Lecture Notes:
Select Topics in Living Systems

Based on lectures by **Dr. Ram Adar** in 2025-26
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These notes were typed in the 2026 winter semester *Select Topics in Living Systems* course taught by **Dr. Ram Adar** at the Technion.

The lectures were given in Hebrew and were live-translated by me to English. The document is provided as is and likely contains many errors.

If you find any mistakes or typos please let me know at `danielbreger@campus.technion.ac.il`.

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Wed, October 29, 2025

1 Phase Transitions in Equilibrium

1.1 Motivation

Phase is a distinct state of matter which can be described by an "order parameter". We'll focus on isotropic polar phase transition. In this case the order parameter is the polarization $\mathbf{p} = \langle \hat{n} \rangle$ where $\hat{n}$ is the unit vector of the particles' direction.

Phase transition in certain areas of physics we see systems which aren't ordered at high temperatures but become ordered at low temperatures. Why? A competition between energy and entropy (Free energy being $F = U - TS$). The temperature at which order emerges is called the critical temperature, marked T_C and the change in the system is called a phase transition.
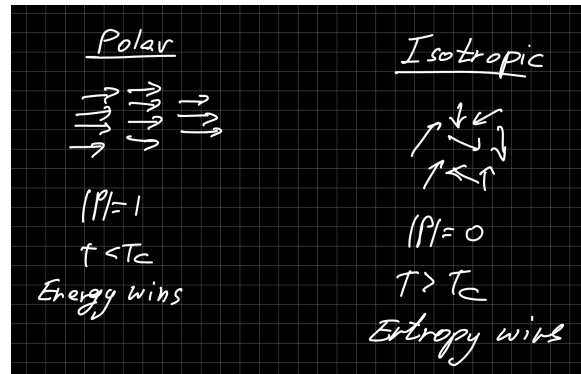

Figure 1.1: Polar and Isotropic ordering of matter.

1.2 Physical Description

We'll assume free energy of the following form:

$$F(T, N, \psi) = F_0(T, N) + F_\psi(T, N, \psi) \tag{1.1}$$

where ψ is the order parameter. ψ is not a conserved quantity but rather a degree of freedom of the system (in contrast to the number of particles, for example). Therefore under equilibruim we find

$$F_{eq}(T, N, \psi) = F_0(T, N) + F_\psi(T, N, \psi_{eq}) \tag{1.2}$$

and the equilibrium value is determined by

$$\left. \frac{\partial F_\psi}{\partial \psi} \right|_{\psi = \psi_{eq}} = 0 \tag{1.3}$$

and the following is true:

$$\psi_{eq}(T > T_C) = 0 \tag{1.4}$$

$$\psi_{eq}(T < T_C) \neq 0 \tag{1.5}$$

We'll focus on phase transition of 2nd order, for which the order parameter changes continuously, thus $\psi_{eq}(T = T_C) = 0$. Otherwise the phase transition is of first order.

1.3   Landau Theory of Phase Transitions

The theory was developed by Lev Landau, for which he earned the Nobel Prize in Physics in 1962. It states that close to the critical temperature T_C the physics of phase transitions is universal. For a phase transition of 2nd order, we will write F_ψ in a phenomenological form as a Taylor series expansion:

$$F_\psi = \frac{a}{2}\psi^2 + \frac{b}{4}\psi^4 \tag{1.6}$$

The terms being of even powers because we don't care about directions. The quadratic term's coefficient is of the form

$$a = a_0 \cdot \frac{T - T_C}{T_C} \tag{1.7}$$

Where $a_0 > 0$. This quantifies the competition between energy (which wins at $T < T_C$) and entropy (which wins at $T > T_C$. The parameter $b > 0$ ensures stability: ψ_{eq} receives finite values and does not approach infinity.
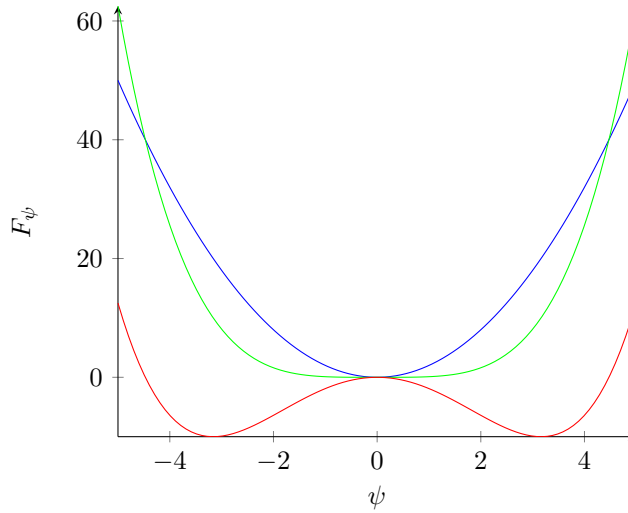

Figure 1.2: The graphs for the functions $2x^2$ (blue), $0.1x^4$ (green) and $0.1x^4 - 2x^2$ (red).

Because of a's sign, we find $\psi_{eq} \neq 0$ for $T < T_C$. Explicitly:

$$0 = \left.\frac{\partial F}{\partial \psi}\right|_{\psi=\psi_{eq}} = a\psi_{eq} + b\psi_{eq}^3 = b\psi_{eq}\left(\psi_{eq}^2 + \frac{a}{b}\right) \tag{1.8}$$

therefore

$$\psi_{eq} = 0, \pm\sqrt{\frac{a_o}{b} \cdot \frac{T - T_C}{T_C}} \tag{1.9}$$

There is no distinction between the positive and negative sign. Either can be chosen arbitrarily (spontaneous symmetry break). There will be a preference given an external field which will add an element of the form $-h\psi$ to the free energy.
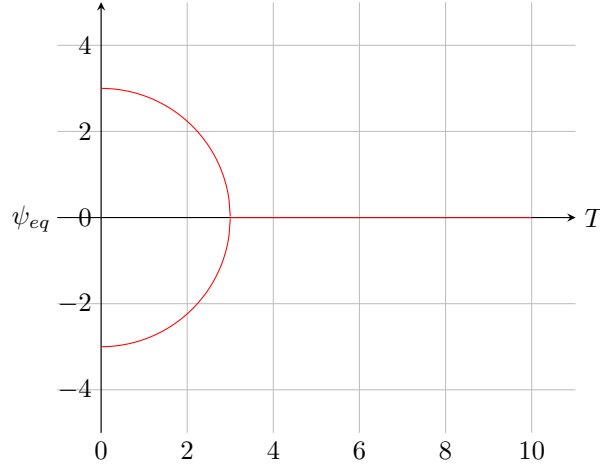
Figure 1.3: The possible solutions from equation 1.9.

Where the parameter a comes from

We'll assume a simple model of polarisation with energy

$$U = - \sum_{i,j \ neighbors} u n_\alpha^i n_\alpha^j \ , \ \ u > 0 \tag{1.10}$$

The energy is lower when i,j point in the same direction. Using mean field approximation the internal energy is

$$U = \frac{u}{2} \sum_{i,j} \langle n_\alpha^i \rangle \cdot \langle n_\alpha^j \rangle = -\frac{u}{2} N Z p_\alpha^2 \tag{1.11}$$

where $p_\alpha = \langle n_\alpha \rangle$, N is the number of particles, and Z the number of close neighbors. As for the entropy, we can find from a taylor expansion

$$S = S(p = 0) - \alpha N p^2 \tag{1.12}$$

and summing it up we find

$$F = U - TS = N \left(\alpha T - \frac{u}{2} Z \right) p^2 = a_0 \cdot \frac{T - T_C}{T_C} p^2 \tag{1.13}$$

where

$$T_C = \frac{uZ}{2\alpha} \qquad\qquad a_0 = \alpha N T_C \tag{1.14}$$

If instead of the vector n_α there is spin ($\pm$) then we find a similar result in the mean field framework. This model is called the Ising model, one of the most basic and useful models in statistical mechanics.

A dynamical description of the phase transition

We described the phase transition from the free energy but we can also describe it from dynamics. For a non-conserved order parameter we can write the dynamics as follows:

$$\dot{\psi} = -\Gamma_\psi \frac{\partial F}{\partial \psi} \ , \ \ \Gamma_\psi > 0 \tag{1.15}$$

the idea being that the dynamics guarantee that the free energy only gets smaller (the total entropy grows). We can see this using the chain rule:

$$\dot{F} = \frac{\partial F}{\partial \psi} \dot{\psi} = -\Gamma_\psi \left(\frac{\partial F}{\partial \psi} \right)^2 < 0 \tag{1.16}$$

this dynamics is called Model A (and often appears with a "noise" parameter). We'll insert F and find:

$$\dot{p} = -\Gamma_p (ap + bp^3) \tag{1.17}$$

the fixed points are $\frac{\partial F}{\partial p} = 0$, the extremums of the free energy. The stability of the solution is determined in phase space.
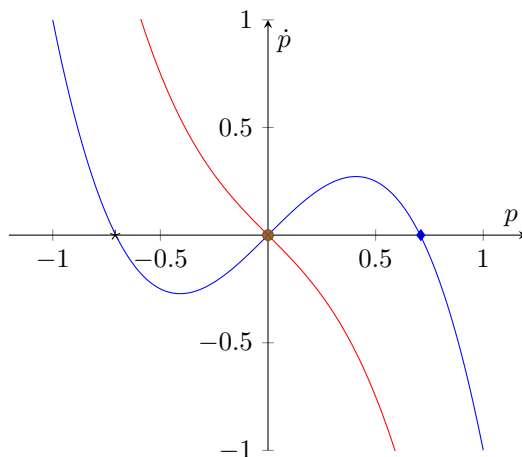
Figure 1.4: The graphs of $f(x) = x - 2x^3$ (blue) and $g(x) = -x - 2x^3$ (red). The positive and negative points marked on the x axis are stable points of equilibrium for $f(x)$. The point marked at $(0,0)$ is a stable point for $g(x)$ but an unstable one for $f(x)$.

in the language of dynamical systems, a phase transition is called a bifurcation. This approach is useful for out of equilibrium systems, for which we can't write down a free energy. Later on in the course, we'll come across dynamical equations of a similar form.

In general dynamical systems, when a function F whose derivative is strictly non-positive with a finite minimum exists (like free energy), then it's called a Lyapunov function.

1.4   Spatial Dependance: The Ginsburg-Landau Theory

We saw that the direction of **p** is arbitrary, in practice different areas of the system will initially develop order in different directions. For simplicity, we'll assume $\mathbf{p} = p\hat{y}$ where p can be either positive or negative. We'll look at the case where **p** points down at $x \to -\infty$ and up at $x \to \infty$.
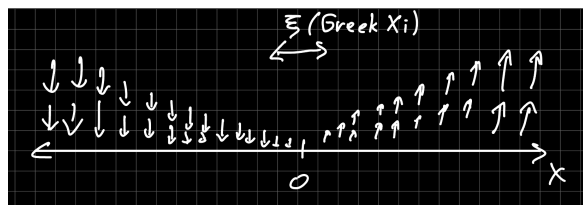
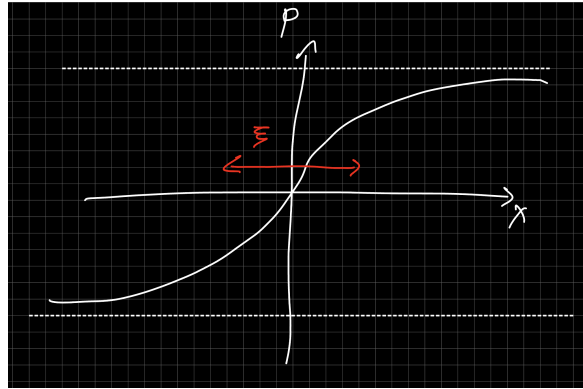Figure 1.5: A smooth phase transition with Domain Wall ξ.

Figure 1.6: A smooth phase transition with Domain Wall ξ.

We'll find a finite range around 0 of width χ in which $|\mathbf{p}| \neq p_{eq}$. This region is called the Domain Wall. To quantify the cost to change $\mathbf{p}$ in space we'll write the density of free energy per length unit:

$$F[p] = \int \mathrm{d}x f = \int \mathrm{d}x \left(\frac{a}{2}p^2 + \frac{b}{4}p^4 + \frac{k}{2}p'^2 \right) \tag{1.18}$$

where $k > 0$. The last element quantifies the cost to change p. In higher dimensions we'll have $(\nabla)^2$. Why is it like this? The cost comes from the gradient, and given there is no preferred direction, a positive and negative change should cost the same.

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The profile $p(x)$ is at the point where the functional derivative (with deltas) of F with respect to p is 0, i.e

$$0 = \frac{\partial f}{\partial p} - \left(\frac{\partial f}{\partial p'}\right)' \; ; F = \int \mathrm{d}x f \tag{2.1}$$

If we apply this to our specific functional we find

$$0 = \frac{\delta F}{\delta p} = ap + bp^3 - kp'' \tag{2.2}$$

which if we simplify it we find

$$\frac{k}{a}p'' = p + \frac{b}{a}p^3 \tag{2.3}$$

if we focus on the case where $a < 0$ and $p_{eq}^2 = \frac{L}{a}$ then

$$-\frac{k}{|a|}p'' = p\left(1 - \frac{p^2}{p_{eq}^2}\right) \tag{2.4}$$

which using dimensional analysis we found a length scale that goes with $\sqrt{\frac{k}{|a|}}$. Looking back at figure 1.6 we guess a solution of the form

$$p_0(x) = p_{eq}\tanh\frac{x}{\xi} \tag{2.5}$$

and use the following identity

$$\tanh' x = \frac{1}{\cosh^2 x} = 1 - \tanh^2 x \tag{2.6}$$

we find

$$p'' = \frac{p_{eq}}{\xi}\left(1 - \tanh^2\frac{x}{\xi}\right)' \tag{2.7}$$

$$= -2\frac{p_{eq}}{\xi^2}\tanh\frac{x}{\xi}\left(1 - \tanh^2\frac{x}{\xi}\right) \tag{2.8}$$

$$= -\frac{2}{\xi^2}p_0\left(1 - \frac{p_0^2}{p_{eq}^2}\right) \tag{2.9}$$

and from this we conclude

$$\frac{2}{\xi^2}\frac{k}{|a|} = 1 \rightarrow \xi = \sqrt{\frac{2k}{|a|}} \tag{2.10}$$

in particular, near the phase transition ($a \approx 0$), ξ diverges. Additionally, the excess free energy in the ξ width layer in comparison to the energy in phases at $\pm p_{eq}$ is called the interfacial tension (γ). It can be shown (in homework) that

$$\gamma = \int kp_0'^2 \, \mathrm{d}x = \sqrt{\frac{-8ka^3}{9b^2}} \tag{2.11}$$

1.5   The Mermin-Wagner(-Hohenberg) Theorem

So far we dealt with spatial change of a binary order parameter p, like spin, such that $\mathbf{p} = p\hat{y}$ but in a more general case $\mathbf{p} = p(\cos\theta, \sin\theta)$ and θ can change continuously. We'll notice that

$$F_p = \frac{a}{2}|\mathbf{p}|^2 + \frac{b}{4}|\mathbf{p}|^4 \tag{2.12}$$

is invariant under a change of θ. This is a continuous symmetry and θ is called the Goldstone/soft mode.

The Theorem Itself

A system under equilibrium at temperature $T > 0$ of dimension $d \leq 2$ will NOT develop a long range order which breaks a continuous symmetry.

Explanation

Let's look at how the free energy is dependent on θ. There is no homogeneous contribution but rather only from derivatives. As before, we'll write

$$F = \frac{k}{2} \int d\mathbf{r} (\nabla\theta)^2 \tag{2.13}$$

Looking at a configuration such as

How much free energy does it cost to create a continuous change in θ in a system of dimension d with characteristic length L? From dimensional considerations $\nabla^2 \sim L^{-2}$ and $\int d\mathbf{r} \sim L^d$ therefore $F \sim L^{d-2}$. For a large L and $d \leq 2$ the cost is low. Therefore change in θ will happen in a spontaneous manner from thermal fluctuations at $T > 0$.

A couple notes

1. In this course we will focus on *active systems out of equilibrium*. The theorem is violated and we can achieve long range order in 2D. This was the first result in active matter which created significant interest in the field.

2. Even in equilibrium a phase transition under the theorem's conditions is possible ($d = 2, T > 0$). In this case the order is not exactly long-range and is characterized by topological defects of the order parameter (we'll deal with defects later in the course). Sometimes such a transition is viewed as a KT transition (Kosterlitz-Thouless, 2016 Nobel prize in Physics)

2   Matter Out of Thermal Equilibrium

2.1   Types of Systems Out of Thermal Equilibrium

In general two main types of systems out of thermal equilibrium exist:

Non-Equilibrium Steady State In this case an outside energy source is injecting energy into the system, mostly through the system's edge, which dissipates. Examples:

1. Shear flow - Energy flows through the externally driven surfaces and dissipates through viscosity.

2. Electric circuits - A source is driving an electric current through the circuit, which dissipates at the resistor.

Near-Equilibrium Systems The system was previously in thermal equilibrium, which it left due to a controlled change in a thermodynamical parameter (T,P) or of an external potential. We can characterize the dynamics of the system arriving at the new equilibrium (relaxation). Examples:

1. Moving an optical trap - a colloid is captive in an optical trap which is suddenly moved. How does the colloid's position change over time?

2. Polymers stretching - a polymer is held on both ends by a force f. The force is suddenly changed. How will the edge-to-edge length of the polymer change?

In this course we will focus on active matter in which the energy is consumed locally by the system's elements. We will mostly focus on the case of a steady state. In general, given a system with measurable dynamics (such as a biological system), we will want to know whether it is in thermal equilibrium or not.

2.2   Macroscopic Description of Movement

In thermal equilibrium a closed system can only carry out a combination of uniform linear motion and rigid body rotation. Therefore, movement out of equilibrium will mostly be characterized by:

Wet Active Matter A closed system with non-uniform motion, such as a liquid with shearing. a relevant active system example for this case is active particles in a liquid.

Dry Active Matter An open system with uniform or non-uniform motion. A relevant active system example for this case is active particles on a surface.

Intuition

A system in which heat is created is out of equilibrium, since its entropy grows. In the first example heat is created due to viscosity and in the second due to friction. In an active system heat can be created from the active mechanism itself too (e.g a chemical reaction).

2.3   Proof of the statement in 2.2

We'll section a body into small but macroscopic parts with mass, energy, and momentum m_i, E_i, p_i respectively. The entropy results from the internal energy, and not from the kinetic energy of the center of mass.

$$S = \sum_i S_i(E_i - \frac{p_i^2}{2m_i^2}) \tag{2.14}$$

In a closed system the angular and linear momenta are conserved:

$$\sum_i \mathbf{p}_i = const. \qquad\qquad \sum_i \mathbf{r}_i \times \mathbf{p}_i = const. \tag{2.15}$$

In thermal equilibrium the entropy is maximized while conforming to the constraints.

$$0 = \frac{\partial}{\partial \mathbf{p}_i} \sum_j \left[S_j \left(E_j - \frac{p_j^2}{2m_j} \right) + \mathbf{a} \cdot \mathbf{p}_j + \mathbf{b} \cdot (\mathbf{r}_i \times \mathbf{p}_j) \right] \tag{2.16}$$

$$= -\frac{1}{T} \frac{\mathbf{p}_i}{m_i} + \mathbf{a} + \mathbf{b} \times \mathbf{r}_i \tag{2.17}$$

$$\rightarrow \mathbf{v}_i = \mathbf{u} + \boldsymbol{\Omega} \times \mathbf{r}_i \tag{2.18}$$

where

$$\mathbf{u} = T\mathbf{a}$$

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