E&M 2025 HW4

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Technion

(Date: May 16, 2025)

1 Question 1

A real scalar ϕ is given. We will develop the equations of motion of a real field given by the following Lagrangian density:

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi \tag{1}$$

Note that even though ϕ is a Lorentz scalar, $\partial_\mu\phi$ is not.

- Develop the equations of motion using E-L or variation on the action
- We now add an element to the lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 \tag{2}$$

  Find the new equations of motion.

1.1 Part 1

The E-L equation as found in the training sessions is

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\right) = 0 \tag{3}$$

But we'll notice that

$$\frac{\partial\mathcal{L}}{\partial\phi} = 0 \tag{4}$$

Thus we'll calculate

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \partial_\nu \phi} &= \frac{1}{2}\left(\frac{\partial \partial^\mu \phi}{\partial \partial_\nu \phi}\partial_\mu \phi + \frac{\partial \partial_\mu \phi}{\partial \partial_\nu \phi}\partial^\mu \phi\right) && (5)\\
&= \frac{1}{2}\left(\frac{\partial\left(g^{\rho\mu}\partial_\mu \phi\right)}{\partial \partial_\nu \phi}\partial_\mu \phi + \delta_\mu{}^\nu \partial^\mu \phi\right) && (6)\\
&= \frac{1}{2}\left(g^{\rho\mu}\frac{\partial \partial_\mu \phi}{\partial \partial_\nu \phi}\partial_\mu \phi + \delta_\mu{}^\nu \partial^\mu \phi\right) && (7)\\
&= \frac{1}{2}\left(g^{\rho\mu}\delta_\mu{}^\nu \partial_\mu \phi + \delta_\mu{}^\nu \partial^\mu \phi\right) && (8)\\
&= \frac{1}{2}\left(g^{\rho\nu}\partial_\nu \phi + \partial^\nu \phi\right) && (9)\\
&= \partial^\nu \phi && (10)
\end{aligned}$$

Therefore the equations of motion are

$$\partial_\nu \partial^\nu \phi = 0 \tag{11}$$

1.2 Part 2

We'll notice the only difference is that this time

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi \tag{12}$$

Therefore the equations of motion are

$$\partial_\nu \partial^\nu \phi + m^2 \phi = 0 \tag{13}$$

2 Question 2

The following Lagrangian density is given:

$$\mathcal{L} = \frac{1}{2}\rho\dot{y}^2 - \frac{1}{2}\tau y'^2 \tag{14}$$

The action is

$$S = \int\int_0^l \mathcal{L}\,\mathrm{d}x\,\mathrm{d}t \tag{15}$$

- ignore surface terms and develop the equations of motion in two ways:
  - Using variation on the action
  - Using E-L equations
- Use the fact the Lagrangian is not explicitly dependent on time or space and find the energy-momentum tensor.

2.1 Part 1

2.1.1 variation

We'll perform variation on the action:

$$\begin{aligned}
\frac{\delta S}{\delta y(x',t')} &= \int\int_0^l \mathrm{d}x\,\mathrm{d}t \frac{\mathcal{L}(\delta y', \delta\dot{y})}{\delta y(x',t')} &(16)\\
&= \int\int_0^l \mathrm{d}x\,\mathrm{d}t \frac{\frac{1}{2}\rho(\delta\dot{y})^2 - \frac{1}{2}\tau(\delta y')^2}{\delta y(x',t')} &(17)\\
&= \int\int_0^l \mathrm{d}x\,\mathrm{d}t \frac{1}{2}\left(\rho\frac{\partial_t(\delta y)^2}{\delta y(x',t')} - \tau\frac{\delta(\partial_x y)^2}{\delta y(x',t')}\right) &(18)\\
&= \int\int_0^l \mathrm{d}x\,\mathrm{d}t \frac{1}{2}\left(\rho\partial_t y\partial_t(\delta(x-x')\delta(t-t')) - \tau\partial_x y\partial_x(\delta(x-x')\delta(t-t'))\right) &(19)\\
&= -\frac{1}{2}\left(\rho\partial_t^2 y - \tau\partial_x^2 y\right) &(20)
\end{aligned}$$

Therefore the equations of motion are

$$\rho\ddot{y} = \tau y'' \tag{21}$$

2.1.2 Euler-Lagrange

$$\frac{\partial\mathcal{L}}{\partial y} = 0 \tag{22}$$

and

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu y} \right) = \partial_t \frac{\partial \mathcal{L}}{\partial \dot{y}} + \partial_i \frac{\partial \mathcal{L}}{\partial y'} \tag{23}$$
$$= \rho \ddot{y} - \tau y'' \tag{24}$$

Therefore the equation of motion is

$$\rho \ddot{y} = \tau y'' \tag{25}$$

2.2 Part 2

The definition of the momentum-energy tensor is

$$T^\nu{}_\mu = \frac{\partial \mathcal{L}}{\partial \partial_\nu y_i} \frac{\partial y_i}{\partial x_\mu} - \mathcal{L} \delta^\nu{}_\mu \tag{26}$$

Therefore via direction calculation

$$T^0{}_0 = \frac{\partial \mathcal{L}}{\partial \partial_0 y_i} \frac{\partial y_i}{\partial x_0} - \mathcal{L} \delta^0{}_0 \tag{27}$$
$$= \frac{\partial \mathcal{L}}{\partial \dot{y}_i} \dot{y}_i - \mathcal{L} \delta^0{}_0 \tag{28}$$
$$= \rho \dot{y}^2 - \left(\frac{1}{2} \rho \dot{y}^2 - \frac{1}{2} \tau y'^2 \right) \tag{29}$$
$$= \frac{1}{2} \rho \dot{y}^2 + \frac{1}{2} \tau y'^2 \tag{30}$$

$$T^0{}_1 = \frac{\partial \mathcal{L}}{\partial \partial_1 y_i} \frac{\partial y_i}{\partial x_0} - \mathcal{L} \delta^1{}_0 \tag{31}$$
$$= \frac{\partial \mathcal{L}}{\partial y'} \dot{y} \tag{32}$$
$$= -\tau y' \dot{y} \tag{33}$$

$$T^1{}_0 = \frac{\partial \mathcal{L}}{\partial \partial_0 y_i} \frac{\partial y_i}{\partial x_1} - \mathcal{L} \delta^0{}_1 \tag{34}$$
$$= \frac{\partial \mathcal{L}}{\partial \dot{y}} y' \tag{35}$$
$$= \rho \dot{y} y' \tag{36}$$

$$T^1{}_1 = \frac{\partial \mathcal{L}}{\partial \partial_1 y_i} \frac{\partial y_i}{\partial x_1} - \mathcal{L}\delta^1{}_1 \tag{37}$$

$$= \frac{\partial \mathcal{L}}{\partial y'} y' - \left(\frac{1}{2}\rho \dot{y}^2 - \frac{1}{2}\tau y'^2\right) \tag{38}$$

$$= -\frac{1}{2}\tau y'^2 - \frac{1}{2}\rho \dot{y}^2 \tag{39}$$

Therefore

$$T^\nu{}_\mu = \begin{pmatrix} \frac{1}{2}\rho\dot{y}^2 + \frac{1}{2}\tau y'^2 & -\tau y'\dot{y} \\ \rho\dot{y}y' & -\frac{1}{2}\tau y'^2 - \frac{1}{2}\rho\dot{y}^2 \end{pmatrix} \tag{40}$$

3 Question 3

A real scalar field $\phi(x)$ is given, as well as the following action:

$$S[\phi] = \int \mathrm{d}^4x \left(\frac{1}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi) \right) \tag{41}$$

where the potential is given by

$$V(\phi) = \frac{\lambda}{4} (\phi^2 - v^2)^2 \tag{42}$$

with $\lambda > 0$ and $v > 0$.

- Derive the E-L equations for a general potential.
- Use the given potential.
- Find all the constant solutions of the equation $\phi(x) = const.$ which satisfy the equations of motion. These solutions are called vacuum solutions.
- Assume a solution of the form $\phi(x) = v + \eta(x)$ where $\eta(x)$ is a small disturbance around the constant solutions from the previous part. Linearize the equations of motion and determine what the effective mass of the field η is.
- The given action is invariant to spatial transformations, $x^\mu \to x^\mu + a^\mu$, where a^μ is a constant. Using Noether's theorem:
  - Find the conserved current resulting from this symmetry.
  - Write the momentum density and energy density down explicitly.
  - Calculate the momentum-energy tensor for the constant solutions you found in part 3

3.1 Deriving the E-L equations

We notice that the integrand of the action is the Lagrangian density:

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial_\mu \phi - V(\phi) \tag{43}$$

Therefore the E-L equations are

$$\frac{\partial L}{\partial \phi} = -\partial_\phi V(\phi) \tag{44}$$

and copying the process in equation 5

$$\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = \partial^\mu \phi \tag{45}$$

Therefore the E-L equations are

$$\partial_\mu \partial^\mu \phi + \partial_\phi V(\phi) = 0 \tag{46}$$

3.2 Applying the potential

Inserting equation 42 into equation 46:

$$\partial_\mu \partial^\mu \phi + \lambda\phi(\phi^2 - v^2) = 0 \tag{47}$$

3.3 Finding constant solutions

Since ϕ is constant we set $\phi = c$ and solve the differential equation:

$$\begin{aligned}
\partial_\mu \partial^\mu \phi + \lambda\phi(\phi^2 - v^2) &= \partial_\mu \partial^\mu c + \lambda c(c^2 - v^2) && (48)\\
&= \lambda c(c^2 - v^2) = 0 && (49)\\
& && (50)
\end{aligned}$$

therefore

$$c_1 = \phi_1 = 0 \qquad c_2 = \phi_2 = v \qquad c_3 = \phi_3 = -v \tag{51}$$

3.4 Finding effective mass

Inserting a solution of the form $\phi(x) = v + \eta(x)$ into the equations of motion we get

$$\begin{aligned}
\partial_\mu \partial^\mu \phi + \partial_\phi V(\phi) &= \partial_\mu \partial^\mu \left(v + \eta(x)\right) + \lambda \left(v + \eta(x)\right)\left((v + \eta(x))^2 - v^2\right) && (52)\\
&= \Box\eta(x) + \lambda \left(v + \eta(x)\right)\left(v^2 + 2v\eta(x) + \eta^2(x) - v^2\right) && (53)\\
&= \Box\eta(x) + \lambda\left(2v^2\eta(x) + v\eta^2(x) + 2v\eta^2(x) + \eta^3(x)\right) && (54)\\
&\approx \Box\eta(x) + 2\lambda v^2\eta(x) && (55)
\end{aligned}$$

Next we fit this into the form of equation 2 and find

$$m = 2iv\sqrt{\lambda} \tag{56}$$

3.5 Spatial transformation

3.5.1 conserved current

Since the action is invariant for spatial transformations, so is the Lagrangian density, therefore using E-L:

$$\frac{\partial \mathcal{L}}{\partial x^\mu} = \frac{\partial \mathcal{L}}{\partial \phi_i}\frac{\partial \phi_i}{\partial x^\mu} + \frac{\partial \mathcal{L}}{\partial \partial^\nu \phi_i}\frac{\partial \partial^\nu \phi_i}{\partial x^\mu} \tag{57}$$

$$= \frac{\partial \mathcal{L}}{\partial \phi_i}\frac{\partial \phi_i}{\partial x^\mu} + \frac{\partial \mathcal{L}}{\partial \partial^\nu \phi_i}\frac{\partial^2 \phi_i}{\partial x^\mu \partial x^\nu} \tag{58}$$

$$= \left(\partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi_i}\right)\frac{\partial \phi_i}{\partial x^\mu} + \frac{\partial \mathcal{L}}{\partial \partial^\nu \phi_i}\frac{\partial^2 \phi_i}{\partial x^\mu \partial x^\nu} \tag{59}$$

$$= \frac{\partial}{\partial x^\nu}\left(\frac{\partial \mathcal{L}}{\partial \partial_\nu \phi_i}\frac{\partial \phi_i}{\partial x^\mu}\right) \tag{60}$$

on the other hand

$$\frac{\partial \mathcal{L}}{\partial x^\mu} = \frac{\partial \mathcal{L}}{\partial x^\nu}\frac{\partial x^\nu}{\partial x^\mu} = \frac{\partial \mathcal{L}}{\partial x^\nu}\delta^\nu{}_\mu \tag{61}$$

combining we get

$$\frac{\partial}{\partial x^\nu}\left(\frac{\partial \mathcal{L}}{\partial \partial_\nu \phi_i}\frac{\partial \phi_i}{\partial x^\mu}\right) = \frac{\partial \mathcal{L}}{\partial x^\nu}\delta^\nu{}_\mu \tag{62}$$

rearranging

$$\frac{\partial}{\partial x^\nu}\left(\frac{\partial \mathcal{L}}{\partial \partial_\nu \phi_i}\frac{\partial \phi_i}{\partial x^\mu} - \mathcal{L}\delta^\nu{}_\mu\right) = 0 \tag{63}$$

and recalling the definition of the momentum-energy tensor

$$\frac{\partial T^\nu{}_\mu}{\partial x^\nu} = 0 \tag{64}$$

Which is our conserved current.

3.5.2 Explicit momentum and energy

The energy density is $T^0{}_0$ therefore

$$T^0{}_0 = \frac{\partial \mathcal{L}}{\partial \partial_0 \phi_i}\frac{\partial \phi_i}{\partial x^0} - \mathcal{L}\delta^0{}_0 \tag{65}$$

$$= \frac{1}{2}\partial^0\phi_i\partial_0\phi_i - \frac{1}{2}\partial^i\phi\partial_j\phi + V(\phi) \tag{66}$$

$$= \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\partial^i\phi\partial_j\phi + V(\phi) \tag{67}$$

$$= \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\partial^i\phi\partial_j\phi + \frac{\lambda}{4}\left(\phi^2 - v^2\right)^2 \tag{68}$$

$$= \frac{1}{2}\left(\dot{\phi}^2 - \partial^i\phi\partial_j\phi\right) + \frac{\lambda}{4}\left(\phi^2 - v^2\right)^2 \tag{69}$$

The momentum density is $T^0{}_j$ therefore

$$
\begin{aligned}
T^0{}_j &= \frac{\partial \mathcal{L}}{\partial \partial_0 \phi_i} \frac{\partial \phi_i}{\partial x^j} - \mathcal{L}\delta^0{}_j && (70) \\
&= \frac{\partial \mathcal{L}}{\partial \partial_0 \phi_i} \frac{\partial \phi_i}{\partial x^j} && (71) \\
&= \partial^0 \phi \partial_j \phi && (72) \\
&= \dot{\phi} \partial_j \phi && (73)
\end{aligned}
$$

3.5.3 momentum-energy tensor for constant solutions

When $\phi = 0$ then

$$
T^0{}_0 = \frac{\lambda}{4} v^4 \qquad T^0{}_j = 0 \qquad T^i{}_j = \frac{\lambda}{4} v^4 \delta^i{}_j \tag{74}
$$

thus

$$
T^\mu{}_\nu = \frac{\lambda}{4} v^4 \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \tag{75}
$$

when $\phi = \pm v$

$$
T^0{}_0 = 0 \qquad T^0{}_j = 0 \qquad T^i{}_j = 0 \tag{76}
$$

therefore

$$
T^\mu{}_\nu = \mathbf{0} \tag{77}
$$

4 Question 4

The following Lagrangian is given

$$\mathcal{L} = -\frac{1}{16\pi c} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_A A_\mu A^\mu \tag{78}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and m_A is a real constant.

- What are the units of the constant m_A?
- Is the given Lagrangian symmetric under gauge transformations?
- Derive the equations of motion using either variation or Eular-Lagrange.

4.1 Units of m

The equation must satisfy that the units of both elements must be equal, so we'll check the units of the first element and deduce the units of m.

$$\left[\frac{F_{\mu\nu} F^{\mu\nu}}{c}\right] = \left[\frac{(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)}{c}\right] \tag{79}$$

$$= \frac{[(\partial_\mu A_\nu - \partial_\nu A_\mu)]^2}{\left[\frac{length}{time}\right]} \tag{80}$$

$$= [\partial_\mu A_\nu]^2 \frac{[time]}{[length]} \tag{81}$$

$$= \frac{1}{[length]^2} [A_\nu]^2 \frac{[time]}{[length]} \tag{82}$$

$$= \frac{[time]}{[length]^3} [A_\nu]^2 \tag{83}$$

but on the other hand

$$[m_A A_\mu A^\nu] = [m_A][A_\mu]^2 \tag{84}$$

therefore

$$[m_A] = \frac{[time]}{[length]^3} \tag{85}$$

4.2 Lagrangian symmetry?

We'll look at the Lagrangian after changing the potential such that

$$A_\mu \to A_\mu + \partial_\mu \lambda \tag{86}$$

therefore

$$A_\mu A^\mu \to (A_\mu + \partial_\mu \lambda)(A^\mu + \partial^\mu \lambda) = A_\mu A^\mu + A_\mu \partial^\mu \lambda + \partial_\mu \lambda A^\mu + \partial_\mu \lambda \partial^\mu \lambda \tag{87}$$

We can immediately notice that any gauge transformation that isn't trivial (i.e $\lambda = const$) will result in the lagrangian NOT being symmetric under said transformation.

4.3 Equations of motion

I'll use E-L. First We'll rewrite the Lagrangian:

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{16\pi c} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_A A_\mu A^\mu && (88)\\
&= -\frac{1}{16\pi c} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) + \frac{1}{2} m_A A_\mu A^\mu && (89)\\
&= -\frac{1}{16\pi c} (\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu - \partial_\nu A_\mu \partial^\mu A^\nu + \partial_\nu A_\mu \partial^\nu A^\mu) + \frac{1}{2} m_A A_\mu A^\mu && (90)\\
&= -\frac{1}{8\pi c} (\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu) + \frac{1}{2} m_A A_\mu A^\mu && (91)
\end{aligned}$$

Next we'll raise all indices for the potentials and lower indices for derivatives

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{8\pi c} (\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu) + \frac{1}{2} m_A A_\mu A^\mu && (92)\\
&= -\frac{1}{8\pi c} \left(\partial_\mu \eta_{\mu\alpha} A^\alpha \partial^\mu A^\nu - \partial_\mu \eta_{\nu\beta} A^\beta \partial^\nu A^\mu\right) + \frac{1}{2} m_A A_\mu A^\mu && (93)\\
&= -\frac{1}{8\pi c} \left(\partial_\mu \eta_{\mu\alpha} A^\alpha \eta^{\mu\gamma} \partial_\gamma A^\nu - \partial_\mu \eta_{\nu\beta} A^\beta \eta^{\nu\alpha} \partial_\alpha A^\mu\right) + \frac{1}{2} m_A A_\mu A^\mu && (94)\\
&= -\frac{1}{8\pi c} \left(\eta_{\mu\alpha} \eta^{\mu\gamma} \partial_\mu A^\alpha \partial_\gamma A^\nu - \eta_{\nu\beta} \eta^{\nu\alpha} \partial_\mu A^\beta \partial_\alpha A^\mu\right) + \frac{1}{2} m_A A_\mu A^\mu && (95)\\
& && (96)
\end{aligned}$$

and after changing indices so it's less horrible

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{8\pi c} \left(\eta_{\nu\alpha} \eta^{\mu\beta} \partial_\mu A^\alpha \partial_\beta A^\nu - \delta_\beta{}^\alpha \partial_\mu A^\beta \partial_\beta A^\mu\right) + \frac{1}{2} m_A A_\mu A^\mu && (97)\\
&= -\frac{1}{8\pi c} \left(\eta_{\nu\alpha} \eta^{\mu\beta} \partial_\mu A^\alpha \partial_\beta A^\nu - \partial_\mu A^\alpha \partial_\alpha A^\mu\right) + \frac{1}{2} m_A A_\mu A^\mu && (98)
\end{aligned}$$

We can now use E-L:

$$\frac{\partial \mathcal{L}}{\partial A^\nu} = \frac{1}{2} m_A \frac{\partial}{\partial A^\nu} \left(A_\mu A^\mu \right) \tag{99}$$
$$= \frac{1}{2} m_A \frac{\partial}{\partial A^\nu} \left(\eta_{\alpha\mu} A^\alpha A^\mu \right) \tag{100}$$
$$= \frac{1}{2} m_A \eta_{\alpha\mu} \frac{\partial}{\partial A^\nu} \left(A^\alpha A^\mu \right) \tag{101}$$
$$= \frac{1}{2} m_A \eta_{\alpha\mu} \left(\frac{\partial A^\alpha}{\partial A^\nu} A^\mu + A^\alpha \frac{\partial A^\mu}{\partial A^\nu} \right) \tag{102}$$
$$= \frac{1}{2} m_A \eta_{\alpha\mu} \left(\delta^\alpha{}_\nu A^\mu + A^\alpha \delta^\mu{}_\nu \right) \tag{103}$$
$$= \frac{1}{2} m_A \left(\eta_{\nu\mu} A^\mu + \eta_{\alpha\nu} A^\alpha \right) \tag{104}$$
$$= m_A A_\nu \tag{105}$$

And the second part

$$\frac{\partial \mathcal{L}}{\partial \partial_\rho A^\sigma} = -\frac{1}{8\pi c} \frac{\partial}{\partial \partial_\rho A^\sigma} \left(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu \right) \tag{106}$$
$$= -\frac{1}{8\pi c} \frac{\partial}{\partial \partial_\rho A^\sigma} \left(\partial_\mu \eta_{\nu\alpha} A^\alpha \eta^{\mu\beta} \partial_\beta A^\nu - \partial_\mu \eta_{\nu\alpha} A^\alpha \eta^{\nu\alpha} \partial_\alpha A^\mu \right) \tag{107}$$
$$= -\frac{1}{8\pi c} \frac{\partial}{\partial \partial_\rho A^\sigma} \left(\eta_{\nu\alpha} \eta^{\mu\beta} \partial_\mu A^\alpha \partial_\beta A^\nu - \eta_{\nu\alpha} \eta^{\nu\alpha} \partial_\mu A^\alpha \partial_\alpha A^\mu \right) \tag{108}$$
$$= -\frac{1}{8\pi c} \frac{\partial}{\partial \partial_\rho A^\sigma} \left(\eta_{\nu\alpha} \eta^{\mu\beta} \partial_\mu A^\alpha \partial_\beta A^\nu - \partial_\mu A^\nu \partial_\nu A^\mu \right) \tag{109}$$
$$= -\frac{1}{8\pi c} \left(\eta_{\nu\alpha} \eta^{\mu\beta} \frac{\partial}{\partial \partial_\rho A^\sigma} \left(\partial_\mu A^\alpha \partial_\beta A^\nu \right) - \frac{\partial}{\partial \partial_\rho A^\sigma} \left(\partial_\mu A^\nu \partial_\nu A^\mu \right) \right) \tag{110}$$
$$= -\frac{1}{8\pi c} \left(\eta_{\nu\alpha} \eta^{\mu\beta} \left(\frac{\partial \partial_\mu A^\alpha}{\partial \partial_\rho A^\sigma} \partial_\beta A^\nu + \partial_\mu A^\alpha \frac{\partial \partial_\beta A^\nu}{\partial \partial_\rho A^\sigma} \right) - \left(\frac{\partial \partial_\mu A^\nu}{\partial \partial_\rho A^\sigma} \partial_\nu A^\mu + \partial_\mu A^\nu \frac{\partial \partial_\nu A^\mu}{\partial \partial_\rho A^\sigma} \right) \right) \tag{111}$$
$$= -\frac{1}{4\pi c} \left(\eta_{\nu\alpha} \eta^{\mu\beta} \partial_\mu A^\alpha \frac{\partial \partial_\beta A^\nu}{\partial \partial_\rho A^\sigma} - \partial_\mu A^\nu \frac{\partial \partial_\nu A^\mu}{\partial \partial_\rho A^\sigma} \right) {}^{1} \tag{112}$$
$$= -\frac{1}{4\pi c} \left(\eta_{\nu\alpha} \eta^{\mu\beta} \partial_\mu A^\alpha \frac{\partial \partial_\beta A^\nu}{\partial \partial_\rho A^\sigma} - \partial_\mu A^\nu \frac{\partial \partial_\nu A^\mu}{\partial \partial_\rho A^\sigma} \right) \tag{113}$$
$$= -\frac{1}{4\pi c} \left(\eta_{\nu\alpha} \eta^{\mu\beta} \partial_\mu A^\alpha \delta_\beta{}^\rho \delta^\nu{}_\sigma - \partial_\mu A^\nu \delta_\nu{}^\rho \delta^\mu{}_\sigma \right) \tag{114}$$
$$= -\frac{1}{4\pi c} \left(\eta_{\sigma\alpha} \eta^{\mu\rho} \partial_\mu A^\alpha - \partial_\sigma A^\rho \right) \tag{115}$$
$$= -\frac{1}{4\pi c} \left(\partial^\rho A_\sigma - \partial_\sigma A^\rho \right) \tag{116}$$
$$= -\frac{1}{4\pi c} \eta^{\rho\zeta} \left(\partial_\zeta A_\sigma - \partial_\sigma A_\zeta \right) \tag{117}$$
$$= -\frac{1}{4\pi c} \eta^{\rho\zeta} F_{\zeta\sigma} \tag{118}$$

[1] This transition was done because we can change indices and find that in both expressions the summed elements are the same. Proof is provided in appendix 1.

Therefore

$$\begin{aligned}
\partial_\rho \left(\frac{\partial \mathcal{L}}{\partial \partial_\rho A^\sigma} \right) &= -\frac{1}{4\pi c} \eta^{\rho\zeta} \partial_\rho F_{\zeta\sigma} && (119) \\
&= -\frac{1}{4\pi c} \partial^\zeta F_{\zeta\sigma} && (120) \\
&= -\frac{1}{4\pi c} \partial^\rho F_{\rho\sigma} && (121) \\
&= -\frac{1}{4\pi c} \partial^\rho F_{\sigma\rho} && (122) \\
&= -\frac{1}{4\pi c} \partial^\nu F_{\mu\nu} && (123)
\end{aligned}$$

Inserting this into the E-L equations

$$m_A A_\mu = -\frac{1}{4\pi c} \partial^\nu F_{\mu\nu} \tag{124}$$

rearranging

$$\partial^\nu F_{\mu\nu} + 4\pi c m_A A_\mu = 0 \tag{125}$$

5 Appendix 1

If we start with

$$\left(\frac{\partial \partial_\mu A^\alpha}{\partial \partial_\rho A^\sigma} \partial_\beta A^\nu + \partial_\mu A^\alpha \frac{\partial \partial_\beta A^\nu}{\partial \partial_\rho A^\sigma}\right) \tag{126}$$

and do

$$\beta \to \mu \qquad\qquad \alpha \to \nu \tag{127}$$

then it becomes

$$\left(\frac{\partial \partial_\mu A^\nu}{\partial \partial_\rho A^\sigma} \partial_\mu A^\nu + \partial_\mu A^\nu \frac{\partial \partial_\mu A^\nu}{\partial \partial_\rho A^\sigma}\right) \tag{128}$$

which are the same. For the other element

$$\left(\frac{\partial \partial_\mu A^\nu}{\partial \partial_\rho A^\sigma} \partial_\nu A^\mu + \partial_\mu A^\nu \frac{\partial \partial_\nu A^\mu}{\partial \partial_\rho A^\sigma}\right) \tag{129}$$

with the change

$$\nu \to \mu \qquad\qquad \mu \to \nu \tag{130}$$

in the right element only becomes

$$\left(\frac{\partial \partial_\mu A^\nu}{\partial \partial_\rho A^\sigma} \partial_\nu A^\mu + \partial_\nu A^\mu \frac{\partial \partial_\mu A^\nu}{\partial \partial_\rho A^\sigma}\right) \tag{131}$$

which are the same.