

Quantum 2 HW1

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Technion

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1 Question 1

1.1 Part 1

Show that

$$\left| \int_0^t dt' e^{i\omega t'} \right|^2 = \frac{\sin^2\left(\frac{\omega t}{2}\right)}{\left(\frac{\omega}{2}\right)^2}$$

$$\int_0^t dt' e^{i\omega t'} = \left(\frac{1}{i\omega} e^{i\omega t'} \right) \Big|_{t'=0}^{t'=t} \quad (1)$$

$$= \frac{1}{i\omega} (e^{i\omega t} - 1) \quad (2)$$

$$\therefore \quad (3)$$

$$\left| \int_0^t dt' e^{i\omega t'} \right|^2 = \left| \frac{1}{i\omega} (e^{i\omega t} - 1) \right|^2 \quad (4)$$

$$= \frac{1}{i\omega} (e^{i\omega t} - 1) \cdot \overline{\frac{1}{i\omega} (e^{i\omega t} - 1)} \quad (5)$$

$$= \frac{1}{i\omega} (e^{i\omega t} - 1) \cdot \frac{-1}{i\omega} (e^{-i\omega t} - 1) \quad (6)$$

$$= \frac{1}{\omega^2} (2 - e^{i\omega t} - e^{-i\omega t}) \quad (7)$$

$$= \frac{1}{\omega^2} (2 - 2\cos(\omega t)) \quad (8)$$

$$= \frac{4\sin^2\left(\frac{\omega t}{2}\right)}{\omega^2} \quad (9)$$

$$= \frac{\sin^2\left(\frac{\omega t}{2}\right)}{\left(\frac{\omega}{2}\right)^2} \quad (10)$$

1.2 Part 2

Calculate the limit

$$\lim_{t \rightarrow \infty} \frac{\sin^2(\alpha t)}{\alpha^2 t}$$

for both $\alpha = 0$ and $\alpha \neq 0$. Conclude that

$$\lim_{t \rightarrow \infty} \frac{\sin^2(\alpha t)}{\alpha^2 t} \propto \delta(\alpha)$$

and find the proportional constant using

$$\frac{1}{\pi} \int_{-\infty}^{\infty} d\xi \frac{\sin^2 \xi}{\xi^2} = 1$$

1.2.1 $\alpha = 0$

Since the limit when alpha is zero isn't well defined, we'll take the limit of alpha approaching zero first.

$$\lim_{t \rightarrow \infty} \frac{\sin^2(\alpha t)}{\alpha^2 t} \Big|_{\alpha=0} = \lim_{\alpha \rightarrow 0} \lim_{t \rightarrow \infty} \frac{\sin^2(\alpha t)}{\alpha^2 t} \quad (11)$$

$$= \lim_{t \rightarrow \infty} \frac{(\alpha t)^2}{\alpha^2 t} \quad (12)$$

$$= \lim_{t \rightarrow \infty} t \quad (13)$$

$$= \infty \quad (14)$$

The transition between 11 and 12 being because the sine is very small.

1.2.2 $\alpha \neq 0$

$$0 = \lim_{t \rightarrow \infty} -\frac{1}{\alpha^2 t} \leq \lim_{t \rightarrow \infty} \frac{\sin^2(\alpha t)}{\alpha^2 t} \leq \lim_{t \rightarrow \infty} \frac{1}{\alpha^2 t} = 0 \quad (15)$$

Therefore

$$\lim_{t \rightarrow \infty} \frac{\sin^2(\alpha t)}{\alpha^2 t} = \begin{cases} \infty & \alpha = 0 \\ 0 & \alpha \neq 0 \end{cases} \quad (16)$$

which is the exact behavior of the delta function. To find the proportion constant we solve

$$\int_{-\infty}^{\infty} \frac{\sin^2(\alpha t)}{\alpha^2 t} d\alpha = [x = \alpha t] = \int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2} dx \quad (17)$$

$$= \pi \quad (18)$$

therefore

$$\lim_{t \rightarrow \infty} \frac{\sin^2(\alpha t)}{\alpha^2 t} = \pi \delta(\alpha) \quad (19)$$

1.3 Part 3

Use the previous parts to show that

$$\lim_{t \rightarrow \infty} \frac{\sin^2\left(\frac{\omega t}{2}\right)}{\left(\frac{\omega}{2}\right)^2} \sim \pi t \delta\left(\frac{\omega}{2}\right)$$

if we insert $\alpha = \frac{\omega}{2}$ into 19 we get the correct expression.

1.4 Part 4

Calculate the limit

$$\lim_{\lambda \rightarrow 0} \frac{\lambda}{\alpha^2 + \lambda^2}$$

for $\alpha = 0$ and $\alpha \neq 0$ and conclude that

$$\lim_{\lambda \rightarrow 0} \frac{\lambda}{\alpha^2 + \lambda^2} \propto \delta(\alpha)$$

and find the proportion constant using

$$\frac{1}{\pi} \int_{-\infty}^{\infty} d\alpha \frac{\xi}{\alpha^2 + \xi^2} = 1$$

$$\lim_{\lambda \rightarrow 0} \frac{\lambda}{\alpha^2 + \lambda^2} \Big|_{\alpha=0} = \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} = \infty \quad (20)$$

$$\lim_{\lambda \rightarrow 0} \frac{\lambda}{\alpha^2 + \lambda^2} = [\alpha \neq 0] = 0 \quad (21)$$

Therefore

$$\lim_{\lambda \rightarrow 0} \frac{\lambda}{\alpha^2 + \lambda^2} \propto \delta(\alpha) \quad (22)$$

and to find the proportion constant

$$\int_{-\infty}^{\infty} \frac{\lambda}{\alpha^2 + \lambda^2} = \pi \quad (23)$$

therefore

$$\lim_{\lambda \rightarrow 0} \frac{\lambda}{\alpha^2 + \lambda^2} = \pi \delta(\alpha) \quad (24)$$

2 Question 2

The matrix representation of the σ operators in the eigenbasis of σ_z is

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (25)$$

1. Show that you can write any 2×2 hermitian matrix as a superposition of Pauli matrices.
2. Show that for a unitary matrix $M^\dagger M = \mathbb{1}$, $|n_0 \pm \sqrt{n_x^2 + n_y^2 + n_z^2}| = 1$
3. Show that no $M \neq 0$ exists such that $\{M, \sigma_\alpha\} = 0$ for all σ_α

2.1 Part 1

Let

$$M = \begin{pmatrix} a_1 + ib_1 & a_2 + ib_2 \\ a_3 + ib_3 & a_4 + ib_4 \end{pmatrix} \rightarrow M^\dagger = \begin{pmatrix} a_1 - ib_1 & a_3 - ib_3 \\ a_2 - ib_2 & a_4 - ib_4 \end{pmatrix} \quad (26)$$

Since $M = M^\dagger$:

$$\begin{cases} a_1 + ib_1 = a_1 - ib_1 \\ a_2 + ib_2 = a_3 - ib_3 \\ a_3 + ib_3 = a_2 - ib_2 \\ a_4 + ib_4 = a_4 - ib_4 \end{cases} \rightarrow \begin{cases} b_1 = b_4 = 0 \\ b_2 = -b_3 \\ a_2 = a_3 \end{cases} \rightarrow M = \begin{pmatrix} a_1 & a_2 + ib \\ a_2 - ib & a_4 \end{pmatrix} \quad (27)$$

which we can decompose into

$$M = \frac{a_1 + a_4}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{a_1 - a_4}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + a_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - b \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (28)$$

therefore

$$M = \begin{pmatrix} n_0 + n_z & n_x - in_y \\ n_x + in_y & n_0 - n_z \end{pmatrix} \quad (29)$$

2.2 Part 2

I assume the matrix M in this part is still the hermitian matrix we decomposed in part 1. Since it is now also unitary, all of its eigenvalues lie on the unit circle in \mathbb{C} .

$$|\lambda \mathbb{1} - M| = \begin{vmatrix} \lambda - n_0 - n_z & -n_x + in_y \\ -n_x - in_y & \lambda - n_0 + n_z \end{vmatrix} \quad (30)$$

$$= (\lambda - n_0 - n_z)(\lambda - n_0 + n_z) + (-n_x + in_y)(n_x + in_y) \quad (31)$$

$$= \lambda^2 - 2\lambda n_0 + n_0^2 - n_z^2 - n_x^2 - n_y^2 \quad (32)$$

$$= (\lambda - n_0)^2 - (n_x^2 + n_y^2 + n_z^2) = 0 \quad (33)$$

rearranging

$$\lambda = n_0 \pm \sqrt{n_x^2 + n_y^2 + n_z^2} \quad (34)$$

and since we know all the eigenvalues are on the unit circle, then

$$\left| n_0 \pm \sqrt{n_x^2 + n_y^2 + n_z^2} \right| = 1 \quad (35)$$

as needed.

2.3 Part 3

I continue to assume we mean M is a hermitian matrix as we decomposed earlier. In that case:

$$\{M, \sigma_\alpha\} = \{\boldsymbol{\sigma} \cdot \mathbf{n}, \sigma_\alpha\} \quad (36)$$

$$= \{n_0 \mathbb{1} + n_x \sigma_x + n_y \sigma_y + n_z \sigma_z, \sigma_\alpha\} \quad (37)$$

$$= n_0 \{\mathbb{1}, \sigma_\alpha\} + n_x \{\sigma_x, \sigma_\alpha\} + n_y \{\sigma_y, \sigma_\alpha\} + n_z \{\sigma_z, \sigma_\alpha\} \quad (38)$$

From the exercise session we know that

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij} \mathbb{1} \quad (39)$$

So we can immediately notice that the only way for 38 to be zero is for each element to be zero, which is only possible if all the coefficients $n_\alpha = 0$, which means the only matrix for which the anti-commutative relation holds is the 0 matrix.

3 Question 3

Prove the following identities:

1. $\text{Tr}(\sigma_\alpha \sigma_\beta \sigma_\gamma) = 2i\varepsilon_{\alpha\beta\gamma}$ and $\text{Tr}(\sigma_\alpha \sigma_\beta) = 2\delta_{\alpha\beta}$
2. Given $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ calculate $(\boldsymbol{\sigma} \cdot \mathbf{a})(\boldsymbol{\sigma} \cdot \mathbf{b})(\boldsymbol{\sigma} \cdot \mathbf{c})$ in two ways: first the two elements then the last 2 elements. What does the result mean?
3. Write down $e^{i\sigma_x}$ and e^{σ_x} as 2×2 matrices.
4. Given $\theta \in [0, 2\pi]$ and $\mathbf{n} \in \mathbb{R}^3$ a unit vector show that

$$e^{i\theta \mathbf{n} \cdot \boldsymbol{\sigma}} = \cos \theta + i(\mathbf{n} \cdot \boldsymbol{\sigma}) \sin \theta$$

3.1 Part 1

From the exercise we know that

$$\text{Tr}(\sigma_\alpha \sigma_\beta) = \text{Tr}(\delta_{\alpha\beta} \mathbb{1} + i\varepsilon_{\alpha\beta\gamma} \sigma_\gamma) \quad (40)$$

$$= 2\delta_{\alpha\beta} + 0 \quad (41)$$

and

$$\text{Tr}(\sigma_\alpha \sigma_\beta \sigma_\gamma) = \text{Tr}((\delta_{\alpha\beta} \mathbb{1} + i\varepsilon_{\alpha\beta\kappa} \sigma_\kappa) \sigma_\gamma) \quad (42)$$

$$= \text{Tr}(\delta_{\alpha\beta} \mathbb{1} \sigma_\gamma) + \text{Tr}(i\varepsilon_{\alpha\beta\kappa} \sigma_\kappa \sigma_\gamma) \quad (43)$$

$$= \text{Tr}(i\varepsilon_{\alpha\beta\kappa} \sigma_\kappa \sigma_\gamma) \quad (44)$$

$$= i\varepsilon_{\alpha\beta\kappa} \text{Tr}(\sigma_\kappa \sigma_\gamma) \quad (45)$$

$$= 2i\varepsilon_{\alpha\beta\gamma} \quad (46)$$

3.2 Part 2

$$[(\boldsymbol{\sigma} \cdot \mathbf{a})(\boldsymbol{\sigma} \cdot \mathbf{b})](\boldsymbol{\sigma} \cdot \mathbf{c}) = [\sigma_i a_i \sigma_j b_j](\boldsymbol{\sigma} \cdot \mathbf{c}) \quad (47)$$

$$= [a_i b_j (\delta_{ij} \mathbb{1} + i\varepsilon_{ijk} \sigma_k)](\boldsymbol{\sigma} \cdot \mathbf{c}) \quad (48)$$

$$= a_i b_i (\boldsymbol{\sigma} \cdot \mathbf{c}) + i a_i b_j \sigma_l \varepsilon_{ijl} (\boldsymbol{\sigma} \cdot \mathbf{c}) \quad (49)$$

$$= a_i b_i \sigma_k c_k + i a_i b_j c_m \varepsilon_{ijl} \sigma_l \sigma_m \quad (50)$$

$$= a_i b_i c_k \sigma_k + i a_i b_j c_m \varepsilon_{ijl} (\delta_{lm} \mathbb{1} + i\varepsilon_{lmp} \sigma_p) \quad (51)$$

$$= a_i b_i c_k \sigma_k + i a_i b_j c_m \varepsilon_{ijm} - a_i b_j c_m \varepsilon_{ijl} \varepsilon_{mpl} \sigma_p \quad (52)$$

$$= a_i b_i c_k \sigma_k + i a_i b_j c_m \varepsilon_{ijm} - a_i b_j c_m (\delta_{im} \delta_{jp} - \delta_{ip} \delta_{jm}) \sigma_p \quad (53)$$

$$= a_i b_i c_k \sigma_k + i a_i b_j c_m \varepsilon_{ijm} - a_i b_j c_i \sigma_j + a_i b_j c_j \sigma_i \quad (54)$$

$$= (\mathbf{a} \cdot \mathbf{b})(\mathbf{c} \cdot \boldsymbol{\sigma}) - (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \boldsymbol{\sigma}) + (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \boldsymbol{\sigma}) + i(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \quad (55)$$

on the other hand

$$(\boldsymbol{\sigma} \cdot \mathbf{a})[(\boldsymbol{\sigma} \cdot \mathbf{b})(\boldsymbol{\sigma} \cdot \mathbf{c})] = [\sigma_i c_i \sigma_j b_j](\boldsymbol{\sigma} \cdot \mathbf{a}) \quad (56)$$

$$= [c_i b_j (\delta_{ij} \mathbb{1} + i \varepsilon_{ijk} \sigma_k)](\boldsymbol{\sigma} \cdot \mathbf{a}) \quad (57)$$

$$= c_i b_i (\boldsymbol{\sigma} \cdot \mathbf{c}) + i c_i b_j \sigma_l \varepsilon_{ijl} (\boldsymbol{\sigma} \cdot \mathbf{a}) \quad (58)$$

$$= c_i b_i \sigma_k c_k + i c_i b_j a_m \varepsilon_{ijl} \sigma_l \sigma_m \quad (59)$$

$$= c_i b_i a_k \sigma_k + i c_i b_j a_m \varepsilon_{ijl} (\delta_{lm} \mathbb{1} + i \varepsilon_{lmn} \sigma_n) \quad (60)$$

$$= c_i b_i a_k \sigma_k + i c_i b_j a_m \varepsilon_{ijm} - c_i b_j a_m \varepsilon_{ijl} \varepsilon_{mpl} \sigma_p \quad (61)$$

$$= c_i b_i a_k \sigma_k + i c_i b_j a_m \varepsilon_{ijm} - c_i b_j a_m (\delta_{im} \delta_{jp} - \delta_{ip} \delta_{jm}) \sigma_p \quad (62)$$

$$= c_i b_i a_k \sigma_k + i c_i b_j a_m \varepsilon_{ijm} - c_i b_j a_i \sigma_j + c_i b_j a_j \sigma_i \quad (63)$$

$$= (\mathbf{c} \cdot \mathbf{b})(\mathbf{a} \cdot \boldsymbol{\sigma}) - (\mathbf{c} \cdot \mathbf{a})(\mathbf{b} \cdot \boldsymbol{\sigma}) + (\mathbf{b} \cdot \mathbf{a})(\mathbf{c} \cdot \boldsymbol{\sigma}) + i(\mathbf{c} \times \mathbf{b}) \cdot \mathbf{a} \quad (64)$$

The two expressions are identical (the vector product element is different in writing but the vector operations result in the same).

3.3 Part 3

$$e^{\sigma_x} = \sum_{k=0}^{\infty} \frac{\sigma_x^k}{k!} \quad (65)$$

we'll find the eigenvalues and eigenvectors:

$$\begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} \Rightarrow \lambda = \pm 1, v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (66)$$

Therefore

$$e^{\sigma_x} = e^{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} = \begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix} \quad (67)$$

and

$$e^{i\sigma_x} = \begin{pmatrix} e^i & 0 \\ 0 & e^{-i} \end{pmatrix} \quad (68)$$

3.4 Part 4

$$e^{i\theta \mathbf{n} \cdot \boldsymbol{\sigma}} = \sum_{k=0}^{\infty} \frac{(i\theta \mathbf{n} \cdot \boldsymbol{\sigma})^k}{k!} \quad (69)$$

$$= \sum_{k=0}^{\infty} \left(\frac{(i\theta \mathbf{n} \cdot \boldsymbol{\sigma})^{2k}}{(2k)!} + \frac{(i\theta \mathbf{n} \cdot \boldsymbol{\sigma})^{2k+1}}{(2k+1)!} \right) \quad (70)$$

And we notice that

$$(\mathbf{n}^2 \cdot \boldsymbol{\sigma}^2)^k = \mathbb{1} \quad (71)$$

because of the identity with pauli matrices and their relationship to deltas and epsilon. This gives us

$$e^{i\theta \mathbf{n} \cdot \boldsymbol{\sigma}} = \sum_{k=0}^{\infty} \left(\frac{(i\theta)^{2k}}{(2k)!} + \frac{(i\theta)^{2k+1}}{(2k+1)!} \mathbf{n} \cdot \boldsymbol{\sigma} \right) \quad (72)$$

$$= \sum_{k=0}^{\infty} \frac{(i\theta)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(i\theta)^{2k+1}}{(2k+1)!} \mathbf{n} \cdot \boldsymbol{\sigma} \quad (73)$$

$$= \cos \theta + i(\mathbf{n} \cdot \boldsymbol{\sigma}) \sin \theta \quad (74)$$

4 Question 4

For a two-dimensional Hilbert space we'll define the bases $\mathcal{B} = \{ |+\rangle, |-\rangle \}$. This is the eigenbasis of σ_z .

- Any matrix representing an operator in the space spanned by the basis can be singularly represented as the linear combination

$$A = \frac{1}{2}(\alpha_0 \mathbb{1}_2 + \boldsymbol{\alpha} \cdot \boldsymbol{\sigma})$$

where $\boldsymbol{\alpha} \in \mathbb{C}^3$, $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, and $\mathbb{1}_2$ is the identity operator. Show that $\alpha_0 = \text{Tr}(A)$ and $\boldsymbol{\alpha} = \text{Tr}(A\boldsymbol{\sigma})$

- We'll define the operator $P = |\psi(t)\rangle \langle \psi(t)|$, where $|\psi(t)\rangle = a_+(t)|+\rangle + a_-(t)|-\rangle$ where $a_{\pm} \in \mathbb{C}$. Calculate P^2 . Write down P as a matrix in the basis \mathcal{B} .
- Let $P = \frac{1}{2}(M_0(t)\mathbb{1}_2 + \mathbf{M}(t) \cdot \boldsymbol{\sigma})$. Show that $\text{Tr}[P] = 1$ and conclude $M_0(t)$.
- From the connection between P and P^2 and P 's form show that $\|\mathbf{M}(t)\| = 1$
- Show that for a general operator O : $\text{Tr}[|\psi(t)\rangle \langle \psi(t)| O] = \langle \psi(t)| O |\psi(t)\rangle$
- Use the previous parts to find a connection between $\langle \boldsymbol{\sigma} \rangle$ and \mathbf{M} .
- Assume the Hamiltonian H describes the time development of the system. Use the relation between \mathbf{M} and $\langle \boldsymbol{\sigma} \rangle$ and the Schrödinger equation to prove that

$$\frac{d\mathbf{M}}{dt} = \frac{i}{\hbar} \langle \psi(t)| [H, \boldsymbol{\sigma}] |\psi(t)\rangle$$

- Assume that H has a trace of zero, i.e it can be expressed as $H = \frac{\hbar}{2}\boldsymbol{\omega} \cdot \boldsymbol{\sigma}$ with $\boldsymbol{\omega} \in \mathbb{R}^3$. Show that \mathbf{M} fulfills $\frac{d\mathbf{M}}{dt} = \boldsymbol{\omega} \times \mathbf{M}$.

4.1 Part 1

$$\text{Tr}(A) = \frac{1}{2} \text{Tr}(\alpha_0 \mathbb{1} + \boldsymbol{\alpha} \cdot \boldsymbol{\sigma}) \quad (75)$$

Since the trace of all pauli matrices is zero, the element including it vanishes. Since we're in a 2 dimensional hilbert space, the trace of the unit matrix is 2, therefore:

$$\text{Tr}(A) = \alpha_0 \quad (76)$$

$$\text{Tr}(A\boldsymbol{\sigma}) = \frac{1}{2} \text{Tr}(\alpha_0 \boldsymbol{\sigma} + (\boldsymbol{\alpha} \cdot \boldsymbol{\sigma}) \boldsymbol{\sigma}) \quad (77)$$

$$= \frac{1}{2} \text{Tr}((\boldsymbol{\alpha} \cdot \boldsymbol{\sigma}) \boldsymbol{\sigma}) \quad (78)$$

$$= \frac{1}{2} (\alpha_1 \text{Tr}(\sigma_x \boldsymbol{\sigma}) + \alpha_2 \text{Tr}(\sigma_y \boldsymbol{\sigma}) + \alpha_3 \text{Tr}(\sigma_z \boldsymbol{\sigma})) \quad (79)$$

$$= \frac{1}{2} (2\alpha_1 + 2\alpha_2 + 2\alpha_3) \quad (80)$$

$$= \boldsymbol{\alpha} \quad (81)$$

4.2 Part 2

$$P^2 = |\psi(t)\rangle \langle \psi(t)| |\psi(t)\rangle \langle \psi(t)| = |\psi(t)\rangle \langle \psi(t)| = P \quad (82)$$

since the state ψ is normalized.

$$P = |\psi(t)\rangle \langle \psi(t)| \quad (83)$$

$$= (a_+(t) |+\rangle + a_-(t) |-\rangle)(a_+^*(t) \langle +| + a_-^*(t) \langle -|) \quad (84)$$

$$= |a_+|^2 |+\rangle \langle +| + |a_-|^2 |-\rangle \langle -| + a_+ a_-^* |-\rangle \langle +| + a_- a_+^* |+\rangle \langle -| \quad (85)$$

$$= \begin{pmatrix} |a_+|^2 & a_- a_+^* \\ a_+ a_-^* & |a_-|^2 \end{pmatrix} \quad (86)$$

4.3 Part 3

We notice that the trace of P is $|a_+|^2 + |a_-|^2$ which is identical to $\langle \psi(t) | \psi(t) \rangle$, and since the state is normalized it's equal to 1. The trace of $\mathbf{M}(t) \cdot \boldsymbol{\sigma}$ is 0 because the trace of each of the pauli matrices is zero. In total the trace of P is 1. Since the trace of P through the previous section must match the trace in the new writing, we find that $M_0(t) = 1$.

4.4 Part 4

We know that $P^2 = P$ and $P = \frac{1}{2}(\mathbb{1}_2 + \mathbf{M}(t) \cdot \boldsymbol{\sigma})$. We'll square the decomposed form of P :

$$P^2 = \frac{1}{4}(\mathbb{1}_2 + \mathbf{M}(t) \cdot \boldsymbol{\sigma})^2 \quad (87)$$

$$= \frac{1}{4}(\mathbb{1}_2^2 + 2\mathbb{1}_2 \mathbf{M}(t) \cdot \boldsymbol{\sigma} + (\mathbf{M}(t) \cdot \boldsymbol{\sigma})^2) \quad (88)$$

$$= \frac{1}{4}(\mathbb{1}_2 + 2\mathbf{M}(t) \cdot \boldsymbol{\sigma} + \|\mathbf{M}\|^2 \mathbb{1}) = P \quad (89)$$

so we must have

$$\frac{1}{4}(\mathbb{1}_2 + 2\mathbf{M}(t) \cdot \boldsymbol{\sigma} + \|\mathbf{M}\|^2 \mathbb{1}) = \frac{1}{2}(\mathbb{1}_2 + \mathbf{M}(t) \cdot \boldsymbol{\sigma}) \quad (90)$$

$$\frac{1}{4}(\mathbb{1}_2 + \|\mathbf{M}\|^2 \mathbb{1}) = \frac{1}{2}\mathbb{1}_2 \quad (91)$$

$$\|\mathbf{M}\|^2 = 4 \left(\frac{1}{2} - \frac{1}{4} \right) = 1 \quad (92)$$

4.5 Part 5

$$\text{Tr}[\psi(t)\langle\psi(t)|O] = \langle+|\psi(t)\rangle\langle\psi(t)|O|+\rangle + \langle-|\psi(t)\rangle\langle\psi(t)|O|-\rangle \quad (93)$$

$$= \langle\psi(t)|O|+\rangle\langle+|\psi(t)\rangle + \langle\psi(t)|O|-\rangle\langle-|\psi(t)\rangle \quad (94)$$

$$= \langle\psi(t)|O|\psi(t)\rangle \quad (95)$$

4.6 Part 6

$$\langle\psi(t)|\boldsymbol{\sigma}|\psi(t)\rangle = \text{Tr}[\psi(t)\langle\psi(t)|\boldsymbol{\sigma}] \quad (96)$$

$$= \text{Tr}[P\boldsymbol{\sigma}] \quad (97)$$

$$= \frac{1}{2} \text{Tr}[M_0(t)\boldsymbol{\sigma} + \mathbf{M}(t) \cdot \boldsymbol{\sigma}^2] \quad (98)$$

$$= \mathbf{M}(t) \quad (99)$$

4.7 Part 7

$$\frac{d\mathbf{M}}{dt} = \frac{d}{dt}(\langle\psi(t)|\boldsymbol{\sigma}|\psi(t)\rangle) \quad (100)$$

$$= \langle\dot{\psi}(t)|\boldsymbol{\sigma}|\psi(t)\rangle + \langle\psi(t)|\boldsymbol{\sigma}|\dot{\psi}(t)\rangle \quad (101)$$

$$= \frac{1}{-i\hbar} \langle\psi(t)|H\boldsymbol{\sigma}|\psi(t)\rangle + \frac{1}{i\hbar} \langle\psi(t)|\boldsymbol{\sigma}H|\psi(t)\rangle \quad (102)$$

$$= \frac{i}{\hbar} \langle\psi(t)|H\boldsymbol{\sigma}|\psi(t)\rangle - \frac{i}{\hbar} \langle\psi(t)|\boldsymbol{\sigma}H|\psi(t)\rangle \quad (103)$$

$$= \frac{i}{\hbar} \langle\psi(t)|[H, \boldsymbol{\sigma}]|\psi(t)\rangle \quad (104)$$

4.8 Part 8

$$\frac{d\mathbf{M}}{dt} = \frac{i}{\hbar} \langle \psi(t) | [H, \boldsymbol{\sigma}] | \psi(t) \rangle \quad (105)$$

$$= \frac{i}{\hbar} \langle \psi(t) | [\frac{\hbar}{2} \boldsymbol{\omega} \cdot \boldsymbol{\sigma}, \boldsymbol{\sigma}] | \psi(t) \rangle \quad (106)$$

$$= \frac{i}{2} \boldsymbol{\omega} \cdot \langle \psi(t) | [\boldsymbol{\sigma}, \boldsymbol{\sigma}] | \psi(t) \rangle \quad (107)$$

$$= -\omega_i \langle \psi(t) | \varepsilon_{ijk} \sigma_k | \psi(t) \rangle \quad (108)$$

$$= \varepsilon_{jik} \omega_i \langle \sigma_k \rangle \quad (109)$$

$$= \boldsymbol{\omega} \times \mathbf{M} \quad (110)$$

5 Question 5

Assume that spin is the angular momentum of the electron spinning around itself and that its magnitude is $\hbar/2$. Use the fact we know from measurements that the electron's classical radius is smaller than 10^{-18} m to find the electron's speed of rotation. Does the result make sense?

The electron's mass is 9.109×10^{-31} kg, and the formula for its classical angular momentum is $S = I\omega$ where I is the moment of inertia of a sphere: $I = \frac{2}{5}MR^2$ and $\omega = \frac{v}{R}$. Plugging this all in we find

$$\frac{\hbar}{2} = \frac{2}{5}m_e R v \implies v = \frac{5\hbar}{4m_e R} = \frac{5 \cdot 1.054 \times 10^{-34} \text{ m}^2\text{kg/s}}{4 \cdot 9.109 \times 10^{-31} \text{ kg} \cdot 10^{-18} \text{ m}} \quad (111)$$

$$\approx 1.446 \times 10^{14} \text{ m s}^{-1} \quad (112)$$

Which is significantly above the speed of light. This of course makes no sense, meaning that electron spin cannot be viewed as classical angular momentum.