E&M 2025 HW8

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Technion

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This worksheet was solved mostly in my bomb shelter, if there are mistakes, I blame the neighbor's kids who could not stop screaming. It was 80dB on average.

1 Green's function

An infinite plane is placed at $z = 0$ and held at potential V.

- Write the problem's Green function
- Above the plane there is a finite wire charged with charge density λ. It is parallel to the XY plane above the X axis so that its edges are at $(\pm L/2, 0, d)$. Find the potential above the plane. You may leave unsolved integrals.

1.1 Green Function

Since Green functions only depend on the geometry of the problem, we just need a Green function for an infinite plane. We already solved this in the lecture, thus

$$G_D = \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} - \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z+z')^2}} \tag{1}$$

1.2 Finite wire

We'll convert the line charge density to volume charge density

$$\rho(x, y, z) = \lambda\delta(y)\delta(z-d)\Theta(L/2-x)\Theta(x+L/2) \tag{2}$$

The potential is given by

$$\phi = \iiint\limits_V \mathrm{d}V'\rho G - \frac{1}{4\pi}\iint\limits_{\partial V} \mathrm{d}S' \cdot \phi\nabla G \tag{3}$$

where V is the top half of space (z>0) and thus the surface is the XY plane. Therefore

$$\iiint\limits_V \mathrm{d}V'\rho G = \int\limits_{-\infty}^{\infty} \mathrm{d}x' \int\limits_{-\infty}^{\infty} \mathrm{d}y' \int\limits_{0}^{\infty} \mathrm{d}z' \lambda\delta(y)\delta(z-d)\Theta(L/2-x)\Theta(x+L/2)G \tag{4}$$

$$= \lambda \int\limits_{-L/2}^{L/2} \mathrm{d}x' G(x,y,z,x',0,d) \tag{5}$$

$$= \lambda \int\limits_{-L/2}^{L/2} \mathrm{d}x' \frac{1}{\sqrt{(x-x')^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{(x-x')^2+y'^2+(z+d)^2}} \tag{6}$$

Which I won't solve.

As for the boundary,

$$\iint\limits_{\partial V} \mathrm{d}S' \cdot \phi\nabla G = -V \int\limits_{-\infty}^{\infty} \mathrm{d}x' \int\limits_{-\infty}^{\infty} \mathrm{d}y' \left.\frac{\partial G}{\partial z'}\right|_{z'=0} \tag{7}$$

$$= -V \int\limits_{-\infty}^{\infty} \mathrm{d}x' \int\limits_{-\infty}^{\infty} \mathrm{d}y' \frac{2z}{\left((x-x')^2+(y-y')^2+z^2\right)^{3/2}} \tag{8}$$

therefore the solution is

$$\phi = \lambda \int\limits_{-L/2}^{L/2} \mathrm{d}x' \frac{1}{\sqrt{(x-x')^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{(x-x')^2+y'^2+(z+d)^2}}$$
$$+ \frac{Vz}{2\pi} \int\limits_{-\infty}^{\infty} \mathrm{d}x' \int\limits_{-\infty}^{\infty} \mathrm{d}y' \frac{1}{\left((x-x')^2+(y-y')^2+z^2\right)^{3/2}}$$

2 Cartesian separation of variables

Two parallel grounded planes are given with separation d. Between the two plates there is a plate charged with surface charge density σ which is placed perpendicular to the other plates.

- What are the boundary conditions the potential needs to fulfill between the grounded plates?
- Using separation of variables find the potential between the plates.
- Find the charge density on the top and bottom plates.

2.1 Boundary conditions

First it's clear the following must be true:

$$\phi(x, 0) = \phi(x, d) = 0 \tag{9}$$

Second, we expect the potential to approach 0 at infinity thus

$$\lim_{x \to \pm\infty} \phi(x, y) = 0 \tag{10}$$

for all y between the plates. Third, We know there is a difference in the electric field across charged plates, which is the gradient of the potential, thus we also expect to see

$$-\left.\frac{\partial \phi}{\partial x}\right|_{x=0^+} + \left.\frac{\partial \phi}{\partial x}\right|_{x=0^-} = 4\pi\sigma \tag{11}$$

and finally the potential must be continuous

$$\phi(0^+, y) = \phi(0^-, y) \tag{12}$$

2.2 Finding the Potential

From the previous subsection we have a set of equations we need to solve. We'll start by setting the Ansatz

$$X''Y + XY'' = 0 \tag{13}$$

with the following boundary conditions

$$Y(0) = 0 \qquad\qquad Y(d) = 0 \tag{14}$$

which gives us the following, using the known solutions for Sturm-Liouville problems

$$Y = A\sin(\sqrt{\lambda}y) \tag{15}$$

and since Y(d) is zero, but we don't want the trivial solution, we find

$$\sqrt{\lambda} = \frac{\pi n}{d} \tag{16}$$

Therefore

$$Y_n(y) = \sin(\frac{\pi n}{d}y) \tag{17}$$

And thus

$$\phi(x,y) = \sum_{n=1}^{\infty} X(x)Y_n(y) = \sum_{n=1}^{\infty} X(x)\sin(\frac{\pi n}{d}y) \tag{18}$$

but we also know that

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -4\pi\sigma\delta(x) \tag{19}$$

therefore

$$\sum_{n=1}^{\infty}(X_n'' - \lambda_n X_n)\sin(\frac{\pi n}{d}y) = -4\pi\sigma\delta(x) \tag{20}$$

We can treat the entire X part as coefficients of an infinite sum of functions we'll develop using the inner product as in a fourier series:

$$\begin{aligned}
< -4\pi\sigma\delta(x), Y_n(y) > &= -4\pi\sigma\delta(x)\int_0^d \mathrm{d}y\sin(\frac{\pi n}{d}y) && (21)\\
&= 4\frac{\sigma d}{n}\delta(x)\cos(\frac{\pi n}{d}y)\Big|_0^d && (22)\\
&= 4\frac{\sigma d}{n}\delta(x)((-1)^n - 1) && (23)\\
&= \begin{cases} -8\frac{\sigma d}{n}\delta(x) & n = 2k+1 \\ 0 & n = 2k \end{cases} && (24)
\end{aligned}$$

and we find the normalization required

$$< Y_n(y), Y_n(y) > = \int_0^d \mathrm{d}y\sin^2(\frac{\pi n}{d}y) = \frac{d}{2} \tag{25}$$

therefore

$$-4\pi\sigma\delta(x) = -\sum_{k=1}^{\infty} \frac{16\sigma}{2k+1}\delta(x)\sin(\frac{\pi n}{d}y) \tag{26}$$

therefore

$$\sum_{n=1}^{\infty}(X_n'' - \lambda_n X_n) = -\sum_{k=1}^{\infty} \frac{16\sigma}{2k+1}\delta(x) \tag{27}$$

Whenever n is even this simplifies to

$$X_n'' - \lambda_n X_n = 0 \tag{28}$$

Whose solutions are

$$X_n(x) = A_n e^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x} \tag{29}$$

but from our boundary condition that demands the solution goes to 0 an both infinities, both A and B must be 0, making X also 0. For odd n we get

$$X_n'' - \lambda_n X_n = \frac{16\sigma}{n}\delta(x) \tag{30}$$

Which is another ODE whose solution is, outside x=0:

$$X_n(x) = A_n e^{\sqrt{\lambda}x} + B_n e^{-\sqrt{\lambda}x} \tag{31}$$

but since the solution could be different at positive and negative x values, we also have

$$X_n(x) = C_n e^{\sqrt{\lambda}x} + D_n e^{-\sqrt{\lambda}x} \tag{32}$$

for negative x.

Since the solution goes to 0 at both infinities we find $A_n = D_n = 0$. thus

$$X_n(x) = \begin{cases} B_n e^{-\sqrt{\lambda}x} & x > 0 \\ C_n e^{\sqrt{\lambda}x} & x < 0 \end{cases} \tag{33}$$

From the demand the solution be continuous we find that $B_n = C_n$ thus

$$X_n(x) = A_n e^{-\sqrt{\lambda}|x|} \tag{34}$$

inserting this back into equation 30 and integrating over x we get

$$X_n'(x) - \lambda_n A_n \int_{-\infty}^{\infty} \mathrm{d}x e^{-\sqrt{\lambda}|x|} = -\frac{16\sigma}{n} \tag{35}$$

therefore

$$2\sqrt{\lambda_n}A_n = \frac{16\sigma}{n} \rightarrow A_n = \frac{8\sigma d}{\pi n^2} \tag{36}$$

so the solution for X for odd n is

$$X_n(x) = \frac{8\sigma d}{\pi n^2}e^{-\frac{\pi n|x|}{d}} \tag{37}$$

and therefore the solution for the potential is

$$\phi(x,y) = \sum_{n=1}^{\infty} \frac{8\sigma d}{\pi(2n-1)^2}e^{-\frac{\pi(2n-1)|x|}{d}} \sin(\frac{\pi(2n-1)}{d}y) \tag{38}$$

2.3 Charge Density

Recalling the property of the electric field that it "jumps" at conductors, we'll find the electric field perpendicular to the plates from the potential and continue from there.

$$\begin{aligned} E_\perp &= -\frac{\partial\phi}{\partial y} && (39)\\ &= \sum_{n=1}^{\infty} \frac{8\sigma d}{\pi(2n-1)^2}e^{-\frac{\pi(2n-1)|x|}{d}} \cos(\frac{\pi(2n-1)}{d}y) \cdot \frac{\pi(2n-1)}{d} && (40)\\ &= -8\sigma \sum_{n=1}^{\infty} \frac{1}{2n-1}e^{-\frac{\pi(2n-1)|x|}{d}} \cos(\frac{\pi(2n-1)}{d}y) && (41)\end{aligned}$$

for the bottom plate y=0 since both approaches to 0 give the same:

$$E_{\perp\downarrow} = -8\sigma \sum_{n=1}^{\infty} \frac{1}{2n-1}e^{-\frac{\pi(2n-1)|x|}{d}} \tag{42}$$

therefore, since the field outside the area we're interested in is 0:

$$\begin{aligned} \sigma &= \frac{1}{4\pi} \cdot -8\sigma \sum_{n=1}^{\infty} \frac{1}{2n-1}e^{-\frac{\pi(2n-1)|x|}{d}} && (43)\\ &= -\frac{2\sigma}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1}e^{-\frac{\pi(2n-1)|x|}{d}} && (44)\end{aligned}$$

and for the top plate we can conclude from the symmetry of the problem that it must be identical:

$$\sigma = -\frac{2\sigma}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1}e^{-\frac{\pi(2n-1)|x|}{d}} \tag{45}$$

3 Green function between two spherical shells

In class you saw the Dirichlet Green function for a spherical shell with radius R_0:

$$G_D(x,x') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{1+2l} Y^*_{l,m}(\theta',\varphi')Y_{l,m}(\theta,\varphi)\times \begin{cases} \frac{1}{r}\left(\frac{R_0}{r}\right)^l \left[\left(\frac{r'}{R_0}\right)^l - \left(\frac{R_0}{r'}\right)^{l+1}\right] & r > r' \\ \frac{1}{r'}\left(\frac{R_0}{r'}\right)^l \left[\left(\frac{r}{R_0}\right)^l - \left(\frac{R_0}{r}\right)^{l+1}\right] & r < r' \end{cases} \tag{46}$$

In this exercise we will find the Dirichlet Green function between two concentric shells with radii a and b.

- Write the Green's function as a sum of spherical harmonics in the following way:

$$G_D(x,x') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm}(r|r',\theta',\varphi')Y_{l,m}(\theta,\varphi) \tag{47}$$

  Explain why you can write the coefficients using this separation of variables

$$a_{lm}(r|r',\theta',\varphi') = g_l(r,r')Y^*_{lm}(\theta',\varphi') \tag{48}$$

  and find a differential equation for g_l.

- Show that the solution to this equation is

$$g_l(r,r') = \begin{cases} Ar^l + Br^{-(l+1)} & r < r' \\ A'r^l + B'r^{-(l+1)} & r > r' \end{cases} \tag{49}$$

- Insert the appropriate boundary conditions for the problem and write g_l using only 2 constants.

- We'll define $r_< \equiv \min\{r,r'\}$, $r_> \equiv \max\{r,r'\}$. From the symmetry of g_l we can show that

$$g_l(r,r') = C\left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r>^l}{b^{2l+1}}\right) \tag{50}$$

  find C and write the full Green function

- Write the Green function at the limit $a \to 0, b \to \infty$

3.1 Green as Spherical Harmonics

First we know that the following is true

$$\nabla^2 G_D(x,x') = -4\pi\delta(\mathbf{x}-\mathbf{x}') \tag{51}$$

if we rewrite the delta function using spherical harmonics

$$\delta(\mathbf{x}-\mathbf{x}') = \frac{1}{r^2}\delta(r-r')\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^*_{l,m}(\theta',\varphi')Y_{l,m}(\theta,\varphi) \tag{52}$$

combining we find

$$\nabla^2 G_D(x, x') = \frac{-4\pi}{r^2}\delta(r - r')\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^*_{l,m}(\theta', \varphi')Y_{l,m}(\theta, \varphi) \tag{53}$$

We also know that spherical harmonics satisfy

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_{l,m}}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y_{l,m}}{\partial\varphi^2} = -l(l+1)Y_{l,m} \tag{54}$$

If we look at the given instruction, we can see

$$\nabla^2 G_D(x, x') = \nabla^2\left(\sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm}(r|r', \theta', \varphi')Y_{l,m}(\theta, \varphi)\right) \tag{55}$$

$$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \nabla^2\left(a_{lm}(r|r', \theta', \varphi')Y_{l,m}(\theta, \varphi)\right) \tag{56}$$

and using the laplacian in spherical coordinates

$$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{r}\frac{\partial^2}{\partial r^2}(ra_{lm}Y_{l,m}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}(a_{lm}Y_{l,m})\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}(a_{lm}Y_{l,m}) \tag{57}$$

$$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{Y_{l,m}}{r}\frac{\partial^2}{\partial r^2}(ra_{lm}) + \frac{a_{lm}}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_{l,m}}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y_{l,m}}{\partial\varphi^2}\right) \tag{58}$$

Using equation 54

$$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{Y_{l,m}}{r}\frac{\partial^2}{\partial r^2}(ra_{lm}) - \frac{a_{lm}}{r^2}\left(l(l+1)Y_{l,m}\right) \tag{59}$$

$$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{r}\left(\frac{\partial^2}{\partial r^2}(ra_{lm}) - \frac{l(l+1)}{r}a_{lm}\right)Y_{l,m} \tag{60}$$

Next we compare this to equation 53

$$\frac{-4\pi}{r^2}\delta(r - r')\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^*_{l,m}(\theta', \varphi')Y_{l,m}(\theta, \varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{r}\left(\frac{\partial^2}{\partial r^2}(ra_{lm}) - \frac{l(l+1)}{r}a_{lm}\right)Y_{l,m} \tag{61}$$

$$\frac{-4\pi}{r}\delta(r - r')Y^*_{l,m}(\theta', \varphi') = \frac{\partial^2}{\partial r^2}(ra_{lm}) - \frac{l(l+1)}{r}a_{lm} \tag{62}$$

From the lefthand's variable dependance we expect to be able to write

$$a_{lm} = g_{lm}(r,r')Y^*_{l,m}(\theta',\varphi') \tag{63}$$

therefore

$$\frac{-4\pi}{r}\delta(r-r')Y^*_{l,m}(\theta',\varphi') = \frac{\partial^2}{\partial r^2}(ra_{lm}) - \frac{l(l+1)}{r}g_{lm}(r,r')Y^*_{l,m}(\theta',\varphi') \tag{64}$$

$$\frac{-4\pi}{r}\delta(r-r') = \frac{\partial^2}{\partial r^2}(rg_{lm}) - \frac{l(l+1)}{r}g_{lm} \tag{65}$$

but since this equation doesn't depend on m, we can also say g doesn't depend on m, therefore

$$\frac{\partial^2}{\partial r^2}(rg_l(r,r')) - \frac{l(l+1)}{r}g_l(r,r') = -\frac{4'pi}{r}\delta(r-r') \tag{66}$$

3.2 finding solution

Since we're looking for solutions outside $r = r'$ then we can solve the following:

$$\frac{\partial^2}{\partial r^2}(rg_l(r,r')) - \frac{l(l+1)}{r}g_l(r,r') = 0 \tag{67}$$

We'll define the following helper function

$$f_l = rg_l \tag{68}$$

thus the equation becomes

$$\frac{\partial^2}{\partial r^2}(f_l(r,r')) - \frac{l(l+1)}{r^2}f_l(r,r') = 0 \tag{69}$$

rearranging

$$r^2\frac{\partial^2}{\partial r^2}(f_l(r,r')) - l(l+1)f_l(r,r') = 0 \tag{70}$$

This is an euler ODE. The characteristic polynomial is $r^2 - r - l(l+1) = 0$ whose solutions are

$$r_{1,2} = \frac{1 \pm \sqrt{1+4l(l+1)}}{2} = \frac{1 \pm (2l+1)}{2} \to r = \begin{cases} l+1 \\ -l \end{cases} \tag{71}$$

therefore the solution to the ODE is

$$f_l = Ar^{l+1} + Br^{-l} \tag{72}$$

therefore

$$g_l = Ar^l + Br^{-(l+1)} \tag{73}$$

We remember that we need different coefficients for both outside the shell and inside therefore

$$g_l(r, r') = \begin{cases} Ar^l + Br^{-(l+1)} & r < r' \\ A'r^l + B'r^{-(l+1)} & r < r' \end{cases} \tag{74}$$

3.3 Boundary Conditions

We'll require that the Green function becomes zero at both shells, i.e

$$g_l(a, r') = g_l(b, r') = 0 \tag{75}$$

therefore

$$Aa^l + Ba^{-(l+1)} = 0 \rightarrow B = -a^{2l+1}A \tag{76}$$

$$A'b^l + B'b^{-(l+1)} = 0 \rightarrow A' = -b^{-(2l+1)}B' \tag{77}$$

renaming B' to B for convenience, thus,

$$g_l(r, r') = \begin{cases} A\left(r^l - a^{2l+1}r^{-(l+1)}\right) & r < r' \\ B\left(-b^{-(2l+1)}r^l +' r^{-(l+1)}\right) & r < r' \end{cases} \tag{78}$$

3.4 Finding C

We'll integrate equation 66 around r' and take the limit $\epsilon \rightarrow 0$:

$$\int_{r'-\epsilon}^{r'+\epsilon} \mathrm{d}r \frac{\partial^2}{\partial r^2}(rg_l) - \int_{r'-\epsilon}^{r'+\epsilon} \mathrm{d}r \frac{l(l+1)}{r} g_l = -\int_{r'-\epsilon}^{r'+\epsilon} \mathrm{d}r \frac{4\pi}{r}\delta(r - r') \tag{79}$$

noticing that the middle element is even in a symmetric segment thus

$$\int_{r'-\epsilon}^{r'+\epsilon} \mathrm{d}r \frac{\partial^2}{\partial r^2}(rg_l) = -\int_{r'-\epsilon}^{r'+\epsilon} \mathrm{d}r \frac{4\pi}{r}\delta(r - r') \tag{80}$$

thus

$$\left[g_l + r\frac{\partial}{\partial r}g_l\right]_{r'-\epsilon}^{r'+\epsilon} = \frac{\partial}{\partial r}(rg_l)\bigg|_{r'-\epsilon}^{r'+\epsilon} \tag{81}$$

$$\left[g_l + r\frac{\partial}{\partial r}g_l\right]_{r'-\epsilon}^{r'+\epsilon} = -\frac{4\pi}{r'} \tag{82}$$

Now we take the limit.

$$\lim_{\epsilon\to 0}\left[g_l + r\frac{\partial}{\partial r}g_l\right]_{r'-\epsilon}^{r'+\epsilon} = \lim_{\epsilon\to 0}\left(g_l + r\frac{\partial}{\partial r}g_l\right)\bigg|_{r'+\epsilon} - \lim_{\epsilon\to 0}\left(g_l + r\frac{\partial}{\partial r}g_l\right)\bigg|_{r'-\epsilon} \tag{83}$$

$$= C\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right)\left[\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right) + r\left(-(l+1)r^{-(l+2)} - l\frac{r^{l-1}}{b^{2l+1}}\right)\right] \tag{84}$$

$$- C\left(r^{-(l+1)} - \frac{r^l}{b^{2l+1}}\right)\left[\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right) + r\left(lr^{l-1} - (l+1)\frac{a^{2l+1}}{r^{l+2}}\right)\right] \tag{85}$$

inserting into equation 81 we must find that

$$C = \frac{4\pi}{(2l+1)\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)} \tag{86}$$

Therefore the full Green function is

$$G_D(x,x') = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y_{lm}^*(\theta,\varphi)Y_{lm}(\theta,\varphi)}{(2l+1)\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)}\left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right) \tag{87}$$

3.5 Taking a Limit

$$G_D(x,x') = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y_{lm}^*(\theta,\varphi)Y_{lm}(\theta,\varphi)}{2l+1}\left(\frac{r_<^l}{r_>^{l+1}}\right) \tag{88}$$

4 Alternating potentials on proceeding slices

A hollow conductive shell with inner radius a is split into an even number of slices separated by planes coinciding with the z axis, where each plane is separated by an azimuthal angle $\frac{\pi}{n}$ from the next plane. Each slice is held at potential $\pm V$ where the sign alternates between slices.

- Write the potential inside the shell as an infinite sum with separation of variables in spherical coordinates, and write integral expressions for the coefficients of the expression.
- Show that the coefficients of $Y_{l,m}$ are zero unless $l + m$ is even.
- Show that the problem has symmetry of the form

$$\phi(\varphi) \to A\phi(\varphi + \Delta\varphi) \tag{89}$$

  and find A.
- Determine for which values of m the coefficient of $Y_{l,m}$ in the expression from the previous part is zero. Write the non-zero coefficients as an integral on $\cos\theta$ only. What is the $Y_{l,m}$ with the smallest l which contributes to the potential?

4.1 Potential in the shell

The solution for the laplace equation in spherical coordinates is

$$\phi(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{l,m}\left(\frac{r}{a}\right)^{l} + B_{l,m}\left(\frac{a}{r}\right)^{-(l+1)}\right)Y_{l,m}(\theta,\varphi) \tag{90}$$

But since we know the potential is not infinite at the origin, $B_{l,m} = 0$. Therefore

$$\phi(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{l,m}\left(\frac{r}{a}\right)^{l} Y_{l,m}(\theta,\varphi) \tag{91}$$

For convenience we'll define

$$\Phi = \begin{cases} V & \frac{2\pi k}{n} < \varphi < \frac{(2k+1)\pi}{n} \\ -V & \frac{(2k+1)\pi}{n} < \varphi < \frac{2(k+1)\pi}{n} \end{cases} \tag{92}$$

Next, we know the potential at $r = a$:

$$\phi(a,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{l,m}Y_{l,m}(\theta,\varphi) = \Phi \tag{93}$$

And since the spherical harmonics are a complete orthonormal set, the coefficients thus must be

$$A_{l,m} = \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\theta \sin\theta Y^*_{l,m}(\theta, \varphi)\Phi \tag{94}$$

4.2 Coefficients are zero

We know that the following relation holds for Legendre polynomials:

$$P_{l,m}(-x) = (-1)^{l+m} P_{l,m}(x) \tag{95}$$

and we know that

$$Y_{l,m} \propto P_{l,m} \tag{96}$$

Finally we know that the problem is symmetric for rotations of π radians. This is important because $-\cos\theta = \cos(\pi - \theta)$

Therefore we find that

$$Y_{l,m}(\theta, \varphi) = (-1)^{l+m} Y_{l,m}(\pi - \theta, \varphi) \tag{97}$$

Which means that we must have $l + m$ to be even.

4.3 A Symmetry

From the question's conditions we find that

$$V(\varphi + \Delta\varphi) = -V(\varphi) \tag{98}$$

We also note that the laplacian operator is invariant for rotational translations, therefore we must find that

$$\phi(\varphi + \Delta\varphi) = -\phi(\varphi) \tag{99}$$

i.e we find that $A = -1$.

4.4 When it is zero

Since we found the symmetry for the problem, the solution we found earlier must also satisfy it, therefore the coefficients for each $Y_{l,m}$ must satisfy it.

$$e^{im\varphi} = -e^{im(\varphi + \frac{\pi}{n})} \tag{100}$$

therefore

$$m = (2k - 1)n \tag{101}$$

I stop here because I simply can't concentrate enough to finish.