RESEARCH STATEMENT

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My research is in differential geometry, primarily in minimal surfaces in $\mathbb{R}^3$. The study of minimal surfaces intersects that of complex analysis, which figures significantly into my work. My dissertation research involves regenerating screw motion invariant minimal surfaces (SMIMS) from nodal limits, and my goals include contributing and stimulating further scholarship of SMIMS and periodic surfaces in general.

1. Recent and current work

1.1. **Minimal surfaces background.** Minimal surfaces can be defined as surfaces that locally minimize surface area or, equivalenty, with everywhere vanishing mean curvature or a harmonic conformal parametrization. The study of minimal surfaces consists primarily of the questions of existence and uniqueness given specific properties. For example, the famous Plateau problem is concerned with a fixed boundary, whereas others pertain to specific topological or periodicity conditions. Another important goal is understanding how minimal surfaces can be deformed, as well as the topology of their moduli spaces.

Because minimal surfaces can be defined with harmonic conformal parameters, we can use complex analysis to define them. The Weierstrass-Enneper parametrization allows us to obtain a minimal surface from a Riemann surface and the Weierstrass data, that is, a meromorphic function and one-form.

The Weierstrass data usually have many free parameters, for which the minimal image is not embedded. A necessary condition for embeddedness is that periods over homology classes are "closed," that is, of a form that fits the desired geometry and periodicity. If possible, solving the period problem determines the Weierstrass data.

Since the parametrization is generally not explicit, closing periods is difficult, and various tools have been applied to different situations. One method uses flat structures defined using the Weierstrass data. These allow us to translate the period condition into simpler geometric constraints on the flat structures. These can often be solved using Teichmüller theory or other complex analysis techniques. In [DF20], we use this technique to prove the existence of two families of triply periodic surfaces with genus 4, of which few are known.

1.2. **Motivation: Helicoidal surfaces and regeneration from noded limits.** My doctoral research focuses on one-parameter families of surfaces which degenerate to a noded limit. In [Tra96], Martin Traizet introduced a method of regenerating from noded surfaces to get families of minimal surfaces near the limit. Results from algebraic geometry [Fay73, Mas76, Tra02] show that one-forms defined by periods will extend smoothly to regular differentials on a noded Riemann surface, which are generally explicit. As a result, the periods converge to algebraic equations in the parameters. If there are solutions that satisfy a non-degeneracy condition, we can regenerate to obtain minimal surfaces near the limit.

My research thus far has centered on SMIMS invariant under an orientation-preserving screw motion S_t, which vary in the screw motion angle. Among these are some of the most well-known surfaces, such as the helicoid and the family which limits to the genus 1 helicoid [WHW09]. However, this class remains relatively unstudied.

Most examples were initially obtained by twisting translation-invariant surfaces, such as Scherk's saddle towers [Kar88] and Fischer and Koch's singly periodic surface [FK88]. Most SMIMS are asymptotic to a helicoid, and there was heretofore only one known SMIMS with planar ends.

All known examples degenerate to a parking garage limit, which is a foliation of $\mathbb{R}^3$ by horizontal planes joined by helicoidal nodes. In [TW05], Traizet and Weber first used regeneration from parking garage structures in the special case where all nodes lie on a straight line. This produces SMIMS symmetric with respect to horizontal rotations as well as screw motions.

1.3. **Dissertation work.** The narrow scope of results on nodal parking garage limits motivates my current research. Indeed, nodes in a straight line do not recover all known surfaces with such limits, nor can they generate SMIMS with planar ends. My main doctoral work [Fre] allows to regenerate without such constraints.

My construction assumes only an orientation-reversing screw motion whose square is equal to S_t defined above. All known SMIMS satisfy this assumption, which places no symmetry restriction on the locations of the nodes. The balanced configurations can be determined as follows. Let $p_1, ..., p_n$ be points on the complex plane corresponding to the locations of nodes in the limit, and let $\varepsilon_j = \pm 1$ be charges indicating the direction of the helicoid corresponding to each p_j. Then a configuration is balanced if

$$\overline{p_j} + \sum_{l \neq j} \frac{\varepsilon_l}{p_j - p_l} = 0 \quad \forall j = 1, ..., n.$$

For any non-degenerate solution, I use the implicit function theorem on the parameters obtain one-parameter families if SMIMS, including all known cases as well as many new surfaces. Among the new examples of note are:

- a new family with planar ends,
- screw motion invariant helicoids distinct from those known. Some of these are candidates for new genus g helicoids, whose existence of such a helicoid would disprove a well-known conjecture by Meeks and Rosenberg [mps],
- twisted Fischer-Koch surfaces, with $4n$ ends,
- strong numerical evidence for SMIMS without a straight line symmetry, obtained from a non-symmetric balanced configuration.

My result also helps to understand the behavior of SMIMS and how it relates to the configurations. The sum of the charges $N = \sum_j \varepsilon_j$ determines how the twist parameter changes:

- if $N > 0$, the surface has $2N$ helicoidal ends and the twist angle increases, explaining the behavior of the genus g helicoids,
- if $N < 0$, there are $2N$ helicoidal ends, but the angle decreases. This is the situation with the twisted Scherk surfaces, which untwist until the ends become vertical Scherk ends,
- if $N = 0$, the surface has planar ends.

These balance equations also arise in fluid dynamics as the locations of vortices in rigid configurations. This might indicate a meaningful connection between SMIMS and fluid dynamics, which I wish to explore.

Parking garage limits are not unique to SMIMS and appear in other periodicity classes, for example-:

- Riemann's minimal surface, which is singly periodic translation invariant,
- Scherk's doubly periodic surfaces,
- triply periodic families in the Meeks family.

I am currently exploring regeneration in these contexts. In the triply periodic case, I am hopeful to find new surfaces of genus 4.

2. New directions

2.1. **Helicoidal regeneration and beyond.** There is much more work to do in understanding parking garage limits and SMIMS. Some major question I hope to make advancements in answering are the following:

Question 2.1. *Can we obtain all minimal surfaces with parking garage limits using regeneration?*

It is reasonable to expect that these surfaces will be periodic, so I believe answering this question is possible as long as we can find all balance equations.

Question 2.2. *Can we classify SMIMS with prescribed topology?*

This is probably only doable for small genus, but results would be significant even in low-genus cases.

Question 2.3. *What limits can a SMIMS have?*

Even SMIMS limiting to a parking garage structure are known to have other limits in the other direction. For example, the twisted Scherk and Fischer-Koch surfaces limit to other parking garages (in the latter case, different ones [Fre]), whereas the genus g helicoidal SMIMS converge to the genus g helicoids.

I plan to work on the following projects:

- **Study less "vertically" symmetric SMIMS:** The regeneration technique of [Fre] proves the existence of surfaces symmetric with respect to an orientation-reversing screw motion described above. All known SMIMS have this property, but I plan to investigate whether less symmetric surfaces can limit to parking garage structures and if regeneration in this case can yield embedded surfaces. Understanding these less symmetric cases is an important step towards classifying SMIMS with parking garage limits. I believe this exploration will also contribute significantly to classifying all singly-periodic genus 0 surfaces.
- **Investigate degenerate solutions:** There are degenerate solutions to the balance equations I found for SMIMS parking garage limits. The implicit function theorem does not apply to these situations, but it is possible that there are still SMIMS near the limit and that the moduli space is higher-dimensional, even with the orientation-reversing screw motion.
- **Broadly consider uniqueness of regeneration:** We are guaranteed a one-parameter family near a parking garage limit, but it is conceivable that these surfaces can be deformed in other ways. I plan to determine under which conditions we are guaranteed a unique family or if there are multiple parameters.
- **Investigate regenerating SMIMS from other limits:** There are translation-invariant surfaces which we may be able to twist into SMIMS and which would not limit to a parking garage structure. I plan to study other nodal limits with hopes of finding the first such examples.

2.2. **Long-term behavior of helicoidal surfaces.** Since regeneration only guarantees the existence of surfaces near a nodal limit, the question of long-term behavior of SMIMS is open. Numerical evidence suggests interesting behavior in the long term, although regeneration tells us no information away from the limits. Other techniques are necessary to study this behavior, and I look forward to testing or developing other methods to understand what happens. Some related projects I have planned are:

- **Investigate behavior numerically:** I can numerically close the periods of SMIMS near the nodal limit to understand the long-term behavior of various examples. The nature of other limits should be instructive as to how to investigate more rigorously.
- **Apply other techniques:** Depending on the expected behavior of the given example, I will apply other tools to the study of SMIMS away from the nodal limit.
- **Consider long-term behavior for other classes with parking garage limits:** The same investigation can apply to the other periodicity classes.

References

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