Probability

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Review: Probability Space, Event

Example 1.

Random Experiment: Roll two dices. What is the probability that they add up to 10?

The probability space for this experiment has two components to it:

1. The sample space: The set of all possible outcomes.
   (1, 1), (1, 2), …, (1, 6), (2, 1), …, (2, 6), (3, 1), …, (6,6).
2. The probabilities of each of these outcomes.
   Each of the outcome is equally likely and they sum up to 1.
   So the probability for each of these outcomes is 1/36.

Event: A subset of the possible outcomes.

Event A consists of all pairs of the form (x , y) such that $x + y = 10$.

$A = \{(4, 6), (5, 5), (6, 4)\}$. $P(A) = 3/36 = 1/12$.

Random variable

Roll two dice. Let X be their sum.

$$\text{outcome} = (1,1) \quad \Rightarrow \quad X = 2$$
$$\text{outcome} = (1,2) \text{ or } (2,1) \quad \Rightarrow \quad X = 3$$

Probability space:

- Sample space: $\Omega = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$.
- Each outcome equally likely.

Random variable X lies in $\{2,3,4,5,6,7,8,9,10,11,12\}$.

A **random variable (r.v.)** is a defined on a probability space.
It is a mapping from Ω (outcomes) to $\mathbb{R}$ (numbers).
We'll use capital letters for r.v.'s.

Expected value

Expected value, or mean

Expected value of a random variable X:

$$\mathbb{E}(X) = \sum_x x \Pr(X = x).$$

Roll a die. Let X be the number observed.
What is $\mathbb{E}(X)$?

Expected value

x	P(x)	x.P(x)
1	1/6	1/6
2	1/6	2/6
3	1/6	3/6
4	1/6	4/6
5	1/6	5/6
6	1/6	6/6

$E(X) = 1/6 + 2/6 + 3/6 + 4/6 + 5/6 +6/6 = 21/6 = 3.5$

E(X) Example 2

A biased coin which turns head with probability p.

Random variable X. x = 1 if head turns up and 0 otherwise. What is E(X)?

$E(X)$ Example 2

A biased coin which turns head with probability p.

Random variable X. $x = 1$ if head turns up and 0 otherwise. What is $E(X)$?

$E(X) = 1*p + 0*(1-p) = p$.

E(X) : Example 3

Throw two dices. X is the sum of the values.

$$\text{Range}(X) = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

$$\begin{aligned} E(X) = {} & 2*(1/36) + 3*(2/36) + 4*(3/36) \\ & + 5*(4/36) + 6*(5/36) + 7*(6/36) \\ & + 8*(5/36) + 9*(4/36) + 10*(3/36) \\ & + 11*(2/36) + 12*(1/36) \end{aligned}$$

Property of expected value

Let $Y = aX + b$.

Then $E(Y) = aE(X) + b$

Example

You have a coin. When you toss, if Head turns, you assign 30 to variable X. If Tail turns, you assign 10 to variable X.

Now, X is a random variable. (why?)

What is E(X)?

Property of E(X)

Toss result	Probability	value	Prob.*value
H	0.5	30	15
T	0.5	10	5

E(X) = 15 + 5 = 20.

Let Y = 100 * X + 1000. (Note a = 100, b = 1000). Let

Toss result	Probability	value	Prob.*value
H	0.5	100*30 + 1000	2000
T	0.5	100*10 + 1000	1000

E(Y) = 3000. Note that 100*E(X) + 1000 = 100*20 + 1000 = 3000.
Thus, E(Y) = aE(X) + b.

Property of expected value

Let X and Y be two random variables and $Z = aX + bY$

Then $E(Z) = aE(X) + bE(Y)$

Example

You have two different coins: A and B.

If A turns head, you assign 20 to variable X. If A turns Tail, you assign 10 to variable X. If B turns head, you assign 2 to variable Y. If Tail turns, you assign 1 to variable Y.

Now, X and Y are random variables.

$E(X) = 15$. $E(Y) = 1.5$.

Property of E(X)

Let a = 2 and b = 4. Z = 2X + 4Y.

Toss result	Probability	value	Prob.*value
Hh	0.25	2*20+4*2	12
Ht	0.25	2*20+4*1	11
Th	0.25	2*10+4*2	7
Tt	0.25	2*10+4*1	6

E(2X + 4Y) = 12 + 11 + 7 + 6= 36.

2E(X) + 4E(Y) = 2*15 + 4*1.5 = 36.

Independent Random Variables

Two random variables X and Y are independent if
P(x, y) = P(x)P(y) for all x in X and y in Y.

Example

Pick a card from the standard deck of playing cards.

X : the suit (That is, spade, heart, and so on)

Y : value (That is, A, K, Q, J, 10, 9, …, 1)

P(x) = 1/4, P(y) = 1/13. What is P(x, y)?

Given a suit and value, there is only one card. So, P(x, y) = 1/52 for all x in X and y in Y.

Note that P(x).P(y) is 1/4 * 1/13 = 1/52 for all x in X and y in Y. Hence P(x, y) = P(x).P(y) for all x in X and y in Y.

Independent Random Variables

Throw a fair coin 3 times.

X : number of heads

Y : 1 if last throw is head and 0 otherwise.

There are 8 cases when a coin is thrown 3 times.

HHH, HHT, HTH, HTT, THH, THT, TTH, TTT,

$P(X = 0) = 1/8$, $P(X = 1) = 3/8$, $P(X = 2) = 3/8$ and $P(X= 3) = 1/8$.

$P(Y = 1) = 1/2$ and $P(Y = 0) = 1/2$.

$P(X = 0, Y = 1) = 0 \neq 1/16 = P(X = 0)P(Y = 1)$

X and Y are NOT independent random variables.

Independent Random Variables

A bucket has 3 red balls and 7 blue balls.

Case 1 (With replacement). A child picks a ball at random, note down the color and put the ball back in the bucket . Child does it two times.

These two trials are independent.

P(RR) = (3/10)(3/10); P(BB) = (7/10)(7/10).

P(RB) = (3/10)(7/10); P(BR) = (7/10)(3/10).

Independent Random Variables

A bucket has 3 red balls and 7 blue balls.

Case 1 (Without replacement). A child picks a ball at random, note down the color and throws the ball away. Child does it two times.

These two trials are NOT independent.

P(RR) = (3/10)(2/9); P(BB) = (7/10)(6/9).

P(RB) = (3/10)(7/9); P(BR) = (7/10)(3/9).

P(A, B) = A(A)P(B|A) (conditional probability)

Bernoulli Trials

A Bernoulli trial (or **binomial trial**) is a random experiment with exactly two possible outcomes, "success" and "failure", in which the probability of success is the same every time the experiment is conducted.

Note 1. Let Probability of Success is p. The probability of Failure is q = (1 – p).

Note 2. Trials are independent.

Note 3. There is no limit on the number of trials.

Note 4. There is no guaranty that success will happen.

Example : Buying a Powerball ticket.

Result 1

The Expected number (average number) of trials for a success $= \frac{1}{p}$

Proof. The expected number of trials is

$1.p + 2.q.p + 3.q^2.p + 4.q^3.p + \ldots$

Result 1 proof continued.

Let S = $1.p + 2.q.p + 3.q^2.p + 4.q^3.p + \ldots$

S.q = $1.q.p + 2.q^2.p + 3.q^3.p + \ldots$

$(S - S.q) = p + q.p + q^2.p + q^3.p + \ldots = \frac{p}{(1-q)}$

$S.(1 - q) = 1$ (Since $1 - q = p$)

$S.p = 1.$

$S = \frac{1}{p}$

Result 2.

Expected number (average number) of trials for k successes $= \dfrac{k}{p}$

Result 3.

Expected number (average number) of failures before k successes $= \frac{kq}{p}$

Proof. Let x be the expected number of failures before k successes. Let y be the expected number of trials for k successes. Then

y = x + k.

By result 2, $y = \frac{k}{p}$.

$$x = \frac{k}{p} - k = \frac{k-kp}{p} = \frac{k(1-p)}{p} = \frac{kq}{p}$$

In class exercise

Assumptions

1. A is an array of size n filled with n different random integers.
2. One of the integers is 0.
3. You are doing a linear search to find 0.

Question 1. How many comparisons it will take to find 0?

In class exercise

Question 2. Did you get n as the answer?

Question 3. Does it make any sense to you?

In class exercise

Note that in this case array is finite and 0 is in the array. Therefore, the following three conditions in Slide 15 are violated!

Note 2. Trials are independent.

Note 3. There is no limit on the number of trials.

Note 4. There is no guaranty that success will happen.

Pr(i)= Pr(0 is at loc i) = 1/n for i in [0, n-1].

Thus, Pr(i)*P(j) = 1/(n*n). P(i, j) = 0. Hence trials are not independent.