

# Lecture Notes: Select Topics in Living Systems

Based on lectures by **Dr. Ram Adar** in 2025-26  
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These notes were typed in the 2026 winter semester *Select Topics in Living Systems* course taught by **Dr. Ram Adar** at the Technion.

The lectures were given in Hebrew and were live-translated by me to English. The document is provided as is and likely contains many errors.

If you find any mistakes or typos please let me know at [danielbreger@campus.technion.ac.il](mailto:danielbreger@campus.technion.ac.il).

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Wed, October 29, 2025

## 1.1 Phase transitions in equilibrium

### 1.1.1 Motivation

Phase is a distinct state of matter which can be described by an "order parameter". We'll focus on isotropic polar phase transition. In this case the order parameter is the polarization  $\vec{p} = \langle \hat{n} \rangle$  where  $\hat{n}$  is the unit vector of the particles' direction.

**Phase transition** in certain areas of physics we see systems which aren't ordered at high temperatures but become ordered at low temperatures. Why? A competition between energy and entropy (Free energy being  $F = U - TS$ ). The temperature at which order emerges is called the critical temperature, marked  $T_C$  and the change in the system is called a phase transition.

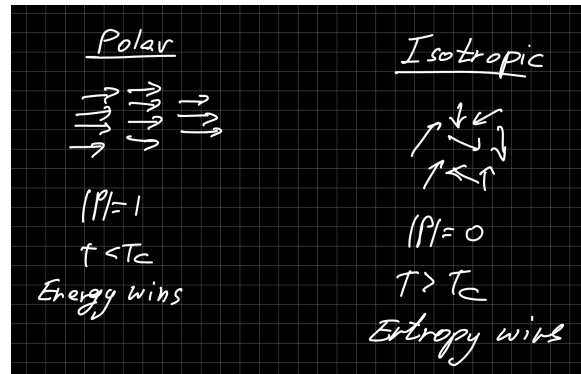


Figure 1.1: Polar and Isotropic ordering of matter.

### 1.1.2 Physical description

We'll assume free energy of the following form:

$$F(T, N, \psi) = F_0(T, N) + F_\psi(T, N, \psi) \quad (1.1)$$

where  $\psi$  is the order parameter.  $\psi$  is not a conserved quantity but rather a degree of freedom of the system (in contrast to the number of particles, for example). Therefore under equilibrium we find

$$F_{eq}(T, N, \psi) = F_0(T, N) + F_\psi(T, N, \psi_{eq}) \quad (1.2)$$

and the equilibrium value is determined by

$$\left. \frac{\partial F_\psi}{\partial \psi} \right|_{\psi=\psi_{eq}} = 0 \quad (1.3)$$

and the following is true:

$$\psi_{eq}(T > T_C) = 0 \quad (1.4)$$

$$\psi_{eq}(T < T_C) \neq 0 \quad (1.5)$$

We'll focus on phase transition of 2nd order, for which the order parameter changes continuously, thus  $\psi_{eq}(T = T_C) = 0$ . Otherwise the phase transition is of first order.

### 1.1.3 Landau Theory of Phase Transitions

The theory was developed by Lev Landau, for which he earned the Nobel Prize in Physics in 1962.

Close to the critical temperature  $T_C$  the physics of phase transitions is universal. For a phase transition of 2nd order, we will write  $F_\psi$  in a phenomenological form as a Taylor series expansion:

$$F_\psi = \frac{a}{2}\psi^2 + \frac{b}{4}\psi^4 \quad (1.6)$$

The terms being of even powers because we don't care about directions. The quadratic term's coefficient is of the form

$$a = a_0 \cdot \frac{T - T_C}{T_C} \quad (1.7)$$

Where  $a_0 > 0$ . This quantifies the competition between energy (which wins at  $T < T_C$ ) and entropy (which wins at  $T > T_C$ ). The parameter  $b > 0$  ensures stability:  $\psi_{eq}$  receives finite values and does not approach infinity.

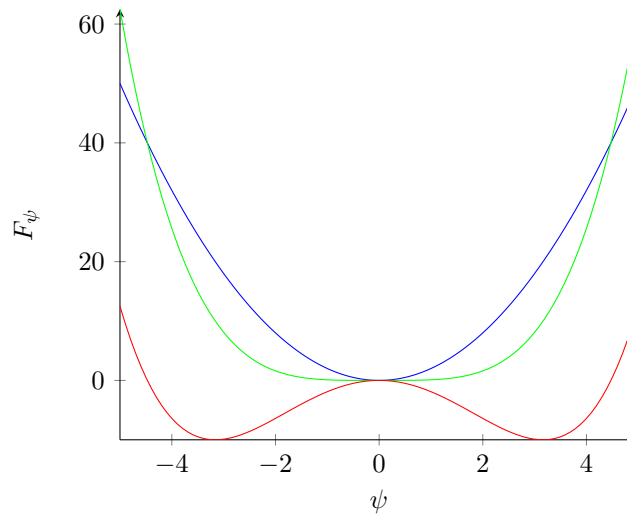


Figure 1.2: The graphs for the functions  $2x^2$  (blue),  $0.1x^4$  (green) and  $0.1x^4 - 2x^2$  (red).

Because of  $a$ 's sign, we find  $\psi_{eq} \neq 0$  for  $T < T_C$ . Explicitly:

$$0 = \left. \frac{\partial F}{\partial \psi} \right|_{\psi=\psi_{eq}} = a\psi_{eq} + b\psi_{eq}^3 = b\psi_{eq} \left( \psi_{eq}^2 + \frac{a}{b} \right) \quad (1.8)$$

therefore

$$\psi_{eq} = 0, \pm \sqrt{\frac{a_0}{b} \cdot \frac{T - T_C}{T_C}} \quad (1.9)$$

There is no distinction between the positive and negative sign. Either can be chosen arbitrarily (spontaneous symmetry break). There will be a preference given an external field which will add an element of the form  $-h\psi$  to the free energy.

#### Where $a$ comes from

We'll assume a simple model of polarisation with energy

$$U = - \sum_{i,j \text{ neighbors}} u n_\alpha^i n_\alpha^j, \quad u > 0 \quad (1.10)$$

The energy is lower when  $i, j$  point in the same direction. Using mean field approximation the internal energy is

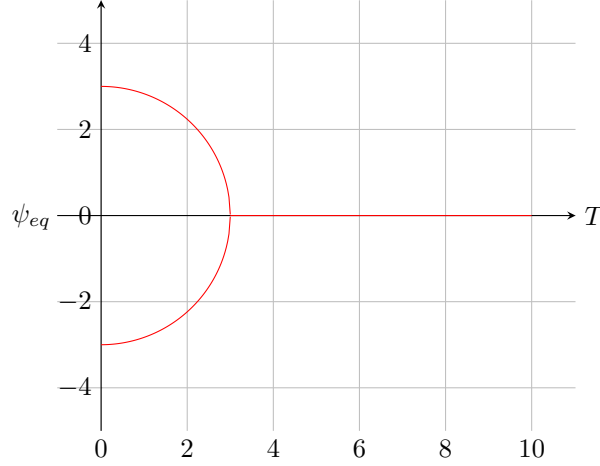


Figure 1.3: The possible solutions from equation 1.9.

$$U = \frac{u}{2} \sum_{i,j} \langle n_\alpha^i \rangle \cdot \langle n_\alpha^j \rangle = -\frac{u}{2} N Z p_\alpha^2 \quad (1.11)$$

where  $p_\alpha = \langle n_\alpha \rangle$ ,  $N$  is the number of particles, and  $Z$  the number of close neighbors. As for the entropy, we can find from a Taylor expansion

$$S = S(p=0) - \alpha N p^2 \quad (1.12)$$

and summing it up we find

$$F = U - TS = N \left( \alpha T - \frac{u}{2} Z \right) p^2 = a_0 \cdot \frac{T - T_C}{T_C} p^2 \quad (1.13)$$

where

$$T_C = \frac{uZ}{2\alpha} \quad a_0 = \alpha N T_C \quad (1.14)$$

If instead of the vector  $n_\alpha$  there is spin ( $\pm$ ) then we find a similar result in the mean field framework. This model is called the Ising model, one of the most basic and useful models in statistical mechanics.

### A dynamical description of the phase transition

We described the phase transition from the free energy but we can also describe it from dynamics. For a non-conserved order parameter we can write the dynamics as follows:

$$\dot{\psi} = -\Gamma_\psi \frac{\partial F}{\partial \psi}, \quad \Gamma_\psi > 0 \quad (1.15)$$

the idea being that the dynamics guarantee that the free energy only gets smaller (the total entropy grows). We can see this using the chain rule:

$$\dot{F} = \frac{\partial F}{\partial \psi} \dot{\psi} = -\Gamma_\psi \left( \frac{\partial F}{\partial \psi} \right)^2 < 0 \quad (1.16)$$

this dynamics is called Model A (and often appears with a "noise" parameter). We'll insert  $F$  and find:

$$\dot{p} = -\Gamma_p (ap + bp^3) \quad (1.17)$$

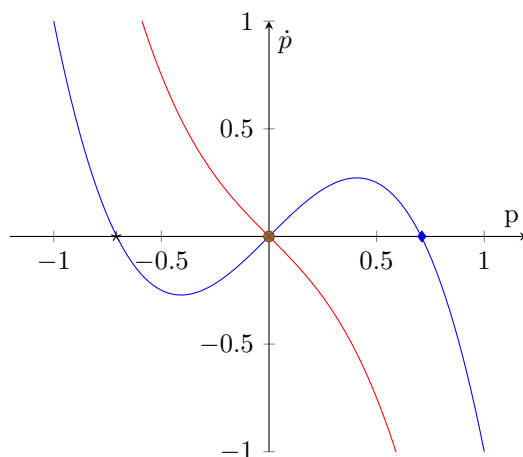


Figure 1.4: The graphs of  $f(x) = x - 2x^3$  (blue) and  $g(x) = -x - 2x^3$  (red). The positive and negative points marked on the x axis are stable points of equilibrium for  $f(x)$ . The point marked at  $(0,0)$  is a stable point for  $g(x)$  but an unstable one for  $f(x)$ .

the fixed points are  $\frac{\partial F}{\partial p} = 0$ , the extremums of the free energy. The stability of the solution is determined in phase space.

in the language of dynamical systems, a phase transition is called a bifurcation. This approach is useful for out of equilibrium systems, for which we can't write down a free energy. Later on in the course, we'll come across dynamical equations of a similar form.

In general dynamical systems, when a function  $F$  whose derivative is strictly non-positive with a finite minimum exists (like free energy), then it's called a Lyapunov function.

#### 1.1.4 Spatial Dependence: The Ginsburg-Landau Theory

We saw that the direction of  $\vec{p}$  is arbitrary, in practice different areas of the system will initially develop order in different directions. For simplicity, we'll assume  $\vec{p} = p\hat{y}$  where  $p$  can be either positive or negative. We'll look at the case where  $\vec{p}$  points down at  $x \rightarrow -\infty$  and up at  $x \rightarrow \infty$ .

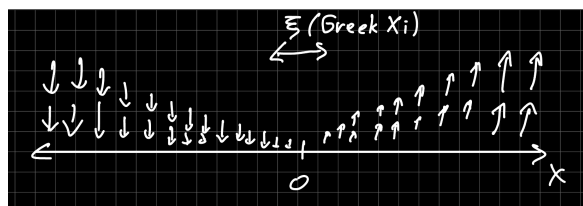


Figure 1.5: A smooth phase transition with Domain Wall  $\xi$ .

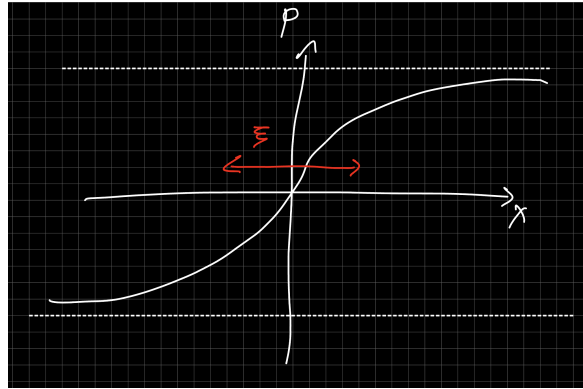


Figure 1.6: A smooth phase transition with Domain Wall  $\xi$ .

We'll find a finite range around 0 of width  $\chi$  in which  $|p| \neq p_{eq}$ . This region is called the Domain Wall.

To quantify the cost to change  $p$  in space we'll write the density of free energy per length unit:

$$F[p] = \int dx f = \int dx \left( \frac{a}{2} p^2 + \frac{b}{4} p^4 + \frac{k}{2} p'^2 \right) \quad (1.18)$$

where  $k > 0$ . The last element quantifies the cost to change  $p$ . In higher dimensions we'll have  $(\vec{\nabla} p)^2$ . Why is it like this? The cost comes from the gradient, and given there is no preferred direction, a positive and negative change should cost the same.

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