Selected Subjects in Robotics and AI HW1

Daniel Haim Breger, 316136944

Technion

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1 Maximum Likelihood Estimation for Non-Gaussian DIstribution

Let $(X_1, X_2, \dots, X_n)$ be i.i.d samples from a Rayleign distribution with unknown scale parameter $(\theta > 0)$:

$$f(x, \theta) = \frac{x}{\theta^2} e^{-x^2/(2\theta^2)}, \quad x \geq 0$$

1. Derive the maximum likelihood estimator $\hat{\theta}_{ML}$.
2. Show that it is a consistent estimator of θ.
3. Determine whether $\hat{\theta}_{ML}$ is biased. If it is, derive the bias expression.
4. Compute the Fisher Information and Cramer-Rao lower bound for estimating θ and check if $\hat{\theta}_{ML}$ attains it asymptotically.

1.1 Deriving MLE

The log-likelihood is

$$l(\theta) = \sum_{i=1}^{n} \left(\log X_i - 2 \log \theta - \frac{X_i^2}{2\theta^2} \right) \tag{1}$$

The definition of the MLE is

$$\hat{\theta}_{MLE} = \arg\max L(\theta, x) \tag{2}$$

To find the argmax we'll differentiate the log-likelihood:

$$\frac{dl}{d\theta} = \sum_{i=1}^{n} \left(-\frac{2}{\theta} + \frac{X_i^2}{\theta^3} \right) = \frac{-2n}{\theta} + \frac{1}{\theta^3} \sum_{i=1}^{n} X_i^2 \tag{3}$$

we require the derivative is equal to zero and find

$$\frac{dl}{d\theta} = \frac{-2n}{\theta} + \frac{1}{\theta^3}\sum_{i=1}^{n} X_i^2 = 0 \tag{4}$$

$$\to -2n\theta^2 + \sum_{i=1}^{n} X_i^2 = 0 \tag{5}$$

$$\to \theta_{\max} = \sqrt{\frac{1}{2n}\sum_{i=1}^{n} X_i^2} \tag{6}$$

verifying the point we found is indeed a maximum

$$\left.\frac{d^2 l}{d\theta^2}\right|_{\theta_{\max}} = \left.\frac{2n}{\theta^2} - \frac{3}{\theta^4}\sum_{i=1}^{n} X_i^2\right|_{\theta_{\max}} \tag{7}$$

$$= \frac{2n}{\theta_{\max}^2} - \frac{3}{\theta_{\max}^4} \cdot 2n\theta_{\max}^2 \tag{8}$$

$$= \frac{2n - 6n}{\theta_{\max}^4} = -\frac{4n}{\theta_{\max}^4} < 0 \tag{9}$$

Therefore it is indeed a maximum, so our MLE is

$$\hat{\theta}_{MLE} = \sqrt{\frac{1}{2n}\sum_{i=1}^{n} X_i^2} \tag{10}$$

1.2 Consistency Check

An estimator is called consistent if

$$\lim_{n\to\infty} \hat{\theta}_n = \theta \tag{11}$$

So we'll look at the limit

$$\lim_{n\to\infty} \hat{\theta}_{MLE} = \lim_{n\to\infty} \sqrt{\frac{1}{2n}\sum_{i=1}^{n} X_i^2} \tag{12}$$

Using the law of large numbers we can say

$$\lim_{n\to\infty} \sqrt{\frac{1}{2n}\sum_{i=1}^{n} X_i^2} = \sqrt{\frac{1}{2n}\sum_{i=1}^{n} \mathbb{E}(X_i^2)} = \sqrt{\frac{1}{2n} 2n\theta^2} = \theta \tag{13}$$

Using the 2nd moment of the Rayleigh distribution. Thus it is a consistent estimator.

1.3 Bias Check

The bias is defined as

$$\mathrm{Bias}(\hat{\theta}) = \mathbb{E}(\hat{\theta}) - \theta \tag{14}$$

thus for our MLE estimator

$$\begin{align}
\mathrm{Bias}(\hat{\theta}) &= \mathbb{E}\left(\sqrt{\frac{1}{2n}\sum_{i=1}^{n} X_i^2}\right) - \theta \tag{15}\\
&= \frac{1}{\sqrt{2n}}\mathbb{E}\left(\sqrt{\sum_{i=1}^{n} X_i^2}\right) - \theta \tag{16}
\end{align}$$

to find the expectation value we'll notice that if $X_i \sim \mathrm{Rayleigh}(\theta)$ then $\sum X_i \sim \Gamma(n, 2\theta^2)$. We can use the formula for the moments of the gamma distribution for the 1/2-th moment

$$\mathbb{E}(\sqrt{\sum X_i}) = \sqrt{2}\theta\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n)} \tag{17}$$

and inserting back we find

$$\begin{align}
\mathrm{Bias}(\hat{\theta}) &= \frac{1}{\sqrt{2n}}\mathbb{E}\left(\sqrt{\sum_{i=1}^{n} X_i^2}\right) - \theta \tag{18}\\
&= \frac{1}{\sqrt{2n}}\sqrt{2}\theta\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n)} - \theta \tag{19}\\
&= \theta\left(\frac{1}{\sqrt{n}}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n)} - 1\right) \tag{20}
\end{align}$$

The expression in the brackets is non-zero so the estimator is biased.

1.4 Fisher Information and CRLB

The Fisher Information is defined by

$$
\begin{aligned}
I = -\mathbb{E}\left(\frac{\partial^2}{\partial\theta^2}\log f\right) &= -\mathbb{E}\left(\frac{\partial^2}{\partial\theta^2}\left(\log x - 2\log\theta - \frac{x^2}{2\theta^2}\right)\right) && (21)\\
&= -\mathbb{E}\left(\frac{\partial}{\partial\theta}\left(-\frac{2}{\theta} + \frac{x^2}{\theta^3}\right)\right) && (22)\\
&= -\mathbb{E}\left(\frac{2}{\theta^2} - \frac{3x^2}{\theta^4}\right) && (23)\\
&= -\left(\frac{2}{\theta^2} - \frac{3}{\theta^4}\mathbb{E}\left(x^2\right)\right) && (24)\\
&= -\left(\frac{2}{\theta^2} - \frac{3}{\theta^4}2\theta^2\right) && (25)\\
&= \frac{-2+6}{\theta^2} = \frac{4}{\theta^2} && (26)
\end{aligned}
$$

And since the Fisher information is based on expectation value it is linear and thus the fisher information of n i.i.d variables is

$$
I_n = \frac{4n}{\theta^2} \tag{27}
$$

The CRLB is therefore

$$
\mathrm{Var}(\hat{\theta}_{MLE}) \geq \frac{\theta^2}{4n} \tag{28}
$$

The MLE estimator is indeed asymptotically efficient as we saw in the lecture, but to show this we can use the asymptotic normalit property

$$
\sqrt{n}\left(\hat{\theta}_{MLE} - \theta\right) \to N(0, \frac{\theta^2}{4n}) \tag{29}
$$

therefore asymptotically

$$
\mathrm{Var}(\hat{\theta}_{MLE}) = \frac{\theta^2}{4n} \tag{30}
$$

which is indeed the CRLB.

2 Maximum A Posteriori Estimation with a Specific Prior

Assume the same Rayleigh distributed observations as in question 1, but now suppose tht prior knowledge about θ is modeled as an inverse-gamma distribution

$$p(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{-(\alpha+1)} e^{-\beta/\theta}, \quad \theta > 0$$

1. Derive the MAP esrtimator $\hat{\theta}_{MAP}$
2. Compare $\hat{\theta}_{ML}$ and $\hat{\theta}_{MAP}$ in terms of bias and asymptotic behavior.
3. Analyze how the choice of prior parameters α and β influences the MAP estimate.
4. Discuss the limiting cases $(\alpha, \beta \to 0)$ and $(\alpha, \beta \to \infty)$ and interpret their effects on the posterior estimate.

3 Cramer-Rao Lower Bound and Efficiency

Let $(X_1, X_2, \ldots, X_n)$ be i.i.d samples from an exponential distribution

$$f(x, \lambda) =^{-\lambda x}, \quad x \geq 0$$

where $\lambda > 0$ is unknown.

1. Derive the ML estimator $\hat{\lambda}_{ML}$.
2. Compute the Fisher Information.
3. Derive the CRLB for any unbiased estimator of λ.
4. Show that $\hat{\lambda}_{ML}$ achieves the CRLB and hence is an effective estimator.