E&M 2025 HW9

Daniel Haim Breger, 316136944

Technion

(Date: June 29, 2025)

1 Potential with Poisson

- Develop the function $\frac{1}{|x-x'|}$ to third order around $x' = 0$.
- Use the superposition solution of the Poisson equation

$$\phi(\mathbf{x}) = \int \mathrm{d}^3 x \frac{\rho(\mathbf{x}')}{|x-x'|} \tag{1}$$

and the approximation you found to reach the expression

$$\phi(x) = \frac{q}{|x|} + \frac{\mathbf{x}\cdot\mathbf{d}}{|\mathbf{x}|^3} + Q_{ij}\frac{x^i x^j}{|\mathbf{x}|^5} \tag{2}$$

where

$$q = \int \mathrm{d}^3 x' \rho(\mathbf{x}') \tag{3}$$

$$\mathbf{d} = \int \mathrm{d}^3 x' \rho(\mathbf{x}')\mathbf{x}' \tag{4}$$

$$Q_{ij} = \frac{1}{2}\int \mathrm{d}^3 x' \rho(\mathbf{x}')(3x'_i x'_j - r'^2\delta_{ij}) \tag{5}$$

1.1 Third order development

We'll mark the function we need to develop as $f(x')$ for convenience.

$$f(x') = (|\mathbf{x}-\mathbf{x}'|)^{-1} = \left(\sqrt{(\mathbf{x}-\mathbf{x}')^2}\right)^{-1} = \left((\mathbf{x}-\mathbf{x}')^2\right)^{-1/2} \tag{6}$$

$$f(0) = \frac{1}{|\mathbf{x}|} \tag{7}$$

$$f'(x')\big|_{x'=0} = -\frac{1}{2}\cdot 2(\mathbf{x}-\mathbf{x}')\cdot\left((\mathbf{x}-\mathbf{x}')^2\right)^{-3/2}\Big|_{x'=0} = \frac{\mathbf{x}-\mathbf{x}'}{(\mathbf{x}-\mathbf{x}')^{3/2}}\Big|_{x'=0} = \frac{\mathbf{x}}{|\mathbf{x}|^3} \tag{8}$$

$$\begin{aligned}
f''(x')\big|_{x'=0} &= \frac{\partial f'(x_i')}{\partial x_j'} = \frac{\partial}{\partial x_j'}\frac{x_i - x_i'}{(\mathbf{x}-\mathbf{x}')^{3/2}} &(9)\\
&= \frac{\frac{\partial}{\partial x_j'}(x_i - x_i')(\mathbf{x}-\mathbf{x}')^{3/2} - (x_i - x_i')\frac{\partial}{\partial x_j'}(\mathbf{x}-\mathbf{x}')^{3/2}}{(\mathbf{x}-\mathbf{x}')^3} &(10)\\
&= \frac{\delta_{ij}(\mathbf{x}-\mathbf{x}')^{3/2} - (x_i - x_i')\cdot\frac{3}{2}(x_j - x_j')\cdot 2(\mathbf{x}-\mathbf{x}')^{1/2}}{(\mathbf{x}-\mathbf{x}')^3} &(11)\\
&= \frac{|\mathbf{x}-\mathbf{x}'|^3\delta_{ij} - 3(x_i - x_i')(x_j - x_j')|\mathbf{x}-\mathbf{x}'|}{|\mathbf{x}-\mathbf{x}'|^6} &(12)\\
&= \frac{\delta_{ij}}{|\mathbf{x}-\mathbf{x}'|^3} - \frac{3(x_i - x_i')(x_j - x_j')}{|\mathbf{x}-\mathbf{x}'|^5}\Big|_{x'=0} &(13)\\
&= \frac{\delta_{ij}}{|\mathbf{x}|^3} - \frac{3x_i x_j}{|\mathbf{x}|^5} &(14)
\end{aligned}$$

Therefore

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} \approx \frac{1}{|\mathbf{x}|} + \frac{x_i x_i'}{|\mathbf{x}|^3} + \frac{x_i x_j(3x_i' x_j' - \delta_{ij}|\mathbf{x}|^2)}{2|\mathbf{x}|^5} \tag{15}$$

1.2 Reaching an expression

We'll insert equation 15 into equation 1:

$$\begin{aligned}
\phi(\mathbf{x}) &= \int \mathrm{d}^3x \frac{\rho(\mathbf{x}')}{|x-x'|} &(16)\\
&= \int \mathrm{d}^3x' \rho(\mathbf{x}')\cdot\left(\frac{1}{|\mathbf{x}|} + \frac{x_i x_i'}{|\mathbf{x}|^3} + \frac{x_i' x_j'(3x_i x_j - \delta_{ij}|\mathbf{x}|^2)}{2|\mathbf{x}|^5}\right) &(17)\\
&= \int \mathrm{d}^3x' \left(\frac{\rho(\mathbf{x}')}{|\mathbf{x}|} + \frac{\rho(\mathbf{x}')x_i x_i'}{|\mathbf{x}|^3} + \frac{\rho(\mathbf{x}')x_i' x_j'(3x_i x_j - \delta_{ij}|\mathbf{x}|^2)}{2|\mathbf{x}|^5}\right) &(18)\\
&= \int \mathrm{d}^3x' \frac{\rho(\mathbf{x}')}{|\mathbf{x}|} + \int \mathrm{d}^3x' \frac{\rho(\mathbf{x}')x_i x_i'}{|\mathbf{x}|^3} + \int \mathrm{d}^3x' \frac{\rho(\mathbf{x}')x_i' x_j'(3x_i x_j - \delta_{ij}|\mathbf{x}|^2)}{2|\mathbf{x}|^5} &(19)\\
&= \frac{q}{|\mathbf{x}|} + \int \mathrm{d}^3x' \frac{\rho(\mathbf{x}')\mathbf{x}\cdot\mathbf{x}'}{|\mathbf{x}|^3} + \int \mathrm{d}^3x' \frac{\rho(\mathbf{x}')x_i' x_j'(3x_i x_j - \delta_{ij}|\mathbf{x}|^2)}{2|\mathbf{x}|^5} &(20)\\
&= \frac{1}{|\mathbf{x}|} + \frac{\mathbf{x}\cdot\mathbf{d}}{|\mathbf{x}|^3} + Q_{ij}\frac{x_i x_j}{|\mathbf{x}|^5} &(21)
\end{aligned}$$

2 Fun with multipoles

Calculate the potential for a distant field in leading order for the three configurations.

First we'll find the volumetric charge distribution, since all three cases look similar we assume can find a general one. All the charges are the same distance from the origin and on the same plane and alternate in sign, and are equally spaced along the φ axis. Therefore

$$\rho(r,\theta,\varphi) = \frac{\delta(r-R)\delta(\theta-\frac{\pi}{2})}{R^2}\sum_{n=0}^{N-1}(-1)^n\delta\left(\varphi-\frac{2\pi n}{N}\right) \tag{22}$$

We divided by R^2 because as we remember from previous homework the 3d delta conversion from cartesian to spherical needs a correction, and the last element being a sum for all the charges, where N is either 2, 4, or 6 depending on the case.

As we know we can develop the potential using spherical harmonics:

$$\phi(\mathbf{x}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi q_{l,m}}{(2l+1)r^{l+1}}Y_{l,m}(\theta,\varphi) \tag{23}$$

where

$$q_{l,m} = \iiint\limits_{\text{everywhere}} \mathrm{d}^3\mathbf{x}Y^*_{l,m}(\mathbf{x})|\mathbf{x}|^l\rho(\mathbf{x}) \tag{24}$$

we'll solve this in spherical coordinates and using the density from equation 23:

$$q_{l,m} = \int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\mathrm{d}\theta\sin\theta\int_0^{\infty}\mathrm{d}r r^2Y^*_{l,m}(\theta,\varphi)r^l\rho(r,\theta,\varphi) \tag{25}$$

$$= \int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\mathrm{d}\theta\sin\theta\int_0^{\infty}\mathrm{d}r r^{l+2}Y^*_{l,m}(\theta,\varphi)\left(\frac{\delta(r-R)\delta(\theta-\frac{\pi}{2})}{R^2}\sum_{n=0}^{N-1}(-1)^n\delta\left(\varphi-\frac{2\pi n}{N}\right)\right) \tag{26}$$

$$= \frac{1}{R^2}\sum_{n=0}^{N-1}(-1)^n\int_0^{2\pi}\mathrm{d}\varphi\delta(\varphi-\frac{2\pi n}{N})\int_0^{\pi}\mathrm{d}\theta\sin\theta\delta(\theta-\frac{\pi}{2})Y^*_{l,m}(\theta,\varphi)\int_0^{\infty}\mathrm{d}r r^{l+2}\delta(r-R) \tag{27}$$

$$= R^l\sum_{n=0}^{N-1}(-1)^nY^*_{l,m}(\theta=\frac{\pi}{2},\varphi=\frac{2\pi n}{N}) \tag{28}$$

Finally before moving to each case we notice that the total charge in all cases is 0, therefore $Y_{0,0} = const.$ in all of these cases so we ignore it.

2.1 2 charges

In this case we have $N = 2$ therefore

$$q_{l,m} = R^l \sum_{n=0}^{1} (-1)^n Y^*_{l,m}(\theta = \frac{\pi}{2}, \varphi = \pi n) \tag{29}$$

and

$$Y_{1,-1} = \frac{1}{2}\sqrt{\frac{3}{2\pi}} \sin\theta e^{-i\varphi} \qquad Y_{1,0} = 0 \qquad Y_{1,1} = -\frac{1}{2}\sqrt{\frac{3}{2\pi}} \sin\theta e^{i\varphi} \tag{30}$$

therefore

$$q_{1,\pm 1} = 2R \cdot \mp \frac{1}{2}\sqrt{\frac{3}{2\pi}} e^{\pm i\varphi} = \mp R\sqrt{\frac{3}{2\pi}} \tag{31}$$

and thus the potential is

$$\begin{aligned}
\phi(r, \theta, \varphi) &= \sum_{m=-1}^{1} \frac{4\pi q_{1,m}}{3r^2} Y_{1,m} &(32)\\
&= \frac{4\pi R}{3r^2}\sqrt{\frac{3}{2\pi}} \cdot \frac{1}{2}\sqrt{\frac{3}{2\pi}} \sin\theta \left(e^{i\varphi} + e^{-i\varphi}\right) &(33)\\
&= \frac{2R}{r^2} \sin\theta \cos\varphi &(34)
\end{aligned}$$

2.2 4 charges

In this case we have $N = 4$ therefore

$$q_{l,m} = R^l \sum_{n=0}^{3} (-1)^n Y^*_{l,m}(\theta = \frac{\pi}{2}, \varphi = \frac{\pi n}{2}) \tag{35}$$

From the symmetry of the problem, specifically that the charges are distributed like a plus shape, and general familiarity with the spherical harmonics, we can tell that $q_{1,l} = 0$ because they are anti-symmetric when observing their behavior in rotations of $\frac{\pi}{2}$. This can be easily seen in any of the online animations of spherical harmonics. When 1 quarter becomes positive, the adjacent ones are negative. In conclusion, we begin our search at $l = 2$:

$$Y_{2,-2} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{-2i\varphi} \qquad Y_{2,-1} = \frac{1}{2}\sqrt{\frac{15}{2\pi}}\sin\theta\cos\theta e^{-i\varphi} \tag{36}$$

$$Y_{2,0} = \frac{1}{4}\sqrt{\frac{5}{\pi}}(3\cos^2\theta - 1) \tag{37}$$

$$Y_{2,1} = -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\sin\theta\cos\theta e^{i\varphi} \qquad Y_{2,2} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{2i\varphi} \tag{38}$$

therefore

$$q_{2,0} = R^2\sum_{n=0}^{3}(-1)^n Y_{2,0}^*(\theta = \frac{\pi}{2}, \varphi = \frac{\pi n}{2}) \tag{39}$$

$$= R^2\left[Y_{2,0}^*(\pi/2, 0) - Y_{2,0}^*(\pi/2, \pi/2) + Y_{2,0}^*(\pi/2, \pi) - Y_{2,0}^*(\pi/2, 3\pi/2)\right] \tag{40}$$

but we notice that $Y_{2,0}(\theta, \varphi + \pi) = -Y_{2,0}(\theta, \varphi)$ therefore

$$q_{2,0} = 0 \tag{41}$$

next, for $q_{2,\pm1}$, we notice the exact same, therefore

$$q_{2,\pm1} = 0 \tag{42}$$

Finally

$$q_{2,2} = R^2\sum_{n=0}^{3}(-1)^n Y_{2,2}^*(\theta = \frac{\pi}{2}, \varphi = \frac{\pi n}{2}) \tag{43}$$

$$= R^2\left[Y_{2,2}^*(\frac{\pi}{2}, 0) - Y_{2,2}^*(\frac{\pi}{2}, \frac{\pi}{2}) + Y_{2,2}^*(\frac{\pi}{2}, \pi) - Y_{2,2}^*(\frac{\pi}{2}, \frac{3\pi}{2})\right] \tag{44}$$

$$= R^2\frac{1}{4}\sqrt{\frac{15}{2\pi}}\left[\sin^2\frac{\pi}{2} - \sin^2\frac{\pi}{2}e^{-i\pi} + \sin^2\frac{\pi}{2}e^{-2i\pi} - \sin^2\frac{\pi}{2}e^{-3i\pi}\right] \tag{45}$$

$$= \frac{R^2}{4}\sqrt{\frac{15}{2\pi}}\left[1 + 1 + 1 + 1\right] \tag{46}$$

$$= R^2\sqrt{\frac{15}{2\pi}} \tag{47}$$

and from the rules of the multipoles we know that $q_{2,-2} = -R^2\sqrt{\frac{15}{2\pi}}$ thus

$$\phi(x) = \frac{4\pi}{5r^3}R^2\sqrt{\frac{15}{2\pi}}(Y_{2,2} + Y_{2,-2}) = \frac{3R^2}{5r^3}\sin^2\theta\cos(2\varphi) \tag{48}$$

2.3 6 charges

Using the same logic as N=4, we find that all moments with $l = 0, 1, 2$ are all zero, as well as $q_{3,m}$ for all $m \in \{0, 1, -1, 2, -2\}$. Therefore

$$q_{3,3} = 6R^3 \left(\frac{1}{8} \sqrt{\frac{35}{\pi}} \right) = \frac{3R^3}{4} \sqrt{\frac{35}{\pi}} \tag{49}$$

Therefore

$$\phi(x) = \frac{4\pi}{7r^4} \frac{3R^3}{4} \sqrt{\frac{35}{\pi}} (-Y_{3,3} + Y_{3,-3}) \tag{50}$$

$$= \frac{3\pi R^3}{7r^4} \sqrt{\frac{35}{\pi}} \cdot \frac{1}{8} \sqrt{\frac{35}{\pi}} \sin^3\theta \cos(3\varphi) \tag{51}$$

$$= \frac{15R^3}{4r^4} \sin^3\theta \cos(3\varphi) \tag{52}$$

3 Quadropole

Calculate the quadropole moment of the configuration

$$
\begin{aligned}
q_1 &= q & \mathbf{r}_1 &= (1,0,0) \\
q_2 &= -q & \mathbf{r}_2 &= (1,1,0)
\end{aligned}
$$

First we convert the distribution to a charge density

$$\rho(\mathbf{x}) = q \cdot \delta(z)\delta(x-1)\left(\delta(y) - \delta(y-1)\right) \tag{53}$$

and inserting this into the formula of a quadropole:

$$
\begin{aligned}
Q_{i,j} &= \frac{1}{2}\int\limits_V \mathrm{d}^3x\rho(\mathbf{x})(3x_ix_j - r\delta_{ij}) && (54)\\
&= \frac{1}{2}\int\limits_V \mathrm{d}^3xq \cdot \delta(z)\delta(x-1)\left(\delta(y) - \delta(y-1)\right)(3x_ix_j - r\delta_{ij}) && (55)\\
&= \frac{q}{2}(3x_ix_j - \delta_{ij})\Big|_{(x_1,x_2,x_3)=(1,0,0)} - q(3x_ix_j - 2\delta_{ij})|_{(x_1,x_2,x_3)=(1,1,0)} && (56)\\
&= \frac{q}{2}\begin{pmatrix} 3-1-(3-2) & 0-3 & 0 \\ 0-3 & -1-(3-2) & 0 \\ 0 & 0 & -1-(-2) \end{pmatrix} && (57)\\
&= \frac{q}{2}\begin{pmatrix} 1 & -3 & 0 \\ -3 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} && (58)
\end{aligned}
$$