Quantum 2 HW1

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Technion

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1 Question 1

1.1 Part 1

What are the probabilities $P_\pm$ to measure a magnetic moment $\mu_z = \pm\mu_0$, when the neutron's position doesn't matter?

Since the neutron's position doesn't matter, the probability of finding a specific magnetic moment will be the sum of the probabilities in all positions, therefore

$$P_\pm = \int_{\mathbb{R}^3} \mathrm{d}P = \int_{\mathbb{R}^3} |\psi_\pm(\boldsymbol{r},t)|^2 \,\mathrm{d}\boldsymbol{r} \tag{1}$$

1.2 Part 2

Calculate $\langle\mu_x\rangle$ in the state $|\psi(\boldsymbol{r},t)\rangle$ in terms of $\psi_\pm(\boldsymbol{r},t)$.

First we notice that $\mu_x = \mu_0\sigma_x$, therefore

$$\begin{aligned}
\langle\mu_x\rangle &= \mu_0 \langle\psi(t)|\,\sigma_x\,|\psi(t)\rangle &(2)\\
&= \mu_0 \int_{\mathbb{R}^3} \psi^*(\boldsymbol{r},t)\sigma_x\psi(\boldsymbol{r},t)\,\mathrm{d}\boldsymbol{r} &(3)\\
&= \mu_0 \int_{\mathbb{R}^3} \left(\psi_-^*(\boldsymbol{r},t)\langle -| + \psi_+^*(\boldsymbol{r},t)\langle +|\right)\sigma_x\left(\psi_-(\boldsymbol{r},t)|-\rangle + \psi_+(\boldsymbol{r},t)|+\rangle\right)\mathrm{d}\boldsymbol{r} &(4)\\
&= \mu_0 \int_{\mathbb{R}^3} \left(\psi_-^*(\boldsymbol{r},t)\langle -| + \psi_+^*(\boldsymbol{r},t)\langle +|\right)\left(|+\rangle\langle -| + |-\rangle\langle +|\right)\left(\psi_-(\boldsymbol{r},t)|-\rangle + \psi_+(\boldsymbol{r},t)|+\rangle\right)\mathrm{d}\boldsymbol{r} &(5)\\
&= \mu_0 \int_{\mathbb{R}^3} \psi_+^*(\boldsymbol{r},t)\psi_-(\boldsymbol{r},t) + \psi_-^*(\boldsymbol{r},t)\psi_+(\boldsymbol{r},t)\,\mathrm{d}\boldsymbol{r} &(6)\\
&= 2\mu_0 \int_{\mathbb{R}^3} \psi_-^*(\boldsymbol{r},t)\psi_+(\boldsymbol{r},t)\,\mathrm{d}\boldsymbol{r} &(7)
\end{aligned}$$

2 Question 2

The matrix representation of the $\boldsymbol{\sigma}$ operators in the eigenbasis of σ_z is

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{8}$$

1. Show that you can write any 2×2 hermitian matrix as a superposition of Pauli matrices.
2. Show that for a unitary matrix $M^\dagger M = \mathbb{1}$, $\left| n_0 \pm \sqrt{n_x^2 + n_y^2 + n_z^2} \right| = 1$
3. Show that no $M \neq 0$ exists such that $\{M, \sigma_\alpha\} = 0$ for all σ_α

2.1 Part 1

Let

$$M = \begin{pmatrix} a_1 + ib_1 & a_2 + ib_2 \\ a_3 + ib_3 & a_4 + ib_4 \end{pmatrix} \rightarrow M^\dagger = \begin{pmatrix} a_1 - ib_1 & a_3 - ib_3 \\ a_2 - ib_2 & a_4 - ib_4 \end{pmatrix} \tag{9}$$

Since $M = M^\dagger$:

$$\begin{cases} a_1 + ib_1 = a_1 - ib_1 \\ a_2 + ib_2 = a_3 - ib_3 \\ a_3 + ib_3 = a_2 - ib_2 \\ a_4 + ib_4 = a_4 - ib_4 \end{cases} \rightarrow \begin{cases} b_1 = b_4 = 0 \\ b_2 = -b_3 \\ a_2 = a_3 \end{cases} \rightarrow M = \begin{pmatrix} a_1 & a_2 + ib \\ a_2 - ib & a_4 \end{pmatrix} \tag{10}$$

which we can decompose into

$$M = \frac{a_1 + a_4}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{a_1 - a_4}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + a_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - b \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{11}$$

therefore

$$M = \begin{pmatrix} n_0 + n_z & n_x - in_y \\ n_x + in_y & n_0 - n_z \end{pmatrix} \tag{12}$$

2.2 Part 2

I assume the matrix M in this part is still the hermitian matrix we decomposed in part 1. Since it is now also unitary, all of its eigenvalues lie on the unit circle in C.

$$
\begin{aligned}
|\lambda\mathbb{1} - M| &= \begin{vmatrix} \lambda - n_0 - n_z & -n_x + in_y \\ -n_x - in_y & \lambda - n_0 + n_z \end{vmatrix} && (13)\\
&= (\lambda - n_0 - n_z)(\lambda - n_0 + n_z) + (-n_x + in_y)(n_x + in_y) && (14)\\
&= \lambda^2 - 2\lambda n_0 + n_0^2 - n_z^2 - n_x^2 - n_y^2 && (15)\\
&= (\lambda - n_0)^2 - (n_x^2 + n_y^2 + n_z^2) = 0 && (16)
\end{aligned}
$$

rearranging

$$\lambda = n_0 \pm \sqrt{n_x^2 + n_y^2 + n_z^2} \tag{17}$$

and since we know all the eigenvalues are on the unit circle, then

$$\left| n_0 \pm \sqrt{n_x^2 + n_y^2 + n_z^2} \right| = 1 \tag{18}$$

as needed.

2.3 Part 3

I continue to assume we mean M is a hermitian matrix as we decomposed earlier. In that case:

$$
\begin{aligned}
\{M, \sigma_\alpha\} &= \{\boldsymbol{\sigma} \cdot \boldsymbol{n}, \sigma_\alpha\} && (19)\\
&= \{n_0\mathbb{1} + n_x\sigma_x + n_y\sigma_y + n_z\sigma_z, \sigma_\alpha\} && (20)\\
&= n_0\{\mathbb{1}, \sigma_\alpha\} + n_x\{\sigma_x, \sigma_\alpha\} + n_y\{\sigma_y, \sigma_\alpha\} + n_z\{\sigma_z, \sigma_\alpha\} && (21)
\end{aligned}
$$

From the exercise session we know that

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{1} \tag{22}$$

So we can immediately notice that the only way for 21 to be zero is for each element to be zero, which is only possible if all the coefficients $n_\alpha = 0$, which means the only matrix for which the anti-commutative relation holds is the 0 matrix.

3 Question 3

Prove the following identites:

1. $\mathrm{Tr}(\sigma_\alpha\sigma_\beta\sigma_\gamma) = 2i\varepsilon_{\alpha\beta\gamma}$ and $\mathrm{Tr}(\sigma_\alpha\sigma_\beta) = 2\delta_{\alpha\beta}$
2. Given $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^3$ calculate $(\boldsymbol{\sigma}\cdot\boldsymbol{a})(\boldsymbol{\sigma}\cdot\boldsymbol{b})(\boldsymbol{\sigma}\cdot\boldsymbol{c})$ in two ways: first the two elements then the last 2 elements. What does the result mean?
3. Write down $e^{i\sigma_x}$ and e^{σ_x} as 2×2 matrices.
4. Given $\theta \in [0, 2\pi]$ and $\boldsymbol{n} \in \mathbb{R}^3$ a unit vector show that

$$e^{i\theta\boldsymbol{n}\cdot\boldsymbol{\sigma}} = \cos\theta + i(\boldsymbol{n}\cdot\boldsymbol{\sigma})\sin\theta$$

3.1 Part 1

From the exercise we know that

$$\begin{aligned}
\mathrm{Tr}(\sigma_\alpha\sigma_\beta) &= \mathrm{Tr}\left(\delta_{\alpha\beta}\mathbb{1} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma\right) && (23)\\
&= 2\delta_{\alpha\beta} + 0 && (24)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Tr}(\sigma_\alpha\sigma_\beta\sigma_\gamma) &= \mathrm{Tr}((\delta_{\alpha\beta}\mathbb{1} + i\varepsilon_{\alpha\beta\kappa}\sigma_\kappa)\sigma_\gamma) && (25)\\
&= \mathrm{Tr}(\delta_{\alpha\beta}\mathbb{1}\sigma_\gamma) + \mathrm{Tr}(i\varepsilon_{\alpha\beta\kappa}\sigma_\kappa\sigma_\gamma) && (26)\\
&= \mathrm{Tr}(i\varepsilon_{\alpha\beta\kappa}\sigma_\kappa\sigma_\gamma) && (27)\\
&= i\varepsilon_{\alpha\beta\kappa}\,\mathrm{Tr}(\sigma_\kappa\sigma_\gamma) && (28)\\
&= 2i\varepsilon_{\alpha\beta\gamma} && (29)
\end{aligned}$$

3.2 Part 2

$$\begin{aligned}
[(\boldsymbol{\sigma}\cdot\boldsymbol{a})(\boldsymbol{\sigma}\cdot\boldsymbol{b})](\boldsymbol{\sigma}\cdot\boldsymbol{c}) &= [\sigma_i a_i \sigma_j b_j](\boldsymbol{\sigma}\cdot\boldsymbol{c}) && (30)\\
&= [a_i b_j(\delta_{ij}\mathbb{1} + i\varepsilon_{ijk}\sigma_k)](\boldsymbol{\sigma}\cdot\boldsymbol{c}) && (31)\\
&= a_i b_i(\boldsymbol{\sigma}\cdot\boldsymbol{c}) + i a_i b_j \sigma_l \varepsilon_{ijl}(\boldsymbol{\sigma}\cdot\boldsymbol{c}) && (32)\\
&= a_i b_i \sigma_k c_k + i a_i b_j c_m \varepsilon_{ijl}\sigma_l\sigma_m && (33)\\
&= a_i b_i c_k \sigma_k + i a_i b_j c_m \varepsilon_{ijl}(\delta_{lm}\mathbb{1} + i\varepsilon_{lmp}\sigma_p) && (34)\\
&= a_i b_i c_k \sigma_k + i a_i b_j c_m \varepsilon_{ijm} - a_i b_j c_m \varepsilon_{ijl}\varepsilon_{mpl}\sigma_p && (35)\\
&= a_i b_i c_k \sigma_k + i a_i b_j c_m \varepsilon_{ijm} - a_i b_j c_m(\delta_{im}\delta_{jp} - \delta_{ip}\delta_{jm})\sigma_p && (36)\\
&= a_i b_i c_k \sigma_k + i a_i b_j c_m \varepsilon_{ijm} - a_i b_j c_i \sigma_j + a_i b_j c_j \sigma_i && (37)
\end{aligned}$$

$$= (\boldsymbol{a} \cdot \boldsymbol{b})(\boldsymbol{c} \cdot \boldsymbol{\sigma}) - (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{\sigma}) + (\boldsymbol{b} \cdot \boldsymbol{c})(\boldsymbol{a} \cdot \boldsymbol{\sigma}) + i(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} \tag{38}$$

on the other hand

$$\begin{aligned}
(\boldsymbol{\sigma} \cdot \boldsymbol{a})[(\boldsymbol{\sigma} \cdot \boldsymbol{b})(\boldsymbol{\sigma} \cdot \boldsymbol{c})] &= [\sigma_i c_i \sigma_j b_j](\boldsymbol{\sigma} \cdot \boldsymbol{a}) && (39)\\
&= [c_i b_j(\delta_{ij}\mathbb{1} + i\varepsilon_{ijk}\sigma_k)](\boldsymbol{\sigma} \cdot \boldsymbol{a}) && (40)\\
&= c_i b_i(\boldsymbol{\sigma} \cdot \boldsymbol{c}) + i c_i b_j \sigma_l \varepsilon_{ijl}(\boldsymbol{\sigma} \cdot \boldsymbol{a}) && (41)\\
&= c_i b_i \sigma_k c_k + i c_i b_j a_m \varepsilon_{ijl}\sigma_l \sigma_m && (42)\\
&= c_i b_i a_k \sigma_k + i c_i b_j a_m \varepsilon_{ijl}(\delta_{lm}\mathbb{1} + i\varepsilon_{lmp}\sigma_p) && (43)\\
&= c_i b_i a_k \sigma_k + i c_i b_j a_m \varepsilon_{ijm} - c_i b_j a_m \varepsilon_{ijl}\varepsilon_{mpl}\sigma_p && (44)\\
&= c_i b_i a_k \sigma_k + i c_i b_j a_m \varepsilon_{ijm} - c_i b_j a_m(\delta_{im}\delta_{jp} - \delta_{ip}\delta_{jm})\sigma_p && (45)\\
&= c_i b_i a_k \sigma_k + i c_i b_j a_m \varepsilon_{ijm} - c_i b_j a_i \sigma_j + c_i b_j a_j \sigma_i && (46)\\
&= (\boldsymbol{c} \cdot \boldsymbol{b})(\boldsymbol{a} \cdot \boldsymbol{\sigma}) - (\boldsymbol{c} \cdot \boldsymbol{a})(\boldsymbol{b} \cdot \boldsymbol{\sigma}) + (\boldsymbol{b} \cdot \boldsymbol{a})(\boldsymbol{c} \cdot \boldsymbol{\sigma}) + i(\boldsymbol{c} \times \boldsymbol{b}) \cdot \boldsymbol{a} && (47)
\end{aligned}$$

The two expressions are identical (the vector product element is different in writing but the vector operations result in the same).

3.3 Part 3

$$e^{\sigma_x} = \sum_{k=0}^{\infty} \frac{\sigma_x^k}{k!} \tag{48}$$

we'll find the eigenvalues and eigenvectors:

$$\begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} \implies \lambda = \pm 1,\ v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{49}$$

Therefore

$$e^{\sigma_x} = e^{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} = \begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix} \tag{50}$$

and

$$e^{i\sigma_x} = \begin{pmatrix} e^i & 0 \\ 0 & e^{-i} \end{pmatrix} \tag{51}$$

3.4 Part 4

$$e^{i\theta \boldsymbol{n}\cdot\boldsymbol{\sigma}} = \sum_{k=0}^{\infty} \frac{(i\theta \boldsymbol{n}\cdot\boldsymbol{\sigma})^k}{k!} \tag{52}$$

$$= \sum_{k=0}^{\infty} \left(\frac{(i\theta \boldsymbol{n}\cdot\boldsymbol{\sigma})^{2k}}{(2k)!} + \frac{(i\theta \boldsymbol{n}\cdot\boldsymbol{\sigma})^{2k+1}}{(2k+1)!} \right) \tag{53}$$

And we notice that

$$(\boldsymbol{n}^2 \cdot \boldsymbol{\sigma}^2)^k = \mathbb{1} \tag{54}$$

because of the identity with pauli matrices and their relationship to deltas and epsilon. This gives us

$$e^{i\theta \boldsymbol{n}\cdot\boldsymbol{\sigma}} = \sum_{k=0}^{\infty} \left(\frac{(i\theta)^{2k}}{(2k)!} + \frac{(i\theta)^{2k+1}}{(2k+1)!} \boldsymbol{n}\cdot\boldsymbol{\sigma} \right) \tag{55}$$

$$= \sum_{k=0}^{\infty} \frac{(i\theta)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(i\theta)^{2k+1}}{(2k+1)!} \boldsymbol{n}\cdot\boldsymbol{\sigma} \tag{56}$$

$$= \cos\theta + i(\boldsymbol{n}\cdot\boldsymbol{\sigma})\sin\theta \tag{57}$$

4 Question 4

For a two-dimensional Hilbert space we'll define the bases $\mathcal{B} = \{\ |+\rangle,\ |-\rangle\ \}$. This is the eigenbasis of σ_z.

1. Any matrix representing an operator in the space spanned by the basis can be singularly represented as the linear combination
$$A = \frac{1}{2}(\alpha_0 \mathbb{1}_2 + \boldsymbol{\alpha} \cdot \boldsymbol{\sigma})$$
where $\boldsymbol{\alpha} \in \mathbb{C}^3$, $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, and $\mathbb{1}_2$ is the identity operator. Show that $\alpha_0 = \mathrm{Tr}(A)$ and $\boldsymbol{\alpha} = \mathrm{Tr}(A\boldsymbol{\sigma})$
2. We'll define the operator $P = |\psi(t)\rangle\langle\psi(t)|$, where $|\psi(t)\rangle = a_+(t)|+\rangle + a_-(t)|-\rangle$ where $a_\pm \in \mathbb{C}$. Calculate P^2. Write down P as a matrix in the basis $\mathcal{B}$.
3. Let $P = \frac{1}{2}(M_0(t)\mathbb{1}_2 + \boldsymbol{M}(t) \cdot \boldsymbol{\sigma})$. Show that $\mathrm{Tr}[P] = 1$ and conclude $M_0(t)$.
4. From the connection between P and P^2 and P's form show that $\|\boldsymbol{M}(t)\| = 1$
5. Show that for a general operator O: $\mathrm{Tr}[|\psi(t)\rangle\langle\psi(t)|O] = \langle\psi(t)|O|\psi(t)\rangle$
6. Use the previous parts to find a connection between $\langle\boldsymbol{\sigma}\rangle$ and $\boldsymbol{M}$.
7. Assume the Hamiltonian H describes the time development of the system. Use the relation between $\boldsymbol{M}$ and $\langle\boldsymbol{\sigma}\rangle$ and the Schrödinger equation to prove that
$$\frac{d\boldsymbol{M}}{dt} = \frac{i}{\hbar}\langle\psi(t)|[H, \boldsymbol{\sigma}]|\psi(t)\rangle$$
8. Assume that H has a trace of zero, i.e it can be expressed as $H = \frac{\hbar}{2}\boldsymbol{\omega} \cdot \boldsymbol{\sigma}$ with $\boldsymbol{\omega} \in \mathbb{R}^3$. Show that $\boldsymbol{M}$ fulfills $\frac{d\boldsymbol{M}}{dt} = \boldsymbol{\omega} \times \boldsymbol{M}$.

4.1 Part 1

$$\mathrm{Tr}(A) = \frac{1}{2}\mathrm{Tr}(\alpha_o \mathbb{1} + \boldsymbol{\alpha} \cdot \boldsymbol{\sigma}) \tag{58}$$

Since the trace of all pauli matrices is zero, the element including it vanishes. Since we're in a 2 dimensional hilbert space, the trace of the unit matrix is 2, therefore:

$$\mathrm{Tr}(A) = \alpha_0 \tag{59}$$

$$
\begin{aligned}
\mathrm{Tr}(A\boldsymbol{\sigma}) &= \frac{1}{2}\,\mathrm{Tr}(\alpha_0\boldsymbol{\sigma} + (\boldsymbol{\alpha}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}) && (60)\\
&= \frac{1}{2}\,\mathrm{Tr}((\boldsymbol{\alpha}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}) && (61)\\
&= \frac{1}{2}\,(\alpha_1\,\mathrm{Tr}(\sigma_x\boldsymbol{\sigma}) + \alpha_2\,\mathrm{Tr}(\sigma_y\boldsymbol{\sigma}) + \alpha_3\,\mathrm{Tr}(\sigma_z\boldsymbol{\sigma})) && (62)\\
&= \frac{1}{2}(2\alpha_1 + 2\alpha_2 + 2\alpha_3) && (63)\\
&= \boldsymbol{\alpha} && (64)
\end{aligned}
$$

4.2 Part 2

$$
P^2 = |\psi(t)\rangle\langle\psi(t)|\,|\psi(t)\rangle\langle\psi(t)| = |\psi(t)\rangle\langle\psi(t)| = P \tag{65}
$$

since the state psi is normalized.

$$
\begin{aligned}
P &= |\psi(t)\rangle\langle\psi(t)| && (66)\\
&= (a_+(t)\,|+\rangle + a_-(t)\,|-\rangle)(a_+^*(t)\,\langle+| + a_-^*(t)\,\langle-|) && (67)\\
&= |a_+|^2\,|+\rangle\langle+| + |a_-|^2\,|-\rangle\langle-| + a_+a_-^*\,|-\rangle\langle+| + a_-a_+^*\,|+\rangle\langle-| && (68)\\
&= \begin{pmatrix} |a_+|^2 & a_-a_+^* \\ a_+a_-^* & |a_-|^2 \end{pmatrix} && (69)
\end{aligned}
$$

4.3 Part 3

We notice that the trace of P is $|a_+|^2 + |a_-|^2$ which is identical to $\langle\psi(t)|\,|\psi(t)\rangle$, and since the state is normalized it's equal to 1. The trace of $\boldsymbol{M}(t)\cdot\boldsymbol{\sigma}$ is 0 because the trace of each of the pauli matrices is zero. In total the trace of P is 1. Since the trace of P through the previous section must match the trace in the new writing, we find that $M_0(t) = 1$.

4.4 Part 4

We know that $P^2 = P$ and $P = \frac{1}{2}(\mathbb{1}_2 + \boldsymbol{M}(t)\cdot\boldsymbol{\sigma})$. We'll square the decomposed form of P:

$$
\begin{aligned}
P^2 &= \frac{1}{4}(\mathbb{1}_2 + \boldsymbol{M}(t)\cdot\boldsymbol{\sigma})^2 && (70)\\
&= \frac{1}{4}(\mathbb{1}_2^2 + 2\mathbb{1}_2\boldsymbol{M}(t)\cdot\boldsymbol{\sigma} + (\boldsymbol{M}(t)\cdot\boldsymbol{\sigma})^2) && (71)\\
&= \frac{1}{4}(\mathbb{1}_2 + 2\boldsymbol{M}(t)\cdot\boldsymbol{\sigma} + \|\boldsymbol{M}\|^2\,\mathbb{1}) = P && (72)
\end{aligned}
$$

so we must have

$$\frac{1}{4}(\mathbb{1}_2 + 2\boldsymbol{M}(t)\cdot\boldsymbol{\sigma} + \|\boldsymbol{M}\|^2\,\mathbb{1}) = \frac{1}{2}(\mathbb{1}_2 + \boldsymbol{M}(t)\cdot\boldsymbol{\sigma}) \tag{73}$$

$$\frac{1}{4}(\mathbb{1}_2 + \|\boldsymbol{M}\|^2\,\mathbb{1}) = \frac{1}{2}\mathbb{1}_2 \tag{74}$$

$$\|\boldsymbol{M}\|^2 = 4\left(\frac{1}{2} - \frac{1}{4}\right) = 1 \tag{75}$$

4.5 Part 5

$$\begin{aligned}
\mathrm{Tr}[\left|\psi(t)\right\rangle\left\langle\psi(t)\right| O] &= \left\langle+\right|\left|\psi(t)\right\rangle\left\langle\psi(t)\right| O\left|+\right\rangle + \left\langle-\right|\left|\psi(t)\right\rangle\left\langle\psi(t)\right| O\left|-\right\rangle && (76)\\
&= \left\langle\psi(t)\right| O\left|+\right\rangle\left\langle+\right|\left|\psi(t)\right\rangle + \left\langle\psi(t)\right| O\left|-\right\rangle\left\langle-\right|\left|\psi(t)\right\rangle && (77)\\
&= \left\langle\psi(t)\right| O\left|\psi(t)\right\rangle && (78)
\end{aligned}$$

4.6 Part 6

$$\begin{aligned}
\left\langle\psi(t)\right|\boldsymbol{\sigma}\left|\psi(t)\right\rangle &= \mathrm{Tr}[\left|\psi(t)\right\rangle\left\langle\psi(t)\right|\boldsymbol{\sigma}] && (79)\\
&= \mathrm{Tr}[P\boldsymbol{\sigma}] && (80)\\
&= \frac{1}{2}\,\mathrm{Tr}[M_0(t)\boldsymbol{\sigma} + \boldsymbol{M}(t)\cdot\boldsymbol{\sigma}^2] && (81)\\
&= \boldsymbol{M}(t) && (82)
\end{aligned}$$

4.7 Part 7

$$\begin{aligned}
\frac{d\boldsymbol{M}}{dt} &= \frac{d}{dt}(\left\langle\psi(t)\right|\boldsymbol{\sigma}\left|\psi(t)\right\rangle) && (83)\\
&= \langle\dot{\psi}(t)|\,\boldsymbol{\sigma}\left|\psi(t)\right\rangle + \left\langle\psi(t)\right|\boldsymbol{\sigma}\,|\dot{\psi}(t)\rangle && (84)\\
&= \frac{1}{-i\hbar}\left\langle\psi(t)\right| H\boldsymbol{\sigma}\left|\psi(t)\right\rangle + \frac{1}{i\hbar}\left\langle\psi(t)\right|\boldsymbol{\sigma}H\left|\psi(t)\right\rangle && (85)\\
&= \frac{i}{\hbar}\left\langle\psi(t)\right| H\boldsymbol{\sigma}\left|\psi(t)\right\rangle - \frac{i}{\hbar}\left\langle\psi(t)\right|\boldsymbol{\sigma}H\left|\psi(t)\right\rangle && (86)\\
&= \frac{i}{\hbar}\left\langle\psi(t)\right|[H,\boldsymbol{\sigma}]\left|\psi(t)\right\rangle && (87)
\end{aligned}$$

4.8 Part 8

$$\frac{d\boldsymbol{M}}{dt} = \frac{i}{\hbar}\langle\psi(t)|\,[H,\boldsymbol{\sigma}]\,|\psi(t)\rangle \tag{88}$$
$$= \frac{i}{\hbar}\langle\psi(t)|\,[\frac{\hbar}{2}\boldsymbol{\omega}\cdot\boldsymbol{\sigma},\boldsymbol{\sigma}]\,|\psi(t)\rangle \tag{89}$$
$$= \frac{i}{2}\boldsymbol{\omega}\cdot\langle\psi(t)|\,[\boldsymbol{\sigma},\boldsymbol{\sigma}]\,|\psi(t)\rangle \tag{90}$$
$$= -\omega_i\,\langle\psi(t)|\,\varepsilon_{ijk}\sigma_k\,|\psi(t)\rangle \tag{91}$$
$$= \varepsilon_{jik}\omega_i\langle\sigma_k\rangle \tag{92}$$
$$= \boldsymbol{\omega}\times\boldsymbol{M} \tag{93}$$

5 Question 5

Assume that spin is the angular momentum of the electron spinning around itself and that its magnitude is $\hbar/2$. Use the fact we know from measurements that the electron's classical radius is cmaller than 10^{-18} m to find the electron's speed of rotation. Does the result make sense?

The electron's mass is 9.109×10^{-31} kg, and the formula for its classical angular momentum is $S = I\omega$ where I is the moment of inertia of a sphere: $I = \frac{2}{5}MR^2$ and $\omega = \frac{v}{R}$. Plugging this all in we find

$$\frac{\hbar}{2} = \frac{2}{5}m_e Rv \implies v = \frac{5\hbar}{4m_e R} = \frac{5 \cdot 1.054 \times 10^{-34}\,\mathrm{m^2kg/s}}{4 \cdot 9.109 \times 10^{-31}\,\mathrm{kg} \cdot 10^{-18}\,\mathrm{m}} \tag{94}$$

$$\approx 1.446 \times 10^{14}\,\mathrm{m\,s^{-1}} \tag{95}$$

Which is significantly above the speed of light. This of course makes no sense, meaning that electron spin cannot be viewed as classical angular momentum.