E&M 2025 HW5

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Technion

(Date: May 18, 2025)

1 Question 1

Start with Maxwell's equations in tensor form

$$\partial_\rho \tilde{F}^{\rho\sigma} = 0 \qquad\qquad \partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu \tag{1}$$

and develop them into Maxwell's equations in cartesian coordinates

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \cdot \mathbf{E} = 4\pi\rho \qquad \frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{B} = -\nabla \times \mathbf{E} \qquad \nabla \times \mathbf{B} = \frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E} + \frac{4\pi}{c}\mathbf{J} \tag{2}$$

We remember that the fields tensor and its dual are:

$$F^{\rho\sigma} = \begin{pmatrix} & -E_1 & -E_2 & -E_3 \\ E_1 & & -B_3 & B_2 \\ E_2 & B_3 & & -B_1 \\ E_3 & -B_2 & B_1 & \end{pmatrix} \qquad \tilde{F}^{\rho\sigma} = \begin{pmatrix} & -B_1 & -B_2 & -B_3 \\ B_1 & & E_3 & -E_2 \\ B_2 & -E_3 & & E_1 \\ B_3 & E_2 & -E_1 & \end{pmatrix} \tag{3}$$

therefore

$$0 = \partial_i \tilde{F}^{i0} = \partial_i B_i = \partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3 = \nabla \cdot \mathbf{B} = 0 \tag{4}$$

and

$$\partial_\rho \tilde{F}^{\rho 1} = -\partial_0 B_1 - \partial_2 E_3 + \partial_3 E_2 = 0 \tag{5}$$

therefore

$$\partial_0 B_1 = \partial_3 E_2 - \partial_2 E_3 \tag{6}$$

but recalling that c=1 we can rewrite this as

$$\frac{1}{c}\frac{d}{dt}\mathbf{B}_1 = -\left(\nabla \times \mathbf{E}\right)_1 \tag{7}$$

and the derivations for the other elements are trivially the same.

Next we look at the second equation

$$\partial_i F^{i0} = \partial_1 E_1 + \partial_2 E_2 + \partial_3 E_3 = \nabla \cdot \mathbf{E} = 4\pi\rho = \frac{4\pi}{c} J^0 \tag{8}$$

and

$$\partial_\mu F^{\mu 1} = -\partial_0 E_1 + \partial_2 B_3 - \partial_3 B_2 = \frac{4\pi}{c} J^1 \tag{9}$$

rearranging and using c=1

$$\partial_2 B_3 - \partial_3 B_2 = (\nabla \times \mathbf{B})_1 = \frac{1}{c}\frac{d}{dt} E_1 + \frac{4\pi}{c} J^1 = \partial_0 E_1 + \frac{4\pi}{c} J^1 \tag{10}$$

We get what's required.

2 Question 2

The following 4-potential is given

$$A^\mu(x) = x^\mu e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)} \tag{11}$$

where $\alpha > 0$.

- Find the electric and magnetic fields derived from the given potential.

- calculate the 4-current for this potential. Show that the current fits conservation of charge.

- Switch the potential to temporal gauge.

- Instead of the above potential, the following potential is given:

$$A^\mu(x) = x^\mu e^{-\alpha((x^0)^2-(x^1)^2-(x^2)^2-(x^3)^2)} \tag{12}$$

what are the electric and magnetic fields now? Explain the result using gauge invariance.

2.1 finding the fields

We'll find the fields tensor instead and derive the fields from it. as we know, the fields tensor is:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{13}$$

thus in our case

$$E_i = F_{0i} = \partial_0 A_i - \partial_i A_0 \tag{14}$$
$$= \partial_0 \eta_{i\alpha} A^\alpha - \partial_i \eta_{0\beta} A^\beta \tag{15}$$
$$= -\partial_0 A^i - \partial_i A^0 \tag{16}$$
$$= -\frac{\partial}{\partial x^0}\left(x^i e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)}\right) - \frac{\partial}{\partial x^i}\left(x^0 e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)}\right) \tag{17}$$
$$= 2\alpha x^0 x^i e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)} + 2\alpha x^0 x^i e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)} \tag{18}$$
$$= 4\alpha x^0 x^i e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)} \tag{19}$$

and

$$B_1 = F_{32} = \partial_3 A_2 - \partial_2 A_3 \tag{20}$$
$$= \partial_3 \eta_{22} A^2 - \partial_2 \eta_{33} A^3 \tag{21}$$
$$= \partial_2 A^3 - \partial_3 A^2 \tag{22}$$
$$= \partial_2 \left(x^3 e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)} \right) - \partial_3 \left(x^2 e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)} \right) \tag{23}$$
$$= -2\alpha x^3 x^2 e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)} + 2\alpha x^2 x^3 e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)} \tag{24}$$
$$= 0 \tag{25}$$

noticing that this behavior is agnostic to which index>0 we choose because it's symmetric for those indexes

$$B_2 = B_3 = 0 \tag{26}$$

therefore

$$\mathbf{E} = 4\alpha x^0 e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)} \mathbf{x} \qquad\qquad \mathbf{B} = 0 \tag{27}$$

2.2   4-current

In the lectures we saw a relation between the 4-current and the fields tensor

$$\frac{4\pi}{c} \mathbf{J}^\nu = \partial_\mu F^{\mu\nu} \tag{28}$$

therefore

$$J^0 = \frac{c}{4\pi} \partial_\mu F^{\mu 0} \tag{29}$$
$$= \frac{c}{4\pi} \partial_i F^{i0} \tag{30}$$
$$= \frac{c}{4\pi} \partial_i \left(4\alpha x^0 x^i e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)} \right) \tag{31}$$
$$= \frac{c}{\pi} \alpha x^0 \left(3 e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)} - 2\alpha x^i x_i e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)} \right) \tag{32}$$
$$= \frac{\alpha c}{\pi} x^0 \left(3 - 2\alpha x^i x_i \right) e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)} \tag{33}$$

and

$$J^i = \frac{c}{4\pi}\partial_\mu F^{\mu i} \tag{34}$$

$$= \frac{c}{4\pi}\partial_0 F^{0i} \tag{35}$$

$$= -\frac{c}{4\pi}\partial_0 \left(4\alpha x^0 x^i e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)}\right) \tag{36}$$

$$= -\frac{\alpha c}{\pi}\left(x^i e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)} - 2\alpha x^0 x_0 x^i e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)}\right) \tag{37}$$

$$= -\frac{\alpha c}{\pi}x^i\left(1 - 2\alpha x^0 x_0\right)e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)} \tag{38}$$

therefore

$$\mathbf{J}^\mu = \frac{\alpha c}{\pi}\begin{pmatrix} 3 - 2\alpha x^i x_i \\ 2\alpha x^0 x_0 - 1 \\ 2\alpha x^0 x_0 - 1 \\ 2\alpha x^0 x_0 - 1 \end{pmatrix}\mathbf{x}e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)} \tag{39}$$

We will show conservation of charge exists by showing the continuity equation results. explicitly the 4-current is

$$\mathbf{J}^\mu = \frac{\alpha c}{\pi}\begin{pmatrix} 3x^0 - 2\alpha x^0 x^i x_i \\ 2\alpha x^0 x_0 x^1 - x^1 \\ 2\alpha x^0 x_0 x^2 - x^2 \\ 2\alpha x^0 x_0 x^3 - x^3 \end{pmatrix}\mathbf{x}e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)} \tag{40}$$

$$\partial_\mu \mathbf{J}^\mu = \frac{\alpha c}{\pi}\frac{\partial}{\partial x^\mu}\begin{pmatrix} 3x^0 - 2\alpha x^0 x^i x_i \\ 2\alpha x^0 x_0 x^1 - x^1 \\ 2\alpha x^0 x_0 x^2 - x^2 \\ 2\alpha x^0 x_0 x^3 - x^3 \end{pmatrix}\mathbf{x}e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)} \tag{41}$$

$$= \frac{\alpha c}{\pi}\begin{pmatrix} \partial_0 & \partial_1 & \partial_2 & \partial_3 \end{pmatrix}\begin{pmatrix} 3x^0 - 2\alpha x^0 x^i x_i \\ 2\alpha x^0 x_0 x^1 - x^1 \\ 2\alpha x^0 x_0 x^2 - x^2 \\ 2\alpha x^0 x_0 x^3 - x^3 \end{pmatrix}\mathbf{x}e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)} \tag{42}$$

$$= \frac{\alpha c}{\pi}\left(3 - 2\alpha x^i x_i\right)\left(1 - 2\alpha\left(x^0\right)^2\right)e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)} \tag{43}$$

$$+ \frac{\alpha c}{\pi}\left(2\alpha x^0 x_0 - 1\right)\left(1 - 2\alpha\left(x^1\right)^2\right)e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)} \tag{44}$$

$$+ \frac{\alpha c}{\pi}\left(2\alpha x^0 x_0 - 1\right)\left(1 - 2\alpha\left(x^2\right)^2\right)e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)} \tag{45}$$

$$+ \frac{\alpha c}{\pi}\left(2\alpha x^0 x_0 - 1\right)\left(1 - 2\alpha\left(x^3\right)^2\right)e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)} \tag{46}$$

$$= \frac{\alpha c}{\pi} \left(2\alpha x^i x_i\right) \left(1 - 2\alpha \left(x^0\right)^2\right) e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)} \tag{47}$$

$$+ \frac{\alpha c}{\pi} \left(2\alpha x^0 x_0 - 1\right) \left(-2\alpha \left(x^1\right)^2\right) e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)} \tag{48}$$

$$+ \frac{\alpha c}{\pi} \left(2\alpha x^0 x_0 - 1\right) \left(-2\alpha \left(x^2\right)^2\right) e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)} \tag{49}$$

$$+ \frac{\alpha c}{\pi} \left(2\alpha x^0 x_0 - 1\right) \left(-2\alpha \left(x^3\right)^2\right) e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)} \tag{50}$$

$$= 0 \tag{51}$$

therefore the charge is indeed conserved.

2.3 Temporal Gauge

We want to find a gauge transformation Λ such that $A^0 + \partial^0 \Lambda = 0$. Explicitly,

$$x^0 e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)} + \frac{\partial \Lambda}{\partial x^0} = 0 \tag{52}$$

therefore

$$\Lambda = -\int \mathrm{d}x^0 x^0 e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)} = \frac{1}{2\alpha} e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)} + C \tag{53}$$

Λ has gauge freedom for constant values so we'll choose $C = 0$ and get

$$\Lambda = \frac{1}{2\alpha} e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)} \tag{54}$$

Next we'll find all the derivatives of Λ:

$$\partial_\mu \Lambda = -x_\mu e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)} \tag{55}$$

raising the index

$$\partial^\mu \Lambda = \begin{pmatrix} -x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} e^{-\alpha((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)} \tag{56}$$

thus the potential is

$$A'^\mu = A^\mu + \partial^\mu \Lambda = x^\mu e^{-\alpha((x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2)} \tag{57}$$

$$+ \begin{pmatrix} -x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} e^{-\alpha((x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2)} \tag{58}$$

$$= 2x^i e^{-\alpha((x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2)} \tag{59}$$

2.4   Switching potential

We notice that the new potential can be written as

$$A^\mu = x^\mu e^{-\alpha \eta_{\alpha\beta} x^\alpha x^\beta} \tag{60}$$

and we notice that this looks similar to the form of the gauge transformation we used in the previous subsection. We'll try to fit a gauge transformation of a similar form and see what we get. We'll try

$$\Lambda = -\frac{1}{2\alpha} e^{-\alpha \eta_{\alpha\beta} x^\alpha x^\beta} \tag{61}$$

This gives us the following derivatives:

$$\partial_\mu \Lambda = -\frac{1}{2\alpha} \eta_{\mu\gamma} \partial^\gamma \Lambda \tag{62}$$

$$= \eta_{\mu\gamma} x^\gamma e^{-\alpha \eta_{\alpha\beta} x^\alpha x^\beta} \tag{63}$$

$$= x_\gamma e^{-\alpha \eta_{\alpha\beta} x^\alpha x^\beta} = A_\gamma \tag{64}$$

We found that we can define a gauge transformation that will just cancel out the potential giving us

$$A'_\mu = A_\mu + \partial_\mu \Lambda = 0 \tag{65}$$

And since the potential is zero, the fields tensor will also be 0, and since it's invariant for gauge transformations, we find that

$$\mathbf{E} = 0 \qquad\qquad \mathbf{B} = 0 \tag{66}$$

3 Question 3

The following interaction of N charges is given:

$$S_{int} = -\frac{1}{c}\sum_{i=1}^{N} e_i \int \mathrm{d}x_i^\mu A_\mu(x_i) = -\frac{1}{c}\sum_{i=1}^{N} e_i \int \mathrm{d}\tau_i u_i^\mu A_\mu(x_i) \tag{67}$$

- use the delta function

$$1 = \int \mathrm{d}^4 x' \delta^{(4)}(x_i(\tau_i) - x') \tag{68}$$

and the definition of current

$$J^\mu(x) = c \sum_{i=1}^{N} e_i \int \mathrm{d}\tau_i u_i^\mu \delta^{(4)}(x_i(\tau_i) - x') \tag{69}$$

to write down the action in terms of current and potential.

- Find the equations of motion for the action

$$S = S_A + S_{int} \tag{70}$$

using variation. Where $S_A = -\frac{1}{16\pi c}\int \mathrm{d}^4 x F_{\mu\nu} F^{\mu\nu}$.

3.1 Action using current and potential

We first notice that in the interaction all the potentials are of at specific points, thus we can use the delta to express them using the continuous space

$$S_{int} = -\frac{1}{c}\sum_{i=1}^{N} e_i \int \mathrm{d}\tau_i u_i^\mu A_\mu(x_i) \tag{71}$$

$$= -\frac{1}{c}\sum_{i=1}^{N} e_i \int \mathrm{d}\tau_i u_i^\mu \int \mathrm{d}^4 x' \delta^{(4)}(x_i - x') A_\mu(x') \tag{72}$$

$$= -\frac{1}{c^2}\int \mathrm{d}^4 x' A_\mu(x') c \sum_{i=1}^{N} e_i \int \mathrm{d}\tau_i u_i^\mu \delta^{(4)}(x_i - x') \tag{73}$$

$$= -\frac{1}{c^2}\int \mathrm{d}^4 x' A_\mu(x') \cdot J^\mu(x') \tag{74}$$

3.2 Equations of motion

The total action is

$$S = S_A + S_{int} = -\frac{1}{16\pi c}\int \mathrm{d}^4 x F_{\mu\nu} F^{\mu\nu} - \frac{1}{c^2}\int \mathrm{d}^4 x A_\mu \cdot J^\mu \tag{75}$$

$$= -\frac{1}{c}\int \mathrm{d}^4 x \left(\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} + \frac{1}{c} A_\mu J^\mu \right) \tag{76}$$

Therefore the variation is

$$\delta S = -\frac{1}{c}\int \mathrm{d}^4 x \frac{1}{16\pi}\delta(F_{\mu\nu}F^{\mu\nu}) + \frac{1}{c}J^\mu \delta A_\mu \tag{77}$$

$$= -\frac{1}{c}\int \mathrm{d}^4 x \frac{1}{16\pi}\left(\delta F_{\mu\nu}F^{\mu\nu} + F_{\mu\nu}\delta F^{\mu\nu}\right) + \frac{1}{c}J^\mu \delta A_\mu \tag{78}$$

$$= -\frac{1}{c}\int \mathrm{d}^4 x \frac{1}{8\pi}\left(F_{\mu\nu}\delta F^{\mu\nu}\right) + \frac{1}{c}J^\mu \delta A_\mu \tag{79}$$

$$= -\frac{1}{c}\int \mathrm{d}^4 x \frac{1}{8\pi}\left(\eta^{\alpha\mu}\eta^{\beta\nu}F_{\mu\nu}\delta F_{\alpha\beta}\right) + \frac{1}{c}J^\mu \delta A_\mu \tag{80}$$

$$= -\frac{1}{c}\int \mathrm{d}^4 x \frac{1}{8\pi}\left(\eta^{\alpha\mu}\eta^{\beta\nu}F_{\mu\nu}\delta\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right)\right) + \frac{1}{c}J^\mu \delta A_\mu \tag{81}$$

$$= -\frac{1}{c}\int \mathrm{d}^4 x \frac{1}{8\pi}\left(\eta^{\alpha\mu}\eta^{\beta\nu}F_{\mu\nu}\left(\partial_\alpha \delta A_\beta - \partial_\beta \delta A_\alpha\right)\right) + \frac{1}{c}J^\mu \delta A_\mu \tag{82}$$

$$= -\frac{1}{c}\int \mathrm{d}^4 x \frac{1}{4\pi}\eta^{\alpha\mu}\eta^{\beta\nu}F_{\mu\nu}\,\partial_\alpha \delta A_\beta + \frac{1}{c}J^\mu \delta A_\mu \tag{83}$$

$$= -\frac{1}{c}\int \mathrm{d}^4 x \frac{1}{4\pi}F^{\alpha\beta}\partial_\alpha \delta A_\beta + \frac{1}{c}J^\mu \delta A_\mu \tag{84}$$

$$= \text{I.B.P} = -\frac{1}{c}\int \mathrm{d}^4 x \frac{1}{4\pi}\partial_\alpha F^{\alpha\beta}\delta A_\beta + \frac{1}{c}J^\mu \delta A_\mu \tag{85}$$

$$= \text{renaming indexes} = -\frac{1}{c}\int \mathrm{d}^4 x \frac{1}{4\pi}\partial_\nu F^{\mu\nu}\delta A_\mu + \frac{1}{c}J^\mu \delta A_\mu \tag{86}$$

$$= -\frac{1}{c}\int \mathrm{d}^4 x \left(\frac{1}{4\pi}\partial_\nu F^{\mu\nu} + \frac{1}{c}J^\mu\right)\delta A_\mu \tag{87}$$

Therefore we require that

$$\frac{1}{4\pi}\partial_\nu F^{\mu\nu} + \frac{1}{c}J^\mu = 0 \tag{88}$$

rearranging

$$\partial_\nu F^{\mu\nu} = -\frac{4\pi}{c}J^\mu \tag{89}$$

4 Question 4

- Use the momentum-energy tensor of the fields in cartesian coordinates

$$T^{\mu}_{\ \nu} = -\frac{1}{4\pi c}F^{\mu\rho}F_{\nu\rho} + \frac{1}{16\pi c}\delta^{\mu}_{\ \nu}F^{\rho\sigma}F_{\rho\sigma} \tag{90}$$

and maxwell's equation with the presence of matter

$$\partial_{\nu}F_{\rho\mu} + \partial_{\mu}F_{\nu\rho} = 0 \tag{91}$$

$$\partial_{\nu}F^{\mu\nu} = -\frac{4\pi}{c}J^{\mu} \tag{92}$$

to show that in the presence of matter

$$\partial_{\mu}T^{\mu}_{\ \nu} = \frac{1}{c^2}F_{\nu\mu}J^{\mu} \tag{93}$$

- Show that the equation $\partial_{\mu}T^{\mu}_{\ \nu} = \frac{1}{c^2}F_{\nu\mu}J^{\mu}$ produces the rules of conservation in presence of matter:

  - for $\nu = 0$:
  $$\partial_{t}E_{em} + \nabla \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{j} \tag{94}$$

  - for $\nu = 1$:
  $$\partial_{t}\mathbf{p}_{em} + \partial_{t}\mathbf{p}_{mech} = \nabla \cdot \overleftrightarrow{\sigma} \tag{95}$$

  $$\partial_{t}\mathbf{p}_{mech} = \rho\mathbf{E} + \frac{1}{c}\mathbf{j} \times \mathbf{B} \tag{96}$$

4.1 Deriving that equation

we'll look at the following (on the next page):

$$\partial_\mu T^\mu{}_\nu = \partial_\mu \left(-\frac{1}{4\pi c} F^{\mu\rho} F_{\nu\rho} + \frac{1}{16\pi c} \delta^\mu{}_\nu F^{\rho\sigma} F_{\rho\sigma} \right) \tag{97}$$

$$= \frac{1}{4\pi c} \left(\frac{1}{4} \partial_\mu (\delta^\mu{}_\nu F^{\rho\sigma} F_{\rho\sigma}) - \partial_\mu (F^{\mu\rho} F_{\nu\rho}) \right) \tag{98}$$

$$= \frac{1}{4\pi c} \left(\frac{1}{4} \left(\partial_\nu F^{\rho\sigma} F_{\rho\sigma} + F^{\rho\sigma} \partial_\nu F_{\rho\sigma} \right) - \left(\partial_\mu F^{\mu\rho} F_{\nu\rho} + F^{\mu\rho} \partial_\mu F_{\nu\rho} \right) \right) \tag{99}$$

$$= \frac{1}{4\pi c} \left(\frac{1}{2} F^{\rho\sigma} \partial_\nu F_{\rho\sigma} + \partial_\mu F^{\rho\mu} F_{\nu\rho} + F^{\rho\mu} \partial_\mu F_{\nu\rho} \right) \tag{100}$$

$$= \frac{1}{4\pi c} \left(\frac{1}{2} F^{\rho\sigma} \partial_\nu F_{\rho\sigma} - \frac{4\pi}{c} J^\rho F_{\nu\rho} + F^{\rho\mu} \partial_\mu F_{\nu\rho} \right) \tag{101}$$

$$= -\frac{1}{c^2} F_{\nu\rho} J^\rho + \frac{1}{8\pi c} F^{\rho\sigma} \partial_\nu F_{\rho\sigma} + \frac{1}{4\pi c} F^{\rho\mu} \partial_\mu F_{\nu\rho} \tag{102}$$

$$= -\frac{1}{c^2} F_{\nu\rho} J^\rho + \frac{1}{8\pi c} \left(F^{\rho\sigma} \partial_\nu F_{\rho\sigma} + F^{\rho\mu} \partial_\mu F_{\nu\rho} + F^{\rho\mu} \partial_\mu F_{\nu\rho} \right) \tag{103}$$

$$= -\frac{1}{c^2} F_{\nu\rho} J^\rho + \frac{1}{8\pi c} F^{\rho\mu} \left(\partial_\nu F_{\rho\mu} + \partial_\mu F_{\nu\rho} + \partial_\mu F_{\nu\rho} \right) \tag{104}$$

$$= -\frac{1}{c^2} F_{\nu\rho} J^\rho \tag{105}$$

4.2   conservation rules

4.2.1   for $\nu = 0$

The equation we start with is

$$\partial_\mu T^\mu{}_0 = \frac{1}{c^2} F_{0\mu} J^\mu \tag{106}$$

Expanding we see

$$\partial_\mu T^\mu{}_0 = \partial_0 T^0{}_0 + \partial_i T^i{}_0 \tag{107}$$

$$= -\frac{1}{c^2} F_{00} J^0 + \partial_i T^i{}_0 \tag{108}$$

$$= \partial_i T^i{}_0 = -\frac{1}{c^2} F_{0i} J^i \tag{109}$$

$$\tag{110}$$

therefore

$$\frac{\partial W}{\partial t} + \partial_i T^i{}_0 = -\frac{1}{c^2} F_{0i} J^i \tag{111}$$

recalling the definition of the 4-current, fields tensor, and momentum-energy tensor we can rewrite that as

$$\frac{\partial W}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{j} \tag{112}$$

as required.

4.2.2   for $\nu = i$

The equation we start with is

$$\partial_\mu T^\mu{}_i = \frac{1}{c^2} F_{i\mu} J^\mu \tag{113}$$

expanding we get

$$\partial_\mu T^\mu{}_i = \partial_0 T^0{}_i + \partial_j T^j{}_i \tag{114}$$
$$= \frac{1}{c}\frac{\partial \mathbf{S}}{\partial t} + \nabla \cdot \overleftrightarrow{\sigma} \tag{115}$$
$$= \partial_t \mathbf{p}_{em} + \nabla \cdot \overleftrightarrow{\sigma} \tag{116}$$

and on the other hand:

$$\frac{1}{c^2} F_{i\mu} J^\mu = \frac{1}{c^2} F_{i0} J^0 + \frac{1}{c^2} F_{ij} J^j \tag{117}$$
$$= -\frac{1}{c^2} E_i c\rho + \frac{1}{c^2} \epsilon^k{}_{ji} B_k J^j \tag{118}$$
$$= -\frac{1}{c^2} E_i c\rho + (B \times \mathbf{J})_i \tag{119}$$
$$= -\frac{1}{c}\left(E_i\rho + \frac{1}{c}\mathbf{J} \times \mathbf{B}\right) \tag{120}$$

combining

$$\partial_t \mathbf{p}_{em} + \nabla \cdot \overleftrightarrow{\sigma} = -\frac{1}{c}\left(E_i\rho + \frac{1}{c}\mathbf{J} \times \mathbf{B}\right) \tag{121}$$
$$\partial_t \mathbf{p}_{em} + \frac{1}{c}\left(E_i\rho + \frac{1}{c}\mathbf{J} \times \mathbf{B}\right) = \nabla \cdot \overleftrightarrow{\sigma} \tag{122}$$
$$\partial_t \mathbf{p}_{em} + \partial_t \mathbf{p}_{mech} = \nabla \cdot \overleftrightarrow{\sigma} \tag{123}$$

5   Question 5

Find the transformation rule for energy density, energy current density, and the elements of the stress tensor under a lorentz transformation

We'll perform a lorentz boost in the 1 direction, and the other directions will be trivial extensions. The boost is

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{124}$$

We'll now perform a boost on the momentum-energy tensor

$$T'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha} T^{\alpha\beta} \Lambda^{\nu}{}_{\beta} \tag{125}$$

$$= \left[\Lambda T \Lambda^{T} \right] \tag{126}$$

$$= \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} W & \frac{1}{c}S_x & \frac{1}{c}S_y & \frac{1}{c}S_z \\ \frac{1}{c}S_x & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\ \frac{1}{c}S_y & -\sigma_{yx} & -\sigma_{yy} & -\sigma_{yz} \\ \frac{1}{c}S_z & -\sigma_{zx} & -\sigma_{zy} & -\sigma_{zz} \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{127}$$

$$= \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \gamma W - \frac{\gamma\beta}{c}S_x & -\gamma\beta W + \frac{\gamma}{c}S_x & \frac{1}{c}S_y & \frac{1}{c}S_z \\ \frac{\gamma}{c}S_x + \gamma\beta\sigma_{xx} & -\frac{\gamma\beta}{c}s_x - \gamma\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\ \frac{\gamma}{c}S_y + \gamma\beta\sigma_{yx} & -\frac{\gamma\beta}{c}s_x - \gamma\sigma_{yx} & -\sigma_{yy} & -\sigma_{yz} \\ \frac{\gamma}{c}S_z + \gamma\beta\sigma_{zx} & -\frac{\gamma\beta}{c}s_x - \gamma\sigma_{zx} & -\sigma_{zy} & -\sigma_{zz} \end{pmatrix} \tag{128}$$

$$= \begin{pmatrix} \gamma^2 \left(W - 2\frac{\beta}{c}S_x - \beta^2\sigma_{xx} \right) & -\gamma^2 \left(\beta W + \frac{\beta^2+1}{c}S_x + \beta\sigma_{xx} \right) & \frac{\gamma}{c}S_y + \gamma\beta\sigma_{xy} & \frac{\gamma}{c}S_z + \gamma\beta\sigma_{xz} \\ -\gamma^2 \left(\beta W - \frac{\beta^2+1}{c}S_x - \beta\sigma_{xx} \right) & \gamma^2 \left(\beta^2 W - 2\frac{\beta}{c}S_x - \sigma_{xx} \right) & -\frac{\gamma\beta}{c}S_y - \gamma\sigma_{xy} & -\frac{\gamma\beta}{c}S_z - \gamma\sigma_{xz} \\ \frac{\gamma}{c}S_y + \gamma\beta\sigma_{yx} & -\frac{\gamma\beta}{c}S_y - \gamma\sigma_{yx} & -\sigma_{yy} & -\sigma_{yz} \\ \frac{\gamma}{c}S_z + \gamma\beta\sigma_{zx} & -\frac{\gamma\beta}{c}S_z - \gamma\sigma_{zx} & -\sigma_{zy} & -\sigma_{zz} \end{pmatrix} \tag{129}$$

from this we can find that

$$W' = \gamma^2 \left(W - 2\frac{\beta}{c}S_x - \beta^2\sigma_{xx} \right) \tag{130}$$

and

$$S' = \gamma \begin{pmatrix} \gamma \left(-\beta W + \frac{\beta^2+1}{c}S_x + \beta\sigma_{xx} \right) \\ \frac{1}{c}S_y + \beta\sigma_{xy} \\ \frac{1}{c}S_z + \beta\sigma_{xz} \end{pmatrix} \tag{131}$$

and

$$\overset{\leftrightarrow}{\sigma}' = \begin{pmatrix} \gamma^2 \left(\sigma_{xx} + 2\frac{\beta}{c} S_x - \beta^2 W \right) & \gamma \left(\frac{\beta}{c} S_y + \sigma_{xy} \right) & \gamma \left(\frac{\beta}{c} S_z + \sigma_{xz} \right) \\ \gamma \left(\frac{\beta}{c} S_y + \sigma_{yx} \right) & \sigma_{yy} & \sigma_{yz} \\ \gamma \left(\frac{\beta}{c} S_z + \sigma_{zx} \right) & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \qquad (132)$$