K)$	Attributed to
2	8	Pfeffermann [11]
3	12	Trump [12]
4	243	Fengchu [2]
5	729	Wen [2]
6	4096	Fengchu [2]
$K \geqslant 2$	$(4K - 2)^K$	Zhang, Chen, and Lei [15]

In this paper we establish via the Hardy-Littlewood circle method the following result.

Theorem 1.2. $N_2(K) \leqslant 2K(K + 1) + 1$ *for* $K \geqslant 2$.

This beats previously known results as soon as $K \geqslant 4$ and shows that $N_2(K)$ grows at most quadratically in K rather than potentially exponential in K. One may prove an analogous statement for prime valued **MMS**(K, N) by reapplying the entirety of the circle method where we detect prime solutions instead of integer solutions. This, however, is not necessary as via an argument due to Granville in [8] one may apply the Green-Tao theorem and deduce the following.

Corollary 1.3. *Given* $K \geqslant 2$ *there exists infinitely many non-trivial prime valued* **MMS**(K, N) *for every* $N > 2K(K + 1)$.

One may potentially generalize these results to multimagic d-dimensional hypercubes given one works out the analogues of sections 4 and 5 for the d-dimensional case. Assuming one does this, we expect that the circle method will yield the bound
$$N_d(K) \ll K^2,$$
where $N_d(K)$ is the d-dimensional analogue of $N_2(K)$.

2. OVERVIEW OF THE PAPER

Our aim in this section of the paper is to present a summary of our proof method. Given $K \geqslant 2$ and $N > 2K(K+1)$ we let $M_{K,N}(P)$ denote the number of $\mathbf{MMS}(K, N)$ with entries satisfying
$$\max_{1 \leqslant i,j \leqslant N} |z_{i,j}| \leqslant P.$$
One easily sees that the number of trivial $\mathbf{MMS}(K, N)$ counted by $M_{K,N}(P)$ is $O_N(P^N)$, thus if one wishes to establish the existence of infinitely many non-trivial $\mathbf{MMS}(K, N)$ it is enough to show that
$$\frac{M_{K,N}(P)}{P^N} \to \infty \quad \text{as} \quad N \to \infty. \tag{2.1}$$

We begin by considering a general diagonal system of equations in differing degrees. Let $C = (c_{i,j})_{\substack{1 \leqslant i \leqslant r \\ 1 \leqslant j \leqslant s}} \in \mathbb{Z}^{r \times s}$ be given, and consider the diagonal system
$$\sum_{1 \leqslant j \leqslant s} c_{i,j} x_j^k = 0 \quad (1 \leqslant i \leqslant r, \quad 1 \leqslant k \leqslant K). \tag{2.2}$$
We define $R_K(P; C)$ to be the number of solutions $\mathbf{x} \in \mathbb{Z}^s$ to (2.2) where $\max_j |x_j| \leqslant P$. This class of problems has been investigated in the past by various authors (see [3], [4]). However, in their application of the circle method they require the $r \times s$ matrix of coefficients C to be *highly non-singular*, i.e., for all $J \subset \{1, \ldots, s\}$ with $|J| = r$ one should have
$$\det (c_{i,j})_{\substack{1 \leqslant i \leqslant r \\ j \in J}} \neq 0.$$
Upon examination of these methods, however, one sees that a slightly weaker condition on the matrix C would suffice. This weaker condition has been used previously by Brüdern and Cook [5], who investigated diagonal systems of a fixed degree k. This is in fact crucial because, upon investigation, one sees that the matrix of coefficients associated to multimagic squares is certainly not highly non-singular.

These considerations lead us to define a notion of when a matrix C *dominates* a function. We establish in section 3 an asymptotic formula for $R_K(P; C)$ provided C dominates an appropriate function. Before stating our results we must establish some notation.

For a given $r \times s$ matrix $C = [\mathbf{c}_1, \ldots, \mathbf{c}_s]$ and any set $J \subset \{1, \ldots, s\}$, we denote by C_J the submatrix of C consisting of the columns $\mathbf{c}_j$ where $j \in J$. For any $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ we denote by $\text{rem}(a, b)$ the remainder of a modulo b considered as an integer between 0 and $b - 1$.

Definition 2.1. We say that a matrix C dominates a function $f : \mathbb{N} \to \mathbb{R}^+$ whenever the inequality

$$\operatorname{rank}(C_J) \geqslant \min \left\{ f(|J|), r \right\},$$

holds for all $J \subset \{1, \ldots, s\}$.

Theorem 2.2. *Let $K \geqslant 2$ and suppose that $C \in \mathbb{Z}^{r \times s}$ satisfies $s > rK(K+1)$. Then, if C dominates the function*

$$F(x) = \max \left\{ \frac{x - \operatorname{rem}(s, r)}{\lfloor \frac{s}{r} \rfloor}, \frac{x - \operatorname{rem}(s-1, r)}{\lfloor \frac{s-1}{r} \rfloor} \right\},$$

one has that

$$R_K(P; C) = P^{s - \frac{rK(K+1)}{2}} \left(\sigma_K(C) + o(1) \right),$$

where $\sigma_K(C) \geqslant 0$ is a real number depending only on K and C. Additionally $\sigma_K(C) > 0$ if there exists non-singular real and p-adic solutions to the system (2.2).

Theorem 2.2 can be seen as a relaxation on the condition that C be highly non-singular, which would be equivalent to C dominating the identity function.

In section 4 we establish the existence of a matrix $D_N \in \{-1, 0, 1\}^{2N \times N^2}$ for which $M_{K,N}(P) = R_K(P; D_N)$ and establish that this matrix dominates $F(x)$ from Theorem 2.2. This is done via a combinatorial argument and understanding the underlying linear system associated to the matrix D_N. In section 5 we establish that $\sigma_K(C) > 0$. This is done by showing that a **DDLS**(N) with distinct integer symbols is a non-singular integer solution to the system (2.2) with $C = D_N$.

Thus from the conclusions made in sections 4 and 5 in combination with Theorem 2.2 we deduce the following.

Theorem 2.3. *For $K \geqslant 2$ and $N > 2K(K+1)$ there exists a constant $c > 0$ for which one has the asymptotic formula*

$$M_{K,N}(P) \sim cP^{N(N-K(K+1))}.$$

Whence by (2.1) we finally establish Theorem 1.2.

3. APPLICATION OF THE CIRCLE METHOD

Our basic parameter is P an, eventually, large positive number. Whenever ε appears in a statement, either implicitly or explicitly, we assert that the statement holds for every $\varepsilon > 0$. In this paper, implicit constants in Vinogradov's notation $\ll$ and $\gg$ may depend on ε, r, s, and the elements of the matrix C. We also make use of the vector notation $\mathbf{x} = (x_1, \ldots, x_r)$ where r is dependent on the context of the argument. Whenever the notation $|\mathbf{x}|$ is used for a vector or matrix we mean the maximal absolute value the elements of $\mathbf{x}$. When $q \in \mathbb{N}$ and $\mathbf{a}$ is an integer matrix we write $(q, \mathbf{a})$ to denote the simultaneous greatest common divisor q and the elements of $\mathbf{a}$. We use $\|x\|$ to refer to the distance to nearest integer of x. We will occasionally define

functions of matrices which will change depending on the number of columns of the matrix. As is conventional in analytic number theory, we write $e(z)$ for $e^{2\pi i z}$. Additionally, we write $[n]$ to denote the set of integers from 1 of to n.

We now proceed to define our basic exponential generating functions necessary for our application of the Hardy-Littlewood circle method. Whenever we make references to a collection of rK many variables, say $\boldsymbol{\alpha}$, we will represent these as an $r \times K$ matrix

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_{1,1} & \cdots & \alpha_{1,K} \\ \vdots & \ddots & \vdots \\ \alpha_{r,1} & \cdots & \alpha_{r,K} \end{bmatrix} = [\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_K].$$

Thus whenever $\mathrm{d}\boldsymbol{\alpha}$ shows up we mean

$$\prod_{\substack{1 \leqslant i \leqslant r \\ 1 \leqslant k \leqslant K}} d\alpha_{i,k}.$$

For any matrix $C = [\mathbf{c}_1, \cdots, \mathbf{c}_s]$ of dimension $r \times s$ we define

$$f_K(\boldsymbol{\alpha}; C) = \prod_{1 \leqslant j \leqslant s} \sum_{|x| \leqslant P} e\left(\sum_{1 \leqslant k \leqslant K} (\boldsymbol{\alpha}_k \cdot \mathbf{c}_j) x^k\right),$$

$$S_K(q, \mathbf{a}; C) = \prod_{1 \leqslant j \leqslant s} \sum_{1 \leqslant u \leqslant q} e\left(\frac{1}{q} \sum_{1 \leqslant k \leqslant K} (\mathbf{a}_k \cdot \mathbf{c}_j) u^k\right),$$

and

$$I_K(P, \boldsymbol{\gamma}; C) = \prod_{1 \leqslant j \leqslant s} \int_{-P}^{P} e\left(\sum_{1 \leqslant k \leqslant K} (\boldsymbol{\gamma}_k \cdot \mathbf{c}_j) z^k\right) \mathrm{d}z.$$

Then by orthogonality it follows that

$$R_K(P; C) = \int_{[0,1)^{r \times K}} f_K(\boldsymbol{\alpha}; C) \, \mathrm{d}\boldsymbol{\alpha}.$$

For any $0 < Q \leqslant P$ we define our major arcs to be

$$\mathfrak{M}(Q) = \bigcup_{\substack{0 \leqslant \mathbf{a} \leqslant q \leqslant Q \\ (q, \mathbf{a}) = 1}} \mathfrak{M}(Q; q, \mathbf{a}),$$

where

$$\mathfrak{M}(Q; q, \mathbf{a}) = \{\boldsymbol{\alpha} \in [0, 1)^{r \times K} : |q\alpha_{i,k} - a_{i,k}| \leqslant QP^{-k}\}.$$

Similarly we define the minor arcs to be $\mathfrak{m}(Q) = [0, 1]^{r \times K} \backslash \mathfrak{M}(Q)$. We also define

$$\mathfrak{S}_K(Q; C) = \sum_{\substack{0 \leqslant \mathbf{a} \leqslant q \leqslant Q \\ (q, \mathbf{a}) = 1}} q^{-s} S_K(q, \mathbf{a}; C),$$

and

$$J_K(Q, P; C) = \int_{\substack{\boldsymbol{\gamma} \in [0,1]^{r \times K} \\ |\gamma_k| \leqslant QP^{-k}}} I_K(P, \boldsymbol{\gamma}; C) \, \mathrm{d}\boldsymbol{\gamma}.$$

Lemma 3.1. *Suppose that $Q = P^\delta$ for some $0 < \delta < \frac{1}{3+2rK}$ and one has the bound*

$$\int_{\mathfrak{m}(Q)} |f_K(\boldsymbol{\alpha}; C)| \, d\boldsymbol{\alpha} = o\left(P^{s - \frac{rK(K+1)}{2}}\right).$$

Then

$$R_K(P; C) = \mathfrak{S}_K(Q; C) J_K(Q, P; C) + o\left(P^{s - \frac{rK(K+1)}{2}}\right).$$

Proof. We begin by defining a slightly larger set of major arcs

$$\mathfrak{N}(Q) = \bigcup_{\substack{0 \leqslant \mathbf{a} \leqslant q \leqslant Q \\ (q,\mathbf{a})=1}} \mathfrak{N}(Q; q, \mathbf{a}),$$

where

$$\mathfrak{N}(Q; q, \mathbf{a}) = \left\{ \boldsymbol{\alpha} \in [0, 1)^{r \times K} : |\alpha_{i,k} - a_{i,k}/q| \leqslant QP^{-k} \right\}.$$

Then by our assumption and the fact that $\mathfrak{N}(Q) \backslash \mathfrak{M}(Q) \subset \mathfrak{m}(Q)$ one has that

$$R_K(P; C) = \int_{\mathfrak{N}(Q)} f_K(\boldsymbol{\alpha}; C) \, d\boldsymbol{\alpha} - \int_{\mathfrak{N}(Q) \backslash \mathfrak{M}(Q)} f_K(\boldsymbol{\alpha}; C) \, d\boldsymbol{\alpha} + \int_{\mathfrak{m}(Q)} f_K(\boldsymbol{\alpha}; C) \, d\boldsymbol{\alpha}$$

$$= \int_{\mathfrak{N}(Q)} f_K(\boldsymbol{\alpha}; C) \, d\boldsymbol{\alpha} + o\left(P^{s - \frac{rK(K+1)}{2}}\right).$$

Via standard methods it follows easily that for $\boldsymbol{\alpha} \in \mathfrak{N}(Q; q, \mathbf{a})$ one has that

$$f_K(\boldsymbol{\alpha}; C) = q^{-s} S_K(q, \mathbf{a}; C) I(P, \boldsymbol{\alpha} - \mathbf{a}/q; C) + O(Q^2 P^{s-1}).$$

Whence upon noting that $\text{mes}(\mathfrak{N}(Q)) \ll Q^{2rK+1} P^{-\frac{rK(K+1)}{2}}$ one deduces

$$\int_{\mathfrak{N}(Q)} f_K(\boldsymbol{\alpha}; C) d\boldsymbol{\alpha} = \mathfrak{S}_K(Q; C) J_K(Q, P; C) + O\left(Q^{3+2rK} P^{s - \frac{rK(K+1)}{2} - 1}\right)$$

$$= \mathfrak{S}_K(Q; C) J_K(Q, P; C) + o\left(P^{s - \frac{rK(K+1)}{2}}\right).$$

$\square$

3.1. **The Minor Arcs.** As is evident from Lemma 3.1, we must establish an adequate bound over the minor arcs. For this we make use of the arguments of [3] without assuming our matrix is highly non-singular. We will instead make use of the assumption that C dominates the function

$$F(x) = \max\left\{ \frac{x - \text{rem}(s, r)}{\lfloor \frac{s}{r} \rfloor}, \frac{x - \text{rem}(s - 1, r)}{\lfloor \frac{s-1}{r} \rfloor} \right\}, \tag{3.1}$$

to derive a Weyl type inequality and a mean value bound.

Definition 3.2. We say a matrix C of dimensions $r \times rn$ is partitionable if there exists a partition $\bigsqcup_{1 \leqslant l \leqslant n} J_l = [rn]$, for which one has

$$\text{rank}(C_{J_l}) = r \text{ for all } 1 \leqslant l \leqslant n.$$

In the proofs that follow we make use of the property that our matrix C contains a partitionable submatrix of size $r \times r \lfloor s/r \rfloor$ and given any $j_0 \in [s]$ we have that $C_{[s] \setminus \{j_0\}}$ contains a partitionable submatrix of size $r \times r \lfloor (s-1)/r \rfloor$. This may be deduced from the property that C dominates (3.1) in conjunction with [9, Lemma 1].

Lemma 3.3. *Let $s \geqslant r$ and suppose $C \in \mathbb{Z}^{r \times s}$ dominates the function*

$$\frac{x - \mathrm{rem}(s,r)}{\left\lfloor \frac{s}{r} \right\rfloor}.$$

Then there exists $j_0 \in [s]$ and $\sigma > 0$ for which

$$\sup_{\boldsymbol{\alpha} \in \mathfrak{m}(Q)} |f_K(\boldsymbol{\alpha}; \mathbf{c}_{j_0})| \ll PQ^{-\sigma}.$$

Proof. By [9, Lemma 1], C contains a partitionable $r \times rn$ submatrix for all $n \leqslant \lfloor \frac{s}{r} \rfloor$. Since $s \geqslant r$ we may take $n = 1$, whence without loss of generality we may assume that the first r columns of C are linearly independent. Let $\sigma < 1/(2K)$ and define

$$\beta_{j,k} = \mathbf{c}_j \cdot \boldsymbol{\alpha}_k.$$

Now suppose that

$$\sum_{1 \leqslant x \leqslant P} e\left(\sum_{1 \leqslant k \leqslant K} (\boldsymbol{\alpha}_k \cdot \mathbf{c}_j) x^k \right) > PQ^{-\sigma}$$

for all $1 \leqslant j \leqslant r$. Then by [10, Lemma 2.4] one has that there exists $q \ll Q^{2K\sigma}$ such that

$$\|q\beta_{j,k}\| \ll Q^{2K\sigma} P^{-k} \quad \text{for all } 1 \leqslant j \leqslant r \text{ and } 1 \leqslant k \leqslant K. \tag{3.2}$$

Since the first r columns of C are linearly independent one has that there exists vectors $\mathbf{v}_i$ for $1 \leqslant i \leqslant r$ which satisfy

$$\alpha_{i,k} = \mathbf{v}_i \cdot (\beta_{1,k}, \ldots, \beta_{r,k}),$$

where $[\mathbf{v}_1, \ldots, \mathbf{v}_r] = \left([\mathbf{c}_1, \ldots, \mathbf{c}_r]^T \right)^{-1}$. It is important to note here that the elements of $[\mathbf{v}_1, \ldots, \mathbf{v}_r]$ are rational, with common denominator, say, $L \geqslant 1$. We then have that

$$\|Lq\alpha_{i,k}\| \leqslant \sum_{1 \leqslant j \leqslant r} \|Lv_{i,j}q\beta_{j,k}\| \leqslant \sum_{1 \leqslant j \leqslant r} |Lv_{i,j}| \|q\beta_{j,k}\| \ll (Q/L)^{2K\sigma} P^{-k},$$

where the last inequality comes from (3.2) and noting that $L \asymp 1$. We deduce that $L\boldsymbol{\alpha} \in \mathfrak{M}(Q/L)$ for large enough Q but since L is an integer this is equivalent to $\boldsymbol{\alpha} \in \mathfrak{M}(Q)$. Thus, by the contrapositive, it must be the case that for some $1 \leqslant j_0 \leqslant r$ we have the desired bound. $\square$

Lemma 3.4. *Let $s \geqslant rK(K+1)$ and suppose $C \in \mathbb{Z}^{r \times s}$ dominates*

$$\frac{x - \mathrm{rem}(s,r)}{\left\lfloor \frac{s}{r} \right\rfloor}.$$

Then one has the bound

$$\int_{[0,1)^{r \times K}} |f_K(\boldsymbol{\alpha}; C)| \, d\boldsymbol{\alpha} \ll P^{s - \frac{rK(K+1)}{2} + \varepsilon}.$$

Proof. By [9, Lemma 1], we deduce that C contains a partitionable $r \times rK(K+1)$ submatrix. Then upon utilizing the trivial bound on exponential sums we may without loss of generality suppose that

$$C = [C_1, \ldots, C_{K(K+1)}],$$

where each C_i is a non-singular $r \times r$ matrix. Then it is enough to establish the bound

$$\int_{[0,1)^{r \times K}} |f_K(\boldsymbol{\alpha}; C)| \, d\boldsymbol{\alpha} \ll P^{\frac{rK(K+1)}{2} + \varepsilon}. \tag{3.3}$$

Via an application of the trivial inequality

$$|a_1 \cdots a_n| \leqslant |a_1|^n + \cdots + |a_n|^n, \tag{3.4}$$

we see that

$$\int_{[0,1)^{r \times K}} |f_K(\boldsymbol{\alpha}; C)| \, d\boldsymbol{\alpha} \ll \max_{1 \leqslant l \leqslant K(K+1)} \Phi_l$$

where

$$\Phi_l = \int_{[0,1)^{r \times K}} |f_K(\boldsymbol{\alpha}; C_l)|^{K(K+1)} \, d\boldsymbol{\alpha}.$$

For a fixed l, the value of the integral is, by orthogonality, bounded above by the number of integer solutions of the system

$$C_l \begin{bmatrix} \Delta_k(\mathbf{x}_1, \mathbf{y}_1) \\ \vdots \\ \Delta_k(\mathbf{x}_r, \mathbf{y}_r) \end{bmatrix} = \mathbf{0}$$

where

$$\Delta_k(\mathbf{x}, \mathbf{y}) = \sum_{n=1}^{\frac{K(K+1)}{2}} x_n^k - y_n^k,$$

and $|\mathbf{x}_j|, |\mathbf{y}_j| \leqslant P$. Since C_l is non-singular this implies that

$$\Delta_k(\mathbf{x}_j, \mathbf{y}_j) = 0, \quad 1 \leqslant j \leqslant r, \quad 1 \leqslant i \leqslant r, \quad 1 \leqslant k \leqslant K.$$

This is simply $J_{K(K+1)/2,K}(P)^r$, where

$$J_{s,k}(X) = \int_{[0,1)^k} \left| \sum_{1 \leq x \leq X} e\left((\alpha_1 x + \cdots + \alpha_k x^k)\right) \right|^{2s} \, d\alpha.$$

By the resolution of Vinogradov's mean value theorem (see [1], [14]) we conclude that

$$\max_{1 \leqslant l \leqslant K(K+1)} \Phi_l \ll P^{\frac{rK(K+1)}{2} + \varepsilon}.$$

Thus we have established (3.3). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Note that since C dominates the function $F(x)$ from (3.1) then by Lemma 3.3 it contains a column index $j_0 \in [s]$ and there exists $\sigma > 0$ for which

$$\sup_{\boldsymbol{\alpha} \in \mathfrak{m}(Q)} |f_K(\boldsymbol{\alpha}; \mathbf{c}_{j_0})| \ll PQ^{-\sigma}.$$

Then $C_{[s] \setminus \{j_0\}}$ satisfies the conditions of Lemma 3.4. Thus by an application of Hölder's inequality one obtains that

$$\int_{\mathfrak{m}(Q)} f_K(\boldsymbol{\alpha}; C) \, \mathrm{d}\boldsymbol{\alpha} \ll P^{s - \frac{rK(K+1)}{2} + \varepsilon} Q^{-\sigma}.$$

Upon setting $Q = P^{\delta}$ for some $0 < \delta < \frac{1}{3 + 2rK}$ we establish via Lemma 3.1 that

$$R_K(P; C) = \mathfrak{S}_K(P^{\delta}; C) J_K(P^{\delta}, P; C) + o\left(P^{s - \frac{rK(K+1)}{2}}\right). \qquad (3.5)$$

3.2. **The Singular Series.** We proceed by establishing the absolute convergence of the complete singular series

$$\mathfrak{S}_K(C) := \sum_{q=1}^{\infty} q^{-s} \sum_{\substack{1 \leqslant \mathbf{a} \leqslant q \\ (q, \mathbf{a}) = 1}} S_K(q, \mathbf{a}; C).$$

We begin with a definition. For any $r \times s$ matrix C we let

$$A_K(q; C) = q^{-s} \sum_{\substack{1 \leqslant \mathbf{a} \leqslant q \\ (q, \mathbf{a}) = 1}} S_K(q, \mathbf{a}; C).$$

It is then immediately clear that one has absolute convergence of $\mathfrak{S}_K(C)$ as soon as one establishes a bound of the form $A_K(q; C) \ll q^{-(1+\sigma)+\varepsilon}$ for some $\sigma > 0$. We accomplish this by making use of the property that C contains a partitionable $r \times rn$ matrix for $n = \lfloor s/r \rfloor \geqslant K(K+1)$.

Lemma 3.5. *Let $K \geqslant 2$ and C be a $r \times s$ matrix with $s > rK(K+1)$. If C dominates $F(x)$ from (3.1) then there exists $\sigma > 0$ for which one has the bound*

$$A_K(q; C) \ll q^{-1-\sigma+\varepsilon}.$$

Proof. We begin by defining a function which will be useful later. Let $\mathbf{b} \in \mathbb{Z}^{K \times s}$ be given, then we define

$$S_K^*(q, \mathbf{b}) = \prod_{1 \leqslant j \leqslant s} \sum_{1 \leqslant u \leqslant q} e\left(\sum_{1 \leqslant k \leqslant K} b_{k,j} u^k\right).$$

Given this definition one may note that the following holds

$$S_K(q, \mathbf{a}; C) = S_K^*(q, \mathbf{a}^T C).$$

Via an application of Lemma 3.4 to the minor arcs $\mathfrak{m}(q-1)$, we see that there exists $j_0 \in [s]$ and $\sigma > 0$ for which

$$\max_{\substack{1 \leqslant \mathbf{a} \leqslant q \\ (q, \mathbf{a}) = 1}} |S_K(q, \mathbf{a}; C_{\{j_0\}})| \ll q^{1-\sigma}.$$

Setting $J_0 = [s] \setminus \{j_0\}$ we obtain

$$A_K(q; C) \ll q^{-\sigma} A_K(q, C_{J_0}). \tag{3.6}$$

Then since C_{J_0} contains a partitionable $r \times rK(K+1)$ matrix one may without loss of generality write

$$C_{J_0} = [C_1, \ldots, C_{K(K+1)}, E],$$

where each C_l is a non-singular $r \times r$ matrix and E is a $r \times (s-1-rK(K+1))$ matrix. Via an application of Hölder's inequality and the trivial inequality (3.4) there exists $1 \leqslant l_0 \leqslant K(K+1)$ for which we have the asymptotic bound

$$A_K(q; C_{J_0}) \ll q^{-rK(K+1)} \sum_{\substack{1 \leqslant \mathbf{a} \leqslant q \\ (q, \mathbf{a}) = 1}} |S_K(q, \mathbf{a}; C_{l_0})|^{K(K+1)}$$

$$= q^{-rK(K+1)} \sum_{\substack{1 \leqslant \mathbf{a} \leqslant q \\ (q, \mathbf{a}) = 1}} |S_K^*(q, \mathbf{a}^T C_{l_0})|^{K(K+1)}$$

Since the matrix C_{l_0} is invertible one has that the condition $(q, \mathbf{a}) = 1$ implies $(q, \mathbf{a}^T C_{l_0}) \ll 1$. Additionally since the matrix C here is fixed one similarly has that the elements of $\mathbf{a}^T C_{l_0}$ are $\ll q$. Thus there exists constants c_0 and c_1 depending on at most C_{J_0} for which the following inequality holds

$$A_K(q; C_{J_0}) \ll q^{-rK(K+1)} \sum_{\substack{|\mathbf{b}| \leqslant c_0 q \\ (q, \mathbf{b}) \leqslant c_1}} |S_K^*(q, \mathbf{b})|^{K(K+1)},$$

where $\mathbf{b} = [\mathbf{b}_1, \ldots, \mathbf{b}_r]$ are $K \times r$ matrices. Applying [13, Theorem 7.1] one obtains has the bound

$$S_K^*(q, \mathbf{b}) \ll \left(\prod_{1 \leqslant j \leqslant r} (q, \mathbf{b}_j) \right)^{1/K} q^{r(1-1/K)+\varepsilon}.$$

Thus we conclude that

$$A_K(q; C_{J_0}) \ll q^{-r(K+1)+\epsilon} \sum_{\substack{|\mathbf{b}| \leqslant c_0 q \\ (q, \mathbf{b}) \leqslant c_1}} \left(\prod_{1 \leqslant j \leqslant r} (q, \mathbf{b}_j) \right)^{K+1}.$$

Since $A_K(q; C_{J_0})$ is multiplicative in q we need only establish the bound for the case when q is a prime power p^h. In this case we may rewrite the above sum by summing over all r-dimensional vectors $\mathbf{e}$ where at least one element is bounded by $\log_p(c_1) =: c_2$ and counting the number of $K \times r$ matrices, $\mathbf{b}$, satisfying

$$|\mathbf{b}| \leqslant c_0 p^h, \quad (p^h, \mathbf{b}) \leqslant c_1, \quad \text{and} \quad (p^h, \mathbf{b}_j) = p^{e_j}. \tag{3.7}$$

Let $\Phi(p^h; \mathbf{e})$ denote this quantity, then we have that

$$A_K(p^h; C_{J_0}) \ll p^{h(-r(K+1)+\epsilon)} \sum_{\substack{0 \leqslant \mathbf{e} \leqslant h \\ \min \mathbf{e} \leqslant c_2}} p^{(K+1)\|\mathbf{e}\|_1} \Phi(p^h; \mathbf{e}).$$

Note that for any given choice of $0 \leqslant e_j \leqslant h$ one has that there at most $p^{K(h-e_j)}$ valid choices for $\mathbf{b}_j$ satisfying (3.7). One then concludes that

$$\Phi(p^h; \mathbf{e}) \ll p^{K(rh - \|\mathbf{e}\|_1)},$$

hence

$$A_K(p^h; C_{J_0}) \ll p^{h(-r+\varepsilon)} \sum_{\substack{0 \leqslant \mathbf{e} \leqslant h \\ \min \mathbf{e} \leqslant C_2}} p^{\|\mathbf{e}\|_1} \ll p^{h(-r+(r-1)+\varepsilon)} = p^{h(-1+\varepsilon)}.$$

Thus we establish that $A_K(q; C_{J_0}) \ll q^{-1+\varepsilon}$. Combining this with (3.6) we establish the Lemma. $\qquad \square$

Whence we conclude that $\mathfrak{S}_K$ converges absolutely and there exists $\sigma > 0$ for which one has

$$|\mathfrak{S}_K - \mathfrak{S}_K(Q)| \ll Q^{-\sigma+\varepsilon}.$$

3.3. **The Singular Integral.** Here we will show that the complete singular integral

$$J_K(P; C) := \lim_{Q \to \infty} J_K(Q, P; C) = \lim_{Q \to \infty} \int_{\substack{\boldsymbol{\gamma} \in [0,1]^{r \times K} \\ |\gamma_k| \leqslant QP^{-k}}} I_K(P, \boldsymbol{\gamma}; C) \, \mathrm{d}\boldsymbol{\gamma} \qquad (3.8)$$

is absolutely convergent. We begin with a slight simplification, note that via a change of variables we obtain

$$J_K(Q, P; C) = P^{s - \frac{rK(K+1)}{2}} J_K(Q, 1; C),$$

which implies

$$J_K(P; C) = P^{s - \frac{rK(K+1)}{2}} J_K(1; C).$$

Thus, to prove (3.8) converges absolutely it suffices to show that the integral

$$J_K(1; C) = \lim_{Q \to \infty} \int_{|\boldsymbol{\gamma}| \leqslant Q} I_K(1, \boldsymbol{\gamma}; C) \, \mathrm{d}\boldsymbol{\gamma},$$

absolutely converges.

Lemma 3.6. *Let C be an $r \times s$ with $s > rK(K+1)$ and suppose C dominates*

$$F(x) = \frac{x - \mathrm{rem}(s, r)}{\lfloor \frac{s}{r} \rfloor}.$$

Then one has that

$$\lim_{Q \to \infty} \int_{|\boldsymbol{\gamma}| \leqslant Q} I_K(1, \boldsymbol{\gamma}; C) \, \mathrm{d}\boldsymbol{\gamma}$$

converges absolutely.

Proof. Since C contains a partitionable submatrix of size $r \times rK(K+1)$, we may use the trivial estimate over oscillatory integrals and suppose that $C = [C_1, \ldots, C_{K(K+1)}]$. We begin by first defining a useful function. For any

$$\boldsymbol{\xi} = [\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_r] \in \mathbb{R}^{K \times r}$$

we let

$$I_K^*(\boldsymbol{\xi}) = \prod_{1 \leqslant j \leqslant r} \int_{-1}^{1} e\left(\sum_{1 \leqslant k \leqslant K} \xi_{k,j} z^k\right) dz.$$

Given this definition one may note that the following holds

$$I_K(1, \boldsymbol{\gamma}; C) = I_K^*(\boldsymbol{\gamma}^T C).$$

Making use of (3.4), we again have that there exists l_0 between 1 and $K(K+1)$ such that

$$\int_{|\boldsymbol{\gamma}| \leqslant Q} |I_K(1, \boldsymbol{\gamma}; C)| \, d\boldsymbol{\gamma} \asymp \int_{|\boldsymbol{\gamma}| \leqslant Q} |I_K(1, \boldsymbol{\gamma}; C_{l_0})|^{K(K+1)} \, d\boldsymbol{\gamma}$$

$$= \int_{|\boldsymbol{\gamma}| \leqslant Q} |I_K^*(\boldsymbol{\gamma}^T C_{l_0})|^{K(K+1)} \, d\boldsymbol{\gamma}.$$

Since C_{l_0} is non-singular we can make a non-singular linear change of variables $\boldsymbol{\xi} = \boldsymbol{\gamma}^T C_{l_0}$. Additionally, from the argument in [13, Theorem 7.3], we deduce that

$$I_K^*(\boldsymbol{\xi}) \ll \prod_{1 \leqslant j \leqslant r} \min\{1, |\boldsymbol{\xi}_j|^{-1/K}\}. \tag{3.9}$$

Hence

$$\int_{|\boldsymbol{\gamma}| \leqslant Q} |I_K(1, \boldsymbol{\gamma}; C)| \, d\boldsymbol{\gamma} \asymp \int_{|\boldsymbol{\xi}| \leqslant Q} |I_K^*(\boldsymbol{\xi})|^{K(K+1)} \, d\boldsymbol{\xi}$$

$$\ll \prod_{1 \leqslant j \leqslant r} \int_{|\boldsymbol{\xi}_j| \leqslant Q} \min\{1, |\boldsymbol{\xi}_j|^{-(K+1)}\} \, d\boldsymbol{\xi}_j$$

$$= \left(\int_{\substack{\mathbf{x} \in \mathbb{R}^K \\ |\mathbf{x}| \leqslant Q}} \min\{1, |\mathbf{x}|^{-(K+1)}\} \, d\mathbf{x}\right)^r.$$

Thus it is enough to show that the integral

$$\lim_{Q \to \infty} \int_{\substack{\mathbf{x} \in \mathbb{R}^K \\ |\mathbf{x}| \leqslant Q}} \min\{1, |\mathbf{x}|^{-(K+1)}\} \, d\mathbf{x}$$

converges absolutely. This follows immediately by elementary analysis as one sees that

$$\int_{\substack{\mathbf{x} \in \mathbb{R}^K \\ |\mathbf{x}| \geqslant Q}} \min\{1, |\mathbf{x}|^{-(K+1)}\} \, d\mathbf{x} \ll Q^{-1}.$$

$\square$

Upon combining the results of Lemma 3.5 and Lemma 3.6 with (3.5) we conclude that

$$R_K(P; C) = P^{s - \frac{rK(K+1)}{2}} \left(\sigma_K(C) + o(1)\right),$$

where $\sigma_K(C) = \mathfrak{S}_K(C) J_K(1; C)$. Since we have shown the absolute convergence of both the singular series and singular integral it follows from the arguments of [4] that $\sigma_K(C) > 0$ whenever there exists non-singular real and p-adic solutions to (2.2). With this, we have established Theorem 2.2.

4. Analyzing the K-multimagic square system

Taking $N \geqslant 4$ we now establish the existence of a matrix

$$D_N \in \{-1, 0, 1\}^{2N \times N^2}$$

for which $M_{N,K}(P) = R_K(P; D_N)$. Note that a matrix $\mathbf{Z} = (z_{i,j})_{1 \leqslant i,j \leqslant N}$ is a $\mathbf{MMS}(K, N)$ if and only if for all $1 \leqslant k \leqslant K$ it satisfies the simultaneous conditions

$$\sum_{1 \leqslant i \leqslant N} z_{i,j}^k = \sum_{1 \leqslant i \leqslant N} z_{i,i}^k \quad \text{for} \quad 1 \leqslant j \leqslant N, \tag{4.1}$$

$$\sum_{1 \leqslant j \leqslant N} z_{i,j}^k = \sum_{1 \leqslant j \leqslant N} z_{j,N-j+1}^k \quad \text{for} \quad 1 \leqslant i \leqslant N. \tag{4.2}$$

One may wonder if these equations are equivalent to those of a $\mathbf{MMS}(K, N)$, indeed it does not seem clear that the main diagonal and anti-diagonal are equal at first glance. One can show that this is implied by the above by simply summing over all j in (4.1) and noting that this is equal to summing over all i in (4.2). Upon dividing out a factor of N one deduces that (4.1) and (4.2) imply

$$\sum_{1 \leqslant i \leqslant N} z_{i,i}^k = \sum_{1 \leqslant j \leqslant N} z_{j,N-j+1}^k.$$

Before we construct a matrix corresponding to this system we must first establish some notational shorthand. Let $\mathbf{1}_n / \mathbf{0}_n$ denote a n-dimensional vector of all ones/zeros, and $\mathbf{e}_n(m)$ denote the mth standard basis vector of dimension n. For a fixed N we define

$$D_1(N) = \{(i, j) \in [N]^2 : i = j\},$$

and

$$D_2(N) = \{(i, j) \in [N]^2 : i + j = N + 1\}.$$

For each $(i, j) \in [N]^2$ we define the $2N$-dimensional vectors

$$\mathbf{d}_{i,j} = \begin{cases} (\mathbf{e}_N(i) - \mathbf{1_N}, \mathbf{e}_N(j) - \mathbf{1_N}) & (i, j) \in D_1(N) \cap D_2(N) \\ (\mathbf{e}_N(i) - \mathbf{1_N}, \mathbf{e}_N(j)) & (i, j) \in D_1(N) \backslash D_2(N) \\ (\mathbf{e}_N(i), \mathbf{e}_N(j) - \mathbf{1_N}) & (i, j) \in D_2(N) \backslash D_1(N) \\ (\mathbf{e}_N(i), \mathbf{e}_N(j)) & \text{otherwise} \end{cases}$$

Let $\phi : [N^2] \to [N]^2$ be any fixed bijection, then the $2N \times N^2$ matrix

$$D_N = D_N(\phi) = [\mathbf{d}_{\phi(1)}, \dots, \mathbf{d}_{\phi(N^2)}],$$

corresponds to the system defined by (4.1) and (4.2) up to some arbitrary relabeling of variables defined by the bijection ϕ. For any subset $J \subset [N]^2$ we define $(D_N)_J := (D_N)_{\phi^{-1}(J)}$ in the sense defined in section 1.

Before we move on it will be useful to define a notion of equivalence between subsets of columns of a matrix. Let M be a $r \times s$ matrix, then we say $J_1, J_2 \subset [s]$ are M-isomorphic if

$$\text{im}(M)_{J_1} = \text{im}(M)_{J_2}.$$

Lemma 4.1. *For any $J \subset [N]^2$ we have that*

$$\mathrm{rank}(D_N)_J \geqslant \min\left\{\left\lceil 2\sqrt{|J|}\right\rceil + 1, 2N\right\} - 2 + E_N(J),$$

where

$$E_N(J) = \dim\left(\mathrm{im}(D_N)_J \cap \mathrm{im}\begin{bmatrix}\mathbf{1}_N & \mathbf{0}_N \\ \mathbf{0}_N & \mathbf{1}_N\end{bmatrix}\right).$$

Proof. Let

$$B = \begin{bmatrix}\mathbf{1}_N & \mathbf{0}_N \\ \mathbf{0}_N & \mathbf{1}_N\end{bmatrix},$$

then by elementary linear algebra, one has the equality

$$\mathrm{rank}(D_N)_J + \mathrm{rank}B = \mathrm{rank}[(D_N)_J, B] + \dim\left(\mathrm{im}(D_N)_J \cap \mathrm{im}B\right).$$

Thus we deduce

$$\mathrm{rank}(D_N)_J = \mathrm{rank}[(D_N)_J, B] - 2 + E_N(J).$$

Thus it suffices to understand the rank of the matrix $[(D_N)_J, B]$ given $J \subset [N]^2$. Via rank preserving elementary column operations we have that

$$\mathrm{im}[(D_N)_J, B] = \mathrm{im}[A_J, B],$$

where

$$A_J = \begin{bmatrix}\mathbf{e}_N(i) \\ \mathbf{e}_N(j)\end{bmatrix}_{(i,j)\in J}.$$

This simplification is the reason we chose this definition of B. Then via the same elementary linear algebra identity used above we further see that

$$\mathrm{rank}(D_N)_J = \mathrm{rank}A_J - \dim(\mathrm{im}A_J \cap \mathrm{im}B) + E_N(J).$$

Thus given m it suffices to establish the largest sets $J \subset [N]^2$ which minimizes $\mathrm{rank}A_J$ and maximizes $\dim(\mathrm{im}A_J \cap \mathrm{im}B)$ with

$$m = \mathrm{rank}A_J - \dim(\mathrm{im}A_J \cap \mathrm{im}B).$$

We do this via a greedy algorithm approach, first note that for $J_1 = [N]^2$, we have that

$$\mathrm{rank}A_{J_1} = 2N - 1, \quad \text{and} \quad \mathrm{im}A_{J_1} \cap \mathrm{im}B = \mathrm{im}\left[\mathbf{1}_{2N}\right].$$

We now wish to find a minimal set $\Delta_1 \subset J_1$ such that for $J_2 = J_1 \backslash \Delta_1$ one has $\mathrm{rank}A_{J_2} = 2N - 2$ and we additionally keep $\dim(\mathrm{im}A_{J_2} \cap \mathrm{im}B)$ as large as possible. We do this by understanding a fundamental relation amongst the columns of $A_{[N]^2}$, namely

$$\begin{bmatrix}\mathbf{e}_N(i_1) \\ \mathbf{e}_N(j_1)\end{bmatrix} - \begin{bmatrix}\mathbf{e}_N(i_1) \\ \mathbf{e}_N(j_2)\end{bmatrix} + \begin{bmatrix}\mathbf{e}_N(i_2) \\ \mathbf{e}_N(j_2)\end{bmatrix} - \begin{bmatrix}\mathbf{e}_N(i_2) \\ \mathbf{e}_N(j_1)\end{bmatrix} = \mathbf{0}_{2N}. \tag{4.3}$$

One sees that for any set Δ_1 of size strictly less than N, J_2 would contain an entire column and row, thus by (4.3) J_2 would be A-isomorphic to J_1. However, if $|\Delta_1| = N$ one sees by (4.3) that the only way one may achieve

$\mathrm{rank} A_{J_2} = 2N - 2$ is if Δ_1 is an entire row/column. Hence, without loss of generality, we may take

$$\Delta_1 = \{(i, j) \in [N]^2 : i = N\}.$$

We do however wish to keep $\dim(\mathrm{im} A_{J_2} \cap \mathrm{im} B) = 1$ if possible. This is unfortunately not possible because $\mathrm{im} A_{J_1} \cap \mathrm{im} B = \mathrm{im}\,[\mathbf{1}_{2N}]$, and it is not hard to see that a necessary condition for the vector $\mathbf{1}_{2N}$ to be in $\mathrm{im} A_{J_2}$ is that J_2 contains an element in each row, but our greedy choice for J_2 cannot have this property. The greedy algorithm is now much simpler as we do not need to consider maximizing $\dim(\mathrm{im} A_{J_2} \cap \mathrm{im} B)$ anymore. We may now have $J_2 = [N-1] \times [N]$ with

$$\mathrm{rank} A_{J_2} = 2N - 2 \quad \text{and} \quad \dim(\mathrm{im} A_{J_2} \cap \mathrm{im} B) = 0.$$

We continue as before and find minimal $\Delta_2 \subset J_2$ such that for $J_3 = J_2 \backslash \Delta_2$ one has $\mathrm{rank} A_{J_3} = 2N - 3$. By the same reasoning as before we may without loss of generality take

$$\Delta_2 = \{(i, j) \in [N-1] \times [N] : j = N\},$$

and obtain $J_3 = [N-1] \times [N-1]$ with

$$\mathrm{rank} A_{J_3} = 2N - 3 \text{ and } \quad \dim(\mathrm{im} A_{J_3} \cap \mathrm{im} B) = 0.$$

This process can clearly be iterated until $J_{2N-1} = [1]^2$.

Noting that the function $\min_{|J|=m} \mathrm{rank} A_J - \dim(\mathrm{im} A_J \cap \mathrm{im} B)$ is monotone increasing as a function of m we deduce

$$\mathrm{rank} A_J - \dim(\mathrm{im} A_J \cap \mathrm{im} B) \geqslant \min \left\{ \left\lceil 2\sqrt{|J|} \right\rceil + 1, 2N \right\} - 2.$$

$\square$

We now establish our final necessary result.

Lemma 4.2. *Let $N \geqslant 4$ and $J \subset [N]^2$. If $|J| > N(N-1)$ then $E_N(J) = 2$.*

Proof. If $N = 4$ then one may through brute force computation establish that for every $J \subset [4]^2$ of size 13 one has $\mathrm{rank}(D_4)_J = 8$ thus $E_4(J) = 2$. Henceforth, we will be working under the assumption that $N \geqslant 5$. We also take

$$\chi_2(N) = \begin{cases} 1 & 2 \nmid N, \\ 0 & 2 \mid N. \end{cases}$$

Let $J \subset [N]^2$ be of size $N(N-1)+1$, and define $S = J \backslash (D_1(N) \cup D_2(N))$ and set $S^c := J \backslash S$. Due to the size of J it must be the case that $|S| \geqslant N+1+\chi_2(N)$. We now split into sub cases depending on the rank of $(D_N)_{S^c}$.

If $\mathrm{rank}(D_N)_{S^c} = 2N - 1$ then $\mathrm{im}(D_N)_{S^c} = \mathrm{im} A_{[N]^2}$ where A is defined in the proof of Lemma 4.1. It is not hard to see that every $\mathbf{x} \in \mathrm{im} A_{[N]^2}$ satisfies

$$\mathbf{x} \cdot \begin{bmatrix} \mathbf{1}_N \\ -\mathbf{1}_N \end{bmatrix} = \mathbf{0}_{2N}.$$

It is also clear to see that at least one of the elements of S does not lie in $D_1(N) \cap D_2(N)$, say (i_0, j_0). Then one may check that

$$\left| \mathbf{d}_{i_0, j_0} \cdot \begin{bmatrix} \mathbf{1}_N \\ -\mathbf{1}_N \end{bmatrix} \right| = N \Rightarrow \mathbf{d}_{i_0, j_0} \notin \mathrm{im} A_{[N]^2},$$

whence $\mathrm{rank}(D_N)_J > \mathrm{rank}(D_N)_{S^c} = 2N - 1$ and trivially $E_N(J) = 2$.

Suppose $\mathrm{rank}(D_N)_{S^c} = 2N - 2$, then by our previous analysis on the matrix A it follows that S^c is without loss of generality D_N-isomorphic to the set

$$\overline{S^c} = \{(i, j) \in [N]^2 \backslash (D_1(N) \cup D_2(N)) : i \neq N\},$$

thus $|S^c| \leqslant N^2 - 3N + 2 + \chi_2(N)$, whence $|S| \geqslant 2N - 1 - \chi_2(N)$.

If S contains one element in $D_1(N)$, say (i_1, j_1), and another in $D_2(N)$, say (i_2, j_2) neither of which belonging to row N then we are done. This is because there exists $1 \leqslant k \leqslant N$ and $l \neq N$ such that

$$\mathbf{d}_{i_1, j_1} - \begin{bmatrix} e_N(i_2) \\ e_N(k) \end{bmatrix} + \begin{bmatrix} e_N(l) \\ e_N(k) \end{bmatrix} - \begin{bmatrix} e_N(l) \\ e_N(j_0) \end{bmatrix} = - \begin{bmatrix} \mathbf{1}_N \\ \mathbf{0}_N \end{bmatrix}.$$

Because the 3 latter vectors in the left hand side are in $\mathrm{im}(D_N)_{S^c}$ we obtain

$$\begin{bmatrix} \mathbf{1}_N \\ \mathbf{0}_N \end{bmatrix} \in \mathrm{im}(D_N)_J.$$

This exact same trick may be done with (i_2, j_2) to obtain

$$\begin{bmatrix} \mathbf{0}_N \\ \mathbf{1}_N \end{bmatrix} \in \mathrm{im}(D_N)_J.$$

Note that the above definitely happens if $2N - 1 - \chi_2(N) \geqslant N + 2 + \chi_2(N)$, which is always true for $N \geqslant 5$, hence $E_N(J) = 2$.

We now consider the case in which $\mathrm{rank}(D_N)_{S^c} \leqslant 2N - 3$. By our previous analysis on the matrix A we see that the largest set $S^c \subset [N]^2 \backslash (D_1(N) \cup D_2(N))$ for which $\mathrm{rank}(D_N)_{S^c} \leqslant 2N - 3$ has size at most $|S^c| \leqslant N^2 - 4N + 5 + \chi_2(N)$, this implies that $|S| \geqslant 3N - 4 - \chi_2(N)$. Note that since $|S| \leqslant 2N - \chi_2(N)$ this is impossible for $N \geqslant 5$. $\square$

Combining Lemma 4.1 and Lemma 4.2 and noting that the for any $1 \leqslant m < N^2$ one must have that

$$0 \leqslant \min_{|J|=m+1} \mathrm{rank}(D_N)_J - \min_{|J|=m} \mathrm{rank}(D_N)_J \leqslant 1,$$

we conclude that

$$\mathrm{rank}(D_N)_J \geqslant \begin{cases} \left\lceil 2\sqrt{|J|} \right\rceil - 1 & 1 \leqslant |J| \leqslant N(N-1) - 1, \\ |J| - N^2 + 3N - 1 & N(N-1) - 1 \leqslant |J| \leqslant N(N-1) + 1, \\ 2N & N(N-1) + 1 \leqslant |J| \leqslant N^2. \end{cases}$$

Thus one may deduce that the matrix D_N dominates $F(x)$ from (3.1) and so we have established the quantitative Hasse principle for $\mathbf{MMS}(K, N)$ for all $N > 2K(K + 1)$. We now go further and focus on establishing the existence of non-singular local solutions.

5. Existence of non-singular local K-multimagic squares

We begin by introducing some notation, for any s-dimensional vector $\boldsymbol{\xi}$, we define $\mathrm{diag}(\boldsymbol{\xi})$ to be the $s \times s$ diagonal matrix which has the elements of $\boldsymbol{\xi}$ as its diagonal entries.

Let us now consider the Jacobian matrix associated to the equations (2.2) defined by the matrix C evaluated at $\mathbf{x}$,

$$\mathbf{J}(\mathbf{x}; C) = \begin{bmatrix} \mathbf{J}_1(\mathbf{x}, C) \\ \vdots \\ \mathbf{J}_K(\mathbf{x}, C) \end{bmatrix}, \text{ where } \mathbf{J}_k = kC\mathrm{diag}(\mathbf{x})^{k-1}.$$

If we replace C with C_J for any $J \subset [s]$, then one certainly has that

$$\mathrm{rank}\left(\mathbf{J}(\mathbf{x}; C)\right) \geqslant \mathrm{rank}\left(\mathbf{J}(\mathbf{x}_J; C_J)\right).$$

Thus if we wish to show the Jacobian has full rank it suffices to show that the Jacobian associated to a submatrix has full rank. This lets us reduce to the case in which the matrix C is a partitionable $r \times rK$ matrix $C = [M_1, \ldots, M_K]$. Next we define the block diagonal matrix

$$\tilde{C} = \begin{bmatrix} M_1^{-1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & M_2^{-1} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & M_K^{-1} \end{bmatrix}.$$

Note that this matrix is clearly non-singular, whence

$$\mathrm{rank}\left(\mathbf{J}(\mathbf{x}; C)\right) = \mathrm{rank}\left(\mathbf{J}(\mathbf{x}; C)\tilde{C}\right) \tag{5.1}$$

This simplifies our problem because one has that

$$\mathbf{J}_k(\mathbf{x}; C)\tilde{C} = k\left[\mathrm{diag}(\mathbf{x}_1)^{k-1}, \mathrm{diag}(\mathbf{x}_2)^{k-1}, \ldots, \mathrm{diag}(\mathbf{x}_K)^{k-1}\right],$$

where

$$\mathbf{x}_l = (x_{1+2N(l-1)}, \ldots, x_{2Nl}).$$

Then by swapping rows/columns and utilizing (5.1) one may deduce that

$$\mathrm{rank}\left(\mathbf{J}(\mathbf{x}; C)\right) = \sum_{1 \leqslant j \leqslant 2N} \mathrm{rank} V'(x_j, x_{j+2N}, \ldots, x_{j+2N(K-1)}),$$

where

$$V'(\mathbf{y}) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2y_1 & 2y_2 & \cdots & 2y_K \\ \vdots & \vdots & \ddots & \vdots \\ Ky_1^{K-1} & Ky_2^{K-1} & \cdots & Ky_K^{K-1} \end{bmatrix}.$$

Comparing this matrix to a Vandermonde matrix it is not too hard to see

$$\det(V'(\mathbf{y})) = K! \prod_{1 \leqslant i < j \leqslant K} (y_i - y_j).$$

Thus we conclude that the Jacobian $\mathbf{J}(\mathbf{x}; C)$, defined by C evaluated at $\mathbf{x}$ has full rank if there exists K many disjoint ordered subsets

$$J_l = \{j(l, 1), \ldots, j(l, r)\} \subset [s]$$

for $1 \leqslant l \leqslant K$ which satisfy

$$\mathrm{rank} C_{J_l} = r \text{ for all } 1 \leqslant l \leqslant K$$

and

$$x_{j(l_1, n)} \neq x_{j(l_2, n)} \text{ for all } 1 \leqslant l_1, l_2 \leqslant K \text{ and } 1 \leqslant n \leqslant r.$$

By [9, Lemma 1] and Lemma 4.1 one has that D_N contains a $2N \times 2NK$ partitionable submatrix because $K \leqslant \lfloor N/2 \rfloor$. Suppose $\mathbf{Z} = (z_{i,j})_{1 \leqslant i, j \leqslant N}$ is a $\mathbf{MMS}(K, N)$, that meaning that it satisfies

$$\left[\mathbf{d}_{\phi(1)}, \ldots, \mathbf{d}_{\phi(N^2)} \right] \begin{bmatrix} z_{\phi(1)} \\ \vdots \\ z_{\phi(N^2)} \end{bmatrix} = \mathbf{0}_{N^2},$$

where ϕ is any bijection from $[N^2]$ to $[N]^2$. From the above work one sees that the Jacobian associated to the K-multimagic square system evaluated at $\mathbf{Z}$ is non-singular if there exists a bijection $\phi : [N^2] \to [N]^2$ for which one has

$$\mathrm{rank} \left[\mathbf{d}_{\phi(1+2N(l-1))}, \mathbf{d}_{\phi(2+2N(l-1))}, \ldots, \mathbf{d}_{\phi(2Nl)} \right] = 2N \quad \text{for all} \quad 1 \leqslant l \leqslant K, \tag{5.2}$$

and

$$z_{\phi(n+2N(l_1-1))} \neq z_{\phi(n+2N(l_2-1))} \text{ for all } 1 \leqslant l_1, l_2 \leqslant K \text{ and } 1 \leqslant n \leqslant 2N. \tag{5.3}$$

First we must fix suitable disjoint subsets $J_l(N) \subset [N]^2$ of size $2N$ for $1 \leqslant l \leqslant \lfloor N/2 \rfloor$ satisfying

$$\mathrm{rank} \left[\mathbf{d}_{i,j} \right]_{(i,j) \in J_l(N)} = 2N.$$

The explicit definitions of these partitions may not seem straightforward but we will also provide a figure to hopefully illuminate what these partitions look like. Given a fixed $N \geqslant 4$ we first define our partition in the case where N is even. For every $1 \leqslant l \leqslant \lfloor N/2 \rfloor$ let

$$J_l(N) = J_l^{(1)}(N) \sqcup J_l^{(2)}(N),$$

instead of writing the definition of these sets $J_l^{(1)}(N)$ and $J_l^{(2)}(N)$ explicitly, we will instead define conditions under which we may determine that a pair (i, j) belong to $J_l^{(1)}(N)$ or $J_l^{(2)}(N)$. If N is even then

$$(i, j) \in J_l^{(1)}(N) \iff i - j \equiv 2(l - 1) \bmod N,$$

and

$$(i, j) \in J_l^{(2)}(N) \iff i - j \equiv 2l - 1 \bmod N.$$

For the odd case we split into further sub cases depending on parity of $\frac{N+1}{2}$. For $j \notin \left[2 - \mathrm{rem}\left(\frac{N+1}{2}, 2\right), \frac{N+3}{2} \right]$ then

$$(i, j) \in J_l^{(1)}(N) \iff i - j \equiv 2(l - 1) \bmod N,$$

if $j \in \left[2 - \operatorname{rem}\left(\frac{N+1}{2}, 2\right), \frac{N+3}{2}\right]$ and $j \equiv \frac{N+3}{2} \bmod 2$ then

$$(i, j) \in J_l^{(1)}(N) \iff i - j + 1 \equiv 2(l - 1) \bmod N,$$

and finally if $j \in \left[2 - \operatorname{rem}\left(\frac{N+1}{2}, 2\right), \frac{N+3}{2}\right]$ and $j \equiv \frac{N+1}{2} \bmod 2$ then

$$(i, j) \in J_l^{(1)}(N) \iff i - j - 1 \equiv 2(l - 1) \bmod N.$$

We define $J_l^{(2)}(N)$ for the odd case in a similar way as follows. For $j \notin \left[2 - \operatorname{rem}\left(\frac{N+1}{2}, 2\right), \frac{N+3}{2}\right]$ then

$$(i, j) \in J_l^{(2)}(N) \iff i - j \equiv 2l - 1 \bmod N,$$

if $j \in \left[2 - \operatorname{rem}\left(\frac{N+1}{2}, 2\right), \frac{N+3}{2}\right]$ and $j \equiv \frac{N+3}{2} \bmod 2$ then

$$(i, j) \in J_l^{(2)}(N) \iff i - j + 1 \equiv 2l - 1 \bmod N,$$

and finally if $j \in \left[2 - \operatorname{rem}\left(\frac{N+1}{2}, 2\right), \frac{N+3}{2}\right]$ and $j \equiv \frac{N+1}{2} \bmod 2$ then

$$(i, j) \in J_l^{(2)}(N) \iff i - j - 1 \equiv 2l - 1 \bmod N.$$

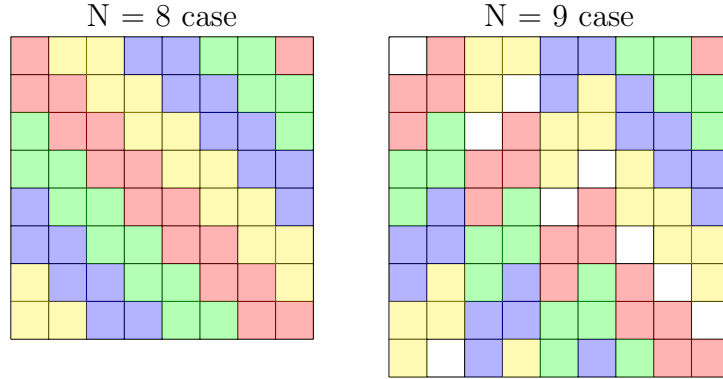We now establish that this partition satisfies the required conditions.

N = 8 case   N = 9 case

FIGURE 1. Example of this partition represented via a coloring on $N \times N$ grid.

Lemma 5.1. *Let $N \geqslant 4$ be given, then the sets J_l define defined previously for $1 \leqslant l \leqslant \lfloor N/2 \rfloor$ satisfy*

$$\operatorname{rank}\left[\mathbf{d}_{i,j}\right]_{(i,j) \in J_l(N)} = 2N.$$

Proof. We begin by recalling the equation used in Lemma 4.1

$$\operatorname{rank}(D_N)_J = \operatorname{rank} A_J - \dim(\operatorname{im} A_J \cap \operatorname{im} B) + E_N(J),$$

where

$$A_J = \begin{bmatrix} \mathbf{e}_N(i) \\ \mathbf{e}_N(j) \end{bmatrix}_{(i,j) \in J}, \qquad B = \begin{bmatrix} \mathbf{1}_N & \mathbf{0}_N \\ \mathbf{0}_N & \mathbf{1}_N \end{bmatrix},$$

and

$$E_N(J) = \dim\left(\operatorname{im}(D_N)_J \cap \operatorname{im} B\right).$$

Note that if we can show that $\mathrm{im}A_{J_l(N)} = \mathrm{im}A$ and $E_N(J_l(N)) = 2$ for all $1 \leqslant l \leqslant \lfloor N/2 \rfloor$, then we are done because by our previous analysis this implies that

$$\mathrm{rank}A - \dim(\mathrm{im}A \cap \mathrm{im}B) = 2N - 2,$$

and hence $\mathrm{rank}(D_N)_J = 2N$.

A few important observations about these sets $J_l(N)$ should be noted, for every $N \geqslant 4$ and $1 \leqslant l \leqslant \lfloor N/2 \rfloor$ the sets $J_l^{(1)}(N)$ and $J_l^{(2)}(N)$ both have size N and contain a single element in every row and column. Additionally $J_l^{(1)}(N) \cap D_2(N) = \emptyset$ and

$$|J_l^{(2)}(N) \cap D_2(N)| = \begin{cases} 2 & 2 \mid N \\ 1 & 2 \nmid N \end{cases}$$

Note that due to the "ladder like" structure of $J_l(N)$ we may, via applications of the fundamental relation (4.3), deduce that

$$\mathrm{im}A_{J_l} = \mathrm{im}A.$$

All that is left to show is that $E_N(J_l) = 2$ for all $1 \leqslant l \leqslant \lfloor N/2 \rfloor$. By the above observations it is not hard to see that

$$\begin{bmatrix} \mathbf{1}_N \\ \mathbf{0}_N \end{bmatrix} \in \mathrm{im} \left(\mathrm{rem}(N,2) \sum_{(i,j)\in J_l^{(1)}(N)} \mathbf{d}_{i,j} + \sum_{(i,j)\in J_l^{(2)}(N)} \mathbf{d}_{i,j} \right) \subset \mathrm{im}(D_N)_{J_l(N)},$$

and

$$\begin{bmatrix} \mathbf{0}_N \\ \mathbf{1}_N \end{bmatrix} \in \mathrm{im} \left(\sum_{(i,j)\in J_l^{(1)}(N)} \mathbf{d}_{i,j}, \begin{bmatrix} \mathbf{1}_N \\ \mathbf{0}_N \end{bmatrix} \right) \subset \mathrm{im}(D_N)_{J_l}(N),$$

thus we establish that $E_N(J_l(N)) = 2$ and we are done. $\square$

With this partition in place we may now fix a suitable bijection $\phi : [N^2] \to [N]^2$ which respects this partition. We begin by first fixing a bijection

$$\tilde{\psi} : J_1(N) \to [2N],$$

we now use the important property that each of our partitions $J_l(N)$ are precisely a previous partition which has been shifted by two rows. Define

$$\psi : \bigsqcup_{1 \leqslant l \leqslant K} J_l(N) \to [2NK]$$

piecewise via the relation

$$\psi(i,j) = \tilde{\psi}(\mathrm{rem}(i - 2l + 1) + 1, j) + 2Nl \text{ for } (i,j) \in J_l(N).$$

Finally we take ϕ to be any bijection from $[N^2]$ to $[N]^2$ which satisfies

$$\phi \circ \psi^{-1} = \mathrm{Id}_{[2NK]}.$$

One may refer to figure 2 for an example of such a bijection. This bijection ϕ satisfies (5.2), additionally we note that for all $1 \leqslant n \leqslant 2N$ we have that the set

$$\{\phi(n), \phi(n + 2N), \dots, \phi(n + 2N(K - 1))\} \subset [N]^2,$$

lies in a single column. Thus, if we can find a $\mathbf{MMS}(K, N)$, say $\mathbf{Z} \in \mathbb{Z}^{N \times N}$, which has distinct values along the columns we immediately have that (5.3) is satisfied and thus establish the existence of a non-singular integer solution and hence trivially non-singular local solutions.

N = 8 case

1	58	51	44	37	30	23	16
9	2	59	52	45	38	31	24
17	10	3	60	53	46	39	32
25	18	11	4	61	54	47	40
33	26	19	12	5	62	55	48
41	34	27	20	13	6	63	56
49	42	35	28	21	14	7	64
57	50	43	36	29	22	15	8

N = 9 case

74	1	58	66	42	50	34	26	18
2	10	67	75	51	59	43	35	27
11	19	76	3	60	68	52	44	36
20	28	4	12	69	77	61	53	45
29	37	13	21	78	5	70	62	54
38	46	22	30	6	14	79	71	63
47	55	31	39	15	23	7	80	72
56	64	40	48	24	32	16	8	81
65	73	49	57	33	41	25	17	9

FIGURE 2. Example of a bijection ϕ satisfying the above properties represented via labeling the (i, j)th entree on $N \times N$ grid the value $\phi^{-1}(i, j)$.

Recall from section 1 that $\mathbf{DDLS}(N)$ exist for $N \geqslant 4$ and are trivially $\mathbf{MMS}(K, N)$ which satisfy this column condition. Hence we are done and have established Theorem 2.3.

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