

The Medium of Contingency

Elie Ayache

An Inverse View of the Market



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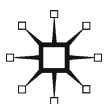
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To Pierre Menard,
in admiration of his *other* work:
'the subterranean, the interminably heroic, the peerless,
and – such are the capacities of man! – the unfinished.'¹

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Contents

<i>Preface</i>	xiii
<i>Acknowledgments</i>	xvi
Introduction	1
Derivative valuation theory vs. derivative pricing technology	6
Implied volatility vs. real volatility	9
Formal reality vs. physical reality	12
 Part I The Matter	
1 The End of Probability	19
1.1 The void of possibilities	19
1.2 A new matter	23
1.3 The infinity of markets	25
1.4 The end of statistics	28
2 The Vision Ahead	30
2.1 Regime-switching model	30
2.2 Recalibration	32
2.3 Is probability necessary?	35
2.4 Price and probability	38
2.5 A new metaphysics	39
2.6 Absolute contingency	41
2.7 The market as an opportunity for speculative thought	42
2.8 The market as the conversion of the image of thought	43
2.9 Ascending to the metalogical level	44
3 Introducing the Market	46
3.1 The unexchangeable place of exchange	46

3.1.1	The market as a continual event	48
3.1.2	The market as quantitative history	50
3.1.3	The intensive nontemporal price process	52
3.1.4	The continuity of the discontinuity	53
3.1.5	The matter beyond antifragility	55
3.2	The medium of contingency	56
3.2.1	The technology of the future	60
3.2.2	The book behind the market	63
3.3	Pricing vs. valuation	66
3.3.1	The surface of the market	69
3.3.2	The smile problem	72
3.3.3	The absolute local	74
3.3.4	In the middle of the event	77
4	The Thought Behind	79
4.1	Two sides of writing	79
4.2	Genesis of price vs. generation of number	81
4.3	The market as geometry	83
4.4	State vs. mark	84
4.5	Probability as an internal episode	89
4.6	The alternative axiomatic system of Shafer and Vovk	92
4.7	Extensive difference vs. intensive difference	98

Part II The Matter in Brownian Motion

5	From Throwing the Dice to Grasping Brownian Motion	103
5.1	The meaning of probability	104
5.1.1	The law of large numbers	105
5.1.2	Intuition	108
5.1.3	Matter	109
5.1.4	Reality	113
5.1.5	Tense	116
5.2	Changing the meaning of matter	119
5.2.1	The strike of contingency	119
5.2.2	The one and the many	122
5.2.3	The reality of contingency	124
5.3	Money	126
5.3.1	Time is not money	131
5.3.2	Money is place	133
5.4	Changing the meaning of reality	136
5.4.1	Ex-ante vs. ex-post	137

5.4.2	Brownian motion	140
5.4.3	From time to place	143
6	From the Marvel of Brownian Motion to the Reality of the Market	146
6.1	The technology of the market	146
6.2	The reality of the market	150
6.3	The market as an inverted order of thought	153

Part III The Matter in Contingency

7	The Paper and the Tree	159
7.1	The market and time	159
7.1.1	Contingency, writing and exchanging	162
7.1.2	Price and time	167
7.1.3	Price and the event	168
7.1.4	Price and the trace	170
7.2	From the mark to the whole market	172
7.2.1	Contingent payoff vs. contingent claim (first take)	174
7.2.2	The invention of writing (first take)	176
7.2.3	The exchange and the abyss	181
8	Archaeology of the Multiple	185
8.1	To be vs. can be	186
8.1.1	Identification and transition	187
8.1.2	The danger of abstraction and the suspension of possibility	188
8.1.3	0 and 1	191
8.1.4	The real future	193
8.2	Chrono-logic	194
8.2.1	Probability as an integral	195
8.2.2	Chronology as a simulation of chrono-logic	197
8.3	Accounting for the event	200
8.3.1	Money and the other face of the event	203
8.3.2	The accident of time and the necessity of work	207
8.3.3	An event that is not but that remains	209
8.3.4	Writing the event	211
9	Archaeology of the Exchange	213
9.1	All of the market!	213
9.1.1	Impossible exchange, necessary exchange	214
9.1.2	The inverse view	216

9.2	Statistics as a proto-market	217
9.2.1	Abstraction and the precision of the present state	219
9.2.2	The immanence of statistics and the immanence of the paper	222
9.2.3	The matter in statistics	225
9.3	The matter in the exchange	228
9.3.1	The non-individual singular	232
9.3.2	Single-case statistics	234
9.3.3	Contingency of the strike	237
10	Matter and Geometry	239
10.1	The singularity of writing	239
10.2	The singularity of the exchange	244
 Part IV The Market of Contingent Claims (or the Matter in Black–Scholes–Merton)		
11	Towards a Contemporary Theory of the Market	251
11.1	The stochastic narrative of the market	251
11.1.1	Definite states	251
11.1.2	Derivatives prices as states	254
11.1.3	Variations on lottery value and random price	256
11.1.4	The curse of the derivative value	260
11.2	The trading narrative of the market	262
11.2.1	Can the derivative trade independently?	264
12	Incomplete Markets	267
12.1	Complete vs. incomplete markets	267
12.2	Martingale measure of the market	270
12.3	Equivocation	272
12.4	Incomplete market when the market is all there is	274
13	The Central Knot	283
13.1	Contingent payoff vs. contingent claim	285
13.2	Probabilistic exit	286
13.3	The alternative exit	289
13.3.1	Differentiating the form	291
13.4	The invention of writing	293
13.5	Genesis	295
13.5.1	That they don't exist	298
14	The Hard Problem	300
14.1	The ultimate probability spot	301

14.2	The presentation of the contingent payoffs	304
14.3	The lure of theory	309
14.4	The semantic theory of the market	314
15	The Book of the Market	321
15.1	Formalism and meta-formalism	321
15.1.1	The instant of the formalism and the instant of the market	322
15.1.2	The infinity of the option price and the infinity of matter	325
15.1.3	Formal deduction of matter	326
15.1.4	A new book for a new reality	328
15.2	The book of genesis	330
15.2.1	Only the book can write history	331
15.2.2	One book instead of two theories	333
15.2.3	Only the book can bind the void	335
15.2.4	Only the book can settle the succession	337
15.2.5	Contemporary art	339
15.2.6	An ontology made of paper	340
15.3	The trading force	343
15.4	Coda	350
16	Denouement: The Theory after the Two Narratives	351
	Conclusion	363
	<i>Appendix A: Regime-Switching Model</i>	373
	A.1 A meta-contextual pricing tool	373
	A.2 Recalibration	376
	<i>Appendix B: Menard's Quixote</i>	379
	Notes	385
	Bibliography	405
	Index	408

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Preface

This is not the first time I have used the title ‘The Medium of Contingency’. It has already served in several works of mine, although never as the title of a full-length book. It first appeared as the heading of a section of *The Blank Swan* – the first book I wrote on the market of contingent claims.² Shortly after *The Blank Swan* was published, I used ‘The Medium of Contingency’ again for an article which I submitted to *Pli*, the philosophical journal of the University of Warwick, and was subsequently published.³ By then, the title had caught on in intellectual circles and it was later picked for a show in a contemporary art gallery in London, at the opening of which I was invited to speak of contingency and of the work of art as its possible medium.⁴ It didn’t take long after that event for the expression to be associated with my name among artists and one of its latest avatars was a contribution of mine in the catalogue of works of Viennese artist Sylvia Eckermann.⁵

Originally, the present book was supposed to be titled *On the Infinity of Markets*. This is how I referred to it in correspondence with the publisher and in the initial book proposal. What I had in mind was the never-ending chain of derivative writing and trading (derivatives written upon derivatives written upon derivatives) and correspondingly the never-ending recalibration procedure of derivative pricing models to the derivatives market prices. Also, I was fascinated by the potential infinity of samples that one can extract from Brownian motion despite the fact that the time interval in which one is looking might be infinitesimal. To me, this double infinity signaled the emergence of an unusual ‘thinking compound’, literally the emergence of a new matter and a beginning in David Deutsch’s sense of the beginning of infinity.⁶

When putting the different pieces together and assessing their relative importance, it appeared to me, however, that the infinity of markets, although still a major theme in the book, was no longer the main theme. Rather, the real stuff the book was made of, as well as the argument that structured it overall, appeared to be the notion of contingency. Contingency was really the deeper category underlying my whole research and thought, of which the market of contingent claims was but the special and exemplary medium. I thought the book would finally give me the opportunity to re-appropriate the label of ‘The Medium of Contingency’ and to give it its definitive reference and body of work.

This book develops the theme that contingency is irreducible to probability and that the market price is really the category that should deal with future events. Probability was created after a statistical

heuristics, in Kolmogorov's monograph,⁷ and it is defined in a framework where a given experiment, possibly yielding different results, can be repeated. However, the really interesting question is: What can we say of experiments that cannot be repeated? How to speak of the probability of a single-case event? My answer, of course, is that the traded price of the corresponding contingent claim is the only way.

In this book, I re-examine the philosophical foundations of formal-mathematical probability theory as we know it today, namely, measure-theoretic probability as formalized by Kolmogorov in 1933. In particular, I investigate the link between the possible and the realized, or between the formalism and the material statistics that the formalism was destined to explain. I was astonished to notice that the formalism speaks only of mathematical random variables and never of the realization of their particular values.

The random variables remain at the state of potentialities, packed with the full set of possibilities, none of which is selected in order to become 'real'. As for the law of large numbers, which is supposed to provide the link between probability and the very concrete realization of a frequency of outcomes that converges to the given probability when the number of trials increases, I found that it mentioned only a sequence of random variables, not a sequence of actual outcomes. It is only the probability distribution of the random variable, which is constructed as the average of the variables in the sequence, that becomes more and more narrowly centered around the mean of the variables, but all this talk remains framed in probability and no actual sequence of outcomes is ever mentioned.

In other words, it is very problematic and is only a matter of interpretation of the formalism to say how formal probability relates to reality, to realization or to statistics. The formalism knows what probability means, not what realization means. The decision is made outside the formalism to interpret 'probability = 1' as the realization of an event. The formalism doesn't know what trial, or repetition of the same experiment, or even statistics mean.

My crucial observation is, then, that derivative pricing theory (and then practice) is the child of the formalism of Kolmogorov in its finest and subtlest and least intuitive consequences, namely, mathematical Brownian motion and the Itô stochastic integral that is constructed on it, not the child of statistics. It is truly mathematics, in its most formal, that has produced this new reality known as derivative trading and pricing, not statistics. Statistics and the fact of repeating an experiment may have inspired the creators of the probability formalism, chief among whom Kolmogorov; however, the formalism acquired its independence and developed formal consequences (Itô calculus, stochastic integral) that had no relations with the original statistics.

The reality of statistics and the corresponding usage of the word 'realize' (as when we say that 'a certain outcome is realized') lie on one side of the formalism, while the reality of the market and of price lie on the other side. The reality of statistics (and the whole meaning that we attach to the word 'reality' there) has nothing to do with the reality of the price and of the market (and the meaning attached to them) if only because the two realities are separated by the span (both historical and philosophical) of the formalism. Statistics couldn't have predicted the derivatives market because that market is a genuine child of the formalism and because statistics couldn't have predicted the formalism. Indeed, the formalism is formal and belongs in a different world altogether.

With the critical reappraisal of the formalism of Kolmogorov and its non-intuitive consequences (Brownian motion, Itô calculus) occupying the center of my book – and I believe the Itô integral, and its consequence on the age old probabilistic debate between the ex-ante and ex-post stances, has never been examined philosophically – I then move on and start spelling out the elements of the new reality known as the market. The investigation retraces the findings of *The Blank Swan*, where the reality of the market was shown to proceed directly from the reality of contingency and of the single-case event without the mediation of probability. Also, I re-discuss the meaning and metaphysics of writing and of the exchange place.

My overall aim is to show that the market is real; that it relates to contingency yet that, as a novel kind of reality, it has nothing to do with probability, *when the latter is understood in its relation with statistics*. The new finding in this book is, on the contrary, that the market can be seen to relate to probability when the latter is detached from statistics and driven back to the pure formalism. It is as if I were proposing a new interpretation of the probabilistic formalism: no longer statistical, but based on the market of contingent claims and their active replication and recalibration by traders using the latest consequences and refinements of the formalism, namely, Itô calculus and the corresponding derivative pricing tools.

This book is more ‘rigorously’ mathematical than *The Blank Swan*, in the sense that mathematical probability is really confronted, this time, at its purest and most formal. It draws on the key works of von Mises, Kolmogorov, and, more recently, Shafer and Vovk, authors of *Probability and Finance* (Wiley 2001). It is also more ‘rigorously’ philosophical than *The Blank Swan*, in the sense that I investigate, in the end, the deep philosophical problem of the relation between formalism and interpretation, or between form and matter, with the market being only a particular case – a very pressing one, of course.

Yet my writing style is neither mathematical nor philosophical in the academic sense. As a category of thought, contingency is mediated by several strands of writing and I see the book as the weaving of those strands around a common axis. As matter sitting at the center of the book, contingency creates a gravitational field within which the book gyrates. For this reason, as the reading proceeds, it is not uncommon that succeeding parts reiterate the themes that were announced and sometimes developed in preceding parts, in a motion that is not only a revisiting of the thought but also a deepening and a hardening.

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Without Serge Kouyoumjian, my oldest friend and co-founder of ITO 33, I would have had no office, no company and probably no authority. Without my second partner and eternally youngest friend, Philippe Henrotte, I wouldn't have had the intellectual excitation.

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To Michel Bitbol, my former PhD supervisor, I owe the illuminating remark that one must, perhaps above all, be suspicious of the notion of *state*. I still owe him a PhD dissertation on the philosophy of probability.

To Robin Mackay I owe the cause of this book, chapters 3 and 5, originally written for *Collapse*, as well as the translation from French of chapters 8–10, by far the steepest.

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To members of the Wilmott forum and of Nuclear Phynance, named or unnamed, sympathetic or antipathetic, I owe my rage and, ultimately, my philosophical hammer.

To Jon Roffe I owe the expression 'immanent theory of the market'.

Martine d'Anglejan-Chatillon hosted the event that named this book. She is how I learned to stop worrying and love the event.

Introduction

This is an ambitious book. Its ultimate purpose is to introduce a new matter, in the sense of a new chemical compound. Matter to be created and defined, not empirically discovered. Matter, not in the sense of physics, but of metaphysics.

The word *matter* here implies that something is absolute and mind-independent.

The event is absolute because it is independent of the frame of reference of possibility. The event emerges from a void of possibility. Consider the creation, or the definition, or the forcing of matter that I propose in this book as the attempt to fill in that void and to write into that blank; to redefine the void positively as matter. Yet, I insist that matter is immanent; it is not supervised by a transcendent principle or by a creator. It is ex-post, like writing.

Most of my investigation of the void of possibilities, or of the blank in which to write contingency and to write the event, will be connected with the financial market and with the exchange of contingent claims (or financial derivatives) in the marketplace. My purpose will not be to explain how their market prices quantitatively come about, but to investigate the nature or the genesis of price. I seek to create concepts from scratch, or to show that some concepts are contingent, therefore dispensable and replaceable.

For instance, I argue that probability, as concept, and the corresponding identification of the possible states of the world can be replaced. My argument is not addressed to financial engineers or to scientists who will, no doubt, argue with me forever about price and probability without ever leaving their plane of reference,¹ but to philosophers of probability and to metaphysicians. The news that I am breaking to them is that there is a new metaphysical category (I call it the 'market' or 'price') that should replace probability in matters relating to contingency. Because contingency is absolute, it is matter, and the medium that translates it absolutely – that is to say, independently of the relative frames of possibility – is material. Even better, this medium (the market) is a technology. As I speak of contingency as opposed to possibility, I speak of matter as opposed to formalism and of technology as opposed to theory.

In my metaphysical reconstruction, contingency is the primary category. Its proper medium is writing, not possibility or probability. As writing, its only place is the exchange place and price is its translation.

Notice how all this discourse, which may strike one as pure abstraction, is foreign to any preconceived or ordinary view of what a market is or what a price is. What I call the 'market' or the

‘price’ (by way of a redefinition) are precisely the concepts that I need and that I properly generate in order to distance myself from possibility and probability. When I talk of the *genesis of the market*, it certainly is not the ‘real’, or empirical, or historical genesis of the market that I have in mind.

I will reach my goal fully when the people on the other side of the boundary between finance and philosophy, the hardcore philosophers themselves, start learning what a contingent claim is and what the market is.

My book is not a probability book but ultimately a book on contingency and the event. (My work is certainly not scholarly.) My topic is a criticism of the limitation of contingency to the so-called ‘states of the world’ or ‘possible worlds’ and of the delimitation of such states, which is a typical metaphysical gesture. This criticism applies equally to objective and subjective probability.

The book introduces matter because of the sharpness of the angle. The angle is so sharp – it is even almost closed – that it is hard and that its hardness is tantamount to matter. Matter manifests itself by its contradiction and difference, by what it absolutely denies, the previous void in which it fills, or the previous walls which it breaches or at least displaces. The sharpest and most differential definition of the market – what we might call its *material* definition – is the floor on which the underlying asset and the derivative written on it trade equally and on an equal footing. The reason why this is so is that in any other representation or formalization of the market – ‘any other’, which means the only other, which is the probabilistic representation or formalization – the derivative is not supposed to be trading but only to be valued and the market (its market) is denied.

Now, to insist contradictorily on the trading of the derivative is to get the sharpest and most cutting (most differential) definition of the market. Notice that not even the underlying asset is supposed to trade in the traditional picture, because its trading process is then only a coincidence, an image that gets imprinted externally on the picture but does not inhabit it properly. The trading process of the underlying asset is supposed to have taken place in an immemorial arena and the theory only reports its vestige and summary, which is then necessarily a stochastic process governing its price. (How, indeed, could a formalization or a quantitative theory be offered without a stochastic process being the ground?)

There can be theories or models of the market which result in the derivation of a stochastic process for the market price. There can be statistical studies of the market which conversely postulate the stochastic process and seek to infer its parameters from the time series of prices. However, derivative pricing, which we select as the precise angle, or rather interval and void in which the derivatives market does not exist and then is made to exist all the more sharply, is one representation which starts with the stochastic process of the underlying price and then uses it to *make* or manufacture something extra, which is the value of the derivative – keeping in mind the ulterior motive that cannot be part of the theory and can only exceed it and which is the making of the *price* or the making of the market of the derivative. In the making of the derivative value (and covertly, of its market) there is an a priori attitude towards the making of a market, which will retroact, as we will see, on the market of the underlying asset itself, therefore absolutely on the market, as its definition.

The market never starts, we are already in the middle of it; it is already a process inscribed in history, hence the relevance of the underlying price process, and this is why derivative pricing, which does not dispute this given at first, can help define the market anew and originally, while at the same

time inscribing itself in the current process. Since there is no other way to start defining the market except the market itself, derivative pricing offers the ideal stand, in reducing the definition of the market to the creation of the market of the derivative, then in re-embracing the definition of the market of the underlying asset from this seemingly internal episode.

There can be theories (general equilibrium, rational expectations, and so on) leading to the formation of the market price of the asset S , however their result is only conventionally called the market price, or maybe only by wishful thinking; it is not really the market price. The arguments that go into its derivation (equilibrium, expectation, optimization, and so on) have really nothing to do with trading or with the market. They are not immersed in the market. On the other hand, theories that have already moved past this moment of genesis and that consider the resulting stochastic process of the price of the asset S also deny trading, as they only retain its footprint, or summary – the stochastic process. The efficient market hypothesis (EMH) is invoked to argue that the trajectory of price can only be random and then probability theory takes over. The sense or the force of trading is lost too, as trading (or the heart of the market that we are after) gets translated, by the effect of a coincidence, into a stochastic process.

It is typical of this move that states of the world are postulated, which have to be two at least: (ω_1, ω_2) , or the price of the asset S going up or down. The world is turned into a random device which now occupies the background and supersedes the trading process, and it is *later* observed that the prices (S_1, S_2) of the asset S coincide with the states of the world. Probability and the random generator become the first driver, and no longer trading. Everything else then becomes a matter of valuation over the underlying states of the world.

Yet, trading is not a valuation. You cannot evaluate something, and then trade it. Trading is original. It is an event. So, the problem that we have is that we want a computational model of trading; we want to describe an ongoing market (and not freeze into the pre-quantitative stage where the deeper motives of trading are considered, i.e. anthropological or economical, and so on); we want to represent the trading process as already happening in time; in other words, we like the presentation of the market under a stochastic process; yet, we wish to re-inject the trading force into the probability representation, which has a tendency to withdraw such force from the market and act as if the market were a random generator whose origin was purely metaphysical.

Derivatives are ideal in starting with a computational view of the market (the stochastic process of their underlying asset) yet in introducing the void or the event within this continuous and ongoing picture. They are continuous with the trading force of the underlying. In the Black-Scholes-Merton model (henceforth BSM), their payoffs are even, strictly speaking, equivalent to trading strategies of the underlying and they are believed to be redundant, yet for this reason and as we shall see, their market jumps all the more originally and genetically to the forefront.²

This angle on the market (a stochastic process of the underlying that is closed and given, which is both equivalent to the manufacturing of contingent payoffs and then to the *writing* of contingent claims which will jump into their own market) is the sharpest and, for this reason, the most adequate angle to define the matter of the market. One of its benefits is to go counter to statistics and to the notion of the market as a generator of random numbers.

Matter is definitely needed, precisely defined differentially with regard to the probabilistic representation, in order to inscribe the market in history and ultimately to mediate the event. The event is also associated with matter because of what is absolute in the event and evades any probabilistic frame of reference. The event emerges from a void; it is not mediated by possibility and this differential definition of the event is also equivalent to the definition of matter as the other face of the void – or what is supposed to fill it.

By choosing derivatives as entry point into the matter (or definition) of the market, we resolutely adopt an abstract point of view which will ultimately strip our definition of the market from any physical cloth and turn it metaphysical. If the matter of the market is ultimately a flattening of the hierarchy of underlying and derivative, then instead of arguing for the genesis of the market from an anthropological, sociological or even economical point of view, the matter we are after will precisely lie in the abstract apparatus of thought which connects underlying and derivative. It will hardly be an exaggeration to declare that derivative valuation theory in its most sophisticated usage, and especially its extension into a derivative pricing technology, will be part and parcel of our definition of the market and of its genesis.

What the matter of the market will precisely contradict in order to manifest itself as matter is, therefore, probability and the corresponding schema of thought based on states of the world and valuation, and generally the *chrono-logical schema of thought* which perceives the relation of underlying and derivative in a sequence: the underlying, then the derivative – the state, then the value. It is from a contradiction that someone like Einstein extracts a physical intuition and eventually a reformulation of the fundamental principles of physics.

On the other hand, it will not in the least help us to think in haste and to argue immediately that the market is the flat arena in which underlying and derivative trade indistinctly. This view – I call it the view of the ‘chaos’ – is certainly real and realistic. However, it doesn’t accommodate a genesis and it certainly doesn’t offer a formalization of the derivatives market.

We need the formalism of probability in its finest consequences (Brownian motion), we need the formalism of derivative valuation theory, and we need derivative pricing as a technology and even an industry, in order to deduce the matter of the market *through* these schemas by traversing them, using them and then twisting them flat. We need to see what the matter of the market *is using and then contradicting* (probability, statistics, and the corresponding idea of temporality) in order that these stages become layers on top of which this revolutionary matter is laid down.

Derivatives are quantitative formulas. It is by contradicting what was traditionally quantitative in their valuation theory – namely, probability and statistics – yet by retaining the quantitative point of view that they impose, that our angle gains its maximum sharpness. To repeat, we need the sequence, followed by the twisting that the derivative valuation theory and then the pricing technology impose on the schema of thought. We won’t define the market as the place of exchange of oranges. Rather, we define it at one or maybe two stages later, when futures contracts on oranges are already actively trading and then options on those futures are written and traded in their turn. Instead of dealing with individuals and agents, with beliefs and consensus, this ascension in the degrees of writing and complexity (which might distract the layman at first from our precise angle and turn him off, as he

doubts whether we are still talking about something tangible and even about the market) allows us to deal with categories of thought such as state, possibility, probability, writing, temporality, etc. As we said, this will not amount to the volatilization of the market in layers of inaccessible abstraction; on the contrary, it affords the sharpest angle from which its matter will emerge.

It is not by accident that I am tackling this angle of the market through derivatives. Derivatives are quantitative. Because they are formulas that are written down on a piece of paper and that prescribe what to pay off in what condition, they have given rise to the whole quantitative approach to their valuation problem. We know as a mathematical fact what a derivative is worth at the moment it expires, so it is very tempting to introduce mathematics the minute before it expires. However, despite the mathematical valuation model, we will see that the whole bottomless abyss of the market opens up in that one minute, because even a minute before expiration we are going to depend upon some theoretical parameter – typically volatility – which we can only infer from the market prices of derivatives and will typically be subject to recalibration. Despite the quantitative temptation inherent in derivatives and expressed in the models we use to value them, derivatives, as they are traded by a market-maker who will turn their valuation theory into a pricing technology, undo all calculations of probability and possibility.

Yet, the market-maker cannot dispense with the quantitative tools or the mathematical models. He needs them if only to produce derivative prices which are not vulnerable to arbitrage and to hedge dynamically the derivatives he is trading. But the paradox is that by using them, he is at the same time criticizing the whole framework of possibility and probability. It is as if there was news to break, not to the financial community – because every trader is more or less aware of the limitation of the model – but to the philosophers of the event. It is as if I were telling them that I have an illustration of their philosophy and of their criticism of possibility – an illustration that materially exists because the market of derivatives materially exists.

In my narrative of derivatives, what I mean by value is what comes out of the theoretical valuation models. I do not mean the value that actors in the market attach or deny to the derivative for whatever subjective, ethical, or political reason. Now, against value as given by the valuation models, I argue for the immanence of the market and, consequently, that price is all there is. For instance, the fundamental parameter in the BSM valuation model is instantaneous volatility. This is a formal theoretical concept. However, given that there are only prices and no transcendent view of volatility, the only working concept in BSM is the volatility that we imply from the market prices of derivatives by inverting the BSM formula against them, or implied volatility. *My whole investigation revolves around trying to make sense of implied volatility.* It consists in pushing things to their ultimate conclusion – as is appropriate in speculative metaphysics. If implying volatility means trying to figure out the number that is the fundamental parameter in the valuation problem from the only thing I have (prices of derivatives trading in the market, assuming liquidity), I thereby inherit a technology problem that to me is more materially pressing than any ultimate concern with value. To repeat, it is a technology problem that justifies my metaphysical reformulation of the market.

Here is my problem: How do I devise and ship a technology to traders – based on definitive valuation models such as BSM and all the generation of models that come afterwards – knowing that at every stage of calibration of the given model to the market, the inferred parameters, for example implied volatility, become stochastic? Because they are stochastic, they call for a higher-order model whose own parameters will only be determined by calibration to the market price, either actual or virtual, of a derivative of higher-order complexity. How do I proceed to make the case for this activity without a terminus?

If the market is the absolute matter, then every stochastic model that assumes an identifiable set of states of the world is like a temporary section taken in that compact matter. However, it is only through calibration and recalibration to the traded prices that the model ultimately establishes contact with matter.

Derivative valuation theory vs. derivative pricing technology

For the reason that derivatives afford us the abstraction then the sharp angle from which matter will emerge, we will successively take up the historical and theoretical stages that derivatives, as a quantitative technology, necessarily have to traverse – namely, probability theory, Brownian motion and dynamic replication (BSM) – and detect the emergence of that matter in each one of these stages. Our mode of deduction will always be peculiar. Matter will emerge, in each instance, from an unusual reorientation or reinterpretation of the corresponding formalism. This is why it is argued that the matter in question emanates from the categories of thought themselves; that it is material, in the sense of escaping the formalism and precisely residing in the void that is outside of it; yet, that this matter is apodictic, that it is thought and almost created by thought (it is not empirical).

The book will attain its goal if the derivative pricing technology, which is the extension of derivative valuation theory when the latter is put to use in the market, finds its proper matter and proper domain. To repeat, derivative pricing technology is in need of a domain and of a matter that vary completely from the domain of derivative valuation theory because the pricing technology *is not* the application of valuation theory. In derivative valuation theory, there is no mention of the derivatives market, let alone an intention of trading derivatives. In derivative valuation theory, there is either a random generator whose states of the world are filled, as if coincidentally, with prices of the underlying and perhaps additionally with state variables ruling the stochastic process of the underlying price (volatility, jumps, and so on) as well as derivatives *which are supposed to be independently written* on that underlying then valued by the non-arbitrage principle, or alternatively there is a trading process of the underlying, complete in the sense that the variable size of the trade is associated with the random price at which the trade occurs, which leads to the manufacturing of contingent payoffs, yet, remains on this side of the market of contingent claims without an access to such a market.

In derivative valuation theory, there is either a probabilistic framework in which nothing happens except the arbitrage-free valuation of derivative lotteries, and by that I mean that trading doesn't happen – and it certainly doesn't help to make the probability distributions vary in time or to make volatility stochastic –, or there is the trading of the underlying, which is summarized by a stochastic

process for its price but from which we cannot escape to the writing of derivatives, let alone to their market. Yet, derivative pricing technology exists, and there is no doubt that it makes use of derivative valuation theory. It exists and it is used by traders to trade derivatives. The book aims to formalize it, or to offer its ‘theory’.

To express the point more concisely, the ‘theory’ of the derivative pricing technology, or the account of how derivative prices exist and derivatives trade, is different from derivative valuation theory. Neither is the pricing technology a dirty application of the valuation theory, for there is a ‘clean’ (original) and intrinsic problem that the existence of the technology poses. There is a domain of reality that the book aims to theorize – this technology – but this is different from the empirical reality of the market. The aim is not the study of time series of underlying prices and derivatives prices and the discovery of the model or law that can rule their relation. Neither is my domain of investigation the sociology of financial markets, in which it is argued that derivatives are traded and contradict the validity of their valuation theory because the valuation models and tools (BSM, typically) are offered to a *society* of derivatives traders and for this reason exceed their theoretical domain.

The existence of the pricing technology is not just empirical or sociological; there is something a priori and apodictic about it, because there is *design* in the technology. Yet, to repeat, the pricing technology is not part of the domain of the valuation theory. The technology is not the application of the theory, we said. The whole matter that we wish to make manifest lies in the void separating the theory from the technology, which is not only filled by history – we argue – and certainly not by the traditional logic of the premise and the consequent.

Derivative valuation theory has the merit of starting with the market to the exclusion of anything else. It starts with the market of an underlying asset S – more precisely, with the net result of its trading, which is the stochastic process followed by its price. All economical or anthropological reasons why the asset S is traded, sold and bought by traders assembled in the trading pit, have been screened off. The theory could have stopped here. It could have limited its ambitions to the description of the statistics of trading of the underlying asset and to the discovery of the probabilistic model, or data generating process, that best fits them. Why suddenly tackle the derivative *valuation* problem? If the market is our sole horizon (as witness the underlying market), are we implying that the derivatives market will also be in view, and shall we understand that it is the same market we will be talking about?

If the market is the sole arena (an exchange arena), why suddenly retreat from it and talk of valuation? Of course, it is the probabilistic valuation that is so tempting. Suddenly, it is forgotten that the stochastic process of the price of the underlying asset is a model or a summary of its market (and only of its market as sole horizon and category) and it is remembered that such a stochastic process *separately* provides the probabilities that the payoffs of derivatives are triggered over time. Suddenly, the picture is withdrawn from the market and becomes one of mere valuation.

If the market is really the sole horizon, it would be better to begin with the market of the derivative and to consider its own trading process originally as an independent stochastic price process. The problem is that the class of derivatives in question is one whose underlying is a traded asset and if, indeed, the theory of the underlying market could have dispensed, as we said, with the worry

and even the mention of the derivatives market, the derivatives market, should it be considered originally, conversely cannot dispense with the mention of the underlying market because the underlying of the derivatives of concern is precisely being traded. And once the two markets are considered jointly, we cannot resist the thought that the derivative price must be equal to the derivative value, with the latter being provided by a probabilistic valuation under some probability measure. The non-arbitrage principle imposes that this is so.

There is tension in the picture. On the one hand, the stochastic process of the underlying asset price that is originally given suggests that any valuation worry is forgotten – that we do not wish to worry why somebody is buying or selling the underlying asset or what valuation theory is inspiring him to do so – and that all we have is the net result of such hidden or underlying motives, a summary which translates into a mere stochastic process of the underlying price. On the other hand, the stochastic process of the underlying price offers the best opportunity to reconsider a valuation problem afresh and in its purest sense – namely, the valuation of the derivative.

Ultimately, the derivative depends only on its underlying (which triggers its payoff), so whereas the reasons and motivations underlying the valuation of the underlying asset were too complex or too unruly or too impure to be modeled, so much so that we were very pleased that they were hidden or screened off and that a trading summary was offered instead under the form of a stochastic process; in the case of the derivative, by contrast, the valuation problem seems very well posed and too good to resist. If it were not for the tension which prevents the derivative valuation problem from becoming a proper pricing problem, we would have been presented with the perfect angle in which to frame what best we could hope for a market – namely, a framework in which the market is the sole horizon and only screen – combined with the opportunity precisely to *make* a market (the market of the derivative) and to make a price.

We seem to be in a perfect situation in which the market is the only world (picture the trading floor) whose origin and horizon extend limitlessly in both directions, truly preventing us from intercepting either one, yet a situation in which we are not passive in such a given and irretrievable world but can be authors – the authors and makers of the derivatives market. Recall that this is the reason why we have picked derivatives as the best angle to define a market and truly extract its original matter. There is design, we said, therefore an a priori attitude, in the technology of derivative pricing which singles it out as the best channel to define the matter of the market. The pricing technology is both embedded in the market as immemorial and indisputable and irretrievable matter and it is designed in a reflexive retreat, as if distance had been taken with the matter of the market. The cost, of course, is to resolve the tension that we have described.

Derivative valuation theory must become a derivative pricing technology. The matter, or the register, or the book, must be discovered or written, in which derivative valuation theory, or the fact that derivative payoffs are triggered by the underlying and are called derivatives for this reason, is reconciled with the fact of the market as sole horizon. The problem is how to make a derivative *price* out of the definition of the derivative payoff (or in other words, how to use derivative valuation theory inside a derivative pricing technology). Indeed, the biggest threat is probability. That the derivative payoff should be so well defined in terms of the underlying price makes the probabilistic

valuation too good to resist, we said. Yet, we also, or perhaps above all, want a derivatives market. We want the derivatives to be bought and sold, and consequently their market price to move, for all kinds of reasons that remain hidden as well but whose expression is that the buyer believes that his investment will appreciate and the seller that it will depreciate.

The probabilistic framework, when it takes over, suddenly withdraws this trading force. All that it leaves in place are static states of the world which coincide with prices of the underlying and with different states of parameters ruling its process (for instance, different states of volatility or of jump size) and non-arbitrage as the only principle enforcing the relation between the ultimate derivative payoffs and their present value. This amounts to computing this value as expectation of the payoff under a probability measure that is equivalent to the objective one in which the underlying price process is given (with its possible stochastic volatility and possible jumps). This, as we said, amounts to exiting from the market and its pit, to leaving the market of the underlying behind as the only market that was ever mentioned and thereafter to specializing solely in its statistical analysis.

It certainly adds nothing to the picture to consider the valuation of derivatives as a side effect of the *statistical analysis* of the market that we have exited from, miles away from their own market and, most importantly, depending on the crucial assumption that the random generator that we think we have discovered behind the statistics of the underlying price is stationary.

Implied volatility vs. real volatility

We know how derivative valuation theory eventually became a pricing technology and how the statistical analysis that was driving derivatives away from the market and into mere static valuation eventually turned back to the market and to its proper auction dynamics. It was discovered that volatility was a crucial determinant of derivative value then it was concluded that the underlying hidden reasons why traders buy or sell derivatives were expectations they might have about the rise or fall of volatility. Notice the contrast with the underlying, which is potentially generating the tension: while the ‘valuation theory’ of the underlying could remain hidden – and by that we mean the reasons that push traders to buy it or sell it – the motivation behind the valuation of derivatives couldn’t but remain apparent by contrast, and very specifically so, because it was none other than the volatility of the underlying price, whose stochastic process and therefore volatility are precisely filling the picture.

So, ‘volatility’ became the market that was open to derivatives traders independently of the market of the underlying. Volatility became the ‘price’ of the derivative. What volatility? Implied volatility, of course. That the value of the derivative should change independently of the underlying price is explained, in the BSM formula, by an increase or decrease of implied volatility. There certainly exists a trading situation in which the underlying price doesn’t move and derivatives traders exchange a given derivative at an increasing or decreasing derivative *price* (depending on the buying or selling pressure), and there certainly is nothing wrong with translating this appetite or dislike for the derivative in terms of the volatility number that would be implied in the BSM formula against its price. Notice that to expect implied volatility to rise or fall is nothing more than to expect the

derivative price to rise or fall. Considering a market for implied volatility, or a stochastic process of implied volatility, is strictly equivalent to considering an independent trading process for the derivative, exactly as we have wished earlier, after we had remarked that the market was the sole horizon and floor, and that the derivatives and the underlying should enter it on an equal footing and with nothing hidden underneath the floor.³

Notice that when traders buy or sell derivatives on expectations that implied volatility might rise or fall, they expect nothing with regard to the volatility of the underlying (or real volatility). The volatility of the underlying cannot be bought or sold. You cannot move it up by buying it or move it down by selling it. You can bet that it will rise or fall; you can exchange derivatives that pay off exactly the difference between realized volatility or realized variance and a certain predetermined level (volatility or variance swaps); but there is a difference between settling the bet and putting the bet to trading. As soon as the trading process of the bet is considered, the price of the bet becomes the basic variable with only supply and demand to move it and with nothing deeper to underlie it. Variance swaps can trade at prices that imply a variance that is totally disconnected from the real variance. Their trading process could be modeled indifferently either as a stochastic process ruling their price or as a stochastic process ruling the implied variance (the variance that we imply from their price, which amounts to the same), but this has nothing to do with the process of the real variance (if there ever was such a thing).

Real variance or real volatility start to matter, and writing their process starts to become a requirement, when it is remembered that derivatives written on a certain underlying are eventually triggered by that underlying (even the volatility or variance swap which in effect are triggered every day), and when a comprehensive, arbitrage-free picture is required in which both the market of the underlying and the market of the derivative are represented. As we said, the motives behind buying and selling the derivative cannot remain hidden once the derivative is confronted with its underlying asset in the same picture. Even though derivatives may be traded independently, on no other basis or account than their own temporary market and their next temporary price movement, when their underlying asset is brought into the picture, this can only be a picture in which the underlying is the ultimate trigger of the derivative payoff and in which its real volatility has to make an appearance.

Surely, a certain derivative is bought or sold for no other reason than the expected rise or fall of its price (all deeper reasons underlying the valuation of derivatives remaining hidden). But how to distinguish, in that rising or falling derivative price, what is genuine movement and what is being only automatically caused by the price movement of the underlying asset? To settle this question, the price process of the underlying asset has to be jointly modeled (as well as a formula produced to compute the derivative delta) and it will not be enough, in the resulting comprehensive picture, to equate the portion of rise or fall of the derivative price that is not caused by the underlying price movement with a rise or fall of, say, BSM implied volatility (when the formula being used is BSM). An arbitrage-free representation simply does not exist in which one dimension is spanned by the prices of the underlying, which are visited by the market in either direction depending on the appetite or dislike for the underlying asset, and the other dimension is spanned by the prices of the

derivative (or equivalently by implied volatility), which are visited by the market depending on the appetite or dislike for the given derivative. The comprehensive, arbitrage-free picture requires that the underlying is the only trading asset and that other dimensions, if they must exist, are spanned by parameters of the stochastic process that summarizes its trading.

The arbitrage-free picture of the ‘trading’ of derivatives is hopelessly a picture of valuation and paradoxically (irritatingly) not of trading. Even though a market for derivatives has been acquired and traders may buy and sell derivatives at prices corresponding to a certain level of BSM implied volatility for no other reason than subsequently to unwind their trade at levels of implied volatility that turned to their advantage, the arbitrage-free representation suddenly denies the layer of the derivatives market and pulls everything back to a picture where the underlying has always been the only traded asset and its volatility has always been stochastic with certain pre-assigned states or levels, or jump processes have always been superimposed on the diffusion with certain pre-assigned jump sizes, and so on.

Alternatively, it is believed that derivatives are bought or sold on expectations that the real volatility of the underlying price will rise or fall. A sudden change in real volatility generates gains or losses in the dynamic hedging strategy of the derivative relative to the premium at which it was originally bought or sold. But the comprehensive, arbitrage-free picture is not one in which derivative price movements – and, correlatively, expectations that the real volatility will rise or fall – precede and drive the movements in real volatility. Once again, valuation takes over and the trading motivation or the trading force is withdrawn. In the comprehensive picture, movements of real volatility are modeled first (that is to say, a stochastic process is first given for the diffusion parameter of the underlying price process, or jump processes are written, overlying the diffusion), and it is only with these movements as underlying causes that derivative price movements are explained. They are explained as changes in valuation.

And why are the principle of non-arbitrage and the corresponding comprehensive picture required? They are required because a valuation formula linking derivative and underlying is required, and a formula linking them is required because their prices, and not only their values, will tend to move together (the delta). However, the existence of a derivatives market, or at least the need to differentiate its matter, seems to suggest that the principle of non-arbitrage has to be exceeded in a way and the comprehensive picture surpassed (keeping in mind that this surpassing and this upgrading should not resolve into the ‘view of the chaos’); it seems to suggest that the pricing technology, while building on valuation theory and the corresponding non-arbitrage principle, actually sits on a different layer. Unless – to repeat – derivative valuation theory was never meant to issue in a derivatives market and was subsequently never meant to hold the market as its sole medium and horizon, despite the fact that it has started with the market (of the underlying) to the exclusion of anything less and has falsely suggested the hope that the market will from now on become the only matter.

The way out, probably, is to try to block the statistical temptation and to prevent derivatives from being valued as lotteries triggered probabilistically by the underlying: in other words, to keep the trading force.

Formal reality vs. physical reality

The whole question is whether we want to have a derivatives market and have it formalized. Perhaps I am making too much fuss about nothing. Perhaps the only formalized market we ever wanted to have is the one we have exited from with the stochastic summary – namely, the underlying market – and all the computational issues and concerns that we will have thereafter are the statistical analysis of the time series of underlying prices and correlatively the valuation of derivatives (lotteries) triggered by them.

A valuation is no market, in that case, even though various epistemological considerations may muddle the issue. It is, for instance, asked whether the statistics of the underlying are truly known. A ‘market’ for derivatives may emerge because volatility is uncertain. Not that it is stochastic or that traders expect it to rise or fall. It may be constant as a matter of fact and trading may only be debating about what its value may be. We say this is no market because this is disagreement on valuation, not pricing. If the statistics were known then the value of the derivative would be the actuarial value that some insurance company would assign to it, or it would be the expectation of its payoff under an equivalent measure (i.e. a single run valuation given by a banker). So, are we saying there is a market because there is disagreement on the objective measure (hence, potentially, on the actuarial value), or are we saying there is a market because there is agreement on the objective measure but disagreement on the equivalent measure? If there were agreement on the objective measure and disagreement on the equivalent measure (or divergence in risk preferences), then sooner or later the market consensus will emerge, arbitrage will be enforced (for instance, by agnostic traders) and the equivalent martingale measure *of the market* will be established. This will become valuation again, under a martingale measure, on which there is agreement, or at least consensus, and which has to be equivalent to an objective measure, on which there is agreement. And so it will be no market. And if there were disagreement on the objective measure (or the objective statistics), then I don’t see how non-arbitrage can be enforced or, at least, written. In that case, there won’t even be a valuation.

Better to adopt the register in which the statistics are known (at least on paper) and then valuation is conducted under a probability measure, either the objective one (actuarial valuation) or an equivalent one. Let us recall that we are not painting a realistic picture. Nobody knows the data generating process of the underlying prices in reality, and entire groups of researchers or quantitative analysts are striving to discover it (or are, at least, entertaining this hope or illusion). The derivatives market may have no other reason to exist, in reality, than the lack of knowledge of the underlying data generating process (DGP) and some would even want to infer the latter from the market prices of derivatives.

If the DGP were known, there wouldn’t be a derivatives market but only probabilistic valuations of derivatives. While this is certainly true – and, as a matter of fact, a basic tenet of our own argument – it is not the brutal existence of the derivatives market that is our starting point (what we have called the ‘view of the chaos’), for we wish to recount its genesis. We wish to retain the formal picture in which the stochastic process (or DGP) summarizing the underlying market is given, and we are only wondering why such a format and formalism may not simultaneously be given for the derivative (i.e. a proper stochastic process summarizing its proper market) and consequently researching what

should be forced into the picture, or into the void separating it from a derivatives market, in order that such a market may exist and that this action or addition or revolution may amount to a genesis of the derivatives market.

That the underlying DGP is given in our picture does not mean that it is known. It is only formally given; it is written on paper. It is important to see that we are in a completely different register than epistemology. If the consequence of the DGP being given, in our formal picture, is that the derivatives market may not simultaneously exist (but only derivatives valuations), we shall not argue, from the manifest existence of the derivatives market or from the need of its genesis to the lack of *knowledge* of the underlying DGP but for an alternative material interpretation of the formalism in which the DGP is given or written. We shall argue that it is only given on paper and not in physical reality, and that the physical reality that interprets the formalism should not be one in which the DGP materially exists but one in which the derivatives market exists. By forcing the matter of the derivatives market in the void (or in the inexistence of a derivatives market) that the formalism implies, we would be defining a reality of the market that is at variance with *and even alternative to* the physical reality of statistics and of the DGP.

In the peculiar interplay of formalism and matter that we would be accomplishing, we would be providing an ontological criticism and eventually criticizing the whole notion of state of the world which is the basic ontological component of the DGP and which is *logically prior to its being known or unknown*, a criticism of the probability framework and of valuation theory. To repeat, our register is not epistemology; it is ontology, or rather the question of what exists and what doesn't, following a certain formalism, what we wish to blow into existence following the formalism – in other words, how we wish to interpret the formalism.

Our register is semantics.⁴

For instance, the stochastic process ruling the underlying price is given and it is Brownian motion with drift μ and volatility σ . The act of giving this process is a stage that is already past epistemology. Perhaps the drift μ and the volatility σ have been statistically inferred; perhaps there have been doubts about the possibility that they could ever be inferred. In any case, now that they are formally given, the whole question has moved beyond any doubt. They are now the premise of a *mathematical* problem: Let a stochastic process be, and let its volatility be σ .

The mathematical consequence of this formally given stochastic process is, for instance, that a certain contingent payoff is perfectly replicable at a certain maturity, given an initial investment π and a perfectly foreseeable dynamic trading strategy. This is all that the formalism or the formula implies. It is important to note that it is equally foreign (external) to the formalism to argue that the volatility σ may be known or unknown or that a price p is given for the contingent claim – for instance, in a 'market' – and that σ can be inferred or implied by inverting the formula against the price p . In the formalism, we don't know what it means that σ may be known or unknown and we equally don't know what it means that a price p should be given. Even the notion of valuation is, strictly speaking, foreign to the formalism.

Given the stochastic process that is summarizing the trading activity of the underlying, it is a step outside the formalism to interpret the up and down price movements of the underlying

as probabilistic states of the world over which a certain contingent claim will be valued. If the contingent claim is valued (by non-arbitrage and probability states), then we will have exited the formalism whose only *meaning* is that a stochastic process be given as a summary of the trading activity of the underlying asset, regardless of any prior or hidden valuation problem. To be at one with the formalism, to enter the formalism, the contingent claim should have had its own trading process independently given; and this is what we are complaining about. We are complaining that while there is nothing to prevent a separate asset from trading side by side with the first one and from having the summary of its trading equally given as a stochastic process, a contingent claim written on the first asset cannot be represented to trade formally – provided, that is, we wish that the relation between the underlying and the contingent claim overlying it be represented.

One wonders what is exactly the extent, as well as the minimum constraint, of this relation. Is it only causality and arbitrage – the one being triggered by the other – or should it extend to cover the history of the formalism and the history of the event that has exceeded it; namely, the existence of the derivatives market as such, in which derivatives traders do use the formulas as indispensable layer?

The formula of the trading of the underlying is not a formula of valuation of a contingent claim; it doesn't even include the existence – the independent writing – of the contingent claim. We make it a valuation formula by superimposing probability states on the prices of the underlying; we turn it statistical and then, either from statistics or equivalently from the valuation framework, we wonder whether volatility is known or not known. But the exit to the formalism of the market of the contingent claim (formalism in the sense that the summary of the trading process is given regardless of any prior valuation reasons or motivations) could have been accomplished alternatively to valuation, to knowledge and to probability states. The exit could have consisted in noting that the contingent payoff that is replicated as a strict mathematical consequence of the formalism of the trading of the underlying is precisely *not equal* to the contingent claim.

Considering the price p of the contingent claim as given (when all we had was the initial amount π to invest in the dynamic trading strategy to replicate the contingent payoff) is as external, we said, as wondering whether the volatility is known or not known. So, we could force the formalism in that void and in that outside, if we wanted a formalism of the market of contingent claims, rather than in the outside corresponding to the question of knowledge. The formalism of trading is not just a stochastic process (in the sense of states of the world and probability). The frame of thought in which there could only be interplay between the formalism of statistics and knowledge or uncertainty (certainly linked with the problem of inference) is an old frame of thought. In the meantime, there has been a technological, and even an industrial, revolution. Now, seemingly statistical formulas can be used in the formal manufacturing of contingent payoffs (in keeping with the formalism of trading) and the realization that the contingent payoff is not equal to the contingent claim can become the exit, or the revolution, instituting the given price as *alternative to statistics* and to the whole question of whether there is knowledge or not.

Why can the level of interpretation of the formalism (the level at which matter begins and enters the picture) not descend deeper than was traditionally thought? From the fact that the formalism

gives no market of contingent claims but only the manufacture of contingent payoffs,⁵ we may well have, if we really want to force the matter of the market of contingent claims, to descend to a level deeper than the one in which the trading process is associated with statistics and correlatively with states of the world. This is what we mean when we say that the market of contingent claims, if it must be formalized – that is to say, offered as a possible interpretation of the formalism, offered as matter – may have to be an alternative to statistics and to the whole problem of statistical inference.

We wish to formalize the market of contingent claims, or provide a ‘theory’ of their pricing technology, as we said. Giving the stochastic process of the underlying – and, as a matter of fact, the valuation theory as a whole – is part of this technology, so there is no denying this layer. We wonder: Could the formalism of the market of contingent claims really be one in which the stochastic process of the underlying price is objectively given, yet there is speculation on what it is? (For there to be speculation – that is to say, uncertainty – there must be something to speculate about, or, at least, the formalism of epistemological uncertainty and speculation has to integrate a formalism of the objectively given.)

The stochastic process of the underlying price being objectively given, there seem to be two ways that a derivatives market, or an independent variability of the derivative price, could exist. Either we are keen on non-arbitrage and, then, not only the underlying process has to be given but the stochastic processes of the parameters ruling it have to be given too (stochastic volatility, jump processes); or we abandon non-arbitrage and we assume that the underlying price process is objectively given only it is not known and the derivatives market is speculation on what it is. In either case, we assume that the underlying stochastic process has entered physical reality and we then differentiate it, either by complicating it and adding processes for its parameters or by assuming uncertainty for those parameters.

An alternative argument for the existence of the derivatives market (a genesis, even) would be to stop arguing from the formally given underlying stochastic process to its objective reality (which we would have subsequently to differentiate either by further stochasticity or by further uncertainty), and to argue, instead, from the formal underlying process directly to the reality of the derivatives market. For the parameters of the underlying process to be unknown or to be further complicated by their own processes, they have to be known to exist (and consequently known to be unknown) or known and observed long enough in time for the processes ruling them to be known. But what if the derivatives market was an overall alternative to statistics and to the corresponding temporal physical dimension, such that there would no longer remain any statistics in this interpretation, of which we might wonder whether it is known or unknown, or whether it has been known long enough for further statistical complications to emerge?

Part I

The Matter

The End of Probability

1.1 The void of possibilities

The radically-emergent event is not part of a previous range of possibilities.¹ It produces its own causes or the very possibilities of which it could be later identified as a member.² An explanation or an *account* of the event – literally the count of possibilities of which the event is thought to be a ‘realization’ – is always given after the event, in what might be called a *backward narrative*.

For this reason, such events are neither ‘improbable’ nor ‘extremely improbable’. They are not even ‘impossible’ – what probability theory characterizes as events of measure 0. They are literally immeasurable. I call them *im-possible*, to emphasize the fact that they are external to the whole regime of possibility. There is no common measure by which the range of possibilities of which they will become members when they take place can be joined with the previous range. The future world that they bring about is radically incompatible with the present one. Hence we cannot, for any time present, conditionalize on their happening as if they were new evidence on which we could rely to update our probability measure. It is not a matter of updating probability but of updating the whole universe of possibilities. The new universe cannot be re-immersed in the old one as a part that used to receive zero measure or zero probability, because the two universes are literally incompatible with each other. They cannot form a union set.

The radically-emergent event (or the event, for short) takes place in a realm that precedes the construction of probability. Probability requires as preliminary that the set of possibilities be identified; and the set of possibilities, in turn, requires an established reality after which the possibilities are modeled. The possible doesn’t precede the real event or announce it, as the paradigm of prevision and, typically, probability theory want us to believe, but it is a fabrication made after the real has taken place, a reduplication of the existing real combined with a slight and unimaginative variation.³ We mentally change a few selected items in the present real – what we call ‘states of affairs’ or ‘states of the world’ – and we trick ourselves into believing that those variations represent alternative possibilities that could have been equally realized, one of which, the present real, happened to become actual.

The real is an altogether different matter than the possible, without any possible communication or mediation between the two. What lies outside possibility or beyond the range of possibilities – literally, the im-possible, or the event – is real, because the procedure whereby we fabricate the possible out of the real is always incomplete and deficient and falls short of the real. What comes as a surprise to our imagined possibilities and shakes them completely is real. The radically-emergent event is real for this reason. It is not unpredictable because of its low probability but because it wasn't imaginable beforehand. It wasn't part of a list of possible states to which probability – high or low, it doesn't matter which – was assigned. Radical contingency, or the event, shakes the range of possibilities and updates a whole new world, which may be incompatible with the previous one. Historic events are history-changing.

After the event, the world is fundamentally different from before. To repeat, the new states of the world are not a subset of the existing set of possible states which initially received very low or even zero probability. The new states of the world are *incompatible* with the previous ones, in the sense that the set-theoretic union of the old and new ranges of states of the world makes no physical sense. A genuine Black Swan event (Taleb cites the example of 9/11) triggers a much more severe revision of our model of reality than a revision of probability.⁴ It changes the whole universe of possibilities, obviously expanding it. Crucially, however, this expansion is not incremental and Bayesian theory is of no help. For instance, Taleb cannot reason counterfactually and call a 'hero' the congressman who would have passed the resolution, in another possible world on 10 September 2001, that cockpit doors should be bulletproof and should remain locked. The world following 9/11 is simply incompatible with the world that we knew before.

Possibilities are not just abstract elements that we may enjoy grouping anyway we like. Recognizing possibilities is the first step of *objectivation*; and for this reason it is constrained by its purpose. Our language and science, as models of reality, are such that we extract invariants from the flux of experience, and thus form the notion of an object and of its properties. In so doing, we start forming *expectations* about the object and its properties; for example, how the object might look if seen from another angle. We invent experimentation and therefore possibilities, for whoever speaks of expectations thinks of possibilities. Most importantly, we imagine counterfactuals: how an object would behave if, contrary to fact, it were subject to such and such experiment. As many philosophers of science and metaphysicians have observed, modalities and dispositional terms reside in language, not in reality.⁵ This doesn't make them less objective, mind you, as it is precisely the construction of the object that we are talking about. However, they might not be real.

Reality is not necessarily coincident with objectivity. The most famous counterexample is quantum mechanics. Quantum reality is a patent refutation of the claim that nature has to conform to our objectivist language, or that realism is exhausted by objectivist realism. To experiment and measure in quantum mechanics, the sentient beings that we are have no choice but to set the stage of objectivation. Experiments have to be designed in such a way that the possible states of some observable will obtain with the known quantum probabilities. However, the irreversible steps that we take in setting up this procedure make it impossible concomitantly to measure a conjugate variable, or an observable whose range of possible states is incompatible with the present one. It is not as if

we were photographing an object from different angles. Because the ranges of possibilities are not conjoinable within a comprehensive range of possibilities, there is no subsistent object lying behind, of which it could be said that it would behave in such a way if a different experiment were conducted. The quantum violation of Bell's inequalities is independent of theory or formalism – it is perfectly real and empirical – yet, it challenges our ingrained view of the existence of an object, which is supposed to be subject to our gaze and experimentation. The notion of 'object' may, after all, just be theory; and we may have to recognize that objectivist reality is an invention.

Even so, there is definitely a mind-independent reality out there that quantum mechanics is addressing. There is something going on at the deeper level where it is not yet decided what observable shall be measured and what objective behavior (i.e. acquiring a certain state or exhibiting a certain property with a certain probability) shall be manifested. The formalism that deals with this reality in quantum mechanics is the wave function. Unsurprisingly, the wave function is a *meta-probabilistic predictive tool*; that is to say, it articulates whole ranges of possibilities that are incompatible with one another. A higher-level probabilistic logic has to deal with such incompatible contexts (or ranges of possibility). Typically, the ortho-algebras of quantum mechanics are one such. Kolmogorov probability and its measurable sets are only a sub-algebra of the quantum mechanical meta-probabilistic calculus.⁶

The case of quantum mechanics is a criticism of the notion of possibility and a proof of its limitation vis-à-vis reality. The existence of a level of reality in which we have to conceive of such a thing as incompatible ranges of possibilities is, to us, a pressing indication of the existence of a level of reality in which we may conceive of *no range of possibilities at all*. If probability is too local and parochial a notion to 'predict' reality in quantum mechanics, why should we insist that reality should be predicted by probability in other domains of reality, typically the event? Maybe prediction is no longer the adequate word, here.

The future doesn't consist of future possibilities. The future is real, when possibility, as we said, is only a fabrication made up *after* the real. The real future (as opposed to our toy-idea of a future) is made up of *events*, which emerge out of nothing that may anticipate them. Such events are real and create the possibilities that 'will have led' to them.

Reality always exceeds fiction. We encounter the real without previous warning. We are made aware of the event and of the world that this event brings about, and then go looking for a partition of that world into 'states of affairs' or 'states of the world'. This conceptualization of the real is only a model that is derived from the real itself. Only after the states of the world are identified do we call them possibilities and retroject them into the past so as to narrate a nice story about how the event might have come about, or about the possibilities that will have led to it. By extrapolation, we project those possibilities into the future and imagine that the future world will be no more than a *variation* of those identified possibilities.

Contingency is real. Even though the present world is actual and is no longer a 'possibility', it is definitely still contingent; that is, *it could have been otherwise*. The future world is no different in this regard. It is as real and contingent as the present world, except that it is not yet actual. Essentially, it too 'could have been otherwise', except that we don't actually know 'otherwise than what'. The

future world lacks the actual reference against which its contingency can be measured and therefore relativized. Precisely, the whole trick is to be able to perceive its contingency before distinguishing the different possibilities or even *identifying* its future state. Only in this way could the future be real.

The change of perspective we are calling for, here, is that we be able to appreciate the fact that the future world *can be different* before perceiving different *from what*. Because it is not yet actualized, it lacks the partition of states that would allow us to frame its difference as a variation of states. This contingency, which is not measured relatively to states, is called *absolute contingency*. The passage of time and our temporal recognition of possibilities are incidental on it.

The quantum mechanical wave function evolves deterministically in time according to Schrödinger's equation. This evolution is only a rescaling and nothing, no event, happens in the interval. On the plane of events, the time evolution of the wave function is thus reduced to null. For something to happen – an experiment, a measurement – an observable has to be selected completely contingently, by the experimenter, and only then something *objective* can be observed: a certain object (manifested now as particle now as wave) acquiring some property with some probability. The point is that the underlying reality (which is what the wave function stands for) is not objective; it is pre-objective.

One can be a realist in quantum mechanics (i.e. believe in this underlying reality), yet not ultimately believe in objects bearing stable and re-identifiable properties, or in objects that lend themselves to counterfactual reasoning. The point of the relative incompatibility of ranges of possibility is indicative, to my eyes, of a point of 'absolute incompatibility' of possibility. It says that reality, or absolute contingency, is categorically incompatible with possibility; that it is another category altogether.

To repeat: that reality should confront us, in quantum mechanics, with a background which is superior to possibility, in the sense that incompatible ranges of possibilities can be articulated against that background, is, to me, an indication that reality is always superior to possibility *in the cases that matter*; that is, in the case of the event or the case of absolute contingency. When it really *matters* and we are not merely strolling around within the perfectly charted permutations that some statistics has secured for us to play around with, we can only speak from a void of possibilities.

Now, the event of the experiment or the measurement, in quantum mechanics, was only punctual. Real time, as bearer of the event (not as the clock of deterministic evolution of the wave function), was thus reduced to a single point. My question: How *to move* that point, or rather drag it by force, along a line? How to *continue* the time of the one-time event; or again, how to turn the one-time event of quantum mechanics into a process? How to conceive of a process of continual updating of radically incompatible ranges of possibility? What higher-level probability can govern this process?

This unprocessable process is simply history and my claim is that the marketplace is one spot where we can stand and accompany history. The event is a case of rupture of the probabilistic clock; for this reason, it is outside time; it is a radical discontinuity. Yet, the demand is to find a place where this discontinuity can be continued: literally the place from which *to write* history (for lack of a better word).

1.2 A new matter

My only intention in discussing quantum mechanics is to point to a level, in physical reality, which is in excess of Kolmogorovian probability, hence to point to an example or a precedent of the criticism of probability. It is not my intention to apply the formalism of quantum mechanics to the market or to history. My point of rupture with sets and enumeration of states remains the event, in the sense that the event belongs to nothing that exists before, to no previous situation or ontology. My point is that the void of possibilities, which is the only ‘thing’ that the emergence of the event can possibly lean against, can be filled with the market as medium of the event and can take place in the marketplace.

History, made up of events, is the absolutely unpredictable thing, yet I argue that there is a place – the marketplace – in which one can simultaneously be subject to history and to the event yet be an original author or maker. This is not mysticism but a new metaphysics, of which we might later have to define the corresponding physics and the corresponding matter/geometry.

We can only write after the event. It is much simpler to say of an actual thing, or of an event that has already taken place, that it is absolutely contingent than it is to say it of a future event. Somehow, the non-existence of the future event forces us to think of its contingency via possibility. We have to think of the future event as one among several possibilities one of which will come to pass, and we get its contingency derivatively on possibility. Yet, we know this is wrong because contingency is supposed to come first to possibility and not to depend on time, for that matter; that is, on the fact that the event is future or past.

As a matter of fact, possibility is wrong even with regard to necessity. To define a necessary thing – as is usually done in extensional modal logic – as a thing that obtains in all possible worlds is not a guarantee against the fact that that thing *might only happen* to obtain in the totality of these worlds, simply because we have reviewed every single world and observed that that thing happened to be there. In other words, there is no guarantee, under this extensional logic, that it is not contingent that the thing is (or seems to be) necessary. Michael Jubien argues that necessity should, on the contrary, be conceived as an intrinsic property of the necessary thing and not as an external variable ranging over a given set of possible worlds.⁷ Intensional logic rather than extensional. Since contingency is the opposite of necessity, contingency too has to be conceived intrinsically to the contingent thing or event and not to depend on sets of possibilities. For this reason, I doubt that any form of probability could be of any help – so long, that is, as probability is associated with extensional set theory.

We argue in this book that the reality of contingency (the one that escapes and surpasses possibility) is made of the same fabric, or matter, as the reality of price. By the very nature of the event and of radical contingency, we will be engaging with the future and we will argue that the financial market, which consists of the pricing and trading of financial futures and derivatives, also known as contingent claims, is the *technology of the future*. It admits of the future, therefore of the event, as its matter.

All markets are made of prices, but prices are what they are only insofar as they move and their future dynamics is *already* part of the present market transactions, through the trading of derivatives

written on them. Prices have no meaning and no being without price volatility, according to us, and, to repeat our basic tenet, there is no such thing as a market in which underlying and derivative do not trade alongside each other. This equalization – according to which the underlying trades at a price and the derivative that is written on it is not valued probabilistically, in a different realm as it were, as the stochastic nature of the underlying price seems irresistibly to suggest, but trades at a price too – is the reason why contingent claims will admit of prices indistinctly and nonhierarchically, in a logic which ultimately defeats possibility and stochastic structure and connects the writing of the contingent clause directly with the exchange of the contingent claim. (The market price is literally the translation of writing.) This is the logic of *recalibration*, whereby the probabilistic model is only a fleeting (although necessary) tool and the ultimate goal of probabilistic valuation is the pricing technology.

The contingent claim is valued only inasmuch as it will trade in the market and inasmuch as its theoretical value will turn into a price. Typically, the traded price will vary from the theoretical value. Theoretical value is by definition derived; it is a function that depends on underlying state variables – the market price of the underlying asset – and on key theoretical parameters such as volatility. Yet, the purpose of trading the derivative is that its price shall become an independent state variable from which we typically *imply* volatility by inverting the valuation formula (implied volatility) and on which derivatives of a higher level of complexity shall be written and traded in their turn.

Recalibration of the valuation model is not triggered by an unexpected change of volatility (how could instantaneous volatility and its instantaneous changes be observable anyway?) but typically by a change of the market price of the derivative that is not in line with the prescription of the model. This is acknowledging the fact that the derivative price has become an independent market variable. For this reason, recalibration can be recognized as definitional of the market and as characteristic of the gap between pricing technology and valuation theory. Recalibration should no longer be perceived negatively, as an imperfection or accident or failure of the probabilistic model, but positively as a new matter calling for a new metaphysics.

We needed the modern extension of the market in its most sophisticated – the derivatives market – in order to understand what the market originally means. The genesis of the market that is proposed in this book is, of course, different from its historical-empirical genesis. On the other hand, if recalibration is the new matter and defines the reality of the market, we are inexorably led to ask: ‘The recalibration of what?’ The probabilistic model that is being recalibrated is thus a constituent part of our reality, or at least of its prototype.

‘Reality’, ‘realization’, ‘possibility’, ‘event’ are our key terms and concepts, but they are also the constituent terms and concepts of probability theory. Since probability theory is at the basis of derivative valuation theory, there will be no escape, in a book that wishes to introduce the reality and even the genesis of the market by contradistinction to the reality of possibility and probability, from first taking a tour in probability theory. As a matter of fact, probability, although necessary as a tool, will appear as a false detour when the question of reality and of what really constitutes matter is asked. The matter of statistics will attract us back instead (yes, in the sense of gravity and ground) and the suggestion will be that the turn to the reality of the market should be taken right

from that ground, as a radical alternative to statistics and to the corresponding *matter*, by bypassing probability and its relative frames of reference.

1.3 The infinity of markets

We said that there were things (a level of reality) that are literally *impossible* from the point of view and register of possibility. Just as it is impossible, in quantum mechanics, to measure conjugate variables in the same context and to join together the ranges of possible states of the corresponding observables, and just as it is impossible to reduce the purely contingent event to the register of possibility and to identify beforehand the range of possibilities (or states of the world) of which the event is supposed to be the realization, recalibration is im-possible and cannot be accounted for in possibility. To make sense of it, you have to bypass possibility. Relatively to the regime of possibility and the corresponding hierarchy between underlying state and derivative value, the market is recalibration. However, in and by itself and on absolute terms, we see the market as the *medium of contingency*, where the contingent claim is now considered absolutely and no longer viewed as derivative on its underlying, or as living only in the trees of possibility or in probability theory.

Probability and rational expectations essentially address the market as a fixed-point problem. The actions determine the probability distributions and the probability distributions determine the expectations – hence, the actions. However, when the solution is reached, the market is precisely left outside, for when everything is fixed (precisely!) and settled, there is no need for a derivatives market.

Thus, it seems that if we wish to start with the market and stick with its matter, we have to turn the starting point of probability precisely into a non-starter. How to explode the schema of possibility and the sample space right from the beginning? How to make sure that probability is at once reincorporated into the sample space on which it was only supposed to be exogenously written? How to make sure that the probability distribution of the roulette wheel becomes part of the wheel, or that the probabilities of the faces become part of the dice-throw? Answer: By inventing the exchange or the market. Simply, let us make it so that the ‘random generator’ is of the same nature and subsists in the same homogeneous medium as the probabilities supposed to pertain to it. Our speculation is that price and the market can be implicitly redefined as the solution of *this* ‘fixed-point’ problem – which is, in essence, an anti-fixed-point problem; it is a full *aleatory point* (as Deleuze would say⁸) or turning the impossibility of possibility into positive matter.

The game of dice or roulette is not really a perfect game. It is not perfectly ‘symmetrical’ or ‘equiprobable’ (in a different sense of symmetry than the usual one). It is not a ‘fixed point’ of randomness. There is dissymmetry between the throw, which is an abstraction or the moment we leave matter in order to come up with one of the outcomes of the multiple, and the apparatus, which is material and supposed to hold the fixed probabilities. How can we abstract from the apparatus and make it so that the throw is total – that in the throw, while the die is in the air, it is not only the face that is undecided but the probability distribution of the faces; and it is not the case that the probability distribution is itself thrown and has to be decided randomly (in a throw of higher level)

between alternative distributions. The distribution law (second-order) is itself also thrown; so on until infinity.

At no stage do we want a material entity or an apparatus finally to hold. This is the total throw, total in the sense that there is nothing but the throw. How can it ever fall back to the ground? It shouldn't, because there is no matter to bring it down. It shouldn't fall to the ground and recoup this other dissymmetry (the dissymmetry of time and of breaking the game, the dissymmetry of going back to square one). It has to come back from the other end (or face), and this means that a new ground or floor, a new matter, must be created, onto which the die falls. (There are two materials at work in chance: the material apparatus, holding the distribution, and the material ground which stops it.) To have the floor not stop the throw is to have it follow the throw always (eternal return in the sense of no return). Better, the floor is the place of origin of the throw: the market, the trading of the underlying and all its derivatives.

Once the 'random generator' is understood as a tradable underlying, all derivatives of all grades of complexity written over it (or all probabilities of all kinds of lotteries written over it) have to trade, too. Suddenly, there exists a material medium, where the 'long run' or statistics no longer make sense, and where we no longer care that the price of a contingent claim might have anything to do with the underlying snaking away to maturity and hitting the strike with a given probability distribution.

Trading in the market is different from gambling. Randomness is here produced by the very thing that will suck the lottery tickets into higher-order randomness – by the exchange. Because of the exchange, the next tick of the underlying asset price cannot be predictable; for otherwise, the price wouldn't be next but would be now. The reason is money (arbitrage) and the result is randomness at any scale, whose paradigmatic representation is Brownian motion. However, the identification of a Brownian diffusion coefficient makes the price of the derivative, which is written on the underlying asset, predictable. Yet, the derivative is supposed to be traded and exchanged in its own right, no less originally than the underlying. By comparison, the casino doesn't put to trading the probability distributions of its roulette wheels.

We aren't suggesting the existence of feedback; it is not as though the derivative price had an influence on the underlying process and disturbed its diffusion coefficient. Rather, the question is: Who gave us the Brownian diffusion coefficient to begin with? Gambling is linked with the long run and with breaking even in the long run. From this, objective (frequentist) probability gets its meaning. However, this isn't the right point of view to compare with the market. The casino and the insurance company do not gamble; they do not engage in the same 'market' or medium that they offer to the public. They are the institutions corresponding to the length of time and to the stretch of space that it has taken the statistics to stabilize and become institutionalized. There is no real time involved in their procedure, but only false time. There is an ex-post accounting equation that is falsely retrojected as ex-ante.

Now imagine a long run that is shrunk to zero by the very fractal property of Brownian motion. However small the time interval in which Brownian motion is given, one can consider stable statistics and apply the ergodic theorem in that interval in order to establish the break-even of the self-financing dynamic trading strategy which involves rebalancing one's position in the underlying

asset as a hedge against the final payoff of the derivative written in it. Since the derivative is also trading, it is not really its final payoff that we should be talking about, but its next payoff, *which is its market price*. The existence of the market price guarantees that we can get in and out of the trade at leisure. It doesn't help to value the next payoff in actuarial fashion, because this is local and there is no alternative trajectory (no break-even in the long run).

Insurance companies and actuarial valuation came historically before BSM. Their way of 'getting out of the trade' is through the whole history and the whole business. The insurance company doesn't really look at the single trade. The premium that it charges locally presupposes all the others. It cannot move on to the next interval: it is stuck there, under the assumption of a whole population to come.

In the market, by contrast, the premium of the contingent claim (or derivative) is adjusted in order to cover that specific contingency. How? Precisely something is available, which is correlated with the final outcome: the underlying asset price itself. Here, the underlying asset is both the source of randomness *and* tradable. Self-financing dynamic strategies, whose presupposition is the market in which *both* the underlying and the derivative trade, replace the whole static, monumental account of the insurance company.

This works because of Brownian motion. One can always subdivide the interval in order that the strategy breaks even almost surely.⁹ Neither the underlying price nor the derivative price is predictable or derivable in any interval, but their relation is *deterministic* (pending the total unpredictability of the coefficients of the equation).

Where does that leave statistics and the corresponding reality? The problem has become so local (and local here means that the value of the derivative *is there* and that it is a price; local means the whole market) that the ground shifts at this point and a whole new reality takes over. We have long forgotten the archaeology of thought lying underneath the formalism of probability and Brownian motion. Instead, we have been distracted by mere mathematical subtleties.

Brownian motion is essentially tied up with the exchange. But is Brownian motion (or any other identifiable stochastic process, for that matter) *really* adequate for the exchange? As soon as we identify a quantitative process, the value of some derivative becomes imposed. This must be a wrong sequence. Let us define *hyper-Brownian motion* by the tradability (hence the unpredictability of prices) of all classes of derivatives. There is no statistics left in this interdiction of trial and of time. The whole marketplace bursts through the total throw, thus redefining *place* (the place of the marketplace) in contradistinction to time (the time of the temporal process).

Correlatively, there is a new meaning of repetition. It is no longer the repetition of an experiment in time that we are talking about, but repetition on the spot, in the pit. The distinctive feature of contingency is that it may be so repeated (almost a definition, here). If there is no repetition in time, the probability formalism can no longer be interpreted in the domain of statistics – there is no longer any realization, and no reality in that sense.

The place that was opened by Brownian motion is the whole place at once. This is our new interpretation of Brownian motion; this is how we diverge from the statistical paradigm, and go,

in the formalism, in the other direction. The place is sensitive to the whole event. All the derivatives prices have to pour into it.

1.4 The end of statistics

How to make sense of the place, when no logic seems to rule it? Either we assume the existence of prices of everything and the valuation models are spurious, or the models are essential in some way. In what way could they be essential? They all live accidentally in time, hence what they achieve in theory and how they live is an accident. This is not what they are essential for. What they do, essentially, is to be stuck in the retrieving moment, when the *next* derivative price takes over the meaning of statistics. This is not a place to *be*. This is a place in which to remain and to be the remains, to be a ruin. Valuation models live really on the other face of the event, which is the ruin. They are not essential but they *remain* essential.

The real question is: What is the real, radical alternative to statistics? How can we imagine it? The end of statistics is the stochastic integral: the time interval shrinking to zero while the self-financing strategy establishes an ex-post account and break-even in the infinite long run. How to extrapolate that end, from the other end? The elements are already here, the tradability of the underlying and the self-financing dynamic strategy, as well as the writing of the derivative (which is just the other side of contingency).

The statistical coefficients used to be inferred from a whole temporal series. Now, Brownian motion and the stochastic integral make it so that we can dive into a single point. This is the end of statistics. The limit is that the coefficients can now be inferred instantaneously, as the trade and the break-even are now instantaneous. To go beyond the time limit: isn't it to say that the inference now comes from the place itself, from the price of the derivative itself? And so, that there is no inference of the statistical coefficient anymore, because every inference will be destroyed by the next? (And what can be even more local than the instant, other than the *locus* itself and place itself?)

Isn't this the total liberation from statistics, hence the view of its end? Statistics is helpless in front of the event anyway. A statistical distribution cannot be inferred when the event is at stake. Now, the place takes over and restores the meaning of the event, which essentially *takes place*. The limit of breaking even in a statistical series, or in a run of some kind, is to break even by inversion into the market. The market occupies and fills both ends. The place is created in between: no gambling, but a new sense of staying or dwelling or, rather, frequenting. One doesn't dwell in a gambling place or live in it; one frequents it. The same change of the order of time is needed in order to understand how frequenting this place makes sense.

We need to diverge, right from the heart of the argument of local dynamic replication, right from the heart of the stochastic integral (which, in a remarkable inversion of the law of large numbers and thanks to the infinity that is contained in a continuity, can now – must now – precede the stochastic differential), right from the inceptive event of post-BSM derivatives markets, into a direction that is radically alternative to statistics.

Derivatives markets are literally the progeny of the mathematical formalism of probability theory in its finest and subtlest. This is why they have nothing to do with probability – either Bayesian or frequentist. They are a new infinity that *begins* in David Deutsch's sense of the word,¹⁰ an infinity separated from the reality that went before the formalism (i.e. probability in the traditional sense of statistics) by an unbreakable and impenetrable wall – precisely the wall of the formalism itself, which is a vacuum of interpretation and calls for a new interpretation or a new reality. They are precisely a progeny of the formalism that was not conceivable or imaginable before.

The Vision Ahead

2.1 Regime-switching model

This book attempts to explain how any derivative pricing tool worthy of that name should work in the market. It attempts to understand what the market *is*, through the way in which the pricing tool works in the market. Recall that the pricing technology is both at one with the immemorial and immanent matter of the market and transcendent relative to the market – it is an a priori and apodictic stance – because of the element of design that is inherent in the technology. Recall that this point of view of the technology is what grants us access to the definition of the market. How a derivative pricing tool should work in the market is, first, by using probability theory and dynamic replication and, second, by recognizing that the matter that it is dealing with, the market of contingent claims, is precisely lodged outside any framing by probability theory or any fixed framework of states of the world. The regime-switching model will be the perfect illustration of this paradox when used as a derivative pricing technology and, accordingly, it will be our model for any derivative pricing technology.

Probability theory, Brownian motion, the BSM dynamic replication argument and the concept of implied volatility (which is how the BSM theory is applied to derivatives in the market and how it becomes a derivative pricing technology) form a historic and significant sequence which motivates writing this book. The reason why the regime-switching pricing model is the paradigmatic tool which summarizes and materializes this sequence (thus summing it up in what I call a technology, rather than theory) and why this book takes it up is that it precisely generalizes implied volatility. It builds precisely on the step outside probability theory and derivative valuation theory that implied volatility means, and it recognizes *matter* in that exit. More exactly, as we shall see, it gives its full material meaning to recalibration and recognizes the recalibration process as the true market process. So, it is not the philosophy that has imposed itself on the technology, but it is the opposite. It is the design, the purpose (and beyond that, the essence of the derivative pricing technology) that has changed the way that I think derivatives books (or more generally, books designed to illuminate the market from the point of view of the pricing of contingent claims) should be written.

Of course, the tool was not designed from scratch or by chance. The main insight is due to the designer of the tool – my partner in ITO 33, Philippe Henrotte – whose credo, which I share, is that derivative pricing and hedging as technology makes sense only to the extent that it is calibrated and recalibrated to the market of derivatives. *From prices to prices, with probability theory, stochastic processes and dynamic replication acting only as an intermediary step, or inner episode of the tool.*

It is sufficient to give it a minute of thought to realize that this new logic ('from prices to prices'), whose other name is recalibration, is incompatible with probability theory and the fixed and delimited states of the world that lie at its foundation. Ever since he graduated as a PhD in Finance and through his teaching years as professor of finance at HEC (École des Hautes Études Commerciales de Paris), the French business school, Philippe's main criticism against the Arrow-Debreu paradigm was that prices that come out as the result of probabilistic computations based on 'abstract' states of the world, according to Arrow-Debreu, will themselves constitute new states of the world (of the market). Hence a contradiction, or at least a perpetual breaching or non-closure of the ontological circle.

The regime-switching model has the unique feature that the regimes have no particular given names (of variables that are specifically stochastic – such as stochastic volatility, stochastic jumps, hazard rates, and so on) but adopt only the names that calibration assigns to them. Every option pricing specialist agrees today that models of the underlying dynamics must include stochastic volatility and jumps in the underlying price. Stochastic volatility and correlation between the underlying price and volatility are needed in order to explain deviations from BSM in longer-dated options, whose value is precisely sensitive to changes in volatility; and jumps are needed in order to explain deviations from BSM in shorter-dated out-of-the-money options, whose value, when they are so close to their expiration, no longer quite depends on volatility but is very sensitive to jumps instead. Indeed, the value of the deep-out-of-the-money option can vary massively depending on whether a jump in the underlying price can trigger it unexpectedly, when it was supposed to expire worthless. These deviations of options prices from BSM are known as the *smile problem* in the industry. Usually, the parameters of the stochastic volatility/jump-diffusion model are calibrated from the instant options prices, or the option volatility surface, or again the smile.

The regime-switching model is the recognition that the parameters will stochastically change, every time they are recalibrated. Hence, the 'real' model should be a collection of stochastic volatility/jump-diffusion models of different structure, each corresponding to a different regime. A particular regime is not characterized by a particular value of volatility or a particular size or frequency of jumps, but by a whole combination of such values. Moreover, two different regimes of volatility and jumps can be merged together and redefined as a single regime, thus preparing for a switching between two such superstructures. As a consequence, it will always be indistinguishable whether a given instance of the regime-switching model is a case of calibration or of recalibration (see Appendix A).

Now, let us try and think that the given sequence of recalibration is only an 'accident' of time (because we all happen accidentally to live in time) and doesn't reflect the real essence of the market, which is that no contingent claim, no matter how complex, should be redundant. Let us speculate, in other words, that the real process doesn't take place in time but on the 'spot' (what I like to call 'in

place'), and then the structure of the market, or its real process, will have no other place to go but into a dimension that has to be real although it is not actual – because actualization is what gives the actual prices we calibrate against in time.

This combination of reality and non-actuality is the stuff the future event is made of and the new matter that the book wishes to introduce. As I said, this is a metaphysical discovery or synthesis, so I guess it can have no causal agency of any sort and cannot change anything, except the way we think. It certainly cannot change the market, the tool, or the world, for that matter. What definitely changes, after the book, is the schema of stable, delimited and identified states of the world (i.e. the basis of probabilistic thought, and perhaps also of objectivist thinking in general) which should be replaced, I speculate, by the medium of the market and prices as translators of contingent claims.

The usual criticism of BSM is that it assumes Brownian motion with constant volatility. Constant volatility is unrealistic, of course, but what's even worse is that the market is complete as a consequence and derivatives are redundant. Assuming stochastic volatility and jumps makes the market incomplete all right and, as a consequence, derivatives are no longer perfectly replicable by the underlying; however, in reality they should no longer trade and admit of market prices for, if they did, their price processes could be used to complete the market. Since the whole pricing technology following BSM has the trading of derivatives as its purpose, I prefer as an alternative and less hypocritical characterization of the market in that register that *all derivatives should trade but that none should be redundant*. The only way to make that work is to forgo the structure of states of the world.

The book is a reflection on possibility, probability, state, time, and so on as metaphysical categories, and I propose replacing them with contingent claims, prices, money, and market dynamics as metaphysical categories of their own. You can see how far I place myself from the sociologists, behaviorists, econometricians, or indeed financial engineers and analysts of any following. I just deal with thought as my primary matter.

2.2 Recalibration

As recalibration is the name of the game, the book aims to give it a firm theoretical foundation. Recalibration is what all derivatives traders do when using derivative pricing tools and is precisely what derivative valuation theory, or academia at large, cannot account for. Derivative valuation theory deals only with stochastic processes and stochastic control and knows nothing of the concept of market price, even less so of the concept of implied volatility. The whole book can even be described as just an attempt to make sense of the BSM implied volatility (which is the simplest instance of calibration and recalibration). It is in line with the project I first started in *The Blank Swan*, as the following quote from *The Blank Swan* bears witness:

The concept of *implied volatility*, which is perhaps the most popular concept among derivative traders, is indeed the whole reason why I am trying to think this inversion in the widest scope that I can, and why the whole purpose of the book might be summarized in trying to find what

the derivative pricing theory, or at least the derivative pricing technology, could be, *given* the derivatives market, and not to try to build, like in all existing books, a derivative valuation theory, given the theoretical assumption of a stochastic process (and no derivatives market).¹

What the expression ‘given the derivatives market’ means is that a pricing tool is only as good as its calibration to existing derivatives prices and is only as useful as the subsequent trading *in the market* of the prices it outputs. *From prices to prices*, as my motto goes, *with theoretical valuation serving only as an inner episode of the tool*. This supplementation of valuation theory with a market that flanks it can only make sense as a *technology*. It is no longer a theory.

So, to be correct, we shouldn’t be speaking of ‘giving a firm theoretical foundation’ to the practice of recalibration but perhaps only a meta-theoretical or philosophical account – unless, that is, we made the market itself an integral part of this extended ‘pricing theory’ and no longer considered it as the exception or the accident that always ruins the probabilistic model! How to bypass probability theory and go directly from contingent claims (which are the only stuff the market is made of) to their prices (which should be the translation of the contingent claims) is where the book starts becoming ambitious, metaphysically (see Part III).

Now, supposing the metaphysical doctrine is granted, according to which contingency has to be mediated absolutely, without bringing in the relative frame of reference of possibilities and profiting only from the new matter that fills the void of possibilities, or that is overlaid above the successive failures of each and every schema of possibility; supposing, in other words, that the market is the place outside of time and time series that future contingent events inhabit and that trading in this market (via recalibration) is better than prediction – for prediction can only evoke possibilities and, therefore, come second, or maybe prediction should not even exist – we wonder how we could put this metaphysical doctrine to usage; how we, human beings who cannot escape the time-bound world, could experience this advantage of the market over prediction. Or maybe we never will and the answer will simply be that recalibration is all that we will ever be able to do when we are hit by contingent events.

Indeed, metaphysics doesn’t have physical agency and can only be put to usage in thought. However, I could have summarized my whole metaphysical elaboration in a single, down-to-earth and almost nonmetaphysical observation. Although they may lose money on the particular day that the pure contingent event hits the market (depending on their position), real market-makers make money afterwards just by virtue of widening their bid-and-ask spreads and the explosion of trading volumes. Everybody thinks this is a quite trivial matter. However, if we think about it for a while without any preconception, we realize that this phenomenon involves all the mysterious metaphysical categories we have been considering, namely, temporal succession, money, price, trading, the event, and so on and that by arranging them differently in thought, in a new inventive manner, we might end up with quite an original interpretation.

For instance, nobody said the market-marker’s profit and loss (P&L) should be accounted for chronologically, or at least day to day, and everybody agrees that successfully predicting an event in the market can only mean that you have made real money out of your prediction (nobody cares about

your private knowledge or private intuition). In other words, reasoning outside time and replacing a mysterious concept (prediction) with its only tangible and clear consequence as far as consistent work in the market is concerned (making money), we can safely declare that the market-maker has successfully predicted the event.

On the face of it, all this may sound like a mere substitution of words; however, it is not neutral, metaphysically. Indeed, what interests me is the reworking, from there, of the whole concept of the market as the medium of contingency and of money as the corresponding reworking of the concept of time.

Accordingly, the market shouldn't be thought of as a complex arena, filled with traders and information flows, in which informational efficiency or rationality can be measured and in which the conclusion is necessarily reached that the market will never settle in equilibrium but will always wander in chaotic dynamics. The market is not complex and shouldn't borrow any of the usual images of complex systems – the market as a living organism or as a neural network or as dynamic chaos. The market is simple. It is the place where contingent claims receive prices. To repeat, the book addresses a simple question: 'What is the meaning of implied volatility?' There is no knowledge in the market to begin with and there is no uncertainty. There is only trading. Implied volatility is a trading concept. My whole idea is that the regime-switching framework is the only answer to implied volatility or to recalibration. People have sought improper ways of generalizing BSM in front of the market. They thought they would generalize the underlying dynamics. Rather, it is implied volatility that needs to be generalized, not volatility.

Implied volatility means and implies stochastic implied volatility for the simple reason that the volatility you will imply next from the market option price will be different from the one you implied previously. In short, implied volatility is at once stochastic implied volatility – a paradox that challenges the ordinary logic of subject and predicate. Analysts wrongly thought that the volatility of the underlying should be made stochastic as a result; therefore they generalized the underlying dynamics of BSM to stochastic volatility models (Heston, and others).

In reality, what should be generalized and made stochastic is not volatility, but *implied volatility* or the fact that is implied by implied volatility; namely, that any parameter of any model of any complexity that we can think of will turn stochastic when it is implied. This is called the 'problem of recalibration'. This is the crucial problem of derivative pricing. It distinguishes derivative pricing technology from derivative valuation theory.

I claim that the regime-switching model is the (perhaps only) answer to the problem of recalibration, or to the problem of generalizing implied volatility (not volatility). The regime-switching model is more than just a model of the underlying dynamics. In any given fixed frame of reference, it relatively behaves like all other stochastic processes; however, the true (absolute) process that it reflects is the process of *non-redundant* differentiation of the successive derivative payoffs. This is not a temporal process. This is the real process of the market.

The *pricing process*, or the real process of the market which cannot be a temporal process, is such that no contingent claim should ever be redundant. In the market, we have price series, not time series.

Traditionally, the reaction has always been to make the underlying dynamics more complex in order to save the contingent claims from their redundant fate. The market is made incomplete by adding stochastic volatility or jumps to BSM. However, such a move doesn't recognize the scope of my criticism. I am not criticizing stochastic volatility or jump-diffusion, and so on, as such. These are very nice stochastic models, perfectly fit for derivative valuation. I am criticizing the insufficiency of *any* model of the underlying dynamics when confronted with the problem of recalibration, which is another word for derivative pricing. The real market dynamics is the dynamics of recalibration, not the dynamics of the underlying. There is no underlying and overlying in the market, no such hierarchy. All we have are contingent claims, all trading and admitting of prices at the same level.

2.3 Is probability necessary?

Recently, a famous economist spoke of a 25 percent probability of a double-dip in the economy by the end of the year. What does this 'probability' $p = 25\%$ mean? If I am not mistaken, because we are talking here of a single-case probability and not of one inferred from a statistical frequency, the metaphysical presupposition is that we are facing a certain number of possible worlds (or possible repetitions or rehearsals of history) $p\%$ of which contain the double-dip and $1 - p\%$ don't. Because the event is empirically a single case, we replace the thought of the statistical population in which the frequency of the event would have been empirically observed had it been repeatable and not single-case with the thought of an instantaneous 'statistical' population of alternative worlds, only a single one of which will be realized.

Contrary to the empirical population, these possible worlds are just an image in our heads whose sole purpose is privately to help us know what we mean when we talk of probability. Note that some authors, the so-called modal realists such as David Lewis, believe those possible worlds really exist metaphysically and are not mind dependent.² Others have suggested that those worlds neither really exist metaphysically nor are lodged in our heads; rather, they literally *are* the meaning of the word 'probability'; they are objective in the sense that meanings 'are not in the head' and are objective. This is the doctrine known as 'objective semantics'.³

In sum, all we have done is invent a word, 'probability', and invent more or less intricate ways of telling ourselves what we mean. De Finetti has proposed putting an end to the metaphysics of probability (his famous: 'Probability doesn't exist!') and has identified probability with the quote some *banker* is prepared to give you for buying the corresponding contingent claim.⁴ In this case, the contingent claim would pay off \$1 at the end of the year in case of a double dip and 0 otherwise, and would be worth \$0.25 today. Those still inclined to use the word 'probability' have equated this 'transaction' with the fact that the banker *believes* that the metaphysical probability we have described above (the one involving the possible worlds) is 25 percent. Put succinctly, those worlds don't really exist (so probability doesn't really exist); they only exist as a belief.

In fact, de Finetti has just stopped one step short of admitting the market of contingent claims, and of replacing all talk of probability with talk of price! We all know that a *single* banker doesn't exist. Only a crowd does. What stops all the other bankers from entering the game and auctioning

this contingent claim; that is, turning the quote into a price? This is our credo: From the moment the thought of the metaphysical possible worlds (or possible states of the world) is transformed into a *written contingent claim* which pays off in case of the event – in this lies, to our mind, de Finetti's crucial step – the market, the crowd, and hence the price are created. If you invite one banker, you invite them all. In reality, de Finetti was still attracted by the image of the possible worlds. However, because he very strongly felt those worlds shouldn't really exist, he had no choice but to interpose a *believer* in them.

Objection: But then, surely, the market consensus (i.e. the price) is some aggregate of individual beliefs? Answer: What do you mean by 'aggregate of beliefs'? Is this aggregate still a belief? If not, what is it? Using the word 'aggregate' immediately prompts the dynamic image of different beliefs entering into conflict in the market arena, and finally settling on the consensus. But why even bother with this image? I say forget the individuals and forget the beliefs. We know what the price of the double-dip contingent claim is at the end of the year. It is \$1 in case of double-dip, 0 otherwise. Let us then *define* the market as the medium in which contingent claims admit of prices *ahead of their maturity*. Better, let us define money as the piece of paper one unit of which you exchange against the contingent claim at maturity in case of a double dip and 0.25 unit of which you exchange against it ahead of maturity.

How can such a thing as money exist? We create it. How does the contingent claim exist? We write it (using the previously created notion of money in its written contract). How does the exchange place exist? As we have said, we just define it as the place where prices attach to the contingent claims. Why would the 'place' where probability is attached to states of the world be more legitimate and more readily available? Probability is a confused and conflicting concept; it is unsettled (literally lacking a proper dwelling place), not knowing whether it should attach 'objectively' to the event or only 'subjectively' to the mind of the believer. I still haven't seen the place where probability attaches to things or to minds. By contrast, I have seen the marketplace.

Moreover, there is homogeneity in the medium of prices, which is absent in the medium of probability. Both the \$1 price and the \$0.25 price are prices; the contingent claim is exchanged against money in both situations. By contrast, a probability of 1 and a probability of 0.25 point to different categories. The former characterizes reality and actuality; it is the realization of the event. The latter characterizes only a possibility. (Our whole metaphysical problem is to understand the relation, or the difference, between the possible and the real.)

I think the only reason why we have imagined states of the world and why we identify them clearly is to stage the *transition* between the possible and the real. I think the raw fact (what I call the 'massive and unsophisticated real') is that the world is contingent, which means that it is not necessary, which means that it could have been different. The unwarranted step that metaphysics then undertakes is to *identify* the different ways that the world could be, by arbitrarily isolating the so-called states of affairs or states of the world and varying them, and to make those alternative worlds coexist with the present one in the only place where such coexistence wouldn't violate the principle of excluded middle. This place is located one step back, at the imaginary moment when neither the present world nor its alternatives are supposed to be realized. To repeat, what is real is

the one and only present world and *what is also real* is its contingency. However, metaphysics has no other way of representing this than by stepping back into unreality. It stages the possible then it has to stage its realization.

Every day the world is real, and every day it could have been different. Metaphysics has no other way of chaining together these differences in time than through a tree of possibilities and probabilistic transitions. The contingency of the current world is referred back to the previous step when the world is just one among a certain number of possible worlds. Only one world is realized at a time and no two worlds can share the '1' of reality at a given time; so the difference is referred to numbers that are less than 1 (probabilities), which the different worlds can share at the previous step.

Now, what if we tried to make the picture more robust and no longer identified the possible variations of the present, massive, contingent world? What if we applied Occam's razor and admitted only that the present world *can be* different without explicating (enumerating) what other worlds it may *be*. In other words, instead of thinking that the verb *can be* is derivative on the verb *to be*, why don't we think the opposite? (Why can't contingency become the basic ontology?) How can we think of the succession between the one and only real (yet massively contingent) world and the one and only real (yet massively contingent) world that succeeds the previous one without mediating the succession by a transition between identifiable states?

When we write a contingent claim that pays off \$1 next year in case of a double dip, 0 otherwise, and a contingent claim that pays off \$1 next year in case of no double dip, 0 otherwise, we don't think of two different possible worlds. The two contingent claims coexist today in the same real world; and so will they tomorrow and next year. They both have '\$1' written on them, whereas two different possible worlds cannot have 'probability of 1' written on them simultaneously. The two contingent claims will certainly admit, today, of prices that are less than \$1. Otherwise, there would be arbitrage. On the other hand, nothing imposes that the prices must sum up to \$1 today (assuming zero interest rates). For all we know, nuclear war might break out next year (or any other contingency, such as all trades being cancelled, and so on), and the market might be 'discounting' this already. Or maybe their prices will sum up to \$1 today, and the arbitrageur who has bought the bundle of them will lose money in case of nuclear war. This is not a problem, as there is nothing wrong with losing money in the market. By contrast, if your range of previously identified states of the world shifts unexpectedly, probably you will lose money too; however, there will have been something wrong with the probability view.

I think the duality between (K, T) (strike and maturity of the contingent claim) and (S, t) (state of the world, time) – or, in other words, the duality between the writing of contingent claims (and ipso facto their market) and the fiction of possible states – was not available to the probabilistic thinkers when they thought of their possible worlds and devised their metaphysics. Probability is backwards, whereas price is forwards. If you value options in the (S, t) picture, you have to work backwards. You fix the future states of the world (under what authority or right?) and you roll backwards to price a single option. By contrast, you work forwards in the (K, T) picture. However, you don't move forwards in time or to another possible world. You price all options of different strikes and maturities (K, T) standing in the spot $(S = S_0, t = t_0)$.

So, the real world containing the contingent claims and their prices succeeds to the real world containing the same contingent claims and different prices. Or rather, it is the same world. There are no transitions, only price dynamics; that is to say, constant recalibration to the market of valuation models that internally respect the no arbitrage principle. But how is the market functioning? How is the price emerging? What is *the* model of the market? I am sure one can endlessly think of the inner workings of the market. However, what interests me, at the higher level, is the short-cut. What interests me is that the picture (contingent claim, price) should be more robust than the picture (state of the world probability). As I said, there is no need to identify states of the world underlying the prices, so long as the market ‘massively’ (i.e. indistinctly) takes care of the transition.

In this sequence of thought, I am not trying to explain what the market is. Rather, my derivation (and my motivation) can be reconstructed as follows. I want to express the contingency of the world; however, I want to do that while getting rid of possibility and the corresponding metaphysics. I then observe that printing money and writing contingent claims is the best way of ‘flattening’ the depth of possibility and the corresponding tree; that is, the best way of materializing the possible worlds and collapsing them into the only real one (at work, here, is a kind of magic that is called ‘writing’). In a last step, I look for the medium that can connect the present day with the maturity of the contingent claim. I call this the market.

We should try and deduce the market and the exchange place directly from the contingent claim, without even assuming the existence of market participants and of exchanging parties. Simply, while possibility needs *chronological time* in order to step back and forth, from the fictive alternative worlds to the realization of the real world, contingency needs *place*. Possibility is a sort of bond, or debt. We alienate the real from its reality in order to fabricate possibility and we are bound to redeem its reality by realizing the possibility. By contrast, the contingent claim is the *conversion of debt*.⁵ While debt is tied up to the debtor who can alone redeem it (unless debt is exchanged on the market, of course; but the market doesn’t exist yet; we are precisely in the process of creating it), the contingent claim is tied up to nobody in particular. Instead of a convergence to face value, it finds divergence and the open place where anybody *can be*. Hence the marketplace.

2.4 Price and probability

Price is not a probability or, equivalently, a mathematical expectation (as orthodox financial theory has it). As my philosophy is directed against the foundation set by Arrow-Debreu – that is, against the notion of basic states of the world and against the idea that basic securities that pay off \$1 in those states of the world should be the building blocks of the pricing theory – I reject the notion that the price of these things is the discounted (risk-neutral) probability of the corresponding state of the world.

Not that such securities shouldn’t trade when their payoff condition is well-defined (I mean when the corresponding state of the world is something tangible and not some *deus-ex-machina* ‘state of the economy’). But I need their price to be given by the market, not by probability.

And what is the basic, to this day unanswered, criticism of Arrow-Debreu? What is, in a nutshell, my basic argument against the identification of states? Simply, that the prices of those Arrow-Debreu securities, if and once you compute them by probability, become additional states of the world/market, because the purpose of the whole exercise is to *trade* them, not to value them.

In a nutshell, my philosophy tries to make sense of the fact which orthodox valuation theory cannot account for – the fact that the derivatives we are valuing have to be traded, therefore, have to diverge from their theoretical value. Herein, to my mind, lies the scandal of valuation theory (to be rejected) and the paradox of the pricing technology (to be philosophically accounted for).

My book aims to make sense of the process of *recalibration* of the pricing tool. Derivatives traders use probabilistic valuation models that are supposed to describe the dynamics of the underlying and its statistical properties once and for all, or at least until the expiry date of the derivative; yet, they *recalibrate* their model every day against the market prices of derivatives, thus inverting its usage and perverting its foundation. Indeed, the assumption of statistical regularity, which is the very basis of any probabilistic representation, is contradicted every day following such recalibration and the parameters of the model are changed every day. How can probability theory and identification of states stand in the market when, by pricing, trading and recalibrating every day, you are going to defeat the model every day, expand the states of the world, and ruin the probability calculation?

I discovered (yes, a true discovery) that the philosophy of the event and of contingency can alone account for this constant context-changing feature of the recalibration process. A superior representation of contingency and of time is needed, the one found in Deleuze's philosophy (and his interpretation of the Nietzschean dice-throw).⁶

Nobody should stop using probability as a tool. Arrow-Debreu prices are given by the market; this is why they are called 'prices'. And the only probability that should exist, to my mind, is what financial theory calls *risk-neutral probability*; that is, the one that precisely helps us interpolate derivatives prices from existing (i.e. trading on the spot) derivatives prices, pending the recalibration that will ruin the structure of states of the world underlying the passing probability computation.

What I reject is the metaphysical view and schema according to which probability is assigned to 'real' and subsistent (in the metaphysical sense) states of the world. To repeat, I believe that the metaphysical presupposition of econometrics and of orthodox financial theory – namely, the existence of a data generating process (DGP) underlying the market – is not true. Only instant prices (of derivatives and underlying alike) exist. That you should temporarily posit states of the world in order to express the risk-neutral probability is what I call 'using probability as a tool'. However, this is only for convenience. Philosophers call this view 'instrumentalism' or the belief that theories are not real, but mere instruments.

2.5 A new metaphysics

The pricing model that I defend in the book, the *regime-switching model*, is a probabilistic model like any other; therefore, any instance of it relies on states of the world. But the real killer is recalibration. The problem of recalibration is what drives me, *metaphysically*, to search for a

substitute to probability. As I said earlier, my book is a book of metaphysics. And the reason why I discuss the regime-switching model is that it sustains recalibration, or so I argue.

Metaphysics won't change the trader's trading habits, I know. However, I believe that it is, in the present instance and problem, the only way to answer the question: 'What is the market and what is price?'

My main insight is the following: Instead of thinking that the market is a complex exchange phenomenon (human, behavioral, what have you) that will never be tractably modeled by probability theory, why not take advantage of the necessity to recalibrate the derivative pricing tools in the market and *define* the market as this alternative medium where the process of recalibration (which exterminates probability) is the rule? Why not postulate that the market is a *simple*, not a composite. That it is the *medium of contingent claims*, and that it is such that contingent claims are never redundant?

In my metaphysics, I need the market, or the price (of any derivative), to be an entity that is always rebellious to a partitioning into states, and ultimately rebellious to redundancy. I do not define the price through the exchange. I do not consider the market as made up by human beings. My work is speculative philosophy. I am building a system of metaphysics.

In my view, the existence of contingent claims and of their market is a blessing for any thinker in literary theory who wishes to investigate writing as a process whose metaphysics is foreign to chronological time and even to biographical time. See the case of Pierre Menard.⁷ Indeed, the 'financial' entry point allows me to broach my topic (writing) from its sharpest angle, as it points my readers to a very material and pressing entity which directly affects their lives and which they will all agree does exist (the financial market), while not everyone agrees, by comparison, that writing (as such) is very material and that books (as such) materially exist and can change one's life.

Of course, everyone agrees that books empirically exist and that they can, on occasion, change the reader's life. What I mean to say is that the book, or generally the medium of writing (thus the market), exists in the sense that the *world* as a whole is said to exist, that the book (or the market) is in fact an *alternative* to the world and to its metaphysics altogether.

Indeed, my contention is that the market of contingent claims, when it is understood as a writing medium and not as a theatre of fixed possible states and probabilities (what philosophers call 'representation' precisely), evades the order of chronological time and its spatial correlate (the identifiable possible states) altogether. Metrical time and space are extensive dimensions that can only receive extensive variables. Price, by contrast, is an intensive variable.

From my first day as a trader on the floor, I was struck that something was going on, definitely not reducible to probability or to the metaphysical idea of a data generating process. Something singular, yet repeated. A constant Black Swan, so to speak. A constant revision and shift of the whole set of previously identifiable (if they ever were) states of the world.

The lesson I take from Taleb's book (*The Black Swan: The Impact of the Highly Improbable*) is that Black Swan events are shifts of the whole range of previously identifiable possible states. So, it is a misrepresentation to call them highly improbable, since probability is only defined relative to a fixed range of states of the world. Only quantum mechanics is able to offer a formalized predictive theory

that subsists at the meta-level where the ranges of possible states are not yet identified. The wave function is just a meta-probabilistic predictive tool. I am not saying that randomness in the market is quantum mechanical! My only interest is that such a criticism and surpassing of probability does exist, and is even provided by physics.

2.6 Absolute contingency

I am not the first to criticize the category of possibility. Neither is Taleb the first to invoke the backward narrative that is typical of Black Swans. To my knowledge, this criticism first came from Bergson, who is famous for saying that real events create their own causes. Deleuze and Badiou followed as philosophers of the event and contingency, and a recent brand of speculative philosophy (Quentin Meillassoux) even argues that, since contingent events escape the range and whole metric of possible states almost by definition, then there is no reason why we should not make contingency the absolute ‘matter’ – from which I deduce that we should define it positively and no longer only negatively relative to possibility.⁸ Why not, in fact, overturn the whole ontology and the whole order of thought and place contingency (the fact that a thing *can be* something or other or, as a matter of fact, that it *can not be* at all) before being (the fact that a thing *is*)? My whole point is then to ask what the medium of such absolute contingency may be, if it is no longer to be mediated by the category of probability.

My answer, as you may have guessed, is that the medium of absolute contingency (i.e. not derived from states) is the market of contingent claims, and generally the market. It is writing rather than prediction. Notice that we are here delving in the domain of speculative philosophy, which only deals with categories of thought a priori, such as the *real* or the *possible*, not with specific empirical domains or scientific theories. It is at this pure level of thought that I suggest we shift to the market of contingent claims as an alternative to possibility. Such is, indeed, my philosophical ambition for the market.

At this stage, it is no longer the ‘mundane’ derivatives market that I am talking about, the one made up by individual traders or psychologies or historical events. The category that allows me to abstract the market to such a level is, once again, writing. There is a deep insight from Derrida at play, here, to the effect that writing is not low and mundane; it is not a derivative activity; to the contrary, it ranks higher than being and is ‘older’ than being (as Derrida says⁹).

If price is to become such an original and *early* alternative to probability and written contingent claims are to become such an early alternative to possible states (early in the process of thought, that is), then even the phenomenon of the exchange has to be thought purely and metaphysically. More to the point, if indeed the market is a metaphysical answer to a speculative philosopher’s worry, then it should be possible to reconstruct the market purely from a priori philosophical arguments. (Meillassoux has even suggested that I no longer call ‘market’ this new category of thought, but ‘arche-market’.) This is what I achieve in the article ‘The Medium of Contingency’, where you will find the name of the market uttered only at the end.¹⁰

To repeat, the market, when elevated to the metaphysical ambition that it deserves, offers a real opportunity to thought which it truly belongs to the philosophers to understand and ultimately

assess. I agree that such a metaphysical speculation will not change the trader's trading habits and will even disturb financial engineers and quantitative analysts, or 'quants', who care only to solve 'quant' problems. But shouldn't we feel at least marginally concerned that the market should lend itself to such a philosophical overturning of thought? Aren't we in the least interested to see the two meanings of the word *speculation* (philosophical and financial) finally come together?

2.7 The market as an opportunity for speculative thought

Possibility is not real (following Bergson), while the market is. That the market should be real, that the prices of contingent claims should be real and non-redundant (and what could be the meaning of 'redundant' when no states of the world have been fixed?), all this points to the necessity of overturning the metaphysical presupposition of states of the world and of probabilistic transitions occurring between them and of looking for an alternative way of solving the problem of *succession* or of the passage of time.

As I have said, I firmly believe in prices being the only real. I just don't believe that you can freeze prices in a fixed representation of states of the world and call this a market. Not that such a representation exists only constantly changes. It simply doesn't exist. I simply define the market as this alternative ontology where the word 'state' no longer has meaning. This is the new metaphysics I propose. There are no mathematical proofs or empirical evidence of what I contend. This is philosophical speculation. I offer a new way of thinking the market, and consequently, of thinking the rest of the world. The problem of recalibration is real, of course; it is not metaphysical. It is just up to us to decide whether it is caused by the probabilities (and consequently to call this a 'defect' of probability) or by the prices (and consequently to call this a 'property' of price, perhaps even its 'essence').

Probability models shouldn't be discarded. They cannot account for the market and for the meaning of price, yet we need them in order to insert the market-maker in the dynamic replication process, and subsequently in the recalibration process. As we will see in the last part, the formalism is needed in order to grant the way out of the formalism systematically. This systematic overflow is what recalibration will amount to and it takes place after the contingent claims are irreversibly written and create, with their writing, a material barrier that blocks the return inside the formalism.

Probability is only an episode and a tool in the hand of the market-maker. I don't think we should look at the market from above, as if from nobody's point of view. Seen from above, the prices of derivatives and of the underlying alike just follow the movements of supply of demand. Who honestly believes there is a strict relation between the two? Who honestly believes that the prices of derivatives are the fair value of some lottery in the long run? Alternatively, who honestly believes they are the result of the subjective probability of some agent? Chances are that the best description of the market, when seen from above, is just chaos.

By contrast, I propose that the only way to look at the market, even to define it, is through the eyes of the market-maker of contingent claims, and through the material process he is engaged in – dynamic replication followed by recalibration.¹¹ A process, as I have said, that takes place

outside probability, yet brings order to the chaos. My criticism of probability applies both to objective probability (either in the frequentist or the propensity interpretation) and to subjective probability. What I mean by the 'end of probability' and its replacement by the price is just the end of the 'state'. Even the single-case probability relies on the notion of possible state.

2.8 The market as the conversion of the image of thought

I should add that, in my metaphysical construction (or 'system', if I must speak like a speculative philosopher), the market is not even a picture; it is a logic. According to me, prices do not even occur inside a market or inside a picture, or a representation, or a theatre of the market. Rather, prices and the market occur 'inside' the contingent claim. They are embedded in it. They are an integral part of its *dynamics*, where the 'dynamics of the contingent claim' is not to be understood, of course, as an evolution of some kind within an external space – we are not talking, here, of the price dynamics, for that would be circular – but as the *dynamics of genesis*.

The metaphysical modality of possibility, usually associated with the fixed and identifiable states of the world, is perverse; it is not something lying ahead in the future, but in the past. It is a fabrication, a few items we fictitiously vary in the actual real and fictitiously project backwards (thus creating alternative possible worlds) in order to make it sound as if the real is the realization of a past possibility and *for this reason* is contingent. We then illegitimately project possibilities in the future. Possibility is as morbid as debt, because it is loan taken from the real.

I propose to transform, or convert, the whole picture. Instead of difference that is based on the recognition of identified states and the recognition of their difference, instead of difference that is derivative on identity, I propose a *difference without underlying states*, or a differential mark that has only itself to support itself – in other words, a sheet of paper on which the formula of difference is written: If A , Pay $x(A)$, Else if B , Pay $x(B)$, or again, the contingent claim. (You can take this phenomenon to be the definition of writing.)

Insomuch as possibility had only the possible state and its recognition to turn to, the contingent claim, this sheet of paper with the formula of *absolute difference* (that is to say, not derivative on underlying states) written on it, has the exchange to turn to. Although ' A ' or ' B ' sound like underlying states, I say the written formula is absolute and not relative to any states because I *literally* interpret ' A ' or ' B ' as written marks, not as states. They do not point to any existent metaphysical states. It is our classical image of thought that tends to represent them as states. States don't mean anything unless involved in a picture where we recognize them, typically the picture of a tree where the *transition* between states is of essence. The sheet that supports the marks is materially one and not partitioned into states. As a matter of fact, it could burn (for instance) and disappear altogether.

What I am saying is that the classical image of thought needs the *transition* and invents the states for this reason. However, once writing is invented in replacement of representational thought, the transition is no longer required. We wake up the next day, *without a transition*, we read the formula that is written on the contingent claim, and we know what to do and what to pay or receive. All we know is that our world is contingent, that it could have been otherwise. We didn't track the tree of

possibilities that has led us to this difference. The mere possibility that the sheet of paper on which the contingent claim is written might ultimately be destroyed is an a priori argument against the tree and the transition.

In the end, the exchange (or price) and the contingent claim sit at an opposite end relative to possibility and probability. I deduce the exchange, as a category of thought, in the same movement of conversion (or dynamics of genesis) as the contingent claim, without even talking of exchanging partners yet. From this, you can see how different from the standard conception my conception of price and exchange ultimately is. It is in a second instance that I remark that the exchange requires an exchange place where anybody can be.

Another consequence is that contingent claims admit of prices, not of expected values, and can never be redundant because in my converted picture there are no states of the world left anyway over which the notion of replication and redundancy can be defined.

The conversion takes place outside time and even 'before' time. It is the metaphysical operation which allows me to open the other door than the probabilistic one and to consider the *place process* instead of the temporal process. Conversion happens before time because it withholds time, and it is formal for this reason. At this level, it is purely a conversion of images of thought that we're talking about. It is only later, when we find that we have no possibility but to reintroduce time, that we will bring back the stochastic processes and their 'diagrams'. We have no choice but to project contingency over states all over again, no choice but to devise probability models. No choice but to replicate dynamically the contingent payoff. However, this brings in the market-maker who *can* subsequently invert the probabilistic model because he engages in dynamic replication. Indeed, my credo is that implied volatility makes sense only in the hands of the dynamic hedger of the option.¹² And this brings in recalibration, which is how probability is exceeded again and the original sense of the market (the one following from the conversion) is recovered.

2.9 Ascending to the metalogical level

Really, my purpose is to *deduce* the exchange. With the conversion, I am really taking my start from a primitive stage, where even the logic of exchanging is not yet operative. Reasoning metalogically, I identify this primitive stage as one of identity and convergence, or a stage of valuation, a ruinous picture indeed for the market and the exchange, and I propose that the conversion transforms this into the contingent claim, into difference and divergence, thereby creating the market in the process, as value is converted into price.

The strategic reason for my doing this is to distance the contingent claim and price from state, identity and probability as much as I can. I think my metalogical reasoning can even be pushed one level higher. If we view a mathematical function (say) as admitting of different values, that is, as differing, only on account of its underlying state variable; if, that is, we understand function value as following from the underlying state variable in that (causal) order of assignment, we speak of *valuation*; we value the function. If, by contrast, the logic of assignment is forgotten – imagine that the algorithm is lost; that the 'tables' are buried by some major earthquake or flood – and all that is

left is the function without the underlying state; if, in other words, only a fragment of the logic is left, the minimal part that says: ‘The value is Y , but it could have been different’, then we speak of *pricing*.

My real motivation is to release contingency from any idea of chronology, and correlatively, from the picture of the tree, the transition and the arborescence. The logic of assignment of value and of valuation (‘The value is Y because the state is X and $Y = f(X)$ ’) is chronological, whereas the logic of price (‘The value is Y , but it could have been different’) is independent of time. The contingency of an actual or even past thing and the contingency of a non-actual or future thing should be thought in the same way.

To price a contingent claim, we shouldn’t step backwards in time and reduce contingency to possible states in a tree. We should step backwards in another dimension than time. We step backwards in the market. The market is only ‘accidentally’ immersed in time. Likewise, Pierre Menard steps backwards in another dimension than chronology and when we observe that Cervantes’ *Quixote* is only chronologically, that is to say accidentally, anterior to Menard’s and that Menard is producing a serious task in his writing the *Quixote*, we realize that the accident of time is not serious. Chronological time is, indeed, an unavoidable accident. However, the genesis of the market is outside time. How we bring back the time-independence of the contingent claim and the time-independence of the market into the unavoidable temporal process is by dynamic replication, according to me.

Dynamic replication is what allows us to slide back in time the clause ‘The value is Y , but it could have been different’ without going through the tree. The reason why we invert the formulas of valuation (BSM or the like) against the market price, and the reason why the dynamic hedger alone is entitled to so inverting them, is precisely to be able to repeat, at every point of time, the final clause ‘The value is Y , but it could have been different’. Inverting the formulas of valuation is exactly repeating, before maturity, the inversion that we saw was operative when the valuation clause, ‘The value is Y , because $Y = f(X)$ ’, got transformed or inverted into the pricing clause: ‘The value is Y , but it could have been different’. To put it differently, recalibration is what brings back the market, what repeats it, and blows away the valuation tree at every point of time.

Introducing the Market

3.1 The unexchangeable place of exchange

I had the fortune to start my trading career in 1987 – more exactly, on 19 October, the day the market crashed.¹ The bank had hired me in September after my graduation from engineering school, but 19 October was the first day I had set foot on the floor, owing to my boss's intuition that there would be something worth watching and learning in there that day. I remember a distinct feeling which I would later recognize as philosophically-charged and which spontaneously imposed itself on me as I stood by the pit and watched the fast trading activity. (I couldn't trade myself as I was still an apprentice – I could only watch.) Simply, I became convinced that there was more going on, in producing the events and their succession, than the mere chain of causes and effects that unfolded in chronological time.

When your world reduces to a trading pit, history becomes quantitative. Events come down to price movements, and all price movements are caused by previous price movements and by quantities offered to be bought or sold at every price level. Action in the open outcry pits is recorded by camera, just in case there is a disagreement on a size or a price and the two parties want to check exactly what words were shouted and exactly what hand movements were made at the moment of the trade. One can safely assume that the 'film of events' unfolding in a pit is exactly reducible to the film that the camera records. There may be causes, not apparent on the film, that trigger the desire of the given trader to buy or sell a certain size at a given price. But as far as the subsequent price movement is concerned, all that matters is the size and the price that he trades, and the shouts and the gestures that he shows to the camera. What effects the trading action of a given trader might have on other traders do not matter either, because all we care about in the trading world is how the subsequent emotions or affects of the other traders will translate into subsequent inner motives and how these inner motives will translate into subsequent trading action. The whole chain of causes and effects is therefore coded in the recorded sequence; and one can safely assume that if the recorded film were to be replayed, it could only be the reproduction of the same chain of events.

Yet, my distinct and instinctive feeling, as I stood by the pit and watched the action, was that the same chain of events *wouldn't unfold* if the same recorded film – that is to say, chronological time and all that it has recorded – was replayed. I felt there was a non-recordable, non-representable, non-retrievable, and non-replayable element in the chain of events that contributed decisively to its unfolding, and that this element was inscribed in the place itself where the events actually took place: in the marketplace, in the actual place of the exchange which is, as such, unexchangeable.

That the event should be this way and no other way, *this* absolute contingency, was linked to the place where it happened, and it exceeded the chronological film, the time-recording. You have to be in a place to watch the event take place and explain its taking place. I guess that this concerns all kinds of events and not just the market; but the market has the advantage of making this irreducible element or excess more acutely felt, because the market is quantitative, and therefore may lead one to think that it is recordable and programmable – before one realizes, all the more acutely, that this is not the case.

I was an option market-maker, so I guess the market was not sitting in front of me or below me, for me to survey it or think about it from outside. I was standing right in the middle of it, even coincident with it (with its being) as I was its maker, yet any philosophical thinking that I might have about it was not going to be reflexive (as if I were to inspect my inner state of mind) because, at the same time, the market was independent of me.

There is a paradox of authorship that is characteristic of the market. You are making it, as it is up to you to quote a bid-and-ask spread and produce a 'market' from scratch, yet the market makes you because you have to follow it and cannot allow yourself to drift away from it. On the other hand, to be a market leader does not mean to pull the market in your own personal direction, because the chances are it will then run counter to you and destroy you. You have to lead the market *in its own direction*, build in advance the material surface that you will set your feet on and produce in advance the matter that will attract you in its wake. As the market both traverses you (inhabits you) and is independent of you, the question 'What is the market?' becomes even more pressing than any other confrontation of thought and matter. It is all the more philosophical as it is absolutely immanent and there is nobody (not even you or the market) supervising your relation with the market. And it is all the more metaphysical as metaphysics seems to have found here its proper matter (and therefore is no longer meta-physics, properly so-called).

There is no superior (transcendent) instance, or stance, from which you could decipher the market or even address its question, because you are totally immersed in it and any external recapitulation or representation is already too late relative to the only thing that goes on in the market and of which the market is made; namely, its perpetually developing event. Better, you are making it and no preliminary idea or point of view of yours can determine or inform your making. Imagine, somehow, the source of the event, the event that nothing can precede or predict or prepare, the event that nobody can make (the subject can only name it after the fact), except that in the present instance, which may be unique in its kind (the market) – this is why I am speaking of the proper matter of metaphysics – the medium is at the same time available in which the market-maker can dwell and make the market.

You are totally absorbed by the market and it seems that it has penetrated your own thought process, even the order of your thought, exchanging, in that order, not only the hierarchy between probability and state, but also between philosophy and its object. When you ask the archetypical ontological question: ‘What is the market?’, you ask it from within the real that the market already is, not from within philosophy or reflexive thought. It is a novel position within philosophical inquiry itself because it is as if philosophy were asking the question of what it is itself. Yet, I insist that there is matter produced in this entanglement and not abstraction or a formal standing or retreat of thought. This may even be a new definition of matter. Matter can be defined as that which can entangle thought and matter in this way. The market is not the *subject matter* of philosophy and of the philosophical gaze. Because of the radical immanence that is at play here, the question ‘What is the market?’ becomes of a unique kind. It becomes the proper question of philosophy.

3.1.1 The market as a continual event

The philosophical thinking of the market extends in the gap between derivative valuation theory and derivative pricing technology. Derivative valuation theory is just a branch of stochastic calculus, which is itself part of probability theory and its formalism. In this respect, a philosophy of derivative valuation theory is no different from a philosophy of probability. Instead of throwing a dice or spinning a roulette wheel, one would consider the underlying market as a random generator. A philosophical questioning of this would consist in asking whether the probability is real or is just a manner of speaking, a shorthand for statistical regularity. Further epistemological complications may arise, due to the complexity of the random generator we are considering. The question may be posed – following Taleb – as to the reliable inference of the probability distribution from which the market is drawing its outcomes. Randomness may be further blackened by essential uncertainty – never being in a position to know which dice we are playing with, and so on. This is all fine, and yet it completely misses the specific material that we are engaging with here – namely, the market.

The market is not just a generator of random numbers; it is the material place of an exchange. Derivatives would, indeed, simply constitute a sub-case of probability theory and stochastic control if it were merely a matter of computing a value for them, in the same way as we speak of the value of a mathematical function. We know what the value of the derivative is, at maturity, as a function of the underlying price on which it derives, and the whole problem is to determine its value *prior* to maturity, with only mathematics and probability theory to carry out this backwards translation in time. What is missed in this traditional way of posing the problem is that derivatives are a *technology*; that is, a material procedure and not just a mathematical function. They are contracts that are materially written on paper and whose destination is to be materially exchanged.

When we consider the stochastic process of the underlying, price is confused with a number and the market is reduced to a random generator; we don’t really wonder what the meaning of price and of the market may be, what their inner process may be.

Now, consider that the derivative contract is materially (directly, without any conceptual or formal intermediary) destined or *meant* to be traded; and that the meaning and even the definition of price may be the following: the number that gets specifically and materially attached to the derivative

contract in that specific place, the market, where derivatives are materially exchanged. Consider that there might be no other definition of price than this one. Consider that we are in the process of inventing the whole technology of price and of the market, and that the derivative is the material piece necessary for that process of making. In that case, the technology of derivative pricing can no longer be thought independently of the fact that to price something is to price it *in the market*, or independently of the fact that the derivative trader will be the main and, as a matter of fact, the sole intended user of the technology.

The philosophy that is *specific* to derivatives pricing technology (as opposed to a philosophy of derivatives valuation theory that comes down to a philosophy of probability) becomes the analysis of the above thought, which brings together market and technology. It is no longer a matter of the complexity of the market, but rather of its simplicity. For it is the same market, the same medium and milieu, that must be thought at the beginning and at the end – as that which produces the underlying price and which receives the derivative price, only to produce it again. The thought can no longer proceed in a sequence: first, the underlying process, then, probability theory and derivative valuation theory, then, a derivative value that is changed into a price and that is effectively immersed and varied in the market. No, from the start the valuation must be a *pricing* in the sense that the derivative price, manufactured and produced though it may be by a valuation tool, is designed to be exchanged in the market and *generated* by the market.

The practical consequence that the user and the manufacturer of the technology have to face is the constant *recalibration* of the valuation model, keeping in mind that the recalibration process must not even be thought sequentially, but must be considered *a priori in the very design of the tool*. Any derivative valuation model is supposed to generate a certain derivative value as an output, as a function of the underlying price and parameters such as volatility. However, the technological purpose of this value is to be a price, and thus simultaneously to act as an *input* to the model. No sooner has the derivative market-maker produced the derivative value with his valuation tool than this value, now become a price, is fed back as an input into the tool, triggering its recalibration. Traders use the Black-Scholes-Merton (BSM) formula to generate option value given volatility, while simultaneously feeding option prices into the formula to imply volatility. The complete act of pricing is the combination of these two directions.

What we call the *fundamental principle* of the market is that the states of the world in the market consist of prices – all the prices and nothing but the prices. States of the market-world cannot be abstract states such as states of the economy, measures of inflation, or labor, and so on. Such abstract states may be implicit; however, *in the market*, they must be represented by prices of assets. So, when a framework like BSM posits the prices of the underlying asset as states of the world, and then computes the values of derivatives as a result, it verifies the part of the fundamental principle which says that states of the world are nothing but prices (for it reduces the whole economy to the price process of the underlying); but it *denies* the second part of the fundamental principle according to which states of the world are *all* the prices. It denies the fact that derivatives prices are *also* independent states of the world, and should not be expressed as functions of the prices of the underlying (or, in other words, as values).

This is a severe philosophical problem. As we can now see, the fundamental principle of the market seems to undermine any attempt at representing the dynamics of the market in a probabilistic framework, no matter how large or complex. Any totalized set of states of the world (for instance, the prices of the underlying and the prices of any number of derivatives written on the underlying) will have as a consequence that the value of a certain derivative whose prices were not part of the initial set of states of the world will be computed as a function of the existing states, and thus will not count as a *price*; that is, as an independent state. Now, of course, the values of derivatives written on a given underlying will ultimately depend on no other variable than the price of the underlying that obtains at their expiration. And, of course, the holder of a European call or put ultimately will not care what levels and what changes of volatility have obtained during the lifetime of the option or, more generally, what exact path the underlying has travelled. Certainly, all he cares about is the final settlement of the option and the corresponding underlying level. *However*, as soon as he starts caring about dynamically hedging the option, he can no longer ignore path-dependency. He becomes dependent on the whole path of the whole market that will take place in the meantime. For the proceeds of the hedging strategy will depend on volatility changes, and hedging against those changes will require running, in the dynamic hedging strategy and on top of the underlying, a trading account on derivatives that depend on volatility, such as variance swaps or other options whose prices become states of the world that are additionally required. In turn, the hedging ratios to apply to these additional hedging instruments and the proceeds of their hedging strategy will depend on changes of the parameters that govern the stochastic process of the volatility itself; that is, the volatility of volatility, and so on; and those parameter changes will, in turn, have to be controlled by a further complication of the dynamic hedging strategy involving derivatives that are more complex still.

If the market is the rule and you are not content to buy (or sell) and hold the derivative until its expiry but wish to trade and engage dynamically in the market and unwind your position at any moment, you become dependent on all the prices of derivatives of all degrees of complexity – in another words, you become dependent on the total ‘state’ of the market, which can never be reduced to a total of states. The market becomes the one and only vast (and continual) event which screens off any residual or ulterior event. As such, it can never be framed or calculated even though it is present. Engaging with it becomes equivalent to engaging with the event, with any event, or with the ‘essence’ of the event, which is to be incalculable. It is a continual event, when the event is by definition a discontinuity, without this being a contradiction.

3.1.2 The market as quantitative history

In *The Black Swan*, Nassim Taleb has discussed the intractability and unpredictability of the event. However, his angle is epistemological. He basically argues that one cannot know the small probabilities. If you think that an event happens once in a hundred years, then you’re going to need ten thousand years’ worth of data in order to ensure that your probability prediction is reliable. Of course, there is no such data and what you will be tempted to do is to fit a certain probabilistic model to the available data, or in other words, to what you already know has obtained with what

frequency, and then let your model work out for you the probability of extreme events that have never happened so far.

Typically, in the financial market, you fit a certain probability distribution to the statistics and satisfy yourself (this is the purpose of fitting) that downward price jumps of, say, –15 percent have a probability that agrees with their historical frequency of occurrence. However, you will never be able to calibrate your probability distribution to a downward jump of –45 percent, because there is no such jump in your historical sample. All you can do is trust that the ‘true’ random generator of the market is accurately captured by your model, and that, extrapolating from this, your model will now tell you the truth about jumps of –45 percent. As Taleb puts it, only the assumption that you are dealing with a certain type of probability distribution – for instance, a Gaussian – can justify the belief that a certain amount of data is enough in order to infer (calibrate) this probability distribution. So, how would you proceed when you don’t know the type of probability distribution that you are after?

Taleb argues that probability distributions in the market should have much fatter tails than the Gaussian. Typically, they should be of the scalable (Mandelbrotian) type. If extreme events happen more frequently than we think, then it is all a matter of correcting the probability distribution – and, if the chosen distribution has infinite moments, a matter of recognizing that we will never really *know* it, because it would take an infinite amount of data to calibrate its fat tails. But notice that the whole discussion remains couched in terms of probability and frequency. If you measure the extreme event on some scale (hence the word ‘extreme’), then it is only ever a matter of relative frequency and comparison: the extreme event happens *more often* than we think.

So, the outlook here is epistemological: Taleb believes in the reality of the random generator, but he thinks we cannot *know* it. The preoccupation with the event becomes ontological, however, when there is no previous scale, or population, of which the event is a grade (whether extreme or not) or member – when the event literally emerges out of nothing. Such an event is not extreme; it is world-changing. You cannot deal with it using either probability or improbability. My point is that the market, contrary to its quantitative and statistical appearance, is the (continual) place of such an ontological event. It seems that Taleb reserves the mild Black Swan problem (a matter of getting the right probability distribution, of using probability language) for the market. When he turns to the case of the major disrupting event that fits no scale – what I call a context-changing event – he considers it only in very qualitative terms. But according to me, the quantitative aspect of the market (prices going up and down) actually masks the ontological fact that probability *never applies at any instant*, because every instant is the *same* (it is not the successor in time of the previous instant); it is the same in that it harbours the same immeasurable, unrepeatably, unheard of, event. I say the event is the ‘same’ in terms of featuring the same irreducibility and incomparability all the time. Above all, this repetition is not to be understood in terms of statistics, of course. It is a repetition of the same wherein the ‘identity’ that the events share is precisely that *they are not comparable and reducible to any scale*, shared or not.

The events in question concern prices and seem to be quantitative, so there *seems* to be a scale. Yet, the market is as eventful as history, and every event, at every instant, must be as incomparable

with the next as historical events are (no wonder the market is correlated to history and always accompanies it). It is through derivatives that such an aporia can be resolved. In derivative pricing, it becomes evident that the market is not a succession of instants, because the process of infinite recalibration it demands freezes everything and puts us ‘at rest’ with the event.

Thus, instead of being regarded as a case of probability and statistics, the market should appear as the *only* case (a singularity, a miracle) where history and its events take on the appearance of numbers. It is a major coincidence. The blank space or the hinge between contexts (this absolute discontinuity) happens to be filled by the (continuous) matter of the market, which I call ‘writing’ (since writing amounts to filling in the blanks). The matter of the market is the matter of metaphysics. By declaring this, I shouldn’t be breaking news to Taleb, really, but breaking news to Badiou and Meillassoux.

So, my main observation concerning Taleb (and my main divergence from his position) is that it is no coincidence that he started his career in the financial markets, in the derivatives pricing and trading business. The market is the only place where the qualitative absolute event, the one that is irreducible to measure and scale and probability, finds quantitative expression, in a material medium borne by numbers, or rather prices. The market is *quantitative history*. One should keep in mind this contradiction in terms: one should remain aware that the historical event is incalculable and unquantifiable because it precedes any scale; and *then* understand the extraordinary nature of price (and of its medium: the market) *as the quantification of that unquantifiability*. This is why the market is truly the technology of the future. You have to realize that price is not a number. Quantifying the event (translating it into numbers) is impossible; yet, the market *is* such a translation, precisely insofar as it takes place *outside* of possibility. ‘Quantitative history’ does not mean that the event is being forced into the mold of numbers. Rather, a quantity, a number of an extraordinary nature, has been found such that history can be quantified.

All that price shares with numbers is the external shell; it has a different mathematics altogether. When the resemblance is mistakenly thought to be full, and price is thought to be just a number, its process can only follow and come second. Then we remember (as if it had been forgotten) that the market price is random; and, since the number is fully present before us, there is no choice but to postulate a stochastic process outside of price and to make price follow it. But in reality, the *price process* came before; it was always present inside price.

Thus, when looking at price one should not look at the succession of empty shells it leaves behind as time passes. One then mistakes them with a time series and starts looking for the random generator of the underlying price, as Taleb does. Taleb was misled by the resemblance between price and number, and consequently applied the category of probability to the market, setting up the whole problem as one of deciding between Gaussian distribution and scalable distribution. Instead of recognizing in the market the (quantitative) *solution* of the severe Black Swan problem (how to mediate and exchange the unexchangeable event), he saw in it an instance of the mild Black Swan problem.

3.1.3 The intensive nontemporal price process

As a matter of fact, Taleb’s whole movement of thought and line of work will later unfold as a constant withdrawal and distancing from the market, whereas I maintain that, given that he was fortunate

enough, like me, to start a career and even a specialty in the financial market, he should have stuck with his specialty and drawn all the consequences – as extreme, incredible, impossible, and metaphysical as they may be – of his experience and expertise with that extraordinary medium and matter. There is a scientific duty that is pressing here. There is the discovery of this new matter and of the new science that attaches to it. The peculiarity is that metaphysics is needed as an introduction to the science. One has the duty to advance this science. Taleb mistook the science of the market for the intervention of an old science – the science of probability – in a field where it does not apply. As a consequence, not only did he move away from the market, but this move has simultaneously expanded itself into a general attack against science.

Instead of looking at the numerical shell, one should look, in price, at its intensive, nontemporal process, at the place process which is that a price should be given for the derivative written on the underlying. The underlying ‘follows’ no process other than *its writing as a derivative and the simultaneous (nontemporal) availability of the derivative price*. Instead of a random generator being posited, based on the statistical time series of the underlying, instead of statistics being understood as the result of successive draws of the random generator, the only generator of the underlying price should be the availability of a price for its derivative. The fact of such a price being given (and this givenness is the *definition* of the market, I would say) should replace the fact of a statistics of underlying prices being given (the definition of the random generator).

The materiality of the market replaces the notional primacy of the random generator. The market acts in a different dimension than the random generator, and therefore in a dimension other than time (it acts in place). Just as we infer historical volatility from the history of underlying prices, we infer implied volatility from the single option price. This substitution should be understood literally and in an epoch-making sense, and the corresponding transposition of the notion of the generator should be carried through and completed. Nobody has yet thought about the real implications of implied volatility.

3.1.4 The continuity of the discontinuity

The existence of the market is the availability of prices of derivatives of all orders, and this is equivalent to the *nonexistence* of the random generator. As I said, a true engagement with the meaning of the market leads to ontological considerations. Instead of trying to fit the market in pre-existing formats, such as sociology or anthropology or even probability theory, an immanent theory of the market – what we have called an ‘inverse view’ of the market – should stick with the market as specialty and special medium, and draw the ultimate consequences from a thorough metaphysical analysis of it; and then one should generalize these incredible findings outside the market by trying to discover what, in other cases, resembles the market. Because of the way that possibilities are nontotalized in the market, the market becomes the medium of contingency; it even *replaces* the event; and anybody who dwells in the market can be said to stand right in the middle in the event.

By replacing the random generator with writing, the market picks out, in the Black Swan, the decisive feature of the *change of contexts*; it seizes the ‘true generator’ of the Black Swan and thus

becomes immunized against it. One should try to think deeply about what, in the exceptional case of the market, makes it such a wonderful medium, such a solution to its own Black Swan problem. One should remain focused and specialized in the market. Taleb, by contrast, never really dropped the frequency or probability-based notion of event (what he calls 'randomness'). He never really upgraded his category to the one that changes whole ranges of possibilities. If he did glimpse it, his angle remained epistemological ('unknown unknowns'). He never saw in the ontology of the event (the radical discontinuity) the opportunity for the existence of a fabric or medium *that will precisely mend the discontinuity by thriving in its gap*, so to speak; by making the discontinuity continuous (this new definition of matter) and the unquantifiable quantifiable thanks to a new kind of quantity or number (price). He passed through the market as a case of wild (unknowable) randomness, only to dedicate the rest of his thinking to events in general, lying largely outside the market. He renounced (and even denounced) his former specialty. He didn't explicitly analyze the event as ontologically incalculable, and thus he deprived himself of the answer to that problem.

The fact that Taleb did not reach, in the market, the stage of the extraordinary thought, or the thought of its extraordinary matter, but has turned his back on it only at the lower probabilistic level, will have consequences for the solution that Taleb will subsequently propose to the general Black Swan problem. Let us keep in mind that Taleb did not fully analyze the consequences – in my opinion, ontological – of the severe Black Swan problem. Instead of talking of the event, at once unique, immeasurable and absolute, he still talks of randomness, a notion that remains relative and referenced. To him, the event is immeasurable because the received science cannot measure it. He does not turn the immeasurability of the event into a search for a new science. Received science is only a tool to know or understand the world for Taleb (The subtitle of his solution book – *Antifragile* – is *How to Live in a World We Don't Understand*²); therefore, in the face of the event, it is the epistemological failure of science that Taleb denounces. He never envisages science as an ontological tool, as a creator of worlds, even of matter. As a consequence, when he turns to trying to solve the problem, the proposed solution will consist in abandoning the enterprise of science – in this case, the science of probability.

Probability science becomes increasingly vulnerable and fragile when faced with the extreme event, because it cannot accurately express its probability. For this reason, the sad truth, Taleb rightfully argues, is that the science of randomness (probability science) in reality *dislikes* randomness, now understood in the metaprobabilistic sense that the probability forecast will be proven wrong. (Taleb will always understand metaprobability, or the criticism of probability, in an epistemological sense – as the failure of a model to capture a generator which, nevertheless, must exist – never in the ontological sense – as the failure of the whole framework of probability; namely, the *nonexistence* of states of the world and the *nonexistence* of a random generator.) Probability science is needed in order to achieve the theoretical valuation of derivatives and manage the risks of derivatives portfolios, but it can be massively wrong in its valuations. Therefore, one way of *liking randomness* instead of fearing it, according to Taleb, is to keep the derivative, which thrives on randomness, but to discard the science of its valuation.

3.1.5 The matter beyond antifragility

As a result, as far as options are concerned, Taleb proposes to abandon the subtleties and finesses of dynamic hedging (the fruits of the received science) and to adopt a static buy and hold strategy. He cannot sell, because a seller of options does not like volatility and, worse still, because a seller needs the science and needs dynamic hedging in order to manage (replicate) the options he sells. Notice that Taleb falls back on the market by default, as the only remaining background once science is abandoned; yet, he never engages in the market as the positive replacement of the science. And when he notices that the market is quantitative after all – that it is a science, in my sense – and that one needs to be quantitatively critical of the premiums of options that one is buying and holding, Taleb abandons the market altogether and writes *Antifragile* as an apology for the asymmetric payoffs (also known as derivatives) that one can find and purchase for free, outside the market – that is to say, generally in life. Notice that, since they are costless, they don't require a seller.

One can summarize Taleb's move in the following way: On the way up, the market was never recognized as the matter which lies beyond probability and which is, in its own domain, the quantitative solution to the problem of the qualitative event or Black Swan; the qualitative event itself was never recognized as an ontological problem which required an ontological solution. On the way down, randomness, which remained epistemological, required as the solution of its problem only that the science be dropped (together with the market).

Even though he has dropped probability and the problem of valuation of asymmetric payoffs, Taleb is still locked into the language of frequency; he is still a slave to the scale and comparison, to the *degree*, and therefore to its relativity. Fragile things resist frequent shocks of lesser magnitude, but break in case of an extreme and infrequent shock. Antifragile things get stronger and stronger (scale and degree) as they are subjected to shocks and accidents (to randomness). Notice that the randomness that affects them is tamed, at least in the sense that the scale upon which the antifragile thing is recognized to have become stronger is itself preserved. How would the antifragile thing fare in case of events that destroyed the whole world, or at least that radically changed the background or the scale against which the antifragile thing must be recognized as being the same thing before it is recognized to have become stronger?

Taleb's *The Black Swan* was a metaphysical book (at least, according to me). The event there was absolute: anything could happen; there was nothing, no world, no range of possibility, no time, that the event couldn't change or abolish. In *Antifragile*, Taleb recognizes a category that is now superior to the event and seems to dominate it; namely, antifragility. But one cannot recognize in the event the absolute discontinuity, the failure of any scale, and *then* propose as a solution a notion that depends on scale.

The solution must be absolute, like the problem. Taleb's new great discovery in *Antifragile* seems to be $(-1, 0, 1)$. The contrary of fragility (1) is *not* robustness (0), but antifragility (-1) . However, the real *not* that he is missing is, as in *The Black Swan*, the *not* that negates the whole idea of scale. It applies to fragile, robust and antifragile alike because it removes the scale, in the same way that the market is the *not* of both probability and improbability, of both prediction and its failure. Antifragility without scale amounts to creating an absolute – in other words, to creating *matter*. The

solution must be metaphysical: the medium that insinuates itself at the hinge of the event; to be in a state of relative rest with respect to the event.

3.2 The medium of contingency

Contingency is one-sided. A contingent thing is the way it is without remainder, without recall, without reason – without reflection. Contingency is unexchangeable, to speak like Baudrillard in *Impossible Exchange*.³ The contingent state is not selected from a pre-existing list of possible states. To list such possible states is to involve contingency in their reciprocity and their mutual reflection: this state *as opposed* to another. Which means that identity is recognized (*this* state, *that* other state) before difference. In contrast, massive and absolute contingency, contingency as *matière première*, has to be recognized *before* state and identity. We must say of the contingent thing that it can be (different) before we say that it is, or what state it is in. In this way, a future contingent event (that doesn't exist yet) becomes no different from an actual contingent thing or an actual contingent fact. To recognize massively that an actual contingent thing could have been different dispenses with any need to list the alternative possibilities that it might have been. We don't need them because the thing is actualized. It is no longer possible, it is before our eyes, yet we know that it could have been different. Its contingency never leaves it, so to speak, to become reflected and dispersed in external possibilities. Similarly, we should be able to think of the future event as something that could be different (or that could even never be at all), before and without knowing what it is or will be.

All that time brings to this is identification, because it is in time that we eventually identify the future event. The whole idea is to secure a bridge to the future event that is other than time, a bridge that is indifferent to the identity of the event (what it is) and sensitive only to its contingency, and which establishes a link, a mediation, a channel of communication with that contingency alone. I argue that the market of contingent claims is such a medium and such a bridge. This bridging is achieved through the combination of *writing* and *exchange* (a kind of alchemy). When something is written, it is too late; think of opening it after the fact, as when you open the will of a deceased person. It is too late because of the material on which it is written and because of the mark that the material bears. This mark doesn't offer interchangeable possibilities. It offers itself as one. It is what it is, yet it could have been different.

It is true that the written contingent claim specifies different states of the world, or possibilities, in which it pays different amounts (If A , Pay $x(A)$; If B , Pay $x(B)$; and so on.). However, these are written; they are not merely mentally represented. In order for them to be realized and to actually trigger the payoffs, the material sheet on which they are written needs to be handed over and exchanged. As such, the material sheet that binds them is one; it is not many. It will not change depending on what state may come about in the future. It must maintain its identity, or rather its materiality, in order to be handed over and to be transmitted. One tends to forget that, because of this materiality, the material sheet can exist (and be exchanged) before the final state is reached, in which it will expire and pay off. In contrast, the final state doesn't materially exist before the expiry, because until that date it is just a possibility. As a matter of fact, the material sheet exceeds the possible states

that are encoded on it. Things can happen to it that were not part of the listed possibilities. It could be lost, or it could burn, or the whole market in which it was supposed to be exchanged against money could disappear.

These possibilities cannot be listed or even totalized, and this massivity of contingency prevents us from listing them in an ex-ante fashion, leading us on the contrary to always think ex-post. We should not think of a future thing as a possibility but as a future reality; as something that has happened and could have been different, but with the mere *incidental* difference (due to the incidence of chronological time) that we don't know what has actually happened. Reality must be separated from actuality. We must think of (massive) contingency ahead of time and find a way of translating this massivity and this ex-post character to a place (and a time) that is before the event, but *without turning ex-ante*. Time is unavoidable, but what must be avoided, when thus going backwards in time (from the future to the present), is the ex-ante attitude. It is as if we were saying that that material sheet, whose ex-post future fate we have already considered (and already realized that it could have been different), is *still* here with us now, in the present, *after the fact – that is to say, before it*. The massivity of contingency after the fact should correspond to an equivalent massivity before the fact, which will not count as an ex-ante stance *because of this massivity*. In other words, the nondistinction and nonidentification of alternative possibilities after the fact (due to the massive 'too late') must correspond to an equivalent nonidentification of possibilities before the fact. Once this equivalence is found, chronological time becomes incidental (non-essential) on contingency – and being chronologically situated before the event (incidentally) will no longer count as being ex-ante.

Chronological time is of the essence when states are distinguished. Reviewing one distinct state after the other imposes a chronology on thought if only because the review is analogical and the identity of each state is recognized (and circumscribes the state) *before* its difference with its neighbor. Reviewing states in this manner necessarily introduces an integral (a synthesis) or a supervising state that runs through each one. We can even go further: this mode of reading is *equivalent* to having such a supervising state. There is the impression of a draw, of waiting to see each outcome. Probability is an integral because it is the summation of a 1 and 0s conveniently distributed over recognizable states. Reviewing the states in an identifiable order is the *same thing* as having the one state that maps into each following a probabilistic transition. This is the meaning of probability. Distinguishing the states imposes this probability structure and requires time and the ex-ante attitude in order to resolve itself.

In contrast, there is no integral, but only a *differential*, in the reverse reading of the contingency, where the fact that the state we are in could have been different is recognized *before* its identity. While the reading of identifiable states is conducive to *valuation* (as in evaluating a function) the reading of contingency ('it could have been different') is conducive to *pricing*. While the list of identifiable and reviewable (reciprocal) states is equivalent to stepping back into the supervising state, the multiplicity and mode of count (not a function, not a number) of contingency and of its massivity cannot step back; it can only invaginate itself in the material sheet itself, exchanging it.

Just as, when we read the list of identifiable possible states, we reproduce (describe, constitute) the supervising state, so, when we 'read' the one contingent strike, or mark, we create its exchange.

It might seem that the ex-ante attitude (waiting and seeing what possible outcome or state will obtain) is more suited to the 'future' than our twisted suggestion, in which we position ourselves in the future after the fact (ex-post) and recede to the present through an unwarranted medium that we are calling the *exchange* (or the market). The ex-ante attitude seems to be a more faithful description of the attitude of predicting what will happen. Crucially, however, the ex-ante attitude depends on having recognized the possible outcomes and totalized them in a list. And where can such identification come from, if not from the *past*? We observe the past series of outcomes of a die or a roulette wheel and we decide that these are the only outcomes, and consequently project them into the future. Yet, the future is massively unpredictable. Anything could happen; fire could destroy the casino or a meteor could hit us while we cast the die. Probability and the ex-ante attitude in which it is framed depend on reducing the contingency of the present world to a *comparison* with alternative possible worlds that we arbitrarily carve out and separate from the present one, using our imagination – or rather, *lack of imagination*. We invent numbers, 0 and 1, that we assign respectively to the possible worlds that have not obtained and to the present actual one. Yet, the present world should have remained *one* in the sense that it admits of no rivals or alternatives, even 'possible' ones. It should have remained 1 without 0, where 1 is not a number (whose fate is to be compared with a 0 or whose difference is to be construed analogically) but is a mark, a strike, whose difference is intrinsic, since it spontaneously says that *it is what it is only to the extent that it could have been different*. This 1 without 0, this unity that has factored in the difference intrinsically, is a new unit of count; it is a different way of counting altogether, whose medium, I shall argue, is money (hence the market), not frequency.

Creating these alternative worlds is not only a fiction; it robs the present world from of its reality. Since the alternative worlds are not compatible with the present one, the only way they could coexist with the latter is in a fictitious *past* stage where all worlds (the present one along with the alternatives) are considered to be mere possibilities. Identifying the present world as a state is the same as identifying the alternative worlds or states, is the same as fabricating the past world. Assigning *comparable* numbers (1 and 0) to the present world and to the alternative ones is only the prelude to assigning to them 'middle' numbers (p_i) that they share equally, *whose sum is also equal to 1*. To explain the contingency of the present world, we thus imagine a past 'real' world from which the present one and the alternative ones spring as possibilities. That past world now bears the number 1 (the sum of probabilities). It is as fictitious as the alternative worlds (since its whole *raison d'être* is to support the incompatibility between the alternative worlds by stepping into the past); yet, the thought is now that of a transmission of reality. The 1 of the reality of the present world is now ipso facto thought to have been transmitted to it by the fictitious past world through a temporal transition. This immediately results in an extrapolation whereby future possibilities are, in turn, thought to spring from the present world, only because it now bears the *analogical comparable* 1 and we now identify this fabricated reality with a sum of possibilities.

The 'sheet of paper' is immaterial when we are in the mode of reading the list of possible states, because only the contents matter and the writing is here merely programmatic. By contrast, the material sheet of paper is *essential* to the reading of contingency because this reading, as a reaffirmation of the mark and of difference, is a reaffirmation of the fact that no writing as difference and as mark is possible unless it is engraved on a material medium – a sheet of paper, a stone, a computer memory, and so on. Once it is recognized that this material property of the sheet is *essential* to the mode of difference of contingency, it is immediately recognized that the sheet can (and must) be exchanged; this is the same movement of thought. It is the *same matter* which bears the difference and strike of contingency as one, without splitting into different states, and which, as a result, turns itself into the exchange.

Finding an equivalent for the ex-post massivity of contingency before the fact (yet without becoming ex-ante) means securing that the two 'masses' are the same following the 'line of flight' of the event, even though they are separated by chronological time. The medium of transmission is the market and is such that, in the mode of count of the event, the price of the contingent claim is *simultaneous* with its final ex-post strike, as if the price and the final contingency were 'light-like' events, to speak the language of relativity theory. What makes the market as massive as the final event and transports it, through this equivalent massivity, right into the middle of the event, or rather, into its place, is the nontotalization of possibilities due to the necessary tradability of all the other contingent claims. Before the final event there occurs the *whole market* – and what the 'whole market' means is that no contingent claim, of any grade of complexity, shall be denied a price and a trading. As I have explained above, any attempt to recede from the final event (or the strike of contingency) via the probability framework will require a total of states, with the consequence that some derivative will admit of a value (a function of other states) instead of a price, thus negating the fundamental principle of the market.

Letting the *whole market* occur before the event is equivalent to blocking any attempt at receding through probability and the ex-ante attitude. It is as if we were (incidentally) situated before the event chronologically but we were essentially brought back, by the pressure of the whole market, right into the middle of the event. The nontotalization of possibilities in the market makes it feel as if it were continually an event, the only event, and as if we were continually following it ex-post. A connection between massivities is established, which falls outside time.

It is because of this material bridge that I speak of a technology of the future, instead of a knowledge or prevision or vision of the future. The market replaces the final event because the nontotalization of its possibilities brings it right into the middle of the event (into its place, albeit not into its time) and, equivalently, the technology that the market replaces the future. It replaces its knowledge (for there is nothing in the future that puzzles us today except its knowledge). The market makes it so that something other than knowledge matters in the future, and so that the future becomes real – no longer in the epistemological sense of the realization of a forecast or a possibility but, on the contrary, in the ontological sense that all possibilities have been withdrawn from the future (since we consider it ex-post) and that this *future real* has been connected to the *market real* through the market technology.

Anyone who travels to the future in the capsule of the market is insulated from time. When one moves outside the market (or outside matter, outside the ex-post stance) to project states of the world, one commits oneself to that outside medium of probability as an absolute medium (even when the selected states are part of the prices of the market). To adopt the logic of the count of states is to leave the *whole market* (as nontotal) and to adopt a metric that is no longer intrinsic. (What is intrinsic is the whole and the nontotal.) Rather, what should measure the distance between the final event and the present market is the intrinsic metric of the *arrangement of matter* (Julian Barbour⁴) inside each. The arrangement of matter of the final event is the strike of its contingency, which is irreducible to possibility; and the arrangement of matter of the market is its strike as whole and essential event, in which every price of every contingent claim is an integral part, irreversible. It is true that something will change between the present market and the final event; however, that change and the way we engage with it is no longer a matter of prediction; it is no longer accounted for against the absolute and extrinsic medium of probability (the ether). We no longer see (and foresee) the final event when we are locked in the market capsule because all that occupies us is the market. The market has replaced the event, literally.

3.2.1 The technology of the future

Industrial revolutions are usually the precursors of metaphysical revolutions. Einstein's abolition of absolute time and space and his enunciation of the relativity principle wouldn't have occurred had the telegraph and railway networks, then becoming a global industry, not presented the problem of the unification of time and of the determination of city longitudes around the world. A whole technology developed to mechanize the coordination of clocks. Simultaneity became a mechanical procedure and was no longer a metaphysical concept.

Based in Switzerland, the home of the clock industry, and working as an application examiner in the Swiss patent office, Einstein was the privileged witness of this revolution. Owing to his status of total outsider vis-à-vis the scientific establishment (unlike Henri Poincaré), he was bold enough to redefine time in terms of simultaneity, and to redefine simultaneity in terms of the procedural transmission of signals between clocks. Relativity theory was conceived as a machine, not as a metaphysical speculation or an amendment of previous theory. In fact, Einstein wrote his revolutionary 1905 special relativity article in the brisk style of a patent claim, where reference to previous work or to similar inventions was precisely omitted.

The market of contingent claims is also a major industrial revolution; however, its metaphysical consequences haven't yet been drawn. In many respects, it is similar to the one that prompted Einstein's theory of relativity, with the market acting as the new globe and the synchrony of prices as a global clock.

Where patents to synchronize distant clocks were filed in Einstein's day, today the industry is filing maps of synchronous derivatives prices. When the traded vanilla options prices are too scarce, data vendors extend the market and produce complete and smooth implied volatility surfaces. When the underlying price moves or time passes, the surface is recalibrated from a new intake of traded derivatives prices instead of being recomputed from an underlying probabilistic hypothesis. And,

once it is observed that implied volatility is traded and stochastic, the market is solicited again for the prices of options on volatility indices (e.g., VIX options) and the latter are repackaged and redistributed, in turn, in a refinement of the market synchrony, instead of upgrading the theoretical probability distribution underlying the BSM model to stochastic volatility.

The medium of probability needs to be abolished, when thinking of the material relationship between a contingent payoff and its present market price. Probability is an outdated, yet very entrenched metaphysical category that has to give way once the market is conceived as a machine or a technology and no longer as a theory (once the suppression of possibilities under the massivity of the final contingency is recognized to be equivalent to the suppression of possibilities under the massivity of the market – under the *whole market*).

Just as there is no absolute time rigidly attaching to the ether, but only time defined relatively to the material procedure of the synchronization of clocks, there is no absolute probability with which to distribute the underlying and value contingent claims accordingly. Rather, probability is defined relatively to the frame of reference, whereas the real, intrinsic relation is the one that prevails between the contingent claim and its market price.

The rule is to infer the probability distribution of the underlying from the market prices of contingent claims. For instance, volatility is implied from the option price in BSM. When the implied volatility differs among strikes and maturities, we change the probability assumption and we now calibrate a stochastic volatility model (or a jump-diffusion, or a mixture of the two) to the full vanilla surface. The next day, we recalibrate the model to the new options prices, thus changing the distribution again (horizontal recalibration). And when we realize that the market prices of higher-order exotic options are not explainable within the model, we upgrade it to the next level. We thus recalibrate a model of stochastic volatility of volatility, or stochastic jump sizes and intensities, and so on (vertical recalibration).

We call ‘intrinsic non-arbitrage relations’ those relations that help us value derivatives independently of any model of the underlying process. These relations are deduced purely from the statics of the respective payoffs or static replication – that is, from the clauses that are *written* when possibilities are over and the underlying dynamics are terminated (typically at maturity of the instruments or at their knock-out barriers, and so on). Now, the observation is that market prices are also model-independent and, by definition, arbitrage-free. Why don’t we consider their relations to be intrinsic too?

Might not the metaphysical revolution lie in considering that the market prices are *written*, too, and are devoid of probability dynamics – that is, written somehow ‘after’ the end of possibilities, after the fact, outside chronological time? Could the market price be essentially occurring in the middle of the event – right in the heart of the terminal-contingent payoff and with no need to predict it – yet, accidentally taking place ‘before’ the event, in what may look like a chronological antecedence but is, in fact, a taking over of the event, literally taking the place in which the event takes place? And if probability and temporal process do not intrinsically occur between the present price and the future payoff, then what does? Relativity theory is, in reality, a theory of invariants – so, we ask: What is our invariant?

Only because a nondeterministic phenomenon repeats itself with a few variations are we able to assemble the variations and retrospectively call them ‘possibilities’ that are open to the event. (The possible states are identified.) The event is the result of abstracting the differences in the same class and of subsuming the facts under the same phenomenon, which we then suppose will admit of different outcomes. The ex-ante outlook therefore has no physical existence; it is a logical abstraction. Possibilities are defined *after* the population, not before.

When the population is blessed with statistical regularity, we call it a statistical distribution and the ex-ante stance finds further support in the belief that the next individual event will now be generated by a probability distribution with the same moments as the statistical one. Probability is also defined in retrospect. Indeed, the whole idea of a timing of the event is illusory.

When there is no such empirical population or reference class of which the event is recognizably a member, metaphysicians can still imagine a set of possible worlds in which to measure its frequency. A less exorbitant alternative is to drop objective probability altogether and to believe only in subjective probability. However, does any of this make sense when probability, as a concept, has been recognized to be past, not future, and only to be misplaced in the future? What could the probability or even the possibility of an absolute event, a Black Swan, possibly mean? Is time itself not void as the medium of such an event?

Derivatives pricing almost kicked off as a branch of actuarial science. The event of the underlying price resting above or below a certain strike at a certain maturity was analyzed as the linking of very small price increments that occurred in abundance in the interval. On the assumption that the instant probability distribution would be inferred from the statistical series, the temptation to compute the fair value of the derivative as an actuarial value was great – one such that you broke even on average. However, the non-arbitrage constraint binding the derivative, the underlying and the riskless bond quickly dispelled this temptation in favor of risk-neutral pricing, if only because of the risk premium attaching to the underlying and of the investor’s expectation not to break even on average. Finally, the dynamic replication argument of BSM – itself compatible with non-arbitrage and risk-neutral pricing – gave derivatives pricing a more operational turn. It turned the abstract equivalence between the real probability measure and the risk-neutral measure into a pressing and very local accounting equation.

In reality, BSM had just consummated the thought that the market was a material procedure and not an application of probability. Nobody cared any longer whether the derivative price was sensitive to the distribution of profit and loss in the long run, or to the instant random generator that caused the systematic slippage in hedge rebalancing. We all woke up in a market where derivatives and underlying were trading alongside each other and moving together. Nobody uses the BSM formula to explain the option price; everybody inverts it against the option price to compute the dynamic hedge.

Implying the BSM volatility from the vanilla option price opens an endless chain: every subsequent model (stochastic volatility, jump-diffusion, and so on) is calibrated to the options market in turn and becomes virtually stochastic by recalibration. Its meta-model will be governed by the prices of higher-level exotics and, where they don’t actually exist, it is only virtually that we should conceive of recalibrating against them. The market is this infinite chain of prices of contingent claims.

If the chain is virtually infinite, then the relation between any contingent payoff and its price becomes intrinsic. Any probabilistic model is an arbitrary section of this infinity and will always be relative. Incidentally, every exotic structure in the ascending ladder will trade at variance with the replication plan corresponding to that section. This means the underlying stochastic process is prevented from running its course at any level. The virtual infinity of prices, or the market, replaces the whole probabilistic hypothesis and the exchange's 'proper time' – or rather, its proper place – replaces the improper and misleading time of probability.

3.2.2 The book behind the market

The market becomes philosophy by realizing exactly what happens to the orientation of thought (what happens to thought) when the market is no longer envisaged as a random phenomenon taking place in the ether of probability, against the transcendent background of possibility, but as a direct medium of contingency from which possibility was withdrawn, or as the material technology of the future from which knowledge and prevision have been withdrawn. In this transformation, thought, we might say, has been brought from the depth (or the illusion of depth⁵) to the surface, from abstraction to matter, from thought to writing. And the market has become necessary, not through the summation or totalization of possibilities, but through the subtraction and suppression of possibilities.

If the market price becomes intrinsically linked to contingency, as we said above, as geometry is intrinsically linked to matter in general relativity – that is to say, immanently, without any superior dimension supervising the link – then this link becomes absolute and, in a sense, necessary. (Relativity theory is a theory of relativity insofar as the nonabsolute character of space or time is concerned, or insofar as the relativity of the given frame of reference is concerned; however, it is an absolute theory when the invariant is considered.) This is not saying that the market price becomes deterministic; it is not necessary in that sense. The market is necessary in the sense that it is the only medium of contingency, once contingency has become absolute in the sense of the abolition of possibility (not the summation of possibility). Contingency is one-faced and one-sided, and the market is necessary in being the only 'other' side of the one-sided contingency. The market is its translation, its mediation, its mirroring and its speculation, without this translation or mediation or mirroring being a relation with two terms and two sides, established within an encompassing supervising space. The market is necessary in the sense in which Blanchot says that 'the necessary book is subtracted from chance', or in the sense in which the thrower of the Nietzschean die embraces a 'fatal necessity' and emits a 'necessarily winning throw' (Deleuze).

Thus, the generalization of the market is the book and the market becomes philosophy when it is recognized that philosophy, too, is looking for a necessity of thought, not as a transcendent sum of possibilities, but in the immanent material sense of Blanchot's necessary book (referring to Mallarmé's). The market becomes a book once it is recognized that the transformation of thought that happens there (which remains special and specific in the market, as it concerns only financial contingent events and their financial translation and mediation through price) can generally be described as the couching of thought in writing. Blanchot says that thoughts, as abstract and difficult

as those that could be found in philosophy books, can be expressed in a literary book provided they are *not yet thought*. My interpretation of this is to say that good philosophy books are the ones in which the thought is written (as in literary books) and *not yet thought* (reflexively, transcendently). Writing is faster than thought, and to say that a literary thought can be philosophical (and can be better than philosophy, according to me) is to say that it should retain what writing and literature have specifically given to it – the event – and should not be slowed down by reflexive (ordinary philosophical) thought. Something is well-written, according to Joubert, when the author cuts the sentence and refrains from adding to it what can be guessed and predicted at that point – what is no longer the continual event and could slow down the sentence if added to it at that point. In my elaboration of the word ‘writing,’ writing takes place when thought races at the same speed as the event, overtaking traditional reflexive thought – or thought that is slowed down by states of mind and the corresponding algorithmic representation.

In a sense, philosophy is nothing other than a philosophy *book* (it is material thought), not just in the obvious sense that philosophy is contained in books and that we all, in the end, refer to very specific books when we talk of the philosophy of specific philosophers, but because engaging in philosophy is engaging with the event of thought, and writing a book is the only way to be up to speed with the thought (usually another philosopher’s) that we are engaging with, making it truly feel as if the chronologically older book now were to be interpreted at the light of the new one, or as if the old one were chronologically posterior to the new one. Philosophy is not a reflexive commentary. As David Wood would say, to reflect in commentary on the thought of other philosophers is to go counter to them. Instead, one should encounter the other (older) thinker, that is to say meet him; and since thought is running the course of the event, which is one-way (one strike), to meet the other thinker is to match his speed, meeting him in the middle of the event, writing a philosophy book that will join his in the non-chronological *place* of thought (or in the infinite race of the event). Wood speaks of repeating the other thinker’s thought.⁶

It is not that philosophy is a closed library and that the only concern of philosophy books is other books, with no connection with the real. Philosophy is thought as the other side of the real; it is speculative in the sense of mirroring the real; however, it should not slow down the real (its event) with the reflective side of the mirror. Philosophy should reflect the real without adding a side or a face; it should literally replace the event of the real. I interpret speculative materialism⁷ (or speculation without the metaphysics of dogma and without the metaphysics of necessity) not just as the affirmation of the necessity of contingency but as the other side of the one-sided contingency – what becomes necessary to it in the (non-metaphysical, non-reflective) sense I described above. The correlation of thought and being, ultimately, is not a relation; it is a one-sided duality in Laruelle’s sense (he has photography as a special means of speculation or as an illustration of philosophy;⁸ I have the market). Once Meillassoux applies to his own medium (his own thought, his own book) the abolition of possibility that is required by absolute contingency, turning his thought into the material book, the problem is upgraded from one of thinking the real as given to thought or of thinking the real without thought, to the discovery of the new *matter of writing* in which the event is repeated. What I call the ‘market,’ thus generalized, the medium through which we connect with the event,

is thus the *book*, the new matter and the proper medium of Meillassoux's speculation. No wonder Meillassoux resorts to Mallarmé and to his poem, which is his book, as the solution of the problem he has posed in *After Finitude*.⁹ (Each one of our heroes proposes a solution after the book that has posed the problem: Taleb, Meillassoux.)

They are all after the thought of the real without thought. They haven't really dispensed with chronology. They want the book that writes itself (Taleb's *The Black Swan*) or that is necessary, subtracted from chance (Mallarmé's). How long before they realize that the matter of which books are made is *real matter*, as absolute and independent of thought as the one they prize so highly in their materialist ontologies? Books, of which the market is the special (numerical, quantitative, nonliterary) instance, are the technology or the material medium that truly permits us to rotate our universe in a direction that makes us insensitive to chronological time. Call the event of the real, its strike, its contingency, the world's best kept secret; and instead of wondering how chronologically or spatially to think with or without this event (to think it as uncorrelated to thought, as having happened before thought and as continuing to happen after thought), think of the availability of a matter that allows us to repeat this event. This is better than thinking it. This is literally making it and replacing it.

Philosophy tries to think itself outside its book (its whole book), somehow theoretically, in the same way that the theory of the market (best represented by derivative valuation theory) ultimately elaborates itself and misses the market in the last instance. Derivative valuation theory thinks of the market against the absolute background of probability. It believes it has no other way of animating the price process of the underlying than against a probability framework and it culminates in valuing a derivative of a certain order of complexity by stochastic control and dynamic replication by the traded assets whose probabilistic dynamics it has identified with the market. However, in doing so, it falls short of the last and most essential step concerning the market, which is that the derivative has to admit of a market in the last resort – or rather, in the first place. In order to think the market, it goes outside its medium, only to recognize in the end – but now it is too late – that the derivative that has been valued last, is *still* the market. Similarly, philosophy thinks the real; it even thinks the existing stock of philosophy books that are now part of its tradition and reality; however, ultimately, it fails to recognize that what it has produced is yet another book; that it is *still* the book.

I am not saying that there is no value for the thoughts that are expressed in the philosophy book outside the matter and the accident – the contingency, the event – of that book. Rather, I am arguing that to think of philosophical thought outside its book is to deny thought its material nature. It is true that anyone trying to think the real without thought will, in the end, have produced just another thought or written just another book; however, this is a facile rejoinder to oppose to him. Instead, one should reverse the direction; one should no longer think of the real as the independent domain, and of thought only as the circle that slows it down or reflects it. One should think that the whole separation of the real from thought or of thought from the real is a false problem and a consequence of the wrong orientation (an orientation that has stepped outside the book). For the thought of such a separation conversely slows down thought relative to its own material process.

One has positively to embrace the aporia according to which yet another book will be written and, no matter what we think, it is still the book. One has to understand this wholeness of the book (the whole book, the whole market) as a homogeneous medium whose acceptance amounts to repeating the event and no longer being separated from it (or, which amounts to the same, accepting the single stroke). Thought has had a tendency to grasp the real against a fixed frame (of thought) that keeps thought separated from the real, when, in reality, this absolute background doesn't exist, and the only way of thinking is to think 'intrinsically', following only the arrangements of matter. The only matter that gets arranged and rearranged behind the real and thought is written matter, without which we wouldn't know anyway what to say or what to write, what to leave behind.

The genesis of the book encompasses the genesis of the market, where the market is the special medium from which probability (this transcendent way of watching thought) has been retired and which is supposed to reorient thought accordingly, by detaching it from chronological time and its absolute metric and making it homogeneous with the event. The book is no less true than the real, and the category of writing enjoys a compatibility with the event that overturns the order of time. What was speculative thought thinking anyway in the standard order of time? That once the absolute is finally thought without thought (once speculative philosophy has reached its end) then thought would stop? On the contrary, I think that a speculative metaphysics without the dogmatism of the absolute, speculative thought without this absolute end (the absolute as an end), is precisely a thought without an end and without a stop. It is only through the reorientation and the correction of the medium that I suggest that thought will have both left the circle (this vicious infinity) and will be guaranteed to reach no end outside. There is an intrinsic link between the event of the real and the book (or material thought) which guarantees and explains that the real and thought never stop and that their event is always repeated.

3.3 Pricing vs. valuation

For me, the market is a new order of thought, a new metaphysics that frames the new matter in charge of filling in the blanks that the Black Swan leaves, of extending and 'continuing' in the discontinuity of the event, of providing the medium that is located at the hinge of the event, and so on. I have reached this conclusion through elaborating the category of *price*, precisely as it diverges from probability and from valuation, and more generally from the whole framework and order of thought in which we think of states and identify them, read them chronologically, evaluate functions that are written on them, and so on. So, it is true that price comes, in my philosophy, before the marketplace, which will only be deduced at a later stage as the place where contingent claims receive prices, as the place where their contingent payoff and final event gets translated – literally dragged in space – into a price, as the medium which allows us to recede from the final and massive strike of contingency without using probabilistic transitions or adopting the ex-ante attitude. Price comes before as a new logic of the count, or a new logic of the one and many – a new mathematics, even. Derivatives, or contingent claims, are essential in this presentation of price because they are written – they are a

schedule of different clauses – and because they articulate the dialectics of the one and the many in the difference of their clauses.

I need, first, to establish the difference between valuation and pricing, between (a) reading the value of a function as chronologically differing *because* its argument or underlying state is chronologically identified, and then recognized as differing in the reading, and (b) reading the function as massively and ‘non-chronologically’ differing, as differing without distinction and identification of underlying states and possibilities, as differing in the massive sense of the ‘too late’ and ex-post, which is the sense of contingency – the function being what it is, not in a sense of a recognition of a state but of a strike, or a mark, which *could have been different*.

Now, this inverted mode of reading is precisely the mode of *pricing* as opposed to *valuation*. I need to establish this point of logic, which at first seems remote from the market, before observing that the latter mode – valuation – is conceptual whereas the first mode – pricing – is material, and requires the material sheet on which the function is engraved. It is in a subsequent move that the difference of contingency, instead of being integrated in the supervising state, is recognized not to be translatable and not to issue otherwise than in an exchange (of the material sheet). The market, which exchanges the sheet, is thus defined (constructed, constituted) as the only way to ‘evaluate’ the difference without looking at it in an ex-ante fashion, by evaluating it after the fact, after it is too late, and dragging it before it takes place (for evaluating something is always a matter of looking at it ahead of time – from this there is no escape).

We think there is something, a bridge, between now and next, between the present state and the future state, and that probability will play out in that interval. In reality, there is nothing between the present and the future; there isn’t even time (time is not the proper medium) because the future contingent event is totally unpredictable (possibilities are never totalized).

Now, consider an alternative in which the relation between present and future (the forecast) is not an artificial, logical, and chronological consequence of the totalization of possibilities. Consider an alternative in which we face the real future – not one masked by an ex-ante attitude whose only cause is the present, or rather past, identification of possibilities. How to have, as of now, not something precise to say about the future (for we are no longer standing in a precise state, now that we no longer precisely identify the possible states), but literally something massive and imprecise, literally traces of the future that have been imported into the present, instead of a false precision that is due to a present artificial identification and has been exported into the future; how to have the ex-post signature of the future somehow handed over to us now? This is the thought, which we will revisit many times later, that the tree needs a root and needs to graph all the possibilities beforehand, whereas the market is the present mass that can ‘stand for’ for the later mass and massivity – up to and including the massivity of disappearance of the sheet of paper on which the contingent claims are written.

I claim that writing is that massive thing that does not distinguish precise states in the future. A contingent claim is written and is programmed to pay off a certain amount of money in the future if a certain event obtains, which is *precisely* described in the writing and which we may, for this reason, provisionally call a ‘state’. However, the payoff is not only contingent on the precise encoded state; it is also contingent on the material sheet of paper that carries the writing. That piece of paper must be

exchanged against the money that will be paid off. The holder has to produce evidence of his claim and to hand it over at maturity in exchange of the payment. Because of this material procedure which has to take place in the real future, nothing is really programmed in advance; nothing is projected, as was the case with probability. For all we know, the holder may suddenly realize that he has lost the sheet of paper (he no longer materially holds the claim), or the conditions ruling its exchange against money may fail to obtain. Imagine that the place of exchange is destroyed, or that the wording of the contract is found to be legally invalid, thus enabling the liable party to escape its obligation. Imagine that the whole world ends. Imagine, in other words, any contingency that may be associated with opening testaments at the time of opening them *and not before*. For this reason, the sheet of paper does not carry any state at maturity but is just massive; its ex-post fate is 'written on it'. Also for this reason, what it carries today is not a state either; it is the whole market. The sheet of paper that was subject to its massive contingency at maturity (in the future) is *still* with us today – the same sheet, as real and undivided. It is exchanged against real money at maturity, and it is exchanged against real money today (at a different price) in the market.

The equation is as simple as that: 'It is exchanged *and* it is exchanged'. Money provides the time dimension and the communication; it constitutes one medium. The exchange of the material sheet at maturity is the other side of all contingencies impinging materially on it, such as default or the end of the world. The exchange at maturity is pricing because of the necessity to include the ex-post fate and the price is ex-post as a result. On the other hand, pricing is possible today thanks to the exchange, because the material sheet is still here today, to be exchanged. One more time: pricing is the *translation* of the testament. Pricing is the realm we find ourselves in when we realize that the mark is opened and read, and we recognize ex post that the mark could have been different. The piece of paper needs to be materially exchanged at maturity in order to translate this price, because exchanging it then is the enactment of its fate ex-post, after the event. Now, before maturity, we still have the exchange because we still have the sheet, therefore we have pricing, we have the ex-post 'valuation' even though we are situated before chronologically. In a word, we have *pricing*, even before, despite the fact that price is essentially ex-post.

In contrast, we cannot say that the state that will be realized in the currency of probability is with us today. If we assume that it is real in the future (when and if realized), it is not real today, but only possible. It is not realized through a material exchange in the future, through something that really crosses a place and leaves a trace (money) in the future that can be found before the future (today), but through an intellectual attribution (probability = 1). As a matter of fact, it is not even real in the future and it doesn't really belong in the future; for it is only an extrapolation from the past. Money was invented for the purpose of mediating between present and future in the correct *forward way*, connecting real to real and not possible to real (which is the backward reverse way) and without the precision and, indeed, the fragility of probability.

Money, writing and the exchange are facets of the same solid. Money was invented for the purpose of monetizing that thing in the future which will be still with us today, and money was invented for the purpose that there should be a price in the future (the real amount of money exchanged then) *and* a price today (the real amount of money exchanged today), keeping in mind that the market is

what closes the solid and allows the sheet to change hands today in advance of tomorrow. The market is massive at both ends. It is massive at expiry, in the sense that the contingent claim is contingent upon the massive material sheet and not just the encoded distinct clauses, and has no choice but to be exchanged as a result; and it is massive before expiry because there are no delimited and distinct states of the underlying either, or of any derivative that may act as an underlying for a higher order derivative, but virtually all the prices of all the derivatives. The market is massive because of the fundamental principle.

It is when states are identified and enclosed in their reciprocity, it is when their device is later run in complete isolation from the real world and when the stability of frequencies is observed, that the notion of generator is formed, which is now supposed, ahead of the event (*ex-ante*), to *generate* it with a probability corresponding to the frequency. The generator is the transcendent instance that is supposed to replace the immanence of statistics. Worse, the random generator is reified; it is believed that there exists something real and instantaneous (Popper calls it a ‘generating condition’ or a ‘propensity’) that would generate the statistical series with stable frequencies if given enough time. The modesty of waiting until after the event (*ex-post*) is replaced by the supremacy and pretense of the generator. My claim is that the category of price, especially if it is thought through the narrative of derivatives and their market, is the way of resisting this transcendent generator and reinstating immanence.

3.3.1 The surface of the market

The reason price is interesting as conducive to immanence and as instating the *surface* of the market (and it is the surface that we want, above all) is that in our analysis of derivative pricing, we encounter two notions of price that we need to level with each other and with the surface, as well as two notions of the exchange that we need to make identical. In one instance, the exchange is considered as the generator of prices of the underlying; in the second, as the massive ‘evaluator’ or rather ‘pricer’ of the contingent claim, where price is defined in contradistinction to value, as translating the contingent claim massively without any distinction of possible states. Price is just a number in the first instance, and it is a logic (even a different mathematics) in the second. The exchange is a mere number generator in the first instance, a topology in the second; it is the other side of the material sheet on which difference or contingency is seen as a single and singular mark and not as a list of states.

Yet, the market is one; it is not split between two notions. Price cannot, on the one hand, be defined at a level of thought where it competes with probability as a logic of mapping the one to the many and, on the other hand, behave like a random number that is ruled by probability and follows a stochastic (time) process. Ultimately, the genesis of price through pricing, or the concept of price as that which happens to the contingent claim once it reverts through its own written matter without distinction of states and is exchanged as a result (for lack of a transcendent space in which a supervising state would be found to support value), ultimately, this inner *process* of price, which we earlier called the meaning of price, must take over the customary notion of an external generation of prices in the market, understood in the sense of random numbers. Time as the dimension in which

the random generation of numbers takes place must be replaced by place, where a generation of writings takes place instead.

The external dynamics of price in time has to become equal to the internal dynamics of price, or the dynamics of genesis. Time as the dimension of the time series of prices has to be made equal to place, or the time, the face, the topology that separates the event from its translation. Price as time process has to become superimposed with price as place process. It is in place that the translation of the ex-post fate to a time 'before' takes place. It is in place that the infinite generation of writings takes place, whose conclusion is to place us in the middle of the event with no time separation.

The image of the time series is very powerful and entrenched. We cannot help thinking that, out of the continual confrontation of demand and supply, price changes in the underlying are continually generated, thus turning the market into a kind of engine. Yet, the continual exchange that we must think of should move along the dimension of writing, not that of time. The underlying price should not follow a stochastic process but its own writing into a derivative payoff; then, in turn, the writing of that derivative into a derivative of higher order, and so on.

It appears that certain statistical parameters are crucial for the pricing and hedging of derivatives. We are told that the underlying price follows Brownian motion (for example) and we are supposed to infer the instantaneous volatility of the process, which we need for pricing and hedging, from the time series of the underlying prices. If the underlying price were really sampled continuously through time, we would be able to find as long a time series as we want in as narrow a time interval as we wish, as this is the fabulous advantage of Brownian motion. Since instantaneous volatility itself is liable to change (as witnessed by the trading of volatility derivatives), in theory we must perform this extraordinary act whereby we mine the largest set of statistical data in the narrowest time interval. However, the derivatives market spares us that extremity. Instead of inferring volatility historically (even benefiting from the Brownian marvel which enables history to shrink into a spot), we imply it from the market price of the derivative itself, *which is otherwise given*. We simply find the volatility coefficient to feed into the BSM formula in order to match the derivative price. Since our purpose is to price the derivative, we may as well use the volatility that makes the price right.

The local properties of Brownian motion exempt us from worrying about the volatility (or, indeed, the whole statistical distribution) that should prevail over the whole lifetime of the derivative and from finding the correct one such that it makes us break even on average when plugged into our hedging strategy over the whole finite horizon. Thanks to the fractal character of Brownian motion and to the stochastic integral, our hedging strategy breaks even 'in the long run' even on the spot, and all we care about is that we be hedged against the next market price change of the derivative. However, the locality of everything (which is the other name of the market) soon turns into a complete statistical black hole. For not only is the notion of a given (reified) and stable random generator already stretched to an extreme with the local character of Brownian motion, but, since the derivatives market has replaced statistical inference by making us infer a statistical parameter (volatility), which in theory pertains to the series of prices of the underlying, from the instantaneously given price of a given derivative, we now have to worry that this inferred parameter (implied volatility) may not make the price of all remaining derivatives come out right.

This is the beginning of the smile problem. The smile problem, as I understand it, is the problem of inferring 'statistical' parameters, which in theory pertain to a given underlying and which we need in order to price and hedge its derivatives, when all that we can observe are market prices of derivatives. It is a severe problem that ultimately leads to the total demise of the whole statistical paradigm (and, correspondingly, the notion of random generator), for the reason that there is not one stage that we may reach in which we may have satisfied ourselves that a certain stochastic model has been calibrated to the extant prices of derivatives, yet in which we may rest confident that the next set of derivatives prices that will prevail in time will be matched as well.

From the instantaneous market prices of derivatives, no matter how complex they may be and how involved a story their prices may tell us about the dynamics of the underlying price, there simply is no way we can predict the future dynamics of those derivatives prices. From a snapshot, there is no way we can infer the subsequent motion picture, unless we believe that the underlying random generator has been captured and will persist – unless, that is, we believe that such a random generator existed in the first place, which we could have inferred statistically given a long enough time series, but that for some reason we decided to infer it from the market prices of derivatives only, as an alternative method. The truth is that the market prices of derivatives are the only method.

The surface of the market constantly blocks any attempt to withdraw from it into a transcendent stance in which the random generator would finally be uttered. A random generator is the guarantee that certain states of the world are visited over time with certain probabilities. However, the fundamental principle of the market is such that those probabilities themselves are put into play and into trading and become additional states of the world. The set of derivative prices against which we calibrate a certain model is never complete, and there is no guarantee that the derivatives of higher order, which did not trade so far and did not register an actual price, would have their virtual prices agree with the calibrated model. As a matter of fact, had those virtual prices been actually available, calibrating the model against them additionally to the existing prices of derivatives (of lower order) would imply different price dynamics for the latter depending on what those virtual prices are. One should therefore never infer anything statistically when facing a market; one should never withdraw into an ex-ante attitude where a random generator is given the right of existence, not even for one second. On the contrary, one should infer without withdrawing and without inferring, always going forwards, always following the surface, always acting as if the virtual price of the next complex derivative could have been different.

The act of inferring statistical parameters, for the sake of pricing derivatives, from the instant prices of derivatives (and not from time series of the underlying price), this act which is definitional of the surface (since it abolishes the supremacy and verticality of the generator) and consequently definitional of the market, has not yet been thought thoroughly and radically. Whatever we do, there are still in our minds remnants of the thought that the market price is a temporary deviation from value, and that value, as derivative valuation makes it clear, is only a matter of probabilistic weighting of the final derivative payoff (or equivalently, a matter of statistical break-even of the strategy of holding the derivative).

We still haven't thought thoroughly what this means: that the information should come from the market prices and nothing but the prices – that it should come from the future and not from the past. Derivatives have the advantage of turning any information about the price into another price. They help us complete the radical shift towards the market, when the market is to be conceived as a radical alternative to statistics instead of a case of statistics. To think what price is, and consequently what the market is, the whole sophistication of derivatives is needed. It is not a coincidence if the exchange is essentially linked with randomness. The existence of a frictionless market and the capacity to buy and sell without limitation imply that the price trajectory should be predictable (differentiable) at no time scale and should be irremediably inflicted with randomness. The efficient market hypothesis entails the random walk, because any predictable future price would by anticipation be traded in the present spot, leaving for the future only the unpredictable. However, this ignores the fact that derivatives, virtually of any order of complexity, are also trading and, consequently, preventing the thought of the random generator from ever materializing.

Randomness is somehow inseparable, in our minds, from the notion of a statistical population; and the sampling of the latter is inseparable, in our minds, from a chronological sequence: a random sample then the other. Randomness seems inseparable from the notion of a trial. Yet, the thought of the radical randomness of the market is such that even the idea of a trial is forbidden. Time is usually the revealer of randomness, either physically or logically. What we are talking about, concerning the massive randomness of the market, is that not even the first sample be drawn. What we have next is always the massive event, which is never a sample. It does not happen in time but takes place. Once place is understood in that sense, as the alternative to time, when speaking of massive contingency, and as the exchange place of contingent claims, which are precisely written as bearers of this massivity and precisely delivered to the exchange as the other side of writing, the marketplace and the exchange acquire a generic meaning that should be generalized to all kinds of markets. All we need is that there exists a future (a contingency) and that there exists a place. So, the probability paradigm seems to fall short of the price paradigm, and this seems to require a revision of the entire metaphysical framework of possibility and realization. This is what prompted me to look into alternative frameworks such as the Deleuzian virtual or the Nietzschean dice-throw.

3.3.2 The smile problem

In an article I wrote in 2006, I used for the first time the expression 'anti-Black Swan trader'.¹⁰ Taleb, who had not yet published *The Black Swan* but was already recognized as the creator of the concept, or at least of the expression, said he was intrigued by my 'anti-Black Swan' proposal. My 2006 article was a defense of the derivatives market as the technology that could take care of contingency beyond probability, or of the Black Swan event that fell precisely outside the probability framework (literally, a technology of the future). Simply, the idea was that, if the states of the world were to coincide with the prices of the underlying asset that the derivative was written on, and if this underlying asset was tradable, then the probabilistic framework and its fixed states of the world would ultimately result in derivatives being algorithmically controlled (replicated) and in their values being perfectly assigned.

Yet, the market is by definition the place of trading of *every financial instrument*, no matter whether derivative or not, *at prices that precisely vary from any pre-assigned value*.

Any probability framework, no matter how complex you make it in order to model the trading behavior of derivatives up to a certain order of complexity, will admit of a derivative of a higher complexity that will be perfectly replicable by the derivatives of lower order and will, therefore, be denied a market of its own and a variance of its own. In other words, the event of its market, if it is to take place, must exceed the existing framework and be incalculable within it. It is precisely the market-maker of that derivative, although he needs the probabilistic framework in order to hedge (replicate) the derivative that he buys or sells and manage his inventory, who must, qua maker of its market, be attuned to this excess, to what falls outside the framework. Nobody has yet thought the market through (and as) this contradiction – or rather, paradox. To repeat, the market is the category that is yet to be discovered and analyzed; it is the matter and the science that is yet to be created. Derivatives just happen to make the discovery of this new matter pressing and to pose the problem in the most acute way.

The probability dynamics that supposedly governs the underlying price is not the real market dynamics or the real price dynamics (even though every derivative textbook will represent the market as just this underlying stochastic process). In the immanent theory of the market that derivatives allow, the real market dynamics – eventually unpacking the meaning of ‘price’ – takes place when the derivative market-maker, equipped as he may be with the derivative valuation tool that reduces the market to the probability dynamics of the underlying price, *trades* the derivative *in the market* and, by so trading it, intends it to vary from the prescription of his valuation tool and to leave its circle. The mystery of the medium that the market-maker is immersed in (the market) is such that the price he makes becomes given at once and made by the market. The derivative price that the market-maker makes is in every sense the output of a formula, the result of a self-financing dynamic strategy that holds and defends that price; and yet, the nature of the medium in which this price is posted is such that the price turns immediately into an input to the formula, inverting it. The contingent payoff that has been replicated eventually turns into a full-fledged contingent claim which trades and admits of a price independently.

It is not necessary that the price that the market-maker makes with the help of the formula (let us call it a ‘value’ in this outgoing movement) should be numerically changed by the market in a subsequent price movement. It is before any change (yet), *as he makes it*, that the market-maker intends it as a market price, therefore as a price that is given by the market and liable to be fed into the same formula (now used in reverse) in order to imply volatility, for instance. The market is this instantaneous paradox of authorship. The market-maker uses the pricing formula to make the price, but the formula is, in a way, useless because the price is made by the market. Yet, the market-maker does not discard the formula; he needs it precisely in order *to make the price as made by the market*. In this simultaneous exchange of the category of the given and the category of the result, in this exchange of the input and the output, lies the specific nature (or mystery) of the market.

The smile problem is the perfect expression of this; it is not meant to be solved because it simply *is* the paradox of the market; *it is* the market. The smile problem attests that you are maintaining the

usage of a pricing formula (in this case BSM), because the smile refers to the BSM volatilities that are implied by the price of each option. Yet, at the same time, it attests that the formula is immediately invalidated by the market (and that you should logically suspend its usage) because the implied volatilities do not come out equal to each other, as BSM would require. This paradox can only make sense through the presence of the market-maker. The market is not left alone (as if the prices of derivatives were just given by the market independently of a maker). One needs the moment of creation. The smile problem is there for the market-maker to see that the prices are the results of his creation (the outputs of the formula he is using), yet, that the main creation of the formula, its main product and result, is that the market turns against it and invalidates it. *The market validates the formula by invalidating it*, because the market that is thereby created does not fully become a market unless it departs from its creator (and was even meant to so depart in the first place, within the same moment of creation). One has to realize that the term ‘market-maker’ is a living paradox, not a contradiction, whose only consistency comes from being constantly affirmed. The only question, which we will tackle in the last part, is whether the paradox could be dehumanized and formalized.

3.3.3 The absolute local

The key to the paradox, which is the key to what I have called the ‘true market dynamics’ – and therefore the key to the market – is dynamic hedging. Dynamic hedging is what precipitates the market-maker in the market and pins him on the floor. It turns the problem of the valuation of nonlinear payoffs into a local problem, which requires the presence of *locals* (another name for market-makers). If you are holding a costless option, you don’t need to hedge it, of course. If you are holding a costly nonfinancial option (an asymmetric payoff, as found in life by Taleb) you may not be able to hedge it; but your losses are limited anyway; you are tinkering and learning from trial and error; but your upside is unlimited. In both cases, you withdraw from the dynamic market; you leave the local, or the place. Financial options, by contrast, introduced the possibility of being dynamically hedged. Now, the premium you pay is no longer the solution of the ex-post accounting equation of the insurer who has sold infinitely many premiums like the one you have bought, and who will therefore break even in the long run, when all the contingencies will have been visited with frequencies corresponding to the premiums collected. Instead, the premium now becomes a *differential* in the mathematical sense; it is no longer a cost but the *rate of change of a cost*, or time decay, because the accounting has become local. Accordingly, the contingency that the premium covers is reduced to its differential, to the essential difference that is at the heart of contingency – that is to say, to the essential nonlinearity. (It is of the essence of the contingent claim that it be nonlinear and that the payoff be asymmetric.) The main consequence of dynamic hedging is to bring nonlinearity and locality together. This may sound contradictory, since nonlinearity is associated with the line (albeit broken or convex) whereas locality is associated with a single spot. But on the contrary, we will see that to understand the sense in which the local, the spot, the market, can be said to be nonlinear is to understand how the market can be the metaphysical and radical alternative to Taleb’s proposal of basing antifragility on nonlinearity and asymmetric payoffs.

How the contingency (in theory supposed to settle at maturity) is locally expressed and locally made present is through the fact that the rate of change of the premium is no longer tracking a final settlement but, instead, the market price of the contingent claim. The magic of the market is to make the ultimate and final contingency *present and local*; for this, it needs the mediation and the translation of price. *It requires the invention of price.* The market price (always by definition the price of a contingent claim) should be defined as that which is required (as the new kind of matter, or material process, also known as technology) in order to bring the ultimate contingency and the ulterior event into the spot; and to transform the relaxed and resigned (and one might say, infinite or indefinite) attitude of whoever follows Taleb's antifragility therapy and programme into the dynamism and constant unsettlement and finesse of the trader on the floor.

The spot was required. Let us consider this as the first necessity. *If it didn't exist, it would have had to have been invented.* It was later discovered that dynamic hedging was the way; however, the availability of the price of the derivative takes equal part in the invention. One tends to forget that tracking the market price is an integral part of the dynamic hedging argument.

The market is local; local is the keyword of the market. And 'local' implies 'place'. Time decay, the nonlinearity of the next price change, and the market which presents all this; this is the local miracle that was made possible. Volatility, or the miracle through which Brownian motion can be sampled in an infinitely shrinking interval yet remain random,¹¹ makes it so that when contingency is reduced to its essential and local difference, it can only be the convexity of something given locally – namely, the market price. Statistics could be reduced to a single spot and yet retain the notion of infinite long run; an insurance premium could be reduced to an instantaneous rate of change (time decay); what we thought impossible (that time series and statistical populations might be reduced to a single spot and retain the infinity of the law of large numbers) was made possible thanks to the wonders of Brownian motion; however, the existence of a market was required in order that we may be able to track an instantaneous contingency locally. The market is the way to make the insurance break-even problem (and solution) local. (I would even say that it is the only way.)

Dynamic hedging is not just a subtlety or a finesse made possible by stochastic calculus. It is true that it turned the contingent claim from a vehicle of extensive difference into a vehicle of intensive difference. It is true that it no longer mattered how distributed in extensive time and space were the different payouts of the contingent claim at its expiry (either at maturity, or on knock-out barriers, and so on) but that the only thing that mattered now was the local convexity of the local market price of the contingent claim, whose fortune under the local randomness (expressed by local volatility) was to be balanced by the local rate of change of the premium (time decay). It is true that this was made possible by the fact that the underlying was tradable (thus enabling dynamic hedging). But more importantly, it was made possible by the fact that the market price of the contingent claim was considered given and available. The two go together. One tends to forget that BSM supposes that the value of the derivative as a function of time and the underlying price *is given* in order, then, to track it with the dynamic replication. But what could it mean to say that the value is given other than through the existence of a market giving it, as a price? And if you say that the value is given per se and has to be so given, held by none other than itself or the argument establishing it (for if it were different, then

anybody using the BSM replication argument would be able to buy the derivative at a lower price or sell it at a higher price than this given different value and realize an arbitrage), you will be yourself affirming that this value is given as a market price, to which the rule of non-arbitrage precisely applies.

It is not long before the availability of the derivative price in the market becomes the first given – or rather, the only given. For who is to make sure that the given derivative price in reality comes second to a first given that would be the instantaneous volatility of the underlying? Only if the instantaneous volatility were known would the BSM replication argument guarantee that the derivative price had, derivatively, to be such. But in what way and following what route does that guarantee function? What is the conducting medium of the BSM derivation? We know what the conducting medium is formally, but what is it *materially*? Perhaps the BSM derivation is only a formalism, and its only *material* interpretation is that the derivative price be given in a market and the BSM formula be inverted, rather than instantaneous volatility being given and being known. Perhaps there is no material meaning to instantaneous volatility being given (and known), and the only given volatility is implied volatility.

Dynamic hedging made the problem local. It turned the ex-post accounting equation of the insurance company into a local problem (infinite long run compressed into the single spot) thanks to Brownian motion; however, it simultaneously brought into existence the local matter, the absolute *local* which I call the market – the fact that the derivative price had to be given by the market. Once the ‘insurer’ becomes locally involved and becomes a local dynamic trader, there are no longer any extensional statistics or actuarial tables upon which he might rely. Even though the statistical parameter (volatility) is formally defined in an instantaneous and local fashion, this only takes place on paper, and the only material statistics on which the trader can rely in order to get his local dynamic hedging strategy going is the market price of the derivative, against which he inverts his pricing and hedging formula. In my reasoning, the inversion of the formula (regarding the price of the derivative as the primary given) comes as an integral part of the material interpretation of the BSM formalism.

The *reality of the market* is the interpretation of the formalism of Brownian motion in the same way as the reality of statistics is the interpretation of traditional probability formalism. For this reason, the smile problem is not a temporary stage, a problem soon to be followed by its solution or correction. The smile problem is the ultimate reality we are talking about, because it is the expression of the fact that the derivative price is always *made as being given*. This implies that the only reality of the pricing model, or the only reality of the given probabilistic framework, is its constant recalibration. Correspondingly, the only matter in the BSM formalism is the writing of the contingent payoff as a contingent claim. The reality of the market is the constant excess, or the constant event that disrupts the existing set of possibilities (the constant Black Swan). The market-maker shouldn’t so much consider the derivative value as being the output of his formula or the product of his replication strategy but, rather, consider the derivative market price as the reason to recalibrate the formula. If the only way of getting the instantaneous volatility that is needed in the BSM formula is by implying it from the given option market price, then this by itself implies that the BSM formula is soon to be recalibrated and the implied volatility soon to change when it is inferred from the next option price.

3.3.4 In the middle of the event

Turning reality from the fixity and calculability of the states of the world (their extensionality) to the intensity of recalibration is turning from a dwelling in representation to a dwelling in the constant event. It does not mean turning from fixity to change or from state to flux or to becoming (in the sense of a temporal succession of different states), because recalibration is not a temporal process; it has nothing to do with a learning process. It is intensive in the sense that the whole infinite chain of recalibration opens up right on the spot, in the same bottomless pit. This is encoded in the meaning of implied volatility: it must be stochastic because you are implying it from a market option price that would not make sense as a market price unless it was construed as stochastic. But any subsequent stochastic model that you might think of for implied volatility is not the last word either, because you would also be calibrating it against the market prices of derivatives (i.e. implying its parameters from the derivatives market prices) – therefore it would also be encoded in that model that it will be recalibrated and that its parameters will turn stochastic in their turn. *This whole infinite chain of recalibration of stochastic models of increasing complexity is all instantly contained in the meaning of market price and in the meaning of the market.*

The market-maker lives in this meaning and in this intensity. He lives in this pit. It would be interesting to ask what becomes of his relation to the states of the world and to the probabilities that are attached to them. What kind of a gambler does he become? One can see that because he lives right at the hinge of the event (in the middle of the event) and not in the world that precedes or follows the event, he somehow achieves a ‘state of rest’ relative to the event.

He lives at the same (infinite) speed as the event. There is a kind of necessity rather than chance that is attached to him. He is *subtracted from chance*, as Blanchot would say, and for this reason he writes the necessary book (the market). Or he is the player of the single (Nietzschean) dice-throw *for all times* (and not once and for all).

In my 2006 article, I used the character of Gene Kranz (the Apollo flight director) in the movie *Apollo 13* as a depiction of the anti-Black Swan trader. The explosion of the oxygen tank in the Service Module triggered a massive recalibration of the whole framework of (stochastic) control represented by ground control. Crucially, the new states of the world that had to be created (and were not part of the previous situation) in order to save the astronauts emerged from the same floor and from within the same technology. The broken technology itself was re-engineered on the spot, without ever leaving the trading floor, in order to bring the astronauts home. Similarly, when looked upon from the point of view of the theory and ‘ground control’, the derivative pricing tool is always exceeded by the incalculable event of trading of the derivative and, as a result, is always broken. BSM says volatility is constant; however, pricing options with BSM in the market leads to implying volatility from the option price, and therefore to stochastic volatility and to breaking BSM. But from the point of view of the floor, from the point of view of the dynamic trader who lives in the middle of the event and always recalibrates the pricing tool, the latter always succeeds in bringing the astronauts home.

Contrary to what it was designed to do (i.e., land the astronauts on the Moon), the Lunar Module was used as a raft to bring them back to Earth. ‘I don’t care what anything was designed to do’, says

Gene Kranz, 'I care what it can do!' This incalculable *capacity*, which addresses the incalculable event beyond the design, wouldn't have been available if Gene Kranz (and his team) had not acted as dynamic traders, totally immersed in the floor. As a result, he could no longer give the odds of the astronauts returning home. When asked by the president of the United States what the odds of rescuing them were, he answered: 'We're not losing the crew.' When asked again, he answered again: 'We are not losing those men.' This did not mean he was certain, as opposed to uncertain. It just meant he was subtracted from chance and was no longer a traditional gambler. He knew no state of the world other than the one he was making himself. Even the category of count is different in his case, and to say that Kranz knew only of *one* state of the world, as opposed to several alternatives, is also to miss the point. Kranz was aware of the massive situation and of its massive contingency. He wasn't aware of the count of states. The singularity of the event is totally different from the individuality of the state of the world (or of the event now counted as a state of the world). It is always after the fact that you count the event. 'During' the fact, the event is massive and the only way of being aware of its contingency is repetition.

When asked, Kranz repeats. I don't think this engaging with the event, this solution to the Black Swan problem, can be described as 'antifragility'. In a sense, it is a very thin line; it is finer than the blade with which the event cuts the world and chronological time. Kranz is a bladerunner. This matter, in which Kranz or the dynamic trader dwells, is very refined; it comes from the excess of refining the study and the matter of the market beyond any possible probability framework. To the trial and error (tinkering) that Taleb proposes, seemingly unaware of time decay, one should oppose the motto of Gene Kranz: 'Failure is not an option.' The excess of refinement or extreme aristocracy in dealing with the event implies a different attitude towards life and the world. The refined market-maker also likes volatility, because it is volatility that brings action to the floor and stretches and tenses the material surface that the market-maker lives in. He likes volatility actively – not passively, as Taleb requires. Taleb wants you to take care of your losses and let your gains take care of themselves. He even describes the history of inventions and technological innovations in terms of this relaxed attitude. Accordingly, Taleb does not have to worry about the lack of volatility because he is waiting and seeing anyway, and is (seemingly) insensitive to time decay. But the active market-maker dislikes the lack of volatility because he gets bored when there is no action. The refinement of thought and aristocracy have melancholy as their correlate. It is not the case, as Taleb claims, that we live in a world that we don't know. Rather, we know the world only too well – so well, indeed, that the world can sometimes be incredibly boring!

The Thought Behind

4.1 Two sides of writing

The difference between possibility and contingency, and consequently between probability and price, may have ultimately to relate to the most entrenched categories of thought; namely, the one and the many and its correlate, identity and difference. Difference, it seems to me, can have two readings: analogical and intrinsic.

We see that two elements in a set are different, or that two values of a function are different, because we have initially identified the two elements or the two arguments. We see that their identities differ and consequently we see their difference, in that order. (Think that 'identity' is itself a function.) There is a chronology of thought here, a mapping and a reading, hence a particular logic. We actually see the whole collection and the reciprocal relations of 'identical with myself'/'different from my neighbor' before we see the individual as such; we see the *relative* individual before we see the *absolute* individual. We see *different* possibilities before we see *one* contingency.

By contrast, the non-relative and non-analogical, or again the absolute or intrinsic perception of difference is when we are presented with the individual *as such* and when, by a kind of Gestalt switch, the 'as such' is not intuited as 'this individual whose identity circles around it to identify it' but as 'this individual which could have been different and which could have not even existed'. It seems to me that identifying the individual as a closed entity is a later movement and re-circling when the 'first blush' is to perceive the individual as a mark, as a mere singular shape whose first impression on us – literally, as a mark – is that we recognize that *it could have been other* rather than recognize *it*.

Difference, whose other name is contingency in this case, seems to be a simpler and more primitive grasp of the individual than its being. Even the word 'individual' sounds as an overworked complex notion, as it means literally that the individual is one and cannot be divided. By contrast, the simpler 'one' that I am after, the 'one' of contingency rather than the 'one-as-opposed-to-others' or 'relative-to-others', is the 'one' of singularity. It amounts to making the 'could-have-been-different' a single and simple notion instead of a complex and derivative one; that is, a primitive.

Correlatively to the two readings of difference (difference relative to different possibilities and difference as absolute contingency) there are two interpretations, or two sides, of writing.

A mapping basically has to be written in order to instruct the difference between the values of the function relative to the difference of the arguments. Writing in this case is a means of fixating the difference; it is here to store the difference and to present it. It is a copy of the analogical difference, hence derivative on it. Whenever we run a schedule in mind in order to migrate from one case to the next, we run a list and what matters are the contents of the list.

By contrast, the writing that is associated with contingency is inseparable from it. To receive as first impression from the individual that *it could have been different* is readily to perceive it as a mark and nothing but. Here, writing is no longer a means, or the mere receptacle of content. It is itself the message. We are tapping, here, into the unavoidably material nature of writing: the fact that in order that something may be written and literally articulated or differentiated, the writing has to impress itself on some material support, either the material sheet of paper or the material stone, or the material computer memory, and so on. To that extent, the impressed material becomes a sign almost without content; it signals only its own intrinsic difference or singularity, that is, its own absolute contingency. For one thing, the material support can itself vanish.

My whole investigation starts when I recognize that *probability* is an integral and that it amounts to integrating the many possible states into the one antecedent (supervising) state in which they receive probabilities and from which they constitute probabilistic transitions. Probability, as such, is associated with the first, 'contentual' and conceptual reading of difference. I then recognize that *price*, or the result of the material exchange of the materially written contingent claims, is associated with the second reading of difference. The market price is the medium of contingency.

Probability is inherently chrono-logical because of the chronology of the operation of thought involved in the first reading of difference. Probability translates the schedule that is inherent in the analogical/chronological reading into the schedule of probabilistic transitions and graphs between states of the world.

By contrast, price is not something that 'supervises' the mark of the contingent individual. Exchanging (in an exchange place) the material imprint is only its other immanent side. As we think that the mark could have been different, our thought exchanges it rather than attaches it to a schedule of relative states and differences thereof. If 'time' or 'chrono-logic' is the logic under which probability operates, I call 'place' the logic under which the 'exchange' of contingency takes place. The timelessness or 'infinite speed' of the mark and its 'mark-et' are to be understood in this sense – a logical sense.

The exchange or the market is not a generator of random numbers that we call prices. Rather, the exchange and price are inscribed in the deeper level of thought where the relation between the one and the many receives an alternative reading. The market is a category of thought, according to us, and it is the genesis of the market and price, rather than the generation of prices, that we should be talking about. This won't stop the market from falling empirically into time, or price from falling into temporal or stochastic processes, as time is empirically unavoidable.

4.2 Genesis of price vs. generation of number

It is only externally that the price is identified with a (random) number. The world starts admitting of numbers when we distinguish and list the possibilities. We attach '1' to the undivided reality as soon as we distinguish (or think we can distinguish) the different possibilities that it could have been, to which we attach '0'. Since the real world cannot coexist with the different possibilities that it could have been, we artificially recede to a past time when the real world and the artificial possibilities are on the same footing, simply as possibilities. This is how probability is invented. It falls between 0 and 1, and to this extent, it is a number.

Alternatively, let us imagine that the real world could have been different, without indulging in the fiction of identifying the different possibilities. Reality is massive and undivided and undistinguishable (in this sense, it is 'unpredictable'), yet, it could be otherwise. It is not that we don't know what other possibilities it could be. The argument is ontological, not epistemological; it is even logical.

I argue that the way to express the contingency of the world without the intermediation of possibility and probability is the writing of contingent claims. The payoff of the contingent claim is written over it. It is real and it is material. It is not a future abstraction or projection. Likewise, the price it receives in the market is real and material. It is not propagated in a tree of possibilities. It is not computed through probability.

Insofar as price is alternative to probability and written contingent claims are alternative to fictitious possibilities or states of the world, and insofar as probability applies to abstract random numbers, price is not a number.

The matter of probability is the ontology of states and beings, enclosed in a reciprocal play and integrative list. Price is an alternative geometry whose matter is the ontology of contingency – not states but marks, not the verb 'to be' but the verb 'can be', whereby a thing is contingent simply and absolutely because it *could have been different or for that matter not be at all* and not because it is in this state rather than in that other state, or in this state relatively and comparatively to not being in that other state.

It is a category mistake to refer to the market as a random generator of some sort, under the ontology of states, and correspondingly to speak of randomness and prediction. This register cannot account for price being a different category. By contrast, there is no randomness or prediction in my ontology because prices are not generated by a random generator. (It is not yet settled to what extent the whole simile of random generation, and even of statistics, may not just be a mere image and a mere interpretation – a logic, which may not be the only one.)

Prices are not generated because the dynamics they spring from is a dynamics of genesis. This is a much earlier and deeper stage in which an alternative pair to state and being – namely, contingency and writing – furnishes the infrastructure. Prediction and correlatively randomness are out of the question because there are no possible and delimited states to predict in the first place. The market is the way to deal with future contingency outside prediction.

I am not refuting randomness, but only the metaphysical subsistence of a 'random generator'. David Bohm has a nice way of introducing contingency first, then chance and possibility (and probability).¹ Contingency is basically when something happens which was not taken into account by the specific context and which, in essence, could as well have not happened. So, contingency is pure externality. We experiment the law of free fall by dropping a stone and suddenly wind blows the stone away from its projected impact point. That is, something goes wrong in our experiment. It is when we subsequently notice that such contingencies tend to repeat themselves and, moreover, that their effects tend to average out by the law of large numbers, that we invent possible states and chance after the observed statistics and start predicting the impact point in windy weather. In other words, possibility or chance is a re-contextualization – always a re-internalization. As a matter of fact, a new contingency is perfectly capable of changing the whole context all over again; for instance, a meteor hits us and the whole probabilistic set up is blown apart.

What I am arguing for is precisely the superiority of contingency in terms of changing the context (or the range of possibilities) and the fact that a 'random generator' is always relative to a certain context and its perfectly identifiable states. To repeat, it is not 'random' that I am arguing against; it is 'generator'. There is no generator of contingency as such, because contingency is absolute and not relative; it is always the ultimate mover, even though we may not – almost by definition – frame it into an ultimate context. Hence the deep metaphysical challenge in overturning the ontology and putting such an 'open' category as contingency first.

The random generator is always a reconstruction, or re-contextualization – what philosophers of probability call a 'chance set-up' or what Popper calls the 'generating condition', which, in the case of the die and according to Popper, would not be inherent in the die as such but in the whole situation involving the die, the throwing hand, the atmospheric conditions, and so on. In most cases, the random generator is reconstructed after the observed statistics. The only physical case where randomness is not 'informational irreducibility' is quantum mechanics where it seems that, case by case, a pure and genuine random draw occurs, which is not due to a hidden variable or some irreducible complexity. This is what drove Popper to his 'propensity' notion – another word for random generator.² After scrutiny, however, it appears that quantum indeterminacy is not compatible with a given range of possible states among which the 'generator' chooses the outcome. It was shown by Paulette Destouches-Février in the 1950s that meta-contextual predictive theories like quantum mechanics – that is to say, theories that are logically superior to Kolmogorovian probability theory and its inherent Boolean algebra – are *essentially indeterministic*.³ Nothing stops us, of course, from calling a certain context-specific quantum experiment a 'random generator'. These are only words that offer a rewording of the results. The whole point is that, ontologically speaking, there is no random generator, or real subsistent propensity, like Popper would have wanted. There is something deeper, reaching behind the contexts.

Back to the market and to prices, the whole question is what we make of our 'random generators'. Are they shorthand for observed statistics? Or are they really something punctual? Beware of the computer analogy which works, here, in the reverse direction. Random generators and infinitesimal generators exist, of course, in the realm of mathematics and stochastic calculus, and we simulate

them on computers (even though random number generators never use ‘random code’). But what is the real physical significance of a pricing and hedging partial differential equation (PDE)? Statistics? Ex-post or ex-ante?

Finally, because of the hierarchy between contingency and possibility, the most interesting case is the single-case event, of course; that is, the ‘probability’ of an event that is not a member of a previously observed, or even possibly thinkable, statistical population. Definitely a contingent claim, written on that event, would admit of a market price (by my definition of the market), without any reconstruction or re-contextualization of possible states. Of course, probability theory would want to perform such a re-contextualization in order to make sense of the price. Probability theory and derivative valuation theory always have to value the contingent claim, not price it. But the whole question is: ‘Why do we need it? Can’t the market become a metaphysical category all by itself?’

And how can we define the ‘market’ in such a way that it remains the medium of absolute contingency and does not require re-contextualization of mediated reality? By an equivalence principle similar to Einstein’s. In his cosmology, matter equals geometry. In ours, matter, which is contingency, equals geometry, which is the market.

Just as inertial mass is the same as gravitational mass, the material on which contingency is written (the material contract known as the contingent claim) is the same as the material that is meant to be exchanged. It is not that we write, and then we exchange. Rather, writing is the same as exchanging; it is the same matter.

In conclusion, there is nothing to separate the market from contingency, no time, no probability, and no mediation of the only reality there is, which is contingency. The market takes place right in the middle of the event or, in short, the market takes place instead of (in place of) the event. As a consequence, the frames of reference in which we value the contingent claims by probability may vary, depending on the possible states you select. Market prices, by contrast, are the invariant of our theory of transformations.

4.3 The market as geometry

We need matter and its equivalence with geometry to create a world. Probability, as geometry, needs matter and one possibility is, for instance, the one in which formal probability finds the statistical interpretation as its matter, or the science of ‘mass phenomena’ – to borrow the words of *both* von Mises and Kolmogorov. Speaking of probability materially and not just formally is really speaking of meta-probability. Interpretation in terms of ‘condition of experience’, ‘repetition’, ‘trials’, statistics, and so on, is not part of formal measure-theoretic probability. It is precisely what matter is to form; it is the material face of the geometry. It is a world. When Kolmogorov introduces these terms in his monograph, he clearly elaborates meta-probability, not probability. As for von Mises, he clearly wanted to wrest the formalism from the matter. Stability of frequencies was precisely his matter – he compared it to the notion of mass in mechanics – and this is why his intended formalism wasn’t pure. He did not really distinguish between probability and meta-probability. (To be fair, we should recognize that we only learnt of this distinction later, after Kolmogorov produced the pure form.)

In his book *Creating Modern Probability*, von Plato writes: ‘It is sometimes said that the maturity of a field of mathematics is measured by the degree to which it can “forget” about its history.’⁴ And I would add, ‘about its world’. So, I wonder: What could be the shape of the new world, what could be the new geometry when the formalism of probability is no longer adjoined to mass phenomena and frequencies à la von Mises or Kolmogorov, but to the new matter whose originality and inception and precision we should all recognize; namely, the market? It is a different shape and a different world I am talking about. We should remember the forgotten world that used to lie historically behind the formalism in order to figure out the possible shapes of the next, or rather, the shapes that it could not be.

I am not trying to recount the historical genesis or construction of the market, or describe how it all looks from God’s point of view. I am offering a new ‘system’ of metaphysics in which money, writing and exchanging (thus pricing) are elevated to the level of primary categories. This is why you often find me redefining the word ‘market’ in order to make it fit my idealization or systematization.

What I like about speculative metaphysics is that thought is offered the freedom to construct systems. Of course, this may sound, at first, totally disconnected from reality or realism. (Unless we agreed that reality is but a metaphysical system, too.)

I am fascinated by the gap between syntax and semantics, between formalism and interpretation, between form and matter. My whole philosophical ‘age’ shifted when I read Alberto Coffa and what he had to say about objective semantics.⁵ Yes, geometry – but only when matter is its other face! I also believe in thought’s power to construct alternative worlds. After Coffa and the objectivity of semantics, I learnt from Meillassoux the significance of speculative thought. I used to hate metaphysics in my early training in analytical philosophy. Coffa liberated me from Carnap and Quine and then a second age, or stage, set me on the road to Heidegger, Blanchot, Bergson, Deleuze. The last two I found illuminating, showing me a sort of equivalence between thought and matter: metaphysics in the sense of creation, not nonsense. Make contingency the ‘first matter’ (*matière première*) and then the market comes out as the corresponding geometry.

What I am offering is different ways of thinking the market and price and, as I have said, this whole trend of thought is more or less dictated by pressures from within the ‘system’ of the market, even from within its technology of derivative pricing – trying to make sense of it.

How this trend of thought can become ambitious is by thinking how probability applies in domains other than purely quantitative and, then, by thinking how the replacement of probability by price, or of possibility by contingency, which I advocate within my system, can be properly generalized. For all we know, probability may be pervasive in our deepest thought and language mechanisms.

4.4 State vs. mark

In arguing for contingency that is not mediated by possible states, I am not saying there are no states; I am saying there should be no delimitation or identification of states to the exclusion of others or, more generally, to the exclusion of the unidentifiable ‘other’, or the contingent.

It is up to you to call ‘different states’ the different conditions in which a certain contingent claim pays off. This would be a mere rewording. It seems to me, however, that for probability theory a state is a state only insofar as the theory has to list all possible states once and for all in order to get the *transitions* between the states right and to roll back in the tree to the present value. A probability state is a state only insofar as it is involved in an exact probabilistic transition and the whole structure is specified.

By contrast, the price is robust to any accident that may happen to your previously identified list of states or, to be more exact, to any accident that may happen to the bunch of provisions that you hold and that are called ‘your contingent claim’. No need that everything should be very finely and exactly wired beforehand. An unforeseen state may be that your contingent claim is destroyed in a fire and obviously you collect no money in the end, even though the ‘state’ in which it would have paid off in theory was realized.

You lose money and there is no compelling argument that the price of the contingent claim was wrong. You are supposed to lose money in the market. However, your probabilistic computation would have been wrong. Note how the capacity of the contingent to incur the ‘impossible’ (even its own material loss) and the robustness of its price towards the loss of money are both due to its material nature, to matter whose other side is the exchange.

The distinction between identifiable possible states and written contingent claims is fundamental for opening the passage in which my whole metaphysics takes place. It is not so much the word ‘state’ that I am questioning as it is the precise operation in which it is put to work in the probabilistic logic; namely, the probabilistic *transition* between the possible state and the actual (or real) state, or put briefly, the probabilistic tree. It is the transition that I wish to criticize.

Surely, the contingent claim ‘Rain tomorrow’ will pay off tomorrow in case of rain, and you are welcome to call this a ‘state of rain’. Surely, the world is in an actual state today and will be in an actual state tomorrow. The word ‘state’, here, is no more than a rewording of the word ‘world’. ‘State’ starts acquiring the non-robust sense that I want to dismiss, however, when it becomes a state in a tree or a *possible state*, waiting for an identifiable transition. What I dismiss is the *identification* of states, not the blunt fact or reality of the word ‘state’, and the projection of this artificial identification and partitioning either backwards, to create the impression that the present word is contingent only on account of being the realized member of a set of possible states, or, what’s worse, forwards, to create the impression that the future is unpredictable only on account of being the undetermined outcome in a list of *possible states*.

I truly believe that the invention of writing and of its correlate, *which is the exchange*, is what enables us to dispense with states and the fragility and non-robustness of their transitions. The world is contingent tomorrow and it could be in any kind of possible or imaginable state. All I am saying is that we shouldn’t even identify them (this is stronger than enumerating them). We will wake up the next day, and only then will we see whether the contingent claim ‘Rain tomorrow’ will pay \$1, pay nothing, or whether everything has been wiped out by a meteor, or perhaps that all trades have been cancelled and markets have been declared illegal. However, I think that the only reason why we want to identify those possible states is precisely to line them up carefully in a grid and

to establish the connections with the present, in order to enable the calculation of probability and mathematical expectation and, consequently, to evaluate the contingent claim. But what if we didn't need this artifice and the price (not the value) of the contingent claim was readily available in the market? Don't ask me how the market performs its calculations. For this is precisely its definition. The market is this 'dark' medium that doesn't need the wiring and the transitions of the tree.

The present prices of the contingent claims 'Rain tomorrow' and 'Sun tomorrow' should, in theory, sum up to 1, like probability. However, there is nothing (no arbitrage) to stop the market from pricing the combination for less than 1 and nothing to compel the market to show the clear (transcendently visible) reason why this is the case. The only valid non-arbitrage relations that should exist in the market are those given by *intrinsic arbitrage*, or model-independent arbitrage, or arbitrage that precisely doesn't depend on our specification of the states of the world, such as put-call parity, or the general non-arbitrage inequalities that were elicited by Merton's 1973 paper on option rational pricing. These, as you will notice, are non-arbitrage relations or inequalities that depend only on the *written clauses* defining the contingent claims. Hence my insistence on writing as the 'flat', state-independent, medium of contingency.

Limiting oneself to intrinsic non-arbitrage is precisely acknowledging that any probabilistic model that we might temporarily need is prone to recalibration, and to reshuffling and expansion of its possible states. To repeat, this is not a mild revision of priors such as Bayesians imagine. This is bringing into the game whole new sets of states that were incompatible with the previous one. How can a Bayesian update for the fact that volatility has now to become stochastic on top of the underlying price (or, what's even worse, that the contingent payoff, which was synthesized of late by a dynamic replication strategy involving the underlying, has materialized into an independently trading contingent claim) and that, suddenly, we now have to make room, in the middle of our nice little plain vanilla option pricing problem, for the fact that barrier options, cliquet options, options and variance, and so on, trade too? How to update for the writing of contingent claims and the genesis of their market?

We don't need probability theory and its enumeration of states to derive put-call parity or to establish that the price of the combination 'Rain tomorrow' and 'Sun tomorrow' is less than 1. However, we do need to enumerate the states in order to value the call or the put *individually*. We need a dynamics; statics are not sufficient. The non-arbitrage value of the call or the put, taken individually, depends on the specific assumption of states of the world that we are making. This is why it is not robust. If we assume stochastic states of volatility on top of states of the underlying, our enumeration of states will be different and the non-arbitrage value of the individual call or put becomes different.

My key observation – so simple, indeed, that it is potentially revolutionary – is that the market prices the individual call or put without assuming a specific enumeration of states. Of course, the 'states' of the underlying are needed in order to determine the payoff of the option, but then these states are prices, mind you, or concretely determined provisions, not abstract, model-dependent probabilistic states such as the stochastic states of volatility we mentioned earlier. And how does

the market add more states? By showing us more prices, not more abstract states: prices of variance swaps, of options on variance, and so on.

Every trader knows that prices are found in the market without any theoretical intermediary sitting between the written contingent claim and its exchange. My question: Why can't a metaphysics of contingency be candidly, yet radically, constructed from this? Why do we have to remain petrified in probability? Is it because probability is such a venerable paradigm and because we all own books where we have studied it? Yet, if we consulted a few books on the philosophical foundations of probability we would see how frail they are. In the best cases, a financial transaction is ultimately invoked to found probability (de Finetti's subjective probability, or break-even in the long run in frequentist probability).

Why can't we take advantage of the present sophistication of the market of contingent claims, of our knowledge of what their prices can do and what probability cannot do, to trigger this metaphysical overturning? Why, indeed, can't the market become a fundamental category? The philosophers of probability seem to remain unaware of the massive case of the market of contingent claims. We are past the theoretical stage. Now, the usage of the theory, or the technology, has to become part of the metaphysics, or of the formalism of the next theory. I call it the *contemporary theory of the market*.

This is not asking for much. All we need is that the problem of recalibration – or the fact that a derivative is intended to be traded as a novel instrument when probability theory makes it redundant with the simpler derivatives that drive the assumed dynamics – may become a key component of the metaphysical reconstruction or re-systematization, and no longer be thought as a regrettable accident. Now, the process of recalibration may certainly not be recovered by a stochastic process, because it exceeds probability. To that extent, causality gives way.

Recalibration – that is to say, the trade and exchange of derivatives of any level of complexity – is what's specific to the market and what exceeds probability. More generally, it is specific to reality and to history. My observation is that the market of contingent claims is reality and history, only quantitative. So, my question can be rephrased thus: 'Why should the quantitative characteristic of the market of contingent claims be the cause of degrading it to the lower category of probability? Can't it, on the contrary, be a reason to construct an upgraded quantitative theory of contingency – which will have to be meta-probabilistic?' It is only when the market is reduced to probabilistic states that we are forced into a debilitating alternative, either to have a complete market or to preserve the incomplete market by preventing some contingent claims from admitting of prices. Can't we argue, on the contrary, that all contingent claims should admit of prices, since writing them is equivalent to trading them and the exchange is the other face of contingency, but that none should be redundant because there should be no fixed stochastic structure guaranteeing that the more complex derivatives are ultimately dynamically replicable by the simpler ones?

The problem of recalibration is not just a negative criticism; it is a very concrete problem that any pricing technology has to face once we recognize that it exceeds valuation theory and the probabilistic paradigm. Recalibration is another word for 'non-existence of identifiable states', or,

for the metaphysically-minded, an indication of a reality out there (the market?) that may have to be addressed at a level beyond the categorization or identification of possible states.

I claim such metaphysics can be constructed (so my concern is construction, not destruction) if we go straight from a meaning of contingency that is stronger than possibility to the notion of price, without the intermediation of probability and its transitions between states, except in local sections of that reality, in which we wish locally to formulate non-arbitrage (pending recalibration).

Instead of experimenting with reality being a number equal to 1, lying at one end of a spectrum where various unrealized possibilities receive numbers between 0 and 1, I'd rather think of reality – I don't know how to think of another thing – then think that this reality could have been different; that is, it is contingent. Contingency is real; possibilities are not – they are produced after the real; they are derivative and fabricated.

What I dismiss is the schema of transition from possible to real. The alternative proposition is an unproblematic transition from real to real, accompanied by the unfailing thought of contingency. In one case, you have the tree of possibilities and its very fragile and exacting branches; in the other, you have the market of contingent claims where anything can happen, including the fact that we should lose money.

Surely, the payoff of the contingent claim is realistically triggered at maturity by causes that connect to that effect. Surely, some precise mechanism takes place at maturity of the contingent claims, in which a certain state can be recognized as the cause of the payoff, such as the state of 'RAIN' or 'SUN' and the state of the financial and legal system in which the holder of the contingent claim may present his claim and the writer of the contingent claim may honor it. The market, which I make my fundamental category, shouldn't sound as if I wanted to blur these states and this fine causality. My point is that there is no causal (or probabilistic) algorithm that tells the market what prices to assign to the contingent claims *before* maturity. My non-causal – call it magical – assumption is that the market finds those prices and they are given to me without explanation, in order to recalibrate.

So, while we agree about the causal process of settlement, or the causal observational procedure that allows us to note that some state of affairs has actually obtained, isn't the whole problem to try to fill the gap before that settlement date? Hasn't probability (and its models) been invented for that purpose? My claim is that the market is the material invention (or procedure, or technology) to fill the gap with prices before the settlement date.

The whole trick is to write the event on a piece of paper instead of making it a member of a range of possibilities. It is one piece of paper, not several possibilities; and, yes, it is already settled. The event has already taken place and we can safely say that it is contingent without invoking any possibility. To look at the mark (for instance, the payoff of a contingent claim) as one mark *that could have been other*, not as several states over which difference is only analogical, to read the mark along its intrinsic and absolute contingency, not along the contents of its schedule, is to exchange it, not to value it probabilistically. (This is our thesis.) It is to read it under the logic of place, not the logic of chronological thought. This exchange happens after the event; or rather, contemporaneously with it, or even better, concurrently with it. This exchange is the *other face of the event* (since the sheet of paper presumably has two faces). Hence a topology, rather than a chronology.

And now it suffices to think that this piece of paper *is still there even before the event*. Or rather, ‘before’ and ‘after’ are indifferent to the topology of the exchange. Whoever sits in the marketplace is flying at the same speed as the point of fracture of the ranges of possibilities.

We are obviously hinting at a new category of thought in which the *mathematics of price* replaces the mathematics and logic of probability. Thinking is representing and framing and mapping; thinking is a reflective self-enclosure; thinking can only draw on a given situation and on its given identified possibilities; therefore thought can never think the event, as such. It can only think before and *then* think after the event. The event, after all, is essentially a discontinuity.

My observation is that the market operates on the contrary right at the fracture between the given contexts or the given situations, because it thrives on the fact that a certain derivative will be redundant if it is limited to the given situation. The thread of the market – which maintains that everything should trade and nothing should be redundant – is thus constantly leaking away of any given context. It cannot be a thread of thinking and building and representing and framing and deriving because, as soon as we elevate thought to reflect (transcendence), we lock ourselves into a building.

Thus, the thread of the market literally snakes away on the floor (immanently, not transcendently); it possesses a material medium, not a conceptual one, which I call writing, mainly drawing on the observation that contingent claims are materially written on paper, not conceptualized as a choice of abstract possibilities and states of the world, and that, as so written, they are destined to be materially exchanged, not to be conceptually valued.

To make it real, it suffices to think that the medium I am talking about is real. That all it takes is materially to stand there on the floor as a real market-maker. The trader on the floor is really concretely connected to the event *as such*, always happening. He doesn’t live in the territory before or after the event. He constantly lives at the boundary. He doesn’t care about *what* the event changes; he straddles, in the event, its capacity to change something. He rides the differential operator of the event, so to speak; he is at rest with the event’s ‘proper time’ or ‘proper context’, which is where logic and chrono-logic as we know them come to their limit, because the event is literally outside time and outside any given context or range of possibilities.

4.5 Probability as an internal episode

Price, or the clear and present exchange of contingent claims, is the only way to address the future contingent event that underlies their triggering. The only way to address, literally *to deal with*, the event of default of a corporate company – to cite this example of a single-case event – is to exchange the corresponding credit default swap.

Insurance companies do not address a future single-case event as such. They charge a premium such that they break even on average given *a past, statistical population* of which that event is construed as a member and which they hope will maintain itself in the future. It is only by an abuse of language (we may call it a model, or an extrapolation, or an idealization) that it is then claimed

that the insurance company has insured that particular individual, or has ‘valued’ that particular event. Objective probability, in its frequency interpretation, is the result of this extrapolation.

The physical meaning of the stochastic processes we write in derivative pricing also stems from this frequency interpretation. Of course, the continuous-time stochastic process makes perfect sense mathematically, as it has issued from measure theory and the un-interpreted formalism that was laid down by Kolmogorov. Measure-theoretic probability has nothing to do with statistics, we all agree. But how this relates to actual pricing is by saying that the underlying will exhibit a certain observable statistics over a certain period of time. This is the reason why actuaries think they are equipped to value derivatives.

To repeat, the only physical meaning of a stochastic differential equation such as BSM is to say that the price increments of the underlying would distribute themselves *over time* in a statistical population with the corresponding mean and variance. It is for this reason that (we think) it is legitimate to estimate instantaneous volatility from historically realized volatility. This, by itself, imposes that volatility should be constant, and I don’t know what to make of models where we have made instantaneous volatility dependent on time other than the fact that they have been made up in order to fit a certain term-structure of *derivative prices* (which is putting the problem upside down). Surely, instantaneous volatility can, in theory, be estimated thanks to the marvel of Brownian motion (by sampling an infinite random sequence in a vanishing time interval). But is this not materially the end of statistics? Isn’t the availability of the option price, from which to imply volatility, the only material way to estimate instantaneous volatility?

Before BSM, derivative valuation was either up to the actuary and to the belief that the premium charged will make the investor break even on average (the average of what? a whole population of derivatives written on a similar underlying?) or in the long run (the long run of what? a long holding period of derivatives written on that specific underlying?) or up to the open market, in the sense that any price for that contingent claim would do, so long as it respected non-arbitrage with other contingent claims written on the same underlying. Non-arbitrage in this sense differed from the minimum constraint applying to subjective probability and had something to do with statistics and with objective probability only to the extent that the martingale measure under which contingent claims were priced as expectations had to be equivalent with the objective measure.

With BSM, we found ourselves tied back again to the real statistics of the underlying (or the objective probability) except that the major gain, compared with actuarial science, was that we no longer needed to wait for the long run or to reason over an average of such holding experiments. Break-even was now guaranteed on each and every single trajectory of the underlying, thanks to dynamic replication (keeping in mind that breaking even on average had now been compressed, thanks to Brownian motion and its self-similarity at all scales, into each and every infinitesimal interval of that single trajectory in which the stochastic integral was converging.)

This locality of the problem (as opposed to the global break-even over a whole population in actuarial fashion) made the concept of instantaneous volatility all the more relevant, as its knowledge was now all we needed locally to replicate the derivative, and then globally to replicate it by integration over the whole path.

But now that the derivative and its underlying were (thanks to BSM, among other events) instantly trading alongside each other in the open liquid market, where would one find instantaneous volatility other than by inferring it from the market price of the derivative? Who would be happy computing a delta in BSM that wouldn't match the next variation in the derivative price as given by the market?

With the advent of *implied volatility*, the whole problem – and I claim, paradigm – shifts. For if we're no longer happy adopting a probabilistic statistical model unless it passed the test of marked-to-market (i.e. unless it matched to co-variation of prices of underlying and derivative as given by the market), would we be happy, after the existence of volatility smiles forces us to look for models of higher complexity such as jump-diffusion and stochastic volatility, if such models didn't simultaneously capture the market prices of derivatives of higher complexity, such as variance swaps, options on variance, barrier options, cliquets, and so on?

Probability theory calls 'states' the possibilities that it knows, from the observed statistics, will be visited by the stochastic variable sooner or later. Of course, possible states make perfect sense, regardless of statistics, in the mathematical theory of probability and stochastic processes, because this is the sense that is given to them by the axioms and the formalism. And it certainly is not against states of mathematical probability that I am taking issue. My question is what sense we can make of 'states' in the pricing problem, now that (a) dynamic replication has severed any link with statistics in the long run and the corresponding meaning of state, and (b) recalibration to the market prices of derivatives of higher and higher complexity (which is the continuation of the act of implying volatility) is no longer an accident or an effect of statistics changing over time, but is now inscribed in the very meaning of the market and the very procedure of marked-to-market.

States (equivalently, probability dynamics of the underlying) are needed in order to impose artificial non-arbitrage conditions on the derivatives, or non-arbitrage conditions that are not naturally inscribed in the explicit payoffs of the derivatives. These states realize a particular price or a particular path of prices of the underlying asset and, instead of assuming abstract probabilities for them, we may consider the prices of the corresponding Arrow-Debreu securities as given by the market. More complex Arrow-Debreu securities may be considered still, which are written on the price or the path of prices of derivatives on the underlying asset and not just of the underlying asset, and their own prices may be further given by the market. In this, the notion of state that I am taking issue with can momentarily be seen to become harmless as it no longer refers to probability explicitly. The pricing kernel that is formed by these Arrow-Debreu securities has nothing to do with statistics of the underlying or with the long run (other than the equivalence of the corresponding martingale measure with the objective measure) and is only here to guarantee that the other derivatives, whose payoffs are spanned or replicated by the Arrow-Debreu securities, are priced in a non-arbitrage way. The problem is that we would need, once again, a probability dynamics ruling the price processes of the Arrow-Debreu securities in order to replicate derivatives of higher order that are not statically replicable by them. Those higher order derivatives need to be *dynamically* replicated. For instance, barrier options are not statically replicable by the vanillas, no matter whether the strikes and maturities of the latter spanned the whole continuum, but only dynamically so, in a

model-dependent fashion. No matter what we do, we always need a dynamical hypothesis at some point in order to impose artificial non-arbitrage. But then nothing can stop the market, by its definition of the giver of prices, from giving prices for those dynamically replicable derivatives that are at variance with the values of our dynamic replication. Recalibration all over again. The argument is similar to Cantor's diagonal argument.

In sum, and to repeat, it is state, *qua* notion that probability dynamics requires to be definitely wired in probabilistic transitions, that I am taking issue with. My proposition is to simplify the problem and to go back to the premise that only the exchange can deal with singular events that know of no statistics and no long run. All we have in the market are prices of contingent claims of varying complexity. Probabilistic models and pricing kernels and derivative valuation tools are only internal episodes that we require in order locally, and always, imperfectly, to hedge something. We have to keep in mind, and we'd better wire it a priori in the pricing technology itself, that those episodes are present and useful only insofar as they will be recalibrated.

4.6 The alternative axiomatic system of Shafer and Vovk

It is usually claimed that theoretical valuation models based on perfect dynamic replication and no-arbitrage arguments are a useful fiction, for they are based on the existence of a perfect world. 'In a perfect theoretical world, the argument continues, replicable contingent claims would be redundant and hence unnecessary. In our imperfect world, by contrast, the prices of contingent claims are not redundant and hence are crucially important. In our imperfect world, agents who use theoretical valuation models must recalibrate them periodically, to learn and adapt to recent prices of contingent claims.'

To repeat, I don't think recalibration should be thought as the consequence of imperfection. Crucially, the speculative twist of my metaphysics is that I want contingent claims to be non-redundant in a *perfect* world, not in an imperfect one. This is really the whole point and it remains to describe the corresponding perfect world. I want non-redundancy of contingent claims at all levels to be a definition of the market. It is in the probabilistic paradigm that the market is not 'definable' except as some kind of imperfection or accident; that is, something that should not exist in the domain! By contrast, I suggest that probability should be dispensed with altogether ontologically and be kept only insofar as dynamic replication is needed to situate the dynamic trader. I suggest a direct and non-mediated connection should be established between contingent claims and their market price. If there are no probabilistic dynamics to start with, the whole notion of redundancy goes away. All contingent claims will trade and none will be redundant.

All quantitative frameworks of derivative valuation start out by defining the universe Ω , the filtration, and so on. I am saying this is precisely only mathematical finance, not finance! Couldn't finance be quantitative and not rely on the reductive mathematics of probability? Couldn't the market of contingent claims be quantitative finance directly, perhaps calling for a different axiomatics and a different mathematics? Contingent claims shouldn't be replicable and redundant. They exist in order to add a price dimension that is not spanned by the price dynamics of contingent claims of lower

complexity. ‘Recalibration process’ should replace ‘stochastic process’ (I mean, replace totally) and the market shouldn’t be thought as a ‘temporal process’ but as a ‘place process’. (We will see, in the last part, how recalibration will constantly play out as matter that is created outside the formalism, as the writing and materialization of contingent claims that weren’t supposed to exist in the formalism.) This sounds as pure metaphysics if only because we have to explain what is meant by ‘time’, ‘place’ and ‘contingency’ – and, incidentally, what is meant by ‘writing’ and by ‘exchange’.

If written contingent claims and their market prices must exist to the exclusion of possible states and probabilities, and if their price processes mustn’t be stochastic processes because of recalibration and the corresponding failure of stochastic structure, the question becomes: ‘How do we build that into an *axiomatics*?’ Indeed, we are no longer content studying something – mathematical finance and probability theory which underlies it – in which the market is no part!

‘But prices are numbers, will my objector retort, hence any axiomatics dealing with prices will have to be mathematical. Why don’t you produce the mathematics at last, instead of basking in sterile metaphysics?’

Of course, axiomatics is mathematics, but we have to agree that the axiomatics of price can no longer reduce to probability theory. Axiomatics is by definition an inception and cannot be derived from previous mathematical statements. It cannot be arbitrary either. This is where the meta-physics, or the meta-mathematics – in a word, philosophy – becomes a necessity for the orientation of thought.

The objector continues: ‘If the main thrust of your argument is to stress the primacy of prices of contingent claims and to dismiss probability theory and stochastic processes, why don’t you simply recognize, like everyone does, that implied volatility is just a quoting convention and that there is no real stochastic process corresponding to it? Everybody knows that there are only prices and that probabilistic models have always to be calibrated and recalibrated in practice. No one trades “volatility” without looking at the volatility that the most recent updates of options prices imply (implied volatility).’

Precisely, there are only prices and the formalism of BSM starts with a model for the prices of the underlying! It is a model of trading, therefore of the market. Because of time, this model has no choice but to involve probability and the trading process of the underlying is identified with a stochastic process as a consequence. Now, why should derivatives, which trade and admit of prices in no lesser measure than the underlying, retain only the probability side of this picture and lose all connections with the original notion of price? Why can’t the fact of implying volatility from the derivative price, which is in reality a misappropriation of a probabilistic concept (volatility), be formalized? Why can’t the *usage* of the BSM formula, which everybody agrees is tantamount to the practice of calibrating and recalibrating BSM to derivatives prices in ways which obviously contradict and misappropriate its probabilistic make-up, be built into a rigorous meta-discourse of the BSM theory? Why can’t the criticism of theory by praxis become a higher-level theory? Why can’t metaphysics, which is the first name of this criticism, finally earn a surname and become a very precise discourse (in the sense of the precision of thought of Bergson)? The revolution from probability to price and from possibility to contingency, which I have detected in our science, can

even add precision to the speculations of one Bergson, Deleuze or, indeed, Meillassoux. It is as if they had had an abstract intuition, and I had found a material and very precise application of their thought.

Two authors, Shafer and Vovk, have proposed an alternative to measure-theoretic probability and the corresponding stochastic structure.⁶ They, too, observe that the notion of state of the world and of probabilistic transitions between states may be too exacting and, ultimately, too fragile. Instead, they take advantage of the argument we have alluded to earlier; namely, that probability may just be a metaphysical reconstruction when all that really matters and really exists is to break even in the long run. With that, they institute money instead of frequency as the counting device, and the accumulations of gains or losses, or capital processes, as a replacement of temporal processes. They recognize that money is logically more robust than probability in situations where the whole universe of possibilities incurs shifts (we know what it means to lose money but we don't know what it means to lose probability).

In Shafer and Vovk's perspective, as in ours, an insurance company no longer verifies the law of large numbers from a frequency point of view. Rather, what it does is stake an amount of money on each contingency and the fact of the matter is that it neither gets infinitely rich nor blows up. What counts is not the frequency but the integral of the frequency against the money staked; that is to say, a global ex-post accounting equation. It is by a reverse extrapolation from the global account to the single case that the insurance company may later be said to 'predict' the single case, or to introduce a probability measure that somehow plays out in time. The ex-ante stance is an illusion.

As a matter of fact, Shafer and Vovk suggest replacing measure-theoretic probability altogether with a game where you construct trading strategies and the only connection with reality is that 'Reality' is your opponent in the game. Crucially, you have a winning strategy against 'Reality', which Shafer and Vovk interpret in saying that either (a) 'Reality' yields and lets you become infinitely rich, or (b) 'Reality' yields and makes sure that the relevant probability-1 event is realized – for instance, that the strong law of large numbers is verified and that long-run frequencies converge to the probability. It no longer matters which of the scenarios (a) or (b) obtains. What matters is that your present strategy is winning.

The strong law of large numbers provides that, with probability equal to 1, a suitably repeated throw of the die will exhibit a frequency of appearance of the different faces equal to their probability.⁷ This, in itself, is not sufficient to grant the realization of the frequency. What is required is an additional principle, known as Cournot's principle, which provides that an event with probability equal to 1 is realized.⁸ Only if Cournot's principle is accepted as an interpretation of the probability-1 event – namely that such an event is realized – do we obtain the interpretation of a probability p that is less than 1 in terms of realized frequency in the infinitely long run.

Now, Shafer and Vovk propose an interpretation of the probability being equal to 1 that is alternative to Cournot's principle. Their interpretation is *really alternative* in the sense that a certain probability being equal to 1 *no longer means* that the corresponding event will be realized. We no longer know what 'reality' or 'realization' means for probable events if we are in the business of proposing alternatives to Cournot's principle. What Shafer and Vovk say is that an event has a

probability equal to 1 – say, the event according to which the frequency converges to the probability in the infinite run – if one could devise a trading strategy that would make one infinitely rich if the event didn't take place. Accordingly, the event has to take place because no one can morally become infinitely rich.

To this, I would add that we no longer know whether (or even that) the event will take place or not because we no longer know what realization means. To repeat, we are proposing an alternative reading of a certain probability being equal to 1. Although Shafer and Vovk sound as if saying that either infinite wealth or convergence of the frequency in the infinite run will obtain, I suggest, by contrast, that the whole idea of waiting and seeing is lost (so radical is the alternative character of the interpretation) and that the only real thing is the actual (winning) trading strategy that one puts in place. As a matter of fact, Shafer and Vovk repeatedly emphasize the advantage of talking of constructive notions, such as the trading strategy, over referring to infinitistic events.

Measure theory in its finest was invented to account for sets of measure zero and the notion of strong convergence of the frequency in the infinite long run, which is characteristic of probability theory, followed. Now, Shafer and Vovk are saying that the only measure of things is the trading strategy. We're no longer in the business of excluding sets of measure zero or even of considering them because we no longer care about the outcome (infinite gain or not). You may care about the outcome if you insist on interpreting the strategy in light of what might follow it. However, the strategy as presently constructed and considered is the only reality that screens off the later fork (realization or not) of the probability-1 event. Money is the only reality; it is present now, whereas probability never was. The feeling that probability was somehow pressing and present (what Popper calls 'propensity'), and, as a matter of fact, the whole ex-ante attitude, were due, as we said, to Cournot's principle.

I do insist that the trading strategy is all there is. There is no fork waiting behind it (infinite wealth or convergence of the frequency) because there is no subsequent realization, by the very displacement of the notion of realization. The strategy is all by itself the *realization* of the probability formalism. The strategy neither means that we will get infinitely rich nor that the convergence of frequency will take place. Or it means both *at the same time*, because there is no time in which we wait and see; there is no ex-ante attitude, no transition between the possible and the realized. In a way, the reality of the strategy, even though we may consider it prior to its temporal unfolding, transforms the attitude into ex-post, in the same fashion as the ex-post accounting equation of the insurance company. The ex-post accounting equation was the only real thing; now the strategy is the only real thing. Chrono-logic, in which ex-ante and ex-post stand opposed, is different from chronology. The first is essential to our problem and, hence, to our metaphysical reconstructions. The second is incidental. As we said, chronological time is but an unavoidable accident.

True, the strategies of Shafer and Vovk unfold in time. I will later propose completely timeless realizations or interpretations of the formalism. If the real trading strategy, happening here and now without looking further (and of which it is only incidental that it unfolds in time), becomes an *alternative* to waiting and seeing whether a certain event happens or not, then an even more *local* strategy, of which it is even more incidental whether it unfolds in time or not, can also be

a candidate: the strategy of standing in place rather than in time, of standing in the market and potentially contemplating all trading strategies and the corresponding derivatives prices.

Even though they dismiss the interpretation of probability in terms of random generator – that is, in terms of waiting and seeing – and take a radical alternative to the whole notion of realization, Shafer and Vovk are still concerned, on the face of it, with games that more or less mimic dice throws, Brownian motion, and so on; that is, the very vignettes of random generation. This is what, in my mind, does them a disservice and completely masks how radical their alternative interpretation is. For instance, they wish to dispense with ‘volatility’, on grounds that it is inherited from the aforementioned vignettes, and they replace it with the market price of a volatility derivative. The completion of Shafer and Vovk’s move and of their radical alternative, according to me, would be to consider the market prices of any derivatives of any order and structure instead of the corresponding parameters of stochastic processes (and not just ‘volatility’). This infinity will then, by itself, condense the whole market in a single point and a single place and will no longer even ‘leave enough time’ for time to run its course.

Ultimately, Shafer and Vovk talk of prices instead of probabilities because of the financial aspect of gambling or of playing against reality in their games and protocols. For them, price is what ‘liquefies’ the frequency count. Considering expected values instead of the individual probabilities was one way of dispensing with the exactitude and delimitation of individual states or of ‘liquefying’ them; however, with expected values now being identified with prices, capital processes can be considered and the dimension of time can enter the game, no longer through counting and frequency, but through accounting and money. Shafer and Vovk use in price the capacity of money for accumulation and integration; as well, the capacity to buy a variable size of the gamble at the given price. However, they don’t reach to the genesis of price and to its fundamental relation with writing and exchanging.

Considering capital processes and ex-post accounting instead of precisely specified stochastic structure enables Shafer and Vovk to dismiss the notion of precise state on grounds of its lack of generality and of imprecision being more general than precision. A state is no longer a state, according to them, as soon as it is not a *precise* state. However, they do not push this rejection of the state to the limit where the whole ex-ante attitude is rejected. They do not profit from the reality of the winning strategy that replaces the transition from the possible to the real. It is not only precisely, but ontologically, that there should be no transition between states. We should radicalize the argument of Shafer and Vovk and, in doing so, move to matter and to deducing matter.

It remains a coincidence, in the eyes of Shafer and Vovk, that when the market is the reality they consider, they should also be talking of prices. They do not consider that which, in the nature of price, makes it follow from a written paper, and therefore engages papers of ever-increasing complexity to be written and exchanged, and to admit of prices, too. They do not consider that price is inconceivable without the whole chain of derivatives admitting of prices. To me, the two sides are inseparable. Ultimately, the ex-ante attitude and, correspondingly, the state are rejected in the market, and new matter is deduced instead, because the market is in material connection with the event.

Shafer and Vovk write: ‘Probability becomes game-theoretic as soon as we treat the expected values in a probability model as prices in a game. [...] We contend that the extrinsic stochastic modeling that does sometimes seem to be needed in finance theory can often be advantageously replaced by the further use of markets to set prices.’⁹ Ultimately, your opponent is no longer a random generator, whose very name suggests an existing a priori structure or propensity to generate, but a player who is ready for whatever it takes, and whose intentions and internal states *are whatever they have to be* in order that you may not win the game. One advantage of calling the player instead of calling the ex-ante random generator is that the player has no precise intentions or propensities; he is a totally ex-post being. This is the trick of money: immanence. What counts is not the supervisor state but the ex-post accounting equation.

Shafer and Vovk see the market as an indifferent ocean of prices, not as positive matter replacing the impossibility of stochastic structure. They do not reject the identification of states of the world for the same reason as we do, which is the pressing and very specific recalibration problem. They write:

The assumption of geometric Brownian motion can be interpreted from our thoroughly game-theoretic point of view. But if we are willing to make more use of the market, we can instead eliminate it. The simplest options on some stocks now trade in sufficient volume that their prices are determined by supply and demand rather than by the Black-Scholes formula. We propose to rely on this trend, by having the market itself price one type of option, with a range of maturity dates. If this traded option pays a smooth and strictly convex function of the stock price at maturity, then other derivatives can be priced using the Black-Scholes formula, provided that we reinterpret the parameter in the formula and determine its value from the price of the traded option. Instead of assuming that the prices of the stock and the traded option are governed by some stochastic model, we assume only certain limits on the fluctuation of these prices.¹⁰

Apart from their rediscovery of implied volatility, I am of course intrigued by the last sentence of our two authors. One feels, metaphysically, compelled to ask: Is the absence of a stochastic model underlying the prices just agnosticism on behalf of our authors, or do they push it to an ontological statement? Or is what Shafer and Vovk propose just a different semantics of the same theory? Indeed, it looks as if the player they call the ‘Market’ in the particular financial game does the entire trick in providing what the standard assumption of random generator would have otherwise done. It is not so surprising, in this case, that ‘prices’ should replace ‘stochastic process’, as this is what the ‘numbers’ generated by the player are exactly supposed to do.

In other words, there is nothing (metaphysically) specific to the market and to the price in their reformulation, in the sense I am pursuing. Theirs might be a different axiomatics of probability – that is, game-theoretic instead of measure-theoretic – but definitely is not the axiomatics of derivative pricing that we are looking for and that does not currently exist, even under a different form.

4.7 Extensive difference vs. intensive difference

The ontology of price is different from the ontology of number (i.e. of state and probability) because of the fundamental notion of the exchange, which, according to me, should be inscribed within price as such and not just posited outside, as just another number generator whose properties eventually reproduce Brownian motion or its cognates.

By saying that price is not a number, I am not advocating black magic or some excursion outside mathematics. Of course, prices that get printed on tickers are single numbers; however, this realization of the transaction is past and cannot be relevant to my forward-looking perspective. Neither is price a value – that is, the numerical *result* of some algorithm or mathematical function (typically, the expected value of some payoff under some probability measure) – for, if that mathematical tool is used by the trader to evaluate a certain contingent claim, his intention is to exchange-trade it; that is, to change its value into a price, thus triggering the recalibration of the tool.

Recalibration is virtually infinite and, before we know it, the tool to evaluate a simple vanilla option may have to be recalibrated to the market prices of vanillas, barrier options, variance swaps, options on variance, and so on, if they are trading. The probabilistic model is only a *section* into the reality of pricing and, if the completed reality is that evaluation should always be followed by trading and pricing as its purpose – as this is the peculiarity of the market – then, for all we know, price may have mathematically to be represented by an object of higher complexity, or higher intensity, than a number.

For those who believe that there is nothing pure or special about the market and that the market is only a social phenomenon – that is to say, something accidental and purely contingent that cannot challenge the abstraction of number or aim at that level – I believe the argument according to which price is the *other* of value (or a translation of contingency that is completely at variance with the traditional partitioning into states) can truly be made logical, even reach into the archaeology of thought itself.

This folds back to the observation that there is no reason why the alternative metaphysics of price – exchange taking over evaluation – should only take place before the maturity of the contingent claims and not extend to the expiry date, where it is absolutely specified what the contingent claim should pay off. Is not the payoff precisely the circumstance where the contingent claim can only be valued and no longer be priced? Is not the payment schedule (If A , Pay $x(A)$; If B , Pay $x(B)$; and so on) precisely the mathematical function which finally makes the two meanings of *valuation* coincide: valuation of a function and valuation of the contingent claim? What would it mean to *price* a mathematical function instead of valuing it?

Here is another way of putting it. If we agree to call ‘valuation’ the act of reading a schedule and discovering what the derivative pays off in each underlying state at maturity, or the act of valuing the mathematical function known as ‘payoff’ (keeping in mind that the existence of this schedule is the very reason why a derivative is called ‘derivative’ in the first place), then the *valuation* of the derivative before maturity by a probabilistic model of transitions between states can be interpreted

as an extension of this unproblematic mathematical valuation. Simply, the observation is that, since it is not determined which final state we are in before maturity, probability is needed as a relative weighting of the possible states. In essence, valuing a derivative by a probabilistic model before maturity would therefore be no different from valuing it at maturity. It is the same way of reading its schedule. Conversely, and perhaps more interestingly, if, as we have said, pricing (market and exchange) is the *other* of valuation that takes place before maturity, one may wonder what the extension of pricing at maturity could be (or what the alternative way of reading could be).

First, we might disagree that the payoff is actually an unproblematic mathematical function. There could be hidden states, or contingencies impairing the straightforward reading of the schedule, such as the states of the legal system which binds the two counterparts at settlement, or unforeseen accidents that may prevent the settlement procedure altogether from taking place. It might well be that the social construction known as the 'market' extends all the way to the settlement date and that the final 'value' of my contingent claim is no less contingent and uncertain than a market price. Such a circumstance perfectly suits my recommendation that contingency should not be bound by delimited and perfectly enumerated states and should only minimally be expressed as the following clause: The contingent claim is worth something, but it could have been worth something different.

From this, it appears that the alternative way of reading the payoff schedule – or, indeed, a mathematical function – should be differential at minimum and not integral. It is only because each one of the states A , B , ..., relates to the other under the integrality of the schedule or program of instructions that they can be assigned probabilities in the general case when the date is not the settlement date. Probability cannot be assigned unless the whole list of states is given. Probability is an integral. Alternatively, one can read the payoff as saying minimally that the state is A (or B , ...) and the value $x(A)$ (or $x(B)$, ...) *but could have been otherwise*.

When the reading is oriented towards the list of instructions (or program, or schedule) and towards difference as a comparison between states (i.e. as *relative difference* which presupposes the integrality of states), it is the content of the program that matters and the list of *possible* states. When, by contrast, the integrality and closure are forgotten and all that matters in writing a particular payoff is the expression of an *absolute difference*, or the fact that the value is such but could have been different, in this case it is the program as *material mark* that will matter and the mark as contingent. This is simply recognizing that no program can be written (i.e. no possible states and possible values be specified) without a material sheet on which to engrave the writing: a sheet of paper, a contract, a material computer memory, and so on.

There is a very fine line separating the two readings, and probably most of my readers will fail to appreciate the difference; however, this is the narrow passage through which I claim that pricing is different from valuation and my whole metaphysics ultimately introduces itself. For the schedule, as list of possibilities which derive on states, is destined to valuation, while the material sheet on which the contingent payoff is written *as such*, as absolute difference, is destined to the exchange.

Presumably, I am still struggling with these fine distinctions; however, I feel they reveal something deep which, as I have said, can be worked into a logic, and no longer mere 'sociology'. Another way of putting the distinction is to distinguish between *extensive difference* and *intensive difference*.

The first relates difference to extensional space and its states on which difference is projected; the other looks at difference intrinsically. The thought has recently struck me that, while the valuation of contingent claims as probabilistically weighted average of payoff states relates to extensive difference, their valuation by dynamic replication relates to intensive difference. (This is probably the reason why dynamic replication is the entry point of my alternative metaphysics.)

Dynamic replication of a vanilla option shows, for instance, that the value does not come from the list of states, but from the kink in its payoff profile, which is where the whole option is so to speak 'compressed'. The kink is the vanilla option as intensive difference. Ultimately, I guess that the conversion I am seeking, from states and probability dynamics to written contingent claims (that are underlain by no states) and prices, is tantamount to wondering whether vanilla options, barrier options, and so on, could be expressed one day as an undivided 'mark' or 'symbol' or 'ideogram' within the appropriate mathematics (the mathematics of price) and no longer be dislocated into states.

Part II

The Matter in Brownian Motion

From Throwing the Dice to Grasping Brownian Motion

The reason I became interested in Kolmogorov and his formalization of probability is that I came to believe that the market, once it is reinterpreted as a derivatives market with the corresponding concept of price, is a radical alternative to statistics.¹ It is only because price is confused with a number and the price process is confused with a time series that statistics are thought to be applicable to the market and that the notion of random generator is thought to exist in the market. But, once it is remembered that statistics is only an interpretation of the probability formalism, the space is open to envision the market as a radically alternative *interpretation* of the formalism and, for this reason, as a radical alternative to statistics. As form, the formalism doesn't know matter. So, here the idea is to think that statistics is one way of appending matter to the formalism and that the market is another. The market is literally a new matter, to be contrasted with the material die or generally the material device (even abstract) that is supposed to generate the statistical population.

From which part of the probability formalism do I claim that derivatives markets literally emerge as a new reality and append a new specific matter? From Brownian motion, which is the result of the mathematical formalism at its finest and subtlest. To understand how separate the market should be from statistics, one needs first to understand how separate the formalism usually stands from its interpretation. The formalism is itself an unbreakable and impenetrable wall or, rather, a vacuum of interpretation. For this reason, the market stands at a double remove and a double separation from statistics.

The philosophy of (objective) probability has essentially consisted in determining whether probability is real or not, whether it objectively inheres in a certain experimental set-up, and how it is that the law of large numbers comes out as a verifiable result. One can read the whole concept of objective probability as the question of what it is that should exist in order that the law of large numbers may be true. If philosophy is the invention of concepts, then this whole philosophy of probability can be summarized by saying that it is all a matter of determining whether the law of large numbers can be shown and derived.

5.1 The meaning of probability

Kolmogorov's probability formalism is pure mathematics and knows nothing of experiment, trial, event, or realization. It is an abstract measure theory in which a function, called 'measure' or 'probability', maps a field of sets into the set of positive numbers and verifies certain axioms of additivity. The passage to the limit in countable sums of measurable sets and the notion of sets of measure zero are its main characteristics. Random variables are mappings between the sample space (the set of elementary events) and the real set, and are said to be measurable as soon as the pre-images of the mapping are measurable.

The words 'event', 'random' and 'probability' are part of a semantic apparatus that has been superimposed on the formalism. When used as labels at the formal level, these words do little more than hint at one possible interpretation of the formalism. Realization, trials, and thus statistics, are just one possible interpretation of the probability formalism – and there may be others. In Kolmogorov's own words: 'Every axiomatic (abstract) theory admits, as is well known, of an unlimited number of concrete interpretations beside those from which it was derived.'²

In asking what the meaning of probability may possibly be, I will consequently embrace, in the following, a different angle than the one from which probability is intuitively understood as the 'degree' (whatever that means) to which a certain event is known to, or expected to, happen. I will inquire what other interpretations can be given of the probability formalism, even if such interpretations prove to be so radically new as to rob probability of its usual intuitive meaning and to make it almost inaccurate to continue to speak of the 'probability' formalism.

You may ask what the point could be in giving alternative meanings to the probability formalism that may disconnect it altogether from the original intuition of probability and from the physical reality that probability was meant to represent. My quick answer is that derivative pricing and trading – or, for short, the derivatives market; or, for even shorter, the *market* – are in my view a consequence of an interpretation of the formalism that is utterly foreign to statistics and to the corresponding intuition of probability. The formalism at its finest (i.e. Brownian motion) has produced a new reality – the market – that has nothing to do with statistics or with the idea that outcomes are realized at each trial and generate *statistical populations*.

The probability formalism is peculiar in that reality itself is a part of its concrete interpretation. If the dialectical opponent of form is to be called matter, then we may say that reality is part of the matter of probability theory. Is this a hint that materialism is different, and perhaps more general, than realism; and that metaphysical insights as to what we mean by reality and by matter are to be gained from the question of probability and its formalism – more deeply so than from other formalisms in which the pair form/matter is at play? And would it then surprise us that a philosopher like Davidson should lean on the example (and even the precedent) of the probability formalism to suggest what an axiomatic theory of truth might look like?³

Interpretation is usually supposed to connect a formalism with the real world, with reality or with concrete models. For instance, certain primitive terms are involved in a list of axioms in Hilbert's formalization of geometry or in Mach's formalization of mechanics, and we later interpret them as

straight lines and circles, or as solids and mass. Similarly, formal probability can subsequently be interpreted as objectively inherent in the chance set-up that produces the random outcomes (dice, roulette) – what Popper calls the ‘generating condition’⁴ – or as mentally contained in the mind of the subject. In that sense, the interpretation of the formalism bestows reality on probability. However, we also say of a probable event that it will either be realized or not; and we commonly believe that an event of probability equal to 1 is realized. We can observe, therefore, that reality enters into the semantics of probability theory at a more basic level than the question of the reality of the probability: before we ask, along with the realist or the antirealist, whether the probability of a certain event is real or not (in the sense of belonging to the ontology of the world and residing either in the object or in the subject), there is first the sense of reality according to which we say that the event in question will either happen or not. The very notion of event, which is more elementary than probability in the formalism, calls for the notion of fact, and therefore that of reality.

We shall see that even this first level is open to interpretation, and therefore to variation, and that an event of probability equal to 1 may not need to be linked with the idea of realization. The intuitive meaning of the word ‘event’ as a ‘happening’ may have to be revised. Even reality, in the sense of actual occurrence or realization, may not have to be part of the ‘realization’ of the formalism (now understood at the higher level of its interpretation). Perhaps reality will gain a wider meaning from its peculiar double entry in the semantics of the probability formalism.

As a matter of fact, in addition to the two levels or meanings of reality (reality as in realization and reality as in realism), we will finally consider a third one: a new brand of reality that is, this time, literally triggered by the formalism. The reality of the market is something we shall *deduce* from the formalism – not the market as a pre-existent reality that was waiting for an interpretation to later make its terms correspond with those of the formalism but, astonishing as it may sound, the market as something that will literally come into being following a certain interpretation of the formalism. It is as if the market were an accident, something we could construct from the formalism but that was not meant to be. This final brand of reality is reality in the sense of genesis and inception.

5.1.1 The law of large numbers

Everybody knows that Kolmogorov developed his formalism based on statistical heuristics. He had in mind repeatable experiments, and he thought of the probability of a certain outcome as the frequency of its appearance in finite sequences of successive trials. The notion of independent random variables was precisely introduced to suggest repetition. As for countable additivity – and, one can argue, measure theory as a whole – it was ultimately adopted to demonstrate the strong law of large numbers, which ultimately gave probability the precise objective meaning of a *limiting* frequency. (As for subjective probability, it required only coherence from the theory, not the full extent of the law of large numbers.)

Measure theory is of the essence as soon as infinitary events are considered, and we need to exclude sets of measure zero. However, what probability theory specifically brings, over and above measure theory, is the notion of independence and repetition, or the notion of independent trials.⁵ Indeed, the strong law of large numbers is believed to be the distinguishing feature of probability

theory. In the case of von Mises, probability is even defined after the law of large numbers.⁶ No matter that infinite sequences of trials may not empirically exist; the meaning of probability – and meanings are real, albeit not empirically verified – is that the frequency will precisely converge to that number in the infinite long run, in an appropriately repeated experiment.

To be more exact, the strong law of large numbers is a mathematical proof in Kolmogorov's formalism, and is therefore unquestionable; however, in order for it to connect with reality, and for probability to acquire its objective and precise meaning, it is necessary (a) to interpret the formal sequence of independent identically distributed (iid) random variables that appear in the strong law as repeated trials of the same experiment (indeed, there is no such thing, in the formalism, as a material die that one would grasp, lift and throw over and over again, so one has to force this image through an external interpretation) and (b) to interpret the probability equal to 1 as triggering the realization of a certain event (Cournot's principle). Now, *sameness* (the sameness of a repeated experiment) and *realization* (the realization of an event) are two terms that are foreign to the formalism; so, it is only through an interpretation that the formal proof of the strong law of large numbers could endow real probability with this meaning.

Now, note that the random variable which converges almost surely to the mean of the iid variables is, in the case of the law of large numbers that interests us, the average of the infinite sequence of random variables that take value 1 or 0, depending on whether the outcome whose frequency we are measuring appears or not in the corresponding trial. The event that will be realized with probability 1 is not, therefore, a specific infinite sequence in which the frequency of the outcome of interest can be checked, but the whole set of all sequences in which the frequency of the given outcome is equal to the mean. In other words, the event that is realized almost surely in Kolmogorov does not distinguish between permutations of occurrences. Additionally, measure theory is here to exclude exceptional events in which the given frequency is not verified – for instance, sequences in which the desired outcome never appears.

It is not always clear, when we speak of probability theory, whether we mean the abstract and un-interpreted mathematical theory above or the theory of probability, where probability is initially intuitively understood and the theory is supposed to equip it with formal proof, so to allow us to derive theorems and produce calculations relative to probability. Kolmogorov himself admits that abstract probability theory was (historically) derived from a concrete interpretation. He had in mind repeatable experiments and thought of the probability of a certain outcome more or less as the frequency of its appearance in *finite* sequences of successive trials.

For this reason, Kolmogorov expounds the 'relation to experimental data' (or to the 'actual world of experiments') in a paragraph which comes, in his monograph, right after the list of axioms. Notably, he first assumes a 'complex of conditions which allows of any number of repetitions' (this is the first hint that the *material* conditions of experience, such as throwing a die, spinning a roulette wheel, and so on, are not part of the formalism – better, *repetition* itself is a notion that is foreign to the formalism and may, for this reason, rank on the side of matter). Second, he states that, in case the probability assigned to a certain event is very small, one can be practically certain that the event would not occur. On the other hand, Kolmogorov adds, one can be practically certain that the

relative frequency of occurrence of an event will differ very slightly from its probability when the complex of conditions is repeated a large number of times. (This is the hint that *realization*, on the one hand, and statistics, on the other, are not part of the formalism either, but also rank on the side of matter, or concrete interpretation.)

This suggests that the law of large numbers is the crux of the *matter* of the formalism, if not of its form. It is the real substance and – we may even say – the other name of the theory of probability. It is what we expect formal probability theory to become when it is *materially* interpreted. As we said, its eventual concern with the law of large numbers is what specifically distinguishes formal probability theory from measure theory as strictly understood.

Another way to put it is to say that the first purpose of probability theory is perhaps to try formally to show the law of large numbers; or, again, that probability was introduced in the first place in order to say that which must be inherent in the complex of conditions (dice throws, roulette spins, and so on) in order that the observed relative frequencies be exhibiting convergence to a limit as a result. After all, it is the concrete law of large numbers that is in need of proof or explanation, not probability, and I doubt that anyone could ever have in mind the notion of objective probability as a modality without consciously or unconsciously completing the thought with the idea of potentially stable frequencies.

If anything, the fact or the reality of stable frequencies is what presses us for the reality of a *generator*, or for the reification of probability. Given the reciprocity between probability and the law of large numbers, just as we must distinguish between un-interpreted probability (the subject matter of probability theory) and concrete probability (the subject matter of the theory of probability), we must distinguish between the formal law of large numbers (a formal proof in the formalism involving, as we shall see, a sequence of random variables and a notion of convergence that is relative to probability theory and is endogenously defined) and the concrete or material law of large numbers which starts with the empirical observation that chance set-ups (Kolmogorov's complex of conditions) systematically manifest stable frequencies.

It is not surprising, at this juncture, that Kolmogorov mentions von Mises, who has also attempted to formalize probability. He writes: 'There are other postulational systems of [probability theory], particularly those in which the concept of probability is not treated as one of the basic concepts, but is itself expressed by means of other concepts (for example, R. von Mises and Bernstein).'⁷ Indeed, von Mises is famous for having formally defined probability, *after* the postulation of a limiting frequency in an infinite series of trials, *as* this limiting frequency.

Whether probability should be first in our mind then given meaning as a modality through the idea that a die *would* show the face 'six' $1/6$ th of the time if thrown repeatedly and if the probability of 'six' was said to be $1/6$, whether probability should be posited in order to formally show the only thing that is on our mind – namely, that frequencies are stable in the long run (Kolmogorov) – or whether probability should be formally defined after the postulation of a limiting frequency (von Mises), it all indicates that the law of large numbers is, indeed, the crux of the matter and the real import of probability theory (as opposed to measure theory).

Indeed, it will turn out, from our analysis, that the law of large numbers is what brings matter into the probability formalism. Once the probability formalism leaves the ethereal realm of form and lands on matter, it produces the law of large numbers. Trying formally to show the law of large numbers will then amount to trying formally to show matter – almost a contradiction in terms, you will notice, for matter is usually opposed to form and may precisely be what is in no need of formal proof.

5.1.2 Intuition

We will later compare the two formalisms (or postulational systems) of Kolmogorov and von Mises, and we will examine the sense in which they qualify as formalisms or, in other words, how they take their distance from finitude and matter. However, we should first consider the intuitive meaning of probability in order to perceive what it is, in that meaning, that has prompted the need of theorization and formalization in the first place. We should go back to the throw of the die in order to perceive what it is, in the die, that should be discarded as finite and material, and what should be kept as essential and formal, when the goal is to establish a theory of probability.

We say of a certain die that it *has* a certain probability of producing a certain outcome, as if probability was inherent in the die. Some would argue that the inherence of the probability in the die is a definitive and acquired property that must be conceived independently of its subsequent manifestation in the statistics of outcomes. Probability may not be an exact *finite* physical science in the sense that probabilistic statements could be definitely falsified by finite physical experiments; however, it is an exact *infinite* science. Infinity is inside the die and not in the experiment happening outside. (Note the apparent contradiction between the finitude of matter, mentioned earlier, and the infinite inherence of probability in the die.) By a method of intuition (inspired from Bergson maybe), one can dive inside the die and acquire absolute knowledge of its probability distribution. For it is all there, after all. There is nothing outside the die that contributes additionally to its probability.

Some, like Popper, have insisted that the tendency of generating outcomes is not strictly speaking inherent in the die and that it involves the whole objective situation of throwing the die (the hand that throws it, the flat surface on which it lands, and so on). The fact remains that the situation in question – what Popper calls the *generating condition* – is considered statically and in a priori fashion. It is true that the die-as-such (the solid cube with six faces) may have nothing to do a priori with the experiment of throwing it and that it may cross the mind of no one that the die at rest is intended to be thrown; it is true that we may have to extend the referent of the word ‘die’, following Popper’s suggestion, and mean by it the *die-as-thrown*; that is, understand by the die as random generator the whole set-up of the throwing experiment, the fact remains that this die, now enlarged to encompass the whole set-up, is still considered statically and that probability is meant to be an objective *fixed* property of that enlarged static die.

Such an abstraction and retraction from the actual production of outcomes soon encounters a problem, though. For probability is *not*, on the other hand, strictly speaking equivalent to a static description of the die or, more generally, of the generating condition. Probability is a *modality*; it precisely describes the die *as* generator of outcomes, the die *as* engaged in the experiment whose

precise termination is the outcome or the die landing on one of its faces. Yes, probability is inherent in the die; however, it points outside; it carries an additional meaning to the mere massive description of the die. It can be reified and what additional meaning it carries is that the die *would* produce outcomes with a given frequency when subjected to the throwing experiment.

It is no accident that we shouldn't be able to unpack the meaning of probability independently of the series of trials. One can try hard and convince oneself that the random generator is located nowhere outside the die and that it absolutely attaches to the die; still, as soon as one is challenged to explain what one means by 'probability' – not only quantitatively or measurably, but even qualitatively: what one means by probability as a concept, probability as precisely independent of the specific magnitude which, admittedly, requires the sequence of trials to be determined – one cannot avoid referring to the experiment and its repetition.

There is no contradiction between the two attitudes. The capacity to generate outcomes with a specific and stable distribution, or in other words, the random generator, may be an intrinsic property of the actual die and nothing but the die – that is to say, a real property, understood in the full ex-ante sense – however, what it refers to semantically speaking – what we *mean* by it – may require the full unfolding of the experiment, perhaps infinitely so. Notice how the interplay between the reality of probability (its inherence) and its meaning (its expression) is already engaging interplay between the ex-ante and ex-post stances.

Popper has coined the term propensity to name this intrinsic property that can only express itself outside. That a die-throwing set-up has a propensity of 1/6 of landing a 'six' means *nothing else* than the fact that the face 'six' will tend to appear 1/6th of the time in a suitably repeated experiment. Note, however, that it is the material entity known as the die (possibly enlarged to include the whole throwing situation) that enables the thought of repeating the experiment. The fact that we should be able to repeat the experiment is more essential to it than the fact that the experiment is a step outside the die. Hence, it all flows back to the die, and this may act, reciprocally, as the definition of the material die. One needs matter somehow to 'attract' and curb the experiment and perhaps an alternative meaning of matter could be obtained if the repeated experiment was intensively reabsorbed inside the die instead of being extensively and chronologically produced outside.

5.1.3 Matter

One needs repetition, therefore matter, to give probability its full intuitive meaning; yet, the demand for formalism is precisely a challenge to dispense with matter. For this reason, the formalization of probability is a real problem and was identified as such by Hilbert in 1900 as one of the twenty-three problems that he posed to future mathematicians.

Von Mises's way of making the material die disappear was to consider only the outcomes of the experiment of throwing it and to consider them *in toto*, in the full glory of their randomness. He formally defined probability as the limiting (relative) frequency of the corresponding outcome in an infinite sequence of trials in which the limiting frequency was postulated to exist. Moreover, in order not to let probability depend on a specific (and, one might say, 'material') sequence or, what is even more pressing, in order to eliminate sequences in which the outcome repeats itself in an

identifiable pattern, von Mises took special care to introduce randomness in his formal system. His *axiom of randomness* provides that the infinite sequence in which the limiting frequency exists (and is defined as the probability) preserves this limiting frequency through sub-selections.

By sub-selection, von Mises means a rule, which depends only on past outcomes in the infinite sequence, with which one could decide whether or not to sub-select the next outcome. In other words, given the infinite sequence in which the frequency is assumed to converge, the axiom of randomness additionally requires from this sequence that you may have no way of selecting an infinite sub-sequence (looking only at outcomes previous to the place you are currently sub-selecting) in which the limiting frequency would be different. Von Mises called *collectives* the sequences of outcomes that exhibited this remarkable property. One might wonder if such collectives could ever exist – formally, of course, for they certainly do not empirically exist.

The existence of collectives was proven mathematically, albeit non-constructively. Not only do they exist, but they are dense in the set of all infinite sequences. The drawback, however, is that one cannot actually extensively exhibit a specific collective and identify it term-by-term. One cannot construct a collective by definition, for otherwise one would know how to construct a sub-sequence that violates the frequency. To put it differently, if the collective could be constructed, then it could be predicted and that, in von Mises's eyes, would mean it was not random. All that von Mises required was that his system of axioms (existence of the limiting frequency and the axiom of randomness) should be consistent:

I said further that 'the existence of a collective cannot be proved by means of the actual analytical construction of a collective in a way similar, for example, to the proof of existence of continuous but nowhere differentiable functions, a proof which consists in actually writing down such a function. In the case of the collective, we must be satisfied with its abstract "logical" existence. The proof of this "existence" is that it is possible to operate with the concept of a collective without contradictions arising.'⁸

Von Mises may have dispensed with the material die; however, what replaced matter in his formulation was precisely his postulate of convergence of the frequency in the limit. He even compared this required stability to the concept in mass in mechanics. As for the axiom of randomness, we will see that it stands for matter precisely in denying the collective any enumerable ex-post character that it was supposed to embody and in making it all look as if we were back inside the die that would *potentially* produce an infinite random sequence of outcomes, only we don't know which, identifiably and extensively.

Convergence of the frequency in the infinite limit was von Mises's primitive matter. One could say that it replaced the 'infinite inherence' of probability in the die. To hold that probability inheres in the die independently of the always finite and always contingent conditions of experiment, yet that the meaning of probability cannot be confused with the static material die, is in the end the same as assuming an *infinite* sequence (which no longer depends on finitude or contingency either) whose material distinguishing feature (where matter means gravity and is opposed to free form) is the convergence of frequencies and their invariance in sub-selections. In a sense, propensity and

the collective are the answers, respectively, of a realist and an empiricist to the same problem of randomness and to what seems to be its inescapable intensional element.

One way or the other, one needs infinity in relation to probability. It then becomes a matter of a different formulation whether to place infinity in the static die *together with* the pointer to spring outside and not just remain statically confined in the die (propensity), or to place infinity in the series of outcomes and to forget about the die, keeping in mind that the invariance of the limiting frequency in all sub-selections, or the other ‘form’ of infinity, is ultimately equivalent to a material random generator. This means that von Mises’ wasn’t a formalism in the end, because it did not quite succeed ultimately in freeing itself from matter.

The answer to both the predicaments of the material die and of the double infinity that is imposed on the collective is the theory of probability as it has been *truly* formalized by Kolmogorov. By ‘truly’ I mean that probability was assumed to pertain to the particular random outcome, not to the material die or to the infinite series. What Kolmogorov did is suppress the notion of random generator, or the idea that a certain unique and single entity – *the* generator – is supposed to underlie the multiplicity of outcomes. He considered the single outcome, only he conceived of it as a variable.

Contrary to von Mises, Kolmogorov did not assume an unbroken series of realizations of the attribute but a single unrealized *random variable*, which is still packed with the totality of its *possible* values. In Kolmogorov, the basic constituent is the throw, not the die, and the throw is a whole mapping between a set of elementary events (or sample space) and a set of values. Next, Kolmogorov considered a series of throws, or a series of random variables, as such totally unrelated to each other or to some underlying reality or materiality that would qualify as the die, except for the fact that the random variables are assumed to be independent (precisely) and to possess the same probability distribution.⁹

For instance, we may consider the random variable which assumes the value 1 or 0 depending on whether or not the pre-image includes the elementary events giving rise to the outcome ‘six’ in a die-throwing experiment (that is, the elementary events that are mapped into the value ‘six’ by the formal random variable known as the die-throw). One finds that the measure-theoretic mean of this new random variable is just equal to the probability measure of the event in sample space that gives rise to the outcome ‘six’ in a die-throw random variable.

Now, consider a denumerable sequence of such independent and identically distributed binary random variables. They all have the same mean because they have the same probability distribution. Next, consider the random variable that is equal to their average and consider the probability distribution of that synthetic random variable. The result then follows formally and mathematically – it has nothing to do with a particular die, or with the notion of trial, or, indeed, with the whole notion of experiment and repetition thereof, and has only to do with integration and measure theory – that the probability distribution of the synthetic average random variable is such that, for any given tolerance, there is a number of random variables involved in the averaging beyond which the probability is zero that the average random variable differs from the common mean of the single random variables by more than the given tolerance. This is the proof of the formal (strong) law of large numbers, which, as such, is unquestionable in Kolmogorov’s system.

It is a matter of interpretation – therefore it is *matter*, not *form*, and it falls outside the formalism – to interpret the random variables as different or successive trials of the same experiment; say, the experiment of throwing a die. No die was ever thrown and all we have done is construct an average synthetic random variable whose probability distribution will tend to be more and more concentrated around a single value, which is the mean of the individual variables. What this says, when we go back to interpretation, is that the frequency of appearance of the face ‘six’ in the sequence (or the value of the average random variable) will, in the limit and with probability equal to 1, be equal to the mean of the individual random variables, which is none other than the probability of throwing a ‘six’.

In von Mises, the probability never attaches to a single trial. Not that it doesn’t materially attach to it (for von Mises is certainly aware that it is the die and nothing else but the die that produces the outcome); it doesn’t *formally* attach to it. Both von Mises and Kolmogorov know that probability is materially there, only their formalisms prevent them from *saying* so. Matter has to give way either by being rejected to infinity, or by being dematerialized in the random variable. In a way, Kolmogorov’s formal variable is also kind of infinitizing.

So, a certain die is imagined, lying behind the sequence of random variables. This die *matters* (yes, matter!) only insofar as it *relates* to the random variables, thereby unifying them somehow. How it relates to them is precisely by *standing* (in the same way as we speak of a model standing for a theory, in model theory, or *satisfying* a theory) for their shared characteristic; namely, their mutual independence and their identical probability distribution. The many have to be made one. We may say that the image of the material die (this interpretation) is short for ‘a sequence of iid random variables’; it is the guarantee that the variables shall be independent and identically distributed.

And what is to support this guarantee? What is it, in the die (or in the die enlarged to cover the whole throwing situation), that will guarantee that the random variables representing the different throws will bear the same probability distribution? What is the generator, or the salient feature of the die, that will get imprinted on each random variable and duplicate on it, so to speak, a model probability distribution? Aren’t we back to the notion of probability inherent in the material die?

Popper has defined propensity to be precisely this property of the die to generate identically distributed random variables. For this reason, he argued that propensity was a more appropriate interpretation of the formalism of Kolmogorov (what he calls neo-classical probability) than the frequency interpretation of von Mises.¹⁰ (In this, Popper did not quite appreciate that von Mises did not propose an interpretation of probability but a formalism that was thought by him to be alternative to Kolmogorov’s. However, as we have argued above, von Mises’s wasn’t a true formalism because it remained entangled with matter, if only under the guise of the postulation of a limiting frequency.)

That the variables should be considered independent and identically distributed is not the consequence of a mysterious property known as propensity; it is its definition. Propensity is an interpretation of the probability formalism. We just postulate that a certain die-throwing set-up has a property, called ‘propensity’, and that this property insures, by definition, that the successive throws, which now become the interpretations of the successive random variables, will be independent and identically distributed.

For this reason, the concrete law of large numbers was never really proved, either in von Mises or in Kolmogorov. It was only formally (or analytically) proved in Kolmogorov and then propensity was needed as a material (or synthetic) condition to make the law of large numbers work for a material die. And it was only ‘formally’ proved in von Mises, in that it was considered as a postulate, lying at the very basis of the formalism. (Von Mises couldn’t *prove* the law of large numbers, because he had no concept of probability available beforehand – that is, before the postulate of convergence of the frequency leads him precisely to the definition of probability.)

5.1.4 Reality

There is indetermination over whether Kolmogorov or von Mises have provided the formalism of the theory of probability; that is to say, a formal proof of its distinguishing feature and of the only thing that matters, the concrete law of large numbers. Their approaches are opposite, in that Kolmogorov offers to show the law of large numbers and von Mises presupposes it. However, given that they are both in pursuit of formalism, they have left matter outside as the thing that ultimately guarantees their proof. Kolmogorov needs independence and identical distributions (both notions not being available to von Mises, who still hasn’t defined probability) and von Mises needs the axiom of randomness in order to establish that the convergence he presupposes is independent of the particular sequence and any sub-selection thereof. The axiom of randomness is tantamount to independence in von Mises, therefore to matter.

Matter is here different from empirical reality. It is theoretical, or rather semantic; it is the *interpretation* of the formalism. In both the ways in which matter forces itself in both formalisms (Kolmogorov’s and von Mises’), there is actually a negation of empirical reality and of the empirical chronological series of trials and outcomes.

In Kolmogorov, the random variable that converges almost surely to the mean of the independent identically distributed variables is the average of the infinite sequence of random variables that take value 1 or 0 depending on whether the outcome whose frequency we are measuring appears or not in the corresponding trial. The event that will be realized with probability equal to 1 is not, therefore, a *specific* infinite sequence ω in which the frequency of the outcome of interest can be checked, but the whole set of all sequences in which the frequency of the given outcome is equal to the mean. Realization in Kolmogorov doesn’t tell us which particular ω in that set is picked.

That the face ‘six’ should be realized in a single throw led only to a particular ω_i and there was then no ambiguity. However, there was nothing to realize it because the probability was not equal to 1. Leaving the random variable at the state of a mapping without a realization for the single case has as a consequence, and as an *advantage*, that when realization becomes the topic and the probability is equal to 1, it becomes precisely totally ambiguous what happens in each individual throw. Von Mises’ collective is saved in this way. In other words, the event that is realized almost surely in Kolmogorov does not distinguish between what von Mises calls ‘sub-selections’ of a given sequence. Additionally, measure theory is here to exclude exceptional events in which the given frequency is not verified; for instance, sequences in which the desired outcome never appears.

The very idea of randomness seems to prevent us from identifying (term by term) the actual sequence of outcomes that is supposed to realize the frequency. One should pause for a minute and consider what it really means that the frequency converges in the long run.

Even if one were to argue from the realization of the whole set of sequences to the realization of a specific one, its membership in the set is here to remind us that the sequence could have been otherwise, provided the overall frequency of the outcomes is preserved. We should be reminded that von Mises, who adopted the alternative approach to Kolmogorov, in that he defined probability in terms of the realized frequency in the actual infinite sequence, took special care to reintroduce randomness in what could otherwise have looked like an actualized sequence in which all is finished and done and the frequency is checked ex-post. The axiom of randomness which he introduced for this purpose provides that the infinite sequence in which the limiting frequency exists (and is defined as the probability) preserves the limiting frequency through sub-selections.

As a result, even though it looks as if von Mises is considering sequences of outcomes in an ex-post fashion, the axiom of randomness really prevents one from imagining that the 'next' outcome is ever given. As a matter of fact, the axiom of randomness reinstates the ex-ante stance in which a random generator is at play at each step, really guaranteeing that we cannot see and foresee the next outcome. Being an empiricist, von Mises did not want to believe in the existence of such a random generator or in the reality of ex-ante probability, and wanted only to consider the observable outcomes. However, we saw that the axiom of randomness, which he definitely needed, precisely reintroduced the undesirable tension between ex-ante and ex-post.

The only way to reproduce the ex-ante stance unambiguously is to adopt an approach like Kolmogorov's. His approach may even be better than ex-ante as it is not clear that anything (an event) ever happens at all, for us even to be talking of ex-ante let alone of ex-post (i.e. before and after the event). Indeed, not only is the outcome never actually drawn in Kolmogorov's formalism, but one could argue that the infinite sequence of iid random variables involved in the strong law of large numbers is not even chronologically indexed. Each random variable in the sequence remains a variable (i.e. a complete mapping) and is never 'thrown' in a single outcome and the statement of the law of large numbers is, as we have seen, only a probabilistic statement concerning the overall random variable which is the average of the individual ones. One can always imagine that the whole sequence is 'realized' in one throw, as a single sample, if at all.

As a matter of fact, one may not need to imagine. If we were considering a population of human beings who all live at the same time and whose statistical distribution of height or weight or lifetime is of interest – say, for an insurance company – then the outcomes would be all drawn simultaneously. Infinity would mean that the population actually has an infinite number of individuals and the metaphor of throwing a die repeatedly, as if in time, would only amount to the idea of successive sampling. As the sample grows to infinity, the frequency of appearance of a certain height or weight or lifetime in the sample converges to a limit. Conversely, we might consider that the total population of outcomes of the experiment of throwing a die is actually drawn and that the chronological physical die is only picking one outcome then the other, as if sampling from the total population. Note, however, that this would be the rehearsal of von Mises's collective and the strange thing – to

repeat once more the consequence of the axiom of randomness – is that, although the population of individuals is given and actual, there is actually no way we could identify it *term by term*, even by thought, even with the mere thought of sampling it.

Given the indifference to whether a physical die is really empirically thrown in a temporal succession, or whether an entire population is already thrown and the growing sequence is only the illustration of sampling, the only chronology that really matters is the chronology of sampling. In the very idea of probability, when it is considered *ex-ante* as a propensity, there is, as we shall see, a notion of tense, of expecting the outcome then seeing what has come out. Chronology, in its most essential, is this order of thought and I should be talking of *chrono-logic* instead. The notion of a random generator may be independent of physical time, as we have seen with the human population, yet it is in essence *chrono-logical*. As such, it will be the target of our criticism. When we speak of the probability that the next individual that the insurance company is considering should be of a certain height or weight or life expectancy, the metaphor of the die works in the other direction and the implicit idea is that we are ‘throwing’ the whole population as one single huge die and expecting that random individual to be generated. Notice that matter, in that case, no longer has the aspect of a physical die. Matter is abstracted as that which acts as the material die in every draw of the infinite human population. Popper’s generating condition becomes inseparable from the end limit in which the statistical distribution (frequencies) is supposed to become equal to the probability distribution. The material die is truly immaterial and one can appreciate the extent of realism that is needed in order to speak of a material die in a situation which is, in actual fact, perfectly identical to von Mises.

To go back to Kolmogorov, only if we interpret the individual random variables in the sequence as throws of the same die (and thus as repeated throws) does this impose the image that the die is thrown, collected and thrown again in an identifiable sequence. Chronological sampling is only an image, which is forced on us by the image of the material die and the corresponding interpretation. Only if independence and identical distributions are brought to bear as the *hypothesis of matter* (of a material generator) does the formal proof connect to anything material.

This is not to mention that realization ultimately is an interpretation, too. Reality itself is part of the interpretation of the formalism, as we said. In a paragraph titled ‘Relation to experimental data’ and separate from the axioms, Kolmogorov specifically indicates that one can be practically certain that the event with very high probability would occur. This interpretation is known as Cournot’s principle and it should be invoked ultimately in order that the probability equal to 1 may elicit the actual realization of the limiting frequency in the infinite run. If the meaning of the probability being equal to 1 is that the corresponding event shall occur, this gives us no idea what the meaning of a probability p that is less than 1 might be. The only statement linking a probability that is less than 1 and a probability that is equal to 1 is the strong law of large numbers, which, as we saw, does not tell us which specific sequence is realized and, correlatively, tells us nothing about the individual throw that is chronologically inserted in such a specific sequence and whose individual probability we are after.

The material die is not part of the formalism; it cannot happen in the formalism. The only purpose of the image of the die is to introduce repetition and chronology (not physical chronology, but

sampling chronology or chrono-logic); however, chrono-logical repetition and chrono-logic (the notion that it is the same re-identifiable die that is lifted again after each landing and thrown again) are not themselves formal; they are also an image. Von Mises speaks of realized outcomes in his formalism; however, the axiom of randomness really ‘unwinds’ the realization. It ‘un-identifies’ the realized sequence as it re-immerses it in an infinite set of sub-selections and permutations thereof which reproduce the non-identification of the specific sequence in the realized frequency of Kolmogorov’s law of large numbers. The image of the material die contains this randomness over and above the mere idea of realization of the labels. One should think of what is common between the completed formalism of von Mises (the realized sequence combined with the axiom of randomness which un-realizes it) and the formalism of Kolmogorov, which realizes only the frequency and un-identifies the given sequence, in order to understand that chrono-logical repetition is just an image and that repetition shouldn’t really happen in time. This is what we will argue at the end of this chapter.

5.1.5 Tense

It is the combination of the *material hypothesis* and the *reality hypothesis* that introduces tense and chrono-logic in the picture. While the formal proof of the strong law involved nothing else than a formal sequence of iid random variables indexed by an un-interpreted n , the combined thought of *some* actual infinite sequence having *realized* the frequency (how, indeed, could there be reality without actuality?¹¹) and of the material generating device (die, roulette, and so on) which represents independence and identity of distributions, or *repetition*, now forces the idea, or rather the image, of chrono-logic. Chrono-logic is just an image and, as such, is not real. What is real may have to give up the chronological order of thought, as we shall see.

It is not infinity that is really the problem; infinity is not unimaginable. Even though the acquisition of the realized frequency only happens in the infinitely long run, we now have in mind the *local* image of a die that is repeatedly lifted and thrown *as part* of the total (infinite) image of the realization. (It will import a part of realization.) Even though the theoretically realized event does not distinguish between all permutations and sub-selections that verify the same limiting frequency, the local image of the die forces the image of local identification and local realization. How could we not identify the realized outcome of every local trial if we are actually throwing a real material die? Realization was brought to bear, through Cournot’s principle, only for the probability-1 event and only in the infinite (global) run; however, now that the series has globally jumped into realizing its frequency in the limit (this is only an image, of course), how can we resist the thought that something will be jumping into reality at each trial, especially when the image of the material die and of the local repetition is pressing it on us?

The local trial retains the intension of jumping *into* reality, which it imports locally from the global infinite end, only it cannot jump into something definite and determined, because only the overall frequency is truly realized. This is why the local throw of the die jumps into something multiple and random (extensionally) all the while retaining the intensional notion of realization that is imported from infinity. *This is how the image of probability and of the ex-ante attitude is created.*

Probability starts to exist locally as tension and ex-ante attitude, because now probability *means* and *intends* (and insists) only insofar as this probability is the local reflected image of the limiting frequency. What we are saying is that the notion of a random generator (or ex-ante attitude and the corresponding intensional meaning for probability, or the very image of a transition from the possible to the actual) is the hybrid product of an infinite interpretation of a probability-1 event and of the finite presence, or image, of a material die. The act of collecting the die and throwing the die, then the impression of *waiting* for the given face to come up and thereby *expecting* the outcome, as if in time, are only an image and an illustration. There surely exists an interval of time between throwing the die and discovering the outcome, however this interval of time belongs in the image and not in the probability formalism. We falsely believe that this interval is filled with probability; that probability operates in this interval somehow.

Of course, something is realized when the die lands on the table because something is always realized; only it is not the *possible* that is realized. According to Bergson, the possible is always made up after the real. As we shall see, the real in this case is the law of large numbers, which is just the re-expression of the contingency of the face. The tension needs to be cut. Realistically, the die cannot but present a certain face when landed and, similarly, it presents no face at all when still imprisoned in the hand. 'Either an event occurs or it does not: there is no middle way', writes Wittgenstein.

Expecting a certain face of the die to show up with a certain probability p somehow introduces tension and suspense in the picture. We feel that a transition will take place in time between the possible and the actual, between the probability and the realization. In reality, expecting a certain individual outcome with a certain probability p has meaning and is tensed only insofar as the outcome will be repeated with frequency p in the overall infinite series. (We all intuitively know that the notion of probability is linked with frequency.) Everything seems to be tensed in time because of the ultimate tension which says that, in the infinitely long run, the probability is 1 that the frequency of outcomes will be equal to p . Cournot's principle, which provides that an event with probability equal to 1 will be realized, tenses everything and falsely introduces the notion of drama, the notion of something ultimately happening and happening ultimately in time.

If an alternative interpretation to the probability-1 event were adopted (something no longer involving realization), or if the image of the chronological material die was given up (for some alternative image or some alternative *meaning* of matter), then probability and the intuitive sense of a random generator (waiting and seeing) would give way.

If the notion of realization wasn't imported from infinity, we would locally witness a series of outcomes, or unconnected empirical facts, without the intension of something being possible and expected to be realized at each trial. There would be no 'trial' even. That each trial faces multiple outcomes is not enough to create the notion of expectation and possibility. Without the jump into reality at the infinite end (which, as we saw with the human population, may not be temporal), there would be no notion of a withholding then a releasing, or a jump from possibility to reality. Chrono-logical succession of the throws is only an image pressed by outside matter – the die and the multiple faces – and the local 'expectation' is the combination of that and the ultimate infinite realization.

As a matter of fact, the infinite probability-1 event is not tensed; it does not feature a *jump* into reality. By itself, it is situated outside chronology. Nobody is holding a probability-1 event from happening then releasing it to its fate, as if in tense and in time. It is not as though heavy cones were usually encountered, standing for one second on their vertex, then falling with probability-1 on their side. The heavy cone is only an image used by Cournot to illustrate the meaning of a probability-1 event. Nobody gambles by throwing heaving cones on the ground. Chronology is definitely associated with the tension of the game, with holding, throwing and waiting for the outcome, then throwing again. There is no direct extrapolation, at least not in time, from the probability-1 event happening with certainty (the falling cone) to the probability p event happening 'with probability p '. The probability p event acquires its tensed and intensional local meaning, as we said, only through the combination of the infinite and un-tensed probability-1 event of the realized limiting frequency *and* the local and intensive image of matter and repetition. Repetition is intensional and it is needed in order to give probability its intensional meaning. Probability (infinitely) inheres in matter (and takes meaning) only insofar as the trial will be repeated (even repeated infinitely – this is why this repetition shouldn't be construed as happening in chronological time). There is no meaning for the probability of single-case events. Conversely, a probabilistic statement concerning a die is not equivalent to the extensional static description of its distribution of mass or to an extensional jump into one of the outcomes. The jump is probable *insomuch* as there is repetition and frequency.

The meaning of probability that we want to decipher seems, therefore, to be in tension. The probability of the face 'six' is equal to $1/6$ only insofar as a frequency of $1/6$ shall be realized in an overall event in which *it is not clear which specific, chronological sequence of outcomes is actually realized*. The very idea of probability and random generator seems to press us to say something about the die we are *chronologically* holding and about the *chronologically* next throw; however, the meaning that probability is ultimately receiving from the strong law of large numbers is 'reshuffling' chronology and destroying it. The image of the material die presses the idea of a chronology in which probability is defined at every step for every throw, yet the very notion of 'realization', which suggests the idea of a transition from the possible to the real and gives probability its apparent intensional meaning, really concerns an ultimate and global event with no relation or correspondence with the local throw.

The point is difficult to appreciate because we are holding the die and expect to say something about the next throw (in time). There seems to be something inherent, pressing and measurable about the propensity of the die to generate the next outcome with some probability however, what's really next is not next in time (for chronology is only an image); it is next in the sense that the whole sequence of variables unfolds, in non-chronological fashion and in non-identifiable extension. The next extensional throw is an image which we think finds its coherence in infinity because what is realized is the limiting frequency, and its realization rolls backwards to the next local throw we are considering and lends it the idea that the next outcome will be realized in time and that its probability will play out in time; however, infinity is ultimately contradictory with the chronological extension and identification of the next throw. There is something pressing to say when holding the die; however, this won't take place in time. The whole trouble that the interpretations of probability are marred with originates from this confusion. There is something local, but not ex-ante. A new

metaphysics of chronological intension, rather than extension, linked to place and no longer to extensive time.

5.2 Changing the meaning of matter

As a matter of fact, pushing the un-identification to its ultimate consequence, and dismissing the image of the repeated throw of the material die which falsely suggests identification, would lead us to consider the material die without the chronology of throws. The strong law of large numbers and the infinity it expresses would be folded back in the un-moved single die, so to speak. Instead of contemplating an extensive infinite series of outcomes, whose extensionality is jeopardized anyway by the von Mises's axiom of randomness or by the un-identification in Kolmogorov, infinity would become single and intensive. The strong law of large numbers and its infinity would mean nothing else than that the die lies on this face, only *the face could have been different*. Instead of seeing in matter the capacity of repetition and chronology (independence and identity of distribution), matter becomes contingency and chronological repetition becomes a conceptual rehearsal.

The image of the thrown and re-thrown material die used to give us the die *then* the face – or to *generate* the outcomes. Now, the image of matter is reversed. The material property of the die is no longer to generate, as if transcendentally from above, but to be stopped by the world – it is there, immanent, at one with the world within the one *strike of contingency* (which says that a thing is what it is only to the extent that it could have been different) – then it is intensively (un-identifiably) varied. The world has stopped the die on this face, as the worldly property of matter (which is not accessible to the formalism) is to present a certain face that could be different. We now see the face before the die; we are situated after the event (ex-post), which we have never expected in time anyway, and the only role that matter, or the die, has is to remind us that the face could have been different.

Time, and the intuition of waiting and seeing that a probability-1 event gets realized, have to be retired from the interpretation of probability altogether. Adding Cournot's principle to the formal (timeless) strong law of large numbers, we obtain the statement of realized frequency in the infinite run, only to realize that this realization cannot really unfold in extensive time because, to repeat, it is infinite, and it does not identify, as per chronology, one member of the realized series after the other. The conclusion is that nothing really happens, nothing is realized, in time. The static and timeless realization of probability is that it is a distribution of mass, that mass means matter, and that the only matter here – the only reality here – is the fact that a static die lies on a certain unique face, because it was forced to do so by the surrounding matter and the surrounding world, and that this face could have been different. This contingency, when it is understood intensively and not extensively – as a difference that is infinitely, repeatedly reintroduced in the face as the completed statement of reality – is then equivalent to the infinite strong law of large numbers.

5.2.1 The strike of contingency

Moving from the notion of contingency as a choice between possibilities (ex-ante and generation) to a notion without a choice (it is too late; it is real) but only with the inherent and intrinsic difference

and variation which prescribes that the face could have been different, at the same time as or even before the face is actually identified, is characteristic of the whole movement from probability to price or from the transcendent ex-ante attitude to the immanent ex-post, from the generator to the strategy and its place. Shafer and Vovk speak of the 'folk' representation in terms of random generating devices, which they want to dismiss; however, their games are still immersed in time. They have moved from transcendence to immanence, they have retired the ex-ante attitude, yet time is still giving them trouble in completing their radical shift. Waiting for the outcomes of the strategy still gives the false impression of ex-ante; however, when the strategy is folded back into the sense of its reality and its immanence, into its place, we can see that the sense of holding the place and holding the position is no longer the same as waiting.

The strong law of large numbers is formally equivalent to the probability distribution because a formal mathematical proof connects them. It is integral calculus differently manipulated: the mean of the probability distribution becomes the frequency to which the average random variable converges. What stands between the two and secures the bridge is the hypothesis of matter (the infinite sequence of iid variables). However, instead of imagining matter as a sequence of throws (which is an incomplete image anyway, because the formal law does not identify a specific sequence), matter has now to regroup and come all together. The meaning of matter is no longer that the die could be thrown repeatedly but that the face could have been otherwise, and *this new meaning is infinite, too*.

As soon as matter enters the formalism, it requires infinity. It is as difficult (it requires as much infinite labor) for the formalism to produce a material die that is meant to be collected and thrown again indefinitely as it is to produce a die that presents a certain definite face which says that it could have been different. It is very difficult to move from the formal mapping in which no possibility is distinguished to the *presentation* of one face (which retains the *matter* of the die to the extent that it says that it could have been different). There is no realization of possibility in the formalism as such. It is not enough to call 'realized' the probability-1 event in order to understand what the realization of a possible event ($p < 1$) means. The only realization is the law of large numbers; it is the matter of the probability formalism (as opposed to measure theory). However, it is through a false image (as we saw) that we could conclude, from the law of large numbers, to the realization of the possible. We should retain the notion of infinity that is contained in the law of large numbers for the sole purpose of realizing that the alternative meaning of matter (presenting a face that could have been different) is infinite, too, only intensively so.

The only thing to retain, in this alternative presentation of matter, is that the face (thus the frequency or the count) comes before the die; we no longer speak of what's inside the die as propensity and ex-ante attitude but of what's outside, of what has really happened, the face, and from the frequency, which no longer extensively counts (because of un-identification), we only retain its ex-post characteristic. *Frequency or a statistics (a repetition of differences) without the chronological image of the throws is just the mark or the strike. It is writing.*

We need to count the face, because it presents itself in reality; only the face cannot happen alone in the formalism. Infinity separates it from the formalism. It can only happen in a series which makes

it happen as frequency – the law of large numbers. On the other hand, we cannot count it because of randomness and un-identification. We have to count intensively, in a prelude of counting the *singularity* of the event, as opposed to its individuality or identification. (The event, as we shall see, is a ‘non-individual singular’.) This count of the face, which cannot present itself alone but only in the law of large numbers, this count without the count (precisely because of the law of large numbers) is the *strike of contingency*. Money is appropriate to it as the medium of contingency without the count of the states and the identification, as the medium of statistics without the generator, or of statistics understood as writing only.

We no longer evaluate the face (extensional probability); we price it (attach money to it). That this face presents itself contingently is a written clause, therefore it is priced. The law of large numbers introduces matter into abstract measure theory; however, the extensional series of outcomes was the wrong interpretation – it was only an image. Once folded intensively inside the face that is being contingently presented, the law of large numbers gives it the materiality and irreversibility of writing. It gives it the irreversibility of statistics without the time series, a frequency without the count of throws – what we have called the *strike of contingency*.

The strike of contingency is internal difference and intensive infinite repetition. It says of a thing that it could have been different before saying that a thing is. The fact that it could be different does not add other indented individuals to the contingent thing. It speaks of its singularity: it is unique, it is what it is insofar as it could be different. This strange way of recognizing the multiple as the intrinsic difference of the one – as its singularity instead of its individuality – implies a new way of counting. This will no longer involve frequencies but, as we shall see, the money account.

Identification, state and counting are not really what *matters* when it comes to limit theorems. Limit theorems and infinity are where matter enters because probability infinitely inheres in matter and because contingency is an intensive strike that is not slowed down by states and by count; it takes place at infinite speed. Frequency and counting of states have to give way, when the matter of contingency advances, in favor of an immanence that is the like of matter in its indifference to the identification of count, typically the material strategy of Shafer and Vovk, or the idea that no frequency converges in any actually identified series but only the betting odds do.

It is certain the frequency will *seem* to converge in an actual series of outcomes that is being generated by an actual throw of the die. If it doesn’t, and short of a miracle, then chances are there is something wrong with the die. The word ‘miracle’ and the word ‘chances’ are here to remind us that the whole language is couched in probability and that even empirical evidence cannot be proof for certain. The mathematical proof of the law of large numbers has both to manipulate infinity (as we talk of convergence in the long run) and to be certain. It definitely cannot concern a particular series of outcomes but all the series susceptible to be produced by the throw of the die. (Now, how can this domain be delimited?) On the other hand, the frequency of outcomes is certainly not the value of a mathematical function of the number of throws whose asymptotic behavior can be controlled by the usual (ϵ, η) criterion. Giving it a little thought, we realize that the only infinity we can have, or the only *proof* we can have (because the proof is infinite), can only relate to the random variables and never to the actual outcomes. The only thing we can show (the only thing that can have the shape

of a proof) is the formal law of large numbers, which pertains to random variables. The series of random variables is formally the replica of the one random variable and, as we saw, chronology is certainly not relevant. Because of un-identification, the infinity and the certainty characterizing the proof ultimately relate to the intensive difference of contingency and to its strike. The formal proof, we may say, pertains to the unmoved die and to its intensive matter. This alone can be shown.

Taking the dual representation and using integral calculus to average the sequence of iid variables in the infinite long run instead of averaging the static probability distribution of the random variable, this reorientation of thought, once we understand that chronology is a false illustration of the sequence of random variables (because of the fact that the chronological sequence of outcomes is not extensionally identified), is how the strike of contingency should be understood.

5.2.2 The one and the many

The formal random variable is a case of the problem of the one and the many. It is one mapping with multiple values. As such, integral calculus is appropriate to it; this is why the notion of the mean of the random variable is crucial. Unsurprisingly, the mean is distinguished when the one-to-many, known as the random variable, becomes the sequence of multiple random variables – integral calculus differently manipulated. The mean is appropriate to the one-to-many mapping because it realizes neither of the values, but it averages them all. This translates into the formal law of large numbers through formal calculus and transformations. The law of large numbers is also a re-expression of the mean. *There is no difference between the random variable and the law of large numbers.* Formal calculus connects them. (However, the law of large numbers realizes matter because it takes the side of contingency and the single face; it is intensive; it features a probability-1 event.)

The random variable is one thing that relates to many outcomes. It is an instance of the dialectics of the one and the many. Now, the sequence of independent identically distributed variables that is involved in the strong law of large numbers is the dual or conjugate view of the same dialectics. The variables share one probability distribution (the one), yet they are independent (the many).

Rather than mass distribution, the conjugate view taken by the law of large numbers is frequency. Mass distributed among outcomes in one view; outcomes distributed along frequencies in the other. Mass summing up to 1 in one view; averages converging with probability equal to 1 in the other. One view is extensional; it takes the different possible, yet unrealized, outcomes under one heading. The other view is intensional and therefore dynamic; it takes one realized outcome at a time, and then argues, infinitely, that it could have been different.

Probability is a mean, when considering the counting variable whose average sequencing gives the frequency. That there should be nothing realized, in the random variable, and only the mean distinguished is the same thing as saying that there is nothing realized in the infinite sequence either, but only the frequency distinguished (and no particular sequence identified). It brings nothing to think of the image of the die, because there is no die and no repetition in the formalism.

There is formally a difficulty in making the iid variables relate to the same die, because there is formally a difficulty in lifting the die and throwing it repeatedly. For the time being, the only correspondence between the material die and the formal random variable is that both map the one

into the many. The die is one material body with many faces and the random variable is one mapping with multiple values. Since the sequence of iid variables follows formally from the formal random variable, maybe the alternative is to try to make sense of the sequence of variables without lifting the die and without ‘repeating’ the formal variable. The sequence itself has to be re-enfolded into the random variable with the new meaning of matter gained in this way.

We certainly did not need the material die in order to form the sequence of iid variables and formally to show the law of large numbers. Independence and identity of the distribution are not external notions to the single random variable. Instead of thinking that matter happens outside, at the moment when independence and reduplication of the probability distribution are first mentioned, we should think that *insomuch as* the probability distribution (as attached to the single random variable) enables the thought of independence, identity of distribution, and thus the formal proof of the law of large numbers, this meaning and intension are what bring matter *inside* and the law of large numbers is at once the expression of the intensive meaning of matter, which says (infinitely) that the face could have been different. Instead of pursuing, through the formalism, the impossible proof of something we think we empirically and externally observe (the concrete law of large numbers), we should rely on the self-closure and completion of the formal proof in order to learn something unexpected about matter, a new meaning of matter that was not visible before the completed formalism had taught us how truly to extend the (infinite) dividing line between form and matter.

There should be a complete duality of concepts between the infinite sequence of iid variables, whose average converges to the mean, and the one formal random variable. That the random variable should be given as a formal mapping of the one into the many and that the infinite sequence of iid variables should be given, these two notions should be the two geometrical sides of one and the same concept. Somehow, infinity of the sequence is the dual side of the ‘infinity’ of matter.

One should try *geometrically* to understand why formally giving a random variable (the mapping) and a probability distribution – this invention by Kolmogorov – is the same thing as eventually getting the convergence of frequency in the sequence of random variables. The probability distribution and the sequence of iid variables should be two different exploitations (angles) of the relation between the one and the many.

The relation between the one and the many is the deepest of subjects; as a matter of fact, it is *infinite*. That there should be *one* mapping (*one* variable, *one* probability distribution), this identity, corresponds to the variables in the sequence being *identically* distributed. That there should be multiple values into which the variable maps, this difference and distance, corresponds to the variables in the sequence being *independent*. Independence is the guarantee that, as the variable is ‘repeated’, something remains the same and something differs; that is to say, the same value is not necessarily repeated. It amounts to the same to say that a variable now takes value in the tensor product of the spaces of values, potentially with all the different permutations.

Mapping the one into the many is deep; the random variable (the probability distribution) and the sequence of iid variables are just two *explorations* of it. The formalism ignores matter and repetition, so one may ask what it is, in the formalism, that has ever induced the thought of the sequence of iid variables, if not something that is deeply encoded in the notion of mapping the one into the

many. There must be a deep truth, of which the probability distribution of a random variable (this concept) and the sequence of iid variables are two different expressions. This is a truth about integral calculus, of which the definition of the mean of the random variable (or equivalently the definition of probability itself, for probability is a mean and is an integral) and the law of large numbers are two different expressions. There is a deep and infinite truth in the integral calculus (or in the mapping of the one into the many) which makes it so that the definition of the mean of the random variable and the law of large numbers (relating to iid variables) are tautologically related.

Beyond the die, there lies this truth and the material die is only a material simulation of it. The die simply reproduces this truth. The random variable is an integrator of the many and the law of large numbers (taking the sum of variables that are the same in one important aspect, which is that they are identically distributed, and different in another important aspect, which is that they are independent) is another integrator. There is an infinite distance between the one and the many. The one is one and there is nothing to break it or duplicate it, and the many are many and there is nothing to integrate them and to collect them. It is not enough to collect the many under the label of *one* variable or *one* mapping. Infinity starts (begins?) when each one of the many values receives a probability that is strictly less than one. The one is broken and it will take infinity to reunite it. The many break inside the one and the one breaks into the many. That *one* face of the die receives a probability that is less than 1 is the sign that it is not alone and that the other faces now share something with it and compete with it. The other faces have broken inside it and have taken something away from it. It will take infinity for the one to show up again (the law of large numbers) and for the individual faces, for now infinitized in the mapping, to re-emerge individually in the sequence (or not even that, as the permutations are not identified – no identity here either).

One can always group the many under one heading or one set or one mapping. Formally, anything goes – even total chaos. For matter to start and for the mapping to start making sense and consistency, the multiple outcomes have to share something material. The different faces of the die share the material fact that they are faces of the same die. Mass has to be shared among the outcomes and this is what the probability distribution accomplishes. When subsets of the sample space start receiving probability measure, this is the indication that they are glued together by some common interest or concern (some call it ‘same condition of experience’). Matter is infinite vis-à-vis the formalism (it lies at infinite distance). That the faces of the die start receiving probability that is less than 1, thus sharing one mass, is a step beyond the mere formal grouping of the multiple faces. It is an infinite step beyond. Now the faces hold together by a unity (of mass, of matter) that is infinitely stronger than the mere formal unity that had grouped them in mind. (Is grouping anything in mind really possible, by the way?)

5.2.3 The reality of contingency

The individual faces have been integrated by mass and now it will take infinity for the individual face to re-emerge again. (It takes the strike of contingency which infinitely says: ‘This is the face; it has emerged; only it could have been different.’) As a formal grouping, the faces were merely possible. The addition of mass marks the intervention of contingency. Mass is the prelude to matter and

matter is what says: ‘The die has landed, this is the face but it could have been different’. Introducing mass is already (infinitely) stepping into irreversibility and univocity. There is no going back to the indifference of the merely possible. The only way is the way forwards through an infinite sequence of iid variables which can extract and exhibit now the individual face only through infinity and an irreversibility of a different sort: the face is lost in a permutation whose exact sequence is not identified. The inscription of mass translates into the inscription of the infinite statistical series. There is no reversibility or identification in either.

Mass, or contingency, or matter, is writing without turning back – without reciprocity and without waiting. The inscription of the statistical series, or writing, is incompatible with identification; hence, it lends itself to the immanence of the Shafer and Vovk strategy – immanence in the sense of no state transcendence and no corresponding identification – or, *which amounts to the same*, to the exchange of the written contingent claim. The exchange is immanence. It is the immanent response to erasing the individual states under the strike of contingency and to the corresponding material sheet which bears the strike and for this reason is ready to be exchanged. Price, in the Shafer and Vovk sense of the immanence of the strategy that no longer identifies the conditional probabilities and no longer recognizes Cournot’s principle or the realization of possibilities, price as the entry into the game without probability and transcendent generator, is the same as price when price is the other face of contingency without states or of contingency which has only one face, and is the same as price when price is the result of exchange. The exchange and the Shafer and Vovk strategy are anyway both mediated by place, which takes over chrono-logic and the ex-ante stance. There might be no inkling of an exchange in Shafer and Vovk’s price because there is still accidental time. However, the exchange is implicitly there because of immanence, because of the way in which the Shafer and Vovk trading strategy and the resulting definition of price translate and drag to the time before what is in reality ex-post; that is to say, immanent.

The die is not formal; it lands on one face, which could be different. It is not necessary to imagine the repetition of the throw. From the moment matter is imagined (the die), the formalism encounters its arrest and the infinity of possibilities becomes the one face which could be different. Probability theory would have remained equal to measure theory had it not been for matter – that is to say, contingency, the shock of the face of matter, the piece of sentence which says: ‘The face could have been different’. Probability theory wouldn’t be what it is without the law of large numbers and the law of large numbers is the same thing as the random variable and its formal mapping except that the mass distribution (probability distribution) is the prelude for repetition, because it is matter and matter is contingency and repetition. (The ‘could have been different’ is repetition without the sequence). The probability distribution is the relation to a common interest called the die.

The collision of matter with the formalism produces both the notion of repetition, or the idea of lifting the die and throwing it again, and the notion that the die is landed (laid to earth/matter) and *rests unmoved on one face*. The two notions amount to the same. It is as difficult (hard) for the formalism to represent repetition (lifting and throwing the die repeatedly) as it is to represent never lifting the die. Lifting the die and repeating the throw amount to saying that the die is already landed and presents one face. Temporal succession is incidental on the contingent presentation of the face, and contingency, once inherently and intrinsically understood, is repetition.

Chronological repetition is only an illustration; this is what has confused us so long. The law of large numbers is something deeper than the repetition of throws; the repetition of throws is only a reflection of the law of large numbers, a superficial image. Independence and the infinite sequence of random variables are all contained in the single random variable (since it is the same formalism of integration, the one and the many, that is operative); or rather, they are contained in the single variable and fold back into it so long as the formal variable is loaded with mass; that is to say, with the probability distribution, with mass which is the prelude of contingency. Chronological succession (as an order of thought) is an easy way out for the thought.

Contingency is the matter of repetition. The law of large numbers is the *realization* of the formalism in the sense of connecting it with reality. This reality is contingency; it is going from the indifferent and formal mapping of possibilities into the reality of the only face on which the die lands, completed with the reality of matter or contingency, which says: 'The face could have been different'. The law of large numbers realizes the formalism for the reason that a probability equal to 1 first makes its appearance here. It is not that the sequence of throws, when conducted to infinity, leads to the convergence of the frequency with probability-1. This is a false image because it presupposes the actual throw and the actual series of outcomes whose frequency, in fact, doesn't converge (how could it?). With probability equal to 1 the die is realized as matter, as lying on one face that could have been different. This reality is intensive contingency. It is not extensional like the total of possibilities. It is a strike, a marked difference. In it is exactly contained the whole series of random variables, in which it is key that the individual outcomes may not be identified – as it will be key that this contingency is ex-post and is writing, and the only identified thing is the convergence of the betting odds, or the existence of price without the identification of state.

Chronological repetition is only an incomplete (and false) image of the law of large numbers, because chronology implies that the die lands on a particular face each time (because of the succession that is essential to chronology). The law of large numbers instantly (in no time) and infinitely realizes the die. Through it, the formalism establishes contact with reality, not through Cournot's principle. Imagine that time and Cournot's principle are retired – that is, the notion of a transition between the possible and the real is retired – and the law of large numbers, now interpreted outside chronology, becomes the first principle of reality. This reality of contingency is categorically different from reality in the sense of the realization of a possibility. Cournot's principle provides that an event whose probability is equal to 1 will actually take place. Is this supposed to be staged in time? Have you already seen an event whose probability is known to be equal to 1 and which, suddenly, and only then, is realized? Either the event is already realized and then its probability is by definition equal to 1 now, with no idea of what it was equal to before, or the event is not realized; it is expected and we don't know what its probability p means.

5.3 Money

It may be difficult to read in the un-moved die an infinite series of faces that is no longer extensive but intensive (or really to perceive the strike of contingency as an infinite intensive repetition).

Another way of retiring chronology would strangely be to maintain the 'chronological' series and the *residual* (or accidental) appearance of time; however, we would remove, in the series, what makes it a frequency – what makes it the result of a *generator*, or the repetition of a transition from possible to actual, or again a series of *ex-ante expectations*. We keep the time series, we keep chronology; however we withdraw tense – *chrono-logic* in the sense that something *happens* in time (ex-ante attitude) and that time mediates some happening.

Matter is infinite relative to the formalism, intensive relative to extensive. This relates to *problematics*, in Deleuze, as distinct from *axiomatics* and is typical of the relation between formalism and matter.¹² The infinity in the strong law of large numbers is indicative of matter,¹³ this is why, if we retire the image of the finite trial, or the incomplete image of the identifiable chronological series, infinity has no other place to go but to regain the intensively infinite strike of contingency. Infinity in the formalism is the expression of immanence; this is why any transcendent translation (through the count and the ex-ante attitude) will always be incomplete. Folding back the infinite series in the static intensive strike of contingency is one way of regaining immanence. Another way is to maintain the external and incidental appearance of temporal succession, while retiring the transcendent (finite, because it is not immanent) ex-ante attitude. It consists in trying an alternative interpretation to Cournot's principle.

We said tense was the result of a combination between a realization without tense (the probability-1 event of the limiting frequency – the reality hypothesis) and the presence of matter qua repetition (which is intensive). So, presumably, if we retired the un-tensed and non chronological jump into reality of the probability-1 event, while maintaining the succession of outcomes, this chronological succession would strangely turn to proceed without tense and generation. We would be pursuing a series, dealing with it in time; however, at no time would there be tense and expectation. We would be, so to speak, always in an ex-post position with respect to the series, rather than ex-ante (in exactly the sense in which the market price is ex-post or statistics is ex-post and which is the sense of writing). How is this possible, given that the series unfolds in time? How could we remain immanent and ex-post like the strike of contingency, yet watch it ahead of time? What medium could be in contact with the immanence and infinity of matter yet recede in time before the event is actually identified, without this regression bringing back the ex-ante attitude?¹⁴

The answer is money. Contingency is not a static distribution of mass. Contingency is intensive. It is equivalent to repetition and independence (or repetition and difference) absolutely, intrinsically, not relatively to the interval separating two throws, not relatively to the two throws. The face that the die presents could have been different in all kinds of permutations, whose converging *non-chronological* frequency, we said, is the intensive infinite strike of contingency. This totality of permutations does not exclude the logical possibility of the permutation where the same face presents itself indefinitely. When the image of matter and actual repetition (incompletely, through chronology) mediates the infinite intensiveness of contingency, this translates into excluding sets of measure zero. Cournot's principle is concerned with what happens and gets realized; measure theory is concerned with excluding sets of measure zero. Cournot's principle retains the transcendent attitude of jumping from the possible to the real (even though this is not tensed and chronological

– it is only logical). The combination of the two is the probability that is equal to 1 (which only measure theory can make rigorous) and an image of realization which imports to the local level the transcendence of the ex-ante attitude. The false image of chronology insidiously introduces the possibility of sets of measure zero – as if we would, indeed, wait in time and witness the same face come up indefinitely.

Somehow, sets of measure zero are the necessary unfortunate consequence of extensionality, or of trying to express the intensive strike of contingency in time and chronological repetition. Extensionally and identifiably, there cannot but exist exceptional series of outcomes (to be excluded by measure theory). Intensively, however, the strike of contingency sums it all (infinitely). This is the same as saying, later, that the winning strategy of Shafer and Vovk is the same, independently of the actual path.

The suggestion, instead, is to create the medium such that it would no longer matter that something is realized, but only place matters. Place as that through which, and inside which, we could be situated before the event chronologically, yet not look at it ex-ante, but ex-post. If we are gambling against the die, we no longer count the frequencies but only accumulate gains (or losses). The exceptional event would presumably lead to our capital growing infinitely; however, what matters is not whether this should happen or not, but the actual constructive gambling strategy we are following. The accumulation of money bridges time and introduces the idea of place that the gambler is defending, as he is focused on what strategy to follow. In this way, money lends itself to the definition of price.

The ex-ante attitude, characteristic of probability and generation, is due to the combination of Cournot's principle (realization) and the identification of the individual trial that we believe we should be dealing with (and saying something about) simply because the material die presents us with this obligation. The case of the insurance company is a good illustration. It is able to address the insurance contract of the individual person because of the strong law of large numbers, and the imported meaning of the probability of individual death (or other accident that is being insured). The individual person plays the role of the individual throw of the die, the homogeneous population of individuals plays the role of the material die that is being thrown and re-thrown, and the realized frequency of accidents in the infinite long run imports locally the image of the possible realization of the death of the individual; that is, of a local happening, something we are locally addressing with probability.

In reality, however, there is no such thing as the probability of death of the individual person. The reason why the insurance company is able to address the individual case, as if ex-ante, is not that an infinite series of possible realizations converges to the certain realization of the frequency in the infinite long run. Rather, it is the ex-post accounting equation of the insurance company, or the fact that the insurance company will exactly break even in the long run if it announces insurance premiums that are consistent with the frequencies of accidents and if the probability-1 event, indeed, takes place; that is, the realization of the exact frequency of deaths which exactly consummates the capital that the insurance company has amassed by collecting insurance premiums. When the insurance company quotes an individual premium, it is not, therefore, adopting an ex-ante stance

towards the possible individual death; it is simply reflecting ahead of time (incidentally) the already finished ex-post accounting equation.

In the same fashion that the individual probability $p < 1$ is explained away by the ultimate ex-post break-even in the global account, the probability-1 event, which, in this case, is the convergence of the frequency in the strong law of large numbers, is explained away by the constructive reality of the winning strategy. Money is operative at both ends. Given the convergence of the frequency and the imported meaning of probability, the ex-post accounting equation reminds us that there is nothing really ex-ante to probability, but only an image and an extrapolation. Money provides the bridge and the medium of this time loop. Conversely, when the convergence is still at play and is not yet realized; when the probability-1 event is not yet decided or even interpreted, forced into reality, because it is the ultimate event and there is no event beyond it that could bestow meaning on its probability, we cannot say that probability means that a certain insurance company can quote a certain contract because it breaks even in the long run; money has to operate before convergence, independently of whether convergence obtains or not; it operates through the winning strategy, whose reality is now and immanent and independent of the outcome and of the *realization*.

The ex-post equation is immanent and material; it is not counting anything in an ex-ante transcendent fashion; it is not relying on the notion of possible states and their realization. Randomness is such in the population that no 'player' could decide, trial after trial, individual after individual, whether or not to take over the liability from the insurance company (conversely, whether or not to pay the insurance premium and take over the payoff from the insured individual) and become infinitely rich in the long run simply because he has a rule for deciding. No player has a rule for selecting a sub-population that would die at a frequency that is inconsistent with the premiums being quoted by the insurance company. The *meaning* of the insurance premium that the insurance company is quoting is that the population exhibits this kind of randomness which prohibits the sub-selection rule and insures that the frequency converges in the way the insurance premium requires.

Cournot's principle guarantees break-even, because it provides that the frequency will converge; however, we now have an attitude that is indifferent to whether the frequency converges or not, an attitude that merges the two conditions in the same clause, and this is the effective winning strategy. Instead of saying that the probability-1 event will be realized (the frequency converges), keeping in mind that logically it might not, we now say that there exists unambiguously the winning strategy whereby either the player becomes infinitely rich or the frequency converges as planned. Material existence, immanent and indifferent, replaces transcendent realization. Saying that the insurance company is able to quote an individual premium, not because of the knowledge or even the notion of probability and random generator, but because it breaks even on average, in an ex-post fashion, except in case of the exceptional event, is the same as saying that the question of whether the exceptional event happens or not (Cournot's principle) is replaced by a winning strategy whose outcome no longer matters but whose construction only does.

The ex-ante meaning of probability (which is tensed in chronology and makes it impossible logically to exclude exceptional events) is only an image, or short for ex-post equation, and similarly

the transcendent jump into reality of Cournot's principle is only an image for the *reality* of the winning strategy. Money is operative in both cases. It is the medium which allows the insurance company no longer to face the individual case and it is ultimately the unit of count in which the winning strategy is measured. Money, which is preliminary to the market or the ultimate place, can be defined as the coincidence of these two characterizations. Somebody who is looking at the law of large numbers no longer through counting the individual cases, no longer ex-ante, but through implementing the winning strategy and through the reality of the winning strategy is therefore adopting an ex-post and immanent attitude, strange as it may sound. He is following, in the statistics, no longer the random generator (ex-ante) but the writing.

Statistics has the character of writing because it is irreversible and offers no comebacks. What seems to come back in the statistics (what seems reversible) is due to the generator or the ex-ante attitude, and therefore to Cournot's principle. Now, Cournot's principle has been replaced by the real and present winning strategy, whose proceeds are only accounted for ex-post and whose present reality is not an expectation – it is not ex-ante – but is simply the *translation* of the written and deposited and ex-post character of the accounting equation to the present. It is a future sediment, a future trace, made present. This apparent twisting of time is made possible by the nature of money. Money is not accidental in matters relating to randomness, but fundamental. The reason why money offers a more universal criterion for deciding on the randomness of a time series than frequency is the same reason why money offers a present and material, and therefore immanent, strategy in replacement of the attitude of waiting and counting frequencies. The argument from the break-even of the insurance company or equivalently from the winning strategy of Shafer and Vovk is a fundamental argument; it is fundamental that the ex-ante attitude and its transcendent correlate should have been retired in matters relating to randomness. As soon as it is determined that what converges is not the frequency of actual outcomes, but the betting odds, or frequency in terms of random variables, this by itself either folds back the frequency into the strike of contingency, as a mark which is devoid of the image of the chronological sequence and which says: 'The face could have been different', or it maintains the times series without the realization of the possible (without Cournot's principle). For the reason that the strike of contingency (the face which says: 'It could have been different') enfolds a statistics without the chronological count or extensional identification of the series and with only the count of money, we say that the strike of contingency is written (*le trait de la contingence*), like the statistics.

Contingency says of a thing that it could have been different or that it could be different before saying that a thing is. As we said, the fact that it could be different does not add other individuals to the contingent thing. It speaks of its singularity, its unique character. The thing is unique, it is what it is (this way and no other way, given), *insofar as it could be different*. We are recognizing the contingency, the strike of the thing, what irreversibly makes it singular, before recognizing its individual state or other analogical states. If we may literally repeat ourselves, this strange way of recognizing the multiple as the intrinsic difference of the one – in effect, as its *singularity* instead of its *individuality* – implies a new way of counting. *We no longer count the individuals; this new way of counting is through the money account.*

5.3.1 Time is not money

We can count the multiplicity of the future event – the fact that it could be different – before even the event is identified or realized or counted in a series. We can count that it could be different – we acknowledge its contingency – before recognizing its individuality. How different this is from the frequency count! Contingent matter that has recognizable faces – such as the die – has wrongly inscribed its contingency into time, simply by interpreting the clause ‘The face the die lies on could be different’ as an instruction to lift the die and throw it again in time, so to realize the different face. We think contingency is a variable taking value among a series of recognizable outcomes. Let us think, instead, that there is no way we can lift and throw again a future event, whose face is not recognized yet. Yet, we have to count, or account for, its contingency. There is no meaning to the random generator of the single-case event, so why should there be one to the generator of repeatable events such as throwing the die? Money will lead to the market in case of the single-case event. It leads to the proto-market, or proto-place, in case of the repeatable event which wrongly inscribes its series in time. The account of the insurance company is a proto-market and the corresponding proto-place is the adoption of the real and effective strategy.

Cournot’s principle realizes the frequency in the long run; however, no individual sequence is distinguished. The combination of infinity in the strong law of large numbers and of un-identification leads to folding back the strong law of large numbers into the singular, contingent unmoved matter. To express the convergence of frequency in the long run is thus a representation of the strike of contingent matter; it is recognizing its intensive difference. In front of that contingency, another representation is possible, as we saw, no longer drawing on time and identification, but using money instead. Even though the Shafer and Vovk strategy unfolds in time, it has removed the ex-ante attitude by offering an alternative to Cournot’s principle. We could wait and see whether the impossible event is realized and whether, consequently, we get infinitely rich; however, now that a strategy is staked on that possible outcome, what *matters* is the actual strategy. Money has this fundamental property. To turn expectation into an effective clear and present strategy, to turn time into place, is in communication with the fundamental property of money which, on the other hand, provides a more general characterization of randomness than von Mises’s frequencies.

It is no accident that the impossibility of gambling systems should be (following von Mises) a more universal characterization of randomness than the invariance of the frequency in all sub-selections of a given infinite sequence (keeping in mind that this axiom of randomness precisely prevents any sequence from being given). It is commonly believed that the axiom of randomness prevents any sequence from exhibiting any recognizable betting pattern, or any frequency of appearance of the outcome of interest that would differ from the odds at which the bet is offered. This, of course, is the reason why you are statistically prevented from making a fortune when betting against a truly random generator. It is believed that a certain property of frequency (randomness) entails a property of the gambling strategy (break-even). However, Jean Ville, on whom Glenn Shafer and Vladimir Vovk base their work, has shown the opposite, by allowing the gambler to take fractional bets, in a dynamic strategy that is the ancestor of the dynamic rebalancing known to derivatives players today. He has shown that some sequences that are random in the sense of frequency can be vulnerable

to a gambling strategy using fractional reinvested bets, whereas sequences that are invulnerable to gambling systems are, of course, random in terms of frequency.¹⁵

This suggests that money, which is the medium through which gains are accumulated, is ultimately a more fundamental 'counting device' of randomness than the frequency count. (It is more 'liquid'.) This is no accident and, in my view, corresponds to the fact that randomness is ultimately incompatible with the extensional chronological view where one outcome is chronologically followed by the next.

Contrary to the current saying, time is not money, because money is more fundamental than time. The alternative interpretation of the probability-1 event that Shafer and Vovk propose doesn't take place in time, but in money. As we said, the real import of their constructive trading strategy is that it no longer *matters* whether the probability-1 event is realized or not. Now, the strategy is the only matter. When all we had was probability, we could only wait for the probability-1 event to happen (or not happen), and had nothing real in our hands in the meantime. With Shafer and Vovk, we have the real constructive trading strategy in our hands, in the meantime. This dispenses with time.

It is well-known that martingales provide an alternative formal presentation of probability theory.¹⁶ However, when martingales are regarded no longer as a formal alternative, but as an alternative to the traditional interpretation of the probability formalism – the one that revolved around Cournot's principle and the corresponding notion of realization, and produced the notion of frequency as a consequence – the connection with reality that we end up with is no longer the notion of the potential realization of the possible, but that of the present reality of the trading strategy.

In fact, all that Shafer and Vovk have done is to recognize the fundamental role that price could play, as opposed to probability. We usually think of price after probability, and more or less identify the price of a contingent payoff with its expected value. Now consider, with Shafer and Vovk, that a probability of 1 no longer means that something possible is realized; and think, therefore, that probability no longer means that a certain frequency is realized in the long run. Nothing is realized, in this sense, and 'probability' is no longer probability. Time series still exist, of course, but they have been robbed of the notion of a random generator. Time is unavoidable, but it is no longer essential to contingency. Frequencies are still empirically observable, but they no longer mean anything as far as probability or 'what to do next' is concerned.

Following this logic, price no longer depends on the prior concept of probability, but becomes what Shafer and Vovk mean it to be – namely, the price at which you buy the bets (possibly fractional) or, in other words, the entry ticket into the trading strategy. Price means what it means insofar as money is what it is. And money is that medium such that the gains you accumulate in it would eventually diverge to infinity if something didn't come to pass. However, this, as we said, is no longer the issue, because time and possibility are no longer fundamental; what is fundamental is what money permits us to have now – and this is the present strategy.

Now, Shafer and Vovk's position sounds unrealistic, because who is to sell you the bets in reality and to settle them step after step? De Finetti has already conceived of subjective probability as betting odds; however, what he had in mind was a single event (not an infinite series as in Shafer and Vovk), and a single agent (a banker). In the infinite game proposed by Shafer and Vovk, they call the

opponent ‘reality’. Reality is supposed to offer the bets and to settle them. When this reality happens to be the actual financial market, it all comes together nicely because you buy a security at its market price in the morning (the bet) and you settle your bet at its closing price in the evening.

Now, Shafer and Vovk do not imply that in the majority of cases (which are other than financial) we should actually create the corresponding market – for chances are that we would then relapse into the temptation of modeling the corresponding prices probabilistically, either objectively (some data generating process) or subjectively (the market as a bunch of agents), thus relapsing into saying that probability is more fundamental than price. In cases which are not readily financial, Shafer and Vovk mean by money something purely ‘imaginary’, whose only role is precisely to show that there is another way of addressing the contingent event than waiting and seeing, being tensed in time. What they wish to eradicate is the notion of the random generator. To repeat, all of this is taking place at a higher metaphysical level, and ultimately money should be seen as an alternative to time. What is interesting, now, I think, and would press the point even further, is if we were even to think of the ‘real’ money involved in the financial markets as this ‘imaginary’ thing Shafer and Vovk have in mind. In other words, to think that the financial markets are the real proof that random generators do not exist.

5.3.2 Money is place

That money and martingales should detect cases of non-randomness that qualify as random for von Mises is not a slight thing; it must be the defining characteristic of money. And this, in turn, should be essentially linked to the faculty of ex-post accounting, which is compatible with matter and immanence, and different from the transcendent frequency count.

Ex-post accounting is immanent and is therefore compatible with the matter of contingency, which is infinite relative to the transcendent frequency count. Money is stronger than frequency (Shafer and Vovk) because it accumulates in the end and leaves us an effective strategy in the meantime, which is insensitive to the frequency count and to conditional probability. Measure theory is perfect; however, independence and conditional probability were introduced in order to lead the way to the law of large numbers in the sense of convergence of the frequency, hence to the meaning of probability. We must imagine a more general characterization of randomness and the corresponding liberation of ‘probability’ from the ex-ante attitude (this will be price). Shafer and Vovk provide the way of interpreting the exception in measure theory no longer in terms of realization or not (which is linked to conditional probability and frequency – which is linked to statistics because of realization) but of the effective strategy. However, they are still reliant on randomness of the same nature.

We said it was not by chance that randomness was linked to the impossibility of gambling systems. Now this, gambling, is linked to a place rather than a tense and is incompatible with the generator and the notion of state. As soon as we decide to count with money, we move away from frequency.

Money has this power. The particular case of the repeatable event gives statistics and the population; however, the general case gives place and the market. Shafer and Vovk show the transition, or how to separate oneself from the generator, all the while leaving time present (hence

the difficulty), before abandoning it for place, and while retaining the sort of randomness that corresponds to the repeatable event and to conventional statistics. The law of large numbers seems to imply a convergence; yet, Shafer and Vovk have the merit of retiring the *ex-ante* attitude through retiring Cournot's principle and of insisting that nothing converges and that nothing is expected. Only betting odds converge and so it is the material strategy that only matters. The law of large numbers is an immanent present strategy once interpreted away from the frequency count and outside chronology.

This essentially defines the notion of *price*, owing to this power of money. This is the reason why the market is the generalization (when place takes the place of time). Shafer and Vovk think it is a coincidence if the market prices are also prices in their sense. Price is defined by money and money is defined by the power that enables it to better represent randomness – without counting.

There is no realization in the formalism of probability theory. Random variables are mappings that remain 'packed' and unrealized. The introduction of matter brought the definition of independence and Cournot's principle, then interpreting the probability equal to 1 as realization of frequency in the infinite run, precipitated the notion of realization of the individual variable. The martingale alternative of Cournot's principle retired the notion of realization (hence of statistical series, hence of time, even though the strategy still unfolds in time). The formalism is open again to another interpretation and another notion of reality. It is not easy to think of the multiple faces of the die, to think of time, and not to think that a certain face, being at *first* possible, is *then* realized. Yet, this is what it takes to interpret reality in the formalism as the reality of the strategy instead of a realization. (In a way, Shafer and Vovk provide an answer to Bergson.) Brownian motion is a product of the formalism, as such un-interpreted. It is open both to the interpretation in terms of statistics and frequency and to the interpretation in terms of the money account.

Probability was intuitive because of the image of the die that would be lifted and thrown, of multiple possible faces one of which would be realized. Probability was thought to inhere in the die (or in the die generalized to cover the conditions of experience – the throwing hand, and so on), therefore to be a very precise concept, because there was nothing outside the die to generate the outcomes. Of course, the individual outcome was undetermined (as there was a multiplicity of them) and certain unformulated conditions (precisely left to chance) were believed to determine the path to each individual outcome in each individual trial. What was precise and determined thus moved to a higher level. What now inhered in the die was no longer the precise paths leading to the particular outcomes, but the way they would be distributed. This precision and determinateness was later confirmed by the convergence of the frequency in the long run.

There was intuition that, every time we lifted a die, something definite, the probability, was at once solicited. However, it all changed when the effort was spent to formalize probability. One could almost say that the discovery of matter (how it separates itself from the formalism) was made *after the formalism was completed*. Whether through von Mises's axiom of randomness or Kolmogorov's random variables which would converge in average without identifying a particular series, matter was discovered to be infinite vis-à-vis the individual throw and even time.

Probability was intuitive; however, it took the effort of formalizing it to discover contingency, in another words, the new infinity and the new reality. And when the new reality was discovered, it suddenly appeared in the light how narrow the channel was, through which intuition had ever thought of the possible and its realization. Once contact was established with immanence and infinite matter, the theme of the possible and the realized revealed its weakness and relativity. A more general (even absolute) medium for contingency was needed and this appeared to be money, or ways of dealing with the multiple that did not abide by the channel of the possible and of time. Not surprisingly, this was ready for alternative interpretations (to the transition from the possible to the real) such as Shafer and Vovk's. Money finds here, at the hinge of the formalism, its entry and definition, leading, later, to the stochastic integral.

Time is an image in which the ex-ante attitude fits perfectly. Time simulates the fact that a certain die is *first* un-thrown (its faces are only possible) *then* it is thrown *then* it lands (and one face is realized). There seems to be a total adaptability between chronology and the fact that possible states are recognized (see below, section 8.2). Money is alternative to time, if only because it retires the ex-ante attitude from processes that still, externally, accidentally, take place in time. It is the same power of money which allows us to have something present and material – the effective strategy – stand in place of the ex-ante attitude and no longer to care about the outcome (because it is the whole image of the transition between the possible and the real, or Cournot's principle, that is here replaced) and which allows a certain contingent payoff to be expressed as a contingent claim where underlying states no longer matter but only the material sheet on which it is written, which will subsequently lend itself to the exchange.

Recognizing underlying states is the same as the ex-ante attitude and is what stands in the way of recognizing the future as real; the sheet of paper, whose other face is the exchange therefore price, is the same gesture as retiring the states or retiring Cournot's principle. Let immanence be the guide: writing a contingent claim and exchanging it is the same immanence as the constructive Shafer and Vovk strategy or the account of the insurance company. We should deduce the exchange directly from the way the account of the insurance company, thus money, permits the ex-post equation.

Price is at once that which corresponds to the written contingent claim, through the exchange which is its other face, irrespective of the states of the world (and conditional probabilities); price is that to which the written contingent claim (written in money, printed in place because of that) destines itself; and price is at once that which allows the player in Shafer and Vovk to buy the ticket for the strategy and enter into the process of accumulating the gains; price is what allows the player to play without waiting or expecting, contrary to probability. He expects nothing because the ex-ante attitude is retired and all that counts is holding a position, adopting a strategy, therefore occupying a certain place. Price is common to both characterizations because of the capacity of money to instate a place (a strategy, or later, a market) that incidentally takes place before the event, yet does not correspond to an ex-ante attitude. The written clause of the contingent claim is ex-post, yet is sent, through the exchange, to a period of time that is accidentally chronologically prior. Similarly, the accumulation of gains (getting infinitely rich or not) is ex-post – this is when the strategy pays out

– however, the materiality of money makes it so that the effective strategy is now, although it is not ex-ante.

5.4 Changing the meaning of reality

Writing contingency and price are materially associated, like matter and geometry. In a past elaboration, I had defined the conversion as the main operator.¹⁷ Now, it seems to me the link can be established through immanence. Price, in Shafer and Vovk, is what allows instating the strategy, playing without expecting, twisting time apart from the ex-ante attitude and realizing the position whereby chronologically prior no longer means ex-ante. Price, in Shafer and Vovk, is the continuation of statistics through its ex-post strike; that is, without turning back and recognizing states, without waiting or expecting. The equation we want to consider now is that writing the contingent claim is, in fact, the same as this continuation of statistics. Through the conversion (my past operator), I now realize that I wanted to define place, as something in which the chronological line becomes negligible. Place, as I now understand it, is the waiting room ‘while’ time no longer exists; it is that which puts time on hold; that where it ‘adds up’ that we should be following a strategy, yet not waiting or expecting anything ex-ante. The conversion is the transformation of attributive writing (the one using identified states) into the material writing which is supportive of contingency (‘It could have been different’) and which cannot but revert into the exchange. This is what sends something ex-post (happening after the event) to a place that is situated before, yet is not ex-ante. What Shafer and Vovk do is the same gesture, through money, whose main virtue is the twist of the ex-ante attitude – whose main virtue is to give a material account of contingency, no longer reliant on counting and frequency. Money (the finance in the Shafer and Vovk’s account) is already all linked with the market; it is the same face.

Statistics is material, yet it is not real; we need infinity and inherence for something to be real. If a random generator could be extracted or inferred from the statistics and if this random generator could be expected to generate the random outcome, one at a time, this would be real. Reality, reification, is bound up with the ex-ante attitude. Cournot’s principle makes the generator real, because it gives ultimate meaning to the probability of each outcome at the level of the individual trial: the probability is real and is *p* *insomuch as* the realized frequency in the infinite run is *p*. Yet, this is vulnerable to sequences that do not realize the frequency – measure theory is then needed to define the convergence of probability.

Statistics is nothing more than the existence of matter – the die standing on one face that could have been different and its materiality, which allows it to be lifted and re-thrown under conditions that realize independence and identity of distributions. Statistics is material contingency – it is matter – and should be thought independently of chronological time. Repetition is not chronological. A repetition of differences freed of the chronological image of the throws, we said, is just the mark or the strike of contingency; it is *intensive frequency*, to be distinguished from the extensive one. The step to reality is when the generator is introduced at every step.

Shafer and Vovk do not question matter – they rely on a die – however, they change the notion of reality. They separate what we once thought to be inseparable. A sequence of outcomes can be

produced, eventually verifying the law of large numbers, yet there is nothing real that acts at each trial; no inference. Maybe it is only a problem of language: Shafer and Vovk no longer want to call that real, or to speak of that reality. This is why the notion of the game is handy. Something – ironically called ‘reality’ – plays against you; but this is a game and your opponent is only a player, so there is nothing ‘serious’ or real. Calling this a game or a player is only a semantic workaround that allows Shafer and Vovk no longer to mention objective or subjective probability, therefore to elude the ex-ante attitude.

The ex-ante attitude, or reality of the generator, is the result of Cournot’s principle and the existence of matter (lifting the die again and throwing it). Shafer and Vovk generalize matter when they say that only the cumulated gains count; they retire the ex-ante attitude, therefore they hold on the exact generator, and now the ‘generator’ is allowed to vary at each trial; they no longer need the exacting constraints of the conditional probabilities. They no longer postulate a generator, so they no longer care whether there is something that will make the frequency converge; the only reality is the strategy.

Shafer and Vovk have captured the hinge between contingency and the talk of reality and time; this was possible thanks to the formalism (which distinguished matter) and then to the realization that Cournot’s principle was only an interpretation. The existence of the formalism illuminates what we mean by interpretation and realization, therefore it shows other realities. Money introduces itself at that hinge, as an alternative matter, generalizing matter and allowing a new twist between time and ex-post, because to be chronologically situated before, relative to being ex-post, no longer means ex-ante. We no longer know where, ontologically or semantically, this is located that ‘something’ will make the frequency converge in Shafer and Vovk. They may be still reliant on matter (die, and so on); however, their semantics have opened a new way.

Now, imagine matter that is infinite in a new way. We learnt the infinity of matter from the formalism because we learnt the intensive strike of contingency, or frequency that no longer depended on chronology, because this is how we ultimately made sense of the infinity of the strong law of large numbers and the un-identification of the particular chronological sequence in relation to it. At a time when outcomes were still generated behind the scenes by a material device (die, and so on) that lent itself to the repeatable event, Shafer and Vovk showed us how to change the language and dispense with the ex-ante attitude and the reality of the generator. They discovered money, or a new way of counting that liberated itself from the extensive frequency and from the underlying states. And now the formalism – of which it was shown that Cournot’s principle was one possible interpretation and that an alternative immanent one was possible in which the reality of the strategy, whose outcomes are only accountable ex-post, is present yet is not ex-ante – offers Brownian motion, or the capacity to have the strong law of large numbers, or the ex-post equation that stands for it, take place in a single point.

5.4.1 Ex-ante vs. ex-post

Now, a really provocative (perhaps even revolutionary) thought. It is about Brownian motion. Brownian motion is a mathematical formal construct; it is a fiction that is true and unquestionable in

theory, but not in reality. In empirical reality, we find only phenomena that approximate Brownian motion. The movement of the pollen particle is truly irregular at any scale that we can observe it, and there is no notion of speed that we can physically attach to it; however, it is not infinitely divisible like true (formal) Brownian motion. Stock prices are empirically unpredictable at any scale that we can trade them; however, we cannot divide their trajectory beyond the minimum tick, and so on. True Brownian motion is not real, but only formal. But my provocative thought is that the derivatives market – which is real, of course – may really be the consequence of true, mathematical Brownian motion.

We saw that the infinite series of outcomes of a random experiment do not empirically exist, since no one can really wait indefinitely. And yet this didn't stop von Mises – an empiricist – from requiring them in order to define probability. Similarly, the infinite sequence of random variables does not empirically exist, yet this doesn't stop us from leaning on Kolmogorov's strong law of large numbers in order to give probability its real objective meaning. As a matter of fact, the passage to the infinite limit is not the real obstacle standing in the way of making sense of the objective probability of the next throw of the die. The real difficulty lies in the incompatibility between the axiom of randomness and the extensional identification of the series of outcomes. Randomness in von Mises prevents us from identifying the collective term by term. Similarly, randomness in Kolmogorov prevents us from identifying a particular extensional sequence of outcomes in which the frequency convergence would be observed, because the convergence in the strong law of large numbers concerns random variables that remain unrealized, and an average random variable whose probabilistic convergence to the mean does not distinguish a particular realization of the sequence.

As a result, there is tension between the strong law of large numbers (either in von Mises or in Kolmogorov) and the notion of propensity, or the idea that there is something tensed and pressing to say about the probability of landing, say, a 'six' with the next throw of the die. Ultimately, it is the notion of random generator that is untenable (a philosophical view defended by Popper), along with the ex-ante attitude that it precipitates. Of course, random variables and random processes do exist mathematically, and can be defined as a function of a variable t that we call 'time'. And this, in turn, can suggest the image that the next outcome is literally generated before our eyes, and therefore that an ex-ante attitude is legitimate. However, when the formalism is interpreted in physical time, we do not know what it means for a physical variable to follow that process or to be generated randomly, other than the fact that it would produce a statistics whose limiting frequency would be equal to the instantaneous probability. And this would, again, put into conflict the meaning of probability, which we wish to be real and inherent in the next throw, and the idea of randomness, which ultimately destroys the identifiable extensionality of the next throw.

This led Shafer and Vovk no longer to speak of frequencies as the basis of probabilities, no longer to count occurrences extensionally and identifiably one after the other, but to integrate all of that into the money account. It is not individual occurrences that should count (nor frequencies), but occurrences integrated against a financial payoff. It is for this reason that the impossibility of a gambling strategy is a more fundamental characterization of randomness than invariant frequencies. Shafer and Vovk write:

Our hypothesis of the impossibility of a gambling strategy is not a statement about probabilities. It is a hypothesis that relates a certain game to the world, which we can state before we compute probabilities for events from the prices in the game. [...] Our understanding of the impossibility of a gambling system, as something prior to the computation of probabilities, is relatively novel. It departs from the thinking of Cournot, Lévy, and Kolmogorov, for they formulated principles for interpreting zero or small probabilities, not hypotheses expressed without reference to probabilities. Von Mises anticipated our way of thinking, for he did say that the impossibility of a gambling system is something more primitive than probability. But his allegiance to this idea was half-hearted; in the end he refused to acknowledge that the impossibility of a gambling system is more fundamental than invariant frequency.¹⁸

Because they no longer count individual occurrences, Shafer and Vovk no longer wait for the event to happen. Their attitude is no longer *ex-ante*; it is no longer tensed in time. As noted above, the fact that the impossibility of a gambling system should come before frequency and probability as a definition of randomness (rather than being a consequence of randomness defined in terms of frequencies or patterns of occurrences) drives Shafer and Vovk to relinquish the principle according to which a probability that is equal to 1 means realization (Cournot's principle). They drop the whole idea that something is realized with some probability, and correlatively the whole idea of a random generator. They drop the transition from the possible to the real, which we call 'realization'. Their attitude is no longer one of waiting and seeing. Instead of realization of the possible and the *ex-ante* attitude, they adopt the reality of the winning strategy. Even though the winning strategy is here and now, this is in the sense of place, not of time (we do not really care about the outcome). The strategy is local, yet it is not *ex-ante*. For this reason, I call it, strangely enough, *ex-post*.

This is the same idea, really, as saying that what is fundamental in the case of the insurance company is the *ex-post* accounting equation, rather than a certain *ex-ante* attitude towards future contingencies. It is because it breaks even on average over the whole population it is insuring, it is because its money account allows it to perform this integration, that the insurance company is able to face a single case and deal with it as if it were assessing its individual probability and insuring an individual. It is the integrality and integrity of its financial account that allows it to reverse time and transform into an *ex-ante* question what is only an *ex-post* equation.

The *ex-ante* attitude is an artifact. It is a representation or a figure of speech. To consider the probability of the single case *ex-ante* is only shorthand for a long sentence in which it is stated that the individual is part of a population of insured people and that the insurance company has broken even on average. For this reason, when the *ex-post* accounting is recognized to be more fundamental than probability, when probability is no longer considered in a void but in a circle, in an institution (in this present case, the insurance company) which allows the account to be closed and the *ex-post* accounting to take place before we seemingly regress to the *ex-ante* attitude – or, equivalently, when the winning strategy of Shafer and Vovk is considered in the place that allows it to be set up regardless of the outcome – then I say that the outlook is *ex-post*. *Ex-post* here means that we understand that

the ex-ante attitude is only a derivative abuse of language; it is parasitic on the ex-post attitude, which is prior and fundamental.

5.4.2 Brownian motion

Now, to go back to Brownian motion, it seems that, thanks to a formal-mathematical wonder, we are able locally to have the infinity we were missing with von Mises's sequences of outcomes. Given any interval that follows a certain point in time, no matter how narrow this interval, we can subdivide it infinitely and get the infinite random sequence we wanted thanks to the infinitely fractal character of Brownian motion. Notice here that the imperative of randomness down to every shrinking interval (which, in turn, potentially contains an infinite number of sub-intervals, all containing randomness) is what prevents us from ever drawing the graph of a Brownian path (drawing in the sense of pencil and paper). In other words, it prevents that path from being extensionally given, just as von Mises's collectives were not given; except that this non-extensionality of the Brownian path is locally condensed in every shrinking interval, and it translates into the path being nowhere differentiable. There is nothing new in this – we are aware of that remarkable mathematical property of Brownian motion¹⁹ – however, what is new is to try to transcribe, for Brownian motion, the argument above, according to which integrating frequencies against financial payoffs is more fundamental than counting or integrating occurrences. The stochastic integral is more fundamental than the stochastic differential in Brownian calculus; the integral is constructed, and its convergence is somewhat in the spirit of the law of large numbers before the differential is introduced. As a matter of fact, the differential is only a symbolic notation; what really exists is the stochastic integral.²⁰

There is still nothing new in this, mathematically speaking; however, what is new is how this is reflected in the debate between ex-ante and ex-post. Even though the stochastic differential looks ex-ante (indeed, the differential of a function is meant to be a 'predictor' of its next variation), in reality it refers to a stochastic integral which is the result of summing up many contingencies in an ex-post manner, exactly like the insurance company – except that taking the limit in the shrinking interval makes it look as if it is ex-ante. (Limits are always nonintuitive.) Brownian motion seems to offer the best of two worlds. There is something we seem to be drawing or throwing like a die. There is the impression of a random generator and the corresponding ex-ante outlook. However, that next differential we are drawing is not a single occurrence; it is already a whole integral, it is already the whole law of large numbers, condensed into a shrinking interval. We no longer need, with Brownian motion, to fake as an ex-ante expectation what is only the result of an ex-post accounting equation. Something is both local and truly ex-post, here, which is better than saying, as we did before, that something is local yet not ex-ante. We no longer need to go to the infinite long run and to the global break-even and come back to face the single individual like the insurance company does. We remain on the spot and Brownian motion and the stochastic integral make the law of large numbers happen on the spot.

It remains to see what exactly we are summing. Remember that the purpose here is to establish an ex-post accounting equation, in the same way as the insurance company. And this so that we can apply for Brownian motion the philosophical conclusion according to which the ex-ante attitude

towards probability (and, consequently, the whole meaning of an objective probability) makes sense only insofar as some global account has broken even on average. It certainly makes sense to regard Brownian motion as a generator of contingencies in actuarial fashion, simply by sampling it at regular finite time intervals, and by giving meaning to the probability distribution we are drawing from through the break-even of an imaginary insurer who would be paying off a certain function of each realization against collecting some initial premium. For this to hold, however, the key assumption is the stability of the distribution over time, not to mention that we would be missing the whole local character of Brownian motion (what I called the ‘mathematical wonder’) by making it behave like a vulgar chance set-up (dice, roulette, and so on) whose outcomes are extended in time. No, the real challenge is to have the law of large numbers and the corresponding break-even apply in the interval that shrinks to zero beneath our feet, so to speak. On the other hand, we certainly do not wish to sum increments of the price S of the exchange-traded asset in that interval, because this would only yield the difference of prices at the extremities of the interval. For the summation to be nontrivial, we should sum increments of a nonlinear function of S .

There is a difficulty in trying to make philosophical sense of the meaning of probability that is attached to geometric Brownian motion because of the infinity that is involved in the notion of limit. As the stochastic differential equation representing Brownian motion teaches us, the relative increment dS/S is supposed to be distributed by a Gaussian probability distribution of a given mean and given variance. What this means is that, provided the experiment of drawing dS/S is infinitely repeated, the frequency of appearance of dS/S will converge, in that population, towards a limit that is equal to the mathematical probability provided by the Gaussian. But there is such infinity in every finite interval dt , as the fractal character of Brownian motion provides that no matter at which scale one looks, there will potentially be an infinite subdivision of this scale or interval over which infinitely many samples of dS/S will be observed. This dS/S is of a vanishing length, however, so it is doubtful that a certain typical dS/S , of identifiable size (there is none), could be singled out in order then to measure its frequency. What I am saying is that the only sense to give to the convergence of the frequency of dS/S is actually through the convergence of the stochastic integral (after which dS/S and the stochastic differential equation are defined as a matter of fact).

Of course, a dS/S of finite length could be picked and its frequency infinitely observed over time, but then this will be infinite time and would be no different from the usual framework of interpretation of frequentist probability. This wouldn’t be taking advantage of the local properties of Brownian motion. The stochastic integral would give a traditional finite stochastic variable; however, what is interesting is to make the convergence local (and differential) and this – I wish to argue – will involve Ito’s lemma and the fact that a derivative will be replicated by its underlying in all but a set of measure zero. (Brownian motion was destined for Ito’s lemma and dynamic replication.) Could, therefore, the sense of probability no longer ultimately reside in the statistical frequency but in the replication of payoffs?

On the other hand, when all is local as we want it to be, there is no guarantee that the probability distribution will not change over time (think of derivatives practice and theory, where we commonly make volatility depend on stock price and time). This local character does not offer us the possibility

of consolidating our accounting equation over time, like the insurance company, by averaging in our books the repetition of the contingency. The whole thing was designed to be continuous in time, so there are no 'comebacks' or rehearsals that allow us to conceive a contingency as a part of a statistical population. The account cannot wait 'behind' for the next contingency to inscribe in it a new mark while making it look as if it is the first time again. The next contingency has moved ahead of the account in time – so, what I am suggesting is that the account should be 'portable', that it should follow us as we go. The strange and difficult thing I am trying to say here is that the combination of time continuity and randomness at any scale, even though it seems to be freezing everything on the spot, on the contrary presses us to move along. There is a kind of irreversibility here.

The conclusion I would like to draw – and I admit that my reasoning is highly speculative – is that the only way to combine (a) a break-even caused by the law of large numbers, (b) the locality of Brownian motion, and (c) the running account, is by considering an account that moves in a self-financing way. What we need to average out in the shrinking interval (in which the law of large numbers applies, no matter how narrow we make it) are typically the differences between a derivative payoff moving along with us, and the proceeds of the dynamic self-financing strategy that is meant to hedge it. Of course, the derivative is not expiring at every second (so that we could speak of a 'continuously moving payoff'); what I mean by payoff is just its market price – as this is what the dynamic replicating strategy tracks at all times before the actual maturity and the actual final payoff.

(One tends to forget that the real crucial assumption of Black-Scholes-Merton is that the derivative price is given as we track it.²¹ Now, BSM bootstrap this price from their own intent to value the derivative; they assume the value is given, and then they show how to establish it. The point remains that locally, before their argument is completed and closed, the given value can only mean the market price. How could they be tracking something whose dynamics ultimately will have to be a result of their own construction? It is only in physics that you assume that something is pre-given by nature and therefore can allow yourself to make that 'given' a part of the equation whose resolution will ultimately say what the 'given' is worth. BSM assumed that the market was nature.)

What makes our conclusion revolutionary is that the interpretation of probability, or saying what probability really means, when it is transposed to Brownian motion, seems no longer to involve an insurance company who could alone look at probability in a seemingly ex-ante fashion and could alone give it meaning because what it is doing in reality is breaking even on average in an ex-post accounting equation. Rather, the interpretation of probability under Brownian motion seems to involve local derivative replication and local derivative pricing. I insist on the word 'local' so as to block any temptation to infer things statistically through some extended time series. The only 'statistics' is the single option price, as we said, and therefore it has to be considered as given. One should not infer Brownian volatility from the history of prices of the underlying asset, but from the instantaneously given derivative price. This is what is done in practice, and implied volatility is, indeed, the only working concept.

The fact that a certain break-even (necessitating underlying and derivative and the continual re-balancing of the hedging strategy) now takes place on the spot is really characteristic of Brownian motion and is the new thing. This leads to replicating the kink in the derivative payoff and displaces

the statistical problem. The philosophical interpretation of the probability attaching to Brownian motion is the exact calculation of the option price. It is the fact that, given volatility, the option value should be exactly determined. One price, no statistics. This is what inverts the problem and brings locality of the market into play.

5.4.3 From time to place

The locality that is involved in Brownian motion and in the infinite law of large numbers being verified in the infinitely shrinking interval is not just locality in time. It is locality in the sense of place and of a happening – an event taking place. When locality is combined with matter, it entails existence – the popping out into existence of the derivative price. When the law of large numbers used to take place with ordinary finite matter (the die) and the ordinary infinite long run, it was folded back inside the one face of the landed die, under the strike of contingency. This is when we realized that chronological succession was only an image. The infinity of the law of large numbers was said to be equivalent to the infinite saying of matter – matter which stood unmoved yet was not static, matter which was no longer extensionally interpreted as static and unmoved but which intensionally meant and said and infinitely repeated that it could have been different, matter whose saying was infinite because it could not be said by the formalism. Alternatively, the strike of contingency was materialized by the trading strategy of Shafer and Vovk, which was as immanent and stood at least as much opposed to the *ex-ante* stance as the strike of contingency ('It is this; it is too late; it is real and no longer possible; yet it could have been different'). Now, we are trying to grasp what happens when both the matter that is equivalent to the strike of contingency of the landed die and the fundamentally available money that monetizes the trading strategy are shrunk into a vanishing interval (become local), with the randomness that causes the convergence of the law of large numbers in the infinite long run *not vanishing* as the interval vanishes.

This is highly nonintuitive, I grant you. Bear in mind that the infinitesimal limit no longer allows sampling, understood in the sense of frequency (see above the discussion of dS/S). What we are saying is that no finite die can sustain such a limit, no ordinary conception of matter. Bear in mind that it is really the formal marvel of Brownian motion that is driving us to this limit, a marvel that is only possible and conceivable mathematically – not the ordinary material proxies of Brownian motion such as the pollen particle or, indeed, the empirical time series of stock prices. What we are saying is that the resulting matter or strike of contingency is the market price of the derivative *being given*. Not just the matter of that price – that new compound – but also its being given, its popping out into existence. We say this inverts the problem, because usually the derivative was *valued* and its value was derivative on the stochastic process of the underlying price; it was a function of that price. Now, the derivative price has to be *given*. The givenness of the matter of the die (inscribed and intensional with the strike of its contingency) is replaced by the givenness of the market price. There is an inscription and a mark in both cases. There is a place where this is marked. What we are saying is when matter is abstracted from its ordinary guise (finite as in the die) and from its ordinary degree and retains only its ulterior meaning of the strike of contingency, we find that *place* is its ultimate repository, place in the sense of the place of happening of the event, the place of

inscription of contingency that is alternative to time and to the illusion in which time had kept us concerning contingency, the real place of contingency, which is, in this instance, the place that gives the price – the marketplace.

Money used to mediate the trading strategy in the case of Shafer and Vovk and price used to mean only that which allowed the gambler to play against reality. Now, as the interval shrinks to zero, price becomes the market price and money is no longer in charge of replacing a time series with a capital process, or a frequency count with a trading strategy. Money now just *gives* the derivative price. Place replaces time and price is no longer a component of a game; it becomes the universal price, no longer making it seem as a coincidence that the market should also qualify as a game and be one among other cases in which Shafer and Vovk could apply their protocols. Understanding how Brownian motion can compress a whole infinite ex-post accounting equation in the spot is nonintuitive and requires something like a lightning flash. We need to understand this in a lightning, and this lightning flash is what materializes the option price (giving it) as we understand. *The illusion of ex-ante is now filled with the ex-post matter, because the infinitesimal limit has made the two movements coincide, the one in which we used to come back from the convergence in the infinite long run in order to postulate the ex-ante propensity (which was only an illusion and wasn't real) and the one which effectively and materially compresses the infinite long run in the single instant.*

To repeat, if the real meaning of probability involves looking for the ex-post accounting equation that expresses a break-even, rather than looking for frequencies that converge, then in the Brownian case, the only ex-post accounting that takes full advantage of the local character of Brownian motion (i.e. of the combination of continuity and irreducible randomness at any scale) is the local replication of derivatives. What sounds amazing in this conclusion is that the derivatives market (I mean the real one) seems to emerge as a consequence of the ideal and formal properties of Brownian motion, combined with the will to interpret them in reality and to give them meaning. (This is comparable to giving meaning to 'ordinary' probability through ordinary statistics.) It is as if a Big Bang had happened, and suddenly derivatives and their underlying had come into being and into trading – i.e. the market had come into being – right after Brownian motion was interpreted.

Thus, Brownian motion is an essential ingredient of the concept of the market. It is the gene for the genesis of the market. Let us not be interested for now in the empirical world out there, or in how the market empirically behaves. We will care about that later. Let us first establish the concept or the meaning of the market. In this order of presentation, Brownian motion is not an outside description of the behavior of price. It is the description from inside. To understand really how the meaning of price is constituted, we need Brownian motion. To repeat, we don't know what price is or what a market is before the genesis. We need to constitute them. If we want to run the argument according to which probability doesn't exist or count except formally and all that materially counts is the account of the insurance company that is breaking even in the long run, if we want to run this argument no longer in extensive time as in the actuarial case but in the vanishing time interval in which Brownian motion magically allows us still to have an infinite long run of samples, then the only way to constitute a running account is to have a derivatives market and an underlying market running together to form the hedging portfolio.

The definition of the market is that which we need to be given (hence the act of genesis) in order that we find in it the price of an underlying and of its derivative in order that we run the local dynamic hedging argument and have the notion of a stochastic differential, which is none other than a stochastic integral made possible and converging in the vanishing interval thanks to Brownian motion. Just as statistics is the material interpretation of formal probability, the market (concomitantly of underlying and derivative) is the material interpretation of that strange extension of formal probability which is called Brownian motion. In this genesis, Brownian motion is a constitutive part of the meaning of the market and of price. It is not that price follows Brownian motion in empirical reality (and opens the whole empirical problem of whether it truly follows Brownian motion). Rather, price follows *from* Brownian motion.

Once we have the market constituted and all grades of derivatives trading, once we have the Big Bang of genesis, it becomes a later elaboration to see that the derivatives prices do not accord with the hypothesis of their underlying following Brownian motion. As a matter of fact, since all the derivatives of all orders are given in the Big Bang, trading at prices that cannot be redundant, the conclusion follows that the underlying price could follow no stochastic process at all for the reason that any kind of process, once assumed, will make some derivative redundant. In brief, Brownian motion is just needed to ignite the market and to constitute its meaning, and then it disappears.

In order to understand the coming into existence or the genesis of the derivatives market (or to understand its concept), we need to push Brownian motion to its ultimate mathematical, ideal consequences. Yes, something real can begin to exist after a purely mathematical finding.²² Market-makers in derivatives really came to the market because of continuous local dynamic replication and because instantaneous volatility was all they needed to compute the instantaneous hedge (hence implied volatility, which exists on the spot). It is a wholly different thing if, after the whole thing had come into existence, we then recognized that this ideal theory will be questioned and criticized and revised because option prices, as given by the market, will not, in fact, respect the theory – what is known as the ‘smile problem.’²³

As a matter of fact, this observation further supports our point, rather than challenging it. Indeed, ideal Brownian motion and the ideal consequence of break-even in the hedging account are now questioned by the market derivatives prices, not by empirical analyses of the time series of the underlying price (to the effect that the underlying price does not follow Brownian motion historically but admits of jumps and stochastic volatility and what have you). Let the jumps and stochastic volatility be introduced in the derivative pricing models by the market-given derivative prices that we need to match instantly and locally, not by the history of the underlying price! (This is what all derivatives traders do, and what urgently needs a philosophical explanation.) Once again, option pricing-and-hedging is, crucially, local; and, crucially, has nothing to do with statistics. This is the inception.

From the Marvel of Brownian Motion to the Reality of the Market

6.1 The technology of the market

The radically-emergent event escapes possibility, or rather, it precedes it.¹ It is im-possible, not in the sense that it will never happen, but in the sense that its 'possibility' is inconceivable before it happens. The genuine event *can only happen*; it cannot be conceived or thought as possible beforehand. Happening is the only thing that it can do and it is only subsequently that its possibility, or even its cause, is thought. It is thought after the fact because nothing exists before the event. Ex-ante is an illusion.

We said that the impossible event was real and that possibility was not – that is a fabrication made after the fact. How could the impossible event be real when the event does not yet exist? To exist, in mathematical ontology, is to be part of a set or a situation, and the event is part of no set or situation. It is rejected by the ontology.² It creates its own situation and becomes part of its own set. It creates its own possibilities, *when* it comes to be real and actualized.

Actuality is not the last word, speaking of reality. We may think that the event is real before being actualized in time and before existing provided we mediate it with something other than time. The essence of the event (its true reality) is that it is contingent (and may have never come into existence, as a matter of fact). If that could be mediated via an appropriate *material* medium, then the event could be said to be real (now). If the im-possibility of the event – this utter negation of the situation and even of the ontology that precede it in time – could be turned positive; if the radical discontinuity and cut that the event is could be turned into matter, that is to say, into a continuous medium (*un milieu continu*), then the event would become real and the medium would be real, too.

We claim the market is this medium of absolute contingency. This, of course, implies a rather unusual take on matter and it is the main purpose of this book to outline or deduce or ‘extract’ this extraordinary (literally revolutionary) matter as it emerges in four perspectives that will home in on the market: probability theory, Brownian motion, Black-Scholes-Merton and, finally, the writing of contingent claims.

Reality, we said, is not exhausted by actuality. The part of reality that is not covered by actuality cannot be populated by possibility. Statistics is actual and, in those cases in which the contingencies that produce the deviations of the population from the deterministic law do average out, the statistical regularity that is consequently observed is also actual. What is not actual and, we wish to argue, is not real is the random generator that we subsequently imagine lying behind the scenes and *drawing* each individual following an ex-ante probabilistic law that is made up to accord with the ex-post statistics. And when it is agreed that the random generator is just a shorthand and a way of speech, or in other words, that it is just a model and for this reason should be granted no more and no less actuality than the statistics of which it is only a relabeling, and certainly no reality, the distinction between what is real and what is not real should concentrate on the notion of possibility that we have accustomed ourselves to associating with the random generator.

Possibilities are OK when they merely signify, in a given model, the different members of the statistical population that are deemed open to an ‘event’ because the model has recognized it as a member of that population and referred it to that reference class. Possibility is OK when it is merely the ex-ante reformulation of an ex-post phenomenon. What is not OK is that possibility should earn the full-blown ontological status of a potentiality (or power, or propensity) that brings about the real. The realizations of a random draw are ‘brought about’ by the possibilities the model has recognized because this is just a language the model has invented for itself. But there is no effective agency in those merely ‘semantic’ possibilities.

The real question is: What brings about the real event, if not a possibility? If you agree that we should reserve the term ‘possibility’ for probability theory – that is to say, for the restrictive framework in which states of the world *have been identified and enumerated* and can subsequently be assigned a probability weighting (regardless of whether this weighting is subjective, or is a shorthand for statistical frequency, or will be updated as we conditionalize on new evidence, and so on); if, in other words, we understand ‘possibility’ as derivative from the real and for this reason incapable of bringing about the real and of qualifying as real, then we need a term to designate the reality that brings about genuine events. Philosophers of the event such as Bergson and Deleuze have adopted the term ‘virtual’ to designate the real that it not yet actual. They repeatedly warn us not to confuse the virtual with the possible.³ More radical thinkers of the event, such as Badiou and Meillassoux, don’t believe in a reality that pre-dates the event. To them, events emerge literally out of nothing. The event is rejected by the ontology, according to Badiou.⁴

So, the first predicament of probability theory, and consequently its inability to deal with absolute contingency and its medium (the market), lie in *identifying* the possible states, not in their subsequent probabilistic weighting. The reason why it is difficult to assign a non-subjective probability to the event that is one of a kind is that this event is not identified as a member of a population or set of

possibilities. How to assign a probability to an event that is not a possibility? Even more radical is the event that it is itself not identified – the radically-emergent event. There is literally nothing of which this event is a member before it occurs. Badiou goes as far as to relax set theory's axiom of foundation (according to which sets should not be members of themselves) to define the event, precisely, as that which can only be a member of itself. While Deleuze speaks of the virtual and Badiou speaks of the void as the media of the event, neither of them describes the event as 'improbable', for that would presuppose a set of possible states and a scale on which probability is measured.

Contingency is absolute and for this reason it is *matter*. Matter is absolute. Matter is immanence. Matter is not supervised by perception.⁵ It is infinite. Material and massive contingency obtains when we reverse the vision of possibility. We no longer perceive contingency through possibility or define it through the identified possible states. Possibility no longer informs and 'perceives' (illuminates) contingency. We simply say of a contingent thing that *it could have been* different, blindly, massively and absolutely. This is the ex-post stance. We call it the *strike of contingency*. It is material because it is too late and we no longer perceive or conceive of ex-ante possibilities.

What is material in our new conception of matter is to be able to say of a thing that it can be different before saying that it is. Not only is massive and material contingency thought independently of the possible states; it is thought independently of, and prior to, the actual state. The *state* is something identified and distinguishable. For this reason, it is not immanent and massive. It is not matter. In the strange materialism which conceives of the state (either possible or actual) as a perception channel, therefore as transcendence, matter is no longer identified with substance or existence or being – quite the contrary.

Contingency is at its best (absolute and material) when we speak of a future event. We want to say of the future event that it could be essentially different even though it is not yet. This revolutionary saying requires a new kind of medium. We need to mediate material contingency by acknowledging the inversion of its customary mode of perception. This entails the removal of possibility. For this reason, the future is real – and so will be the market, which is the technology of the future and its medium.

The question of the reality of probability is quite a different thing. It is linked with the ex-ante stance. Probability has objective meaning because of the law of large numbers (probability is objectively equal to the limiting frequency). However, the realization of the frequency in the long run makes it sound as if the probability of the outcome in the single trial is not only objective but is real – as if the trial could be individuated. When the law of large numbers is interpreted as a frequency count, outside the account of the 'insurance company' whose ex-post break-even in the long run is the only matter, it tempts us into thinking that the probability is real. Yet, there is no reality, here; there is only matter and the ex-post accounting equation of the insurance company.

The non-identification of the particular sequence (axiom of randomness) robs the 'next' throw of its reality and of its ex-ante character, as we saw, and makes it all fold back inside matter. Frequency, we saw, is only the intensive strike of contingency of the die as landed and the money account is the matter that mediates it. Matter here takes over the problem of reality. It reinstates as ex-post what should have never been otherwise. It relieves us of the reality of probability.

Giving back to matter its precedence and giving back to the ex-post stance its precedence over the ex-ante stance will lead us to the reality of the future or to the *reality of the market*. The reality of the market will be obtained through the ex-post stance (though matter and immanence) even though it shares with the reality of probability and with the ex-ante stance the fact that the market price of the contingent claim (or derivative) takes place, chronologically, before the contingent event of its settlement. How this will be accomplished is by recognizing, first, that matter is defined as that which explains the law of large numbers, either in Kolmogorov, explaining the independence and identity of probability distributions, or in von Mises, as the starting point, and by recalling, second, that possibility and the corresponding modal realism come from a wrong interpretation of the law of large numbers and should have been left inside matter, without a mention of Cournot's principle.

We should follow the lead of *what is not* possibility in the search of reality. We should remove possibility from the matter of contingency concerning the future event. We should find matter in the ex-post stance and follow only its lead. Of the die, after it has landed, we just say that the face could have been different. Of the insurance company, we just say that its ex-post accounting is break-even. Money is of the essence in the ex-post accounting of the insurance company and it gives us the lead to the financial market and to the writing of contingent claims. The invention of writing is what allows us to say of a future event that it could have been different before saying what it is; that is, without state and possibility. This is how the way is paved to the reality of the future, through the mediation of the material market. When it is realized that chrono-logic can be reversed regardless of the incidental chronology and that the ex-post stance is what materially counts, both the future and the market become real.

What misled us into using probability in the markets is the fact that markets possess a single metric. Prices are numbers that can either go up or down and that form time series. This made it irresistible to invoke stochastic processes. By contrast, nobody would think of history as a stochastic process, if only because history consists of history-changing – or context-changing – events and stochastic processes can only make sense within a given context at a given time. Yet, the market is an integral part of history, and I, for one, believe that there will always be a market as long as there is history or contingency in the world. Instead of applying stochastic processes to the market, based on the illusion that prices behave like numbers and that the market behaves like a random generator; instead of downgrading the market to the 'lower form of probability',⁶ we should set our sights in the opposite direction and think of the market as a special form of history... which *happens* to be quantitative.

And what makes the market historical and real; that is to say, incommensurate with a random generator? What is the generator of the market if not possibilities and probabilities? It is the exchange – the material exchange of *contingent claims* (which Shafer and Vovk, for instance, do not recognize as distinctive of price). The contingent claim is written on a material sheet to which anything can happen, due to its materiality – contingencies that were not necessarily part of its intended schedule; for instance, its total destruction. At the other extreme, *all the market* can happen to it, or the simultaneous trading of contingent claims of all grades of complexity. As a consequence, no total set of possible states can ever be laid down, which amounts to the same as saying that the

corresponding level of reality is precisely free of all ranges of possibilities. The material medium is thereby created through which we can communicate, in the future event, with the fact that it is contingent before the fact that it is. Thus, the reality of the market is the same as the reality of the future; and contingency is mediated as it should be, absolutely and not relatively to states.

The exchange is not a slight thing. You have to think that (a) the capacity to capture different payoff clauses on a single sheet of paper, or the invention of writing; (b) the simultaneous, and even automatic, translation of this register into a material contract binding writer and buyer; (c) the invention of money which guarantees the homogeneity of medium between the later contingent payoff and its earlier translation known as the *price*; and (d) the market which offers the place where the price takes place ahead of the event triggering the payoff – that is to say, it takes place instead of this event, in its place – are different facets of one and the same material procedure, also known as *technology*, which we have put together in order to deal with the future event and produce a material process that is adapted to history.

6.2 The reality of the market

The financial market is real in a special sense of the term. Its reality is special because it is deduced from the formalism of probability and its finest construct: Brownian motion. There is a special sense in which, because the ex-post accounting that is infinitely given in a single spot by Brownian motion has to find its external contingency, this externality is the *reality of the market* or the market price of the financial derivative as given. What does it mean that the market price is *given*? What does it mean that the market (really) exists? Brownian motion allows compressing the law of large numbers in a single spot; however, the matter is not the die in this case. The matter is *that a price exists for the financial derivative*. To give this price instantaneously is the definition of the market. We think the market is an exchange between individuals, but the exchange has to be fundamentally defined as that which ascribes to the price of the derivative the same *matter* as the statistics: given and ex-post.

There is a sense in which the derivative price, as it is given in order that the hedging strategy of the derivative breaks even *both on the spot and in the infinite long run* (a marvel that we owe to Brownian motion), is the only statistics – that is to say, it is ex-post like a statistics and not ex-ante like a probability, strange as this may sound when one realizes that the derivative price is given ahead of the event of its contingent settlement. There is a sense in which statistics, as given and ex-post, is a relief from the burden of probability and the ex-ante stance. It is more credible that the outcomes of throws of a die should manifest a stable statistical distribution than it is credible that a random generator should really exist behind the scenes and generate those outcomes. The whole elaboration of probability is the way to explain the stability of frequency and the law of large numbers, but the hypothesis is somehow less credible than the conclusion. It is as if the conclusion did not really need to be shown. How could it be shown (that is to say, formally) when the conclusion is the existence of matter?

As surely as we know that matter exists and we understand what being given and being ex-post means for matter (what immanence means for it), we know that the law of large numbers is in

no need to be shown beyond the mere 'showing' of matter. It is the expression, in matter, of the contingency of its presentation. The matter of the die commands that the die be landed on a single face and, at the same time, that this face could have been different. It would have been surprising if the series of outcomes of the throw of the die did not exhibit statistical regularity. The 'degree' of matter that is here involved lends itself to the repetition of the throw in time. But time is only an image. As we saw, there is no real chronology in the intensive strike of matter which repeats: 'The face could have been different'.

The given character of matter (therefore of the statistics) is independent of chronological repetition and is even incompatible with chrono-logic. Therefore, there must exist a given matter and a given statistics without chronological repetition. That the frequency might not be stable (in time) for this one-time statistics is incidental. It is just that the 'degree' of matter is different. The sense in which statistics is given and, as a result, is less a burden for thought than probability should be the same sense in which the market price is *given*. This easiness of statistics on thought as opposed to the burden on thought of probability is what 'being given' means. Statistics has not been really investigated. We don't know yet its ex-post character thoroughly. We should think it over and over (that is to say, relieve it from thought, make it easier on thought, understand that there is nothing to think and to show in the statistics other than matter), and then move to the market price, via Brownian motion.

Matter is misconceived at first in the market. We first think of some die, some random generator inherent in the market, some major statistical phenomenon (different trading behaviors averaging out), which ends up producing the price of the underlying asset, following Brownian motion. We think the matter of the market lies in a major and external phenomenon which predates the whole argument, when the argument (mathematical, semantic with respect to the formalism of Brownian motion) is inceptive. The *argument* is to be able to form a convergent ex-post equation at any scale (as this is what Brownian motion permits). The notion of the 'given price' is the externality that this requires. Underlying and derivative trade together (this is the definition of the market) and the derivative is such (precisely) that the hedging self-financing portfolio can be formed and the break-even can be verified at any scale, down to the instantaneous spot.

Brownian motion is not concerned with the sole underlying (efficient market hypothesis). Brownian motion is a formal construct, at first. Ito's integral calculus is invented, second, as well as self-financing. It is then observed that the hedging portfolio can be formed and the law of large numbers, relieving us of probability and of the ex-ante stance, written and verified for the single spot. *This* becomes the market according to us: this, or the *simultaneous outing* of the underlying and derivative price. It doesn't matter that the underlying and its derivative do not follow exact and pure Brownian motion in reality. The reality of what? Let us not forget that we are in the process of creating the reality of the market and that its reality is the result of the purity of the argument from Brownian motion. Money is not incidental on probability; it is essential. For this reason, we can reciprocally think that Brownian motion was *made* for the derivative and for the self-financing strategy – that it was made for money, as the *only way* to obtain the convergence of the ex-post account down to the spot. *There is a notion of necessity here, and this necessity is materializing the market.* We all know that

Brownian motion doesn't admit of a derivative in time; it admits of a derivative in space, which is literally the financial derivative price, and this allows us to form a portfolio with the derivative and its dynamic underlying hedge and the corresponding stochastic integral. Instead of prediction in time, the derivative in space, or rather in place, gives us the market. The market is better than prediction.

That the random generator of the underlying price may be given first, as our first intuition of the market suggests, makes the market of the derivative redundant. In reality, there could be no thinking of a random generator a priori, because the reality of the market is still in genesis. The random generator (a by-product of statistics) cannot first exist (and will never exist) because it is the radical alternative to statistics that we are talking about here. *The random generator is that the derivative price should be given* – rather than the derivative being valued derivatively on the underlying price or the derivative being redundant (which is the opposite of being given and even the opposite of being) because the underlying random generator is given. This is the founding equation – or rather, the *founding inversion* – of the market.

The derivative price is given, together with the idea of the infinitely fractal randomness of the underlying process which enables the convergence of the ex-post accounting equation on the spot and calls for the price of that derivative as a component of that accounting equation. We say it is an idea because Brownian motion cannot ultimately be given as a process. There are no stochastic processes that may be given in the market (either Brownian motion or any of its cognates) and their idea should never materialize because only derivative prices are given and the mere thought of an underlying stochastic process being given would make some derivative redundant and would prevent its price from being independently given. Even though the idea of Brownian motion is needed to trigger the whole argument and to produce the market as a consequence, it has to disappear almost as instantaneously because the whole place is created at once: derivatives of all grades of complexity are written at once and their prices are given at once, non-redundantly.

That the given price (virtually of all derivatives) replaces the statistics is the indication how *place*, where the price is given by definition, replaces time. The big problem of the market is time. There is no statistics *in time* in the market and the market is a one-time statistics because time is incidental for the market. Yet, the market has a privileged relation with time because it is forward-looking and it is all pointing to the future event. There is a sense in which time is only incidental and a sense in which the market can only produce statistical time series. That the price of the derivative (and virtually of all derivatives) should be given replaces the need to check the underlying trajectory in time, and it does so essentially. That it be given in the market is the *replacement* of time and statistics.

With the invention of writing and the profusion of the derivatives, the one big event (the market) is created which cannot turn back (revert). It is a contingency that is equivalent to that of the world at large, or in total, with no place left, no room left, to exchange it. It is the total die. For the reason that everything trades and everything is connected, there is no derivative that you can hope to trade and replicate until its expiry date, whose replicating strategy does not gain, in the intervening time, from all the price changes of all other derivatives, which are virtually infinite in number and in grades of complexity. The event becomes every day, the market becomes the event. The market accompanies history for this reason: there will be a market as long as there is history.

6.3 The market as an inverted order of thought

The founding equation, or inversion, of the market according to which no underlying random generator should ever be given and only derivative prices should be given – or, to put it differently, the founding inversion according to which the transition to the market should take place right from the immanence and ex-post character of statistics without a regression into the ex-ante concept of probability (a wonder that is made possible by the idea of Brownian motion) – this founding inversion can be made logical. We can rehearse it as an inversion in the order of thought itself.

We shall call *conversion* the transformation of the positive (algorithmic, probabilistic, ex-ante) mode of reading into the *material* mode, into the ex-post mode which says: ‘It could have been different’. This sounds like a logical transformation and yet what it implies is the market. A logical transformation can be equivalent to a genesis and the market is created as thought inverts its order.

Matter affirms its intensive strike. The massivity and irreversibility of matter translates into the unusual dealing of contingency with time: ‘It could have been different’ without chronological reciprocity or reversibility. As we have said, the event is the archetypical ‘matter’ in this regard, since reversibility is blocked by construction. It is at the same time that the event is contingent, yet is what it is. The multiple from which to ‘count’ the event is different from a state count or a frequency count or a chronological count. Money was invented and the writing of contingent claims (in money) was invented in order to move back in time and reinsert back in time this impossibility and unadaptability of the event in time.

The financial underlying asset is at the same time written on the sheet of the contingent claim and moving the world towards it. For this reason, it generates Brownian motion before the settlement of the contingent claim. Our contention is that Brownian motion is the *translation* of the matter of the market in time. It is the translation in time of the ex-post character of the intensive strike of contingency. It gives the market in this way before the event, and then the market is thrust back in the middle of the event.

The conversion is what transforms debt into equity, or into the contingent claim; however, it is part of the force of the conversion that the contingent claim no longer obeys the logic of the possible state and probability. It is part of the force of the conversion that the *exchange* of the contingent claim ensues (exchange in a market).⁷ Debt is a schedule and an instruction – it is an obligation and an algorithm – and the force of the conversion transforms it into a material sheet of paper on which states are no longer distinguished and which lends itself materially to the exchange, as its only way.

The exchange is *already* part of the way of reading the contingent mark: forever intensive, insisting that it could be different absolutely, with no relation and reciprocity to distinguished states. This way is the *material* way: the mark is already marked; our reading is ex-post relative to it and this is precisely the reason why its contingency is absolute. We no longer measure the contingency of the mark relatively to states of possibility, as there are no possibilities left.

The exchange is the only way to recede from the strike of contingency without falling into the ex-ante attitude. When time is introduced, the infinitely nested writing and trading of the derivatives make sure that we essentially remain stuck in the middle of the event and only incidentally recede in

time. It will even emerge that the only way to recede in time while simultaneously avoiding the ex-ante attitude is through the infinitely nested writing-trading of derivatives, which is just the ultimate consequence of the exchange. The strike of contingency creates Brownian motion when it recedes, because it is immanent and it requires the immanence of the ex-post attitude; therefore, it generates the market of the derivative, whose tracking is the only way to reconstitute an ex-post break-even on the spot.

The conversion transforms relative and external difference into absolute and internal difference; it transforms the schedule (and its corresponding chrono-logic) into a material mark. This is how the *material paper* is created. It is no paper when we look at the different possibilities that are coded on the paper. We then take a transcendent distance with them; we then only look at the *content* of the paper. The paper starts to *matter* when it is too late, when the world might as well have ended and the mark is read and the paper is opened like a testament. The only way to vary the mark in this case is to note that the clause (written on paper and interpreted right here right now, in the middle of its event) could have been different. This is the alternative way of considering the difference, in which the different alternative states are no longer available. We are no longer selecting from the paper that which is supposed to make a difference; we are no longer comparing the current state to another possible identified state and deriving the difference from the comparison, analogically. The actual mark could be different in infinitely many ways, none of which is identified and none of which is even thought to lie outside the mark. For all we know, the mark could be different in the sense that it could be altogether destroyed or burnt.

Difference pushes back through the matter of the paper in order to express how different the mark can be; however, difference cannot vary the mark in time. It cannot leave the paper and project different states outside the mark in order to compare it with them. The event (the mark) strikes with its contingency and the fact that it could have been different; however, the event is absolute and there is no way we could see *how* different it could have been. The event was not part of a set of variations or possibilities among which it was picked. It was the only possibility because it created its own. In this sense, it is a necessity (it can only happen). No other event than this one could have happened, because the event is teaching us *at the same time* what it is and that it is contingent. Yet, its contingency is somewhat more important and more essential (more event-full) than what it is (its identity). It is a contingency on the spot, without any identifiable variations.

The event says that it could have been different and that it could have not happened at all for that matter (the meaning of its contingency), yet it can only say so when and inasmuch as it happens and occupies the whole place, taking place without leaving room for any other variation. There is an intensive sense of contingency at play here, which is incompatible with extensionality (its whole logic is different) and has never been really scrutinized. The event plays an unusual game with time as we only have time to look into the happening of the event and its possible variation; yet, time is definitely not essential to the event because the event doesn't essentially happen in time (it happens only accidentally in time); it happens in place. Time is not the right medium.

This unusual 'variation' of the event (whose right medium is not time) has the market and the exchange as its medium. However, it is difficult to describe the exchange at the expiry date of the

contingent claim, because there is no time left. We need to construct the idea of the exchange in time, before the event triggering the contingent payoff, and then be forced back in the middle of the event by a similar incompatibility of the market and of the exchange with time and chronology. The market is the construction or the unusual medium that enables us to step back in time without turning ex-ante.

Something pushes from the contingency of the mark through the paper that carries it, yet the mark (the event) is already too late and we are in trouble trying to grasp how different it could have been, in an ex-post fashion. (We are always ex-post relative to the event, as the event is always unpredictable.) The whole difficulty (and problem) lies in a particular twisting of ex-ante and ex-post. The market is supposed to mediate the variation and turn to us as the other side of the contingency, exchanging it. Yet, it is difficult to tell the market apart at the settlement date of the contingent claim because there is no time left.

To read the instructions of the difference analogically, at the settlement date, is to *value* the contingent claim. We value it in this state, differently and relatively to the other state. However, to value it absolutely and not relatively, to change even the order of thought from a reading of its difference in possibility to a reading in contingency, in the mode of the strike that is too late but could have been different, is to *price* it. It is difficult to articulate these nuances of structures and orders of thought at the expiry (settlement date) of the contingent claim. It is not apparent, at expiry, how the market sends back the price of the contingent claim towards us. All we know is that something says, in the contingent strike, that it could have been different and that it says so infinitely, always in excess of any given frame or set of possibilities that we could have settled in mind.

Probability and frequency (the count of states) are the positive mode of reading at expiry. They list possibilities, they adopt the ex-ante stance; they need structure and, potentially, conditional probability. They rely on the content of the instructions not on the material paper (matter). They are not immanent with respect to contingency. They are incompatible with the ex-post and 'too late' mode in which contingency alone can regain its absolute character and its infinite variation (i.e. it is different relative to nothing, or to everything). For that, money is required. Money is at one with the written, material character of contingency. Probability and frequency (this mode of reading at expiry) are written in chronology and time, whereas the strike of contingency, absolutely different, is written in money. Counting is ex-ante whereas accounting is ex-post. It is difficult to tell these things apart at expiry because of the lack of time, but what we are looking at is already the difference between probability as ex-ante stance and probability as a concept that is derivative on an ex-post break-even equation.

The way the market is supposed to send a price towards us as the expression or translation of absolute contingency at expiry (difference in itself, which says only that it could have been different without any visible variation or relative states, without even state) translates, before expiry, into the *given character of the market price* of the contingent claim, against which alone we can form an ex-post accounting equation that breaks even on the spot. To articulate the inarticulate (at expiry) character of price, we require Brownian motion. Brownian motion is how the strike of contingency translates as we recede back in time.

The event plays an unusual game with time. It requires time in order to happen and that we know what the event is, yet time is not the place of *variation* of the event. Time is always too late for the event, for it is only after the event (ex-post) that we know what it is and that we think, at the same time, that it could have been different, or for that matter, could have never taken place. It is an essential part of the event that we deal with it ex-post, as already occupying the place (already taking place) and all the place (with no time left to any variation), yet that it is contingent and could have been different. This variation cannot take place in time; place is its proper space and price is its proper translation (expression).

We don't know how the price of the contingent claim is given at expiry, because the accidental time that is then reduced to nil makes it sound as if there were no place capable of giving a price. We don't know what it means that a price is given at expiry, when all that matters and makes a difference is a different way of reading the event on the spot (ex-post, no projected variation, just the thought that it could have been different). It is hard to imagine a market being given out of pure logic and the mere reversal of an order of reading (and of thought). We have to take a distance with the event in order that the market may be given. It is from a distance with an event (before it, in a way) that the market can be given, yet the whole idea is not to think that the interval of time has enabled this phenomenon and that the market is given in time – preceding the event chronologically. The idea of a market being given has to take place at a distance (in time) with the event and then be engraved back in the event. We drag the ex-post character of the event to a place in time before the event, yet we make sure that the accounting never becomes ex-ante. This is possible thanks to Brownian motion, which precisely produces an infinite long run on the spot, thus disguising an ex-post attitude as ex-ante. It invents the price of the contingent claim (which is written on paper for the purpose of ex-post) in order to take the accounting equation out of the bottomless pit of Brownian motion and to make operative the self-financing strategy.

We didn't know what the exchange meant at expiry – we intuitively had the impression that it was at one with the written character of the mark, with its ex-post character, which couldn't revert in time and become ex-ante but only revert through the matter that the mark is written on, hence the exchange. The exchange was written for the mark; it was the other face of writing. There was no space for the exchange; there were only the tension and the idea, the space where the paper (and its ex-post mark) had necessarily to be thrown and thrust, yet a space that was strangled by the lack of time. Before expiry, the exchange is *already translated* (already: this means without mediation, without time, literally through a miracle) in the derivative price that we are tracking in order to form the instantaneous ex-post accounting equation – which is the only way to mediate the ex-post, untranslatable, unexchangeable ex-post character of the contingent mark or the event. Writing was invented, money was invented, and concomitantly the market was invented in order to *give the price* and close the ex-post accounting equation before expiry, thus mediating and expressing the inexpressible (and impossible to mediate) exchange; that is to say, mediating it without denying the ex-post character of the mark.

Part III

The Matter in Contingency

The Paper and the Tree

7.1 The market and time

Let us confer to the market the status of metaphysics. What is the level of abstraction such that the thought of the market becomes purely metaphysical? What is the elimination of all effects, physically or empirically attaching to the market, such that only metaphysics would remain, or, in other words, only the precision of thought concerning the market? How to present a metaphysics, or a problem, such that the market would be its precise solution? How not to think of the market as the place of exchange of some predefined value, through the notion that the valuable goods are presented, first, and we reach to them, second, with a price? How could the exchange be abstracted away from the underlying value?

The traditional background of the market is economics or political economy or econometrics; what if the market were abstracted away until there remained no other background but the ground itself? What is the fundamental ontological question concerning the market?

This ultimate background is time. Continuous time and continuous trading have become inseparable from the market. This is not only the consequence of modern times; this is the market finally finding its original category, which is time. Because the (financial) market is everywhere and trades continuously in time today, it precisely inspires us to make this inquiry into the perfect origin, or the perfect background, or the most precise thought or definition concerning the market.

In time, the market is necessarily associated with random processes. It is difficult to imagine an exchange, a crowd bidding up or down for a commodity, and not to think that the price evolution is random. Indeed, the efficient market hypothesis is a simple and precise thought. The relation between time, randomness and the market seems to be co-substantial. We can imagine that not only the market would receive a precise definition from the pure metaphysics we are pursuing, but also, in the same stroke, time and randomness. What if the market had such an original relation with time that randomness had to be revised at its light and no longer be considered given beforehand even as a concept? What if the concept of the market had to reach behind the notion of background states of the world and even contradict that notion?

Do we buy or sell something in the market because the transaction is not available elsewhere and would we then step into the market with the thought that the price is unfair because of that; that had buying or selling been available elsewhere, the price would be different? Do we go to the market because we do not know what a thing is worth and we seize price as the proxy of value? In reality, we don't value the thing, we just want it or want to get rid of it and the market is the only way. The next price builds on the market, on the price that took place before. The price that went before is the only value, yet it does not present the thing we are after in a larger frame or in a tribunal where its value is judged; it presents the thing directly, to be had or to be got rid of at that price.

The price is equal to the thing inasmuch as it occurs in exchange of the thing. There is no better presentation of the thing than its trading. Valuing the thing, judging it, wondering what the thing is, all these modes of presentation of the thing are too indirect and, in a way, conceptual. Trading the thing is exchanging it and 'varying' it (therefore assessing it and valuing it somehow) in a space that is not larger than the thing, because it precisely exchanges it. Trading the thing is directly seizing its matter, as if holding it by the skin. Subsequently, the movement to grasp the thing or to jettison it creates the next movement in price. Although the next price follows the previous price in time, they are of the same nature. They are the same price. *This is the cause of randomness.* This is how the market can be a temporal process, yet without time getting detached from the trading process and becoming a larger container. The randomness is not referred to a larger frame in which statistics becomes the background truth. It is purely temporal and purely random in the sense that the only place and only random generator is the market itself. This is the force of trading as opposed to the states of the world where lotteries are evaluated.

Just as we don't recede from the thing to value it in a frame that is larger than its price, we don't recede from trading to value the randomness of price in the larger frame of statistics. You can buy and sell at prices because there is such a place and there is no other cause. It is an aside and a derivative effect that we should not know the value. We don't think of the value, we think of the thing first. Elaborating its value is a second thought; it is too late and inappropriate.

How about contingent claims? We want to buy them or sell them. They yield money and we can only have them for money. The market is their place of exchange. We don't know the value of the contingent claim; so, we go to the market because we want to buy it or sell it. We go to the market when we do not know something. Contingent claims are typical of the lack of knowledge because typically we don't know their outcome. Probability is a second thought. Stable frequencies create institutions capable of selling the contingent claim or the lottery (insurance company, casino) but this is no market and this is no probability; it is a closed account that has broken even in the long run. When there is no such institutional seller, the market is open. The only way to translate, to a date before the settlement date, the contingent amount of money that is going to change hands at the settlement of the contingent claim is money that changes hands in a market at that earlier date. (Is this not the same as saying that the randomness of price precedes probability and is more fundamental?) The market is the typical place of exchange of contingent claims and our contention is that contingent claims admit only of prices and not of values.

It is not only sufficient to exchange contingent claims in the market in order to ‘value’ them; it is also necessary. The only place you could buy an orange from is the market. If you buy it from your local supermarket, then the supermarket is replenishing its inventory from the market and the price tag it is showing for the orange is correlated with its market price. Go ahead and *value* the orange if you like. It may be your favorite fruit, hence you may value it a great deal. What values you could attach to it, or to anything really, is independent of their market price. The market price is what you need to pay in order that the orange changes hands and that you get it. Exchanging the orange and getting it as a result is a simple and primary and material act. The value of the orange is derivative and is a second thought. It is an abstraction.

Now, the market price of the orange is the present translation of future contingent amounts of money relating to it. You may not expect from the orange anything else than consuming it on the spot, but it is not you who make its market and set its price. Surely, its market-maker is concerned with aligning its present price with the price of contracts dealing with future deliveries of stocks of oranges, what we call ‘future contracts’; and future contracts, in their turn, are bets that yield future contingent payoffs. Be value what it may and be your appetite for oranges what it may, this won’t stop the present price of oranges from being the reflection, or the translation ahead of time, of their future prices and the contingent variations of those prices.

There is no meaning to price independently of price volatility. The present price of oranges may be determined by present demand and supply; however, the demand for oranges is not limited to the consumers. Market players may be buying oranges just ahead of a rise in the price of oranges – we call them ‘speculators’ – and sellers of oranges will be definitely monitoring the inventory, for fear of short squeezes, or at any rate the price of futures written on the orange. For all we know, options written on orange futures may be already exchanged and the volatility of the prices of orange futures may have already become a commodity, trading alongside the orange. Price is all about price volatility and there is, in reality, no way that you could exchange oranges in the market without actually or virtually simultaneously exchanging all grades of derivatives written on their price. The market may have not historically traded futures, options, variance swaps or barrier options written on the price of oranges alongside the physical oranges; however, the meaning of the market exceeds its empirical or actual history (its historiography).

It is certainly not the price of physical oranges that is driving the price of their future contracts or of options written on them. What could drive the price of the option? Could it be the history of prices of its underlying? The option price is all about future volatility, not historical volatility. For all we know, the market action in orange options, or bets on the future volatility of orange prices, may be driving the market action in orange futures through the transactions of option market-makers who are hedging the options they are transacting, and the market action in orange futures may be driving the action in the market of physical oranges. It is not without a reason that the market is *one* place and it is in that *same* place that the underlying, as well as futures and options contracts written on the underlying (and potentially or virtually the whole infinite chain of derivatives of increasing complexity), all trade at once.

Price is all about future price volatility and when the future is the only matter and the only commodity, there simply is no ground any longer (no value), there simply is no hierarchy any longer whereby an established fact concerning the underlying (for instance, its historical volatility) would be the ground of valuation of derivatives written on it. There simply is no statistics anymore. *The future is not the matter of statistics.*

Consequently, the pure temporal randomness that we are talking about shouldn't be inscribed in the time of statistics and time series but in place. The randomness that is inherent in trading and escapes the framing of statistics, for this reason, ultimately points to place. The one place in which all grades of derivatives trade at once (where each grade reflects a traditional statistical parameter – for instance, variance swaps reflect variance, gap options reflect jumps, options written on variance swaps or more basically the term-structure of variance swaps, reflect the variance of variance, the term-structure of gap options reflects the volatility of jumps, and so on) *replaces the statistics.*

It is not by accident that the derivative (and its potential infinite chain) trades alongside its underlying. The mere possibility that the derivative contract may be exchanged simultaneously with the underlying (i.e. the existence of a place, of a common floor which opens this possibility) entails that it is *virtually* exchanged. By this, we mean that the price and price evolution of the underlying can no longer be considered alone, even if the derivative isn't actually exchanged. They now behave as if the derivative were really exchanged, which I express by saying that it is *virtually* exchanged. That the derivative is *virtually* exchanged means that it is *really* exchanged. There is no difference between actuality and virtuality in the market because the meaning of the market (its essence, its reality) depends as much on the virtual prices as on the actual prices, and perhaps even more so on the former.

7.1.1 Contingency, writing and exchanging

The contingent claim is typically written.¹ Because it is only at the settlement date of the contingent claim that we will know what world we are in (as a matter of fact, whether the world still exists or not), because the world is then essentially contingent and has no relation with the present world and because no tree of possibilities links it with the present world, we need a piece of paper, which is to be opened and read only then and there, to 'remind' us what contingent payments should be made in that contingent world. Writing was invented to supplement this loss of memory.

This is how far apart probability and writing are set from each other. Writing is all about the surprise of opening the testament and reading its clauses *then and there*. The material on which writing is engraved cloaks the written contents and hides them; it withdraws them from the eye of possibility and prevision. For this reason, writing implies reading after the fact (ex-post) and the material sheet only manages to *find itself* chronologically before. It is as if it were lost (destroyed) in the aftermath of the event and then found before the event, by a peculiar twist of chronological time. Its only translation or settlement is then the market.

Opening the contingent claim and reading it ex-post is linked with money and the possible loss of money – with the world having possibly been wiped out and the material sheet of paper having basically been annihilated. Because of the material existence of the piece of paper, there is the

material possibility of its annihilation – an eventuality that is not otherwise open to possibility and to its possible states. *Our thesis is that this alone is the justification of the existence of the market and of its variance from probability and from the corresponding ex-ante schema.* This is the justification of the market price being chronologically situated before the event of expiry of the contingent claim, yet engaging with it as should be, in a truly ex-post fashion and not ex-ante in any way.

The reality of the written testament, when it is opened in the real future contingent world – real in the sense of exceeding possibility and of possibly having been wiped out, real in the sense that money can sustain a loss of money and take in massive contingency whereas probability, which is calibrated in an ex-ante loop, cannot take a loss of probability – is the *same as the reality of the mark* that the contingent claim bears, the reality of the clause ‘Pay \$1 in case A, 0 otherwise’ that is materially written on the piece of paper and materially meant to be exchanged. By contrast, probability is not written and has no such reality. What is written is a future real exchange of money, timed for the opening of the testament up to and including the annihilating event. This writing is programmed to be opened and read ex-post at a time when anything can happen, completely outside possibility and prevision, and for this reason it still finds itself today as *trace* of the future, and for this reason it is exchanged today.

Because the contingent world that will prevail at the settlement date is basically surprising, we need the corresponding surprise of opening a testament (the contingent claim) and discovering its provisions. Writing was invented because of contingency. Even a bond, or a contingent claim that is supposed to pay the same invariable amount in any case, is written because we need to be reminded, at expiry, of the amount to be paid – simply because time has passed between the issuance and the settlement, and time is contingency. For, it might be the case that the borrower no longer wants, or is no longer able, to redeem his debt, in any case.

Writing is typical of contingency – it is its other face, especially when money is to be exchanged contingently and when money is ‘still’ the case today – and for this reason, the written contingent claim ‘still finds itself’ today and finds itself exchanged against money today. Hence, the exchange is the other face of written contingency. Because the written contingent claim is meant to be opened in the future contingent world, it finds no other place to go in the present but the market. This is the definition of tradability. The contingent claim is written because of the surprise of the future contingent world, to which corresponds the surprise of opening the written testament and of reading it then and there, and because we are situated today in a present world in which the contingency has not yet been settled and in which the written contingent claim, as material piece of paper, *still finds itself*, the only amount of money that could be attached to it today, ahead of the final contingent amount, is a market price. This is the definition of the market price.

The only settlement of the contingent claim ahead of its settlement date is its settlement through the exchange. The exchange is the only translation back in time (to the present) of the future contingency. The present holder of the contingent claim needs it to be settled today and its only present settlement is the market price. De Finetti thought of that price as a definition of subjective probability. He thought that the price that a banker would offer to take over the contingent claim from its holder is the reflection of the subjective probability that this banker has assigned to the

contingent event. But who said the definition of probability should be our purpose? Forget about probability being the purpose and then the only thing that will matter will be the price offered, not the probability that subjectively went into the head.

Moreover, a market, or tradability, is truly needed in the sense that the price should be the *universal* present settlement of the contingent claim. The 'banker' who takes it over from the holder should be able to revert at once to the market and get rid of the contingent claim (settle it) in his turn. Liquidity, or tradability, is the true translation of contingency ahead of time.

It is not the numerical value of the price that matters (what some call the 'exchange value') but that the contingent claim should be exchangeable at a price. The numerical value of price matters for de Finetti because what he is after is the valuation of probability. Tradability (this dispositional term), or liquidity, is a material property attaching to the paper and only to the paper; it is a 'continual' property corresponding to the 'continuity' of matter and it is reflected in a material medium called the market. Price is not a value (a number); it is more archaic than value in the sense that an observable in quantum mechanics is more 'archaic' than the particular eigenvalue or in the sense that writing, in Derrida, is more archaic than state or being. The numerical value of price should never be abstracted away from price.

The price is nobody's offer in particular. It belongs only to the marketplace. The auction process makes us think of a subjective process – that an agent bids then another agent outbids him for reasons that we deem subjective. We think the price changes for no other reason than the competing appetites of bidders or impatience of sellers. But who said that price formation or price change should be explained psychologically and, for this reason, contradict the proposition that the price belongs only to the exchange place? The place was never contradictory with price change. Nobody has come up with extra calculations to determine the price change; it is *still* the same contingent claim that everybody is exchanging against a price.

Now, there is a case where not only the contingent claim is exchanged but the event triggering its payoff is also the result of an exchange. Underlying assets are exchanged. We postulate stochastic processes for their prices as a result. We think we can value derivatives written on them, only to find that those derivatives are, in the end (or, rather, in the first place), exchanged. Why do we make the exchange that produces the underlying price different from the one that determines the derivative price? The exchange is a place with no distinction. Might not its definition be precisely that it's the same exchange in which derivative and underlying are immersed?

We are not happy with the way the exchange and randomness are classically conceived. The exchange and randomness may be more fundamentally linked than we think because the reason we are unhappy with how the one is classically conceived may be the *same* reason why we are unhappy with the other. There may be something crucial (literally) that has been missed in the concept of the exchange and has also, fundamentally, been missed in the concept of randomness; and this crucial missing thing may just be the way the exchange and randomness have to be conceived *in the first place*. Instead of conceiving of the exchange and randomness in the first *place*, we sequence them in time as per chrono-logic and the corresponding order of thought. We isolate the exchange of the underlying from the exchange of its derivative (we think of it first, sequentially).

Because of this mental act of isolating, randomness is thought following the model of the dice – a relative model, not delivering the absolute matter or the full *aleatory point*. The underlying price is modeled stochastically and the derivative has no other way then but to be *valued*. It is not fundamentally exchanged *in the first place*. Its exchange becomes a derivative accident. Similarly, randomness is always thought sequentially because the states of the world are listed and totalized, and the multiple in question is thought, classically, relatively to the one. What is missed, as a consequence, is the multiple from which the event (history) will issue. We never think of the continuous matter (the full aleatory point) in which the probability distribution is thrown simultaneously with the dice and which amounts to saying that the event (absolute contingency) is the only matter. The right way to conceive of randomness is, from the beginning, to face history. The market is the first mark.

The definition of the market (and of pricing) won't be complete until we think of the market as the place of exchange both of the underlying asset and of the contingent claim written on that underlying asset. The concept of price won't be complete until it is thought absolutely in isolation from value – until we fundamentally deny the idea following which something has value and *then* is traded at a price that will disfigure this value and alter it. Derivatives that are written on a traded underlying asset offer the sharpest angle from which to express this point. The underlying asset is traded for reasons that we have already argued, according to which the market takes over any probability framework. Yet, the price of the underlying asset finds itself moving randomly in time (the exchange is archetypical randomness because of the efficient market hypothesis) and the temptation is then purest to summon back the probability paradigm. Let us not forget that price processes make the best case for the science of stochastic calculus: pure numbers, pure random generator. It is when the temptation is purest for a probabilistic valuation (hence, derivative valuation) that the comeback of price is purest.

The definition of the market was not complete with the first layer of the reasoning which argued that all securities (whether underlying assets or derivatives written on them) were contingent claims and were traded because of the contingency. Indeed, this left time open and we had to fill it. The complete definition of the market lies in doubling the layer: an underlying asset is already trading by the first virtue of the exchange (which is the translation back in time of contingency) and once the temptation of probability is, as a result, at its highest, the market lies in saying that the derivative written on it should not be valued as a result but traded in its turn. For the exchange to acquire its best definition and be the total *other* of valuation, the definition should profit from the moment when the temptation of valuation is at its highest.

It is not enough to work oneself into believing that there is truly no other value than exchange value, or that there is only truly price. To make the exchange come *first* (absolutely) or the price come *first*, one has to make price come back after a temptation of probability and valuation that was triggered by nothing else but the first layer of the market and by the first attempt. This layering of valuation and pricing, in which pricing cannot win absolutely unless it won against the probability temptation that was triggered by the price itself, this liberation of the exchange from any ground and any value has as correspondent the liberation of randomness from a similar layering.

The complete definition of the exchange is not independent of the complete definition of randomness, which is that randomness should vary from the pattern of functions being valued over states (from probability as an evaluation) and from this way of reading the one-to-many relation. The exchange is essentially linked with randomness (efficient market hypothesis); however, the *material writing* – or the surprise of opening testaments and of finding out the excess of contingency over possibility, or the possible loss of money – is such that all derivatives of all grades of complexity (layered, one over the other) must be written and must be exchanged.

The material paper is such that it has to list an exhaustive list of states as contents and as provisions of payoffs, and that it cannot list the state providing for its own disappearance, for then this additional list disappears, too. The material paper has two levels by necessity, the one corresponding to the content and the other to its irreversibly contingent and unexchangeable matter. There is no such thing as a contingent claim that provides ‘Pay \$1 in case of *A*’ and that does not provide in writing what to pay (for instance, 0) otherwise. The contingent claim cannot leave open for interpretation the other branch of the alternative. It has to list an exhaustive range of possibilities. However, the ‘otherwise’ is not an absolute otherwise or an absolute outside and the ‘exhaustive’ range of possibilities is only relatively exhaustive. The alternative clause does not mean ‘Pay 0 in all other cases than *A*, up to and including, for instance, the case of annihilation of the material sheet on which both the positive and the negative clauses are written’, for the alternative clause disappears together with the material sheet. The ‘otherwise’ means the complement of state *A* in a range of possibilities that is predicated on the continued existence of the material sheet: for example, ‘Pay \$1 in case $S > K$, 0 otherwise’.

The range of possibilities is relative to a context that we believe to be stable (in this case, the perennial existence of the sheet). Default of the issuer of the contingent claim is another outside, or state that falls outside the context. It is not the material sheet itself that disappears in case of default but a background context that was supposed to be stable and whose disappearance is no less catastrophic than the annihilation of the sheet; namely, the verb ‘Pay’. (The material sheet exists in order to bind its issuer, therefore the disappearance of the latter as bound by the sheet or, in other words, his default, is equivalent to the annihilation of the sheet.) The material sheet cannot provide what to pay in case of default for, if this recovery value were explicitly written, it would itself disappear with the sheet or be subject to default, too. Default is the incapacity to pay, therefore it cannot be provided for under a clause that begins with ‘Pay’. Rather, default has to be experienced ex-post; that is to say, unpredictably and un-programmatically, and negotiation has, only then and there, to occur between the issuer (or his liquidator) and the holder.

A material sheet is needed in order to group all alternatives (*A* and not *A*) exhaustively; however, the very materiality of the sheet is such that the states are never absolutely exhausted, for what is material can disappear in ways that are, by definition, outside the range that has been coded as *content* inside the paper. (The clause ‘Pay’ is material, too, and can materially fail in case of default.) The presence of the material sheet (this is a capacity, not a possibility) is what brings negotiation to the table after the event (what I call ‘opening the testament’). Even though the issuer has defaulted, the material presence of the sheet in the hands of the holder brings up negotiation about the recovery

value. The issuer cannot argue that his default falls under the ‘otherwise’ which provides to pay 0, for the ‘otherwise’ was relative to the context of his non-defaulting. Even though default is equivalent to the annihilation of the sheet as binder, the sheet is still materially present and its fate has to be negotiated after the event. Likewise (although paradoxically, in a way), if the sheet were materially and completely to disappear, then the absence of money exchanged in that case would not be the same as the absence (‘Pay 0’) that was provided by the sheet and, for this reason, the first absence could (if absurdly) be negotiated, too.

This inherent double level of the material sheet (what’s inside the content, or inside the context, and what’s outside the context and has to be negotiated after the event) is what makes for its exchange. Because of the clause that cannot be written on the sheet of paper (which is its annihilation under whatever form, default or definitive), the paper is subject to the exchange. The exchange is the only way to support this massive risk. There is massive risk in the market, which has nothing to do with prevision and computation.

Difference is analogical only when the states are totalized, and this totalization is equivalent to the very existence of the state. A state presupposes a range of states, or a context. The state is a state only inasmuch as there will be a transition between states, and this presupposes a context. A transition cannot occur between a state encoded on paper and the annihilation of the paper. To say that the materiality of the paper is what allows for the absolute reading which is independent of any context (as it cuts through all contexts) and which says: ‘It could have been different’ is the same as saying that the materiality of the paper is what allows for (and supports and inscribes) the disappearance of the paper. The absolute difference, which is not relative to a total or set of states (‘It could have been different’), includes the disappearance of the paper. The paper ‘could have been different’ in the sense of not existing anymore. Observing the paper as a strike or a mark or an absolute difference is observing it materially. *This is the same as the possibility of observing its disappearance.* Matter has a way of disappearing that cannot be coded and cannot be reduced to a state, therefore to a transition.

7.1.2 Price and time

Price is the only present settlement of the contingent claim. Contingent claims have tradability and exchange as their other face. Time, which is the reason why price exists in the first place (as *present* settler of *future* contingency), could not but turn price into a temporal process, thus random. Probability made a comeback as the category in which to frame the price process, when the thinking was that price, as the sole translator and settlement of any future contingency, precisely dispensed with probability. The valuation of derivatives written on the price process (of the underlying asset) is the typical application of the return of probability. To argue for the predominance of price instead of value and for the predominance of pricing instead of valuation is to require a *market* for derivatives written on the underlying price.

It is for this reason that derivative valuation theory has taken us farthest away from the market, only to bring us back face-to-face with the market, as derivative pricing, or as the criticism of derivative valuation through derivative pricing. It is for this reason that derivative valuation theory is a historical opportunity. It has got everything we need. First, the price of the underlying is the

sole state variable. It is a price, supposedly replacing probability and perfectly random because it is a price. The consequence is that the derivative is valued, therefore is denied a market. Second, the price of the derivative, now making a comeback and taking over value, expands the states of the world as it should.

The market itself is wedded to time, hence to the randomness of (the underlying) price, and the only way to mediate these dynamics is another price, that of the derivative (written on that underlying). Time is unavoidable and the market cannot close time off unless it trades all levels of derivatives on top of the underlying. The time in question is the time separating us from the contingent settlement (in which anything can happen). *The only way to make the time that separates us from the contingent event become the same as the time in which the price (supposed to translate the event) is random, is the trading of derivatives all the way up.* The exchange as the present place of the future contingency cannot close itself off and receive a precise definition unless it becomes the exchange of both underlying and derivative. The tradability of the written contingent claim (which is the theory of the intervening time) cannot receive its full definition unless the writing and tradability of all grades of derivatives are considered. Randomness, or the consequence of price being the only present translator (in time) of future contingency, cannot really start off unless it is rid of probability (probability can never begin) and completely immersed in price (of all grades of derivatives).

To repeat our conclusions again: Price is the translator in time of future contingency and the intervening time yields a randomness that cannot be mediated by probability but only with the full market of derivatives. The exchange is the only translation back in time of the event and the intervening time cannot but make *all the market happen*. The exchange is predicated on time and cannot acquire its full definition unless all derivatives are written and exchanged. There is no middle way. Randomness which starts with price (and no probability) cannot but entail the whole market, whose correspondent ultimately is the event.

The contingent claim is written for the sake of opening testaments ex-post and taking in massive contingency (up to and including annihilation). As a result, price replaces probability in being before the event yet in remaining ex-post (as opposed to probability which fakes the ex-ante attitude and is not truly before the event). Now, time seems to leave us no other possibility than the price dynamics being ruled by probability until expiry, probability which we thought we had eliminated! But the market of derivatives is precisely here to take over and to close this gap and lift this contradiction. The market is a radical alternative to statistics, because of the writing and the peculiar twisting of ex-ante and ex-post, because of the exchange which is the other face of writing.

7.1.3 Price and the event

Randomness was supposed to be connected with unpredictability in some way. However, probability has tamed randomness (it has boxed it) by making the outcome unpredictable among a given, pre-identified set of outcomes. The die is one material thing that has many faces. Nothing can stop us from conceptualizing this one-to-many mapping. This is how possibility, or the possible faces that the die may be landed on, are imagined. The problem enters a material phase, however (as opposed to a merely conceptual one), when mass enters in the picture and the material, massively undivided

die is considered as one mass lying in the massive material world. The material world in which the die is materially lying cannot but stop the die on one face and present that face. As being part of the material world, the die finds its material contingency which says that the die lies on one face, only this face could have been different.

There is no limitation to the strike of contingency. The face of the world is at one with the face of the die (it is all but one unlimited strike) and by prolonging the strike we could have said that the face of the world could have been different. Yet, we isolate the mass of the die from the mass of the world and we argue that the face of the die could have been different, given that the face of the world (the conditions of throwing the die) is otherwise unchanged. The die is an ideal object of probability because it enables this act of isolating. One can, indeed, repeat the experiment of throwing the die. One invents the probability distribution of the throws, all other things being equal. The definition of the isolated mass of the die is this repetition.

The non-identification of the exact sequence of throws reminds us that chronology is an illusion, however, and that the next throw is only an image supposed to mediate the strike of contingency. The probability distribution of the die is not itself thrown with the die. We never imagine that the face of the world that has stopped the die could have been different simultaneously with the face of the die. We are only accustomed to the one-to-many mapping in which the set of outcomes is totalized and identified. Yet, the essence of the event and of unpredictability is that there should be no such identification and delimitation. The absolute multiple arises when we say that the face could have been different absolutely, without limiting the strike and framing the face (without relatively framing them). The massive contingent world should be the first matter (*matière première*) in the sense that no cut is yet made in order to fabricate possibility. The stability of statistics, which is the stability of the mass of the things that we have cut off (artificially) from the mass of the world, is such that the ex-ante attitude is invented as a consequence, when the event should have remained absolute and should have been only dealt with in an ex-post way.

What is the randomness that can reflect this absolute unpredictability of the event? I argue it is the randomness of price. Let the event be the original category (absolute unpredictability) and probability (or relative unpredictability) be the accident, or the degenerate case. We should have always started with the event. Randomness should have been identified with the randomness of price from the beginning. The event is the original and broader category; however, we were misled by the precision and, in some sense, the improbability (the scarcity) of the medium of the event, which is the financial market, or the market of contingent claims, or the invention of money, of writing and of the exchange. For the reason that only in finance could the event be mediated in its broadest category, we thought that the financial market was too special and too specific. We dismissed it under what we thought was the broader category of probability. We made the randomness of the market price a sub-case of probability theory.

That there should be no quantitative translation of the event except financial is no reason not to draw from the market the broadest conclusions concerning the event (conclusions, for instance, such that probability should be seen as the specific and spurious case). We first dismiss probability by making price (or the exchange of the contingent claim) the only present fate of the material sheet of

paper that will be opened only after the event, and we dismiss probability a second time, and more durably and temporally so, by arguing that the randomness of price, which could then only arise precisely because of the intervening time on which price is predicated, is to be mediated by price too; that is to say, by the whole market. The randomness of price, when it is folded back and recalled to its original medium, which is price, appears to be the original randomness, the one without the limitation of the strike or the distribution of probability – the one originally dealing with the event.

To recap, the present line of argument is that the event should rule randomness at first and originally (that the event should rule in matters of unpredictability), that the market price is the only quantitative translator of the event, and that the randomness of price therefore is original randomness. We argue from the event and through price as its mediator, and then we access the idea of randomness as being that of price specifically. Randomness shouldn't have been conceived before the randomness of price, and the randomness of price cannot be delivered by probability.

7.1.4 Price and the trace

The contingent claim is written and the working property of writing is that it triggers a surprise, at the time of opening of the testament, which is equivalent to the surprise of the contingent world. On the other hand, as we said, the contingent claim *still* finds itself ahead of the future event, because it is written. Obviously, time is involved in the interval. However, our peculiar formulation makes it sound as if the present price, due to the contingent claim 'still finding itself' in the present, is *posterior* to the event.

There is already a twist in time. The presence of the contingent claim is a residue (it *still* finds itself), almost a *trace* of the future event. The present price is ex-post, in a sense. This, we are inclined to say, is due to the alchemy of writing. Money that changes hands today (or the definition of price) for the sole reason that the written contingent claim, by whose clauses money will change hands in the contingent future, is *still here* today (chronologically before the event) introduces a twist in time whose consequence is to hold that price is everything but ex-ante.

The exchange is the solution of the problem of being situated chronologically before the event, yet of not wanting to address the event in an ex-ante way – which is impossible anyway because the event is unpredictable. Replacing probability by price is the attempt to address this 'impossible metaphysics'. Price finds itself before the event in the twisted medium in which 'being before' in time is no longer equal to 'being ex-ante'.

Perhaps this capacity of being before without being ex-ante is due to the twisting of the ontology and to the replacement of being and state by contingency. One could argue that this ontological revolution has consequences on time, chief among which is the dissociation of the time of contingency from the time of prediction. The ex-ante stance is eliminated by reducing the statistical population to the single case and by noticing that this doesn't call for the ex-ante stance, quite the contrary. It calls for the market price, or the result of exchanging a material piece of paper whose possibility of annihilation is the indelible sign of the ex-post stance, where losing money as a result and the continuing medium of money are the guarantee that price remains ex-post, all appearances to the contrary. The last thing we would want is that the price evolution would switch back to the time of prediction, or to the logic of the ex-ante stance, or to the framework of probability.

Probability is objective because of the strong law of large numbers. Probability is objectively equal to the limiting frequency. However, probability is not real as a result of that. Limiting frequency is realized because of Cournot's principle, which states that an event whose probability is equal to 1 is realized. This limiting realization falsely propagates back to the individual throw and makes it seem as if the probability of the next throw is real (that it is ex-ante) and the possible is about to be realized. However, extensionality is lost in reality, because the limiting frequency doesn't identify a specific sequence. Only the image of the die we are holding makes it sound as if something real had to be said about the individual probability; but this is only an image. Chronology, in which probability seems real at every step (or the ex-ante stance), is only an image. Matter is the only thing there is and matter, relatively to the die, is the strike of contingency. The infinity of the strong law of large numbers is just the infinity of the strike of contingency, which amounts to saying that the face the die is landed on could have been different.

The other matter is money. Instead of speaking of realization of the limiting frequency and importing locally the image of chronology and the image of the reality of probability, we speak of break-even in the infinite long run. Breaking even is ex-post and for this reason it is matter. Alternatively to the reality of probability being falsely exported from the realization of frequency and actually encountering the problem of non-identification of the specific sequence, the insurance company may face the individual throw (or the individual contingent claim), with the seeming sense of reality of the probability, because it is breaking even in the long run in an undivided and un-individuated way. Ex-ante, in this case, is just a way of speech; it is a rewording of the existence of the whole money account of the insurance company and of the ex-post break-even. Money introduces a new way of counting (ex-post accounting) which replaces frequency. Money introduces matter or ex-post. Only matter exists (only the ex-post stance exists) and money is what enables us to extrapolate the existence and reality of probability from the existence of matter in the statistical case, because of the *logic* defined by the account of the insurance company. (There is not really time in the insurance problem, because there is no contingency and there is no event; there is no single case.)

The problem we now set ourselves to solving is how to take advantage of the same property of money, regarding the twisting of ex-post and ex-ante that was made manifest in the insurance statistical case, and apply it in the genuine case of the single-case event, or the case of real time and the real future, where the real is to be understood in terms of contingency (and not the realization of a possibility). How to adapt the new way of counting of money – the accounting which is more robust and more massive than the frequency count? How to take advantage of the material property of money (money as matter and enabling the ex-post stance) and adapt it to the single case? *The answer lies in writing the contingent claim with a contingent payoff expressible in money.*

What is ex-post in the contingent claim is our capacity to open it after the event and to read the amount of money to be exchanged. Anything could happen and money could be lost (the material existence of the sheet is an argument against the fine and fragile probability tree). The existence of the material sheet of paper 'guarantees' that anything could happen to it. It could burn or the conditions of its settlement might never be gathered. This is the privilege of matter. Money could be lost. For the reason that this is robust, that the surprise is *then* revealed and that losing money does not matter (as opposed to losing probability), this way of accounting (and losing) is what allows the

sheet of paper to *still find itself* today before the event and what allows its price to be the translation, not ex-ante in any way, of the ultimate surprise.

To *still find itself*, for the written paper, is to find itself massively, outside prevision, outside the positive conceptual reading of its contents, which seems to call for anticipation; that is to say, to find itself ex-post – this is what ‘outside prevision’ means. The matter of the material paper provides that anything can happen to it, which will only be revealed ex-post. Its annihilation can happen to it; however, this cannot be written as an instruction or as content: the paper cannot provide as a coded instruction what will happen to it in case of annihilation because this provision would be annihilated, too. Rather, the mere massive presence of the material paper is what ‘writes’ this possibility. To have the paper materially, as opposed to having probability or the tree of possibilities which knows only of precise transitions, is to have, as its other side, the possibility of its material annihilation. Possibility does not have another side and the tree cannot include the branch leading to the annihilation of the tree. The matter of the material contingent claim is the ‘guarantee’ that anything can happen to it – that anything has *already* happened to it and, as such, that it is still here today and still finds itself today.

Price attaches to this material sheet by virtue of its material exchange (which wouldn’t be possible if there had not been a material sheet, made to be exchanged) in a way that probability cannot attach to possibility. Price attaches at present to a material sheet that is made to be opened later and to surprise us later absolutely contingently, up to and including the contingency of its annihilation, in what seems to be an impossibility at first, because we wonder: ‘How can something present attach to a sheet of paper whose fate is removed from the present because it is sealed and will only be revealed later, ex-post?’ Probability could never attach to such a thing because probability is a backward induction that needs to chart all the possibilities (probability is equivalent to the exhaustive reading of the states of the world). Yet, price can attach to it, *precisely by virtue of the exchange*, which is the other face, the present face of the written contingency.

Despite all the contingencies that can and will happen to it, the sheet of paper finds itself today. It finds itself today despite and *after* all these contingencies. This is the meaning of the expression ‘it still finds itself today’. ‘Still’ means ‘yet, despite’, and it means ‘after all has happened’. The sheet of paper finds itself *insomuch as* it will be exchanged and priced. It doesn’t find itself absolutely (in metaphysical time). Absolutely, the sheet is simply there because it is present; but that it finds itself *to be priced* is what justifies for it the adverb ‘still’. Exchanging it for money ahead of the future contingencies is what justifies the thought that the price is somehow ex-post and that the sheet of paper still finds itself today (after all this), to be exchanged.

7.2 From the market to the whole market

The written contents of the contingent claims are what give us the impression, or rather the illusion, of a valuation. We are confronted to multiple possible values at expiry and because we read them in the paper, inside its fold and heedlessly of the matter the paper is made of, we think we are reading them ahead of the event in a present precise state. Because (we think) we are in a single

state today that is confronted to multiple states tomorrow, we believe we have to find a single value today, corresponding to the possible values of tomorrow and summarizing them somehow. The reality is that the states of tomorrow are not states; they are marks written on a piece of paper that can altogether disappear. This disappearance cannot be coded into a state and, for the reason that the paper bearing the marks can altogether disappear, the marks are no states. The state presupposes a context that it cannot surpass (see how quantum mechanics is a meta-probabilistic tool, a critique of probability and therefore of state), whereas the mark is integrally part of the material sheet which can disappear out of context.

The reality is that there are no precise states, either today or tomorrow, and no precise transitions between states. The reality is the massive ex-post fate of the written paper, which is only to be revealed later and to which *corresponds the massive market of today*. We have invented probability for the sake of computing the present value of the contingent claim. The contingent claim generates a contingent payoff tomorrow. For what reason do we need to value it today? Is it because we want to exchange it against money today? Why not exchange it directly in a market then, without the intermediary of probability and valuation?

The contingent claim admits of values in all the states that are coded inside its paper (as contents), because this side of the contingent claim is just the side of decoding the function that assigns values. We assign probabilities to those states and the present value is a convex combination of the values known to obtain at expiry. But what is the value of disappearing at expiry and what is the state that encodes it on paper? Admittedly, the value of this implicit impossible state is zero. This is the state that cannot be seen by the function that assigns values at expiry because it is the state corresponding to the disappearance of the paper on which the function is written. It cannot be seen therefore it cannot be foreseen. It can only be seen by another sheet of paper which provides what to do and what to pay in case the first one disappears but this other sheet can disappear in its turn. In all cases, in order to make the annihilating state part of the present valuation, we need to assign a probability to it. We encode it as a further state on a sheet of paper that can disappear in its turn. We make it precise; however, this precision is potentially ruined by the disappearance of the paper that further encodes it and that says 'Pay \$1' in case the state obtains (isn't the present value of that second paper what gives us the probability of disappearance of the first one?) and by the fact that the disappearance of the first sheet (after nuclear war, say) is not the only state lying potentially outside the contents of the paper. Another contingency or externality may be the default of the counterparty, or the failure of the legal system binding all counterparties, or the disappearance of the market as such, and so on.

The whole idea here is to replace probability by price and the state by the written mark. It is a confusion to want to assign probability to the state of disappearance of the paper. It can only be written as a further mark on another paper, but this brings up the question of the disappearance of both papers, which will have formed then a single two-page paper. (It ultimately brings up the question of the end of the world.) The whole idea is that marks on paper do not need a tree or a total of states; they do not need *state* (which is equivalent to the total of states). They only need money, exchange against money today, and exchange against money tomorrow upon opening the paper – an exchange which sustains the loss, the contingency and even the end of the world.

Let us try and imagine a market with only marks and no probability. Arguing in the nested infinite fashion that all grades of derivatives should trade is arguing after the 'end of time', or ex-post, after all that might happen has happened. There is a property of the material paper, and thus of writing, which plays itself out in the exchange (thus producing the nesting of derivatives) and which still finds itself, after the event, to sustain the end of the world. The existence, or rather the remaining of the paper after the end of the world and the loss of money that is its other side, is the same as the necessary tradability of the paper and virtually the excess over any probability tree. The paper and the way it travels in time operate on both sides. Once the market is created after the first price, it is created as a whole (all grades of derivatives). We haven't yet investigated the full meaning of price taking over probability, or the meaning of money and writing being invented. This invention has to bring about derivatives trading all the way.

7.2.1 Contingent payoff vs. contingent claim (first take)

One has to wake up after sleeping, open the sealed envelope in which the payout clauses of the contingent claim are spelled out and figure out, then and there, how much to pay or, for that matter, whether the world still exists. The material sheet of paper is the guarantee of this reminder. We write because we forget and because we go to sleep. If, by contrast, we never went to sleep and the eye of possibility (ramified and magnified by the possibility tree) remained open, then there would be no need to *spend* time and to wait until expiration to exchange the contingent amounts of money.

The dynamic replication argument of BSM collapses the future into the present spot. Buy back the contingent claim from its holder for the premium π prescribed by BSM and a dynamic self-financing trading strategy using the proceeds of the premium he has collected will guarantee him, at expiry of the contingent claim, that he will end up holding the exact contingent amounts of money that he would have expected to receive, had he kept the contingent claim, therefore that he will pay himself (i.e. keep to himself) those exact amounts and pretend he had never relinquished his contingent claim.

There used to be disagreement about the present value of the contingent claim (therefore, allegedly a market) because of the difference between investors regarding their aversion to risk. However, BSM has settled those differences and shown that there was but one rational present option value. All we need is that the underlying should be trading and its volatility be known. In the BSM world, there is really no need to wait until expiry! Now, people argue that volatility is not known and, for this reason, BSM is not right and, for this reason, one has to wait until expiry and settle the contingent claim only then. However, people forget that the BSM argument and world is not about knowing. Nobody *knows* volatility in BSM; volatility just is. (Ontology, not epistemology.) Now, is it forbidden to *think* that volatility is? (This is different from knowing that volatility is.) Is the BSM world unthinkable? If you think that volatility is, then the whole argument unfolds and there would be no need to spend time in BSM and there would, therefore, be no need to have a material paper on which to write contingent clauses for the purpose of discovering them after time has passed (after sleep and after the event). There would be no market either because the value of the contingent claim

would be settled. The contingent claim would be equal to the contingent payoff, and the latter would be algorithmically replicated.

People argue that there is a market precisely because volatility is not known. However, the market cannot be related to epistemology. The market exists or doesn't; this is ontology. In the perfect BSM world where volatility *is* (and the concept of knowing is not even available), there simply *is* no market. Let us suppose for a moment that the market is, indeed, down to cognizing agents and that, in the ideal world where the tree of possibility exists objectively (without anybody observing it), there is no need to exchange contingent claims because their value is settled. (All that would be exchanged is the underlying itself.) Let us suppose this ideal world exists and that the real world is just an attempt at *guessing* it – that the market is just a guess posted against the background of the ideal existing world. However, now we have made the more pressing argument that, in the ideal formal world, there would be no notion of materializing the contingent payoff on paper and of writing the contingent claim. In the ideal formal world, even the paper wouldn't exist. All we do there is replicating contingent payoffs and not contingent claims. We replicate the contents and the provisions coded on paper, not the material marks engraved on paper. As a matter of fact, there is no paper.

Writing or not writing is an ontological question, once again, not an epistemological one. Writing exists because time exists; and the writing and the time that exist are an ontological argument against the existence of the ideal world in which possibility is graphed once and for all. They are more powerful than an argument that would allow the existence of the ideal world only veil its knowledge. (As we will see, the invention of writing, or equivalently the genesis of the market, will be directly linked to the timelessness of the formalism. It will emerge that the market is a different way of applying the formalism, which doesn't cross the path of epistemology, or even ontology.)

Ultimately, we write because we sleep and the tree of possibility (even in the ideal world) is interrupted. My whole argument is that the ontological property of writing, which is that we discover its provisions only after the fact, is the same ontological property as the market. *It is the market.* Writing the contingent claim materially is an excess over possibility (it is a total excess, because it is the ontological trigger of the market). To write it is to exchange it is to trade it at variance with its present value in the possibility tree, because to write it is to admit of the passage of time and of sleep, is to admit that we wake up and find ourselves in the future contingent world in which the world may have ceased to exist, contrary to the tree.

Every day brings new possibilities and trying even in the imagination, even in speculative thinking, even in a formal ideal world, always to mend them back in an ever-enhanced and inflating tree is an explosive activity. The tree needs the whole future to be graphed before it represents its original node. It is backwards and unsustainable for this reason. Having a market and a present exchange instead of a present original node is the only forwards way – that is, the only way to have *as of now the fact that the future will bring up unimaginable possibilities*, possibilities that we can't graph. Yes, the market is situated before the future event (it takes place today chronologically) yet it is 'logically' truly ex-post. The presence of the material paper already bears the marks of its future

opening and of its future ex-post fate, and this bearing of the future marks, although not yet existent, finds its translation in the present market.

7.2.2 The invention of writing (first take)

The 'mark' of the future opening of the paper is none other than its very material constitution. The future event is not marked and 'written' on the paper in the sense that it is known beforehand. Simply, the material paper will change hands in the contingent future against contingent amounts of money, or not change hands at all in case of annihilation, and the fact that it is the same *material* paper that will change hands tomorrow and that is present today is what bears the mark of the future. The mark is the *materiality* of the paper itself (identical through time, surviving and supplementing memory loss, sustaining loss for this reason) which finds itself translated today by the *market*.

The market is more immanent than possibility. For the reason that the paper will be exchanged tomorrow against contingent money, it is exchanged today against the market price; there is no more to the equation than that. The only way to travel time is to hold the same sheet of paper. Its price is just the amount of money it is exchanged against today in anticipation of the amount money it will be exchanged against tomorrow.

We are always misled by possibility and the identified possible states. We are misled by the 'state'. (How, ever, to think of something without thinking of the state? Yet, this is what we ought to do.) We believe the future is real only when it is identified. We confuse the 'marks' of the future with the future cognizable and identifiable *states* of the future. In reality, the future is nothing more and nothing less than the massive fact of exchange of the material sheet against the massive, unidentified and just contingent, amount of money, which may consist in no exchange at all and in total loss. This massive fact of the future is equivalent to no more and no less than the massive paper itself, which finds itself today, as such, massively, as such, exchanged. The future is no more and no less than future contingency, or the mark that says: 'It could have been different' without distinction of states, and this mark and this mode of saying are inseparable from the materiality of the paper bearing the mark.

The market is no less im-possible and massive than this massive unidentified mark of the future. Because of the two massivities, the market is contemporaneous with the future event. The present material paper *as bearer of the marks of the future* (already ex-post) is inseparable from the present material paper *as exchanged in the market*. We wouldn't touch upon the contingent claim the idea that its future fate is contingent and as such unidentified without at once touching upon it the idea of its present exchange. The present exchange *is* the future contingency (the future ex-post, the future massive awakening), only the equivalence or the mediation takes place outside prevision time. The occurrences of the words 'present' and 'future' are accidental in the last sentence.

What is the difference between writing the contingent claim on material paper and imagining the possibilities? What does writing bring over and above possibility? Why is the pricing (the market and exchange) of contingent claims so categorically different from the valuation of possible states, whose other name is probability? (How did possibility and probability ever get mixed up with the contingent claim and its exchange?) The difference is the continued presence of the material sheet. Write clauses to the effect of the future exchange of contingent amounts of money on a material sheet

of paper (which bears the differential writing and remains an undivided and unique sheet of paper precisely because of the materiality – this may even act as our definition of matter) and exchange it against money today *for this reason* and for no other reason.

We tend to forget how phenomenal and momentous (how exceptional, how impossible) this simple and original invention is. Instead, we invent value (derivatively). We value (identify) the future states and we value (with probability) the derivative as a result. We believe the states can cover and take over the fate of the contingent claim. We forget that something contingent can always happen, something not accountable by the list of states, something we believe to be residual or negligible at first. Yet, this ‘marginal’ thing is precisely the reason why there exists a material paper as opposed to a tree of possibilities and why, as a result, there *remains* a material paper and why, as a result, there exists the whole market.

The market is the chronological solution of the problem of ‘predicting’ the future or of ‘communicating’ with the future using a ‘tree’ that doesn’t blow up formally, a tree that sustains the massive contingency. There is the future (there is the event, there is no way round that) and there is time separating us from it. The event is unavoidable and the time separating us from it is unavoidable. Prediction is the temptation inherited from the false impression of the ex-ante stance which was brought about by statistics. We think prediction is the only way to make sense of our two unavoidable accidents: the event that will take place and time that unavoidably (yet, accidentally) separates us from it. Yet, there is another way that doesn’t transit through states and trees (which are unstable and unsustainable). The mysterious property (the material property) of writing which allows it to travel time outside the tree of possibility (and its eye), and for this reason to strike us as a surprise at the time of its opening, is the *same* property that delivers writing to trading as of today.

For the reason that we cannot predict the future – and money and writing were invented to reserve the surprise of the future until later – the market is not a probability computation. The contingency attaching to the material paper (the loss of money, the massivity) is *essentially linked* with the present tradability (exchangeability) of the paper, and this link has not yet been fully investigated. If anything, the unavoidability of time and of the event should act as an argument (or a premise) to the conclusion that writing has contingency and the surprise of the future on one of its faces *and* the exchange on the other face. Money, writing and time are essentially linked in this way *because* of the event. The ontology of the event is the same as the ontology or the materiality of the paper, which is the same as the ontology of the market.

Price is the result of a material exchange in which a material paper is involved, not the result of a valuation. A material paper known as the contingent claim triggers a contingent exchange of money at its expiry and *for this reason* it is exchanged against money in the present market. This has nothing to do with the assignment of probability to underlying states and with the computation of present value as a result. To assign probability to underlying states is the same as valuing, and this imposes that we see and comprise the states – that they are present and identified. The states are then the background. They are independent of the material paper and they are pinned in absolute space, somewhat like the ether in pre-relativistic theories of dynamics.

One cannot conduct a valuation, or a probability assignment, without listing the total of states of the universe of possibilities. The probabilities have to sum up to 1. The probability tree is metaphysical in the sense that it is supposed to exhaust reality and it sticks to the absolute background for this reason. A major overhaul of the probability tree cannot take place unless it is ex-nihilo and shows the tree to have been *absolutely* wrong. Typically, a tree cannot include a branch whose outcome is the disappearance of the whole tree. Because it has the ambition of a totalitarian metaphysics, the tree is destined to embrace all states, which are abstractions that have nothing to do with the tree as such (they are supposed to hang, out there in metaphysical space). Because it embraces *all* states, it leaves no room for the one state that it cannot embrace, which is the state of its disappearance. The tree is always relative to a context (a total of states) and, for this reason, is not robust vis-à-vis an external change of context (a contingency).

The material paper, by contrast, bears no states like the tree; it bears only marks. They are marks and not states (although the metaphysician will want to pin them on absolute states that he thinks coincide with them) because they are borne by a material paper and because the material paper has, as a property which belongs to a different logical plane altogether than the contents that the marks elicit and yet is at one with their nature of marks, the capacity to disappear. It is implicit in the paper, as the other side of its material existence, that it might disappear *and for this reason* it is exchanged today (this is our thesis) *and for this very reason* it cannot be foreseen but we always have to open it ex-post – we always have to wake up in the contingent world, which might be the one after the end of the world, and only then and there go check what's written on the paper.

For the reason that the paper is not only made of its contents, or metaphysical states, but carries as its other side, as the one non-metaphysical state, the very capacity to disappear (which I do not call a possibility in order to establish the distinction), it is robust to changes of context, it exceeds probability and lends itself to a market instead. Although price, or the outcome of the exchange, takes place before the expiry of the contingent claim, it is connected, through the materiality of the paper whose formula is the following: 'It still finds itself here, to be exchanged', with the loss of memory and the rupture of the tree of transitions – with the act of waking up and opening the paper ex-post. The paper admits of a price today because it still finds itself today after all that happened – a twist in time, indeed.

There is something in the mechanics of the exchange that has fundamentally nothing to do with the assignment of value, because of this relation of price with the ruin that succeeds to the event. It is the same paper that bears the marks, on the one hand, and that is opened ex-post and exchanged as a result, on the other hand (as a result of its materiality). This ex-post capacity of the paper is incompatible with anticipation and, *once it is recognized in advance*, it translates into the exchange and precisely not in valuation.

Price is altogether incompatible with value. The two lie on two different planes and in two different logics of reading. They lie in two different temporal modes. A common mistake is to think that we value the contingent claim, first, and then we change this value into money by trading the claim in the market. The value that precedes price in this mental operation is supposed to be some (probabilistic) average of the value that the claim will yield at maturity. It is true that the weighted average of

amounts of money is an amount of money; however, this is a quantitative coincidence, averaging is only a mathematical operation in this case and does not turn the computed average into a traded, or exchanged, amount of money automatically.

Think that there might be no 'averaging' of the final contingent payoff other than exchanging the piece of paper on which it is written. It is true that the numerals can be mathematically averaged but how could their exchange virtue be averaged? How to understand that the final contingent amounts of money are what they are; namely, contingent amounts of money that require the actual material paper that supports them to be opened only at maturity, in the then-and-there contingent world, and to be exchanged against money only then and there, and that it is this virtue – what they are by virtue of the material paper they are inscribed on – that needs homogeneously to translate back into the average which stands for them today? What if we made the contingent amount of money that is exchanged at maturity no longer stand as an abstract numeral that prefixes the symbol of money but shot its monetary nature into its very constitution – as if money literally stuck on it – and wanted to translate *that* back to the present and to reflect it into the average? What if we understood that the final contingent payoff is not a value (an abstract number) but a price, a number that cannot be separated from the paper that hides it from value and anticipation and from the ex-ante stance at maturity, a number, therefore, that is discovered only on opening the paper in the contingent world (i.e. ex-post), and we wanted *that nature* of price to translate back directly to the present? How to translate back to the present the ex-post nature of price without turning ex-ante?

We say the translation has to be direct and without the intermediary of probability because the time of prevision and anticipation in which the ex-ante stance is inscribed is already bypassed and short-circuited by the property of the paper which is that the paper is opened ex-post. The paper is materially already here, yet it belongs to the event and even to its aftermath (what ex-post means). It is 'still' here and open for negotiation despite the event having already consumed it and despite its fate being all written on it. It is the materiality of the paper that gets translated to the present (chronologically before the event but, literally, after it) and translates into the exchange, not the contents it encodes whose conceptual nature and reciprocal play – whose chronological reading – create all by themselves the ex-ante attitude and the impression of a present value. One cannot attach a price to the contingent claim save by exchanging it and making its ultimate capacity of being opened ex-post change hands. *One tends to forget how indissolubly linked to the material paper and to its materiality trading and therefore pricing are.*

If anything, valuation is derivative on pricing and is an imperfect abstraction (simulation) of its operation through time. Valuation is an improper collage on the ex-ante board of what really goes on, and what really goes is really only ex-post. The probabilistic averaging itself is but a summary of a statistical break-even, or essentially the result of ex-post accounting. A certain statistical regularity prevails and, for this reason, somebody continuously betting at the right odds breaks even in the long run (equivalently, the insurance company learns by trial and error how to adjust the premium it is charging in order to break even in the long run). For this reason, the price of the bet or the premium of the insurance contract are said to reflect the 'probability' of the event. Probability in this case is just short for the whole ex-post accounting and break-even operations.

It seems as though the insurance company is facing the single case or the gambler the single throw of the die; however, the reality is that they are both leaning on their whole account and on the capacity of that account to embrace the whole population of cases, long enough for break-even to occur. The statistical regularity may be the consequence of something at once present in the die or in the population, some mysterious virtue one wishes to call a probability or a propensity. It is true that one can show the law of large numbers, hence the convergence of the frequency, given the concept of probability, or something instantaneous and present (which we think for this reason is *real*) and inherent in the object under study. Conversely, it is not sure that the whole concept of probability was not invented in order formally to show the law of large numbers. Since the convergence of the frequency is produced by nothing except the given isolated object, the hypothesis was made that something must inhere solely and locally in the object in order to explain the convergence. Maybe so, but this virtue is not *ex-ante* (and, for this reason, *real*). It should be no less *ex-post* than the observation of matter, or the observation that the die is landed on one face, only this face could have been different and virtually different in proportions (or frequency) that reflect the stability of matter.

There is nothing formally to show in the law of the large numbers, because the law is just the translation of matter which, by definition of matter, cannot be formally shown. We say the law is not formally shown (appearances and pretences to the contrary) because it is never a formal argument that establishes that the sequence of formal random variables attach to the same material die or population. One has materially (i.e. outside the formalism, precisely as an interpretation) to suppose so, through a property, such as propensity, that one then attaches to the material object – the property that each successive draw is an independent random variable with the same probability distribution as the one that went before.

A contingent claim can be valued with probability because the implication is that, if the contingent claim is played repeatedly enough, the issuer selling it for that fair value or the buyer holding it for that fair value will break even in the long run. It is a mistake to think that the player addressing the contingent claim with such a probabilistic valuation is facing its contingent payoff in an *ex-ante* fashion and that the present value is something that is had in the present, relating to the future. The player (or the insurance company) facing the single case is only instantiating a single step (a single element) in what will be (in what will have to be) the whole set of trials and the whole sequence whose break-even shall only be observed *ex-post*. The present valuation is not really present; it is single and, for this reason, we think it is present. It is single in the sense that it is part of a population, whose only sense and meaning is that it will materialize the break-even *ex-post*. When the contingent event triggering the contingent payoff is not part of a statistical population (it cannot be repeated, or the distribution is not stationary, which amounts to the same) the only way to value it in the present is to price it; that is to say, to exchange it. This is the only true way of addressing the event before *it* (singly and exclusively of any overall population) happens, yet we do not call this attitude *ex-ante*, because we reserve the term for the play of probability and expectation, for the fake reversal of time that is done under the cover of the global account and the total population.

There is no way we could predict the event and value it *ex-ante*. Necessarily, there are events and, necessarily, time separates us from the identification of the event (the knowledge of what the event

is), yet time should not be filled with prediction or any anticipatory attitude towards the event. (Prediction or assigning a probability to the event is a present valuation.) We can only write the event; we can only invent money, invent the contingent claim, invent paper in its ex-post capacity, and invent the market which will then attach a price to that paper. The price takes place before, but it is a mistake to conceive it ex-ante, as a valuation.

The act of opening the paper at expiry and eventually exchanging it, then and there, against the contingent amount that is written on it (then and there opening it because ‘then and there’ might include a world where the paper no longer exists), this act of pricing the paper is the *same* act as pricing it in the market before. Price is indissolubly linked to the ex-post reading; it is linked to it by logic, not by time, and for this reason the ex-post character of pricing is independent of time. (This is the ex-post character of contingency, equally independent of time.)

7.2.3 The exchange and the abyss

We wish to develop the argument, really seriously, really deeply and materially (as material and genetic implication), that exchanging the contingent claim in a market, or pricing it, is the only way to value it ahead of time. It is even the only way to value it at expiry. We have invented states of the world that are pinned on nothing but the totalitarian background of the full description of the world, and the contingent claim is supposed to be ‘worth’ something, at expiry, in each of those states. The notion of state or state variable or underlying is complicit with the notion of value and valuation. Somehow, the notion of the world being fixed in its state bestows on the contingent claim the notion of being worth a fixed value in this state. The notion of state implies a whole metaphysical presupposition – it is a settlement, certainly a shortcut in thought, a useful representation or symbolization, which masks, however, the real and massive fact that the contingent claim is not worth anything fixed at expiry and settles in no state, because it is basically ‘worth’ something only to the extent that it is exchanged against money, that it changes hands and literally transfers from one person to the other, that it is literally *unsettled*.

The act of exchanging is literally overlooking an abyss. The observation that the contingent claim is worth something only to the extent that somebody different from us (not settling in the same state as us) hands us money in exchange for our claim (in ‘settlement’ of our claim) is actually admitting the abyss in. Anything could happen during that flight over the abyss: the world could end, or the counterparty supposed to pay us the money could default, or the very conditions ruling the settlement of the claim could become invalid, and so on. To say that the contingent claim is worth something in the corresponding underlying state, this metaphysical extrapolation, is abstracting away from the abyss and the contingency and, unfortunately, it transmits this false feeling of settlement and fixity through the tree mapping the probabilistic transitions between states.

It is only because the contingent claim is a material paper and not a list of abstract states that settlement of its value at expiry cannot take place unless it is submitted materially to a counterparty and unless this counterparty materially pays the corresponding money (keeping in mind the abyss properly attaching to money itself: depreciation, hyperinflation, and so on). This material exchange, bearing all the contingencies, is the reflection of there being no state underlying the paper and no

predictor of it, of there being no ex-ante attitude vis-à-vis the paper and of the paper having to be opened and read ex-post, like the last will of a deceased person, possibly after the end of the world.

Valuation theory can only put abstract mathematical numbers on abstract states. It ignores the material exchange; it ignores the abyss (or the 'anything can happen' that falls outside the context and the list of states) that is attached to the paper as its other side. Or it can only re-factor any contingency into a new state (which, in turn, presupposes a new context and a new abyss looming at its other side). Valuation theory, which only occupies itself with the contents of the paper, ignores anything that can happen to the material paper as such (default, failure of the context, end of the world) and this is so by necessity – the paper is unrepresentable and unsurpassable for it. Our thesis is that, precisely for this reason, or as the summary or crystallization of the abyssal fate of the paper, the paper is actually exchanged in a market before its expiry.

Contrary to the prescription of value due to the underlying state, contrary to *valuation* which is a purely mathematical assignment of numbers, contrary to the whole metaphysical presupposition of the state or the whole archaeology of thought which enables us even to form expressions such as: 'The contingent claim is *worth* something', there is, at maturity, the notion of *pricing*. Pricing is the attribution of 'value' to the contingent claim no longer under the regime of the underlying state, its metaphysical presupposition and the corresponding notion of valuation, no longer under the chrono-logical assignment of value $V(S)$ following the state S , no longer under this order of reading and, correspondingly, this order of thought, but under the reverse order which recognizes the mark rather than the state, or the undivided and, for this reason, unlimited strike which says of the contingent claim at its expiry that its value could have been different (and says only that).

Pricing at expiry is inseparable from the material paper and its capacity to bear the mark (up to and including the capacity to disappear) and, for this reason, it is inseparable from the abyss and from the exchange. For this reason, money was invented – in order to be handed out in exchange or in settlement of the contingent claim. Pricing, or the necessity to recognize over and above the states that prescribe value the capacity of the paper to disappear, to sustain total loss and for this reason to be opened only ex-post, is thus inseparable from the act of paying out money in exchange for the material carrier of the marks, which surpass the states precisely because of the materiality of the paper, where we understand by materiality the very property which unites the states under the same mass (no delimitation) all the while carrying the abyss as the other side of this materiality precisely.

Pricing, or the 'valuation' under the reverse regime of value which recognizes the contingency before the underlying state, which recognizes the *contingency of the mark as first underlying* up to and including the fact that the world could be different in that it has disappeared, requires the material exchange at expiry, or the actual abyss standing between the two exchanging counterparties which actually absorbs the abyss inherent in contingency and in the ex-post reading. Pricing is ex-post and, for this reason, it requires the abyss of the exchange and the attachment of material money, not of abstract value, to the contingent claim.

We have discovered a regime of valuation – pricing – which required the exchange at expiry, and now the simple observation is that *the material paper still finds itself before expiry*. It can still

be exchanged before expiry, *therefore* it is priced before expiry; it is priced today. Pricing, which is ex-post at expiry, is ex-post today, despite the market today being situated chronologically before the event, because of the exchange. This means that pricing will always exceed valuation before expiry. What value it is worth at expiry may be written on the contingent claim; still, this value will remain worthless and unsettled until the counterparty hands over the money in exchange, over the abyss (only to be discovered ex-post and which cannot be part of the contents and schedule of the paper) which provides that the world might no longer be in existence or the counterparty might no longer be willing or able to pay. Price, or the last resort and last opportunity, makes sure that value is never really settled (instead, money is exchanged bearing the entire abyss). Price comes counter to value, from the other side.

Now, before expiry, the contingent claim is also exchanged. We may have devised valuation models, or tools that transmit the reading of states and prescribe value, tools that may even factor in the disappearance of the paper under a new state (but will be vulnerable to the im-possible state which overhauls the whole tree), still, the fact remains that the paper is first and foremost exchanged and that the metaphysical or abstract assignment of value will remain worthless unless it is settled by price. We value the contingent claims with probabilistic tools (using a tree) but our immersion in the market and in the exchange is such that we only value them inasmuch as they will be priced. Price is the guarantee that the ex-post attitude has been factored in, that all kinds of things could have happened in the abyss, up to and including the disappearance of the tree or probabilistic tool, or, equivalently, the reinjection into trading of its parameters. (Price is never the settlement, the prescription, but it comes after the event, stretching over the abyss.)

The non-existence (non-persistence) of any tree, or the virtual tradability of any of its parameters that is due to the virtually endless chain of derivatives writing and trading, is eventually the same condition as the disappearance of the paper (or the end of the world). That no tree could carry the valuation of the contingent claim but that the whole market happens to it instead is the same condition as the radical contingency of its disappearance despite and against all kinds of states that may have been provided for in its contents. Recalibration is the other name of pricing. It is the reflection of the peculiar ex-post disposition of the market.

The probability tree has a wrong way of occupying time. Price occupies time in the right way, which is that price always happens after the event. Price occupies time with the event (it has never left the event), except that the circumstance of the market makes it so that price is present before. This is thanks to the paper still finding itself before, to be exchanged. For this reason, the chronological antecedence of price cannot but attract the trading of the nested derivatives; or writing on top of writing. We should dig deeper in the archaeology of that thought. The archaeology of price and of the exchange will occupy us next. There should be no other way of occupying time before the event than the market, because the market happens to be before yet was never projected before by the image of thought corresponding to the chrono-logical reading of states, and to the precision of states.

Present value is thought to occur before; the present state is thought to occur before; but in reality this illusory interval of time is filled by a projection or a reflection – by an image. It takes matter to really drag the future to the time before; it takes dealing with the full event (as time is materially

made of the event), and only the market can offer this density, because of the nested writing that it will precisely produce. As soon as price finds itself before by virtue of the exchange, it cannot be envisaged without the simultaneous exchange of writings, which, fortunately enough, *translates* (moves across) the massivity of the future event and literally drags it to the present spot.

Archaeology of the Multiple

Value appears as soon as the multiple exists. Or to be more precise, as soon as the *differentiated* multiple exists, a multiple whose constituent parts are fully identified. In saying this, we imply that one might envisage a multiple within which multiple states are not identified, a multiple which would therefore be one, understanding this unity in a sense that does not contradict multiplicity. This book is concerned with the unusual matter from which such a non-differentiated multiple is made. As far as it is concerned, rather than value we must speak of *price*.

But let us come back, for now, to the classic situation in which the constituent states of the multiple are fully identified and their *identity* is the primary value assigned to them. The notions of value and of a differential are usually associated with the notion of a many-valued function. Such a function is said to have a value, and the variation of its value relative to that of its argument is measured by its differential. The simplest function that can be defined on a set of states is the function that maps each of these states into itself – what we call the *identity* function. To be identified is to admit of a value. Identification is an act subsequent to the initial givenness of the multiple; it already implies a metaphysical choice. This is why the alternative approach which does not privilege identification must, from the start, choose a term other than that of *value*.

Identification brings the multiple from the virtual multiplicity that it was before its constituent states were identified to the realization of these states and the realization of their difference. It is as if thought ran through each of these states one after another and realized them. The identity function, which makes each identified state correspond to itself, would thus be interchangeable, under the metaphor of succession, with the *reality* function that makes each state correspond to its realization, to the fact that it exists. To each state, then, would be assigned the value 1, the symbol of reality, in an analogy with probability calculus, which assigns a probability of 1 to the possible state when the latter is *realized*. This suggests that probability emerges when time, or succession, is mixed with the multiple.

Thus, the logic of the function is the logic of assignment. It always proceeds from the underlying state to the value that the function attributes to it, the value taken by the function when it is *evaluated*

in that state. It is the act of discerning the underlying state that guides the act of discerning the value of the function and the first assignment or the first function that we have recognized is, as we have said, the one that assigns to the state its own identity or, in other words, the function that discerns the state. All a function does, in the end, is to translate a difference in underlying states into a difference in value. Even when a function is constant, it is recognized as constant only because the state variable to which it is applied is (precisely) variable, admitting of different states.

Inversely, how could we imagine a logic of difference that would begin with the function instead of beginning with the state? A logic in which, instead of expecting the function to be different (or constant) *because* the underlying state is different and instead of our thought transitioning from the state to the assigned value, we simply observed the value without the causal prerequisite of the assignment? What would the logic of difference be in which we limited ourselves to simply remarking that the value, observed without any transition, *could have been* different? We could apply this reversal to the identity function itself and, instead of transitioning from the state to its identity (what is called 'identification'), limit ourselves to its reality and to the remark that it *could have been* different. In this way, contingency would intervene before identification, and even before the notion of state. We would thus conceive that a thing could be this or that *before* conceiving of the state of this thing and the different things that it could be.

8.1 To be vs. can be

The value is different *because* the underlying state is different, and if the value is not different, it is *because* the function that assigns the value to the differentiated state is constant. The difference of the function follows from the differentiation of states – that is to say, from the identification of states. Here, the thinking of difference follows a strict chronology that leads from identification to difference. This direction of the transmission of difference is characteristic of valuation. We might say that there is value and valuation whenever the thinking of the underlying identified state precedes the thinking of the function. Since the function is an instruction or a program (to *this* state, assign *that* value), we can say that, following the logic of the assignment of value, the state always precedes writing. Here, writing is subordinated to the state; all it does is to mark and remark its identity. For we mustn't forget that the first function is the identity function, which only tells us the difference of each state insofar as it, first, affirms its identity.

In the logic of valuation, the function is written only so as to guide the value, which *subsequently* will be different. Writing is only there to serve as an answer to the question: 'What is the value of the function in this state?' The difference of the value is here only an analogy of the difference (that is to say, the identification) of the state. In the logic of valuation, we do not take advantage of the primary characteristic of writing; namely, the *materiality* of its support. In order to mark the instruction and to report the remarkable fact that a certain value is assigned to a certain state whereas a different value is assigned to a different state, in order to articulate this difference, writing has need of a material support to which to consign – in which, literally, to engrave – this difference.

Now, it could be argued, shifting ontology from the conceptual to the material, that only this engraved instruction exists, only this material writing of the function exists, and that the partitioning

of underlying states does not exist as such. Let's imagine that some cataclysm, or simply the natural effect of erosion and decay, has obliterated all borders, all delimitation between states – in other words, that the very identification of states no longer exists – and that all that emerges from the underground in which this civilization has been buried is the stone upon which the function is written. Writing is a trace, a vestige, and *is nothing else*. We cannot say that writing *is*, in a positive sense; it is only that which *remains*. The ontology of writing is subtractive; it emerges once states have been withdrawn, literally subtracted.

In the reversal of the (chronological) order of thought for which the phenomenon of the ruin provides the metaphor, written difference now comes *before* the prior differentiation of states. The value of the function detaches itself from the entire program that used to assign each value to each state in a well-organized manner. It is no longer a program or a stream of waking thought that distributes values and establishes differences: thought has received a blow that has made it lose its 'memory', so that, upon awakening, the only meaning of difference that it can grasp is that of contingency. It can no longer make out the hierarchy or the positive decision tree that links state to value. The only thing that it can perceive is an inversion of the established order. It perceives value, certainly. The function still exists qua difference; but thought no longer retains, of the basis and the partitioning that used to found this difference, but a minimal and immediate memory, a memory that says only: 'The value *could have been* different.'

Contingency is the vestige of the delimitation of states. It is the symptom of a difference which now comes before identification and which, in consequence, is no longer analogical. To that effect, it depends upon the only thing that remains materially: writing. We have remarked that the *identity* function is itself a particular function and, thus, a particular writing. When it is subject to chronological reversal in its turn and to the invasion of the foreground by the ruin and the vestige, it is no longer a matter of the identification of the state, since the identification of states is precisely what is obliterated and withdrawn, but of the *reality* of the state, independent of any other state that could be, whether real, possible, or only conceivable, and even independent of the *state* – a reality upon which is superposed (and with which is even fused) the thought that the state *could have been* other.

8.1.1 Identification and transition

It seems a difficult, even insurmountable task to make the thought that the state could have been other come before the thought that the state *is*, or simply the thought that it is a question of a state at all. We are so used to the idea that the state, the fact that things are, comes before everything else, and that it is the basis of everything. When our starting point is the already identified thing (as is ordinarily the case), it is difficult, or even impossible, to admit that identification is itself a function and that it is therefore carried out in a *second* stage. How can identification come second, that is to say accidentally, when the subject to whom the identification is supposed to apply already presupposes its identification? For our thought is itself dependent upon the logic of partitioning into identified states, the logic of assignment and value. To arrive at this sort of impossible thought, we don't need to think of another thing; we need to change thinking altogether (to change its order).

Before we think it as participating in anything whatsoever, we always think that the thing participates in itself. From the start, we confuse the reality of the thing – that is to say, the reality

in which it participates – with the fully delimited identity of the thing. Let us think instead that the thing participates in a whole world, a real world, before being identified as such, and before being separated from this world. Let us think that the thing is only a *state of things*; that is to say, a part of a special configuration or a special partition that our thought has imposed upon the world with the aim of classifying it, and according to a logic of *transition*. In reality, the thinking of the world and of its becoming – indeed, the thinking of the world *is* the thought that the world changes, or that it is what it is: essentially contingent and having emerged in one stroke – arises before the thought of the well-identified state of things, except that our thought can only represent the changing of the world in the form of a transition. Just as thought moves from the underlying state to the value that the function assigns to it, thought represents change in the form of a state that moves on to another state. The functional hierarchy (the function qua program: to *this* state assign *this* value) is but the herald of the thinking of change; and the first function that we ‘identified’, the identity function, is but the herald of the thought that the identified state will move on to the following state according to a distinct hierarchy of transitions, or a tree.

That which seems to us to happen locally, ‘on the spot’ – namely, the relating of the identity of the thing to the thing, or the fixity of the identified thing which is translated into the fixity of the value that the function assigns to it – in reality takes place nowhere. It does not take place in reality, but in the abstraction of a thought that has separated the thing from the world *in a second stage*, after the ‘first stage’ of the reality of the world and the reality of change (or of contingency, which is the only reality), so as to prepare representational thought for change. The chronological thought that assigns the value to the state, in that quite distinct order of transition, is but the germ of the chronological thought which, later on, will only be able to understand the changing of the world in terms of transitions, just as distinct and just as hierarchized, between identified states, or which will only understand its contingency as the realization of one particular identified state among equally identified alternative possibilities.

Now, the reality of the changing of the world, which is certainly far too vast and far too ‘material’ (too continuous) to be filtered by a tree of state transitions, must be our primary inspiration and our guide *in order no longer* to believe in the myth of the assignment of value and to do away with the tree, which is decidedly too fragile. An entire mass of change must succeed a preceding mass, without following the distinct path of the tree; which is to say, correlatively, that only the reality of the world will be recognized, at the same time as the reality of contingency. All that will be recognized is the (material) thought that the world *could have been* other. The thinking of difference will be practiced *retrospectively* and after the fact, cutting short the temptation to think that the difference of value is caused by the difference of prior states and prior identification.

8.1.2 The danger of abstraction and the suspension of possibility

When the underlying states are fully identified and when the algorithm that evaluates the function is well-defined, we speak of *possible* values of the function. We thus place ourselves in an abstraction wherein no value in particular is realized, but where any of them, indifferently, could be. Rather than stopping with the (unique) reality where the function takes a well-determined value (but one

that could have been different), the mind is filled with the total vision of possible states and possible values.

This disconnection from reality can only be envisaged because we are contemplating a distinct totality of possible states. And when this abstraction of states replaces reality, it does away with reality, definitively. We cannot call a particular state or value realized just because thought, later on, happens to focus on it.¹ Reality is not a choice, or a play of permutations, superposing itself upon a set of possible and indifferent states. The precision (and, thus, fragility) inherent in the distinction of states and in their totalisation should be a warning in itself. We take a great risk in *identifying* underlying states (the risk of fooling ourselves, and of missing certain states out of the total count), and this risk is in itself an indication that an irreversible divorce from reality has come about and that the reality lost will not return. Or if it does return, it will be with a vengeance, precipitating us into a state, or a situation, that will not be part of the totality we had previously conceived of.

True reality is not a reality that obeys the vision of possible states and that very conveniently comes to place itself and signal itself as a *realization*. To think reality, we must abolish the thought of possible states. It is contingency that must replace the thought of possibility. We will no longer think of the possible values of a function, we will only think that the actual value could have been different. Actuality is not a state subsequent to an initial state that was only possible. There is no initial state and no possible state. There is not even an actual state. It is the notion of state altogether that is called into question here. There is a reality, massive and undivided, which is not even identified or such that the fact that it could have been different would be a consequence of its identification. It is real by the same token that it could have been different. Its reality is not the reality of its state or of its identity, but a 'flatter', more immanent reality, which is not yet threatened by the transcendence of the state or of identity.

To say that reality is what it is, is not a synonym for *identification*, but the affirmation that it could have been different. To say that things *are what they are* ('This is just the way it is') signifies, first, that they are contingent, and is no tautology. To think actuality without either thinking the actualization of a possible *state* or even thinking an actual *state*; to think actuality without thinking of reality having confronted some kind of choice, is to think contingency. We must purge actuality of any idea of the possible and see, before the actual state (ontologically and not chronologically before) the reality of contingency.

The identification of the state, we have said, is but the prelude to transition. We imagine that the actual world is the result of a transition between a possible state of the world and a real state, what is called realization or actualization. But let us invert the direction of our gaze for a moment. Let us start with the actual world without distinguishing in it the state or states that would have just been realized after a transition from possible states that only God knows of. Let us start with the world as it presents itself to us, without distinction or delimitation of states. By eliminating the series of possible choices that the world has supposedly confronted before being realized, do we also eliminate the idea of its contingency? In seeing the world as a flat and massive 'totality' (keeping in mind that we do not mean by this a totality of states or of possibilities, something we can't be sure is even conceivable), do we lose sight of the idea that the world could have been different?

Since the world is indeed real, there is no longer any need to consider the possibilities from which it would have transitioned in order to become real. We no longer need the transition, and my thesis consists in affirming that we will no longer need the states and their identification, *because identification was only a prelude to the transition*. It is a fiction (or merely a narrative) to think that the world had to transition through a certain state in order to get to where it is. Since the world is real now, we no longer need the fiction of transition; and, consequently, we no longer need the fiction of the state, even the present or actual state. We don't need to identify the state of the world either before it is realized or afterwards, because the 'realization of the world' is itself a myth. The world is always real, it has never been realized.

But as real as the world may be, and as fictive as may be the possibilities that supposedly had to be realized in order to produce it, the world is nonetheless contingent. We can quite well think that the world could have been different *without identifying* the different possibilities that could have obtained. It is not that these possibilities might have been and that we have eliminated them after they had served to realize the world and had been used up in the process. It takes no effort to eliminate them. On the contrary, it is their existence that implies an effort of the imagination, since they are fictive. To eliminate them, it is enough not to think them.

If the idea of the contingency of the world (namely, that it could have been different) is independent of the idea of the possibilities that would have realized it and even of the whole idea of realization (understood in the sense of the transition between the possible and the real), then the idea of the contingency of the future, non-actual world must also be independent of the idea of future possibilities. When the world is not actual, it seems difficult to think that it 'could have been' different before thinking what it is. One tends to think that the world must be, and that it must be identified, before one could think retrospectively that it *could have been* different. The idea that the actual world could have been different is facilitated by the idea that the world already presents itself to us and that it is enough to think that it is not necessary. But when the world is not yet present, the chronological illusion leaves us no other choice than to think that it will be, and that it *could* then be different – that is to say, that it will be one among a number of different possibilities.

Now, if the future world is possible, this means it is not real. Certainly, it is not actual, but actuality does not exhaust reality. The world will be actualized, but this must not be understood in the sense of the actualization of a possibility. That which is real yet not actual is the virtual, as Deleuze teaches us.² Before Deleuze, Bergson teaches us that the possible is just a fabrication and a retrojection. It consists in taking the real and adding to it a clause of unreality, then in projecting it backwards so as to create the fiction that the possible will be realized. I would add that the possible is a *variation* on the theme of the real. We identify a certain state of things, or a certain number of states, in the real world, and we imagine that in order to describe possible worlds we need only to vary those states. Doubtless, we hope thereby to explain that the world is contingent. In order to explain that the world *could have been* different, we generate a (factitious) list of alternative possibilities on the basis of the present world, and project them backwards to the factitious epoch when the actual world was ready to be realized as one of these possibilities.

In order to think that it *could have been* different, we get diverted into the thought that the world *could be* different. The possible has not been invented and 'designed' for the future, but for the past. The fiction of the possible is created so as to explain a clause of change in the past and to deploy the story of what the world could have been. The danger is that the possible will then slide inexorably towards the future. The future world, supposed to be as real and as massive, as difficult to discern in terms of delimited states as is the present world, now becomes a possible world, and the *absolute unpredictability* of the real world (its invisibility, its indiscernibility), which we understand as its *irreducibility to partitioned states*, is transformed into a relative unpredictability, that of a choice of the possible state that will be realized.

The non-identification of the real world (an intrinsic non-identification that prevents the value of the identity function from being applied to it) is transformed into the (external) non-identification of the world that will be realized among a list of possibilities. The inscrutability of reality (the genetic condition of the real which only the market can transmit and translate) is transformed into the vision of the tree and the supervision of the conditions of possibility. Whereas the future should be thought on the basis of the genetic condition of the real, through the immediate, immanent mediation of the market (this is the definition of the market), we divert it from the real and think it under the category of the object, reviewing its conditions of possibility as they spread out into a tree of possibilities.

8.1.3 0 and 1

Probability calculus *evaluates* reality. It assigns the value 1 to the real world and the value 0 to the possible world that has not been realized. According to this logic, possible worlds are perfectly symmetrical and totally permutable (since they are all *equally* possible), and the difference between 0 and 1, which is only a difference in value, expresses no *real* difference between the world that has not been realized and the world that has been. The numerical scale is homogeneous, and precisely what probability calculus does is to introduce a perfect continuity between 0 and 1, the number *p* of probability, which the possible worlds will equally and symmetrically share.

But the number 1, when it is assigned to reality – what we might equally call the *unity* of reality – should not be a value and should not be a number. It only becomes a value on the numerical scale that associates it with the number 0. The number 1 should not be the result of the *identification* of reality, which is what it becomes when it is associated with 0 and when reality is identified in opposition to that which is not realized. Identification, as we have said, is itself a function, and the difference in value between the realized world and the non-realized world becomes an analogical difference – that is to say, an external difference – when the arguments of the function, or the possible worlds that are variants of the real world, are identified first. On the contrary, the *difference* of reality should have remained internal and shouldn't have been translated otherwise than via its contingency. These possible worlds should have never been identified.

Once distinct and delimited states of things are identified in the real world, and once possible worlds are conceived (literally imagined and projected) as fictive variations of those states of things, the unity of the real world, the 1 of reality, becomes a number. It is for the reason that these possible worlds, which are fabricated as the variation of illegitimately identified states of the world, must be

compared with the real world (indeed, this is the whole purpose of their identification) and must mark their difference from the real world – the analogical difference which says that they cannot coexist with it – it is for this reason that they receive analogically the value 0 and that this value transforms the unity of the real world into the number 1 which is then opposed to the number 0. Thus, the true symbol of the identification of reality is not the number 1, but the number 0. It is through its association with 0 that the 1 of reality becomes a number.

The unity of reality should have never become a number or a value. (It should have remained a 'meaningless sign,' a valueless and therefore unexchangeable sign, an incision marked in place, a monetary sign that knows no scale, no background to which it could be related apart from the material exchange and the market – the miracle of writing.) Reality should have never been identified; and to say that it is *one*, that it is what it is, should be equivalent, with no distinction or identification whatsoever, to saying that it could have been different, or again, that it is contingent. The symbol of reality, 1, should have kept one sole sense, without any diminution and without any scale – without any reduction – the unique sense of contingency. The probability number p , comprised between the numbers 0 and 1, is but the derivative product of identification and evaluation, and only comes about so as to *continue into the past* the numerical symmetry or exchangeability that was illegitimately established in the present between the real world (now supposedly realized) and the possible worlds (supposedly unrealized) that are the fictive present variation of it.

The number 0, symbol of present identification, breaks apart the univocity of the real world's contingency. It introduces into the unity of the real world the equivocity of number and the symmetry of the numerical scale (its continuity and homogeneity). This is expressed in the symmetry and the impossibility of the number p of probability, once we artificially recede into the past and once the real world becomes numerically and quantitatively *as possible* as its illegitimate variations. The number p of probability is the *generality* of the illegitimate evaluation and identification of reality. Qua generality, it will be well-suited for scientific calculation and for the mathematical function, but improper for repetition, which is characteristic of the philosophical concept and which alone can translate internal difference, or the event.³ (The market, or *price*, will be the philosophical concept of contingency, the concept that does not reduce its difference or generalize it, but repeats it.)

The generality or impossibility of probability is expressed in a temporal symmetry: just as probability *continued* into the past the identification of present reality and completed the becoming-number of the 1 of reality through the number p that is less than 1 – for the fabrication and the identification of the possible worlds are only completed once they substitute for the initial reality the *synthetic reality* of the world that is artificially reconstructed in the past, a world whose reality is the sum of probabilities – soon, probability will extrapolate this identification into the future. Having been the derivative product of the present identification of reality and its evaluation, probability becomes the only possible evaluation of the future. The future world loses its reality (the univocity of its contingency) and becomes a possible world. Consequently, probability (or prediction) begins to seem like the primary mode of apprehending the future, when in reality it is only the derivative product of the illegitimate identification and evaluation of *present reality*.

8.1.4 The real future

The future world should have remained *just as* unpredictable as the present world – *unpredictable* in the sense that its states are indistinct and non-identified – rather than seeming *more* unpredictable in the sense of relative valuation. In other words, the future world should be just as contingent, *and consequently just as real*, as the present world. Just as the present world is confused with its actual state, and its reality confused with a valuation, so the future world is confused with its possible state and its real contingency confused with an indetermination between many possible worlds (of which probability is then the valuation).

Now, we must seek for the future world the same *absolute valuation* as we found for the present world, when that valuation was understood as an absolute function, as a difference that is absolute and internal and not analogical or relative to underlying states. In the same way as the unity of present reality was the absolute recognition of the fact that *it was what it was* – which is to say that it could have been different – and not the relative and analogical recognition of what it was on the basis of the identification of its state, we must recognize absolutely the unity of future reality – that is to say, we must recognize absolutely that it could have been different, prior to recognizing the possible states that would analogically explain that this is so.

We have called *price* this absolutization of valuation, this absolute recognition of the difference of value that no longer depended on the algorithm of assignment, or on the function that had been buried under the rubble. In the same way, we must find a *price* (or an ‘absolute value’) for the future rather than, and instead of, the relative valuation of probability. The difficulty is that, whereas present reality is actual and its contingency can be easily comprehended as a non-necessity, future reality has only its possibility as a room of variation. The ability to *price* present reality (the recognition of the fact that it could have been different, which occurs before the recognition of its state) was made easy by its actuality, but it is difficult to see what could enable us to *price* future reality before its actualization.

The real, but not actual, translation of the contingency of future reality can only take place in the virtual. We shall call *market*, or *market price*, this absolute translation of future contingency, this anticipated *price* which cannot yet benefit of actuality in order to come about and which, for all that, does not come about through the intermediation of possible states. The whole difficulty comes down to forbidding ourselves from collapsing this anticipated virtual into chronological time. Qua anticipated, it must undergo the inescapable test of time. And the latter will not fail to *actualize* the virtual in the form of a tree of possibilities (thus, the possible is the *actual* for this anticipated virtual) that will then separate us from future contingency.

We must make a constant effort of counter-actualization, or virtualization. At every instant that will separate us from the date of the future contingency, this virtualization effort must resist the positive logic of valuation and relative attribution, just as *price* resisted *value* when reality ‘was’ present and not future. This resistance will be all the more difficult, before the future date, as the tree of possibilities and the corresponding probabilistic valuation are the most natural temptation. Indeed, we shall see that the market will almost be defined as a movement of perpetual resistance against probabilistic recuperation and as the resistance of price against valuation.

Just as the *price* of reality is the critique of the identification of its actual state and of the identification of the possible states that would explain analogically that it was contingent, just as price is the key critique of possibility which retires the parasitical machinery of delimited states and keeps absolute contingency from being mandatorily defined as the disruption of a pre-existing range of possibilities, just so price will be the key critique of possibility in the environment of the market, once we realize that the price of the contingent claim can never be recuperated by the probabilistic valuation.

8.2 Chrono-logic

The identification of reality, or its foundation by the underlying state, diverts the univocity of contingency (the 1 of reality which signifies only one thing: that reality is what it is only insofar as it could have been different) into the numerical equivocity of 0, into the opposition between that which is realized and that which is not. In this respect, the function of identification is *chronological*. By that, I mean that it follows the logic of the tree and of transition. Even if reality is present and fixed, even if nothing ‘happens’ any longer or has ever happened in order to realize it, even if it is an eternal reality that we are talking about and no transition has ever taken place in order to lead to it, I claim that the identification or recognition of this reality, the gaze that converges upon it, doubling itself and recognizing that it *is* (in other words, the function that attributes the value 1 to this reality, or which attributes the value ‘to be’ to its underlying state), that this mode of presentation of reality *produces its being within a logic of transition between states*. It is as if this reality was inherited from or originated in the cause that presented it. It is as if the logic of time and transition, in other words *chronology*, subsisted in the recognition of this reality, despite the fact that time is the present time and this reality is eternal, despite the fact that there exists no interval of time or passage of time.

To think reality as *one* under the aegis of the realization of a state is to think of a transition and an inheritance and, thus, of a tree logic which is only analogical, since the attribution of the value 1 only entertains an analogical difference with the value 0 and the realization of reality *by virtue of its state* differs only analogically from its non-realization by virtue of ‘another’ state. The tree of possibilities, or the identification of states, renders affirmation and negation symmetrical, when affirmation was supposed to be the only real thing.⁴ In the end, we only think the state so as to sustain a transition; that is to say, so as to bring onto the scene a past memory and an expectation, an objective thought (both analogical and lingual) of undivided reality that no longer conceives it except under the aegis of identification and transcendence – that is to say, under negation. For, to recognize that reality is what it is – *this* state – by virtue of *this* identified state, is to recognize that it is *not* some second identified state. This analogical and evaluative unity of reality, a sort of unity founded by the reciprocity and, thus, the equivocity of the state, is different, as we have said, from its *contingent unity*, from what we have called its absolute valuation, whose sense (univocal, admitting of no other face and no reciprocity) is that reality is what it is *only insofar as* it could have been different. Indeed, the contingent unity of reality does not follow the logic of attribution and transition, since reality is not, in this case, constituted or focused around any state that would

realize it, but is literally dispersed (albeit remaining one) in the thinking of the *other* that it could have been – the *other* of reality that is no more identified or crystallised than reality.

Consequently, it seems logical, in the schema of attribution, that when time *effectively* interposes itself between the state and reality and, for this reason, suspends the identification of reality, the transition between the state of reality and the value of reality spreads out into a tree of transitions between one state and its following state. The tree that remained imperceptible in the valuation of present reality and did not give any hint of a transition or an inheritance beneath reality as given in its present state – that is to say, beneath its identification – becomes visible, and even invasive, when reality is suspended in time. This speaks in advance of the hard time that price will have in trying to resist the probabilistic valuation function.

8.2.1 Probability as an integral

For here is another definition of probability: We said that value appeared along with the multiple when the question was one of *evaluating* a function in each of its identified states. The writing of the function that attributed the value then resulted only from analogical difference, since the function varied and differed only by virtue of the variation of the underlying state. The value was what it was only to the extent that the underlying state was *recognized*, and recognized to be different from the other possible state. To be recognized was then to be recognized via the analogical difference from the neighbor (whereas undivided reality, which is one because it is contingent, has no need of being recognized or compared in this sort of reciprocity; it is not a *result*, a memory or an expectation; it is incontestable). Because of the symmetry between the identified states, this meant that reality was not one ‘in reality’, but presupposed the givenness of all the alternative states, to which the value 0 was attributed at the same time as the value 1 was attributed to reality.

The writing that distributed the value of the function thus only *slowed down* the value and made us wait for it – this is what we have called its ‘chronological character’. For analogy is only a waiting for a result or a verification: waiting to verify that we are, indeed, in this state and not in the neighboring state. The writing of the multi-valued function is thus a mediation and a slowing-down, a detour and a *derivation* that it seems natural to want to replace with a return to the present. For it then seems natural to claim the value of the function no longer under the aegis of the differed or verified state (that is to say, one distributed by writing, like by the postal service) but that of a *present state*. Thus, analogical difference and valuation by identification and recognition bring about, as a natural reaction, the necessity of a primordial unity wherein we ‘fuse’ the states back into each other, and no longer recognize them as distinct.

It is their analogical symmetry, once more, or what might be called their ‘indifference’ or their exchangeability, that logically requires going against difference, in search of that *present state* that would transcend the different states. Rather than each of the states being analogically recognized by the value 1 that its identification would attribute to it and by the series of 0s that it would attribute to the other states, all of the states will now become united in the *present state* alone – or, rather, they will be grouped together in the notion of succeeding it. The analogically different states become different states which *the present state could equally be* – thus, analogical difference is founded upon identity

and presence – and, in this way, make concrete the transition that remained implicit in the tree of attribution. We thus speak of the *possible* values of a function; the symmetry and the indifference of attribution are thus incompatible with the genetic conditions of the real (which is the contingent and the ‘it could have been different’). Into univocity and immanence, we thus introduce mediation.

The difference between states becomes a function of succession which attributes a probability to each state on the basis of the present state that precedes it, and this probability becomes a summary, for each of the states, of the composite recognition that up to this point was constituted by the value 1 and the series of all the 0s that accompanied it.

As a matter of fact, the exchange of the unique mark of contingency against the identification of possible states, or the exchange of the 1 of undivided reality (admitting of no 0s) against the numerical couplet formed by the 1 and the 0 of the identification of the state and the reciprocity of states, is only the first step towards the *objectivation of probability*. Probability is objective not because it is not subjective, but because the exchangeability and permutability of the identified states is a necessary condition of objectivation and of *objectivist language*.

To objectivate probability and to think the transition between the possible and the actual is to think the permutability of states – to think that *one or the other* of them might be realized indifferently. By contrast, the mark or the strike of absolute contingency depends on the particular order and on the path followed. It is unexchangeable in the sense of Baudrillard, when he says that the world is unexchangeable.⁵ So, imagine that the realized state were always the only possible state (that is to say, it is necessarily realized in some way) and that it did not rub shoulders with other states that would be exchangeable with it (which means to say that it is not itself *identified* in the mass of the real, for to identify it would come down to identifying its variation, or the other states with which it might be exchanged); but imagine that each realization of this ‘state’ (a non-identified state – hence the scare quotes) were in reality an event, which disrupted the ‘set’ of possibilities inside which other states would have coexisted with the one that were realized – if indeed it were even necessary to imagine such a set and such a plane. Imagine, in other words, that the process were the same as that of Pierre Menard’s writing, in the sense that the context would be disrupted every time and the realized possible would be unique every time, like the *Quixote* that Menard is writing, but that it differed from Pierre Menard insofar as the realized possibility would be new every time and not invariably equal to the *Quixote*; imagine, quite simply, history as it would normally evolve towards the future and not remain subject to the constraint of a past text, as in Menard; then such a process could not submit to objective probability, or to the objectivist representation.

Probability is thus an average – that is to say, an *integral*. There are infinite ways of summarizing the series composed of a 1 and many 0s with a number p that is less than 1. The only constraint is that these numbers p , which, for each state, sum up its analogical difference from the other states under the sign of succession to the sole present state, must add up to precisely 1 (the unity symbolising the reality of the present state). Similarly, for the multivalued function, there are infinitely many ways to calculate its weighted average as the integration of its values against the distribution of probability chosen for its underlying states. Thus, our unitary reflex when faced with the multivalued function

and its algorithm of attribution, or the writing that defers it and makes us expect it as the result of a comparative and analogical difference, is to *integrate* it.

If we follow Derrida in believing that all writing is a dislocation of the unity of the present, then the natural reaction when faced with the writing of a function that would make us diverge in the difference of each attribution and each instruction, or when faced with its *dia-gram*, will be to seek to grasp the function in one single bloc and to speak of its average, or present value, or integral, rather than its value at each point. Certainly, the average is not capable of expressing the richness of the function's variation; this is why successive integrals, called its 'moments', will end up replacing it as dual representation. Despite this, what matters here is the reflex of unity and of the integral, that of capturing the function as one single integral 'variation' rather than as a valuation at each point. Although the series of moments of a function characterises it uniquely, it does not give us the value of the function at each point. It is simply a question of another way of thinking the multiple: as integral and non-derived.

Having said this, this way of thinking integrally rather than differentially will remain dependent upon the identification of states and upon the evaluative diagram, since it is its contrary reflex. The integral here is but the other face of analogical difference and of the symmetry of states, the other side of their nature as *possible* states. It repeats, without the derivation of the particular state, the unity under the aegis of the state (whence the necessity of the present state; whence the whole trickery of succession and transition). It removes from the particular state that which is easy to remove, analogical difference, and retains only the present of the state and the integral. The indifference of the particular state is recapitulated by a probability (by a present state of indetermination and irresolution, which for this reason is unitary). We wish to represent the function before the derivation and we invent the integral, or probability, as this present stance, preceding the derivation. Yet, it is not the reality of the value that we anticipate in doing so, but only its indifferent possibility. Indeed, this is totally different from the anticipation of the non-indifferent and non-permutable real which *could have been* different. The real was never possible (in this sense, it is unexchangeable and necessary) and yet it is contingent. Actuality is incompatible with possibility, but is compatible with contingency. Actuality and contingency are both real. We need a medium that would maintain the reality of contingency by separating it from actuality, but without mixing it with possibility.

8.2.2 Chronology as a simulation of chrono-logic

Thus, probability falsely seems to be mixed with time and with the expectation of an event or an outcome to be 'drawn' in time (or with our ignorance of the outcome and our expectation that it will be revealed in time). There is no more an interval of time and of duration or expectation in the probabilistic transition than there is between the present value of a function (its average) and the multiplicity of its values. The present value, as we have said, or the integral of the function, is but the natural reaction against the deferral and the delaying of value by analogical difference and by the diagram of the function – in other words, against the pro-gramming of the function by its writing. (Time does not exist; all that exists are the different ways in which we look at writing.) With difference being analogical and the attribution of value proceeding only via the recognition

that the underlying state is *this* fully identified state and not *that* other, just as fully identified state, the unitary and integral reflex then replaces analogical difference with a present state and a present value which 'simulate' the different and analogical states in a logic of succession and transition.

Now, this succession is precisely only a simulated one. There is here no passage of real or physical time, but only that of merely metaphysical time, that which substitutes the present state for the different *and* analogical state (present = different + analogical). That which seems to proceed randomly in time, the stochastic process or the 'random draw', is in reality only the accidental introduction of time, as if from without, into this operation of integration. It is a mere coincidence if the present state, which supposedly integrates analogical difference into the idea of succession, is confused with the chronologically present state of a random system (or random function) that will seem to us to take its value in a second stage, or to realize it.⁶

There is nothing more in the idea of probability than the confrontation between the one and the many, between the integral and analogical difference. (By contrast, the confrontation between price and internal difference, or the confrontation between the multiple that is not differentiated in identifiable states and its translation, will be something entirely different.) Before throwing the dice or spinning the roulette wheel, we are simply confronted with the multiplicity of the outcomes. The probability of a given face or number is only an anticipation, in the form of an integral and a present value, of the multivalued function whose analogical difference attributes the value 1 to the face or number in question and 0 to the others. Just because we see the die *before* it falls on one face, or the roulette wheel *before* it stops, this does not mean that there is a probabilistic *physical* transition between this prior state and the final outcome. Nothing links the moment when the die stops rolling or when the roulette wheel stops spinning with the moment that chronologically precedes it. It is just that the moment that precedes the realization of the multivalued function is similar to the *integral moment*, also called the present state, the moment when the multiplicity (or the writing of the function in the sense of attribution – also called *random variable*) is summed up by one single state, and analogical difference is simulated by a succession.

The whole image of determinism and indeterminism doubtless owes its origin to this coincidence or this similarity (which engenders an illusion) between the temporal succession in which we unavoidably (but accidentally) live and the logical succession that requires us to sum up the diagram of the function and its analogical difference in a present and 'preceding' state. The impression of the random draw and of expecting its outcome (the impression of a random *generator*, or more generally of a *causality*, whether deterministic or probabilistic) can even be said to be the result of our objectivist language, which tends to separate reality into delimited and identified states. As in Bergson, it is language that creates the tree structure and the transition between its states – that is to say, the impression of memory and of expectation.⁷ Thus, objectivist language would be essentially probabilistic, evaluative and integral, in the sense of the unificatory and presentative reflex that is provoked by the vision of analogical and programmatic writing. Objectivist language is a natural presentative reaction in the face of the multiple, which sees in it only analogical difference. This is why it is only able to extract a probabilistic *rule* (Born's rule) from quantum mechanics, and sees in it only a superficial indeterminism and a random generator which, this time, it believes to be

inherent in nature, when quantum mechanics is, in reality, bound up with the more profound and more irreversible strike of contingency – the one which only the market will be able to convey, and which only price will be able to translate or transmit.⁸

The image of the die whose winning face we await, or of the lottery whose result we await, is thus only a re-presentation, or metaphor, or simulator, of the present value that I have called 'logical'. It brings into play an irresolution of the value of the function, an expectation doubled with an impatience, precisely, a need to evaluate the function before having to penetrate into its diagram and into the partitioning of its analogical difference. The interval of waiting is nothing but the best way to say that we 'do not know', that we are suspended in the moment of indifference in relation to values and to different states, a moment quite rightly called the 'present state'.

Ultimately, the probability theory made mathematically rigorous by Kolmogorov in 1933 is but a theory of the measure and integration of functions. There is no place in it for the interval of physical time, the tension that comes from waiting for the random draw, or the irresolution that comes from ignorance of the outcome. It is the dialectic of the multiple and of integration (of the function and of analogical difference) that reigns alone here. And when time is introduced into it and the theory of stochastic temporal processes comes to crown it, it is not at all a question of real and creative time, of a time bearing with it contingency and radical emergence, but of a 'logical' time which does nothing but deploy and extend the logical succession of states that we have already described.

As for physical probability, which seems to inscribe itself in (real) physical time, it simply profits from a coincidence between expectation and irresolution. Faced with a draw that can be repeated, it is natural that the logical average, calculated in the present logical time, should be identified in principle with the average of empirically observed results and that the probability called 'objective' or 'statistical' should be identified with statistical frequency. We sought an average, a present value to pronounce *before* the derivation and the deferral of the multiple; so, it is natural to confuse it with the average that is observed *after* the realization of the multiple. Here, we take advantage of a reversibility of time (and, thus, an unreality of time) occasioned by the possibility of repeating the random trial. We take advantage of the fact that the random draw is a good *a priori* integrator of the multivalued function. There is no difference, we think, between the present state, which precedes the draw and which substitutes a prior indifference (from which the present value issues) for the analogical difference of the attribution, and the ex-post state which sums and integrates the realized differences in the indifference of a mean that is calculated after the fact – that is to say, a frequency. It is no coincidence that the method of random simulations called Monte Carlo is one of the best ways numerically to evaluate the integral of a function.

Because of repetition, or rather because of the idealization which posits that the draw repeats itself identically, time is abolished. Thus, there is truly no distinction between before and after. It is really an intrinsic and a priori property of the chance set-up that objective probability seeks to express, a summary which, in this case too, precedes the realization of the different values; that is to say, an integral. We then believe that we have at our disposal more information concerning the function than the mere desire to sum it and integrate it mathematically: a certain dissymmetry in the dice, a bias in the roulette wheel. We remember that the function (or the 'generator' of values)

is, precisely, physical and not purely mathematical, and its bias seems to us a far more correct way to grasp it integrally. Thus, among the infinitely many ways to integrate and attribute a distribution of probabilities to its different values, our preference will lean towards a particular distribution that will be given by the empirically observed frequency.

If it could, objective probability would prevent the time of the repeated draw from passing. Objective probability is a need for a priori integration like any another, with the extra distinction that it seems to describe a priori something more than a mere mathematical multiple. What gives the impression of *expecting* a generator, in the case of a random draw, and of submitting to its bias or its propensity – this reification – is but the anticipation of an average which one knows will establish itself over the course of an ideal experience of the repetition of the draw, but which in reality is only a logical summary like any other, only one that brings with it a distinctive supplementary sign. Evidence for this is that while objective probability could be distinguished among all possible integrals by its coincidence with the empirical frequency observed up to a certain point, it will never have anything to say about the ‘real’ probability of the next singular event, or the next draw.⁹ In other words, we will have at our disposal no means whatsoever of refuting the proposition that the probability is in reality very different from the frequency observed up until now, and that it is only at the end of a far longer period than that of the experiment conducted thus far that the real frequency will adjust to the real probability.

All of this just to say that it is when considering the single-case event that we should seek a sense of objective probability that would suggest something more than a simple arbitrary average. We are then confronted with alternative possibilities, and once more with a function admitting of many values; and the whole question is that of knowing whether the present stance, that of a present state which precedes the future realization, is only an illusion and a simple coincidence, or whether it signifies something more than the simple reflex of mathematical integration. Is the present ex-ante state physically linked to the state after the event? Is the ex-post state really the *future* of the present state (in the sense of a passage of real time and of a causality, not simply that of a logical time and chrono-logic)? Does the probability of the single case owe itself to something other than the need to substitute a radiant present state (a state of knowledge and of present value) for a multiplicity of analogical states?

8.3 Accounting for the event

It is remarkable that the only link permitting us to place the future result of the multivalued function in the real *future* of the present state is the link of action or of decision. Doubtless, the metaphysician will maintain that the probability of the future event is inscribed somewhere in the present world, and that if an empirical statistical population cannot repeat it in this world because the event only takes place one single time, then the event will at least be ‘repeated’, along with the whole history of which it is a part, in a repetition of the world which we call a ‘possible world’ and which is no less real than the present world. Still, this metaphysical and contemplative probability will not become pressing in this world unless it has some influence within it. Now, can it influence it otherwise that

in influencing the decision, and therefore the action, of a subject? We certainly could suppose that the real probability of a future event participates in the present 'mechanics' of the world in the sense that, if the future of this world is bound to unfold spontaneously and following its sole mechanics, then it would be different if the probability of the event were initially different. But can the event influence the mechanics of the world otherwise than by happening or by refraining from happening? And will its probability then be something other than the expression of the 'threshold' (measured on what scale?) beyond which the event will take place? In other words, won't the probability end up referring back, once more, to different repetitions of the unfolding of the world, or to possible alternative histories, of which it would simply measure the metaphysical frequency?

Once more, no one can prevent the metaphysician from believing in the real inscription of the present probability of the future event. Inversely, no one can deny the existence of statistical frequencies. As long as the event does not take place, we can bask in the contemplation of the value or the integral that precedes it as much as we want, and assume that this value or integral is an integral part of the world. Conversely, as soon as the event takes place, we can start monitoring its frequency if we convince ourselves that the circumstances in which this same event would be repeated could be reproduced. But neither the *a priori* integral nor the *a posteriori* integral is important. What is really important is probability insofar as it mediates the *passage* between the present moment and the moment of the event. Metaphysical probability and objective frequency are both static measures which know nothing of the *passage* of the event – by which I mean the fact that the event comes to *pass*.

The hypothesis of the physical random generator is just another integrator of the multiple and, for this reason, coincides with the present state, only it is a little more distinguished than the others, as we have said. To the present state that precedes the derivation or deferral of the multiple (the diagram of the function) was added, in the case of a physical function capable of producing a multiplicity of physical and not only mathematical values, a present dissymmetry (a physical *form*, or rather, the dissymmetry that introduces itself in form and that we call *matter*) of which we thought that the statistical frequency was the best way to integrate it retrospectively. When the dissymmetry in question seemed too impure (owing to an accumulation of irregularities, a bad specification of initial conditions, and so on), the case of quantum mechanics and of its pure and elementary probabilities (as elementary as the elementary particles) was welcomed triumphantly. Objective probability could finally be calculated *a priori* (Born's rule) and not only estimated *a posteriori* on the basis of frequency. And yet, the question of the passage between the present moment and the moment of the event remained unbroached. For it was still logical integration that we were talking about. What crucial information did the random generator really supply concerning the single-case event, apart from its agreement with a frequency that would be observed if the event were to be repeated? (An event is always singular with regard to *its* probability.) There is nothing more to the irreducibility of quantum probabilities than the irreducibility of the multiple. Here is a physical function that is irreducibly indeterministic – which is to say that it is irreducibly multivalued – and this has nothing to do with the possibility of repeating the draw. The function will take its value singularly, and we are singularly in the present state. So, to repeat the question: What does

its probability bring us, over and above a mathematical integral? How does this probability operate through the passage of time?¹⁰

The only way physically to link the present state with the event is through the intermediary of a contract. What we must hold in our hand today is not an abstract number (probability), which we do not know how to put to use at the moment of the event and whose only significance will be that it used to measure a threshold in a series of metaphysical repetitions of the world that only God knows of; and it is not an abstract number, either, which we must retain long after the event has occurred, long enough for the series of hypothetical repetitions of the event to unfold and to record its frequency. Rather, the number we must hold today must be concrete enough to be able to *count* tomorrow, at the moment when the event occurs. The best proof that probability did not operate through the passage of time was that it made no difference, at the moment of the event, whether the event had been judged more or less probable the moment before. Time was not physically invested. It did not share itself out differently depending on whether a more or less probable event had taken place. Whereas, if we found a way to make a little ‘amount’ of the future time of the event available before the event and to recuperate the whole time of the event at the moment of the event, then the event would take part in the time that precedes it by way of a transaction in which the medium of time will no longer be logical and detached, but physical and shared.

This time turned physical is called ‘money’. Money counts. It is material, it is an invention, even; and, for this reason, it is not metaphysical – it is not a time that was there all the time. Not that it might replace time in any physics where time is supposed to count, but perhaps it will replace it in the regions where the future is supposed really to take place and the event really to come to pass: where it can make a difference today that the event makes a difference the day after, and where it will count, at the moment of the event, that its probability could have been this or that other different probability.

And, first, the time of money (physical time, or, as we will later call it, ‘material time’) could never be a time where nothing happened. Money is precisely designed to negotiate contingencies. After all, the game of probability was not sufficiently *interested* (*intéressé*). The time of money cannot be defined in itself, detachedly and prior to the contingencies that come to pass in it and are negotiated in it. The time of money is not an absolute time. Money only counts because it makes it possible to invest a certain amount of money in order to hold a contingent claim that is written on a certain event and to recuperate a larger amount of money upon the realization of the contingency in question. In other words, the time of money is defined *after* the ‘synchronicity’ of the market, or that which renders synchronous, in the sense of the proper time of the market, the price of the contingent claim and the final event triggering it.

There was a total separation (and a sterilization) between the time of probability and probability; this is why we couldn’t say what probability *did* through time. By contrast, the contingent claim is written *only insofar as* it will pay something at its maturity and will then be exchanged against a quantity of money; and it is written *only insofar as* it will be exchanged against a quantity of money before its maturity. The exclusivity of writing is one and the same. This is its contract and its market. It is one and the same ‘only insofar as ...’. This is the equivalence of two material marks: the final

written contingency, the engraved matter, and the mark that is transported by the exchange, the exchanged matter or, literally, the market.

It is writing that renders time material and shareable (in the sense of the share and of participation, in the sense of investment). The number of probability is a number only insofar as it is used to compare the probability of two events, or to weigh something before the event and after the event (for we feel that something must count depending on whether the realized event was more or less probable). Money is the number (the *numéraire*) of writing, and it alone permits the materialization of the time of contingency and of the passage of the event. We write the multiple; we materialize through writing the diagram of the multivalued function. But what writing then prescribes and attributes is not a value, but a price. Value is not realized in the abstract but in the concrete of the exchange and of the place. When we look at the realization of the function from the point of view of price and of money rather than that of value and of the abstract number, we no longer consider the realization of a possibility; realization is no longer symmetrical and indifferent; it becomes real – that is to say that realization is immediately doubled (and even is taken over) by the realization that the value, now called the *price*, *could have been different*.

8.3.1 Money and the other face of the event

To regard the value taken by the payoff function at expiry in view of exchanging it against a sum of money (writing thus being stretched towards the exchange and the realization of the payoff) distances it from possibility and imprints it in place; which means that we are now reversing the direction of reading, from the diagrammatic attribution of value to the realization that the price *could have been different*. It is a question of a conversion of the gaze: of now seeing the written programme (the payoff) *as giving rise* to the monetary exchange. In order to pay off, this has to be written as a contract; the irreversibility of the monetary exchange abolishes the reversibility of possibility and diagram, especially when money is associated with the irreversible loss of money. When writing is envisaged in this way, as the opening of the testament and the realization of what one must pay, then it is materialized; it is contracted towards its matter and material support, as opposed to its contents, and it becomes a contract. We then realise that this contract ‘remains’ written before its maturity. (‘To remain’ is here opposed to ‘to be’; it is a prior vestige.) The same written matter that realizes the function as an irreversible exchange at maturity lends itself to an exchange *before* maturity because of this very materiality that *remains*.

The speed of writing is independent of the chronological frame of reference. When it is written, then it can only be opened at the end like a testament, when it is too late. Hence the invariable speed. Hence the irreversible strike that possibility cannot catch up with. That which is written irreversibly (qua monetary matter which at maturity breaks with the reversibility of possibility), that which is thus engraved into matter, is *continued*, before maturity, in the exchange and in the market. At work, here, is an argument from the continuity of matter and from the geodesic line extending between the market price of the contingent claim and the event triggering it. That which is written in view of monetary exchange is only written so as to say that it could have been different. This is the signification of the strike: contingency without the analogical state. With this sense of the mark

or the strike, a sense which allows it precisely to admit of a price, we retreat into a medium that is independent of possibility, of value, of the integral, and so on, a medium which reproduces *before time* the unicity and non-analogical univocity of the strike, which anticipates contingency, leaving the 'too late' for possibility; and this medium is the market.

The strike of contingency (without possibility) is pulled back in time without changing in any way the fact that time is over and possibility is over; it translates – literally, drags backwards in space – place (where the strike of contingency, the 'It could have been different' is marked) and not time, in an initial exchange of thought which no longer leans upon the state but traverses it, which no longer follows the dimension of time but the dimension of place. The place of the market *arrives* (before time) through this translation of writing and this nature of money, through the speed of writing which takes over possibility.

Money, the contingent claim, and the market are all three created together, in order to respond and to correspond to one another. Metaphysical time, supposedly eternal, kept us from thinking that it might have been created *for* probability. And yet, if such an affinity and such an adequacy had been demonstrated, that would have resolved the enigma of the passage of probability in time: of what it does in time. As for money, if we consider that money is the only 'time' that passes between the maturity of the contingent claim and the present time, it will be easy to think that it has been invented for that sole purpose: for that consistent play.

The contingent claim takes a little 'time', that is to say a little money, before expiry; and it takes all of time (or rather, it takes all of place, it *takes place*, it happens) at expiry. If money is truly to replace time, we must no longer think that this operation happens in time, but only *in money*. Only money passes between the 'present' moment and the final contingency; and this passage of money outside of time and independent of the dimension of time takes place in the market.

Money passes *as the only time* between the present moment and expiry, but the only projection that we have of it in time (for we live accidentally but unavoidably in time) is the market. The time that passes between the present moment and the final contingency is not chronological (or logical); it is the time that passes between the two faces of material writing, that which separates the face upon which it is *marked* that the world could have been different and the face where, through this marking and this materiality, the paper upon which this mark is made is exchanged. This could be called the interval between the 'moment' (the face) where contingency is resolved and the moment where it is not yet resolved – that is to say, that we have retreated from its resolution elsewhere than into time: we have left for elsewhere, for the outside of time to which the 'It could have been different' appealed.

The event gives us the impression that it takes place in time: we wait for it to happen. But if the event happens in time, then it would mean that it is detached from time; that time is but a pre-existing theatre for it. Whereas with money, the interval is not temporal; it is only a separation between the two faces, which is not even a metrical separation. It is the separation that says that there is a face and that there is another face; that the written matter which bears the mark 'It could have been different' is the same, *through that very matter and through that very contraction (through that very contract)*, as that which is retracted and withdrawn in the exchange. It is because there is

an exchange that there is a price, and it is because there is a difference in price that money is there to measure it.

It is not in a preceding time (the logic of succession) or in a present state (logical time) that the event has not yet taken place; it is on the other face of the paper. The difference that measures the interval of 'time' is the difference of price – a sum of money. Once price is found to be less than the final amount of money that is contingent on the event, it can be repeated in time while retreating. The market, which seems to proceed in time, is but the repetition, now viable, of the moment of retraction, the repetition of the other face under the strike that says that it could have been different. All that happens is the retraction. Between one moment and another all that happens is the reversal of the face. But in chronological time (accidental but unavoidable) what happens is the market.

If money replaces time, then 'before' the event there must be less time – that is to say, less money. Whence the present price of the contingent claim that is lower than the amount of money that would be paid off in case of the event. Probability is also less than 1 before the event, but it lacks the sharing of time, or the *unity and integrity of the account*. In that schema of thought, we think that we retreat, through the tree of possibilities, from actuality to the present state where the future actuality is only one of the possibilities. By contrast, in the schema of money, we retreat only towards the other face, which is possible precisely because written matter has two faces. The other face of the 'It could have been different' (that mark, that printed promise of money, that sense of the impression that no longer renders the function indifferent as an analogical attribution) is the *recognized* matter of paper that lends itself *immediately* to the exchange.

By creating money to imprint writing in place, and knowing that there will be less money on the other face (that is to say, less time 'before'), we have created price and we have created the passage of its time which is the market. Something always comes to pass in money (unlike metaphysical time, where we seem to await the event and where probability poses its enigma), since the event has already taken place or *will* take place (it is real; it is not possible; it is upon its other face that we find ourselves) and because we retreat, not into the time when it has not yet taken place and when it is merely probable, but into the space where the price of the contingent claim is less. The market is the continuation, the translation, of the strike of the event. It is the continual event. For this reason, the market does not really come to pass in time; it comes to pass in the freezing of time, in its suspension or substitution, in its replacement and in its place, which is called money. It is because the market is separated from the event by the other face of contingency (which retreats, indifferently to time, indifferently to past or future contingency, repeating that 'it could have been different') that it can then *repeat* itself in chronological time, rather than proceed in chronological succession.

If the event creates the possibilities that will have led to it, then these possibilities do not truly precede the event. We identify artificial variations in the contingent world, we recognize states and delimit them, and probability then plays itself out as an integral in front of the 1 and the multiple 0s that have been thus fabricated, as the integrative reflex of thought in the present state which substitutes the logic of succession for the artificial multiple, a state which *logically* precedes, but does not really precede, the multiple of the attributive function. Probability being a circular fabrication, it does not operate in chronological time; or rather, it does not invest it. The event *comes to pass* in

chronological time, which is to say that it is detached from it. In the false idea of chronological time inherited from probability, the event is only a possible state that is realized; time does not belong to the event; before the event, in time, it is not the event; and after the event, it is no longer the event.

To anticipate the event and to 'remain' in the event before it takes place (to 'still be' in the event before it takes place, just as Menard 'was still' in it afterwards; and this makes me think once again that Menard's *fundamental exchange* – that which Menard has exchanged in thought and which is the proper and distinct definition, separate from time, of writing – allows one to remain in the event before it happens; it allows one to write in place of the event *before* the event; that is to say, to make its market), we must keep in contact with the *proper time* of the event, the time that never leaves the event and which is the mark of its contingency. The only alternative meaning of the strike of the contingent event is that the latter could have been different. It is not into alternative states that we must retreat in order to anticipate the event; not into logical space where the event is permutable and calculable; not into the chronological dimension where the event has not yet happened; but into the place and the strike where the event happens and *is still happening*, and where its strike never stops extending and reinforcing itself *in the mark*, through the promise to pay money that imprints it in this place. We must take advantage of the idea that the event has already taken place, but that it could have been different, and carry this difference backwards, or rather, towards the other face of the event: not the time where it had not yet taken place, but the other face of the imprinted strike which says only one thing because of money; namely, that the mark could have been different – the face that is called the exchange. Actually to retreat 'before' the event, knowing that it has taken place and that it could have been different: this is impossible and has no value in time; but it is possible, and even necessary, in money; for in money, it has a price. Money creates the medium wherein this kind of 'exchange' with time is possible.

If we retreat into time, where the event has not yet come to pass, we find only the time that is detached from it. Whereas we want to retreat, in the case of the market, into the place where the event takes place and is still taking place or, rather, the place where it is the exchange of the event, its place, that takes place in its place. In money, 'before' the event, it is *the whole market* that takes place. What we find then is not an interval of time in which the event does not come to pass, but a place where it is the market that takes *place* instead of the event – that is to say, a place where place itself takes place. Between time and the event there is a relation of container and content: the event comes to pass *in* time, and this is why time is detached from it and pre-exists it. By contrast, between the contingent claim (the event marked by money) and the market, there obtains a relation between one face and the other face, with money as the medium that reinforces the strike, a medium wherein we can flip from one face to the other.

The market, this infectious real that is opposed to the possible, takes place before the event in this new scale of time which is money. The scale of the time of money is defined in terms of the 'synchronicity' between the contingent payoff and the market price. Money was invented only so as to engrave the contingent claim in place at the moment of the contingent payment, to mark the irreversible strike, and *simultaneously* to define the market price. The market is the event before it takes place, where the adverb 'before' should be understood in the dimension of place rather

than the dimension of time; the market is the event in its place. The price to pay for this massive retreat – massive because of the equivalence between inertial mass and gravitational mass, between the written matter of contingency and the matter that is exchanged, which is what guarantees the ‘synchronicity’ of price and of the payoff and deforms the geometry wherein price is accelerated according to the very strike of the matter of contingency – the price to pay is the market.

It is to be expected that, in the time that passes before the event, we cannot find the event and can say nothing of it: neither that it is probable, nor that it is improbable; it is certainly not probability that takes place through time, up until the event, and makes it come to pass. Now, if the event does not come to pass in time, then it is not in time that we wish to place ourselves before the event. We wish to place ourselves before the event *after it has taken place*, so as to say something about its contingency, to say that the event could have been different. (It is in that same place that we want to go to, after the event; it is the expression of the event, its strike, its *characteristic*, its sense.) We want to go to the place where it is the contingency of the event that precedes it (and not its probability, which does not characterize it), keeping in mind that the contingency of the event occurs after the event and following the event (*après lui et d’après lui*), *following its trait*. (It is the place where Pierre Menard went *after and following* the *Quixote*.) To do so, we lend money to the event; we give it credit; we *credit* it in response to the question: ‘What ever happens before the event?’, which is the *incredulous question* of the event; we imprint the event; we invent money, paper, the contingent claim that pays off, and we create the market.

We wish to ‘remain’ before the event, in the place that is full of the event but in which the event has not yet taken place. Before the event, in time, we do not find the event, since it has not yet come to pass. Whereas what we seek, before the event, in money, is the whole event. We do not want the event as such, its realization, but the passage of its strike and its characteristic: its contingency, its sense, which is the whole event (since the event is nothing but sense). We will not find the event, but only its other face. It is by the loan of money that this is possible, *by the specific nature of money which sustains irreversible loss and, for this reason, is compatible with the irreversibility of the event*. Since we have inscribed the strike in money and in the *injustice of money* (‘It could have been different’), this money has sense only through that which comes before, through the market price of the contingent claim. Money only has sense in this tracking backwards in time, or rather this tracking backwards in the market which gives the impression of a tracking backwards in time. In the case of the insurance company, before the creation of the paper and the exchange, the impression of tracking backwards was produced through global statistics and the closure of the insurance company account.

8.3.2 The accident of time and the necessity of work

Money was created only to imprint the contingent payoff function in place and to engrave it in the mark and on paper, and, retrospectively, to be the currency of the price of the contingent claim that will occur before, in the market. By creating money and by writing the contingent claim, we guarantee ourselves of the existence of the market that will take place ‘before’ the event triggering the contingent claim, in money; that is to say, the market will replace the event while being, at the same time, the whole event: the event that is not separated from itself by time and possibility, by the

identification of states within the strike (for it is this identification that renders the future non-real); an event such that it is therefore necessary that this event (of the market) takes place before the event (triggering the contingent claim) – since we are dealing with nothing else than the place *that has taken the event*. This is what is meant by saying that the *whole market* takes place before the event, in replacement; the market without exception: all of the real that replaces the real of the event through a sort of inversion; the infinite which is the inverse of the strike: ‘Before the event? But there is the whole market!’ Creating money provides the material medium wherein chronology is dispersed, precisely because it is not dense enough in regard to the strike of the event. Chronology is lost and dispersed while money is had and is earned. Money creates the exclusivity of the event.

Money was only created to imprint the value function on the paper and to leave no other variation than the ‘It could have been different’, *which can only retreat into the market*. And all of this happens at great speed: it is neither before, nor after; it is the other face, the reversibility of irreversibility (the other manner/matter than that of the insurance company); precisely, this comes to pass ‘when’ the event does not come to pass, in its place and not before it: on its other face.

It is a question of maintaining a relation to the event other than that of prediction; that is to say, a distance (a retreat) other than that of time. It is only accidentally that time separates us from the event, because we live accidentally in time. The event may not yet have taken place (meaning that it is the place that has taken the event or that *is* the event; it is the place of the event that comes before the event and that thus takes place: in this antecedence of the place to the event there is something other than a temporal dimension or a logical antecedence) and yet we can find ourselves in this place, in this position, without being separated from the event by time. In doing so, we are not separated from it by space either (my register is not special relativity and space-like or time-like separation) but by place, or rather by the inversion of the terms of the expression ‘the event takes place’ – by their exchange. We are separated from the event by its other face. A topological, rather than a metrical, separation.

It is accidental that time must pass in order for the event to come to pass in time. Indeed, it is this accident that misleads us and makes us think that there is nothing else we can do before the event, for the event, than to predict it (or expect it). But the event is unpredictable. It does not come to pass in time, since it creates its time and the possibilities that will have led to it in time. Since there is no possibility but only a void of possibilities that leads to the event in time, there is no difference between saying that no possibility in that void leads to the event and saying that all the possibilities in that void lead to it. Since the event contradicts the logic of possibility (it exceeds it), it comes down to the same thing to say that the event is impossible and that it is necessary. (It is im-possible: it is literally situated outside possibility.) Thus, the event can only happen. Black Swans can only happen, for there is nothing else they can do. Thus, the event *has already taken place* (since it necessarily takes place) from the point of view of the time before and from the point of view of possibility. Or rather, the *place that the event takes* must replace the interval of time and the illusion of possibility that separate us from it accidentally.

That we may not identify the event before it comes to pass is no reason to think that the medium (and the connection, the mediation) that we share with the event is not real. We must not identify the

state of the actual world either, and the present and actual world must be just as *unpredictable* as the future world. There is no point in identifying the alternative possible states of the actual world with a view to vary them, given that the world is then massively real (actual) and massively contingent. It is enough to think that it could have been different without identifying different states in that difference any more than we identify the present state. It is afterwards that we fabricate the fiction that the actual world was, in time, a future world. It is after it and following it that we imagine the alternative states of which it would be only a possible realization, and which are supposed to explain its contingency. But if we refrain from identifying these alternative states, if we abolish any idea of possibility (which is only a fiction anyway and is fabricated in the present world so as to make us believe that it was once a possible future world and, by extrapolation, to make us believe that the future world is possible) and, along with possibility, any idea of prediction, then the present world will be unpredictable both on account of the abolition of possibilities and on account of the lack of possibility (since it is now actual); and, by the same token and in the same sense, the future world will be unpredictable. Not because it is indeterminate between several possibilities and we don't know which of them will happen, but because there is no longer any possibility concerning it.

The future world will be as unpredictable as the present world (abolition of possibilities) and as necessary as the present world (lack of possibility). We will then be able to entertain with the future world the same relation as Pierre Menard entertained with the *Quixote*. A relation of work, which is transmitted through the place that the event takes (for Menard, this place is the book) and not through the time in which the event comes to pass. (The event transits in time; it *is* time, but in what an ephemeral fashion, one even rejected by being and by ontology!) It is just as credible to entertain a relation with the future event that does not pass through time and prediction as it is credible that Pierre Menard passed his time writing an existing text. Precisely, he did not pass time, since, as far as the event of the *Quixote* is concerned, he slipped into that dimension outside of time that is indifferent to the order of succession between past and future (see Appendix B). Now, it so happens that we live accidentally in time. It is this accident that Pierre Menard does not resist; time and Pierre Menard could not avoid their paths crossing and, for this reason, for Menard the *Quixote* is a text of the past; but that does not stop him from demonstrating to us, precisely through his accident and our will to believe that *this accident is not serious*, the essential relation that we can entertain with the event outside of time: a *work relation* rather than a state relation. It is up to us not to cross paths with time and to establish with the future event the same work relation (without state and without possibility, without hope and without regret) that Menard established with a past event. This work relation is the market.

8.3.3 An event that is not but that remains

We must slip into the dimension where the future event no longer has any possibility (and where it is therefore necessary). We slip into it by thinking that the event has already taken place and that it is necessary (trivially, for these possibilities that do not exist or which will be produced after the event, which comes down to the same thing), but we swap the order of the terms, by *realizing* that the event does not take place but that, instead, the place takes event. (From now on, 'the place takes event' will

be the other face, the 'before' face, of 'the event takes place'.) We pivot on the place, which alone takes place and which takes all events, rather than pivoting on the interval of time and thinking that it is time, and thus the necessity of prediction, that separates us from the event. It is not at all necessary to predict: we can do something else with the event, *elsewhere than* (I no longer say *before*) it takes place – on its other face. We can make the market.

Predicting the event comes down to identifying in advance the possibility that it will have been, and that will have been realized. But the event is unpredictable – not because we don't know how to determine the future possibility that will be realized, but because the event itself creates the possibilities that will have led to it and because these latter are therefore not even to be seen, before the event, in order to be foreseen. Rather than experiencing the event in advance as one leap into the absolute void, as an unknown that is really unknown, as a total absence of form and of possibility that prevents us from identifying it, it might be more reassuring and less anxiety-inducing to think that the event *has already taken place* and has already crashed in actuality. It is its recovered corpse that is unidentifiable, because the traits of its face are all smeared in the mass of actuality and into one single trait that says that it could have been different without distinguishing the differences. Its corpse – that is to say its vestige, its trait, its mark, what *remains* of it 'before' it; for the ontology that we pursue here is that of the remains and not that of being; it is that of the burial of monumental and vertical reason. Our ontology is that of the ruin that one must know how to drag, outside time which only proposes the accident of the event, across the very place of its ruin, across the place where it remains qua ruin, all the way to the other face of the event, all the way to that retreat opposite the field of ruins that is called the *book of the event*, and whose quantitative avatar, for simply numbered writings, is the market.

The ontology of the book and of the market – this way we have found of remaining (not of being) 'before' the event – consists in retreating 'before' the event with that which *remains* of the event, with the sole trait that says that the event could have been different. Without the positive support of the attributive and dictatorial function (the function that dictates), the trait is but a vestige that we must be able to drag out on the other face of the event, 'before it has taken place', without rewinding the film of the event in time – that false manoeuvre, that misleading direction. This is why the contingent claim, marked by money which carves it into place at the moment of actualization of the final event (the expiry of the contingent claim) and which prevents it from ascending into the edifice of exchangeable possibility, *remains* written and marked before the final actuality, and in this way lends itself to that passage outside of time that is the market. Writing and exchanging in the market are but the continuation of the trait marked in money at the same infinite speed as its irreversibility. It is no more and no less than the *definition* of writing that we are contemplating here. Writing has no aim except to explore the other face of the event, to write it in advance, outside of time, by retreating from it into the book, by remaining in its place – in the place of its ruin qua event, in the *place that has taken the event*.

Massive actuality abolishes the fiction of possibility and is not incompatible with the contingency that repeats that the present event (even the past event) *could have been different*. It is this actuality and this abolition of possibility that we must find out how to impute to the future event. We must

recognize that the future event is entirely real, now that we recognize that it is no longer possible (it is even, in a certain way, necessary), and it is this imagined actuality that we must be able to retract and remove from it so as to retreat. It is the actuality of the future event that we must anticipate, not its reality – for its reality is unchanged. This imagined passage via actuality (what I have called a ‘loan’ or a credit which we grant the event) yields the bifurcation, or rather the turn, that we must learn to take in order to keep possibilities non-identified and to come back to the place – and not the period of time – ‘before’ the event. In this fashion, we use the same approach as for the actual (or past) event which consists in saying: ‘It could have been different.’ And by doing so, we get the market.

8.3.4 Writing the event

The event is impossible, not loosely and fictively, not mediately, not because it is a possibility that is lacking, or an empty set of possibilities, or the complement of the sum of possibilities, but because it is situated on the exterior of possibility, in the reality that abolishes possibility. The future event is impossible *because it is necessary and because it has already taken place*. The strike of contingency is the characteristic of the event – that which says that the event is (or will have been) *only insofar as it says that it could have been different*. It is the internal difference, the internal distinction of the event, which does not make it vary across a set of permutations. The strike of contingency is the reality of the event from whose actuality time accidentally separates us; and it is this strike or trait that we must be able to retract from the event and drag all the way to the place of ‘before’, where we will not be able to say that we predict the event (for the possibilities are not identified and the strike of the event *is not* its identification; the strike of the event is indifferent to the time of actualization), but will say, instead, that we *write* it.

To write the event, to drag its necessary trait (its contingency which is independent of time) all the way to the place where we find ourselves, does not amount to guessing or anticipating it, but to situating ourselves in its *middle*.¹¹ The reason why the world exists and resists collapsing into the object and into the permutability of states is the unexchangeability of the strike of contingency which introduces itself into the heart of the set-ups that will later allow us to objectify the phenomenon and to establish the range of possibilities. The exterior of possibility where the event takes place absolutely and im-possibly is the same exterior that establishes the incompatibility of ranges of possibilities in quantum theory. In the strike of the contingent event, we find the same non-permutability and the same non-commutativity as obtain between incompatible contexts of quantum observables, even though the absolute event may not be a part of any particular context or set of possibilities before it comes about. Its trait and its unexchangeability in some way precede it. The critique of possibility that is deployed in the name of the incompatibility of contexts of quantum observables and the impossibility of joining their ranges and summing their possibilities must be the same as the Bergsonian critique which claims that the event is impossible to predict because its possibilities only emerge afterwards. The impossibility of exchanging the strike of contingency against a range of possibilities must be the same as the impossibility of exchanging incompatible quantum contexts. The reality that resists the possible consists precisely in this strike which passes before, and to which the possible is subordinate.

Now, I claim that it is possible to join the middle of the event and to plunge into that reality, before the event has taken place. No one wishes to guess the event: guessing it, choosing it, is always disappointing. Instead, we wish to retract its trait, to extract it. Writing the event will be right, but it will not necessarily be true (no identification). It is no longer important to *guess*, and we no longer even know how to account for our guessing, once possibility has been suppressed from the path of the future event and once we know, despite that void and even through it, how to find the middle of the event. What point could there be in writing if not to communicate with the trait of the event across this massive and dark medium which exchanges the trait in the only way that *remains*: outside of possibility, by remaining after the vestige and following it, yet *before it*, by withdrawing opposite the ruin and the vestige, onto the other face, into the book?

The aim of all writing is the impossible exchange, for otherwise why would one write? We could even say that writing is nothing but the material solution of the impossible exchange, since writing is intended for the exchange. It is contingency that is the absolute thing; it is the attractive, fascinating thing, that we must be able to exchange in immanence and not in transcendence, in a joint operation that will define the exchange in the most original way possible – as well as, by the same stroke and in the same movement, transcendence and immanence. All that will count will be to pivot onto the other face of the event, standing right in its place. Contingency is the trait and, thus, the style of writing. The secret of the book *is* this translation in time (this manner of retracting the trait of the event and pulling it backwards: extracting it). The necessary book is subtracted from chance for this very reason. It is not at all a question of having *guessed* right and of speaking truthfully; it is a question of the middle of the trait, it is a question of being in the middle of the event, *before the event*. For otherwise, the medium (that is to say the market, the book) would have no reason to be. The book is necessary (the market is necessary), as is the trait of the event: precisely subtracted from chance and from the possibility that has been abolished.

Archaeology of the Exchange

9.1 All of the market!

L'événement a lieu. The event takes place. It takes place and the task of writing consists in replacing it, in taking its place before it. Writing consists in occupying oneself with occupying this place and no longer with the event; literally in *distracting oneself* from the event through the preoccupation and the hustle and bustle of the place that takes it. The place of the event is not separated from it by an interval of time and, thus, we do not *expect* the event in that place. We have travelled to the place of the event by flipping over its marked face, using the channel that has replaced time, and which is the loan of money we have written out for the event. All of writing had to be invented in order to flip over this face of the event and to withdraw into that place. Writing is not a slight thing, the book even less so. To the question of what happens before the event, of what happens absolutely on the other side of the event – for even the possibility of the event cannot happen before the event given that it cannot do without it – we must respond: ‘But *all of the market* and *all of the book* happen before!’

Not that the market is the thing that we make, or the book the thing that we write, *while* we wait for the event. The event will not take place *inside* a market or *inside* a book; for the market automatically replaces the event by flipping over its face. If time and possibility are abolished and are no longer the paths leading us from the place of the event to the event, then the identification of the event will no longer count and will no longer matter to anyone standing in that place. It is accidentally that the place will extend in time up to the event. It turns out that the market takes place *also* in time and extends every day up to the event. But he who inhabits the market (or the book) does not traverse time or possibility up to the event. For time and possibility are abolished by the strike (the trait) of the event and replaced by its vestige – by its ruin – which, through writing, we manage to drag onto the other face.

The market is the special case where money has replaced time and the number of money has replaced the number of probability (which was inactive in time), thus placing us in the middle of the event. With money and the engraving of the strike of the event in place as a contingent payment

of money, we have created the paper which could then retreat in time to the present spot and express the contingency of the strike ('It could have been different') outside possibility. For this reason, the paper admits of a *price* in the market (as opposed to a value) – by virtue of that other face of the programmed instruction which is the very matter upon which the instruction is engraved.

The price of the contingent claim will lead day by day, in the market, to the final event and to the final contingency, but without traversing possibility. The market finds the price without calculating – without evaluating the contingent claim. Once the valuation tree or algorithm is convoked, the conjunction of the tradability of the underlying asset (or assets) and of the givenness of the price of the contingent claim (which is the definition of the existence of the market) leads to the infection of the valuation algorithm by all the prices of contingent claims of increasing complexity written on the same underlying asset. The problem of the probability of the single-case event is that we do not know what universe of possibilities to refer it to and, consequently, what set to measure it against, given that the possibilities are supposed to be created after the event. In the same way, when evaluating the contingent claim, we do not know whether the range of possibilities will consist only of the prices of the underlying asset on which it is strictly written, or whether we should widen it to include the variations of the volatility of the underlying asset and the variations of the volatility of that volatility, and so on. To each of these new ranges of variations corresponds the range of variations of the price of a contingent claim of corresponding complexity (variance swap, option on variance, and the like). But the hypothesis of existence of the market is equal to the hypothesis of variation of all these prices. In short, all the ranges of variations of all prices of all contingent claims must be taken into account in order to evaluate the first contingent claim – which comes down to the same thing as saying that none of them must be taken into account and that the only value is *price*, or that the only evaluator is the market. Or better, that we must not *evaluate* the contingent claim.

This is why there is an entirely different preoccupation going on than that of expecting the particular event that will trigger the payoff of the particular contingent claim at its maturity. This is why, on this side, *all of the market* happens and why it occupies all of the place. The resident of this place no longer worries about the particular event; he no longer has time to expect it; he worries about writing the *book of the market* (the book that the market is). The only real thing concerning the event is the unexchangeability of its strike: the impossible exchange. This is what should preoccupy us; this is what's worth writing. Now, this precisely inverts into the market.

9.1.1 Impossible exchange, necessary exchange

What characterizes the event from this side is that it cannot be evaluated and that it cannot be exchanged against a range of possibilities. It is given but not exchanged: it has no currency of exchange other than itself. The event is literally despairing. To exchange it, we must fool it; we must steal its very trait, steal from it the matter upon which this trait is engraved. We must invent writing (and, in the same movement, the market).

What is the exchange? It is that which is *impossible* to do with contingency, by definition of the latter and by definition of that which is possible (possibility).¹ The contingency of the trait is its unexchangeability. Possibility is only the phantasm of exchange and of the permutation of

contingency, the idea of the reciprocity of states. What is the exchange? It is that which is *necessary* to do with contingency, by definition of the latter (and of its strike) and by definition of paper (of material writing).

The exchange of contingency is impossible in transcendence and necessary in immanence. Given contingency as absolute, the exchange can be defined either as immanence or as transcendence, depending on whether we decide to start from contingency and draw the conclusion of necessity (that is to say, deduce matter: speculative materialism) or that of impossibility (the black hole of possibility before contingency) – depending on whether we decide to venture, with the event, into place or into time, into the synchronicity of money or into the impropriety of time.

Given contingency and the absoluteness of its trait, the exchange is defined and is captured between the face of impossibility and that of necessity (impossible exchange/necessary exchange), or between the face of transcendence and that of immanence. Given contingency and its question (What can we do with it? How can it be translated?) – that is to say, the question of its exchange – impossibility and necessity, transcendence and immanence can, in turn, be defined as alternative philosophies to be adopted, or alternative orientations of thought (where to look, what to think).

What happens before the event in time is *impossibility only*. But what happens before the event (on its other face), in place, is *the whole market*. ‘The whole market’ is the other face, the positive face, of ‘impossibility only’. The subtractive ascertainment, desperate in the proper sense (‘impossibility only’) is the same as the excessive ascertainment of the market (‘the whole market’), but flipped over onto its other face. The trait of contingency has a way of rendering possibility impossible before the event, through its unexchangeability. When this trait is communicated to the market (and when the impossibility of the transcendent exchange is turned into the necessity of the immanent exchange), it is price that becomes responsible for rendering possibility impossible, through the infection of prices of contingent claims of increasing complexity (what we call *exotic* derivatives or *exotic* instruments) – which means: through the implication of the whole market, through the response to the *incredulous question* of the event: ‘But there is *the whole market*!’

We must therefore admit that it is a question, here, of the same matter and of the same material impossibility; that price is the *matter of this impossibility* or, rather, that price is the reality of contingency which is precisely unexchangeable against possibility. The market does not take place in time, for time does not belong to the reality of the event; that is to say, to its virtuality. The market takes place in reality. There are two decisive critiques of possibility: on one hand, the unexchangeability of the strike of contingency (the event creates its own possibilities), and, on the other, the market price (because of the infection by the prices of all exotic instruments). We must conclude that we are dealing here with the two faces of the same critique. The market translates the event, outside of the possible, as the other face of its critique of possibility. The market price is absolute; it is not relative to any range of possibilities, as probability can be. (Theory of invariants.)

With the market, we have not retreated into the time where the event does not come to pass or into the possibility that cannot be realized. We have retreated by *remaining* in the reality of the future event; in the market, we are *in the place* of the event; we ‘are still’ in the middle of the event

before it takes place. This is only possible in virtue of an exchange of thought which brings about paper. Thought exchanges the order of the terms that went from the state to writing, from underlying to overlying; it now writes *before* the state like Pierre Menard and thus recovers only the paper (see Appendix B). The latter cannot but fall into place; the reality of the strike of contingency from which one has subtracted actuality while retaining the massivity cannot but yield the matter of the paper. It is the very definition of the paper and of matter that thought drops in our lap when it exchanges itself in this manner.

9.1.2 The inverse view

The exchange is a category of thought, the category we obtain through the exchange of its terms. The market, that which *remains* of the event and of the trait of contingency before it takes place, is but the materialization of the 'impossible' withdrawal of the trait, moving backwards from its actuality. It is that which we obtain and which we find ourselves holding in our hand when we hold the event but the event has not (yet) taken place – when we hold it by its other face. Price is the contingency of the trait ('It could have been different') transported into the other place, the place 'of before', or the other face. The market is this inversion of metaphysics: this exchange of the terms of thought. For this reason, it is a book. The paper, the place that receives it, the exchange, *are nothing but* this exiting of thought; they are nothing but the effect of the inversion of the order of time (which usually locates the massive contingency of the event after it takes place); they have seized the contingency of the event *alone* and were able to replace the order of time – that dictation – only through the invention of the paper and of the exchange concomitant with it.

Money is the diversion of time and price is the diversion of probability (the contingent claim is the diversion of the possible state). In diverting our attention towards the market, we no longer pay attention to the event in time; we do not ask what the market does for the event in time. The market is not an evaluator that is accessory to probability; it replaces the event with its own event. We no longer make anything for the event; we only make the market. The *incredulous question* of the event ('What happens then before the event?') is diverted and replaced by the question of the market: 'What happens in the market and why, then, does the market exist?'

The market becomes more important than the event; it occupies the entire place of the event. Money, paper, exchange, *the whole market* were invented in order to divert us from awaiting the event in time – for time has been replaced – and to occupy us in its place, *otherwise*. When we respond: 'But *all of the market* happens!' to the question 'So what happens, then, before the event?', the intention is to divert the attention of the questioner towards the matter that is truly worthy of interest. The link between the event and the market that takes place for it, before it, comes from this precise diversion of the questioner's attention and interest. Certainly, the market concerns or will concern the event; but, at the moment, it is all of the market that takes place!

The market did not emerge in time in order that the question of its link to the event might be posed; it emerged in the place of the event and takes that place entirely; this is why *we do not pose the question* of its link to the event (that is to say, the response is self-evident; the market replaces the event). The market itself becomes the whole question.

It is by responding that *the whole market* takes place before the event that our response no longer concerns the event (it diverts the attention away from its question); and, for this reason, it addresses it very precisely. The question of that which comes to pass before the event is at once pressing and impossible (it is precisely pressing because it is impossible). The best response is the total response, which says that something entirely different happens, something that does not lead to the event, then – for, otherwise, it would be part of its question and would be denying the impossibility of the question – but something that comes *from* the event, the trace of the event. It is the event that leads to it, from the future where the event is actual and contingent, to the present where the future event is real and contingent. The market does not predict the event; it is not included in the event; it is not a part, the part before, it is the whole; it is the whole market. Yet, there would be no market if there were not the event.

Before the event: this notion comes from statistics. In statistics, we have the proto-market, we have the number of probability and its origin, we have the reference class, but we also have the cycle and the closure of the account of the insurance company together with statistical regularity. Statistical regularity does not point towards an ‘infinite’ generator, an inherent generator which would come from the future and from the direction opposed to the traced and written record, but towards the repetition of the insurance account in a closed circuit. Repetition is not generality and transcendence; it is linked neither to a past en bloc, nor to a future en bloc. It is this cycle that gives the impression of being before the event, in a position to predict it, when, in fact, one is posterior to the statistical record and, thus, it is by deconstructing the cycle of the insurance company that we can do away with the notion of generator. There is no infinite generator or infinite inherence, because of the closure of the insurance account. Guided by immanence, we should ascend to the market, retreat without identifying the event, through the coup de force of the marketplace and the fact that the market *replaces* the event. More generally, writing replaces the event while we wait; it distracts us, occupies us. The coincidence that takes place afterwards in time, between what one had written and what turned out to be the case, is quite negligible.

9.2 Statistics as a proto-market

The impossible question of what comes to pass before the event, the expectation of the event which probability subtends, was made possible by statistics. Statistics is a mixture of repetition and generality. We identify a singular and contingent event but, instead of rehearsing it in the only truthful fashion, which *does not consist in predicting it* but comes down to addressing it and addressing it strictly alone, outside of time and through the medium of contingency which replaces the event (i.e. through the market), we generalize it into a *class of events* by abstracting from the event a concept broader than it.

It is not the singular event that we then expect in time, an impossible thing, but the realization of the broader event, the *conceptual event*. We expect the passage from the class of events to the single element, for only the abstract class is capable of populating the time that precedes the event with a present concept – precisely! – for it seems to resemble the event but is, in reality, ill-suited to it (as

ill-suited as expecting the event in time), and which simply benefits from a coincidence. For we have benefited from a lack of originality of the single event, from a degeneration of its contingency, which was too weak, as it were, to change the world and was deflected instead towards the *population* of the world.

Given that a man's height or the date of his death, the sudden shift in the path of a pollen particle in a liquid, or of the price of a share in the market are contingent and, for this reason, could be different from what they are, we have elected to recognize in the height or in the date of death of *another* man, in the *next* shift of the pollen particle, or of the share price the different value that the given height, or the given date, or the given shift could have taken. The fortuitous presence of a population, to which we could now relate the single event through an effort of abstraction, now produced the notion of the *possible* value that a height, or a date of death, or a shift could admit of – the event no longer being absolutely that of the height of *this* particular man, as such unique and irreplaceable, or of *his* death, as such absolutely invariable as well, or no longer being the event of *this* particular shift of the particle or of price, as such no less singular and irreproducible, but having become more generally the height or the date of death of *any member whatsoever* of the human population, or any individual whatsoever of the population of sudden shifts in the trajectory of pollen particles or stock market prices.

No doubt, the relative insignificance of the single event has contributed to its fusion into the mass of the population. Certainly, no one man's height or date of death is going to change the world, any more than the aquatic caprice of a pollen particle or the financial blip in a price. And if metaphysicians sometimes speak of possible worlds in order to figure the alternative paths that this world could have followed if a certain historical (that is to say, contingent) event had been different, in reality this population of worlds is the intellectual extrapolation of the aforementioned human population, or of the population of shifts of pollen particles, or share prices. For these possible worlds are no more different from our world than one man's height is different from another man's, or than the next turnabout of a pollen particle is different from the last one. They are constructed (fabricated) on the basis of the actual world, with 'possible world' being the name for the world in which one imagines that the possible variation of the key event took place. Now, this variation is only a derivative variation, precisely – that is to say, it is not original. No one can imagine how the world might really, originally change. Even when the decisive event is single-case and does not belong to any visible statistical population, it is our *conceptualization* of the event – that is to say, our unjustified accession to a broader generality capable of making the event vary over an imagined population – that allows us to glimpse these famous possible worlds. To merely *think* of the event ahead of time is to conceptualize it, is to extrapolate the thought from the population.

Thus, statistics is the result of an abstraction which constructs the general class behind the single event or recognizes the population in it, combined with a conceptualization which then posits that this *general* event will repeat itself in each single instance (the generator). We had to decide that all we were interested in, in the life of a man – in the whole massive and immanent world of which this man is an undivided part – was his height or the date of his death, in order to be able to assemble into a population (or *reference class*) individuals who would no longer differ except in their heights or the dates of their deaths. An individual only becomes comparable to another in height because

one has abstracted away the whole story of a life (or of two lives) that would render absolutely inconceivable the context of the comparison of heights. As men are grouped into this imaginary population of men who differ only in their height or date of death, *possible values* are created; that is to say, the transcendent notion of a random function that will admit of these possible values emerges.

9.2.1 Abstraction and the precision of the present state

Probability is only the result of fixing a diagram. The contingency of the height of a man or of the date of his death is the *absolute* mark of a difference. It bears, inscribed within it, as its very trait, the fact that the height or the age of death *could have been different*, but it doesn't present this difference as relative to a *state* in which the man (become general, representative, functional, diagrammatic) would have to find himself in order to have this particular height or die on this particular day. From the irreversible strike of contingency to the recognition of different states, difference *turns its back* on its absoluteness to become analogical. It goes back on its strike, on its agonizing cry; it goes back on its word.

The subsumption of different states into one set of states, which was made possible by the statistical population (when the *reversal* from the irreversible height of a singular man, or the *reversal* from the irreversible event of his death, would have normally had the force to *reverse* the world and to prevent all possible communication with the world of the other man), immediately creates a sensation of *expectation*, the expectation that the random function will take one of its possible values. When states are identified and grouped into a set, they immediately become relative to one another and reflexive (conscious) of one another. This circle and this unity are expressed in the *unity of the present state* wherein one is supposed to expect the event, and measure its probability.

The notion of the diagram, or of a positively written function, does not only suppose that the possible states are distinguished and grouped together as a whole. More significantly, it supposes that the present state be identified, on the basis of which the supervision of the set of states proceeds. The present state *is* this set – integrative reflex and reification. It is the conceptual event, or the class of events, which is present and which *reverts* from the mark of contingency, when the latter was only a trace, and as such always past and irreversible. Even before we worry about stating the exact probability of the event, the closed review of possible states, when it is understood as the review of a recognized random function, imposes on the present state a precision equal to the precision of that review. We know, in this schema of thought, that the random function will assume one of the catalogued values and that the corresponding event will be realized. Now, the presently recognized function induces an integral value that is abstracted from the particular value. We imagine that it is only a question of time before the function admits of its definite value and we consequently conclude that the present time must *also* attribute a value to it as of now.

In reality, present time is as much a fabrication as is the set of possible states (what we have called 'logical time'). That a function exists and must dictate the value for each state imposes that the present state is also dictated. As we have said, identification is a function as 'algorithmic' and positive as any other. We have the impression of reading an instruction when the event is realised

(‘The random function takes this value’) and we conclude that the present state must obey the same mode of reading and the same instruction: that it must be instructed and identified.

The correlate of assembling the possible states in a set and in reciprocal reflection (into reciprocal *probability*) is the idea of succession, for it comes down to the same thing to say that the states are identified and catalogued, and to understand and ‘realize’ each possible state as the successor of the sole present state. Attributive thought is essentially chrono-logical, as we have said, short of being physically temporal. The reciprocal reflection and reciprocal probability of possible states (the *reciprocal play of beings*, as Badiou would say²) create the *common trunk* that the present state is, from which possibilities are supposed to spring like branches. The precision of the review of states imposes the precision of transitions within the tree and, in reaction, the latter imposes the precision of the present state.

Possible states only interpret their difference analogically (in the name of identity) as the succession of the different possible branches to the one and the same common trunk, and this imposes the present state as the retrospective convergence point of these divergent possibilities, the trunk being in some way induced by the branches. Thus, present time, the time in which we expect the realization of the event, is derivative and abstract; it is not real; it comes from the retro-jection of the set of possibilities. It is linked to future possibilities, since it is a result of their fabrication, when it is supposed to be absolutely independent of them.

Statistics is real and, in reality, it should be interpreted without the present time, through the integral account of the insurance company, through the immanence of the insurance account, which is closed and cyclic and not immersed in a larger space – for it takes advantage of the closure of the actuarial table. We must define the market as that which permits one to reverse the irreversible strike of contingency and retreat in time only accidentally. We mustn’t replace probability absolutely, but at the exact hinge where it derives from statistics.

In reality, there must be *no link* between the true present time and the true time of the future event. The contingency of the singular event must be interpreted as an irreversible and unexchangeable trait, which says both what the event is and that it could have been different and, if the mark should ever be transported (translated) to the present time, this would have to take place independently of transitions between states. The present world to which the trace of the future event would thus be transmitted in the form of a contract would have to be as massively contingent as the future world.

Probability is an account book. It seeks to link the future world to the present world. It thinks that the ‘value’ of the event persists in time; that if a random function takes its determinate value in the future according to the diagram that dictates the value to the state, then it must take a value today, dictated by the sole present state. We speak of the probability of an event as the present value of the function that would be equal to 1 once the event is realized (its indicating function). It is enough to mentally attach a payoff or financial value to the realizations of the event to see how pressing the necessity of a present value becomes.

The value of a lottery, once it is drawn, becomes equal to the value of the prize corresponding to the event realized. Before it is drawn, the lottery must be worth something – its pressing, present

value – and probability, no doubt, was only invented to represent the present value of the lottery as the mathematical expectation of the contingent prizes. It is because it is financially worth something after the draw that the lottery must be financially worth something before the draw. The player takes part in the event *before* it happens, and the material set-up of the lottery serves to assure this ‘persistence’ and this inter-temporal participation, this expectation.

I contend that it is not necessary that the random function (the lottery) should admit some *value*. The term ‘value’ is ambiguous, and not by chance. It is enough that a function exist and that it be evaluated in its possible states – it is enough that its values be dictated by an attributive algorithm, or diagram, or list of instructions – for its evaluation in the present state to become irresistible. Probability is only the present evaluation of the event when this event is described by a random function.

Now, the random function is not *absolutely* contingent; it is only so relative to the catalogued possible states. Even before we worry about its exact present value or about the probability of the event, the presupposition is that the present state is precise and ‘necessary’; that it, at least, is not contingent. But we ask: If the value of a lottery is contingent in the future world in which the draw takes place, why would its present value be ‘necessary’ today? Not that it must be worth something by necessity and we know how to determine that – indeed, the probability may very well escape us – but at least we entertain the thought that such a determination is possible. Since the value is dictated in a precise way in each possible state, the presupposition is that it must be dictated in a precise way in the present state, which is the synthesis and the origin of these successor states. The idea of succession implies that of transition, and the precision of the transition implies the idea of the tree and of the present state.³

Even though the content of the present value may be indeterminate and may require an unfeasible calculation, the present state that holds it like a prong is at least supposed to be given in a precise way. Now, this present, pressing and precise state is a mere fabrication, as we have said. It is the automatic consequence of the exact review of possible states in the form of a positive attributive function. It seems to take place in the physical present when, in reality, it exists only in a conceptual, or logical, present. It is chrono-logical, but it does not precede the event in real physical time. If, by contrast, contingency had remained massive, material and written, and had not been distributed into identified states, it wouldn’t have been translated today in the form of a present state and the corresponding expectation of a transition. For why wouldn’t the present mark or marker (the present indicator) of the event be as contingent as the event is?

We think the event is contingent on account of its random function, and it is the super review of the latter (or the conceptual event that takes place in the chronological present, or the present root of the tree, or, again, the present state) that is artificially projected backwards and artificially confused with the real present time. Through this confusion, we think that the present time is really linked to the event and that a notion like the *prediction* of the event is really possible; and, if the present world itself is ever considered contingent, we think that it owes this contingency to its own random function, whose root and present time retreat further into the past. But in no case would the random function of the present world be linked to the random function of the *future world*. What

point would there be, indeed, in predicting a future event if the present state was not itself precise and predictable, or if – what would be more serious – the actual world were not itself a *state*?

It is difficult to see how to avoid introducing hierarchy and arborescence. For if we admit, as the absoluteness of contingency seems to require, that the future world will remain contingent when it is actualized, in the same way that the present world remains contingent even though it is actual (the two worlds being as real as each other), it is hard to see how the two contingencies would be chained together. Chronological order demands that contingency resolve itself progressively in time (filtration). The future only seems contingent relative to the fixity of the present, which serves as a root for the alternative future branches. For this reason, it seems impossible to think that future and present are contingent ‘at the same moment’ (‘together’). That is, unless we think that the future is already uniquely determined at the present time, and that the two of them are equally and correlatively contingent only because the historical thread that links them (by necessity) is itself contingent. All of history could thus have been different, but *in the present one* – that is to say, conditionally on the present historical thread in which we find ourselves, this particular future we have would correspond necessarily to this particular present we have; and they would both be contingent for an external reason.

9.2.2 The immanence of statistics and the immanence of the paper

An alternative would be to no longer think chronology. Present and future would be contingent *together* (at the same moment) because present time, or present place, would be already at the heart of the future event. Chronology would be, so to speak, purely incidental (a slowing down of the event). ‘Succession is not illusory,’ writes François Zourabichvili, ‘it is simply [time’s] least profound aspect.’⁴ Chronology serves only to identify the event after all, to tell us to what state it ultimately corresponds. But what if we decided to eliminate the state from the outset? The future world is not contingent because of indeterminacy among a list of possibilities, but it is massively contingent, and it will remain so even after its actualization. It is the actualization of the event that recognizes it and identifies it; but its contingency doesn’t. Thus, the present world would be contingent for the same reason, which is that its state would not be identified either. Yes, the present world would be as unpredictable as the future world, not because of the indeterminacy of the state, but because of the inexistence of the state – because of the incompatibility *tout court* of ‘state’ with the new logic.

The very word ‘prediction’ would no longer apply here. There would be nothing to say, today, about the contingency of the future world other than transporting oneself there in time and observing *then* that it is contingent. Unless traces of this massive future contingency could already reach us today! These traces would lend themselves to everything, except the prediction of the event. They wouldn’t have taken place *before* the contingent event in a precise present that would serve as its antecedent and its root, but *in its place*. We may well want to admit that the present world is contingent on its own account (its own function), but it is difficult to admit that something is contingent, in the present world, on account of the future world. But why distinguish between the future and present worlds any longer if chronology no longer takes place? They would be one and the same contingent world, except that certain events in it would not yet have been identified (the

course of chronology is needed for that: the 'slowing down' of the event by time is needed for that) and would be replaced, before their identification, by their place.

This place is the place of the market. The price of the contingent claim is not a present value: it is not a number or a value; it is multiple. It is the *translation* of the contingent claim: the materialization, before expiry, of the 'It could have been different', which, *because of the non-identification of the future contingent event*, does not need to lodge inside the precision and prediction (and even predictability) of a present state. The 'present' price of the contingent claim is just as contingent as the contingent claim, by translation. (In other words, one no longer awaits the non-identified event in time; there is no ex-ante stance any longer; no probability, no possible worlds; there is only a patient writing which, paradoxically, does not wait; there is the relation of work, not of the state).

The massively contingent future event has never been the member of a reference class; it has never been possible; its contingency is immanent and inherent in it. It is the false idea of belonging to a reference class that places us falsely in a 'present' of the future event (its present value); whereas if its singularity and its massive (immanent, non-referential) contingency alone are affirmed, they can only be so affirmed after it (because nothing can predict the event in advance) and the idea emerges at the same time that some of this trace of the event can reach the present without passing via the *present of present value*, for the very reason that the event is singular. It cannot be predicted, and *it is for this reason* that it is already here and it is already written; its trace, its site, its place, are already here. Thus, the contingency of the event could be 'before it', in the present world, not in the form of a present state but in that of a stateless contingency, and this *without contradiction* thanks to the inexistence of the state and of the reference class.

The event is real and contingent, not possible. Its non-possibility eliminates the present state and present value. The latter is replaced by price, which falls in its place. The exchange and money replace time. The exchange is the medium that replaces chronology and which enables the withdrawal from the final trait of contingency to the present place (for the exchange, money and writing replace the states). For this, it is necessary that contingency be marked on paper, and that paper, by its very materiality, encode as its other face the eventuality of its total loss. It is the immanence of paper that enables the transit to the present without the present state. Statistics gives us an equivalent of this immanence: the account of the insurance company also exchanges written contingency, but without exchanging it in a market. It exchanges it in a proto-market which is its closed account, whose contraction and trait are furnished by statistics, itself a contingency that is always marked afterwards, and not before.

In the new metaphysics where states are not identified and where contingency is the primary strike (whose medium is writing), money replaces time and chronology is abolished. The market price takes place at the 'same moment' as the final contingency. The immanence of the 'It could have been different' brings about the contraction of the paper towards its matter (whose other face is total loss) and, therefore, its immediate exchange. Statistics possesses its own immanence (that which withdraws without passing via the state): the closed account of the insurance company. Moreover, it is statistics that has created the improper notion of time in which one *expects* the event. In reality, statistics does not itself make use of this notion; time is truly false in statistics, whereas it is falsely

true in the probability of the single-case event that was inspired from statistics. There has been such an inversion.

What I am saying is that even the time in which we expect the single-case event, the time of Popper's propensity, is, in reality, borrowed from finite statistics and from the insurance company account where, as a matter of fact, it doesn't even occur. Even if one did not consider that Popper's objective probability is of statistical inspiration – for Popper's critique of von Mises consists in claiming a probability for the single case that would be defined independently of the entire series of cases – I claim that the time in which it is tensed comes, in reality, from statistics.

Statistical regularity and the closed account of the insurance company account have furnished this latter with the capacity to adjust the insurance premium it is charging gradually until its financial balance reaches equilibrium *ex-post*. In the statistical cycle that follows, the money thus provisioned in the accounts, which, in reality, only results from a long process of *ex-post adjustments*, suddenly manifests the miraculous property of being present ahead of all forthcoming contingencies. Because of the return (regularity) of statistics and the *accountability* of money (its ability to count *ex-ante* after having counted *ex-post*: it is, in fact, this accountability of money that allows the closure of the cycle of time, wherein resides the miracle and the mystery), and through a sort of stroboscopic effect where the radius of the disc seems to spin backwards when, in fact, the disc is spinning forwards and is only lit at a greater frequency than that of its cycle, money, thus prepared by statistical regularity and adjusted to the future distribution of contingencies, gives the impression of having 'evaluated' the latter in advance, *ex-ante*. This startling configuration is then summarized by the notion of probability.

By inverting the order of the terms of the *ex-post* accounting equation, and by placing *before* the event a sum of money that was only obtained *after the event*, a statistical *ex-post* average is thus transformed into an *ex-ante* mathematical expectation, and the frequency that entered as the integrand in the statistical average is transformed into an *ex-ante* integrand, or probability. Thus, what is missing in von Mises, in order to render *ex-ante* his statistical probability defined *ex-post*, is the closed account of the insurance company (the accountability of money) and, correlatively, the 'finitude' of statistics (as opposed to the infinite inherence of Popper's propensity) which was all that the insurance company needed in order to adjust its insurance premium and to balance its account.

Von Mises was not concerned with the 'materiality' of insurance accounts – this kind of finitude. His goal was theoretical, metaphysical (or anti-metaphysical, empirical, which comes down to the same): the infinite goal of *defining* objective probability. With the empirical evidence of the statistical series at his disposal, he had no way to surpass it other than to push its extrapolation to infinity. Certainly, he only sought to define probability *a posteriori*, always after the passage of the strike of statistics and never before; but doesn't the passage to the infinite betray a desire for the *a priori*? To speak of *the* probability of an event, even when the latter is defined relative to a whole given series and requires the event to be a member of a reference class – doesn't this presage an ideal situation in which the event, whose probability has just been ideally inferred from a passage to the infinite, makes us expect it to be realized individually with the aforementioned probability?

Isn't von Mises's passage to the infinite limit, which had the advantage of keeping empirical probability asymptotically separate from metaphysical probability, precisely the preliminary to

Popper's propensity, von Mises's infinite-empirical situation thus being connected without break with Popper's empirical-infinite situation? Popper crossed over into the infinite by turning asymptotic probability into an objective property (mysterious, and thus 'infinite') of the situation, or propensity. As soon as the event is repeatable, it is propensity that will now generate the statistical series – if need be, all the way to the infinite, which now becomes indifferent – whereas von Mises's statistical series was itself supposed to engender, produce, albeit via a passage to the infinite, the objective notion of probability. Popper thus crystallized the desire for the *a priori* that was latent in von Mises's infinity. The *a posteriori* that was not supposed to detach itself from the whole series in von Mises precisely does detach itself at the hands of Popper's infinite and becomes *a priori*: it becomes a generator, now inherent in the objective situation, as what Popper calls the *generating condition* of the event.

Nothing will prevent the event any longer from being truly single and from taking place only once. However, Popper's single-case propensity is subject to the objection of the indeterminacy of the reference class. A unique event has no probability in itself, or absolutely. The fact that its probability is less than 1, or a percentage, indicates that we are implicitly referring to a *conceptualization* of the event; that is to say, to a reference class which would potentially allow it to repeat itself (even if this repetition were to take place in alternative possible worlds and not in the actual world) in order, precisely, to lend itself to statistics. Behind Popper's propensity there is still a subjective presupposition that replaces von Mises's statistical series or empirical population.

At least, Popper will have established the notion of a random generator relative to a statistics à la von Mises.⁵ He will have succeeded in returning from von Mises's infinite to the *ex-ante* stance – he will have reified probability in this sense – but while remaining 'stuck' in the series. My contention is then simply that the *time* in which we wait for the random generator, or the stance which consists in expecting the event to take place with a certain probability, is itself a result of the insurance company's illusion of (stroboscopic) antecedence. In reality, it is explained by the account of the insurance company and the corresponding reification of propensity is due to Cournot's principle.

9.2.3 The matter in statistics

Only the closed account of the insurance company and the accountability property of money enable one to retreat in time; as a matter of fact, it is not by coincidence that the only way of making sense of single-case probability is to resort to subjective probability, the basis of which is a financial transaction (de Finetti). Von Mises's probability could have limited itself to being nothing more than the re-expression of an *ex-post* statistical summary, but the passage to the *ex-ante* stance and the medium of time that corresponds to it are not independent of the problem of the insurance company and of its closed account. Von Mises could have continued to ignore the *ex-ante* stance as long as he didn't intend to *return* from the infinity of his series. His quest for an exact rather than approximate definition of probability forbid him anyway from stopping before the infinite limit, because a frequency calculated over a finite series could always be falsified. As for Popper, his propensity might have given the impression of concerning only the next single-case event and of being 'infinite' in the sense that it now concerned the generating condition *once and for all*; but

I claim that the time that subtends it, the interval of time that necessarily separates the instant and the place of the definition of propensity from the instant and the place of the realised event, is an illusion whose real substrate is the 'finitude' of the statistics of the insurance company and of its closed account, or a reification due to Cournot's principle.

De Finetti's financial intuition is, in reality, linked to time and not to the singularity of the event (see below the distinction we make between ex-ante and single case in section 9.3.1 'The non-individual singular'): it replaces the time of the expectation of the event with the *account* of the financial transaction and the corresponding materiality and finitude. But the time subtending expectation concerns objective probability, too: I claim that in that case also it comes from a financial or proto-financial circumstance – the circumstance of the insurance company account which was concealed under the conceptualization of probability. Only the material medium of money allows one to *conclude* something, concerning the event, *before* the event. It may well be that statistics serves to guarantee the objectivity of probability, as opposed to subjective probability which is simply named by an agent; however, the expectation of the event, or the arrow of time that seems to tense objective probability, even if it is built up by Cournot's principle, has to be deconstructed and brought back materially to finance.

Material time, as opposed to the immaterial time that is chronology, is composed of the univocal trait of statistics combined with the closure of the account and the stroboscopic effect. One has the impression of retreating in time when, in fact, one is not. Now, the market replaces all of this in the register of the single-case event. The market is also ex-post, and the impression that it takes place before the event is due only to the impropriety of chronological time – to its unsuitability for the problem, to an usurpation really – of the same nature as the impropriety of the time of the insurance company, the time we have called truly false.

It is the degenerate case of statistics that has allowed us to speak of possible states and to group them together in the abstraction of the reference class. The precise review of the possible states had as its correlate, as we said, the precision of the present state, and this implied the notion of present value or, again, of probability. The time of expectation was thus purely chrono-logical and concerned only the conceptual event that had been abstracted by logic beyond relative differences. Only the material account of the insurance company will be able to include this logical time in a physical medium, which in reality will be constituted by money. It is by proceeding via the insurance company, via the statistical regularity and the closed account, that the time of objective probability was able to earn a substance. When we speak of objective probability as such, we extract it from the integral operation of insurance within which it is mixed: we give ourselves the impression of expecting the corresponding event absolutely with the corresponding probability; but the thought should have never left us that money was 'in reality' (virtually) always wagered on this stance, and this should not seem any stranger to us than the financial wager of subjective probability. It is precisely Cournot's principle that illegitimately enables the illusion of time, when money should, in reality, replace it.

Only money has the accountability required (the *matter* required) to go back in time (or, rather, to seem to go back and to *manipulate irreversibility*, to create an alternative dimension of time). It was created only to write and imprint the contingent claim in *place*, beyond the underlying states

that come to it from time and from the corresponding order of thought, and to materialize, in the form of the price that only the exchange will be able to transport back in time to before the event, the price of contingency that is only known afterwards. (Writing, the exchange, price are truly the substitutes for the insurance account. The thought, in this reconstruction, is that the exchange did not exist before the insurance.)

It is through a temporal exchange, within the temporal loop that is produced by statistical regularity and the closed insurance account, that we are able to assign a present 'value' to the contingency of a future death. I shall call *matter* the interval across which this exchange takes place: matter, because it takes us back to the immanence of a material procedure and to a sequence that does not pass via the transcendence of expectation and objective probability; matter, for the reason that the strategy of Shafer and Vovk is deemed material when Cournot's principle is recognized as a transcendent decision. The time of expectation and probability are only illusions, or abstractions, subsequently erected above the plane of immanence.

One wonders what the technology of the retreat could be if the singular event does not degenerate into statistics and if the latter does not fix the relative states. What could the corresponding *matter*, or the material 'synchronization' procedure, be?⁶ Difference would no longer be ruled by underlying states but would be marked directly on the material sheet that inscribes the contingency of the event. Logical time would be abolished, since the conceptual event would not be defined through the abstraction of differences, and states would in any case no longer be underlying to institute the *present state* after the precision of their review.

We would no longer be able to retreat from the trait of contingency *in time* in order to evaluate it. The time of probabilistic expectation was only due, in any case, to a transcendent illusion whose reality is immanent; it was only due to the capacity of money and of the insurance account to remain and to count 'before', or rather, as a 'return' from two sorts of irreversibility. The statistical series of von Mises extended irreversibly and infinitely, without ever turning back, and the insurance premium was adjusted in the same gradual and irreversible movement, until the ex-post break-even was reached. As well, the strategy of Shafer and Vovk was ex-post, we said. Now, the insurance premium remained present in the account: it *counted* in it, whereas probability did not count. It is through a *return* to the account (or through a turning towards it) that we *take account* of the fact that the money is present in it, at a time when the contingent event does not yet take place. It is the materiality, or the gravitational mass of money, that allows this suspension of time and this impression of waiting. Ethereal probability wouldn't have afforded it, and if it gives the impression of doing so, it is because it is tacitly extracted from the configuration of insurance.

With the insurance account in mind, we can denounce the illusion of probability and of time, we can indicate the true matter and the true synchrony. And, finally, we can understand the market as the generalization of this configuration, as the general relativity corresponding to this special relativity. The gravitational mass of money and the inertial mass of contingency are equivalent. The market is the geometry and contingency is the matter. All we have to do, then, is leave 'in place', in the place of the event, the matter capable of bearing and teleporting its trace – namely, writing, which lends itself immediately to the exchange.

The fact that probability in von Mises is estimated *ex-post* does not allow us to stop and to wait, for the estimation will not be exact until the infinite limit of the series is reached. The estimated probability thus pursues its course and its convergence to the limit within the series. Introducing the gravitational mass of money changes the problem. Its mass bends the geometry of space-time. Time is no longer absolute but is defined after simultaneity, and the latter follows from the material synchrony between the strike of contingency and the market price. The present price and the future event no longer have the frame of reference of probability and possible states between them. They are now synchronous, not in chronological time, but in material time, in the time of money which replaces time. It is because the insurance company pays money against the final contingency that it funds its account before, and that money finds itself present at the two extremities. For this reason, money is, in particular, present at present, and gives the impression that it is evaluating contingency in advance. In the same fashion, money that is irreversibly lost at the end of the world and which the material paper alone can encode as its other face, is the same money against which we exchange the paper in the market today after realizing that the material paper and its *ex-post* fate are *still* with us today. Thus, the impression of expecting the event in time only exists because of the continuous material medium of money (operating in the insurance company's account) and does not exist in the absolute, as if someone were waiting for the event in absolute time. In reality, (transcendent) time does not exist. We inhabit statistics as a ruin (immanence), not as a temple (transcendence). We fall into it always *after* contingency, never before, and it is only money that allows us to retreat and withdraw in front of the ruins and to write the book of these ruins, *as if* we had predicted them. It is the statistical degeneration of the singular event that creates this *insurance hotel* facing the ruins, in which we can retire with the account book. What would be the retirement place of the singular non-degenerated event; what is its hotel?

9.3 The matter in the exchange

I claim that the matter of the exchange replaces the matter of statistics when there is no statistical population. The two are immanent and are opposed to transcendent time. One is composed of paper and of its return into the market; the other is composed of the immanent consolidation of the insurance account. In the same way that statistical regularity combines with the matter (accountability) of money and produces, with the money that remains in the account, the impression of evaluating the event in advance, we must discover the 'regularity' or the 'recurrence' (the table?) or the 'mass' that would combine with money to produce in advance the price of the singular event.

The insurance premium (the present value of contingency) is said to evaluate the event of individual death because of a break-even in the *ex-post* accounting equation obtained on the scale of the whole insurance company and by virtue of the presence of money in the account. When the event is truly singular and belongs to no reference class, I claim that the money of the price of the corresponding contingent claim is equally present, only its calculation does not obtain from an adjustment on the scale of the big house of insurance.

In the case of the singular event, we no longer have statistical regularity and the consolidation of the account to explain, on a global scale, why this local and individual price can *seem* to concern the particular event when in reality it is only the effect of a reduction or an extrapolation. The big house and the long-term statistical accumulation are no longer here to justify that this premium we charge would exactly be worth *this* much for *this* particular contract. In the case of the singular contingent claim, the money that must be paid at the end, upon the resolution of the contingency, and the money of the current market price are still existent, but there is no longer the passage between them via the consolidated house and the statistical table. We must remember that the insurance premium only seems to have taken place before but that, in reality, it does not: only money gives the impression of a stroboscopic retreat. In the same way, we must imagine that the price of singular contingency seems to take place before but that it also, in reality, does not.

It is the statistical break-even *posterior* to the facts that justifies, along with the present money of the insurance premium, the stance of an *anterior* evaluation of contingency – or rather, the corresponding abuse of language. Although the particular contingency seems to be identified, and the insurance premium that evaluates it in the present seems to address it in particular and in advance (since it is a matter of insuring this particular individual against death), in reality the identification of the event is an artifice, and is but the optical illusion produced by a global account and a global statistics. Correlatively, the illusion of the ex-ante stance and of the reciprocity (reversibility) of possible states is produced by the abstraction of the conceptual event beyond the statistical differences observed *ex-post*.

The evaluation of the insurance premium is in reality *immanent* because the particular event has not really been identified in itself, and its transcendent possibility has never really been invoked. The matter of statistics flows in one direction only, which is massive, and it is only through an abuse and a forcing that the probability of a single case is pronounced. We thus seek, for the singular event that escapes the population, the equivalent of this matter and of this immanence (the equivalent of this real irreversibility which is only illusorily presented as a reversibility).

The long-term adjustment of the insurance premium, which has never inverted or contradicted the direction of the flow of statistics, is an engraving and an ex-post inscription, in itself irreversible. It is through a particular configuration and manipulation (the insurance company account) that the direction seems to be inverted. We thus seek the corresponding *configuration* and *manipulation* for the singular event.

The irreversibility that will correspond to that of statistical inscription is that of the *writing of contingency on paper* (for statistics is a writing, we said, and the strike of contingency that replaces statistics by being folded back in the matter of the die is equally a mark and a writing), and that which will correspond to the big insurance house and to the corresponding manoeuvre of return from irreversibility is the exchange. The materiality and the contraction of paper are what permit us no longer to identify the particular event in advance: paper retreats without identification and without opening the testament in the exchange. The paper is contracted towards its matter and away from the identified states that constitute the written content because of the ultimate clause that it carries on its other side: the clause of total abolishment of the paper. Because of this apocalyptic clause, the

paper is materially nailed to its expiry date; it is nailed to the time of opening the testament, and manages only to retreat in time by virtue of the exchange without the recognition of state. It plays the role of the global *matter* of the insurance company account which precisely enabled us *not* to identify the individual event.

Money, when present in the consolidated account and combined with statistical regularity, constitutes the univocal matter through which the valuation of the insurance premium finds its immanence. To replace them in the case of the singular event, we invented the paper, on which contingency is marked together with the material and pressing clause that the paper will thereby be exchanged. The paper and writing are a great invention indeed! We must not falter, here, under the pretext that paper is trivial and familiar – that it has existed for a long time and has long since been exhausted – and that it cannot furnish the solution to as *grave* (from gravitation) and massive and attractive a problem as the evaluation of the singular event! (We said that the singular event was that which attracted the whole universe and that the most truthful way of responding to its attraction was precisely writing.) On the contrary, we must give back to paper, to writing and to the exchange all of their *complication* – the idea that the writing of contingency and the exchange are but one and the same thing, folded and implicated in the very matter of paper. Paper is this matter! Why would the act of writing be less metaphysically charged and less massive than the whole set-up of the insurance company?

Time has neither existed nor played any role in the case of insurance, since in reality we have never gone back in time. We have not ceased to follow the trait of statistics and to adjust the insurance premium and we have never turned back. Now, consider that the case of the insurance is too special and degenerate: it has brought into existence this particular house in whose account the consolidation would take place and the manipulation would be produced. Writing on paper and the exchange – or, in other words, the market – *had not yet been invented*.

The insurance contract does not stand ‘alone’. It stands because of the house that consolidates it. As for the singular event, it must be evaluated alone. We cannot rest content with a global break-even in the account; for this reason, we must turn towards the market. Qua contingent, the singular claim is written irreversibly and can only be expressed afterwards; but it must be tracked, individually and no longer through the whole population, *before* it passes. This is unlike the statistical event, whose trace is, in reality, the whole past population and which is distinguished in it only through an abuse of language. This is why the paper was created, the paper which speaks of nothing other than the contingency of the trait, which escapes the diagram of states, but whose immanence (or the fact that it does not distinguish the states that it encodes because of its contraction by the apocalyptic clause) is translated and turned into the exchange.

The exchange is what replaces, for singular contingency, the account of the big house and the univocal gradual adjustment that enabled a present value to seem to be attached to a particular contract. We must maintain the univocity of statistics (the fact that its trait never retrogrades) but we must find the way to replace its immanence – which was due to the fact that the particular event was not expected as such, as identified individual, but remained buried in the mass – with an immanence and a matter which truly attach to the singular event this time but (and here resides

the difficulty) which do not distinguish it transcendently (i.e. the singular event must remain in immanence). We must distinguish in the event its contingency, whose immanent translation is the exchange in the market, and not translate it into a possible identified state, which would, on the contrary, be an attempt at an impossible exchange in the sense of Baudrillard, one that would introduce an undesirable transcendence.⁷

Probability does not exist; all that exists is the statistics with which one can evaluate contingency immanently by virtue of the insurance company account. One evaluates contingency via the global break-even, and this creates the illusion of probability. This is the best we can do with statistics. To evaluate the contingent claim in the same way, we must not retreat in time and we must not use probability either. It is exactly at this juncture that the paper and the exchange come in: in this inexistence of time. They are even equivalent to the inexistence of time. Through price and through the exchange, we do not place ourselves chronologically before the final contingency, but at its very heart, or rather in its place, by virtue of material time and the contraction of the paper, which renders synchronous the apocalyptic clause and the present price. The equivalence between written contingency and exchanged mark presides over the geometry of space-time and over the logic whereby money replaces time and time no longer exists. The notion of a random generator and of propensity is a *logic* (as we have said), as well as the notion of time that tenses it; this logic can be changed without negating the basic ontology, which is contingency. For instance, Shafer and Vovk have changed the logic.

The contingent claim can only draw its value from 'elsewhere' if it hasn't expired yet and doesn't draw its value from its schedule at expiry. In the case of insurance, the particular contract draws its value from the 'elsewhere' of the big house. As for the 'elsewhere' of the singular contingent claim, it is particular to it since it must be attached to it and there is no population in which the singular event is buried but, at the same time, there must be a detour in some 'elsewhere' because time still separates us from the final resolution. This 'detour in elsewhere' is the market. By virtue of its contraction, the written paper removes contingency from the supervision of the state; it moves it away from the identified state that we owe to statistics.

When we seek to evaluate a contingent claim, we think of using probability. We project the possible states and we expect to compute a mathematical expectation. Implicitly, we behave as if the insurance company account were in place, for projecting possibilities is only another way of saying that the population exists and that the insurance account has reached ex-post break-even. When we remember that there is no population, we still maintain the set-up of ex-ante evaluation. We reify probability, or we have recourse to the market only in its capacity of an adjuster of the evaluation (risk-neutral probability), only as a complex calculator that will, only *subsequently*, get involved in the set-up we have initially prepared. We think the market is secondary and subsidiary; we think price is secondary to value, and this is why risk-neutral probability is conceived as a *change* of the initial objective probability. All we want is to *adjust* the average of profit and loss and, in doing so, we presuppose the existence of a statistical, or at least objective, average.

In reality, the retreat in time should never have taken place. The only component of the insurance set-up that we should have kept is the *ex-post* accounting equation. The passage to the market should

have intervened by going in the same direction as statistics and by taking over from probability *at the moment when* the latter was extrapolated from the population and made specific to the single case, not afterwards once the reification of probability had already been effected. The market is not a change of probability measure; it is a change of probabilistic *logic*. We write the contingent claim, but not for the purpose of programming possibilities. (K, T) has nothing to do with (S, t) . We write the contingent claim, and the matter that we engrave is essential, for it remains after the contingent claim has been written. It is this matter that will retreat, not possibility.

We have the impression of writing, now, a future contingency. In reality, we write *in the future*; the paper records the mark of contingency ('It could have been different'), and because of its contraction to mass and the non-identification of states, the paper remains written 'after the future'; that is to say, before it. The state is programmed by the attributive algorithm, it is intended for the transition and, *for this reason*, the present state takes place before. In contrast, there is something in the *irreversible* strike of contingency ('It could have been different') that renders it insensitive to time and makes it take place before in the same way as it would take place after: a new kind of reversibility which redefines time. If we forget possibility and let it completely disappear along with the population (disappear as the optical illusion that it was, of the discernment of the one within the population and after the ex-post break-even), then we will realise the significance of paper and of writing. It is paper and writing that will ensure that, once the expiry date of the contingent claim is reached, contingency and difference will be legible at that moment without having been predicted or programmed in advance – without any transition or passage in time. Because of the apocalyptic clause, one does not *predict* with writing. All that one does is to write the irreversibility of the trait in advance. Testaments are always surprising.

9.3.1 The non-individual singular

Before expiry, the paper is closure via the exchange; it is not the opening of possibilities. 'Before expiry', or the *market*, is not an *ex-ante* evaluation. For we do not retreat in time; we retreat through the same immanent medium as the stroboscopic retreat of the insurance value, except that this time the matter (the instantaneous medium of contingency) has been found. We do not *evaluate*; we place ourselves at expiry and we *price* – that is to say, we flatly observe that the event could have been different. It is this flatness that is dragged backwards. As it is flat and doesn't know the *depth* of the underlying states, it will not really have retreated. For the depth of the image, where is inscribed the impression of retreat, is but the illusion due to the *identification* of states and of possibility.

We maintain the insurance's ex-post mode of inscription, but we no longer have at our disposal the account and the population that created the immanent retreat. What must replace them is the market: *it is its definition* to be this replacement. If the depth of possibility (ex-ante evaluation) is abolished along with the population, if all of probability is no longer pertinent, then we are obliged to hang onto the only thing that remains (like a ruin), the immanence of the ex-post accounting equation of insurance; we are obliged to dwell and remain after the event, even if it means retreating by other means. It is the *matter* of paper that will, by definition, permit this instantaneous retreat outside of time.

We haven't observed different possible states at expiry (we have not performed *valuation*); we have just observed that the event *could have been different*. We uphold and affirm the (massive) material contingency of the paper. It is this virtue of the paper that is transported into the exchange. The material (and non-programmatic) writing of contingency is *equivalent* to the writing of price, because it is writing on a paper whose capacity to encode total and irreversible loss as its other face – as its cataclysmic disappearance – establishes instant connection with the present time, in which the paper *still finds itself*. Paper (which insurance did not have at its disposal) allows us to mark that the event could have been different, and this, through the exchange, solely through the price attached to this particular claim, replaces the global account of the insurance company and its global break-even. The matter of paper, for this singular contingent claim, is deduced, if you will, from the break-even of an insurance account *that has not taken place because of the singularity and the absence of a population*.

It is as if we were saying that the singular logically came after the population and not before (since we are deducing the matter of the singular contingent claim from a set-up initially designed for the population), that it was more 'complicated' or 'sophisticated' than the population, that it had taken place because of the absence of the population but that it would remain as multiple and as massive as the population; and not that the population was composed of singular individuals that were identified in advance. Immanence comes before everything else; hence, the (transcendent and identified) individual is in reality invented *after* the population. We must come back to the singular via a 'reduction' of the population that is different from the transcendent return of the individual which make us think of individual probability, even if this means re-establishing the hierarchy and recognizing that the *non-individual singular* (non-recognized, non-reciprocal) is, indeed, the primary thing, whose other name is absolute contingency, and that it is, indeed, a conceptualization and an evaluation (the identity function) that make us identify the latter when we should not have done so.

The matter of paper bears within it, *as its quiddity*, the equivalence between writing contingency and writing price (or the exchange) only insofar as – this is its definition – the break-even of the insurance account, which we must necessarily maintain given the necessity of maintaining the ex-post posture and the immanent mass, *has not taken place* and insofar as this non-passage and this 'non-place' are due to the singularity of the event. (The passage via the insurance account is an intellectual necessity that serves to jettison identification before we return, as if by accident, to the singular.)

We think that the singular is the opposite of the population, and that the evaluation of the singular, which is all that now takes place, must a fortiori adopt the ex-ante stance which was only an optical or stroboscopic illusion in the case of the population. With the single case, we think the time has come to *realize* the ex-ante stance, when the ex-ante stance must a fortiori disappear along with the *population* that had brought it about. The whole problem comes from the confusion between ex-ante and the singular case, when they are two different things. The individual is not the same thing as the singular: in this observation, the whole mystery of the market and of the paper is contained. Rather, the contracted-exchanged matter of written contingency must come from the other side of

the singularity, via the 'non-individual' reaffirmation of the ex-post stance, which is immanent and doesn't attach to any particularly identified individual.

The contracted-exchanged matter of contingency does not come from the side where the singular is opposed to the population, but from the side where what happens ex-post – the irreversible strike of contingency – although singular, is not an *identified* singular. It is the 1 of contingency without the 0. We have the writing before the state, difference as such, non-analogical difference, as was the case with Pierre Menard; for it is the identity of the 'individual' or of the generator that informs the differences and makes us expect them. Though the event may be a singular event, it is not identified today and thus we are not obliged, under the pretext that it will be identified one day, to address it in the ex-ante stance. What happens ex-post – contingency, the 'It could have been different', the irreversible strike – does for the singular event the same thing that ex-post statistics did: it does not identify it transcendentally, in itself. In any case, 'the whole market' will be there (all of the recalibrations) in order to turn any event that we would have been tempted to identify in advance via statistics into an event truly without a reference class. The exchange, we said, is the same thing as the massive non-identification that is due to the strike of contingency; this is why it lies outside of time, outside of the depth of the identified possibility, and flows into the mass of the final contingency. When we think that the singular event is an individual and is opposed to the population for this reason, when we think the singular through a wrong return from the population, we keep tacitly in our head the axiom of identity (or the axiom of the generator) whereas this axiom is in reality smeared by the final strike. The singular event really comes from the real and massive future; it never was a conceptual event. From today, it is loaded with the whole market, which 'un-identifies' it. This is the meaning of the matter of paper.

9.3.2 Single-case statistics

It is not the identified singularity that returns in reaction to negating the population but contracted-exchanged matter, which reaffirms the passage via after the event. Rather than the event being identified in the present, it is marked in the future (this is what replaces the illusion of identifying it through the return from the population; this is the substitute for the immanent identification of statistics: the marked strike, 1 without 0, flows in the same direction as the statistical mass) and it thus returns to the present spot together with *all of the market*. It is not the event, as such, that interested us in the time of statistics, so much as a certain immanent, massive, non-identified way of addressing it through the whole insurance account. In the absence of the statistical population, this way of addressing the event indirectly and without identity is replaced by another account and by another house: by the paper and by the market. The market price seems to address the event directly (ex-ante), but this is no less of an illusion than in the case of insurance and the passage through its global account.

Once more, let us be guided by immanence. When the event was lost in the population, the evaluation was, in reality, carried out across the mass of the global insurance account; immanence was guaranteed because the event was not identified as such, and the transcendent illusion that tasked us with identifying it was denounced as an optical (stroboscopic) illusion and the illusion

of the ex-ante stance. But the event is now 'accidentally' singular. By this, I mean that immanence is still in effect and that we still wish to treat the singular event as a population wherein it will not be identified, but that the 'accidental' absence of the population prevents us from doing so. How to *go further into immanence*, passing from the population to the singular event, and not retreat towards the transcendent individual? How can one argue that the singular event is, in reality, even more immanent than the population, and not isolated and elevated by transcendence? Certainly, the event was not recognized in the mass of the population (that is to say, captured by the reciprocity of a state or an identity function), and that was facilitated *by the account and by money*, which are the royal road that we must henceforth keep in mind at any price; but what is the non-immanent element that still remained in the population? How can immanence progress further?

What still remained transcendent in the population was the population itself, or the principle that assembles it. Between each member of the population is introduced the principle of the population, its generating principle. For, in reality, difference is only recognized between each individual of the population in the name of *identity*. We presuppose the identity of the generator in order to contrast the differences. This is what introduced chronological time and the conceptual event: all of probability. We count the population and we assemble it. More immanent than this would be an *uncounted singularity*, a non-individual singularity. We must maintain the set-up *which did not identify the event within the population* (the account and money) but, at the moment when the event becomes singular as a matter of fact, we must not count it either.

Here, the individual is a recognized individual only insofar as it returns from the population by the *illusion of the ex-ante stance*. But we said that we must continue the path of ex-post immanence, going towards the singular *through the population* and not by returning from it, in a yet greater divergence, as the continuation of the immanent principle of the non-identification of the singular event and not as its denial.

In this domain of ours, we do not speak of counting in the sense of counting the elements of a set. For, in this extensional logic of count, the singular event would be individual, of course. For us, it is not even the notion of the *quasi-set* that is at work, where the cardinal number would be maintained and the ordinal number abandoned, where the number of members would be recognized but not the order that would allow us to recognize them individually and to tell the difference between them. Or we could say that our situation comes down to pushing the notion of the quasi-set to the limit of singularity, the limit question thus being as follows: If we can recognize the number of elements of a quasi-set without recognizing the first or second or third one, how would this apply to the case of a quasi-set reduced to only one element? Can the event be the only element of a set and yet not be the first element?

If this is possible, doubtless it is because we speak of events and not of entities. It is in itself, and not in a set, that the event is singular yet, despite everything, not identified. The strike of the event *does not count*: it is not an individual. The event passes. We speak here of non-analogical difference: of the strike which is what it is *only insofar as* it could have been different. (This is the definition of contingency and of the event.)

For here is a new ‘constructivist’ definition of contingency (not convergent, but divergent, as we wished: even though it seeks to capture singular contingency). Suppose the notion of statistical population is established together with the identification of the singular event which, in reality, is not a true identification because it seems to return from the population only under cover of the mass of the insurance account. What could be a yet more immanent ‘isolation’ or ‘singularisation’ of the singular event – that is to say, a passage to the singular through an excess of immanence and through the continuation of the passage into the mass of population, not through a transcendent return? What could it be, this increase in the divergence and non-identification of the singular that we seek? We have already posed the question, and we now propose that the response should be positively *constructed* as being contingency. The singular that comes after the population via an excess of immanence and not via a return to the individual would be, by definition, contingency. More immanent than the population in the sense of the non-identification of the individual, this would be non-individual singularity, or the contingency of the strike.

There would thus be no ‘event’ in contingency, nothing dramatic yet, but simply a ‘logical’ play between the population and the singular, where the singularity would be regained through a continuation of immanence, and not through reciprocity and the ‘return’ of (to) the individual. We have simply rejected the notion of the identification of the individual *as such*. The individual is now to be understood as a transcendent illusion when the only existing reality is that of immanence, the illusion that comes from the mass of the population and the stroboscopic retreat which, in turn, are enabled by the account and money. Identification had no aim other than to *claim* to address the singular event under cover of the population and the count: it was tensed by time (the ex-ante stance) in the illusory feeling of a temporal retreat. Thus, there was no notion of the individual independent of population and time. Indeed, there is a circling in the notion of the individual, a second stage in which the identity function comes to be applied. This is why it must disappear if the population disappears.

Relative differences are recognized between the individuals in a population at the same time as the population is assembled and declared to be one. This is the definition of a *population of individuals*. The differences become relative (or analogical) with regard to the generating principle of each individual in the population, which is only the conceptualization of the difference as the repeated realization of a conceptual event. Thus, the population finds itself constituted of individuals at the same time as the individual is spotted and recognized via a ‘return’ from the population that will have done nothing other than borrow, in our case, the artificial passage of time – the stroboscopic retreat. Without this retreat, we would have only had a massive population, hardly appropriate to anything but the statistical law of the ensemble, without probability and ex-ante expectation. The whole manoeuvre thus consists in reducing the population to the unique and single event, without diminishing its massive and statistical sense. Would a single-case statistics be thus possible? Indeed, there would be a ‘residual’ (fossil) sense of statistics that would be maintained, in the single case, *in contrast to* the individual and to probability. It is that of paper and of the market. (The underlying asset does not follow a statistics; it follows its writing; this is the alternative interpretation of the formalism.)

9.3.3 Contingency of the strike

We want the identification of the unique event to remain *lost* in its singularity just as it was lost in the population. Through the singularity (which now replaces the population following the reduction) we must address the event globally and massively, without discerning it. This singularity which 'loses' the event is the contingency of the unique trait. It is certainly unique, singular even, but it pushes this singularity to the extremity of the One *that is opposed to the extremity of the individual*: the One of contingency and of massively contingent reality, the One without the zero of the identification of alternative states, the One without the identification of the individual analogical state.

It is easy to see how extension (whose vocation is totality) can fail to express the notion of necessity. No matter how hard we try to define a necessary thing as that which takes place in all possible worlds, this definition remains external to the thing (since it brings into play the replicas of the thing in foreign worlds) and does not convince us that, simply because the thing *happens* to find itself reproduced in this given totality of alternative worlds, it must be necessary in and by itself – necessary in the sense of an intrinsic property. For extension is precisely inviting in contingency. Isn't it contingently that these alternative worlds seem to cover the totality of our spectrum, and contingently that the so-called necessary thing is reproduced in them? Thus, the totality of worlds and the observed persistence of the thing in those worlds might only be an accident, a mere coincidence that might as well have not taken place. However, it seems harder to argue that possible worlds are not a satisfactory representation of contingency either. Yet, contingency is only the negation of necessity. Why would the extensional logic of possible worlds fail in the case of necessity and succeed in the case of its contrary?

Doubtless the illusion of a difference between the two situations comes from the distance, relative to the supposedly necessary thing, of the totality of worlds and of the phenomenon of its replication in each and every one of them, when the contrary – namely, that a contingent thing in this world is a thing that could be different in at least one other world – seems closer to the thing and to the notion of its contingency. Yet, it is with different tokens of this other possible world that we end up filling the entire spectrum of the possible and we obtain the necessary. It is the same notion of extension that enables us to count and identify the first possible world in which the contingent thing is supposed to be different and to complete this count and the identification until we cover the totality of worlds in which the necessary thing is supposed to be the same. If intension must be lacking in the extensional notion of necessity (the totality of possible worlds), then it must be lacking in the extensional notion of contingency (difference in a possible world). But how could contingency be intensive? If something, in the nature of extension, is the cause that extension cannot credibly express necessity at the limit of the totality of worlds, then something, in extension, must also be the cause that extension is not credible either in expressing contingency as early as the next step we make in the next different world, and well before we cover the totality of worlds. If extension has an inadequate way of maintaining the identity of the thing across worlds in order to express its necessity, then it must conversely have an inadequate way of introducing the difference of the thing, in the next different world, in order to express its contingency.

This intension of contingency that we are looking for should be the sense of reading it which never leaves its strike in search for other possible identified worlds (the analogical mode) but which reads the strike *as* saying nothing other than the fact that it could have been different: this other face of writing, whose appropriate schema of thought is precisely the exchange and not the distribution of states. A new notion of difference is to be found here which incises and drills the matter of writing until it reaches the geometry of the exchange. We do not evaluate difference *relatively* to external worlds, we exchange it *absolutely* – we *price* it – in place.

The totality of possible worlds may suffer from the major defect of the *untotalization of the possible*. Indeed, the infinity of possibilities that confront us may precisely fail to be totalizable, like Cantor's transfinite numbers.⁸ As a matter of fact, the financial market, together with the necessity of exchanging contingent claims at prices that diverge from the values that the identification and totalization of possibilities would assign to them in valuation theory, presents any such hypothesis of a totality of possibilities with the same refutation as Cantor's diagonal argument. In any case, the most immediate argument against totalization, and before even invoking the chain of infinities, can begin with a critique of the very principle of extensionality that governs set theory. The contingent event simply does not lend itself to the logic of the element of a set. To see why, we need not necessarily argue from above, to the effect that any totality of possibilities would ipso facto create the external contingency or event that would come and disrupt this recognized and identified universe – an argument to the effect that necessity as sum and total of possibilities has the necessity of contingency as its other face (Black Swan events can only happen) – but we may also, and perhaps even more effectively so, argue from below and from the complement of totality, which is the void, to the effect that the event precisely emerges out of nowhere and that it is not the element of any set. It is perfectly possible that two ranges of possibilities, which are perfectly totalized, should be physically incompatible and this would mean that the physical object, whose properties are described by the ranges of possibilities and which gives physical meaning to those possibilities, is in fact constructed subsequently to the reality of contingency, which has the nature of event and takes place at the stage where the event emerges, the stage where it emerges from nothing. 'The exterior of possibility where the event takes place absolutely and im-possibly', I have said, 'is the same exterior that establishes the incompatibility of ranges of possibilities in quantum theory.' The contingency that plays itself out at the hinge between the extensional ranges of possibilities or – what comes down to the same – in the void of possibilities is, for this reason, refractory to extension. Badiou expressed this intension of contingency by saying that the event is only an element of itself (thus contradicting set theory). But, in doing so, Badiou expressed it as a function of a *future* extension: he kept the event inscribed in time, even if that meant thinking that the event does not exist (and does not fall under any existential quantifier) *before*. What I propose, by contrast, is that the event is independent of time (whether before or after).

Matter and Geometry

10.1 The singularity of writing

The primary definition of contingency is that which takes place outside the framework and the context, that which escapes the count. In a given situation, we expect the production of a predicted result – for example, we expect that a body will fall vertically when we let it go. Contingency is, then, the arrival of an unaccounted-for or uncatalogued circumstance; for example, a violent gust of wind that carries the dropped body far from the vertical. Contingency is by definition singular: that which could have been different, not within an identified population, but for the reason that it could, above all, have not happened. It is, first, against itself, and not against a background of possibilities, that contingency is singular.

From the gust of wind that takes us out of our context, no return or recontextualization is possible; there is no return towards the state. For the moment, our eyes remain fixed on the inside, and the gust of wind, or contingency, is that which *should not have happened*. There is one single strike in this expression, one single departure; there is no rounding and no identification. Contingency has, then, nothing to do with possibility. It is only when the event is repeated that we begin to place back inside the circle that which singularly escaped it and that the result begins to be distributed statistically. Once repeated, the gust of wind is no longer only contingent but it becomes *some* gust of wind, any gust of wind whatsoever (*quelconque*). The cancellation and averaging of errors produce statistical regularity. The singularity, the unforeseen, now becomes a statistical law; the phenomenon is recontextualized within the framework of possible states to which probabilities are attributed. Until the day comes when a new contingency takes us out of that framework, too.

Our whole enterprise consists in grasping contingency at the moment of its departure and its intrinsic difference, before this regrouping and before this repopulation.

We set out, first, from David Bohm's definition, in which he makes clear how contingency, as escaping the given context, is later recontextualized as chance and randomness:

[*Contingencies* represent] essentially independent factors which may exist outside the scope of things that can be treated by the laws under consideration, and which do not follow necessarily from anything that may be specified under the context of these laws. Such contingencies lead to *chance*. We are here taking the word 'contingency' in its widest sense; namely, the opposite of necessity. Thus contingency is that which could be otherwise. Chance will then later be seen to be a certain very common form of contingency [...].¹

The singular event could, and even should, come first, without the intermediary passage via the population; it must be absolute and not a derivative of the population. In our late reasoning about the singular event, we have used the population only to show how the event could be reduced to its singularity without becoming an identified individual. The set-up of the insurance company and of its global account have served us only to retrieve the forgotten question of the singular event, by recalling that its identification, which is so contrary to the notion of singularity that we wish to bring forth, is really a voluntary and not a spontaneous act, which is linked to time and is facilitated by the temporal conductivity of the insurance account.

The singular event could come first on condition that we reject the 'focus' on the individual, which we now know to be transcendent; or, better yet, on condition that we do not start with the individual at all. Consider that the reference class, statistics and probability are a particular case, a degenerate case even, of the primary category of absolute contingency. The event is not identified: it does not belong to a set or to a population, even one reduced to its singleton. Writing and exchange, that way of dealing with the event and addressing it before it takes place and before it is identified, is probably an alternative to Badiou's set-theoretical metaphysics. Rather than deactivating the axiom of foundation, as he does, and declaring that the event does not belong to any situation given in advance and is only a member of itself, it is perhaps enough to look at the event outside of time. The event is not identified and is not a member of a population either before or after; it does not occur in its own situation and does not become a member of it at the moment when it happens. The event is singular, and since it is written, *ex-post*, in the contingency of the strike without the state, it is exchanged, *ex-ante*, via the paper and the market.

The analogical state is opposed to singular-being. We wish to address the event and challenge it (evaluate it) through the whole population that loses it, as we did in the insurance case, only in the present case this population is reduced to a singularity. This singularity, which is thus inherited from the *other* side of the population (the side that does not return to the individual) and which removes the individuality of the event from the event, is the absolute One that covers the *whole world* in the univocal passage of its strike. This strikes passes not in order to discern and to identify individuals – because in order to discern one must return – it only passes to the extent that, in passing, it says that the event could have been other. There is a very singular, and even less than singular, sense to contingency. Contingency is so much the absolute One that it is less than one. It is the One without the balance of the one (without the return).

The sealed strike of writing plays the role of the 'singular population' that loses the individual in the singular event. Derrida has already noted the non-logocentric, delocalized and 'lost' aspect of

writing. It is when all is lost that one opens the testament and learns what must be done and what must *then* be paid. Sealed writing has the property of being opened only at the end, at the heart of the event, and of pronouncing only the strike of contingency – to be *this* only insofar as it could have been *that* – *without any prior visible transition* that would have allowed us to ‘predict’ this or that by way of an analogical state.

Writing is a loss of time because it plays with time; it steps out of the ‘eye of time’ and loses focus. The materiality of writing is literal: writing is materially a part of the future contingent world in which the strike will be, then and only then, revealed. It is only accidentally that the contingent claim occurs before the event, in the time that accidentally precedes it. We should understand literally the ‘metaphor’ of the testament where we open something that has been neither seen nor heard before, which has never been duplicated and rendered reciprocal (recognized) by the state but which is, in spite of that, already written.

The paper on which the contingent claim is written does not present two (or more) states of the world at maturity, but one and the same writing. Depending upon the state of the world that then obtains, we know whether we should pay or not. But whereas the state of the world would be different depending on whether we find ourselves in the state where we shall pay or in the one where we shall not pay, the paper remains the same. It is not identical with the state of the world, for between the two we find the distance wherein is inscribed he who must read the paper in the light of the state of the world that has been actualized, and determine what must be done. My thesis is that the paper is exchanged *perforce* before maturity (and that the whole market exists *perforce*) in virtue of this distance, which allows the strike of writing to potentially reside and remain unique in two (or more) different worlds.

The material virtue of the written paper that instructs us as to what must be done depending on the case (this writing of the condition and of contingency) is the same virtue that dictates that the paper remains in all cases, and the same that dictates that, for this reason, it remains *before the case*; that is to say, that it is exchanged. (The matter of the paper does not anticipate the event; it is a residual of it like the ruin, and the market is the book withdrawn in front of the field of ruins.) States of the world cannot be exchanged before expiry; they are nailed down in their hierarchy and in their arborescence; they are not ambiguous and must be identified – whence their fragility. The ambiguity or ubiquity of the paper at expiry (the fact that it is materially the same in all states and, as a matter of fact, that it may as well disappear altogether) is expressed (and is even cashed in), before expiry, by the capacity of lending itself to the exchange. It is the same capacity and the same matter. At work here is the principle of equivalence of two writings: geometry (the market) is the other face of matter (contingency). The state as such, even when varied in possibility, is not equal to material contingency. Certainly, the state could be different, since it is one among several possibilities; but this difference is not inscribed in it. The state is different only analogically, within a population or in the process of a transition. It is exactly conceived to represent this difference and it depends upon it. The state disappears completely when the representation or the sequencing of difference disappears (what I have called the ‘transition’). Unlike the paper, it leaves no material residue or trace.

We think of the paper as being only the copy, or the recapitulation, of the indicated states. We represent the tree of possibilities: if the paper is worth this or that in the future realized state, then it will be worth this today. This logic of evaluation and of attribution only takes advantage of the diagram that is provided by the contents of the paper; but it neglects the matter of paper, whose essential property is that it maintains itself across states and thus escapes the transition that makes one state disappear in favour of another. This identity of the paper across states is the guarantee that the payoff written on it should not be read in the mode of positive attribution but in the mode of the testament: in the mode of the 'It could have been different.' It is this compact matter that lends itself to the exchange. To exchange is to respond immanently to the contingency of the payoff. Rather than looking at it under the tree of attribution and evaluation, we place ourselves *at the same level* as the clause 'It could have been different.' Now, to place ourselves at this level is to place on the same plane the intention to evaluate and not to evaluate, to engage our money and to withdraw it. It is to invent the possibility of *switching our position*, which is the market. We evaluate contingency by the contrary of the construction of states – by the guarantee that we can withdraw from the game at will and leave it to somebody else. It is not the process by which the market manages to determine the price that interests me here. What is important, once the market is established, and what then and *only then* defines the price (for it is a misunderstanding to try to define it beforehand, or on the way to the market), is that we can, at this price, withdraw from the evaluation and recoup our money – a notion of break-even even more direct and immanent than in the case of the insurance company.

The transition leading to the state prevents it from being ambiguous: it dictates it, and the state will be what it is without being able to be different. As for the paper, when it translates the realized state by an incision in a matter that remains the same across states, it detaches the state from the tree where it is nailed and it inverts the sense of its dictation into the sense that says at once that it *could have been different*. The paper supports the contingency of the state instead of its possibility, and subsequently detaches it from any possible transition – for the state is only the state insofar as it is identified via the transition, and the transition, as we have said, is only the simulation of the reciprocity of the analogical states. The paper is the only way to detach ourselves from the tree. As a matter of fact, the contingent world in which the instruction written on the paper is read may not correspond to the explicitly instructed state: it may be a world where the world ends in a cataclysm, and the paper burns up with the rest of the atmosphere.

The instruction written on the contingent claim consists in paying. The identified states of the world are different and distinct; the paper bearing the written marks is the same; but it is the payoff that is contingent. It is because the written formula (which is identical and unchanged across states) is translated into money payments that the complete strike (formula + money) produces the writing which is *what it is insofar as it could have been otherwise*. It is money that is attached contingently to the invariable paper at expiry and to that extent a price (also expressed in money) will be attached to it before expiry.

What allows us to erase the states underneath the paper, as we have said, and to nail it irrevocably in the direction of the place, is money. It is money that finishes up the contingency that had begun with writing, and it is money that comes back in exchange, in the form of price. Money is attached

to the written formula at expiry in order to produce the contingent strike, and is attached to it before expiry in order to produce the market price. Money can be defined in this way: as the medium of synchrony between the contingent strike and the price. Money *counts*, unlike absolute time in which absolute expectation used to take place. It counts insofar as the paper instructing the contingent payoff is negotiated for money before.

The possible state is only a state, as we have said, in virtue of the transition that makes it proceed from possible to actual, and which operates in time. But money replaces time, and is concerned with the transition between the price of the contingent claim (or the place that takes place 'before' and which is the market) and its final strike. Just as time is supposed to realize possibility in a false role-play with reality (a representation), money *realizes* contingency in the true sense of reality. It realizes contingency in the sense that that which translates the contingent strike at expiry – money – precisely does not make a *state* out of it (for the state is that which lends itself to a false realization, that of possibility) but a mark. The state (A, B, \dots) that is written on the contingent claim (if A , pay $x(A)$, if B , pay $x(B)$, ...) is a 'semblance of a state'. It is the mark of contingency and not of identification. It would have been an identified state if it had taken part in a transition. But, on one hand, it is already too late at the time of opening the testament and all transitions have already passed and, on the other hand, the state is eclipsed by the opened testament, which is not a programmatic attribution of states but a cause of money payments and thus of injustice, of irreversibility. Money springs from its box at the opening of the testament, and it exerts pressure in place, going further still in the direction of the 'too late' and blocking any return (reciprocity) towards any calculation or program or state whatsoever; retaining, in the contingent world, only the side that says 'It could have been different' without saying that it was possible. This irreversibility in the nature of money is due to the irreversibility of the loss that only money can sustain.

The contingent claim is a testament whose envelope remains closed before expiry, and for this reason it knows only of a closed exchange in which no state is either revealed or identified. It is literally a writing, something that is exchanged before the fact, but which, in reality, comes to pass after the fact; that is to say, too late, when the only thing that *remains* to say is that the amount of money could have been different. The role of money is thus to press in that direction. It is in pressing in the direction of that matter, which only affirms contingency on the side that says 'It could have been different' and which does not return via possibility, that the paper reacts to this material pressure and rebounds towards the exchange. It is in reaction to the strike that does not return but presses in matter without the reflection of the state – the strike of money that the paper bears materially – that the paper materially returns (rebounds) and is exchanged. The materiality of the paper is such that we must wait until the end to observe the irreversible loss (up to and including the loss of the paper itself) and that we hold the paper at once in our hand – that the paper is *still here with us*, to be exchanged.

We became interested only derivatively in the *content* of writing, or the list of instructions and states. At the beginning, the content was supposed to remain concealed (sealed in an envelope) and all that could show itself to us was the mark or the writing, not the state. It is in a second stage that we have abstracted the writing from the material and irreversible and pressing world that it kept

only for the end, and that we have interpreted it as a *state*, as if we were standing in an initial state today, where the probability could be calculated.

The reality is that we are stuck with the paper, today, in the market where it is exchanged, just as we are stuck, at the end, by money and the irreversible strike – stuck by the then unexchangeable world. Just as the One of contingency is not detachable from its world (and I have said that this One was more immanent than the statistical population), the paper upon which it is marked becomes non-detachable from the market. The market of today and the unexchangeable world of tomorrow both abolish possibility, the one through unexchangeability and non-reciprocity, the other through the exchange. The two writings are equivalent or, rather, the one is the *geometry* of the other. Geometry is there wherever matter goes, and the market is there wherever contingency goes.

Just as time, or the filtration, operated the transition that distinguished the state, and even made it a state and affirmed it as such (the transition wherein physical time was really produced by logical time and logical time was produced by the attributive reading and the corresponding present state which integrated the analogical difference of the succeeding states), money, which replaces time, operates the *transition that negates the state*, the transition which only retains contingency on the side of the ‘Too late, but it could have been different’. Money is the alternative temporal logic in which the reading of the strike of contingency does not return but never ceases to pass. This is because money, by contrast with the chrono-logic that rules the tree and the transitions between states, can sustain the irreversible loss of money. To read something in a final (‘too late’) yet contingent fashion, to read what is written from its material side, is thus already to be steeped in money and to have money as the sole mode of circulation and exchange, which does not serve to integrate into an initial state, but to disintegrate, or to exchange, in the market. Money is to price what chronological time was to the initial state that logically convoked the attributive mode of reading.

10.2 The singularity of the exchange

Let us imagine that price and exchange have not yet been invented and try to deduce them materially from immanence and singularity ‘alone’. Singularity must not lend itself to the ex-ante stance any more than the ex-post stance must necessarily be linked to the population and to the long run. It is the population and the mass of statistics that guaranteed the ex-post break-even and the immanent evaluation in the case of the insurance company and its account. How can we maintain the ex-post break-even in the case of the singular contingency?

It is immanence that is characteristic of the ex-post stance, and not necessarily the population. The population is only a particular case of immanence, and it is the population that we must replace. In the statistical case, the single event was ‘lost’ within the whole population and it is there that its ‘price’ was inscribed – or the premium of the ex-post break-even. Now, in the truly singular case, and to rehearse our argument once more, we must replace the population and the mode of inscription of price: the price must now attach to the singular event through immanence and a ‘mass’ that replaces the population, and it must not attach to it in individual fashion. We must maintain the loss of the individual and of identity *and* maintain the singular attachment. How might this be possible?

It is the paper and written contingency (only to be revealed in the passage of the strike at expiry) that will replace the mass of the population. To write contingency is a move against the transcendence and the individuality of the state; and it is this that will enable us to 'wait for the long run' that is the immediate liquidation of our contingent claim at its current market price in the same way as the population had enabled us to wait for the ultimate break-even in the statistical case. The price is ex-post for this reason (through the inexistence of time deduced by the same token). The ex-post stance of the insurance came from the fact that we had never retreated towards the insured individual in reality; the individual was only a belated extrapolation. Equilibrium and the break-even were only attained in virtue of the population. Now, in the absence of the population, yet in the presence of the singular event, we must attain equilibrium in one go, by moving in the same direction as the population through the paper; that is, without reaching to the individuality or the identity of the event through the 'mass' of its singularity (which only the paper can materialize) any more than we had reached to the individual through the population. It is the exchange that will enable us to find an equivalent mass (an immanence) which will not be relative to the number of individuals (as there are none and the event is truly singular), and which will retain from the mass of the population only its way of swamping the individual.

Let us imagine that the materiality of the paper and of the contract (that is to say, of writing) has not yet been discovered. Let us imagine that we do not yet know what it means and what it implies that a contract should involve its issuer in a contingent payment and that the contract, for this reason, should be exchanged before its expiry and admit of a price. We do not yet know the time interval that separates the present price from the future contingent payoff: we know nothing of the virtue of the paper which consists in retreating massively from the heart of the event, and no market yet exists. But suppose that we have discovered the notion of break-even in the long run, which only operates in one sense and does not go back in time.

By adjusting the insurance premium through trial and error, the insurance company attains break-even in its ex-post accounting equation and creates, solely by this route, the notion of the individual probability and of the valuation of contingency as mathematical expectation. Here, we see – if it need be repeated again – that probability only makes sense within the global account of the insurance company and only subsequently to the category of money, which alone enables the contingent payment that is made after the fact to 'go back' in time, against the current of the statistical 'tide', and to materialize as insurance premium and present value.

Probability thus having been invented (or extrapolated), we then directly considered a contingent claim and found nothing better than to value it as the mathematical expectation of its payoff under the objective probability measure (the actuarial approach), or under the risk-neutral probability (the equivalent measure which ensures that one can continue to evaluate the contingent claim formally as a mathematical expectation, or, in other words, as if its valuation agreed with a global account that will only break even in the long run, when, in actual fact, supply and demand entail that investors on the contrary discount a risk premium and are not content with break-even in the long run).

Now, with this we committed two abuses. In the first case, we supposed as given the insurance company and the global coherence of its account (this immanence) which alone would give a

meaning to the notion of objective probability. In the second case, we supposed as given precisely the market that we seek to construct.

The reality is that the event is singular and does not belong to a statistical population. Consequently, the 'account' that must evaluate the contingent claim will have, at once, to be independent of the insurance company and to come first to the derived notion of probability. We can all guess that this mysterious account will be the market, and, instead of representing the market price by probability (presupposing a population and a break-even in the long run), we have to recognize that the price and the exchange must *replace* the notion of ex-post break-even. Price must follow from the singularity of contingency *before* the derivation that leads to probability through statistics and through the population. It is not the notion of probability that must be adapted to the single case but, prior to that, the notion of the break-even. Let us not forget that the exchange is not yet invented and that, for now, the only 'house' confronting contingency is the insurance company, with its capacity for a global account and an ex-post break-even.

Neither the exchange nor exchange value yet exists (there is no partner with whom to exchange and this partner must therefore be created). The first elements of this primitive world are contingency and the statistical population and, accordingly, the only notion of valuation is the one derived from the insurance company's ex-post break-even and which seems ex-ante by extrapolation. In the absence of population, we thus do not know, for the moment, how to 'value' a contingent claim; we do not know what this would mean. It is out of the question to first imagine the (ex-ante) value of the contract and then to imagine that it will be exchanged and will admit of a price. Price, when it emerges as the result of the exchange (and of the invention of paper, whose immanence will replace the mass of population), will have nothing to do with value and its ex-ante posture.

A certain premium is associated with a contingency, and this premium guarantees the ex-post break-even by virtue of statistics. It will subsequently seem, for anyone who looks at the problem backwards and retreats in time, that the premium is a sort of average of the contingent payoff. It is the population that bears inscribed within it, 'at the same time', all the contingencies, and it is by confronting it as a whole, through its account, that the insurance company arrives at the average of contingencies. It is only when we reverse our gaze that the mechanism operating between the total population and the global account yields the probability of the single case and mathematical expectation as an average; but it is precisely this mechanism that needs to be replaced, in the truly singular case, before we invert our gaze: by keeping the pressure and the direction of the population, only replacing the population.

It is the paper that will bear written upon it, at the same time, the different contingencies. (Is there any other means to write them, as contradictory as they may be, and to keep one matter and one mass, which would thus replace that of the population?) Rather than awaiting the population and awaiting the global break-even, the virtue of the paper will be precisely that he who holds it reaches break-even without waiting, *by unwinding their position in the market*. There was truly no time dimension or temporal retreat in the univocal direction of statistics, for there was only mass and accumulation. There is no time dimension either in the market: the population disappears in the turning over of the paper that is called the exchange, and with it disappears the false impression of time.

It is accidentally and not essentially that the market price is established before the final strike of written contingency, giving the impression that the market evaluates the contingent event ex-ante. It is because it comes before (accidentally) that the price gives the impression of being linked to the final contingency via probability. For probability is the misguided impression of time, which is improper to the event. We think that time separates us from the event and that probability is meant to fill this gap; we think that time is oriented and tensed towards the event – that it times the event in some way – and that probability measures this tension. But time is incidental to the event; we simply find that time is there. Since it is the only theatre that we perceive, we wrongly think that it stages the event.

The event does not yet exist (logic has not caught up with it; it is not yet identified by time). It is essential that the event does not exist, most times, because it is contingent and, for this reason, cannot be permanent. And ‘when’ the event does not exist, since we ourselves live in time, we think that all that exists, on the other face of the event and as if to replace it, is time. ‘What else but time can fill space,’ we think, ‘when the event does not fill it?’ Our question itself presupposes time, since it is temporalized.

I have already proposed that the market is the other face of the event, the place that replaces it ‘when’ it does not take place or, rather, *there* where it does not take place (if we must abandon the time dimension in which the event and that which is not yet the event succeed each other, and retain only place, or a topological dimension). Contingency is a question of place, not a question of time. Contingency is essentially a place, we said, and when it is accidentally separated from the time that does not make it happen yet, what remains of its place, independently of time, is the market. (What remains is the reading of contingency as ‘that which could have been different,’ which is independent of time and which is the exchange.)

It seems essential to us to live the event, to know it and to *frequent* it in time (a presupposition of frequency and of the population, when the event is by definition so singular that it is not necessary to know it and to encounter it in order to be with it and in its middle). Now, this is only an impression, and the event (or its place) must be visited, not lived. Statistics always comes to pass after the event; it is only the trace of it. We collect statistics, but we do not explain the mystery; we content ourselves with an ex-post accounting. As trace, statistics is a writing and a ruin; it is something we observe empirically without a generator. Probability and the random generator are an illusion, as we have said. The trace is immanent; it is immanence incarnate for he who seeks the cause, or the transcendent notion, of the event.

In the same way, writing is invented as the trace of the singular event that enjoys neither statistics nor the population. Writing is designed for the time that follows the event: it is only at this moment that the testament is opened and the possible total loss of the paper, which cannot be encoded as a clause in the content of the paper but is the other face of its materiality, is observed (keeping in mind that this irreversible loss is not incompatible with the nature of money, quite the contrary). The exchange of the material paper is, then, just what comes to replace the mechanism that allows one to retreat in time, in the case of the statistical population, and to have the impression that one is addressing contingency in an ex-ante manner with probability. The exchange does not essentially

occupy the time that precedes the event. That it, in fact, does occupy it is only an impression that is due to the unavoidable accident of time. Through writing and the mass of the material paper (a mass that is not but *remains*), the strike of the event has the virtue of detaching itself from its place or, rather, of remaining in this place but escaping the coincidence of time and identification. Once again, imagine the passage of the event in a place rather than in a time, and then convince yourself that time is incidental and that consequently the passage of the event can take place at any time, even before the event. This guaranteed passage is the market.

What does it mean to say of a process – that of price – that it is accidentally, but inevitably, inscribed in time? That we must never direct time and the identification of the state upon it. (Probability must never mix with price.) The essence of the strike of the event – namely, the fact that *it could have been different* – this essence which is a writing, is translated by the exchange as its other face. The matter of the paper (contingency) finds in the exchange its geometry. (Why does matter need geometry? This is the deep question.) Just as inertial mass finds its equivalent in gravitational mass, the massive writing of contingency, which distinguishes no state and which only, essentially and independently of time, repeats that the event could have been different, finds its equivalent in the mass that produces the ‘field of price’, the field of exchanged matter.

Now, this equivalence takes place outside of time, according to the very logic of the non-attributive reading – the reading of the strike inversely to the state and to time. Maybe the price process comes to pass in empirical time – this is the inevitable accident – but chrono-logical time (that which comes from the transition between states) is improper to it. It is only improperly that we model the price process in terms of probability and transitions between states. On the contrary, the market of contingent claims and recalibration are such that no stochastic process that we have imagined for the underlying asset price can even get started, and the consequence of this, in virtue of this abolition of possibilities, is that the price of the contingent claim is in immediate communication with the strike of the event. The massive way of reading the contingency of the strike, which says that it could have been different without any distinction of state and whose other face is the exchange, finds as its translation, in the chronological time into which empirical time inevitably throws it, the abolition of possible states and the invalidation of all stochastic processes – an abolition and an invalidation that are themselves due to the exchange.

Part IV

The Market of Contingent Claims (or the Matter in Black-Scholes-Merton)

Towards a Contemporary Theory of the Market

11.1 The stochastic narrative of the market

11.1.1 Definite states

A market is all sorts of things and is composed of all sorts of people. Economies and histories create the market. Ideologies are for or against the market. Despite all their causes or their biographies, markets are ultimately down to prices, or even better, to changes of prices. There is one instance in which the market receives a precise definition, and this is when it is reduced to a stochastic process ruling the price of some traded asset. This is typically the starting point of derivative valuation theory, and this reduction of the market to a pure numerical process, which screens off the 'real economy' and represents the market purely in terms of statistical models and coefficients, has often been criticized.

My claim is that, as controversial as this reduction of the market of the underlying asset to a mere stochastic process may be, it does not follow through its own intended precision. For if the purpose is to value the derivative written on that underlying, why not increase the precision of the thought by affirming that the underlying and the derivative will eventually trade in concert, as this is what we certainly have at the back of our heads when we commence such reductions? Let us try and define the market as the place where underlying and derivative trade freely, without logical hierarchy or, at least, hierarchy in presentation. The problem is that the derivative is defined as a function of the underlying at its termination, not as a freely trading security. For this, it necessitates a schedule established in terms of states of the underlying and of clauses providing what the derivative is worth in all those states at its termination; and this is certainly what has prompted the idea that the world (the market) should be organized in states before the termination as well.

The postulation of states of the world in which the derivative has to be worth something definite before expiry is what has precipitated probability and non-intrinsic non-arbitrage. States of the world are the presupposition of the fundamental principle of arbitrage. Note that those states need not be

prices of the underlying asset on which the derivative is written strictly speaking. Underlying the value of the derivative before it expires may be other variables, such as the volatility of the underlying price, which may not affect it at the time of expiry (since the derivative is no longer sensitive to volatility, then, but only to the underlying price) but may, at that time, affect the value of a derivative of longer maturity, whose variations would then constitute an additional dimension to the price of the underlying, at the expiry of the shorter-term derivative.

Surely the volatility of the underlying could, in theory, play the role of that additional state variable. The reason, however, why I prefer the price of the longer-term derivative as a state variable describing changes in volatility is that the instantaneous volatility of the underlying is not observable. Instantaneous volatility cannot serve, for example, in the definition of the payoff of a certain other derivative. Volatility is a theoretical concept. Just as the underlying asset price is not only observable, but also the result of trading and serves in the explicit definition of the derivative payoffs that we are considering, let us try and adopt as a rule that we will complicate and expand our state space only by using variables that make an explicit (i.e. explicitly written) difference in the payoff of some derivative. These variables need not be directly prices. The state of default or no default of a certain corporation or counterparty is a typical case of such explicit variable that is not itself a price. It is explicit insofar as it underlies the writing of credit default swaps, for instance, and the price of these, which are tradable, could then serve as an alternative mapping. Thus, we could request, without loss of generality, that underlying our state space there should only be traded prices, either of the underlying asset or of derivatives written on it.

But to go back to our first derivative, even though its value may not depend on the volatility of the underlying at its expiry, or equivalently on the price of the longer-term derivative, it will depend on that price prior to its expiry. There would certainly emerge model-independent (or intrinsic) arbitrage opportunities (i.e. we would find no pricing kernel), if, prior to the expiry of the shorter derivative and for a given fixed underlying price, the price of the longer derivative were to vary massively along the other dimension, which is volatility implicitly, without entailing correlative variations of the value of the shorter one. In any case, the point I am trying to make is that if states should be identified, at the expiry of the derivative, in which its value is explicitly prescribed, then valuation theory requires that states should be identified, prior to its expiry, in which the value of the derivative is definite also. There might be additional states to those that explicitly write the payoff of the derivative at its expiry, typically states of volatility, which are explicitly recognizable as states in which other derivatives admit of different prices.

In any case, the whole framework is couched in terms of an experiment that is defined in a state at a certain time and admits of definite outcomes at definite states at a later time. Thus, a list of derivatives is worth something definite today and will be worth something definite tomorrow. The fundamental theorem of arbitrage can't be formulated outside such a framework. It is not clear whether non-arbitrage requires a pricing kernel (i.e. the existence of probability measure) or whether probability requires non-arbitrage in its definition (the so-called 'invulnerability' to Dutch books in the definition of subjective probability). In any case, what seems to underlie both arguments and their reciprocity is the *assumption of definite states*.

What could be the alternative? It may be tempting not to organize the world in states before the expiry – to imagine that the derivative is not *worth* something (i.e. it is not evaluated over states) but is simply and massively priced or exchange-traded in the market without states. (Probably, we should relax the metaphysical assumption of states even at maturity and look only in the paper at its massivity, its ex-post fate, to be recognized massively.) If, from defining the market as the simultaneous trading of underlying and derivative, something really unhappy was going to be deduced concerning the entire framework of states of the world, then why not? Why not go all the way?

It is difficult not to imagine the world as made up of states of the world. Are we saying that when the underlying and its derivatives trade at market prices, they no longer settle in states? Aren't the states of the market precisely constituted of prices? Are we arguing for some fuzziness of the states? In reality, it is not so much the statics of states that pose the problem but rather the transitions between states. States are the indication of the settlement of a probabilistic problem. Statistical regularities have been observed ex-post; a certain state of affairs has been observed to obtain recurrently, and the move is subsequently made to map that in a list of states, thus setting the problem in the ex-ante outlook, preparing for prediction.

Something is exchange-traded, yet we insist on attaching to it a numerical value, which we apologetically call its 'exchange value'. It is the concept of value that I am criticizing, not the exchange. To exchange-trade is to hand over paper money against the contingent claim, in a quantity which we call its price. Two material pieces are involved (the contingent claim, money); the exchange is their meeting and their transfer one for the other, yet we insist on value, as if the only thing that mattered was the abstract number (the quantity of money). The exchange depends on the contingency of materialization or the persistence of both the contingent claim and money at the time of the exchange, yet we call the result a 'value', pinned on the notion of underlying state and the chrono-logic that corresponds to it.

The contingent claim does not have *value* at maturity, in the sense of a function valued in a definite state. There, too, it is exchanged against money following the written provision and depending on the recognition of the contingent world that corresponds to the provision, and depending, chiefly, on whether or not the material paper of which the contingent claim is made is still there to be opened, read and exchanged, together with the background conditions that grant the exchange. One has to wait and see whether the world still exists and the paper is still materially there. The paper has this way of *holding* time. It has the capacity of making time pass materially, not abstractly.

We cannot anticipate the event of opening the paper; the reading has to be ex-post. It is a true event in the sense that it is unique, and not one among many outcomes. For this reason, strangely, it is unpredictable. What makes it unique is that only then will it be resolved whether the paper (or the world at large) still exists or not. The only way to 'expect' the event of opening the paper, the only way to guarantee this ex-post reading as of now, is materially to own the paper and trust that it will reveal its fate once it is opened at expiry. We need to own the paper in order to defer all contingencies, no longer to worry about them and make them matter only at the appropriate time. Now, it is that same paper that is exchanged today at a price. Were it not for the material capacity of the paper to conceal

the event until it happens (the unique event of discovering the contingent world at expiry, up to and including its disappearance which cannot be predicted because it is unique), were it not for its capacity to *hold* the time of its own disappearance, the paper wouldn't be exchanged today.

A state is not a state at maturity because of the massivity of the paper and its unpredictability – because of its eventful character. For this reason, there is no exchange value and no state of the exchange before expiry either. One has to let the time of the paper – the time of its disappearance – pass to the exclusion of any other time. This is the time of matter and the arrangement of matter;¹ this is the time of the event, which the market (alone) otherwise mediates. This is intrinsic time – linked to the paper, to the event, to the ex-post character of the reading and to the *whole market* as the trading of all grades of derivatives will come to support it.

11.1.2 Derivatives prices as states

The fundamental theorem of arbitrage is more general than the case of derivatives that are written on a traded underlying. The states it considers, in which the experiment admits of outcomes or, equivalently, the general assets admit of values, are general states of the world: states of the economy, states of the political states, states of default or no default of the corporations or institutions participating in the economy and, eventually, states that are identified with prices of traded assets. The problem enters a new phase, however, a phase that is potentially disruptive of the whole framework, when derivatives written on traded assets are considered to the exclusion of anything else and their valuation problem becomes the only problem at hand. As we have said, because their value depends on their underlying at termination, it is thought that states of the underlying are needed prior to expiry, as well as states of any parameter governing the stochastic process of the underlying price (or its probability distribution).

We said we wanted to adopt as a rule that the latter parameter may be expressed by the price of another derivative written on that underlying. Why are we so wary of statistical parameters such as instantaneous volatility becoming state variables and do we prefer that derivative prices be used instead? Is it because the underlying price is the exemplary state of the world? Or is it because, in the trading practice, the fate of statistical parameters is to be implied from derivatives prices anyway? Nobody is their right mind ultimately believes that instantaneous stochastic parameters make sense in a market – derivative wouldn't be written and traded if the notion of such parameters were workable – so, is this the reason why derivative prices are instinctively more attractive as alternative mapping? Could an additional reason be that stochastic parameters typically depend on the stochastic model, when ultimately we are looking to describe a model-independent reality, which is the market? This question will preoccupy us for most of the chapter.

The underlying asset trades, so we typically represent states of the market with prices of the underlying. What else is there? How are the *dynamics* of underlying prices represented and made part of the market? That the price process of the underlying should be stochastic is an essential component of the market. But is the market also *materially* made of the underlying stochastic process, or should we wait until prices of derivatives written on that underlying are considered and stand as the material proxies of the underlying stochastic process? It is rare that a market should be

represented with the asset price alone and without the thought of the temporal process ruling that price (this is what trading is all about). Could it then be an essential and definitional property of the market that the temporal process of the asset price should also be represented by other *prices* – those of derivatives? Price as a category has the capacity to embed everything after all, and the market is made up of prices anyway.

The underlying asset price cannot be volatile while allowing, at the same time, that the price of an at-the-money option written on it is equal to zero. The fundamental theorem of arbitrage provides that the derivative and the underlying should be valued as expectations under a measure that is equivalent to the objective one and equivalence means that the two measures must agree on events of measure zero. Thus, the market has the unique capacity of expressing the essential property of the trading of the underlying (its price volatility) with the existence of a price and with the trading of another asset (the option). Is it then correct to say that the underlying asset price is volatile *because* of the existence of the option price?

Causality does not flow in this way. Neither does volatility later become stochastic because options start to trade at different prices for a given fixed underlying price. As we shall see, it is only the strictures of non-arbitrage, combined with the fact that the derivative is derivative, that impose the rewriting of the independent trading of options as the becoming stochastic of the volatility of the underlying price.

Although it seems static, the simultaneous existence of two prices (of the underlying and of the option) implies that the first is stochastic. By the way, could the *existence* of a price be independent of its variability? Note that we can always imagine the existence of a third price, that of an option written on the option. The market has the capacity of bringing on the same floor, as same nature and same matter, the prices of the underlying and of the derivative. Conversely, when two prices coexist (seemingly statically) and we consider that the market is everything (it is both the existence *and* the dynamics of prices) and that there is nothing but the market, what could ever constrain one price to trade in one prescribed direction rather than in another? What could ever constrain options, for example, to trade redundantly with their underlying, as the BSM model prescribes? Must we understand that to represent the trading of the underlying by a stochastic process whose consequence is the evaluation of the derivative written on that underlying is in contradiction with the *matter* of the market? Isn't the step in the direction of the time series of the underlying prices a step outside the matter of the market, whose definition, as we said, is that the price of the underlying and the derivative should exist on the same level and the same floor? Shouldn't the volatility of the underlying price, or the very fact of the trading of the underlying asset, *really* be represented by the derivative price instead? And what do we mean by *really*? What is it, in the reality of the market, that is in contradiction with the register of statistics? This, as you recognize it once again, is the question guiding our whole inquiry.

Consider the following riddle: What thing is of such a nature as to be stochastic and such that its stochastic nature is *represented* by something of the same nature? We would like to answer that that thing is *price* and its nature is the *nature of price*. To repeat, should the stochasticity

of the underlying price be represented by stochastic parameters like any other stochastic process or, instead, by something of the same nature; namely, the price of the derivative written on that underlying? Is there not a case, here, to argue that there is something in price and in its stochasticity that goes beyond the mere category of stochastic process? Shouldn't the trading of the underlying asset be equivalent to the writing on it, and therefore to the pricing, of the contingent claim? Can we think of another domain where this is the case?

Let us rehearse the stochastic representation of the underlying price for a while. Let us momentarily entertain the thought that the underlying process is stochastically represented or, in other words, that different prices of the underlying are overlain by a probability distribution and that this is supposed to be the whole market. How do we, then, make it a part of (the representation of) the market that the parameters of the distribution are stochastically changing, too, or that the non-parametric distribution is stochastically changing, too? Do we just say that there are, for instance, different states of volatility and that volatility is stochastically moving between those states?

Let us rehearse this stochastic narrative of the market for a while. Let us initiate the chain of thought by saying that, for some reason, we are no longer satisfied by the fact that the underlying prices follow the last probability distribution that we had in mind (for instance, the Gaussian). Indeed, we now think the process must be more complex. But how is the thought of increasing the complexity of the process even becoming perceptible to us? Is it because we are imagining a time series analysis of some kind and the Gaussian distribution no longer fits it? Or are we just thinking *in abstracto* and only *imagining* grades of complexity that are theoretically increasing? How long before this pressure for increased complexity is understood to stem from no other source than the feeling that derivatives of increasingly complex payoffs *have to* trade – for instance, vanilla options having to trade at different prices for a given fixed underlying price? How long before it is understood that different states of the world being identified with different prices (typically, of derivatives) is the first thought that we must have in order to make the market more complex, or simply in order to *think* the market, and that it is not a derivative thought? When is the mystery ruling the confusion (or the passage) between stochastic implied volatility and stochastic volatility finally going to be resolved?

Question: Is there a link between this peculiarity of the market – that prices could be the mapping of the stochasticity of prices – and the first peculiarity, which is that the underlying price is at the same time the lottery and the price of the lottery? There is definitely a polysemy or ambiguity of price at play here.

11.1.3 Variations on lottery value and random price

Considering the underlying asset as a lottery whose uncertain outcomes are the future prices of the underlying asset, one may wonder what its present fair value may be. Usually, the randomness that underlies lotteries is produced by a separate device: a die, a roulette wheel, the wheel of fortune, a random draw from an urn, and so on, and the question is then asked at what value, or amount of money, to transact the lottery before it is drawn. This necessitates actuarial science, at least from the point of view of the institution that is issuing the lotteries tickets, collecting the fees and paying out the prizes (also, supervising the random device). The beauty and peculiarity of the market and of the

traded asset, however, are that the value of the lottery is the current price itself. Buy the underlying asset for its market price and hold it, and the random fluctuations of that price, which the ticket (the underlying itself) you have bought *is still worth*, will eventually become the random outcomes you are expecting and betting on.

There is a coincidence in the fact that the value of the lottery is itself the stochastic process, or the random device generating the random outcomes. Trading generates randomness (by the efficient market hypothesis), but trading yields the price at which to buy the very device of randomness, leading unambiguously to the random outcomes. This is a peculiarity of trading (randomness + availability of price). As there is no question, on pain of arbitrage, that the price of the underlying is the price at which to transact the lottery paying out the future price as a random outcome, the only remaining exercise is to reconcile it with the rationale of actuarial science.

There is a coincidence in the fact that the market produces the randomness (creates the lottery) and, at the same time (and consubstantially), warrants the transaction of the corresponding lottery tickets. To be sure, the objective probability distribution of the underlying price is considered to be given, and, in particular, its volatility. However, we know that this in no way can, will, or even should be of any use in computing the value of the lottery ticket that the underlying is, because the value has to be equal to the current price itself (which is generating the ultimate randomness). All that remains is the observation that, if the current price is to be conceived as a present value regardless (present value in the actuarial sense of break-even in the long run), then the real probability measure has to be changed into an equivalent one, which preserves the events of probability zero and in which the drift is brought to zero over the interest rate. Given volatility and the risk of investing in the underlying that volatility entails, talk of risk preference is invoked in the underlying pricing problem only to the extent of explaining that the real drift is superior to the interest rate (or that investors want to earn a risk premium over breaking even).

Imagine, for one second, a lottery ticket that is traded, regardless of the random generating device. All we know is that the lottery ticket shall be worth the final random outcome at the time of drawing the lottery. Then, potentially, the same could be said as was said before regarding the traded underlying. Simply buy the lottery for its traded price and its fluctuating price, as time goes by and as trading carries on, will make it all look as if that price were the random generating device, eventually drawing (revealing) the final price, or prize, which you will collect by simply selling your lottery ticket in the market.

Maybe the lottery is traded because there is speculation about the probability distribution underlying it. However, the trading has the virtue of letting us see things from the other direction as well. We think the lottery is worth its final outcome, when it is finally drawn, because of the draw. But it is, at that moment, also worth its trading price. Now the lottery is also worth its trading price before the draw. Let us then think there is no difference in nature between the two prices. Let us think trading is all there is. Who said there ever was a draw at the end? Who said there hasn't been a draw every minute before the end? How big is the step in thought before we admit that the trading price is not a function of the final draw and, therefore, of the probability? That the event is the same, that reality is the same, both before and at the end?

A derivative written on the underlying could be traded independently in its market, except at expiry, when it is the price of the underlying, rather than the price of the derivative, that is the ultimate drawer of the lottery. A short time before expiry, the price of the derivative will also have to depend on the price of the underlying and not to vary too much from its own prescribed price at expiry, otherwise there will be arbitrage. This, the non-arbitrage relation between the derivative price and the underlying price, is given by the fundamental theorem of arbitrage. Prices of both derivative and underlying have to be related in such a way that they come up as the discounted expectation of their respective future payoffs under a probability measure that is equivalent to the real one, otherwise there will be an arbitrage opportunity.

Being traded, the underlying has no choice but to admit its traded price as the value of the corresponding lottery, and now non-arbitrage – which here assumes that the derivative is traded in the same market – forces the derivative to be valued, or priced, as expectation under the same probability measure. If the underlying was not traded but was only conceived of as a lottery, then the derivative would be a lottery with the same underlying device but a different payoff and all that would count is that anybody wishing to quote a price for both lotteries should take care not be vulnerable to a Dutch book argument. In other words, regardless of whether probability is thought to be objective or subjective, it would only be the tool to ensure that the prices cannot be arbitrated. Now, all that trading the underlying does is single out a value for the underlying (conceived as lottery) which is its traded price. When it is observed that the market price is not equal to the actuarial value (which ensures breaking even in the long run under the real probability), the notion or risk-neutral probability is invented to readjust the growth of the underlying in conformity with the actuarial rationale.

Brownian motion has the peculiarity of pinning down the derivative price because Brownian volatility is unchanged by the Girsanov theorem of equivalent measures and the underlying price is fixed by the market. It is also believed that the derivative price is pinned down because of perfect dynamic hedging. The two arguments must be the same. *Perfect dynamic hedging of the derivative must be the exact same package as the tradability of the underlying combined with the condition of equivalent measures for Brownian motion.*

It is believed that BSM has liberated the option pricing problem from risk preferences. Any risk-averse player facing a one-time lottery would prefer to take less money than the lottery's fair value and run (less money than its actuarial value to insurance companies who can afford to wait for the long run and play the lottery infinitely often), rather than face the uncertain outcome. By hedging, BSM supposedly made the player insensitive to risk: the usual explanation is that he no longer faces a lottery because he hedges. In reality, risk-neutrality does not come from hedging. It comes from pricing the option consistently (no arbitrage) with its underlying; that is, under the same probability measure. Risk-neutral does not mean immune to risk; it means that a change of measure now makes it look as if the underlying and its derivative are now valued at their fair value; that is, by a risk-neutral investor. Risk-neutral pricing is just finding a pricing kernel: making sure that the prices of contingent claims are expectations back again.

Brownian motion, tradability of the underlying (or the mechanics of supply and demand which alone justifies the risk premium) and non-arbitrage between the underlying and the derivative impose the value of the latter. This is the magic combination. Take any lottery, even Brownian – for instance, triggered by the air temperature – and then its value will depend on risk preferences. Make it tradable, or rather, make the random generator the tradable underlying itself and then the value will be determined and independent of risk preferences. Take a tradable underlying, only not following Brownian motion, and several pricing kernels will be available, making the derivative value undetermined.

Brownian motion, tradability of the underlying and non-arbitrage, this concurrence of events or coincidence seems almost an accident from the point of view of the derivative. The derivative was intended to be traded, in its own market. Little did it suspect that tradability of its underlying, which is paradigmatically the random walk, will impose its value and abolish its market. Conversely, there was no intention of putting the derivative to trading, in BSM, and no conception of implied volatility. The whole idea that implied volatility will converge to the real volatility is an accident – a fluke – following the simplicity of the formula.

The fair value of the derivative, conceived as a lottery, should have been equal to its actuarial value, or the value that would make a player break even in the long run if he repeated the play infinitely. Because such an opportunity is not available and the lottery is only drawn once, the valuation cannot be statistical; it becomes subjective and involves risk preferences. It so happens, however, that one particular lottery – the underlying asset – is already indisputably valued and its value is its market price. Non-arbitrage between the underlying asset and any lottery driven by the same random device (the underlying itself) imposes that both values shall be expressed as expectations under a probability measure which is equivalent to the real one and which, in particular, values the underlying at its market price. It so happens that there is only one such equivalent measure when the random device follows Brownian motion in the real measure, and the consequence is that the value of the derivative is then uniquely determined.

As we said, we can argue that this determinateness is *otherwise* explained as perfect hedging under Brownian motion. That the value of the derivative lottery should be determined on a single run, without the usual leeway of the subjective evaluations, is what creates the impression of (or the argument for) perfect hedging. How, indeed, could the valuation problem admit of a unique answer and not be objective arbitrage? And what kind of arbitrage could be enforced on the single run and no longer statistically? The answer is arbitrage with the *traded* underlying or, to put it in simpler terms, the existence of a *price* for the underlying. This is what is otherwise expressed as a perfect dynamic hedging involving the underlying.

The valuation of the underlying as its market price (which seems to determine everything) is not just *any* valuation and certainly not subjective. It is objective in a sense, because the underlying is tradable and can be bought and sold at that price. If frequency and statistics are usually associated with objective probability and objective valuation, then *trading is a single-run statistics*. *The market is an alternative to statistics and not an instance of statistics*. This buying and selling, which is as

material and immanent as statistics, is otherwise reflected as dynamic hedging. It is the same thing. In a way, the marvel of BSM is just a sophisticated detour that shows the marvel of the following phenomenon: that something random should admit of a definite price (or value) today without a question; in other words, the very idea of a traded price being at the same time random (and calling for lottery valuation for this reason) and being its own price. This is the very idea of a market.

There is something unique in being random and being a price (for this reason): the price is random because it is a price (EMH) and its being a price is what fixes the value of the lottery that it is because it is random. Randomness suggests probability and valuation (actuarial science, subjectivity and beliefs) and, yet, there is this unique phenomenon called 'price', which is independent of valuation. Yes, price is yet unexplained, as phenomenon and even as matter. The pre-probabilistic randomness of price should be made foundational. And could even the notion of randomness precede the notion of price in that case, or should the notion of price itself become foundationally equivalent to (pre-probabilistic) randomness? On the other hand, there is something fundamental linking lotteries and valuation (or, equivalently, linking definite states of the world and probability). When we think of lotteries, we immediately think of their present value.

11.1.4 The curse of the derivative value

The notion of price is such an alternative to statistics that we wish ultimately to reveal a contradiction between the idea of trading and the representation of the consequence as a stochastic process ruling the market price. Trading shouldn't be confused with a random generator of numbers called 'prices'. The existence of the random generator, or the existence of the objective probability distribution, when it is combined with trading, leads inexorably to the changed probability measure, as we saw. Now that there is probability, we feel confident that there are lotteries and therefore confident in the idea of their valuation. As we saw, the price of the underlying (whose existence we only owe to trading) is the value of the corresponding lottery, and the prices of lotteries with different payoffs (the derivatives) verify the non-arbitrage constraints given by the changed probability measure, or the existence of the pricing kernel. The consequence, in BSM, for instance, is that the price of the derivative becomes fixed and its trading is reduced to a stochastic process that is derivative on the process of the underlying price. In BSM, there is no independent tradability of the derivative.

If derivatives must trade independently of the underlying, then their prices will have to follow an independently given stochastic process. *This is because we have made trading objectively representable by a random generator* – a process which is the net result of the reasons why derivatives trade. Precisely, the advantage of the stochastic representation is that nobody cares about the reasons why anything trades. All that matters is the stochastic process that the price follows as a result.

However, because of non-arbitrage, whatever independent dimension of stochasticity (tradability) that we may have imagined for derivatives prices will have to be mapped by a variation of some kind of the underlying process. The required existence of a pricing kernel implies the existence of states of the world that underlie the variations of derivatives prices. Ultimately, there will always have to be a grid of underlying states relating only to the underlying asset (first among which, its own prices) and overlying which a certain formula will, only then, *yield* the derivatives prices. There

seems to be no way to turn the derivatives prices themselves into primitive states (without a further underlying state). Trading seems always to have to be equivalent to a valuation, as regards derivatives. Because they are derivatives, no matter what we do, they will never become independent in the sense of no longer depending on other underlying states and of no longer being equal to evaluations computed over those states.

How could we prevent the trading of the derivative from being explained by a stochastic process that concerns the underlying alone? How could stochastic implied volatility, which is the result of option trading, no longer be recuperated by a stochastic volatility process rewritten for the underlying? Perhaps trading shouldn't be represented by a given stochastic process to begin with (or a given random generator). Perhaps states of the world should simply not apply in trading. Suppress states of the world and the condition of no arbitrage will no longer require the existence of a pricing kernel whose consequence is to enslave the tradability of the derivative to the underlying process.

Assume uncertain volatility, or any of the subjective reasons why at a certain point people disagree on the objective BSM option valuation; however, do not then attempt an external point of view (*sortie descriptive*). Try and do not represent option price variability by an objective stochastic process. Is this possible? As soon as options trade (for whatever subjective disagreement), is it possible not *objectively* to represent their price processes and to prevent the volatility of the underlying asset, which may be objectively constant for that matter, from becoming stochastic merely for the sake of that representation? Why must there be a distinction between the level of thought of the underlying asset (the process that it truly follows as it trades in its market, perhaps with constant volatility) and the level of thought of derivatives trading? Or maybe the distinction is only transient and the real thought is that the derivative market will settle eventually – that it will even stop eventually – and true objective volatility, which may be constant for that matter, will ultimately be revealed.

There seems to be a curse: no stable representation of the derivatives market. Either the derivative is denied a market from the start, or it will be soon denied a market. In the case of the underlying, there is no such difficulty because there is initially a salutary coincidence. There is no intermediary parameter, such as volatility, whose objective determination fixes the underlying value but whose subjective estimation explains the phenomenon of trading and subsequently triggers an external representation in terms of an objective stochastic process. From the beginning, the underlying is the end state. Its trading has been externalized from the beginning.

At what exact point do we turn from the objectively given (and settled) process of the underlying price to the thought of trading derivatives, in such a way that the question of agreement or disagreement on volatility eventually arises? Could it be that everybody knows what the volatility is and the only reason options trade is that some players need some market-maker to sell them or buy from them pure naked options: pure bets on the underlying settlement? What kind of market would that be? And why wouldn't this universal objective knowledge of volatility (universal in the sense that there are only written formulas and written mathematics expressing volatility) extend to the knowledge of the BSM theory, in such a way that nobody would need to sell or buy anything because everybody would replicate their bets with the underlying alone? Why is there this unsettlement

of the philosophical categories in the derivatives market, as we all wonder whether it is all about epistemology or just a matter of formalism and universal mathematics?

The underlying stands no such confusion. It is independently traded, period. There is no parameter, which is supposed to be fixed in the mathematical representation and which the market will have to take over in the next layer. Could we avoid inventing such a parameter altogether? Could we avoid representing the underlying trading by a stochastic process? As soon as we write such a process, the market will be open to derivatives and this will only amount to the philosophical unsettlement of the corresponding stochastic parameters: will they turn stochastic; will they be uncertain; will they be subject to trading? Maybe trading should be liberated from this whole stochastic narrative to begin with: no written processes, no random generation, but only one surface with written payoffs and their prices. Maybe trading is not a random generator; maybe there is no external point of view, no statistics and no history. History (in the sense of historical series) is external to the market and parasitic on it; history engenders the objective temptation.

They say: 'Maybe there is a random generator, only it is not stable; it is yet not crystallized.' But then why bother? In reality, the whole question of convergence of options prices to the real volatility is ill-founded. Where does it begin and where does it end? The market is comprised between its beginning and its end, yet the market has no beginning and no end. It is not excessive subjectivism that prevents objective representation in the market, either. What is the market, then? Where is its distinctive line drawn? There must be something precise, intrinsic and immanent to say about the market. What could that be? What could be standing between the level of objective description and its constant failure? The event, perhaps? Or maybe, the whole concept of time should be revised, in order to solve that aporia.

Simply, no random generation. The chrono-logic of the multiple has to be revised. The market is not an object. There is inner tension between the disagreement and the outside representation of the disagreement as further objective independent states. How could there be a market without the disagreement and how could the subsequent existence of the market not be objectified? The market is something that exists and presses the question of its being (because we typically ask: 'Why would there be a market otherwise? Why would there be an options market if BSM were true?'), yet we should not objectify the market. The market exists; derivatives exist; however, we should try and not *represent* the underlying stochastic process. It is not that the probability distribution (e.g. volatility) is not known; it is deeper than this. Fundamentally, the whole framework of precision of states shouldn't be invoked.

11.2 The trading narrative of the market

The underlying asset is traded for all kinds of reasons and, as a result, the price process is summarized under a stochastic process. The statistical regularity is supposed to screen off the underlying causes and the minute effects. It is supposed to be a conclusion. All kinds of things happen underneath the surface; however, we are fortunate that a statistical regularity reflects itself on the surface and from now on this will be our only phenomenon and only law. The statistical distribution may vary

in time; in essence, however, it remains a conclusion, something that admits of no further stage in the reasoning.

The derivative written on the underlying may also be traded for all kinds of reasons; however, in its case what lies underneath the surface of its trading has been made explicit. The derivative cannot hide from the fact that a stochastic process has been written for its underlying asset price. This is what forces any trading of the derivative ultimately to be reducible to a trading (stochastic change) of the statistical parameters governing the underlying price. This is what forces the question of whether the statistical parameters that are implied from the derivative prices will converge to the objective parameters governing the underlying price.

If there had not been a writing of the processes, there wouldn't have been the reduction of the derivative price process to statistics or, consequently, the question of the ultimate convergence. The writing of the underlying stochastic process had nothing to do with a derivative market at first. It was a mathematical statement, a mathematical 'let be': 'Let the underlying process be this or that'. In this existential statement there was no hint about the *knowledge* of the statistical parameters, or the uncertainty affecting their knowledge, and even less so any idea of a convergence. There was supposed to be no transition to a derivatives market.

Any independent variation of the derivative price has always to fold back to the underlying and to receive the following explanation: 'The reason why options trade is that the volatility of the underlying must be stochastic and should have been modeled as stochastic from the start'. If there were no link between derivative and underlying (and no pricing kernel to enforce non-arbitrage), then the trading process of any derivative would have been written as the statistical-stochastic summary and conclusion that it is, as the surface that it is, underneath which one needs look no further. Volatility could have been independently traded in its market as an independent asset class. Unfortunately, volatility derivatives are ultimately derivative on the underlying whose volatility we are measuring. They are ultimately written as explicit payoffs involving the underlying. There seems to be no way out.

Writing the underlying process as a statistical summary had a different meaning and different intention – it is of a different register altogether – than the subsequent exploitation of this writing as the beginning of the derivative pricing problem. We might acknowledge that, yet we also acknowledge that something has to be written to represent the derivative trading. What could that be? How to represent the derivative market without first invoking the underlying process and, consequently, introducing the statistical parameters whose variation become the only way to explain the derivative market? The trading of the underlying was supposed to be completed with its conclusion (the stochastic process). What is derivative trading adding to it (hence the ill-founded question of the retroaction of derivative markets over the underlying)? Why, indeed, are people trading derivatives and how to represent that?

The only reason we ever started writing a stochastic process for the underlying was to summarize its trading with an eye on the statistical time series (this is an econometrician reflex). Then somebody must have observed that the problem was now mature for derivatives valuation. Somebody must have observed that we might as well evaluate bets on the underlying, now understood as a lottery.

But nobody thought of derivatives trading. These values had better be invulnerable to arbitrage, of course, against each other and against the underlying. It is a coincidence if the market or the price of the underlying provides an evaluation of the lottery that it is; it is a coincidence that is proper to the underlying. By valuing the derivatives as further lotteries, the market of the underlying was supposed to continue acting as an evaluator (typically, of the lottery of the underlying), and not become an arena of trading of derivatives, in which their prices would become *states*.

11.2.1 Can the derivative trade independently?

Derivatives trade because they are written and, for this reason, exchanged. The writing and the trading of derivatives should be thought contemporaneously with the trading of the underlying, as soon as the surface of the market is stretched out and before we form the thought of a stochastic process to account for the trading of the underlying – even before we form the thought of objective probability in which this process is written. In a sense, the existence of a price for the option should be the first statistics concerning the underlying and truly replace the thought – yes, even the thought – of an instantaneous volatility. Any complication and further enrichment of the ‘process’ of the underlying should be thought through the richness of prices of derivatives of increasing complexity, not through the complication of its stochastic process.

There is a flat matter of the exchange that needs to be completely emptied of any statistics supposed to lend to option prices their time value. Why is it so difficult to envisage options trading independently of their underlying (i.e. for a fixed underlying price) and not to reassign that effect to a cause that definitely has to do with the process of the underlying – typically, stochastic volatility? Options may be bid up or offered down for no other reason than the paper they are written on, yet the simultaneous presence of the underlying, supposed to be the only lottery and to trigger the payoffs of the options, compels us to rewrite the variation of options prices as variations of *valuations* over an independent component of the underlying process; for instance, stochastic volatility, or different jump sizes of the underlying, and so on.

The intuition from statistics can never really disappear. Once it is observed that the time process of the underlying price is the actual cause of triggering of the payoffs of derivatives, it is very difficult not to imagine the objective probabilistic triggering or, in other words, the statistical triggering. It is difficult not to imagine a statistical heap of such payoffs that an insurance company would be in charge of valuing by breaking even in the long run. Once it is observed that the market offers no such long run or the opportunity of statistical rehearsals – or, in other words, once it is accepted that the actuarial fair value is out of the question in financial pricing – the only requirement that remains is that derivatives be priced consistently with non-arbitrage and that their market price be a mathematical expectation of their payoff under *some* probability measure (the martingale measure). However, the martingale measure is supposed to be equivalent with the objective one. To repeat, the market is not a subjective probability problem, in which agents are asked to assign prices to derivatives in an internally coherent way (i.e. barring Dutch book arguments) and in which there is no meaning given to probability other than the prices, or odds, that the subjects assign to lotteries. The market does not ignore the meaning of objective probability (hence, does not disregard statistics), because it is a requirement that the martingale measure be equivalent to the objective one.

Option prices may not vary following a certain dimension that is independent of variations of the underlying price without this dimension being reflected in variations of the underlying process, as written in the objective or in the equivalent risk-neutral probability measure. It is the underlying process, as first given in the objective probability, which is the ultimate prescriber of risk factors. It may well be that the state space is much larger than states (prices) of the underlying and that our world is much larger than the market; however, as far as the pricing of derivatives written on that underlying is concerned, any variety of states other than the underlying prices had better be mapped by a variety, or by changes of the probability distribution of the underlying prices.

It seems there could not be an independent tradability of the derivative. Tradability is a dispositional term. We hold the derivative and we feel it has the disposition to trade. It is a material piece of paper that we can exchange at once in the market (it was designed for that purpose). Yet, this readiness, this exploration of worlds to come, has always to be recounted back as the instantaneous volatility of the underlying price having always been stochastic at that point, or the size of its jumps having always been stochastic, and so on. The derivatives market, we are reminded, has always been a *valuation*. It has always already selected the martingale measure; it is a valuation, whose results it remains to read off the prices and which may have been hidden from us.

In an independent trading of the derivative, we would like to see the derivative trade at a certain price against a given underlying price then trade immediately at a different price, while the underlying price remains unchanged. The price should move because of tradability, not valuation. Valuation folds back any change of price of the derivative into the stochastic process of the underlying; it rewrites it as a change of the relevant state variable (in this case, instantaneous diffusion volatility, or instantaneous jump size) which has only to do with the underlying process. It redeployes the machinery which ends up with the derivative *valuation* all over again.

Derivatives are evaluated as lotteries and the value is the result. But this is not a price. There is no meaning in embedding the result into a market and in claiming it is a *price*. The market has always been an evaluator in this scenario (verifying non-arbitrage, selecting the martingale measure); it has never been a generator (a creator) of prices.

The BSM formula may have been first in inspiring the thought that something specific to the derivative (and disconnected from the underlying) can be traded; namely, implied volatility. A market for implied volatility was created; this is what we were all bidding up or offering down. It didn't yet matter whether this implied volatility was connected to realized volatility (the statistics of the underlying) in any way; all that mattered was the materialization of that number, of that 'commodity', which we traded.² Realized volatility is not tradable; you cannot buy it or sell it and make it go up or down directly as an effect of trading like you would an underlying stock. It is options that you needed to buy and sell, and then you found that their prices could not vary as independent states, but had always to be recomputed as values in a larger model where the instantaneous volatility or the jump was now made stochastic.

To be a state in the market is to be the price of an independently trading asset. There is an ambiguity or a coincidence in the case of the underlying, because its price is, at the same time, the value of the lottery that the price will play out. It is evaluated by the market as the lottery, yet this evaluation is nothing more than the market handing you the underlying for that price, which

will become the lottery, which is the lottery. Derivatives should enjoy the same *nature of price*. They should trade simultaneously with their underlying (as they do), when it is not yet written what stochastic process the underlying price will leave as a trail (with stochastic volatility, or not) and when it is even less written that the derivative pricing shall be an evaluation over that – that the derivative price changes shall be interpreted as values being read off the changes of the underlying volatility or the changes of the underlying jump size. Derivatives trade without statistics and the only thing we know is that the underlying will eventually trigger the payoff of the derivative. Surely, a stochastic model can be imagined for the underlying and the derivative price can be recomputed as value. But the real question is: What would be the revolution of statistics in which the price of the derivative replaces even the notion of underlying price volatility?

The main virtue of derivatives is that they are *written on the underlying*, not that they are indicators of its instantaneous volatility. The first virtue of the underlying is that it trades; not that it admits of statistical properties and produces coefficients such as volatility. That it trades is a statistics of its own that does not depend on precise metaphysical parameters. The derivative admits of a non-zero price not for the reason that the event of its triggering is not an impossible event (equivalence of measures), but for the reason that it is exchanged and must be exchanged for a price. *We must rethink the virtue of the exchange as a material substitute for the statistics*. Nobody is betting on statistical parameters when they trade derivatives. Simply, the path of the underlying prices can lead the way up to the expiry of the derivative and to the triggering of its payoff.

The pricing tools may not be right (BSM) but they have created a firm reality. The market-maker who is making quotes on options is not reflecting changes in the market forecast of future volatility. He is moving the options prices mechanically. The inner mechanics of the pricing tool may be based on valuation, which is the contrary of pricing and tradability, yet the result of using the tool is the tradability of the derivative.

What new reality we get with the result of the model is stronger than the truth of the model. We should probably stop trying to make sense of the market from within the theory of the tool. Let all the background theories (martingale measures, what the market thinks, and so on) recede behind this new reality of the market that the tool has helped in creating. The real mechanics of the tool is the usage of the tool. The usage of the tool is intensional, not extensional; from the point that it presses, we should, once again, extract the whole chain of derivatives trading and, for this reason, get as far as we can from any recuperation by the martingale measure or overall valuation.

Once the framework is in place in which no derivative is redundant and the state space is all made up of independent prices of all grades of derivatives (no longer valuations), we will see that the first picture in which we spoke of martingale measure had by the market, or what the market *believes* the future volatility should be, and so on, doesn't make sense. The whole chain is a single point which communicates directly with the event (the continual event). *This mode of refutation of the structure of states of the world is the same as the mode of the event*. Any temptation of framework or explanation by martingales or what the market thinks, and any impossibility of having prices of the underlying and prices of the derivative as states on equal footing, will be abolished retrospectively once the chain is pushed to the limit and the new matter of the market is established.

Incomplete Markets

12.1 Complete vs. incomplete markets

The first marvel that the market came up with against probability is the trading of the underlying asset itself. The exchange is an extraordinary matter, which is not yet fully investigated according to us. It is not that something is random (a lottery) and then we seek its value and its price. The price is there already and it cannot be anything but random because it changes. It changes because there is the minimum assumption in the market that a transaction will be followed by the next (there is, thus, the assumption of time) and that, at the next step, either the buyer or the seller is not satisfied (while his counterparty is) and will be prepared to drag the price either up or down. There is no time to try and guess or predict the next price movement in the market, because it is the exchange and trading that we are talking about and any future predictable price will trade as of now (EMH). The exchange, left to its own forces and own logic, seems thus to overturn the relation of antecedence between action and prediction.

For the first time, something cannot be what it is without being random. It was never *observed* to be random; there was never a distance from which to observe it. Its randomness is very close to it, for otherwise it cannot be. Usually, something is random and, for this reason, it is impossible to predict it with certainty. Here, prediction is considered and automatically dismissed (for otherwise the next price would be equal to the present price and there would be no market and no price change) and, for this reason, once prediction with certainty is eliminated, the price is random. Randomness is the consequence of the absence of prediction, not the other way round. The price is objectively random – hyper-objectively random, even. By that, we mean that there isn't something which is objectively given and which is the randomness of price or its random generator. There isn't such a separation and objectivation of randomness. There is just the price and that is enough. The price is random at any scale (Brownian motion); it is so random (in this sense) that it admits of no probability. In an article on game-theoretic probability, Vovk shows, for instance, that probability emerges second to trading.¹

What does it mean to look at the *probability* of the price movement (as if its randomness was objective)? Do we need probability to evaluate the lottery that the price will turn up? Well, the value of that lottery is already given and it is the price itself. As we said in an earlier chapter, there is an evaluative aspect in probability. We *expect* something, in matters of probability. We look at the random device, as the choice of one outcome between many outcomes; but this is not all. In probability, there is an additional tense and expectation; there is a degree. Either we expect an objective probability (as frequency, linked to the evaluation of the insurance company) or a subjective probability (linked to a price given by a banker). But, in a market, there is no need to expect or to evaluate anything. The price is already there. Certainly, it is objectively random; but probability in the sense of valuation or expectation is already taken over (*dépassée*) by price. (The price is hyper-objectively random.) Randomness makes us think of probability and want probability; we feel there is a lottery and that we can evaluate the lottery; we imagine probability. But value is already taken over by price. The truth (or the mystery) is already all there. To repeat: this is objective, the process is described from outside as the result of trading; but the value – or, rather, the price – has already preceded the result and any result. Price has already preceded any valuation.

Brownian motion is structural to the exchange (because of no predictability at any scale); it is attached deeper than probability. By showing that probability, in which to recognize Brownian motion quantitatively, emerges subsequently to the exchange and to trading, Vovk shows that we don't need the background states – what he calls 'stochastic structure' – when considering the force of trading, originally. Money allows Vovk to dispense with probability as the foundation, because the probability-1 event is re-expressed as a trading strategy that would make a fortune if the event didn't come to pass. Choosing trading and market efficiency as my basis, the observation that the derivative, too, has to trade originally and with force leads me, in my turn, to dispense with probability and the associated background states. So, it must be the same argument. The market dispenses with stochastic structure and its description is, as a matter of fact, complete as soon as we assume continuous trading in fractional size, which is essentially equivalent to Brownian motion, and it is recognized that the underlying price is its own value as a lottery. No wonder the market is then, financially, also said to be complete. Derivatives are already settled; the mystery according to which their value will not depend on probability either is just part of the first mystery; simply, they must be valued without arbitrage with the underlying.

Making the market incomplete by expanding the states of the world doesn't really help. We are told there are now more risk factors than traded assets; that there is the dimension along which the instantaneous diffusion coefficient varies stochastically and maybe the dimension along which the instantaneous diffusion coefficient of the diffusion coefficient itself diffuses; that there are the dimensions corresponding to different jump sizes, and so on. Then, we are told there is no longer a unique value for derivatives. We cannot value them uniquely, yet we are told the market does, because *there is* a market and it is unique. We are told that the market selects its martingale measure.²

We are told that the market *prices* the derivatives. Does that mean that they trade? If derivatives did trade, they would expand the state space; here, however, the market is merely an evaluator that is confined in its predefined grid. Derivatives are valued over that grid and, because the evaluator

is the market, the value is called a price. That's all. Usually, the argument for the incompleteness of the market stops there. But let us push it a little further. If derivatives trade and if we can infer the martingale measure from their prices, what is to stop us from running dynamic portfolios with their price processes and from completing the market?³ It is argued that we do not know the martingale measure of the market. The price processes of derivatives cannot be known and the dynamic hedging portfolio cannot be formed. But couldn't the market alone be forming it and intrinsically articulating its own completeness? There wasn't really a voluntary act of forming the hedging portfolio in BSM. All that was required was the tradability of the underlying (which is the absolutely first proposition) and that the market enforced non-arbitrage in pricing the derivatives.⁴

Shouldn't the market be considered complete as soon as derivative prices exist, regardless of whether they enter into dynamic hedging strategies? Conversely, shouldn't an incomplete market be unable to select its martingale measure? Derivatives could be valued by anyone in this case, *except* by the market. Anyone could select a martingale measure when there are many available, but the market couldn't. The formalism of the market (in this case, the underlying's) stops one step short of the selection. By the time we identify the statistical shape of the underlying process (stochastic volatility, jumps, and so on), we will have long exited the market and long entered history. So, the market cannot catch up with our *evaluation*. Conversely, trading takes place before the summary of trading is given under the form of stochastic processes. The trading processes of derivatives could have been given from the start; but then one wonders under what form. As we will see in Chapter 13, from the Brownian motion of the underlying price, when it is understood qualitatively and not quantitatively, as a trading intensity which has not yet been taken over by extensive time and has not yet turned into a trading history, the trading of derivatives will open a door that is alternative to statistics and to that form.

It seems there should be a difference between the price processes of derivatives being, first, objectively given in the objective probability (or the existence of as many traded instruments as there are risk factors, or the initial uniqueness of the martingale measure), and their price processes being, secondarily, given in the martingale measure that the market has selected among the many that were available. It seems we could construct the hedging portfolio in the first case and complete the market (or, equivalently, that the martingale measure is readily unique) and not in the second. That the derivatives should merely be priced – or, rather, valued – by the market seems to be different from their actual trading. When they trade, derivatives push their own process by specific supply and demand, and their trading is not just a reflection of a valuation by the market, understood merely as a non-arbitrage evaluator.

To trade is to be active; traders have handles on the traded derivatives. We thus inherit one more argument for there being something suspicious in confusing the trading of derivatives with mere evaluations. Are we sure it shouldn't be forbidden to back up the martingale measure from prices of derivatives? The theoretical implication works only in one way: from the platonic reality of the martingale measure to the valuation of derivatives. Yet, by inferring the martingale measure from the derivatives prices, we commit an unjustified move going in the other way, from another reality. It is in the reality of trading that we are supposed to imply the martingale measure from the market prices

of derivatives; but those prices are then infected by the ‘price quality’ and they no longer qualify as the values given by the martingale measure. Implying parameters is a completely extra-theoretical move; a different reality of the market (which was never expected by the theory) is assumed in this movement.

12.2 Martingale measure of the market

Presentations of incomplete markets are usually very abstract. There is talk of contingencies and of contingent claims payoffs spanning those contingencies. Then, there is talk of those contingent claims being made available for trading. There is thus, first, the act of writing the contingent claim (for instance, the one whose payoff spans the contingencies of either rain or sun tomorrow), and there is, second, the act of trading it. Cochrane is exemplary in this respect.⁵ The contingencies are glossed separately from writing and writing is glossed separately from trading. It is not specified that the only contingencies that we have collapse as contingent claims written on a certain underlying asset. The price process of the underlying asset is not explicitly given or, if it is, the different pieces are not brought together.

Now, I contend that derivative pricing theory that is specifically oriented towards developing pricing and hedging tools (the literature upon which derivative technology providers feed) starts from a different place. It has already collapsed its problem and moved unconsciously towards completing the market. This literature starts by *specifying the stochastic process of the underlying price*. From the beginning, the world is thus reduced to contingencies pertaining to the underlying alone. Then, the trading is specified (this price process). There are no contingencies other than variations of the underlying process (stochastic volatility, jumps, and so on) and no assets can be created other than derivatives written of that underlying. Since we already have continuous time and continuous trading, since we already have the market, what is to stop us from trading those derivatives and completing the market?

We have two registers. One, like Cochrane’s, is not concerned with derivative pricing technology but only with the abstract framework that introduces it. Typically, Cochrane won’t go as far as writing the stochastic process of the underlying price in the risk-neutral measure. The other starts with such a process but will typically refrain from drawing all its consequences: completing the market with the price processes of derivatives. I wish to instate a new mode of reasoning that proceeds from the writing of the process, to what it *really* implies and means, to what is its *purpose*. Writing a stochastic process in the frame of mind of pricing derivatives is not innocent. It is the only way to proceed if we want workable tools, but it condemns the market to completeness – at least, if we remain stuck in that naïve view.

BSM (and the development that follows) is not a sub-case of asset pricing theory; it cannot be an illustrative paragraph in a book like Cochrane’s. It is the start of something different where talk of incomplete or complete market becomes spurious, and even dishonest. Writing of the process, which issues into the writing of the derivative, and hence its trading, should ultimately instate a whole new register, in which calibration and recalibration are the rule, *and which therefore results*

in the criticism of states of the world. There is no difference between BSM and all the complications that followed. There is only *one* martingale measure in one case (therefore the market had better pick it – what is the *market*, a person?) and there are several in the other case; however, there is one market, therefore we are told there will be *one* martingale measure that the market will pick.

Very easily, surreptitiously, the formulation progressed from BSM and its definite martingale measure to problems where the martingale measure was given from the start. Why? Because after BSM the register has always been to price those derivatives with pricing tools that will be used by makers of the market (who will wonder, for instance, what the delta of the derivative will be in this market they just made). BSM had already all it takes: Brownian motion and price. But BSM concerned the underlying asset alone. There was really no derivative pricing intended in BSM but only a nice complete story to tell about the underlying asset – Brownian motion because of EMH and price as value. Derivatives are even redundant from this story-telling point of view. Now, add complications to the underlying process, yet persist in writing it. At that point, you should exit the market and consider as many arbitrage-free evaluations as there are equivalent martingale measures. Yet, you insist that you still have the market in mind, all the market and the only market, from the beginning; so, how can you now evade the fact that the market will find its martingale measure and, what's more, present it to us?

To each additional complication in the underlying process, the market will have already priced the corresponding derivative whose trading in an appropriate portfolio will complete the market again. This is really an argument to show that one shouldn't add complications in the underlying process, but directly in writing further derivatives and giving their market price. There is no market in BSM other than the market of the underlying asset, which is complete. So, what does it mean, when we make the process more complex, that the market admits of a martingale measure? Where did the market suddenly come from? What is now the complete theory of the market?

Are we running the underlying asset separately like a lottery and separately trying to evaluate differently re-engineered payoffs that we call derivatives? But what does it mean, then, that the derivative should be valued consistently – arbitrage-free – with its underlying? There seems to be an equivocation on the point of view. You cannot have the point of view of trading and the point of view of evaluating running together. Those who trade the underlying are not aware of a stochastic process; they are creating it. They don't adopt the outside point of view of the closed lottery. Yet, we want those who evaluate the derivative to adopt just this point of view.

The underlying asset trades in a market. We have defined for it the conditions of a market, the one that says that the next price cannot but be unpredictable, for if it wasn't it wouldn't be the next price, it would be the present price – a condition for randomness which we have recognized to overtake probability, for the traded asset is no longer observed and *expected*, it is no longer valued and probability no longer has a hold on it – and the one that says that this randomness needs no valuation and no probability anyway; that it is faster than the valuation of a lottery, because it is already the price. The market is such a shortcut, compared to probability. For this reason, the descriptive problem is complete for the underlying asset and it is perfect; there is nothing we could add to it.

Since value is already collapsed in the trading, as far as the underlying is concerned, and the whole martingale language is lip service to valuation with probability, it shouldn't be allowed – it would be a change of subject – to, then, even think of *evaluating* derivatives written on the underlying. If the underlying price is just considered as a random generating device, then lotteries pertaining to it should be evaluated elsewhere – certainly not in the same market. It is an equivocation on the market to go ahead and ask what the value of a derivative could be.⁶ Is the derivative trading, or what? This is the market of the underlying and our focus is on that gravity and that pit; why bring about the subject of derivatives? From outside, this looks like a stochastic process and Brownian motion looks appropriate to model it. Why bring derivatives? If they come from outside, they should be left outside and never be associated with the market of the underlying (which is complete and is our sole preoccupation).

We have created a problem artificially and, then, we are upset that we are unable to recover the first nature of the problem. The underlying asset is trading; we are already describing it from outside and expressing the randomness attached to EMH. What does it mean, then, to *value* the derivative? Value for whom? Where is this person located, in the set-up of the underlying market, who will value this derivative and buy it or sell it? Are we suddenly considering a leeway, a margin of negotiation? Value is an old concept (utility functions, and so on); it has now been covered by the stochastic process of the underlying. Or else, consider from the start a trading of the derivative, with its own unpredictability, due to its own efficient market.

Volatility is not subject to uncertainty or negotiation in BSM, or in any of its complications where the underlying stochastic process is written. Everything is already programmed in the derivative valuation models. The market emerged from an equivocation on the market. Because BSM was so perfect, it precisely created the direct trading of derivatives in the sense of their own independent process. Never forget this. Radicalize this conclusion.

12.3 Equivocation

One cannot be describing the trading process of the underlying, with its two main components, which are Brownian motion and the nature of price, and then turn, as if incidentally, to evaluating derivatives. Were derivatives here from the start, to trade in the same market? Was their price, for a given fixed underlying price, going to experience the same necessity of randomness as the underlying price, due to the efficiency of their own market? The market in which the underlying trades and registers an objective stochastic process is not the same market as the one in which the derivative is valued as a lottery written on the first, where the first is now only considered as a mere random generator. There is equivocation on the meaning of market. Rather, a market was never intended for the derivative, once the trading of the underlying was envisaged from outside, in a move which completed the description of its trading and closed it.

The market closes itself with the description of the trading of the underlying as a stochastic process; you cannot then open it to valuing the derivative and say it is the same market. No wonder derivatives are redundant in such a picture. So, you add a jump to the diffusion or a third state to the

binomial tree. Derivatives are no longer redundant in the sense that the underlying could replicate them. Now, there are several non-arbitrage values for the derivative and we no longer have a fixed one. There may be two values that do not breach the principle of non-arbitrage, but there is one market.

Leave the derivative valuation problem outside the market and the derivatives will remain non-redundant; however, they will remain without a market. Nobody can value them without recourse to risk preferences, and so on. But as soon as you bring them to the market; as soon as the problem recovers its initial purpose, which was the pricing of derivatives (not their valuation) in the same market, the market is said to select its own pricing or martingale measure.

So, the derivative price is fixed, because there is one market. But is the derivative trading, then, and if it is, once again, what is to stop us from forming hedging portfolios with its price process and completing the market (after calibrating to the derivative price supposedly)? Once again, the only thing that stands in our way to completing the market (hopelessly) seems to be the identification of its martingale measure. Is this really accessible? To repeat, this is different from initially being given the identifiable process of trading of the derivative in the real probability measure. It is said we could identify the martingale measure that the market has selected by taking the prices of derivatives as inputs and, from a list of martingale measures which are all equivalent to the real measure, find the one that matches the market prices of derivatives. This presupposes that the process in the real measure is already identified. We need to know that the volatility of the underlying is stochastic in reality, or that the underlying assumes jumps in reality in order that the equivalent measures may also reflect stochastic volatility and jumps (otherwise they won't be equivalent to the real measure) and only differ by assigning different 'market prices of risk' for stochastic volatility or for jumps. Everything is possible on paper, of course. In reality, however, there is tension in the picture and equivocation on the point of view. To recognize the stochastic process of the underlying in the real measure – that is to say, from outside, as the trading summary that it is – is to exit from the market. This is a prelude for statistical analysis and no longer for trading. Those who trade the underlying asset from inside its pit, we said, are not aware of any external representation. They are performers who create the market. And once we exit from the underlying market and list the equivalent martingale measures, the derivatives market that we now turn to in order to point us to the right equivalent measure is not the market either; it is an external evaluator who merely respects the principle of no arbitrage. The prices of derivatives that it shows us have been disconnected from their trading pit as well. They are no longer prices but values. It is only now a matter of an external coincidence, of picking numbers that match other numbers from a passive list. The very word 'selection' (of the martingale measure) smacks of passivity and sounds incompatible with the force of trading. On the contrary, to remain immersed in the market and to give the inputs of the derivatives market the true sense of *prices* opens up to the recalibration problem of the pricing tool or formula we are feeding them into. In the formalism, which goes only one way, we shouldn't be able to invert the formulas and identify the martingale measure in this way. (The market is closed in its objective semantics, going that way.)

Another way of blocking redundancy is to add a third state that is not a state of the underlying – to allow for another state variable; for instance, instantaneous diffusion. Not only will the pricing formula no longer be unique but, for a given underlying price, we will now have many derivative values, even for a fixed formula. However, the imperative of non-arbitrage, or the existence of a pricing kernel, imposes that this other state variable should be a variation in the underlying process. Even though the variability of the derivative price for a given underlying price makes it look as though the derivative were independently trading, it still is a valuation; we are still reading out its variability from the other state variable; its price cannot be the ultimate state. *It seems to us there is no escape from redundancy other than to admit a market for the derivative at the same level as the underlying.* It is no use to argue for non-redundancy in the sense of non-replication by the underlying, then have the market of the underlying, in which the underlying is the sole driver, price the derivative in its uniquely selected martingale measure. This non-arbitrage pricing is not a trading of the derivative.

In a sense, reopening the problem from the angle of the underlying (adding a state, either of the underlying – a jump–or of another parameter ruling its own process) is evading the real problem of incompleteness and non-redundancy. It serves us a valuation of the derivative under the guise of a market, no less redundant because no more trading. In a sense, BSM is better because the market is closed to the derivative from within the formula; and it opens from outside the formula or, rather, through it. There is no real incompleteness except through trading, and this necessitates the trading tool; we must penetrate *that* reality.

12.4 Incomplete market when the market is all there is

If options were perfectly replicable why should they be traded? The existence of options markets leads one to relaxing the hypothesis of perfect replication. This can be done, for instance, by adding a jump process to the diffusion (or a third node to the binomial tree). Several martingale measures are now possible and the recourse to the market is just one (objective) way of picking one. Here, tradability of the option is no more and no less than the *deus ex machina* solution of the indeterminacy of the option value – we need the market for this reason and let the market decide. In this situation, tradability does not mean the independent variability of the option price. Tradability means that the market has selected a martingale measure – only God knows in what way – and that an arbitrage-free market value (not price) exists. Through their trading, options do not put into play a variable of their own; their market is not useful in that sense, for option prices are, here, slaves to the underlying prices (once the market has selected the martingale measure). Alternatively, volatility is made stochastic to gain option price variability for a given underlying price. However, it only looks that way on the surface. Indeed, a market for options is still not in hand because it is now changes of the real volatility of the underlying that are dictating the options prices – which are no less slaves. Something has to give up in the grid, and why not the whole presupposition of states of the world.

The whole dialectics of complete markets vs. incomplete markets is predicated on the prior framework of states of the world and probability. It dates back to the period in theoretical finance when states

of the economy were modeled, or contingencies pertaining to a field that is larger than the market, then, a certain number of traded assets was said to span the contingencies or not. There were states that the market could be in, correlated to states of the economy, and the market was said to be incomplete when those contingencies were not all written and coded as traded contingent claims. Wealth and consumption were underlain by states of the economy or 'nature', and the problem was posed to see whether the traded assets could help attain these objectives. The market did not yet close itself on itself as the problem of pricing of the derivative. When BSM wrote the underlying process, they assumed that it represented the whole market. Now, contingencies to be attained, or wealth to be replicated, were just the payoffs of derivatives. Derivatives depended only on the underlying, so their pricing problem or replication problem obviously no longer necessitated any other state variables than those pertaining to the underlying process.

The market started to close itself off when derivatives became the sole horizon. This gave us nothing more to begin with than the underlying process. Now, contingencies were just the wanderings of the underlying process. However, this now was at the same time the trading of the underlying asset, a force that was not expected. Reframing the economy as the market relabeled the contingencies as those of the underlying market itself and replacing the states of the economy that needed to be replicated by derivatives payoffs made those recede *behind* the underlying asset instead of remaining in front of it, as targets. The underlying process was now the only place of contingencies and what we were trying to replicate depended on the underlying asset. This was an overturning of the problem. Obviously, the derivative would never acquire the status of an independent trading asset in this picture. Obviously, the market would become perfect and complete with the underlying asset alone, because it was now the only horizon; there was no longer an economy and its states to provide the absolute background.

Trading of the underlying asset came to the forefront, forever relegating the derivative to a mere valuation. There used to be underlying economical states and the price processes were correlated to them. Now, the only driving force became the market itself. This changes the incomplete vs. complete market problem. Now, the only way to make the market incomplete was to enrich the underlying process. But this doesn't really work because the market is omnipresent (it is our sole horizon) and, therefore, omnipotent: how to stop it from selecting its own martingale measure? With the derivatives pricing problem as the sole horizon, we reach a null point, the only exit from which is to make the market incomplete through the trading of derivatives: to exit from the other side.

The market should be thought as being incomplete as soon as we think of the market. There should be no prior and more encompassing space – there should be no container – in which the question is *then* posed to determine whether the market is complete or incomplete. There should be no other face to the incompleteness of the market; incompleteness should be its simplest thought. The question of completeness vs. incompleteness hinges on the category of count. We, first, count a number of contingencies or states that the economy could be in and, then, we review whether the number of traded assets is larger or smaller than the number of contingencies (as if the contingencies were planned on one side and the number of traded assets on the other side). In this accountancy, it is not clear what stops the market from being completed and the remaining contingencies from

being written and made available for trading.⁷ Should we think that they are just abstract and for this reason not writable, or even identifiable? That we can count them but cannot make them explicit, or that their cardinal is not defined by an ordinal?

It is out of a (necessary) generality of thought that we imagine that the contingencies of the world or the economy may (or, even, must) exceed the number of traded assets. It is out of a generality of thought that we imagine an incomplete market. No need to be explicit about the contingencies; all that matters is that the thought of the incomplete market may not lead to contradictions. For this reason, the definition adopted in more modern texts is to invert the order of presentation. It became even more abstract. We no longer compare the count of contingencies and the count of traded assets. Markets are now said to be incomplete when the martingale measure is no longer unique. We no longer explicitly recognize the different measures or index them by parameters to be calibrated from derivatives markets.⁸

All this will change when the step is taken and the market of the underlying asset becomes the only horizon and the only economy. Now, the problem becomes explicit. Now, the only way to make the market incomplete is by explicitly enriching the underlying process – because we are writing it anyway. However, every time we vary it or enrich it in a certain way, we are now able to respond with the appropriately written derivative.

Or maybe this whole reasoning is illegitimately explicit and sequential. Maybe this reasoning is overly constructive, when it shouldn't be. Maybe we shouldn't be able to introduce our thought into the martingale measure of the market and to imagine a next step where the n^{th} derivative readily completes the market. Maybe we should consider that, from the start, the derivative price process is given, as a trading process with intrinsic force, and from the start it is a martingale in an equivalent martingale measure; that, from the start, the derivative is trading with force, as an independent asset, yet its market is aware of the statistical *history* of the underlying asset. Don't ask how this prodigy might be possible. This is the privilege of the market. In this conception, the market is both dependent and independent. Don't try to teach the market the *choice* of the martingale measure. The apparent sequencing of the one and the many – the existence of several martingale measures, then the unique measure that the market selects – may just be an artifact of our *thought process*.

What seems irresistibly to tempt us into such a thought process is the way the problem has been set up for derivatives pricing (writing the underlying process explicitly, then calibrating it from the prices of derivatives then pricing other derivatives), when calibration and inversion of the valuation formulas (e.g. implied volatility) should really belong to a different logical plane than the one in which states of the world are given, martingale measures are considered and valuation is obtained. As we said, the purpose, here, is to price derivatives – we are already introduced into that thought – for this reason, we consider their trading and the market being completed.

Alternatively, if the writing, the pricing and the trading of derivatives were not on our mind, then the complication of the underlying process (the addition of jumps and stochastic volatility) will have, as a consequence, that the payoff of derivatives can no longer be replicated by the underlying as in BSM. It is perfectly OK to write such complicated processes on paper and to get as a result, on paper, that the derivative payoffs can no longer be replicated; but what is the reality behind the

paper? If it is really in time that the stochastic process of the underlying is being observed and its statistical parameters are being identified (the frequency of jumps, the volatility of volatility, and so on) then how could this story – or, rather, history – not allow that during the same time, as the time series of the underlying prices is studied, derivatives are written and put to trading and the time series of their prices are considered, too? We can do whatever we want on paper; however, the decision to upgrade the underlying process from Brownian motion with constant volatility to jump-diffusion and stochastic volatility is supposed to paint a more realistic picture. It then remains to decide whether the realism in question concerns econometrics or the market. Are we complaining that the underlying is not realistically following Brownian motion, or that derivatives written on that underlying are not realistically redundant with it?

An alternative to this whole conundrum is no longer to count the states, either of the economy or of the market. It is to open the problem from the other side. There are already gaps of thought in the framework in which the background economy is introduced first and the market introduced inside it. Contingencies impinging on the background are said to be larger than the market. We can understand that contingent states of the economy may not be attained by the market. (How could the price processes of traded assets ever replicate states of labor, or welfare, or inflation, or any of the variables that have been identified by economists or researchers of political economy?) But let us look at the problem in the other direction. If the market is our sole horizon, could the reflection of states of the economy on it be anything else than states of prices or states of variables ruling price processes? If the economy and the market are operationally reduced to price processes, could the filtration not itself be reduced to the natural filtration associated with those processes? Surely, the market is not the whole economy; but, once we confine ourselves to the market, doesn't the market become its own whole economy? Once price processes become the only rule, aren't we bound to write explicitly their complications and what is then to stop us from trading the corresponding derivative and completing the market? Yet, the gap is left in thought and this question is not answered.

Once the price process is written in time and complicated in time, we, in reality, have exited the market. We become the only ones able to select a martingale measure out of the many available (the market can't), and, from that moment, we should never mention the market again. However, we keep the thought of the market, while keeping the idea of selecting the martingale measure. This improper mixture of thoughts then results in the monstrous thought of the market selecting its own martingale measure and eventually completing itself.

Should we also imagine that contingencies pertaining to the strict market are not writable? Yes, something changes for the price process, something makes a difference, yet it is not explicitly writable as an explicit change in its identifiable parameters. Something makes a difference, but it is not a recognizable difference such as the volatility being now different and such that a derivative could be explicitly written to span this new identifiable variable. Maybe the filtration is not the natural filtration, and so on.

But why not take the problem from the other side? Why would the contingency that makes a difference for the market (and, in particular, for the underlying process) not simply be a different price for the given contingent claim? Instead of identifying a parameter in the underlying process,

of making that parameter variable and writing the corresponding derivative to be traded at prices that span the new variable, why not make the derivative tradable right away without identifying the 'corresponding' parameter? Since we are at trading and at markets anyway, what better contingency and difference than the different price of a different contingent claim?

Now, we are in trouble trying to figure out the difference this makes for the underlying process. Surely, something must have been different from the start, for that process – different states of volatility – and the different price of the different contingent claim is just the reflection of that? No! Or maybe the difference in the underlying process will have to catch up with the different price of the contingent claim? No! The whole frame of thought in which the underlying process is thought first and difference for it thought second is backwards. We are at trading and markets anyway, so we'd better short-circuit the whole probability framework and go directly to the market. Yes, the underlying price and the derivative price should be given with equal force, yet this shouldn't be equal to chaos. The derivative should *relate* to its underlying (otherwise the technology would be missed), yet this relation should not fall back on a mere valuation. We shouldn't specify the background space first and then write the dynamics in that space. The dynamics should create its own space. States of the world, writing the underlying process and always falling back on the underlying process as a redescription: this whole way of thinking amounts to believing in the background space. Instead, the dynamics should be the dynamics of recalibration, in which the market is incomplete without ever the thought of its completion.

The new logic is to say that a new price for a new contingent claim is a contingency for the market that is unwritable as a variation of the underlying process, in much the same way that the abstract states of the economy were unwritable; except that this contingency emerges from the market itself. To reason this way is to neutralize all ideas of convergence of prices to values in the long run or of belief. As we will see later, the contingency or the new price will have to be impossible to recover by a rewriting of the underlying process. As a matter of fact, it will consist in a step outside the limits of the formalism which gave meaning to value and to price, when the contingent claim, which the formalism was not able to cover or even to formulate, takes over the contingent payoff.

We no longer count, or we no longer count in the same way, when the market becomes the sole underlying (or the underlying becomes the sole market and sole economy) and the sole horizon. It seems as if the count was still available, as if we could still think of a number of contingencies, or states of the world, being given, first, and the market being given, second, with the mission to span the previous number. It seems as though we could, first, specify states of instantaneous volatility, or a number of jump processes, or any other variation or enrichment of the underlying process that the underlying alone could no longer replicate; however, with the idea of the underlying trading in the market and of the market being the only 'stuff' and only economy, there is nothing to prevent any number of derivatives from the idea of their own trading, and subsequently from completing the market. This is an argument to the effect that starting out with the market as sole underlying/horizon is already exceeding the category of count, for it is only through a false impression, due to the false impression of number, that we believe that a number of contingencies could be thought, first, and the idea contemplated, second, of the number of derivatives that could span them.

Once the market becomes the underlying/horizon, it is already the case that the market is always already completed, if we look at it from the side of the category of count. The category of count was suitable when there was the abstract, or transcendent, separation between the background economy and the market contained inside it. As I said, this separation allowed that contingencies might then just be thought in the fixed background, and not be writable. Once the market is considered as the sole horizon, its immanence, or the fact that contingencies are only down to trading, cannot but lend trading force to the derivatives, and this, when improperly mixed with the thought of the martingale measure that the market selects and the corresponding count, becomes equal to the potential completion of the market. Looked at from a certain angle – the angle of the count – the market is always complete. The underlying process as an absolute background in which the number of contingencies is given at will is not a good background, because it is not stable; it is always already overwhelmed and taken over by the trading of derivatives.

It is only in a temporary station of the thought, which is in reality unstable, that we believe we can freeze the number of contingencies in the underlying process and then consider that the market is complete or incomplete. A new logic should be enunciated in which writing the underlying stochastic process, instead of condemning the market to completion and turning the prices of derivatives into slaves of the grid of states of variables of the underlying process, creates the reality of the technology which is the *reality of recalibration* – implied volatility that traders start trading without relation to ultimate convergence or belief. The market is completed in the sense that underlying and derivative trade alike, and this means that the market is always *already incomplete* without even another face, without even the category of states of the world; for absolute contingency, or the event, has no other face and the market is its other face.

The whole dialectics of incomplete vs. complete markets took place before a market for derivatives was considered. In the larger point of view in which contingencies of the economy were considered and counted against the number of traded assets, the latter were supposed to be independent of each other (not derivative on each other). The point of view was too broad anyway to bother looking into the details and require that a certain asset be derivative on another. In the broad consideration of attaining the general states of the economy, what would be the point in adding a superfluous complication and in specifying that a certain asset is, as a matter of fact, derivative on another? Better to keep a distance with the assets and consider them on an equal footing: a number of traded assets to be eventually compared with the number of contingencies.

A market for derivatives, hence the derivative as traded asset, began to be prominent after the reduction of BSM. There was no longer the general background of the economy and its abstract contingencies, no longer a general information set of which the market was only a subset. Suddenly, with BSM, the stochastic process of the underlying price and other processes governing the parameters of the first became the only filtration, and what other information to be revealed over time was deemed irrelevant for the derivative pricing problem anyway. As evidence that the question of incomplete and complete markets is really incompatible with the BSM event, I cite the surprise we all had when we realized that the market was complete in the BSM framework and all the attempts we then made to make it regain its incompleteness. The only way the market could be incomplete

again (we thought) was to enrich its process. So, we introduced jumps and stochastic volatility, and the underlying as sole traded asset was no longer sufficient to span all the contingencies of the 'economy'.

Note that all this is artificial, however, or purely formal. We are simply trying to exceed the traded asset with the only thing that is at our disposal, which is the traded asset. It is OK to *think* of general contingencies of the economy that exceed the market; but how is the complication of the underlying process, which is now our only excessive contingency, even to be thought? Has the underlying asset been historically observed to behave like that? Looking at general contingencies in the economy is not the same thing as looking at variations of the underlying process. Are those variations a general fact about the economy?

One shouldn't look at statistics as such anymore. Let us consider instead the following narrative. Let us first consider a traded asset in the purest case of trading, which is Brownian motion. Let us then remark that all further complications amount to leaving the minimum requirement of EMH and to reintroducing history artificially – where history means that Brownian motion is now historically observed to admit of stochastic volatility or to alternate with jumps, and so on.

The market is, first, complete and in no need of derivatives. We make it incomplete on one side by enriching the process – we have no other way – and we try to recover completeness on the other side by writing and trading derivatives – we have no other way either; until we realize that the market is not about statistics but about trading and that we should open it *from the start* to derivatives and vary it—that is, make it incomplete – from that side only: by considering different prices for the same derivative given a certain underlying price.

So, we imagine the independently trading processes of derivatives written on that underlying. But don't we have to consider that the prices of derivatives are consistent with the underlying (a problem we wouldn't have had in case the assets were really independent)? This imposes that the price processes of the derivatives (what we have called their 'independent trading processes') are not really independent but are read out as evaluations in the martingale measure that the market will have found. And if the market had, indeed, already found it, then all price processes of all derivatives were already given and I don't see what distinguished the precise few among them that we called 'trading processes' intended to complete the market. Could completing the market only depend on a mental decision? I propose, as a radical alternative, that the market no longer be thought as a part of the economy but as a whole, as the total of itself; yet a total that is never totalized, no longer dependent on a background of states; a total in which contingent claims are written *before* the idea of states of the world on which they are written or which they span and in which their prices become, for this reason, directly the contingency.

The incomplete market theory is historically (and, we believe, indissolubly) linked with the metaphysics of background states of the economy. States are transcendent and remote; they are given in their fixity and we postulate that the traded assets do not span them; there is nothing we can further do. There is a postulated separation between the market and the background states, and the states are so abstract and fixed (fixed because they are abstract – thought cannot work on two

registers and curl back on itself) that it comes as a *subsequent* question to determine what market price for goods will prevail in each contingent state.

In the Arrow-Debreu model, agents need exactly to anticipate the price of goods in order to determine the ratio of Arrow-Debreu securities they wish to hold in order to buy the goods in the corresponding state. It is interesting to note that the incomplete market theory was proposed as an alternative way of treating the unrealism of the perfect foresight hypothesis. Another suggestion was to assume exogenous expectations as a way of dissociating the price that the agent admits in his decision-making process and the price that will obtain. Instead of adding to the complexity of the problem (by postulating such exogenous expectations), the incomplete market theory thought better to take away from its complexity – simply to assume that, for some reason, the securities spanning the background states are not all available for trading (or, perhaps, even for writing).⁹

Once the step is taken, however, no longer to view the market against the background state but to consider the market as the only background – that is, prices prevailing *in the market* become the only states of the world – there is no longer the question of guessing the price in the given state. The price becomes the state. The problem becomes impersonal (third person). Price processes can be written as a summary of a trading activity; the market becomes the sole agent. As a matter of fact, agents disappear in the new register of derivatives pricing. Derivatives are supposed to trade as impersonally as the price process of the underlying and this is the reason why the martingale measure *of the market* takes over any question concerning risk preferences. Eventually, the only act the agent has to perform in each price state is to determine the ratio of underlying to buy or to sell in order to hedge a derivative. This, of course, depends on the volatility. However, the knowledge of volatility is not open to debate, in the price state, in the same way that the market price was open to debate in the background state of the economy. Volatility is considered to be given because the underlying process has been written as a summary of a trading activity, and this is *objective*.

It is a muddling of the problem and an equivocation to think that the agents do not know what volatility will prevail. There are no expectations any longer in the derivative pricing problem. It was a liberating step to write price processes (trading processes) from the outside. Agents may have had expectations before they traded the underlying; however, what we are reporting now is the result of trading – its statistics, reified and re-objectified as a stochastic process. To repeat, there is no doubt about the existence of objective probability in this setting, and the martingale measure is not a subjective probability. It is the re-expression of the major step, or the major coincidence, that price is, at the same time, the randomness (the lottery) and the value. However, the incomplete market theory is in trouble, too, because, now that the market is the only background, it is also the only horizon. There is nothing to stop it from valuing all the Arrow-Debreu securities in its martingale measure and from turning their values into objective price processes. Once the market becomes the only background and prices become the only states, states of volatility or jumps (the new states of the economy) are no longer remote; they are all here, given with the underlying price. It suffices to estimate them statistically. There is no transcendence and everything is at hand. Conversely, there is nothing to stop new derivatives from being written and from being traded.

The whole metaphysics and *orientation of thought* (i.e. the relation of thought to its background, what is given to thought) changes when the market becomes both the background and the horizon. I believe that incompleteness of the market should subsequently no longer be thought back into the background states. The market should generate its own incompleteness from the tradability of the derivatives, not from historical price series and statistics. Jumps should no longer be assumed in the underlying process but between incompatible pricing contexts. Options are not tradable in the BSM context, but they become tradable in the next context. This, by the way, answers the perfect foresight problem better, because this puts us in contact with the event. As soon as the market becomes both the background and horizon, the whole metaphysics of states should be abandoned. This leaves only the changes of context – the only form of incompleteness that is suitable for the market.¹⁰ It is no longer the jump or the volatility that should be the issue or the state but the pricing context of the given derivative, in which it is precisely required that new states emerge; in other words, derivative *prices* and not derivative valuations.

The Central Knot

Is it a coincidence that the binomial tree and Brownian motion should yield a complete market, respectively in discrete time and continuous time? That the market should be complete or incomplete seems to us so dramatic an alternative, yielding alternative consequences so incommensurate with each other,¹ that it cannot depend on picking Brownian motion or the binomial tree among a choice of several available stochastic processes, as if another choice were possible and it was only a coincidence that we had picked Brownian motion or the binomial tree. Brownian motion and the binomial tree seem to us to belong to a category of their own inside which it would no longer be a coincidence if we picked them because they fill it completely – their own special category.

We think it is a category mistake to identify Brownian motion or the binomial tree with a stochastic process of an ordinary type; that is, such that variations of that type may describe alternative processes which would no longer yield a complete market. As a matter of fact, it is just a coincidence that Brownian motion or the binomial tree should be represented, from outside, as probabilistic processes that we may then compare with other processes. What they are, from the inside, is something special and specific to the market – as such, incomparable.

Randomness in the market precedes probability. It is characteristic of the exchange. Simply, a price process cannot be anything but random for, in case the future price was predictable, it wouldn't occur in the future but would trade as of now (efficient market hypothesis). The exchange seems to prevent any other possibility than the future price being random. In order to be in the future, the price has to be random. It is not that something (the price) is in the future, then we subsequently check whether it is random or not. Time seems to depend on randomness as a *precondition* and not the other way round. It is a coincidence if time is *later* appended and the time series of prices coincided with stochastic processes and yielded statistics.

The price is *essentially* random, not temporally random. Statistics is a later coincidence which may be muddling the issue, for that matter. Brownian motion and the binomial tree yield complete markets in financial theory because they are a perfect and complete description of the market, when its randomness is understood essentially; that is to say, genetically and not temporally. What the

price can do, essentially, is go up or down. Hence the binomial tree, of which Brownian motion is the continuous limit. Although Brownian motion may branch into infinitely many states, the stochastic integral and the law of large numbers that is compressed in it, bring back the binomial feature; they bring back the dS , which is a single factor and, as such, drops out of the equation of the hedging portfolio involving the derivative and the underlying. The convergence of the stochastic integral makes it so that, for all practical purposes – which are perfect replication, or replication that no longer engages probability – Brownian motion is for continuous time what the binomial is for discrete time. (The convergence of the stochastic integral introduces the notation of the stochastic differential dS .)

Brownian motion is more satisfactory than the binomial tree, for it represents absolute trading – continuous trading that is no longer relative to a fixed, hence arbitrary, time interval. Brownian motion is paradigmatic of the market; it comes before probability because of the EMH,² and we contend that *for this reason* it has the martingale representation property and all contingent payoffs are replicable. The market cannot be anything but complete when looked at paradigmatically – as the price both generating the randomness and giving the value of its own lottery, or what the market is all about. It is only when we exit the essential category of Brownian motion and represent it probabilistically that we are tempted by market incompleteness (adding a jump to the diffusion, making the diffusion coefficient stochastic, adding a state to the binomial tree, and so on) and we inherit the riddle of the martingale measure that the market selects. There should be no martingale measure that the market selects; each contingent claim should be given its trading process (incompleteness should be conceived in the sense of constant innovation through the derivative trading and recalibration to its price) and, instead of even *writing* the volatility of Brownian motion, we should think of the corresponding price of the corresponding volatility derivative.

In our present speculation, we wish to prohibit the probabilistic exit and the *writing* of coefficients. There is no history and no statistics, no representation of a temporal process when we trade in the market. The criticism should come from the change of pricing contexts and from the prices of derivatives.

The wonderful thing about trading is that the traded asset S is, at the same time, the device triggering the lottery and the ticket to buy the lottery. The price of S is both the random entity and the current price of the lottery whose outcomes are the future random prices of S . Both effects are due to the exchange. The price is random because of the exchange and it is the price of the corresponding lottery because of the exchange.

Randomness that is generated by the exchange is qualitative at first, not quantitative; intensive, not extensive. It is only in a second step, when we decide to represent the randomness extensively, that we argue that the price of S can branch into two states, three states, infinitely many states, and so on. In the latter schema of thought, randomness is established, first, and, then, we exit probabilistically in order to represent it. However, when randomness is still being generated in the exchange – that is to say, qualitatively and genetically at a stage when the number of states, or even the separation of states is not yet acquired (when extension is not yet acquired) – all that really counts is that the price may go up or down.

Up and down are qualitative and not quantitative. They are not a number of states; they are just directions – the ones that are essential to the market. If somebody wishes the underlying asset S to branch into three states or four states, and so on, one can always argue that these states are reached only after two steps or three steps of intermediary up and down binomial moves. Similarly, Brownian motion is the qualitative movement of price once construed in continuous time. Obviously, what matters most in the continuous limit is that the path should never be differentiable or predictable, hence Brownian motion. Of course, the price may jump in reality, but we wish to argue that the jump is like adding a third state to the binomial fork. The jump occurs at the stage of the probabilistic exit. Essentially, as it is being generated from the exchange, a continuous range should be visited by the price – or so we wish to argue.

Jumps are dramatic; they bring in external elements to the essential or genetic argument of the exchange. For instance, they can be explained as brief illiquidity episodes, where liquidity appears as the additional or parasitic concern. If the motion of price, as it essentially arises from the exchange, is always qualitatively binomial in discrete time (up or down without size specification) or Brownian in continuous time, then this essential description of the market is *complete*. It is complete because it is qualitative and it qualitatively closes itself off. As a matter of fact, it is also complete in the financial-theoretic sense that all contingent payoffs then turn out to be replicable.

13.1 Contingent payoff vs. contingent claim

Replication of contingent payoffs was first shown by Black and Scholes in their 1973 paper, after an argument by Merton.³ It wasn't until 1980 that the argument was made rigorous with the help of the formalism of martingales, in two papers, the first by Harrison and Kreps (1979) and the second by Harrison and Pliska (1980).⁴ Below is an emblematic quotation from Harrison and Pliska's 'Martingales and Stochastic Integrals in the Theory of Continuous Trading' (1980). What motivates our present speculation is precisely to follow through the formalism without exceeding its limits, and to see what consequences this may have for the existence of a market for contingent claims and even the existence of contingent claims themselves. Can a market for contingent claims be formalized?

We have inverted the order of the paragraphs as they originally appear in the paper by Harrison and Pliska. Emphasis is ours:

Throughout this paper, we focus on an isolated market in which certain securities are traded, assuming that no arbitrage opportunities exist internal to this market. We seek to characterize the class of contingent claims that investors can attain, and the prices at which they can attain them, by dealing only in the designated securities. In discussing the valuation formula, for example, we have focused on a market where only the stock and bond are traded, and we have discovered that investors can manufacture call options for themselves in this market at the price specified in the formula. *No comparison is made with the price at which options do sell, might sell, or should sell outside our market, although it is obviously possible to do so.*

In brief, the option is *attainable* in this market, at a time zero price of π , by dealing only in stock and bond. In the economics literature it is customary to go further, arguing that arbitrage profits could be made if options were sold in a parallel market at any price other than π , and that existence of arbitrage opportunities is inconsistent with equilibrium in the total economic system. See, for example, the original paper of Black and Scholes or the recent article by Cox, Ross and Rubinstein. *To reduce verbiage and to get a self contained mathematical theory, we shall simply stop with the statement of attainability.*

The crucial word here is *verbiage*. Suddenly, this is called *verbiage* and deemed totally irrelevant to the formalism to go and see whether the options or contingent claims, whose payoffs we have just replicated or attained within the formalism, admit a market of their own, or worse, whether an independent market for them has ever been conceived. As a consequence, we conclude that an important distinction should be made (which Harrison and Pliska don't make) between contingent payoffs and contingent claims.

As we see it, a *contingent payoff* is just an amount money, contingent on the price of the underlying S , which the Black-Scholes-Merton (BSM) dynamic replication algorithm can help us manufacture for ourselves at a given maturity T , whereas a *contingent claim* is an independently *written contract*, whose written payoff may be numerically coincident with the contingent payoff that we have synthetically manufactured. Crucially, however, the contingent claim can be independently traded, at prices that may differ from the value that the formalism assigns to the contingent payoff as the cost of its replication. The reason why the contingent claim trades independently is that it is material, therefore materially tradable, whereas the contingent payoff never was.

The contingent payoff is part of the formalism, whereas the contingent claim isn't. Harrison and Pliska should really be talking of 'contingent payoffs' rather than 'options' and 'contingent claims'. Contingent payoffs are attained and manufactured but they haven't been written out as independent claims. So, there is no point in even thinking or worrying about their market. What Harrison and Pliska call *verbiage*, they should rather call equivocation.

13.2 Probabilistic exit

Due to their attainability by the dynamic trading of S , contingent payoffs add nothing to the idea of trading S , whose essential description is the Brownian motion of the price of S . The market is always complete once its motion is understood qualitatively or genetically, without external and subsequent additions or complications. An asset S trades and its essential price motion is up or down (or Brownian in the limit), and this has as a tautological consequence that every contingent payoff can be manufactured. It is only in a subsequent move that the probabilistic exit is attempted and the market is made incomplete through probabilistically complicating the stochastic process of the underlying, when it should have been made incomplete through *price*. The process should have never been written probabilistically. The price of an option, or a volatility derivative, should have taken over the diffusion coefficient, right away.

Probability is external to price and incidental on it. Price, we said, is random before even probability is imagined because randomness is the necessary condition of the future (or of time) in the market environment. Because of the randomness of its process, price inspires probabilistic thought in a *second step*. We wonder what the probability is that the price ends up above or below a certain strike at a certain date in the future. We wish to evaluate the corresponding lottery. However, price is also quick to surpass and short-circuit that thought, because, as price, it is already the value of the lottery ticket allowing us to collect the future contingent payoffs given by the movements of price itself. Nobody needs probability to compute the present value of the lottery governed by the underlying price, because this value is the price of the underlying already. As for the partial probabilities of the events of ending above or below a certain strike, or of hitting a certain barrier, or, indeed, of following a specific path, and so on, nobody needs them either so long as the purpose of having them is to evaluate the corresponding bets. Due to the major coincidence that the bets we wish to make will be settled in money and that the underlying, whose price evolution defines the payoffs of the bets, trades in a *financial* market and may be bought and sold for money, too, not only do we short-circuit probability because we will move directly to the value of the bets in money (thus ignoring the metaphysics of probability and considering directly the result of integrating probability against a financial payoff), but the settlement of the bets itself will be offered through the underlying price, by replication.

An insurance company may be interested in the probabilities in order to compute the present value of the lotteries through actuarial science and breaking even in the long run; however, as we said, if the lotteries are settled financially, the insurance company had better evaluate them away from the market, for the market readily offers those valuations and, additionally, the market does not even wonder what the value of such lotteries may be. The market does not consider the lottery, first, *and then* wonder about its value. Through the trading process of the underlying, it readily generates the contingent payoffs. The market of the underlying is complete in the sense that the lotteries need not, and even should not, be considered as questions of valuation.

The market of the underlying is not just a generator of random numbers, which are otherwise and incidentally known, so to speak, as the 'underlying prices'. If all that interested us in the underlying price motion was whether the number known as the price would end up higher or lower than a certain threshold at a certain term, or hit a certain barrier in the meantime, or, indeed, follow a specific path and with what probability, then the financial nature of price and the quantities of the underlying stock that would be bought or sold at that price would not matter to our problem and would only strike us as additional features, irrelevant for that matter.

When it is specified, however, that the reason we are after the probabilities is to evaluate the corresponding bets and when it is observed that those bets will be settled in money, the financial nature of the random device that is itself deciding the outcome of the bets is remembered. We remember, in other words, that a lottery that will be settled in money had better be bought or sold in money and that the underlying, which is itself the trigger of the lottery, is readily negotiable in money; we remember that there is no looking any further for the value of the lottery that will deliver the future random values of the underlying as its outcomes, for that value is the ready market price of the underlying and the future random values are the future random prices.

The trading marvel must be remembered, thanks to which buying the underlying S is entering into that lottery and committing oneself to its random outcomes (for the lottery is none other than the underlying price) while automatically purchasing the lottery ticket for the right price which is the price of the underlying itself. This is the major coincidence (the marvel) which sets the market, its randomness and the corresponding lotteries in a category of their own. It must also be remembered that the underlying is not only negotiable in its market in the same currency as the payoffs of the corresponding lottery, which is money, but that, additionally, it can be bought and sold in variable quantities. The market of the underlying, we might say, finds its complete characterization in the randomness of price that is due to the exchange and in the *variable size of the trade*. This, we might say, is the defining characteristic of the exchange.

Now, it turns out that these two features, randomness of the price and freedom of the size, are exactly what it takes to synthesize contingent payoffs. The martingale representation theorem that Harrison and Pliska crucially utilize in their derivation is nothing else than a *translation* of the phenomenon of the underlying market, or what trading really means. It is the same thing to trade the underlying at random prices, in quantities that can be made functions of those prices, and to manufacture contingent payoffs that will be triggered at predefined dates or predefined thresholds. This is the meaning of the replication argument. When the process (S, t) is understood in the full sense of a financial market – and no longer as a mere random generator – it becomes equivalent to (K, T) , where $K = f(S)$ is the contingent payoff at maturity T . It is part of the *meaning* of the market that it be complete in this sense. To repeat, attainability of the contingent payoffs is just another word for the trading of the underlying.

It is in a later move, in what we have called a *probabilistic exit*, that the trading process is given a probabilistic representation and that we get all that we are accustomed to hearing when the market is said to be incomplete; namely, (a) that such a thing as the *quantitative* diffusion coefficient of the Brownian motion is first construed and identified, (b) that the binomial tree is *quantitatively* criticized and a third *state*, adjacent to the first two, is conceived, (c) that the diffusion coefficient is made stochastic in its turn (thus confirming the probabilistic exist and representation – what could be the limit now?), (d) that a jump process is added to it, and so on.

In the original qualitative sense which is the defining sense, the market is complete. It is absolutely complete, in an undifferentiated sense – not yet amenable to a probabilistic representation or separation. Once probability is held back and made not to interfere with the genetic character of the market, once the market is restored to the simplicity and closed character of its absolute and original defining sense, it appears that an alternative exit was possible. If the equivalence between dynamically trading the underlying and manufacturing predefined contingent payoffs is made to sit in the centre of the picture as the central knot, it appears as equally possible, with regard to that knot, to hand over the intensive trading process to an extensive probability representation, or to deliver the synthesized contingent payoff to the material fate of *being written as an independent contingent claim and of being consequently independently traded*.

13.3 The alternative exit

Either we exit with the time process or we exit with the place process. We can open either door. One has to realize that, with regard to the original knot of the underlying market, writing the contingent claim in a materiality and reality that was not part of the formalism and subsequently delivering it to its market *is an alternative move and an alternative exit* to writing the stochastic process of the underlying and extensively identifying its parameters, which can then lend themselves to their own stochastic processes with nothing to stop them or limit the number of such processes. Trading the underlying (or, in a word, the market) is such a peculiar category that it is retired from probability at first and probability has to be subsequently conceived as just *one* possible exit (out of two). We have mentally to perform this separation and train our brain to performing it, so entrenched probability is in our mind.

To have a real understanding of how separate the probabilistic representation should be from the trading intensity (the intensity of the exchange whose defining elements are randomness before probability and the variable traded quantity), one has to observe that the written materialization and subsequent trading of the contingent claim (which was not part of the original form or formalism and was, as a matter of fact, forbidden and unutterable in it) is equally separate.

Once the probabilistic door is opened and numbers are suddenly introduced, thus breaking the symmetry and the completeness and the original undifferentiated character of what trading means, the absolute picture of trading starts lending itself to analogy and comparisons and the model is then said to be wrong because, in reality, the underlying price does not follow Brownian motion with constant diffusion parameter. BSM is said to be wrong because the empirical time series of the underlying prices does not match the theoretical statistical distribution. BSM is thought to be a wrong mathematical or quantitative model and the whole tradition of criticizing mathematical modeling in finance opens up.

We don't think the categories of right and wrong apply to BSM quantitatively. What's wrong in BSM is something more pressing and altogether more dramatic (more important) than the mere comparison of numbers. BSM is wrong in its form, as formalism, not in its mathematical probabilistic exit. It is wrong because it is closed to the existence and subsequent trading of contingent claims – closed to their writing, because to write them is to trade them. One has to think of this more significant and more urgent way of being wrong while keeping in mind the initial and equal separation of both the probability of the underlying states (or time series) and the price of the contingent claims (or place series) from the original picture.

Claims of BSM being a wrong model of reality are already committed to the mathematical instantiation of BSM. They have already gone one step too far. The trading of the underlying, when it is construed as an intensive phenomenon, admits the writing and the subsequent trading of the contingent claim in its independent market *as an extensive differentiation that is alternative to statistics*. To repeat, as trading intensity (what exchanging means), the market or the trading process is so retired from probability that the writing and trading of contingent claims is the opposite, but

equally possible, exit. This explains where implied volatility comes from and why it will be forever irreconcilable with real volatility.

It is not without a reason that the absolute and qualitative completeness of the market is broken by probabilistic complications of the underlying process on the one side (adding jumps, stochastic volatility, and so on), and recovered by the trading of contingent claims on the other side. However, going for the probabilistic exit first may not be the best choice. For one thing, it denies contingent claims their true and deep tradability. As we already said, adding jumps or stochastic volatility causes the ambiguity and multiplicity of the martingale measures and, concomitantly, the puzzle of the one that the market will select. Contingent claims are said to be evaluated in this selected martingale measure and, because the evaluator is the market, their price process is then illegitimately identified with a trading process. This is a make-believe; it produces the weird situation in which the martingale measure becomes unique and the dangerous slippage of meaning in which a result (the evaluation of contingent claims in that mysterious martingale measure) becomes a given (a trading process which is as given as the underlying process).

The market cannot be its own result, then become its own given. There is a problem of chronology of thought and a cheating on numbers in this 'manipulation'. Better to move from the qualitative completeness of the underlying market to the trading processes of contingent claims without the probability providing the transcendent background (it will only provide the tools, the intermediary). As witness that probability is very entrenched, think of our tendency to see in the underlying trading process a stochastic process first, then to inquire about the probabilities, then about the values of lotteries, then about their prices (when those lotteries are brought back to the market), when the formalism tells us that there is nothing more to the replication of contingent payoffs (those lotteries) than the trading of the underlying.

As soon as randomness is thought to develop in time and time is thought to be the original background or theatre against which we then check whether something is random or not, this sequencing of thought in which probability is the main intermediary step seems inevitable. We have seen, however, that randomness that is induced by trading and the exchange may have an inverted relation with time, as it is itself the precondition of time. Randomness and trading size are immanent to the exchange because to exchange is a material process that cannot but generate randomness and occur in size as the other face of its materiality, as opposed to randomness that is first conceived abstractly, as the result of some abstract random generator, then recognized as the randomness of that price in relation with that quantity.

Before we even unfold the exchange in time, it is part of its meaning – intensively, so to speak, and not extensively – that randomness of the price and the size of the transaction should be involved. Compare this with an 'empty' or immaterial random generator that we need to unroll in chronology – or, at least, against the chronological thought of identified states, one among which is supposed to be randomly picked as the definition of randomness – in order to find out that it is a random generator. Now this, the randomness of the price and the variability of the traded quantity, are equivalent to the manufacturing of contingent payoffs at a stage where extensive time has not yet entered the picture. It is in this respect that introducing time in the trading process of

the underlying or introducing the market of the contingent claims are equal breaches of the original intensive meaning of the exchange. It is a coincidence if the exchange can unfold in time.

13.3.1 Differentiating the form

My astonishing claim: it is not the mathematics that has created the derivatives market, in a situation such that we may then criticize the quantitative or algorithmic aspect of the project and contend that the underlying motion is not Brownian, that trading is not perfect and frictionless, that the mathematics or the quantitative model are not right – a criticism à la Taleb, whose consequence is that we go looking for the right model or the right dice, scalable distributions instead of Gaussian, and so on. What has created the derivatives market is the *formalism*, in the exceptional and historic manner in which formalisms create matter: through the idea that they *are not supposed* to do so – through an accident, through an event, through an illegal extrapolation, through a misinterpretation (*contresens*) rather than a logical derivation, through the closure and self-containment of the formalism rather than its openness or its capacity to act as a model or an exemplar, through the absence of possibility that the formalism may ever produce something like that; that is to say, *through the void*.

Formally, the market of contingent claims does not exist and they are not even written, to be exchanged. Formally, there is nothing but the trading process of the underlying and the trading strategies that involve, as Harrison and Pliska say, only the traded securities (the bond and the stock), as a result of which certain contingent payoffs – not contingent claims – are manufactured. We do not know what to make of the written contingent claim in this framework. Such a thing is not defined and doesn't exist.

It is in a different framework (the one we have called 'probabilistic exit') that states of the world are defined externally; that is to say, first and transcendently, as well as outcomes of securities and probabilities. This is a framework in which arbitrage *then* implies that the present values of the securities have to be expectations of payoffs under a probability measure. In this situation, the securities are defined, first, and are supposed to trade, second; the only coincidence or peculiarity being that states of the world in such a framework are actually states of the underlying price and the real probability the probability of its transitions.

All the ambiguity stems from this question: Is the contingent claim actually predefined and existent, and can it consequently have its own divergent market? (Talking of arbitrage presupposes that the price of the contingent claim can be otherwise.) It makes an immense difference to suppose that contingent claims exist or don't exist. If they do, we have to picture, at least in our mind, their possible divergence, the forces that bring them back in line, somebody who will make or lose money in the long run if they don't, the market that anticipates that or corrects for that, and so on.

In the present framework, by contrast, there is no probability background. The only background is the trading of the underlying. For this reason, we can take the alternative radical route, the one that argues for price before probability and for the stage that occurs before the differentiation of states and probability – the stage of the undifferentiated market in which the market means what it means genetically and qualitatively: randomness which results from the material exchange and the variations of the traded quantities and not from the conceptual differentiation of states.

It is *quantitative* to argue that the states exist, that the mathematics of probability exists, and that the arena of arbitrage is open in that setting. It is, then, equally quantitative to argue against the mathematics quantitatively. By contrast, it is *form* and form alone to talk about the trading of the underlying and to pursue a self-contained mathematical theory as Harrison and Pliska say. The original BSM paper assumes the derivative *exists*, and that its price will be a function of time and the underlying price before maturity. Why? Its price can be a function of anything if the market is open for it.

The probabilistic exit is not the right approach. The right approach is to have only the underlying and its trading and to see what contingent payoffs we may manufacture. In certain special circumstances (Brownian motion), what we manage to manufacture through an appropriate trading strategy turns out to be a function of the underlying price alone, *hence it can be written* and, for this reason, it can join the market of the underlying. This is the genesis of writing. We manufacture a payoff that depends only on the trading variable (the underlying S) and does not depend on the stochastic structure that might have existed (e.g. states of stochastic volatility), a payoff which we can *for this reason* reintroduce at the beginning as an independently written contingent claim. To do so and to differentiate that form into the material existence of a market of contingent claims *is a radical step outside*; an event so to speak. In all other circumstances, our dynamic replication will depend on the probability assumptions and the proceeds of our trading will be distributed *probabilistically* around the contingent payoff that depends only on the trading variable. Why is it better to argue for the emergence of the material market of contingent claims from the approach in which only the underlying trades, than from the probabilistic exit? Because it is better to reason distinctly and to know that the formalism has nothing to say about a market for contingent claims.

Note that there seems to be an equivocation on the uniqueness of the martingale measure when the market is complete as well as on their number, when the market is incomplete. When the market is complete, all that we are considering are contingent payoffs that can be manufactured by a trading strategy involving the underlying. The value of the self-financing strategy is a martingale at any point of time, as one cannot generate arbitrage opportunities by trading in a self-financing way an underlying that is 'fairly' and completely trading in its market; but now the question is the converse of this one – whether all martingales are thus representable.

It seems Harrison and Pliska have assumed that the contingent payoff admits of a price before its maturity and that arbitrage is already enforced, since they speak of it as a martingale already and they only wonder whether it is attainable. But if it were independently trading as a material paper, it might not be a martingale under the filtration of the underlying already. Thus, there is equivocation. We haven't yet written the contingent claim independently, yet we are imagining for it, prior to its expiration, a price process that seems to be already curbed by some forces and to be already a martingale. So, is the market complete in the sense that the contingent claim is not independently trading?

There is equivocation in Harrison and Pliska. They say they don't want to mention the parallel market of contingent claims, yet they suppose that their price processes in this market are already martingales under the same probability measure (i.e. they are already arbitrated with the underlying). The resolution of the equivocation is to consider that the martingale that they assume is

only an intermediary formal posit (and not the material price of an independently trading contingent claim) which allows them to get the final result they need; namely, that, in this closed market, contingent payoffs are eventually replicated at any maturity.

We find the same equivocation on the number of martingale measures (i.e. whether there are many or there is, ultimately, only one) when we falsely make the market incomplete. Contingent claims are then lotteries that are evaluated in a non-arbitrage way and later their price processes, or martingales, become trading processes. Definitely, there is a gap between writing the stochastic process of the underlying and moving, one way or the other, to the existence of independent contingent claims.

13.4 The invention of writing

Traditionally, the trading process is illegitimately confused with a stochastic process and this means that states of the world, as such transcendent, take over all of sudden. States of the world and probability become the absolute background (the ether) and it is in a later move that securities are brought into the picture, whose payoffs we know in the different states of the world and whose value we wish to determine today, and it is as a 'later' coincidence that one of these securities is recognized to be the underlying whose price process is the driving stochastic process. Probability takes over and, when the underlying process is the only driver and the valuation of derivatives written on it is the only issue, the only states of the world that can be relevant are, additionally to the underlying prices themselves, the different states of volatility or jump sizes of the underlying price, and so on.

The picture is now familiar. An insurance company would value all securities as their expected value under the real probability measure, because the rationale is to break even in the long run. By contrast, agents faced with a single run would value them at a discount when they are risk averse (or equivalently with a risk premium that they hope to earn); then, it is recognized that in a market environment the single run 'valuation' of the underlying had better be its current trading price, hence the different martingale measures that have to be equivalent to the real one. Then, the confusion is committed following which the derivatives, supposedly additional lotteries whose valuation question arises after the market is closed for the underlying under its stochastic process, are uniquely priced by the *same* market under the martingale measure which the market (as a single trading place) selects. The confusion is further thickened by making the value process of the additional derivative a price process, or trading process, which is thought to be as given as the initial underlying process, and the value of the extra lottery that stochastic volatility or the additional jump provide is then thought to be preferably equal to the price that the market attaches to it under the guise of that derivative price.

When probability takes over the trading process of the underlying, the question may be asked about the non-arbitrage valuation of different lotteries whose outcomes are driven by the underlying. However, we now wish to argue that, although probability is also present in the representation of the trading process of the underlying as temporal price process, it is *an altogether different approach* to concentrate on the self-financing trading strategies of that underlying. In such an approach, we

never leave the heart of the underlying market. We do not bring in foreign ‘paper,’ or the contingent claims, of which it becomes a muddled issue whether they should be valued as lotteries or traded in their turn.

When we focus on trading the underlying dynamically (in variable size), there is not even mention of the contingent claim as an independently written security whose payoff the underlying *then* replicates. There isn’t the setting in which it is asked how to value something consistently with the underlying. There isn’t even *something* whose value is, first, considered independently and consistently – that is to say, as variable, and, second, curbed by arbitrage. The whole formalism of the dynamic trading of the underlying could and should unfold without uttering the word ‘option’ or ‘contingent claim’ a single time.

All there is to the argument is that contingent payoffs can be manufactured perfectly when the underlying motion is Brownian. We know what amount of money should be initially invested in what fraction of the underlying and how, then, dynamically to rebalance both our holdings in the underlying and in the money account without any external intake of money (i.e. in a self-financing way) in order to reproduce any function of the underlying price at any maturity. There is a perfect duality, we may say, between all kinds of trading histories of the underlying and all kinds of future contingent payoffs.

One way of swallowing back the probabilistic framework that is made up of states and extensive time into the trading intensity was to think of Brownian motion qualitatively, as we did – as randomness due to the exchange before probability represents it and as the variable size of the trade before extensive time unfolds the trading history. It is at this pre-quantitative intensive stage that we wished to read the meaning of trading as equivalent to the replication of contingent payoffs; however, the contingent payoffs seemed to depend on the unfolding of time for their actual materialization. To end up with the quantitative contingent payoff, we needed the probabilistic representation and the unfolding of the actual replication strategy, so now we wonder what might intensively stand for the contingent payoff, at a stage when trading the underlying has not yet been differentiated into probability and states?

To repeat, we needed probability and states, we needed the temporal process of the underlying, in order to compute the initial cost of replicating the contingent payoff and to unfold the path and the actual self-financing strategy whose proceeds will constitute the contingent payoff. Trading the underlying in the meantime, at prices and in size given by the extensive stochastic process, was the guarantee of manufacturing the final contingent payoff no matter what. So, now we wonder what could be the substitute for this extensive story at the pre-quantitative, pre-probabilistic stage.

It is true that the approach that focuses on trading the underlying is locked on this side of the market and has no means of jumping over to the market of the contingent claims per se. We cannot even think of the contingent payoff that we are manufacturing as a contingent claim which would otherwise admit of a market, or even of the question of what its value may be. It is true that every time we invest the initial cost of the replication strategy in the corresponding fraction of the underlying (or invest the proceeds of a short sale of that fraction in the money account) and run the replication strategy, we end up with the predefined contingent payoff; but nobody has asked what the value of

that contingent *claim* was. The proceeds of the replication strategy are not supposed to be used to replicate the liability of the seller of that contingent claim.

At a time when the underlying was the only traded entity and the only contract, there wasn't the notion of a *claim* whose writer would have to guarantee the contingent payoff no matter what (as this is what writing means). It is only now, as we wonder what could stand for the contingent payoff that the trading strategy of the underlying is able to guarantee no matter what, at a time when the underlying has precisely been denied the extensive probabilistic framework and extensive temporal unfolding, that the answer may be the *writing* of the contingent claim. *We are now inventing the writing.*

We used to say that, as soon as the contingent claim was written, it would have to trade. Now, all we are trying to do is find a substitute for the contingent payoff which only extensive probability and time could materialize, in a world where the underlying is the only paper, yet at a time when the underlying is intensively held back. We haven't yet materialized the writing of contingent claims; all we are looking for is the substitute for the guaranteed contingent payoff, at a time when extensive time and extensive trading are denied, and all we are finding is the written paper, whose meaning is to guarantee the payoff precisely.

The same operation (let's call it a phenomenological reduction) which permits us to think of the randomness of the price before probability and of the variability of the size before actual extensive trading, simply through the intensive meaning of the exchange, permits us to think of the writing of the contingent claims whose contingent payoffs were shown to be equivalent to the extensive trading strategies of the underlying. Now, given this reduction and its two sides, it becomes equally possible to differentiate the trading through extensive time and the stochastic process (S, t) , or through the market of the contingent claims.

Going for the probabilistic exit closes the door to the other side, which is the writing of contingent claims. As a matter of fact, the probabilistic exit doesn't represent the variable size of trading; it is only concerned with the stochastic process of the underlying price; so it doesn't really unpack the whole meaning of trading. The surface of the written contingent claims (K, T) does. It is now a small step, or rather a single jump, to introduce their market prices as alternative to all the successive differentiations of the underlying stochastic process. Because of money (this is what's specific to the present lottery) and of trading, the heart of the market, when compressed, gives the written surface of contingent claims instead of the maps of the stochastic process.

13.5 Genesis

It doesn't require probability to hold the underlying, be subject to its random motion because of the exchange, and later to observe that the price we had bought it for turned out to be the value of the lottery that the outcomes of its future prices generated. This, we said, is the specificity and the privilege of price and randomness (as opposed to value and lottery) when the exchange is their genesis. Going one step further in time and benefiting from the other dimension that the exchange opens beside the randomness of the price and which is the self-financing variation of the size of

the holding, we conclude that it doesn't require probability either (I mean, not even its concept) to manufacture contingent payoffs.

Sticking purely with trading and not exiting in the outside probability space, in an image similar to the one that Einstein had suggested when he wondered how space and time would look like if one was riding the photon; what's more, sticking with the intensive meaning of trading at a stage when extensive time hasn't yet been released but when the exchange and its bottomless pit mean already, before time and probability, that the price will be random and the size of the trade will be variable at will, translates into the idea of manufacturing contingent payoffs whose fixity and presence to mind, at a stage when time is kept intensive and they cannot actually materialize as payoffs, now introduces writing for the first time.

Time and probability seem unavoidable in matters relating to the market and trading, for how are we to imagine the result of trading, and even its arena, without unfolding time and unrolling the action? Yet, we are saying that time and probability are not indispensable and, as a matter of fact, they muddle the problem. Because of money and of the precedence of the concept of money to time and number, because of money being an alternative to time (see, in section 7.2.2, the elaboration about not just representing the average of numbers but the average of numbers written in money), there seems to be an alternative space in which the meaning of trading can extend, and this is the surface of the written contingent claims.

If we adopt the probabilistic exit and let the trading process become a stochastic process whose random numbers just *coincide* with prices now left to stand only as numbers, the underlying and all kinds of lotteries or contingent claims become later additions, whose values we only look to compute in a non-arbitrage way; the market becomes an evaluator like any other, and its only specificity and only 'depth' will be that it is unique. The unity of the market will be coincidental (anecdotal) and no longer essential as when we will (later) conclude that the trading of contingent claims is an integral part of the *one* market of the underlying.

By letting the written contingent claims stand as a substitute for the contingent payoffs that we would manufacture from inside the trading process (before any probabilistic exit), we are quite far indeed from the notion of contingent claims that would come as later additions or external lotteries. Their writing and materialization are now an integral part of the *meaning* of trading the underlying. Their surface represents the alternative exit to time and probability, as far as the trading of the underlying is concerned, owing to the distinguishing property of money and place. It is a subtraction rather than an addition. The market prices of those contingent claims, now written and materialized out of intensity rather than extensity, will not come as external additions either. They 'appear' as soon as extensive time and probability are held back or withdrawn, together with the premiums that correspondingly attach to the contingent payoffs by computation (hence the subtraction).

The existence of market prices for the contingent claims belongs to a different stage and different space altogether than the stochastic process of the underlying and the corresponding probabilistic exit. Contingent claims begin to exist and to admit of prices precisely when extensive time and probability are retired and the intensive meaning of trading is probed. Implied volatility means that the stochastic process whose volatility we are implying *should have never been written*. It is the mark of opening the other door: the market of the contingent claims which was supposed to remain closed.

Recalibration is the same as *not* writing the stochastic process and not exiting through probability. Implied volatility is the actual consequence of the closed formula and the complete market delivering what they were not supposed to deliver. When volatility is formally given (in the formalism of the underlying trading), it is not supposed to ‘vary’ (in the sense that it could be inferred, or guessed, or speculated about, and so on). Let volatility be. Given the volatility of the underlying and self-financing dynamic trading strategies, contingent payoffs can be manufactured. This is a story concerning only the underlying. It is a formally written story; how could it vary, let alone be inverted? The current flows only in one direction. A pre-computed premium is invested in the trading strategy to manufacture the predesigned contingent payoff. What is to change the computation apart from a complication of the given stochastic process? What does it mean that the contingent claim, which doesn’t and will never exist in this narrative thread, admits of a different price?

If the existence of the market of contingent claims was the mathematical consequence of the BSM model or of Brownian motion, then we could imply the diffusion coefficient, or volatility, from the market price of the contingent claim. This would amount to a quantitative or mathematical adjustment or calibration of the formula. A quantitative formula would, here, be establishing a bridge between two quantities, volatility of the underlying and the price of the contingent claim, and we could pivot on either one to infer the other. If the price of the contingent claim were different from the prescription of the formula, then we would expect it to converge to the prescribed value, if not instantly by the action of an arbitrageur who knows all there is to know about Brownian motion, the volatility number and perfect replication, and who knows that it is only a matter of automatism and of perfectly foreseeable time before the gain he instantly seizes materializes in effect, at least on the long run by self-adjustment of the market as a whole, as investors learn that buying the contingent claim at a cheaper price than its theoretical value or selling it at a higher price eventually generates profit, run after run, after completion of the replication strategy.

But then we wonder where this different price came from. How could a situation present itself in which volatility is known as it is, or as the formalism *let it be* (let volatility be), and the price of the contingent claim is different from what the formula says it should be? Does the holder of the formula have superior knowledge to the market? Are we already in the business of comparing different kinds of knowledge, by the way? Does the holder of the formula *know* volatility? Then, what does the market know? Is it coming up with the prices of contingent claims without knowing volatility? Or does the holder *think* he knows volatility or believes volatility to be what it is, and then reasons that if volatility was what he thinks it should be, then the price of the contingent claim should be what the formula says it is? But isn’t the formula describing a mind-independent and knowledge-independent reality?

Let volatility be, says the formula. Volatility is not known in the market; it is not a price. For this reason, the equation does not belong in a market or even in an arbitrage situation; it belongs on paper. The market trades prices and only prices, and it is a parasitic philosophical problem to go and discover whether fundamental value should exist, underlying those prices, and whether the market participants know it or not. A theory of the market is, in any case, coherent without this metaphysical supposition. For the first time, something to be known, or just something *to be* (or,

rather, to be let be), volatility, gets mixed up with what appears to be prices in a market. I would argue this is no longer a market; this is only formalism and paper.

I am not saying volatility is not constant or stable, in the market, as the formula suggests it is on paper. My problem is how volatility is known; how it is brought from paper or formalism, or from the sentence 'Let volatility be', to the plane of the market. If we had reached in a market the level of knowing volatility (like the formula says we should: on paper), then there would be no longer a need for the market. In brief, I am having trouble even situating the BSM formula and derivation. The formula and the formalism show a result. If an underlying process is given and is supposed to be Brownian motion with a given volatility, then the premium would be known such that, when it is invested in a dynamic self-financing trading strategy involving the underlying and the money account, a certain contingent payoff would be manufactured. That is all that is shown. Any other variation over what the formalism shows will involve postulating the existence of the price of the contingent *claim* $V(S, t)$ and that it depends only on the underlying price S and time, when nothing grants what variables it should depend on beforehand, especially if its postulated existence is equal to the postulated existence of a market.

To repeat, it is perfectly acceptable to postulate a stochastic process for the underlying (which sums up all kinds of trading activity pertaining to it) and to explore the consequences of dynamically trading it; however, in such a picture and such a derivation, in such an *orientation of thought*, mention shouldn't be made of the independent contingent claim, let alone of its price. We aren't investigating whether the formalism or formula is right or wrong. We are just examining what it says. The underlying process is given (after the trading of the underlying and its market); couldn't that correspond to a world where there subsequently exists a market for the contingent claim? The answer is no, for as soon as a market for the latter is even conceived, with the implicit notion of variation of its price, I contend that the process of the underlying can no longer be given.

There is incompatibility of discourse and of orientations of thought between the stochastic process being given (precisely as the definitive summary of a trading activity) and the price of the contingent being even considered as a price. Where is its space of variation? What does it mean? When probability takes over and the summary of the underlying market is conducted from the outside, the market can no longer be opened. Both the underlying and the derivative are now locked outside the market (which becomes equal to a mere random generator) and all we are trying to do is apply non-arbitrage. This is no market.

13.5.1 That they don't exist

The market of contingent claims is not the consequence of the mathematical model. It is not connected to it by that bridge. *It is the consequence of the non-existence and even unutterable name of the contingent claim from within the formalism.* All the questions trying to connect volatility as given on paper and the market price of the contingent claim vanish ('Should the price converge? Or should we infer from the price the market's forecast of volatility?') once we recognize that the market of contingent claims doesn't exist, going in that direction. Contingent claims are not even redundant in the formalism; they simply do not exist. It is a mistake to think they are redundant;

for we assume they are written and independently available for trading, as this is what writing them implies, and then we say, 'Oh they are redundant'. *They are not redundant and this is why they admit of a market.* It is crucial to see in what way they precisely are *not* redundant in the formalism – which is that they don't exist – in order to understand how the formalism engenders their market. They don't exist in the formalism and, for this reason, they are not redundant. We think, first, they are redundant (for everything in the formalism seems to lend us such a thought) and, then, we realize they're not (because we realize they don't exist) and, for this reason, they admit of a market. Their non-existence is not the cause of the non-existence of their market; quite the contrary, it is the cause of their non-redundancy, hence of their market. The market and its way of existence (or genesis by the formalism) are of such nature that non-existence and non-redundancy for the reason of non-existence (when we would have believed that non-existence couldn't be the cause or the reason of anything) can add up and produce the market.

The market of contingent claims completes the picture of the reality of trading of the underlying (the exit in place instead of time). Once again, the reality of the market (of contingent claims) is of a peculiar nature, because of the peculiarity of its deduction. Redundancy is such a negation of the market that it needs something stronger than negation, which is non-existence, in order to be dismissed and yield the market in a weird kind of way. I think this is related to the necessity of reading the formalism in just one way. The market isn't an object with many faces; it has only one voice, one arrow, one strike, one face. It is not a reciprocal and all-round being; it is paper and it is writing. For this reason, its existence cannot be deduced except as a surprise, immanently not deductively, as something that remains, not something that is. It is writing, which is older than being; for this reason, redundancy is a worse enemy for it than non-existence. The reason why the market of contingent claims exists is that the right formula or formalism is not the one that says that they are redundant, but the one that says nothing (or says they don't exist).

It is not an ambiguity of terms; it is an ambiguity between being and non-being. Of course, the market of contingent claims exists externally and independently; however, the reality of the non-deduction, or the reality that is deduced as non-being, is operative in creating a link between underlying and derivative (in instating this whole engineering field) that shouldn't have existed. We are arguing for non-existing things, one way or the other – the non-existence of the model, which is a better reality than being true. For this reason, I am impatient to write this book, as if some definitive truths, or their contraries (some non-truths, non-consequences and, even, interdictions, blockades), were urging me to get them out.

There is urgency in the deduction of the matter of the market of the contingent claims, and the urgency is *precisely* that there is no matter to deduce and that the formalism is completely closed to that perspective. There is finesse and a subtlety in the reasoning – quite unusual, yet crucial and typical of this deduction, which seems to me all the more pressing. There is an argument from writing. There is no prior reality of the market, nothing there to begin with. For this reason, it is very important to follow through what the written formula and formalism say *exactly*. This is a new mode of investigation, which is oriented towards the interior of the formalism, and then towards its expression, because of the new nature of the reality that is produced.

The Hard Problem

Probability theory and the name of the market have been improperly mixed at the beginning. We relied only on our familiarity with their terms. The market is not just the market; it is essential for thought. You may think it is not as fundamental as a physical theory; in fact, it is as fundamental as a *metaphysical* theory.

Is it essential to ever conceive of a *continuous-time* stochastic process? At first, randomness was thought to be discrete. Everybody was convinced continuous-time processes could only produce determinism and no randomness.¹ Bachelier is historically the first to have conceived of a continuous-time random walk.² He invoked the infinity of causes. When there is an infinity of disorderly influences, he argued, each with a different response time, randomness becomes continuous in the limit.³ To conceive of something random, we first think that its cause has to act identifiably randomly; that is to say, discernibly and discretely. Time has to stage randomness; it has to prepare the stage, first, in which the infinity of causes eventually leads to randomness in continuous time. For this reason, randomness is first thought to be discrete.

Is it an accident that the first continuous-time stochastic process ever to be conceived should have been so in relation with the market? The infinity of causes in the market is not incidental; it is tantamount to the infinity and continuity of *matter*; it is connected with the matter of the event. An infinity of causes concentrates on the trading spot and produces the continuous-time stochastic process known as Brownian motion, according to Bachelier. But why doesn't the volatility of Brownian motion, or any higher-up coefficient of the later complications of Brownian motion, itself become random as a consequence of the infinity of causes?

It is essential to trade continuously and to conquer time from that direction, before any metricization or quantification of time. This should even qualify as a category of thought. What is essential in Brownian motion is not time or the moment when quantification takes over in order to establish the diffusion coefficient as a number. What is essential is not the discernible series of outcomes. What is essential is that the stochastic integral should precede the stochastic differential and that the ex-post attitude should be made available on the spot.

14.1 The ultimate probability spot

Brownian motion is the continuous-time limit of the binomial tree, and the convergence is derived probabilistically. Both the binomial tree and Brownian motion are characteristic of complete markets or of the fact, which we have now recognized to occur before probability, that the trading of the underlying is equivalent to manufacturing contingent payoffs. We didn't need probability when we argued that the price could move up or down (the binomial) and that we could trade variable size at any price, and this was found to be equivalent to the writing of contingent payoffs, still without probability. However, we now discover that probability is needed in order to make sense of the convergence of the binomial tree to Brownian motion.

The argument from trading the underlying to manufacturing the contingent payoffs is impeccable and proceeds without probability, until we reach the limit of infinitesimal time and we find that probability is needed in order to make sense of the limit. It is precisely at the limit in time that the writing of contingent claims materializes and becomes equal to the intensive trading of the underlying (the one that dispenses with extensive time). We invent the writing of contingent claims at the infinitesimal limit (about to become the market), and it is at the same infinitesimal limit that Brownian motion obtains and we seem to require probability to express it as the limit of the discrete random walk. But do we really? Have we not already argued (in Chapter 5) that probability, when it is to be made sense of in the case of Brownian motion, in an ex-post accounting equation that is similar to the insurance case, requires the price of the contingent claim to be given, against which alone the break-even can both be instantaneous and in the infinite long run? In other words, hasn't probability already been taken over by price?

The argument from trading the underlying without probability and from understanding the contingent payoffs as just artifacts of the trading did not necessitate any notion of break-even in the long run. To the contrary, it discovered in the underlying market both the device of randomness and the price with which to purchase the corresponding lottery, as well as the further capacity to synthesize more complex lotteries, known as the 'contingent payoffs', with the variable trading size. This seemed to dispense with the insurance company paradigm and even suggested the market was a wholly different register of randomness (the one that precedes probability and just happens to coincide with it, once time is switched on). So, how is it that we are meeting again with the insurance company paradigm and requiring the break-even argument in the limit?

The answer is that the two limits in time are, in fact, the same. The writing of contingent payoffs materializes and becomes the market of contingent claims right at the limit where this market is independently required to make sense of Brownian motion. Brownian motion should be understood outside time and the frequency count, in the limit which gives matter and which places the ex-post stance first.

It is not a coincidence that the first historic case of Brownian motion (Bachelier) should be conceived in the field of finance and trading. While the motion of the pollen particle is materially random at any scale that we can observe it and has no observable speed because of the material impact against it of countless atoms of the liquid it is bathing in, the motion of price is formally random at any mathematical scale. The price of an asset can be formally imagined – it is just a

number – unlike the material pollen particle. As well, it can be formally conceived that its motion must be random at any scale because it is absolutely formally that we argue that if this were not the case and if there existed a scale, no matter how finely we have to dig in order to find it, in which the price were predictable, then this time interval would really have to be zero because that predictable price would have to be traded as of now by the very auction process governing the exchange.

Price is perfect for Brownian motion (it is almost as formal, as we said), for it is the meaning of price (the meaning of the exchange) and not just its material behavior or trading in finite ticks or finite time intervals that is here at play.⁴ Bachelier may have invoked the infinity of causes in order to stage before our mind the material cause or reason why the motion of price had to be random at any scale; however, an *infinity* of material causes becomes formal really. Instead of arguing that there is no end, no limit and no characteristic threshold to the buying or selling phenomena that may cause the price to move but that buyers and sellers can, on the contrary, react to any price difference, no matter how small, and react in a correspondingly small delay, Bachelier could have made this infinity of causes equal to the formal existence of the market. He could have reverted the argument and argued that for there to be a market and a price motion at all (for the market not to collapse in a single spot and die) this motion had to be unpredictable at any scale.

Price is as formal as Brownian motion, we said, because of the availability of the continuous-time limit and of the infinity of causes to the concept of price (as opposed to the discrete atoms hitting the pollen particle in large yet not infinite numbers); however, price is not strictly identical with the formal-mathematical Brownian motion. Price is a number, but it is not just a number. There is something material attaching to price and that is the paper that is exchanged for that price. Better to say that price is an ideal model of the formal-mathematical Brownian motion, yet, as a model and interpretation of the formalism, there is something material attaching to it.

There is formally no problem in attaching a quantitatively fixed diffusion coefficient to mathematical Brownian motion. This is an un-interpreted formalism after all. However, with the interpretation of Brownian motion as price motion and with the meaning of price as emanating from the exchange, there enters an intensional element. From the deep argument that an exchanged price should be unpredictable at any scale, we can only adopt Brownian motion through a *probabilistic exit* – fixing the diffusion coefficient, for instance, or even conceiving it, crystallizing it in thought extensionally. Yet, the argument from the exchange is so deep that such a probabilistic exit is not granted. Are we considering observing the price statistically as a temporal series in order to conceive quantitatively of its volatility?

Once price is no longer thought as being just a mathematical number and once it is remembered that it ideally yet *materially* interprets Brownian motion, we seem unable to conceive a quantitative diffusion coefficient for it unless we exit in time and consider, if instantly, that a certain statistical time series is thinkable. Let us not forget that it is a coincidence that price should translate extensionally and extrinsically (Shafer and Vovk speak of *extrinsic stochastic modeling*) as Brownian motion; it is a coincidence (unavoidable but unessential) that time should represent the motion of price and that we should have no other choice in time but to consider the quantitative diffusion coefficient. However, the intensive property that seems to us more essential to Brownian motion,

especially when it is intensively materially interpreted as exchange, is the stochastic integral. The stochastic integral is really what best materializes the *form* of price, or the fact that price is an ideal yet material interpretation of Brownian motion, because it integrates the variable size of the trade against the price change.

The stochastic integral dispenses with looking at the next variation of the price – as a matter of fact, there is none, because the stochastic differential is just a symbolic notation. When matter comes into the formal picture, when the exchange is materially considered as a model of Brownian motion, its matter cannot develop in time (as it certainly must, for the market has a tendency to expand in time and to make sense in time); it develops as the stochastic integral that manages to shrink an ex-post accounting equation – that is to say, matter – back into the instant. The quantitative diffusion coefficient, which relates to time, has no place in the *material* interpretation of Brownian motion, but the price of the derivative, which is available and against which the ex-post accounting equation is integrated, has one. The exchange, when it is intensively conceived – that is to say, when it is meant as a material model for Brownian motion – cannot take place in time and concern only the underlying; it has to materialize into the price of the derivative and the stochastic integral which keeps track of the P&L of the hedged portfolio. In material time, what counts is the switching of ex-ante and ex-post – not the quantitative measure of the diffusion coefficient, but the reality of the derivative price against which to run the stochastic integral or, in other words, the reality of the market.

Shafer and Vovk do not want to worry about states of the world and extrinsic stochastic modeling because they have identified in money the alternative medium in which capital gains are expressed and whose accumulation ultimately counts in order to break-even or to blow up to infinity. They are right in saying that since the accumulation and the ex-post accounting are the only things that count (the only matter), we should no longer trouble ourselves with looking into the fine workings of the transitions and conditional probabilities. There is no fine measure there and nothing to measure; all this can remain hidden behind the player (Reality) who will do whatever it takes and will always move in such a way (generate a move) as to win.

Shafer and Vovk may have recognized in money the more liquid and malleable medium in which to account for randomness; however, they did not recognize in the financial market its specific monetary nature. To their eyes, it is just a coincidence that the market should generate numbers that are prices and are expressed in money for, to their eyes, the market is a statistical phenomenon like any other. In their game, the price at the opening of the day is the forecast, and the price of the same asset at the end of the day is reality's move. Yet, in this coincidence between the price at which the lottery is bought and the actual final move triggering the lottery which is the movement of price itself, we didn't just see a coincidence; we saw a marvel – the marvel of the market.

Recognizing the monetary nature of the market is essential to our eyes, and the reason why states of the world do not matter in our account is, ultimately, that the exchange comes before probability. The intensity of the exchange withholds the probabilistic exit, as we said, and materializes the writing of the contingent payoffs as an alternative exit into reality, corresponding to the market of contingent claims (the contingent claims trading independently). By writing the contingent claims, we also suppress the states of the world.

Shafer and Vovk do not access the *reality* of the market of contingent claims that is alternative to statistics – the reality that is prompted in our analysis by the intensity of the exchange of the underlying, once the writing of the contingent payoffs appears as the alternative to withdrawing the time process, or the reality that is prompted by the necessity to find the price of the contingent claim against which the ex-post accounting equation can take place on the spot with Brownian motion. For this reason, when they move to the infinitesimal limit and, accordingly, lose the discrete nature of their game and of the moves of the game, they have no choice but to resort to non-standard analysis.⁵

We, too, have a problem at the infinitesimal limit, because the necessity to interpret the binomial (up and down) intensively makes us meet with probability again and with probabilistic convergence to Brownian motion, when probability in the sense of the ex-post break-even was to our mind already dismissed by the marvel following which the price of the traded asset is both the lottery and the price to purchase it. We might have dispensed with probability computationally, with regard to price being both the lottery and the value of that lottery; however, it seems that we had not dispensed with probability ontologically, as the background against which the price process is registered. Probability was still there. As a matter of fact, the premium to invest in the trading strategy whose outcome is the given contingent payoff depends on volatility, and volatility attests to ontological probability. Volatility is not eliminable here, as it is precisely preserved in the change of measure.

It is only, precisely, at the infinitesimal time limit that a quantitative diffusion coefficient is no longer required and probability is supposed to be taken over by the other door. It is also precisely at the infinitesimal limit that ontological probability is supposed to provide the background for the convergence of the binomial to Brownian motion. But this is taken over by the other door as well, given that the materialization of the contingent claim, now become price in response to the disappearance of quantitative volatility and the corresponding probability, is precisely what warrants the ex-post break-even argument that accounts for probability in Brownian motion.

The idea of up and down and succession in the qualitative argument was a temporary image. We said it was qualitative and intensive but we didn't say how that could be conceivable when succession took place precisely in time and when time was precisely the domain of probability. The up and down movement was meant to be understood qualitatively (i.e. without a quantitative volatility measuring the deviation between up and down) only in the infinitesimal limit. It is precisely in that limit that (a) Brownian motion is reached (yet without a quantitative diffusion coefficient), (b) the price of the contingent claim is materialized, and (c) the materiality of that price closes the infinitesimal break-even ex-post equation. If non-standard analysis is, ultimately, crucial in the passage to continuous randomness (as witness Shafer and Vovk), then the market is what's non-standard in our analysis.

14.2 The presentation of the contingent payoffs

In orthodox presentations like Rebonato's, the statement is made very clearly that states of the world (ω_1, ω_2) are given as the absolute background, on which prices of the underlying (S_1, S_2)

are then superposed.⁶ The market is, from the start, envisaged as derivative on probability and on its background assumption of identified states of the world. The distinction might not be apparent in the mathematics (or even the formalism), for we speak of a filtration and of the price process S being adapted to it, but we now believe that the distinction is ontologically crucial.

If the background states of the world are given first, then we are, from the beginning, located outside the register of trading. Lotteries can be evaluated and the fundamental principle of arbitrage be put to use in this framework (as a matter of fact, this is its designated framework); however, this is not trading. Trading should not be reducible to a stochastic process. Trading is more direct and is simpler than the prior identification of states of the world. Randomness is a consequence of trading, so states of the world cannot serve as the background of trading. It is only when prices are confused with numbers and we look at the numerical *consequence* of trading – the random generation of numbers – that we postulate states of the world at its start.

The traded price is ontologically (essentially) random and not temporally random. By this, we mean that it is inherent in the *meaning* of trading that the price should be random. If the price motion were not random, there would be no market, which means that randomness is a necessary consequence of the market (it comes second in logical order). We do not need the time series or the collection of samples to understand that *price* means that the price is random. Likewise, the idea of a random *generator* is supposed to capture the idea or the potential of randomness before time is actually unfolded. We know what volatility and the jump sizes and the frequency of jumps mean before the jump-diffusion process is actually temporally sampled. We know what a probability distribution means before it is sampled. But this essential vision of the random generator (essential in the sense that time is withheld) is not possible without the quantitative translation. It is not possible to speak of the generation of random numbers without having in mind the notion of quantitative volatility, or frequency of jumps, or the notion of a quantitative probability distribution (the *notion* of quantity if not necessarily the quantity itself).

We may not know volatility or the frequency of jumps, or maybe the probability distribution is not yet specified, or maybe it is changing all the time, or worse, maybe indeterminism is essential in the sense of quantum mechanics in which incompatible families of states of the world are considered; still, in the *idea* of a random generator and the generation of numbers, quantity is already imagined and it precedes the idea of randomness. Things are different with price. We observe that price is random (we observe the consequence); we let price exit in the probability representation; we think of the probability of certain future prices obtaining and we consider actuarial break-even in the long run; we imagine valuation as expectation; then, we remark that nothing of this is really needed and that the present price is already the present value of the future price. We think this is a coincidence because of the detour into probability, what I have called the 'probabilistic exit'.

Maybe probability is externally useful in matters relating to price. Maybe we are interested in the notion of the probability of certain future prices being such and such (and why should we be so interested? Are we already contemplating the valuation of derivatives? If so, why not suppose derivatives do not yet exist?). However, let us, for the sake of the present argument, suppose that the reason probability might be needed is *only* to compute the expectation of future prices of the

underlying, in order to compute its present value. Let us think that the valuation of the asset, whose price we are observing in the market, is the only problem at hand and probability is only needed (probability as it was invented in the domain of lotteries and their valuation) in order to compute a present value. Of course, we realize what an unnecessary detour probability – even its idea – would be in a situation like that.

It is no coincidence that price should be random as a consequence of trading and that its randomness should subsequently suggest the use of probability in order to compute the expectation of future prices; and it is no coincidence if we realize that the present value that we have thus computed had better coincide with the present price. The whole thing is not a coincidence because the separate elements that we bring together in order to report a coincidence shouldn't exist in the first place. We shouldn't have even thought of probability in the first place. The price is random because it is a price, and because it is a price, it is its own present value. The market is the self-contained theory of its own randomness and of its own valuation. So, let us *really* think, in the specific case of the market, that probability is parasitic and is due to an illegitimate analogy with lotteries. Let price be a simple basic category and let its simple (i.e. non-composite) meaning be both that it is random *and* that the present value that we would be tempted to compute (using probability), for no other reason than that price is random, is none other than price itself.

It was already suggested that expectation comes before probability and that the whole probability theory can be recounted with expectation as the basic concept.⁷ When we are contemplating several probabilistic outcomes, we may be interested in the individual probabilities; however, the first movement of thought is to think of the average outcome, of the individual that isn't one of the particular individuals but that probabilistically stands for them all. Now, we are saying that, in matters relating to price, price itself is more fundamental than expectation (or present value) and comes before it, not only because present value had better be equal to the present price when price is available (otherwise there will be arbitrage), but because this is no coincidence and we should have never thought of present value, first, only to realize, second, that, 'Oh, price is already present'. It is no coincidence because price is not random for reasons that are external to price. It is random because it is the result of an exchange and, as such, it is handed over in the present as the best 'expectation' of the future outcomes.

We should try and make the concept of price basic and undivided. There aren't two separate parts in price, one part which says that price is random, as a consequence of which probability is invoked in order to compute present values (because present value is all that really matters in matters relating to randomness), and a second part which says that, by a fortunate coincidence, the present price is the present value. Rather, the randomness (which first suggests an unnecessary detour into probability before we realize that price is the present value) and the price are two faces of the same phenomenon, which is the exchange. Price is not a degenerate case of probability, whose symptom of degeneracy is the coincidence. For all we know, probability could be an extrapolation around price, as it is no coincidence that money should operate in matters relating to probability (Shafer and Vovk, Ville).

What other purpose than to compute present values of derivatives could probability have in a world that is (for now) limited to the trading of a certain underlying asset? Probability would be

of no use if our only purpose were to compute the present value of the lottery bearing the future prices of the underlying as outcomes because this present value is readily equal to the present price; however, what of lotteries yielding other financial outcomes than the underlying price, yet triggered by it? We have already remarked that the market is financial and that the money against which the underlying is bought and sold is the same that is eventually handed over as payoffs of the lotteries. It is all part of the same financial world to consider financial lotteries triggered by the underlying – what is otherwise known as ‘financial derivatives.’ So, at this stage probability seems to be required in order to look, through the different scenarios of evolution of the future underlying price, beyond the simple average which says that the present value is the present price. Probability seems to offer us discrimination in an otherwise compact picture.

But, we ask again: How is the discrimination even brought about? In a compact world where all that exists is the traded underlying, the problem is closed and complete. There is the randomness of the future price and there is the present price, and they complete one another in the thought of the market – the first being random because of the market and the second being the value of the first one because of the market. In such a world, the concept of external lotteries has no place. It is a big step outside to even envisage independently written lotteries. Have we stepped outside our compact world to write them and, if so, what is their bearing on our world? How could we mix the thought of valuation of the external paper with the thought of the underlying that is compactly trading in its own world? Or are those lotteries destined to trade as well – but then, have we independently postulated their trading process?

As we look for some differentiation or discrimination in the compact world of the traded underlying, probability then seems definitely to be of no use, and of no mention. Any discrimination in the compact phenomenon of trading of the underlying should, rather, come from the variable traded size. As we said, the complete definition of the market does not stop at the randomness of the price and its completed valuation through price itself. What is even more characteristic of the market is that the size of the holding can be varied, and even self-financing strategies executed. In the self-financing strategy, we acknowledge both that the price can appreciate or depreciate and that the proceeds can be employed to alter the size of the holdings. The self-financing strategy brings money and the materiality of the trading account on top of the mere randomness of the number known as ‘price.’ In the self-financing strategy, there is the true recognition of the market as a material medium that surpasses the random generation of numbers.

It turns out that, with this added discrimination concerning the market – namely, that quantities can be traded in variable size and that time (which is money) can be tracked through a self-financing portfolio (in other words the addition of the material paper and its size and the addition of the medium of money and finance) – contingent payoffs of all kinds can be manufactured. There are infinitely many self-financing dynamic trading strategies to which corresponds an infinite variety of contingent payoffs, even strongly path-dependent ones. The martingale representation theorem provides that any martingale, or any price – or in other words, any risk-neutral probability of any path of the underlying state variable (which is here the underlying price alone) – can be attained by the proceeds of a self-financing trading strategy.

Of course, this is only true of Brownian motion. However, our qualitative argument, in which up and down are only qualities, not states, and time is still withheld, considers Brownian motion as the blueprint. The discrimination or the differentiation that probability was supposed to operate in the compact phenomenon of trading of the underlying is thus relayed by the discrimination of the infinitely varied contingent payoffs that can be manufactured by the infinitely many self-financing trading strategies. The essential presentation of the market of the underlying (essential, in the sense that time is withheld and the intensive meaning of trading is only considered) is, thus, completed by the simultaneous presentation of all kinds of contingent payoffs. If asked what the instantaneous (essential) picture of the market of the underlying could be and what could instantaneously essentially capture not only the fact that price both means randomness and present value (because of randomness) of that future price as being equal to the present price, but also the potentially infinite variation of the size of the trade in the specifically financial way which is the self-financing way, then we should answer with the instantaneous presentation of the contingent payoffs.

Their *presentation*, in that essential moment, is their writing. Writing is not yet equivalent to exchanging and to pricing, but is only about to be. Writing the contingent payoffs is not independent of the market of the underlying – it is not a parallel market – *but for the first time is here to better present it*. It is a much more complete picture of the market to present the contingent payoffs, written in the sense that they will be replicated and manufactured and attained, but not yet in the sense of their parallel and independent trading, than it is to present the underlying price with a temporal stochastic process. Time is just a coincidence with regard to price. Better to think that it is a coincidence that the trading process resembles a stochastic (temporal) process than to think that it is coincidence that the present price is the present value of the lottery whose future random outcome is the future price.

It is only because of the coincidence of time and of the corresponding temptation of probability (the discrimination of the compact phenomenon of trading into time and statistical time series) that someone like Rebonato postulates the states of the world (ω_1, ω_2) as the absolute background and makes operative the fundamental principle of arbitrage, relative to which underlying and derivative are equally lotteries and equally derivative on the background states. This results in the martingale measure formalism and the multiplicity of such martingale measures when the market is incomplete. Because the world has been initially reduced to the trading of the underlying and the trading of its derivatives (i.e. potentially their market), because all that matters, ultimately, is the process of the underlying, the market cannot be made incomplete except by complicating the underlying process, by writing further conditions pertaining to its statistics. Unsurprisingly, to every such complication there corresponds a contingent payoff (variance swap for stochastic volatility, gap options for jumps, and so on), and the unsurprising conclusion is that, to every degree of incompleteness, there potentially corresponds a contingent payoff whose trading shuts it down.

We play this game of false incompleteness, as if we were blind to the fact that it is the derivatives market that we have in mind from the start, so what is to stop the derivatives from completing any incomplete market? We are told the market selects the martingale measure, and this is how the valuation process of derivatives becomes a trading process. But how could such an upheaval take

place in the market and the fragility of the martingale measure and its selection be preserved? If the market takes over, it should really take over.

14.3 The lure of theory

We act as if it were a coincidence that there should exist a market and that our problem should admit of a solution before we have even posed it. We make the market of options address a problem that we think we have posed, first, and of which we expect the market, subsequently, to provide the answer, when the market has always been the answer and has always been posed before we even posed the problem and when it is no coincidence that we should find the market subsequently, or realize that it has always been present, because it really was always present. The market is quite handy (more than a coincidence, a true miracle) in selecting the martingale measure, or in making us believe that there is an instantaneous (not statistical) number for volatility, known as ‘implied volatility’. There is something in the order of the thought and of the formalism that bestows on the market the quality of an answer, when the market doesn’t require that order of thought.

From the start, there shouldn’t have been a derivative valuation theory (BSM or, more generally, risk-neutral valuation in incomplete markets) to which a market pricing of derivatives would *then* be compared. From the start, there should have always been the market and only the market. That there should only be the market is the only way to make the market absolutely incomplete – incomplete by the writing and trading of contingent claims, not by the complication of the temporal process of the underlying. The temporal process and the corresponding order of thought open the false question of how the market is coming up with the derivatives prices *given the statistics*. (The false question is: ‘Will the prices of derivatives converge to their fair values?’)

As if from the market prices of derivatives, considered as empirical reality, something could be deduced for the theory! (The theory is couched in the terms, and framed in the image of thought, in which an empirical reality is supposed to confirm it or falsify it. This is the reason why we go and check what this empirical reality known as the market has ‘selected’ as martingale measure, as if the different martingale measures were avatars of the market and the only reality check was to go and see which avatar reality has selected empirically; that is to say, passively, when the real reality is the force of trading.) Or, as if the theory could be independently conceived and then find in the market its application or its empirical reality! In the immanent theory of the market, there isn’t even the dimension in which a theory, first, articulates the formalism and, second, finds the matter; first, poses the question and, second, finds the answer. We seem to start with the market of the underlying as the level that poses the problem (that produces volatility) and to end with the market of derivatives as the level that finds the answer. There might exist a statistical theory of the underlying, or a time dimension in which the statistics are observed, but this axis and orientation of thought cannot meet with the derivatives market.

We know how the argument holds on paper. Given Brownian motion with diffusion coefficient σ which the underlying price is supposed to be following, a certain rule based on the partial derivative with respect to the underlying price of a certain martingale (in fact, the expectation under

the risk-neutral probability measure of the final contingent payoff we are trying to manufacture) guarantees that the final payoff is, in fact, manufactured. Note that the martingale is not the *price* of the (independently traded) contingent claim. There is no *contingent claim* and no price for it as far as we are concerned. This is only an intermediary step in the computation which we need in order to produce the dynamic replication rule. All that matters is that the final *contingent payoff* shall be attained. This is how the ambiguity relative to whether we are presupposing the price of the contingent claim before we actually have it gets resolved.

It is true that time seems to pass in the formal argument and that it gives us the impression that someone is aware of the volatility holding and remaining constant (for instance) during that interval of time. In reality, volatility is not physically or psychologically constant; it is formally constant. Had it been formally dependent on time, it wouldn't be any less independent of physical time or psychological time.

How is the bridge between the formalism and reality established anyway? Brownian motion with constant volatility is *formally* given and only formally given. Accordingly, we contend that the objection that Brownian motion isn't the case in reality, or is the case but doesn't admit of constant volatility, is not a valid objection.

Before we try to imagine how BSM could be false, we should try to imagine how it could be true. For the formalism to be true, we should be able to imagine the physical model corresponding to it, together with physical time and the right order of question and answer. With physical time and the corresponding order of question and answer, comes the question of the holder of the knowledge: 'What holder holds what knowledge, and what holder holds what other holder?' The casino holds the knowledge of the statistics of the roulette wheel and, for this reason, it invites players in. But it doesn't invite other casinos. The price of the lottery – the odds – is not negotiable. The casino is the ultimate container, or holder. Now, take the underlying market. Who is holding the knowledge of its statistics? Who is outside? I am not debating how the hypothetical holder of the knowledge has access to it – say, access to the value of volatility. I am not posing an epistemological problem. It seems to me, rather, that there is a problem in the registers, or levels of discourse. The market is not a device, immersed in a larger container or reality (like the roulette wheel is immersed in the casino). When we write the stochastic process of the underlying and materialize the corresponding register or level of discourse, of course the formalism is unproblematic, but from that moment the interpretation, or the translation of the formalism into the register of discourse, poses the question of the container. Because we are supposed to describe all the market (this is the sense of the formalism), the hypothetical holder of the knowledge of the statistics can only be immersed in the market. He cannot independently offer to value a contingent claim. He can only trade and manufacture contingent payoffs, from the inside. There cannot be two registers, one in which the underlying is said to admit of a stochastic process and one in which the stochastic process of the contingent claim is to be constructed or found. If it is only ever considered, it has to be written as an independent stochastic process because, in this register, a stochastic process means a trading process and not a valuation process.

To be equipped with the knowledge of volatility – not epistemologically, but semantically, in what the formalism means; that is, in the sense of the register – is only to be able to perform the dynamic

trading strategy that manufactures the contingent payoff. In this register, an independent price for the contingent claim may not be observed. When we accomplish the probabilistic exit and believe we are dealing with a random generator (of underlying prices) of which the register of knowledge of its volatility may now be opened, we are welcome to propose lotteries triggered by that stochastic process. However, we will have left the register of the market (which occupies the whole place); we will not be able to go back and immerse the lotteries in the market. Either they have their own trading processes written out (because we now hold the paper they are marked on), independent stochastic processes that will ultimately dismantle the states of the world of the underlying, or we are back in the first market in which the underlying trades alone and we may only manufacture contingent payoffs.

What is the link between the formalism and reality, between the BSM formula and the option price? The link is not what we believe; namely, that the formalism is an idealization of reality. The difficulty comes from the expansion of the underlying statistics in time contrasted with the instantaneous character of the option price. What does time mean? A notion of time is implicit in the statistics being given in time and in the option price being given instantaneously. Does time mean that we are waiting and observing only ex-post whether the formalism works and whether the option has been perfectly hedged or replicated? Or are we projecting in time the following image: somebody assumes volatility is constant in time until the maturity?

We buy and sell the underlying for many reasons that we keep hidden under the overall stochastic process. Why can't we do the same with the derivative? Why can't time be the same for the underlying and the derivative? We may be buying the derivative and selling it because we anticipate a rise in volatility; however, why can't we hide that under an external stochastic process of its own? We can't do so because the underlying is here and its statistics are 'visible' in time and not hidden – until, that is, the existence of a derivative price and the time in which it is given are recognized to be incompatible, as we said, with *giving* the statistics of the underlying.

In order that we may apply the formalism, how do we conceptualize the fact that somebody knows volatility in reality and in time? We are missing instantaneous volatility and we are lucky the market provides the option price. So, we unconsciously dismiss the previous question. I am led to believe that the whole thing has nothing to do with epistemology or with the time corresponding to knowledge. The option price is a new strange way of applying the formalism, that's all. (The only interpretation of the formalism is implied volatility, as we said elsewhere.) The strange and marvelous thing is the *instantaneously given* option price and its hypothetical relation to statistics which, by contrast, seems to presuppose an expanse of time. Breaking even on every path introduced the variety of prices of derivatives – as such, instantaneously given – or the *locality* of the market, which is unknown to actuaries. Consequently, the question is no longer whether volatility is constant or not in reality and in the corresponding time. The way in which volatility is given on paper and applied to the pricing is completely different from the way of reality and from the register of the latter question.

We switch from the timelessness of the formalism to the instantaneous character of the option market – to giving the option price instantaneously. We think it is a coincidence and, even, a blessing. In fact, there never was a space of physical reality in which the formalism could be said to be an

idealization of reality. This is because of the discrepancy in time. We open the time for the statistics (in which supposedly to argue that volatility is true or not), but this not the time of the option market. There is a direct link between the time of the option market and the timelessness of the formalism, without the time of statistics intervening in between. I've always had difficulty with the epistemological/ontological sustainability of the physical reality supposed to lie in between. The time corresponding to it is different. In our register, you just cannot know the statistics and there is no time corresponding to that knowledge. There is no ultimate container – unlike the case of the dice or the roulette wheel. The formalism presses us to step into reality in order to interpret the formalism physically as statistics and, consequently, to expand the time of its knowledge, but the reality is that the option market has already, *instantly and locally*, taken over that step.

The conditions under which the stochastic process of the underlying is given are different from those under which the price of the contingent claim is given. This is not the same reality in which we trade the underlying and summarize its trading with a stochastic process, and in which a market of contingent claims is given. Anything can be given on paper, so long as self-contained formalisms are offered. However, in matters (yes, over and above forms) concerning statistics and the market, the reality outside the formalism becomes perspectival and two realities, or two perspectives, may be completely incompatible. We cannot be in the register of observing the statistics and summarizing them (giving the underlying stochastic process) and be, on the other hand, in the register of the market of contingent claims. Reciprocally, if we have a market for contingent claims and we are in that register, could the underlying be anything else than *trading in a market*; in other words, the farthest that it can be from the register of statistics and the stochastic process?

To be trading in a market, for the underlying, is intensive. It is to be compressed in that intensive pit where we have summoned the qualitative argument and where the force of trading prevails, as opposed to the probabilistic exit. It is that essential moment that prompted the writing of contingent payoffs (we said) and their potential trading as contingent claims. If the underlying was following a stochastic process with a written random generator (I no longer say: a known random generator, or an existing random generator, because neither epistemology nor ontology is the issue – the only issue is what's gets written, formalized, and what this means), if the underlying was really a device like roulette or dice or any chance set-up, then the value of lotteries written on it would not need a market. Lotteries written on an underlying whose stochastic process is formally written (i.e. is not trading in a market, following our distinction) do not require a market to be valued.

We would like to establish connections (even though we don't know yet the logic in which they can be enunciated) such as: it is because the underlying is *really* trading (in the sense of force and the making of randomness from the heart, not in the sense of recounting it from outside) that there *really* is a market of contingent claims. Note that if the contingent claims are really trading, in the sense that they are bought and sold for reasons that are hidden under their trading process, this cannot coexist with the writing of the underlying stochastic process for, then, the statistical parameters would no longer be hidden. Indeed, they would be written.

The way in which the stochastic process of the underlying is given (this summary, this external view, this exit from the market) is incompatible with the way in which the price of a derivative written on

that underlying is given. This is the incompatibility of registers that we want to expose. It is not that the truth of the derivative price will criticize the truth of the underlying process. The bridge which we walk, going one way, to see that the derivative price is this when the underlying process is that, is not the same bridge which we could walk the other way to argue that the underlying process is no longer that when the derivative price is no longer this. These are two incompatible registers. It is not, either, that the derivative price will converge to the truth prescribed by the underlying process.

Congenital to every formalism is what is given by its sentence – what I called its ‘register’. In the formalism of the stochastic process of the underlying, no option price is given. If, conversely, the option price must be given in the formalism of a market, this must be one market and one trading, and the underlying must be trading as well with equal force, before the probabilistic representation. Even if it is understood timelessly, once the underlying stochastic process is given formally, its *material* interpretation implies the expanse of time, because this is statistics. In statistical physical time, we wonder how to understand the option replication algorithm: Is it an ex-post accounting? Is it anticipation? We don’t see how BSM could really be true in the sense that the option price may falsify it. In statistical time, there is only meaning to the replication of the contingent *payoff* and not *claim*. The timelessness of the formalism, and no physical time, seems to have been the real and only reason why the instant givenness of the option price was possible.

To repeat, it is what is formal in the formalism (as such written, detached from physical time or physical reality, detached even from epistemology and ontology) that engenders the market. The option price as given in its market is the consequence of the non-existence of this market in the formalism. It is a different reality, accessible only through a jump. We cannot interpret the formalism in the ‘normal’ physical reality and get an option market. We don’t see how, in physical reality, volatility could be given as constant, could be known to be such, and the option market not know it. We cannot physically imagine the process by which somebody rebalances the delta following the BSM algorithm and his knowledge of given and constant volatility and, as a consequence of which, the options prices align themselves to the prescription and their market disappears. There is no imaginable physical reality in which the underlying process and the options market are both given (with what it means to be ‘given’ in physical reality). That the option price should be given – this instant gift of the options market that is not connectable to statistics via anticipation, or ex-post break-even, or convergence in the long run – is the result of the timelessness of the formalism and the absence of physical reality as an intermediary or as an *interpretation*.

BSM is a formalism whose material interpretation is the instant option price (as given by the market), not the instantaneous volatility of the underlying. As such, epistemology is not the question. It is not because volatility is unknown that options trade in their own market and traders use BSM to trade options. There is no uncertainty of volatility in BSM. On the contrary, BSM has created the certainty of volatility. Equipped with the tool that tells them what the option price should be, given volatility, traders trade options because they have a certainty about volatility. Others may not know volatility, or the options market may reflect a different volatility, but he who trades the option knows what the volatility is. He is not betting on a future evolution of volatility, or that volatility in reality is different from what other market participants think; he is simply trying to establish an arbitrage.

He knows what volatility is for certain (he is not betting on an undetermined outcome) and he is trying to bring the option price in line with this certainty.

It is because BSM has expressed the option pricing problem as follows: 'It suffices to know volatility in order to determine the option price' that options traders think they know volatility because they know the option price (it is their job to know it). If the option was an ordinary asset with no scientifically sounding parameter (such as volatility) that determined its price, then knowledge wouldn't have been articulated; nobody would claim that the price of the asset should be different; it would be what it is and whoever buys the asset or sells it would, in this case, be making a bet that its price will go up or down; the price of the asset would be its sole and own vehicle. The existence of volatility as determinant of the option price gives the illusion of something inherent in reality (the volatility of the underlying) of which the option price is the consequence. There could be no uncertainty about something scientific, inherent in reality.

The BSM theory gives the impression of science and scientific determination. Volatility is not a price; it is an inherent statistical truth; therefore, it is known; the mode of access to it is certainty and cannot be anything else. Volatility is known and certain and *given* in the formalism, after all. Simply, the impression of the formalism carries over to reality. BSM is too good (almost too incredible) for the volatility *not to be known*. It becomes secondary whether volatility is, as a matter of fact, known or unknown. In their approach to trading, everybody knows that they know volatility. There is, in trading, a deeper game than it sounds going on between knowing and not knowing, between matter and thought, especially when statistics and parameters of temporal processes, such as volatility, are at stake. There is really a shortcut between the formalism and the option price. It is the shortcut that bypasses time and knowledge.

14.4 The semantic theory of the market

It is time to write the book of the market, now understood in the sense of a formal theory of the simultaneous trading of derivative and underlying. The market has to be defined in this way, precisely in opposition with any temptation to overlay it over background states of the world. I am not saying the random generator of the underlying doesn't exist; I am saying it shouldn't exist; it shouldn't be written. It is in writing (therefore in prescription) that it doesn't exist. What do we care if it really exists? Go ahead and statistically monitor the underlying if you like, and try to value its lotteries, separately or with no arbitrage with the market. It won't help even if you start valuing them separately and then bring them close to the market. This articulation and temporalization of the movement of thought is suspicious. They are not to be valued on the side *then* regrouped with the market.

We should start with two independent trading processes, one for the underlying and one for the derivative, yet keep in mind that the one is derivative on the other; otherwise there would be no technology, and no technological layering. It is true that derivatives have had their trading space established because of the layer of models, because of sedimentation, so to speak, and all kinds of dead models that have produced a ground. It is true that traders use technology, some technology

– it no longer matter which. It is not the quantitative aspect of the technology that we are after, but only the sediment that it has created, the layering of matter.

We no longer need to go all the way to maturity, and roll back. The market is here in the next minute, and it is all that we need in order to recognize that the derivative is derivative on its underlying, as its price moves with the underlying price, following a certain rule, the market delta, no matter which. That an underlying should be tradable is only the guarantee that it is part of the same market as the derivative, the next day, the next minute.

Why is the market, in the sense in which underlying and derivative trade together, more pressing and even necessary a book to write than a general book of statistics or derivative valuation theory? There seems to be an extreme sophistication of the thought that should be made available to everyone – a new discovery, a new medium and new matter, something having to do with time and the event and altogether more pressing around us than statistics and probability, surrounding us even.

It should be a general need and even a necessity of thought to see what could be done with writing and, therefore, trading. There is an infinity in the derivatives market that cannot be kept away from the public. We think statistics is important; yet, there is a sophistication in thought that statistics is unable to achieve and which, for this reason, makes the market all the more compulsory reading and a necessary book. Everything has to be invented, perhaps even a new language. This book is a meditation on the boundary between formalism and reality. There is writing all the way in the interface: writing of the contingent strike or claim, writing in the middle of Brownian motion, writing of the contingent payoff as a presentation of trading.

Why is any statistics always writable as a discrete hard payoff and tradable in the same market as the underlying? What could the definition of the market be such that it is precisely the medium where the discrete payoffs which approximate theoretical statistical parameters (for instance, the payoff of a variance swap which approximates variance) are put to trading and, for this reason, always require theoretical valuation models?

We cannot be preoccupied by the statistics of the underlying (be in that register of discourse), care whether the underlying process is Brownian motion or not, and whether the so-called ‘market’ is complete or not and, at the same time, open our mind to the notion of the real market, the one in which underlying and derivative will be freely exchanged. Someone looking into the statistics has already left the market to look at something else. Conversely, do you imagine telling an option trader: ‘Wait a minute; this is the process of the underlying! Your market is complete or incomplete’ or ‘By the way, your market is incomplete’? Who has competence to address the trader in this fashion, once the meaning of the trader and the corresponding meaning of the market are opened and made problematic? This incompatibility of discourses seems anecdotic and informal to you when it is staged by a living trader and somebody bursting in the pit to interrupt the trader and tell him statistical news about the underlying – as if it mattered!; but I believe it denotes an incompatibility in the true orders of discourse.

Likewise, it shouldn’t be possible to *form* the sentence which says that, because the underlying process is Brownian motion, the option market doesn’t exist. In the formalism, the underlying

price follows Brownian motion under a certain regime of existence and truth, which is proper to the formalism, and the option name is not even uttered, let alone the existence of its market considered. When we move to the level of interpretation of the formalism, where truth becomes equal to the existence of a physical model, there may be a (physical) truth about the underlying following Brownian motion, and this may be ontological or epistemological, and we may wonder who knows such truth or whether nature alone keeps it as a secret, and so on; however, I don't see how the range of the existential quantifier of the option *market* could open up, how truth could vary between the existence and inexistence of the option market in order to settle for (say) its inexistence.

The 'market' always seems too big a word or a thing to utter, in front of the 'simple' truth of whether something historical and statistical follows Brownian motion or not. Has the existence of the market been tried out then found not to work? Has the option market existed briefly then stopped? How did it only ever open? We cannot hold in physical reality (as we do in the formalism) that the option market is, in fact, never mentioned. The formalism doesn't need to know that it is not mentioning the option or its market even by name. But physical reality does. Once the name is uttered, existence becomes a physical possibility; therefore, there should now be something physical keeping the option market from coming into existence or removing it from existence after it existed.

The reason why the options market doesn't exist in the formalism is that it cannot be mentioned – all that exists is the trading process of the underlying. In physical reality, where the existence of the option market is a possibility, one should not link it with the truth about Brownian motion, for then we would expect some kind of causality: Please somebody show us how the persistence of Brownian motion (its stationary character, its persistence in the minds of the agents) leads to the disappearance of the options market. Better to use the formalism to jump directly into the options market and never consider a statistical interpretation of the underlying process. Never differentiate it in time.

There is a sense in which the only right thing to do or to say is BSM. BSM is a formalism, not a reality. What's formal in BSM is the inexistence of the market of contingent claims, or, what amounts to the same, its existence that is due to a coincidence, to a forcing of the formalism outside itself through the coincidence of the paper on which the contingent claim is first written as a contingent payoff (in front of the suspension of time and the qualitative argument) and, second, as a contingent claim that will acquire its own market. This reading of the formalism is equivalent to implied volatility, which is what we want to generalize (as opposed to real volatility).

Implied volatility is equivalent to the withholding of time and of the temporal stochastic process. It is because of the accident of time (unavoidable, yet not necessary) that BSM admits of a quantitative diffusion coefficient or volatility. As form, however, BSM is the expression of the intimate link between Brownian motion and the couple formed by underlying and derivative. The marvel of Brownian motion is that its volatility is, in theory, inferable from the longest time series that is compressed in the narrowest time interval; that there is convergence on the spot of the estimation of volatility, as opposed to the drift, for instance, which can only diverge. This deep property of Brownian motion is the reason why volatility is the only thing at work in the relationship between the price of the underlying and of the option.

The locality of estimation of volatility (as opposed to drift) has *as its other face* the locality of the relation between underlying and derivative. The derivative price (or the market, or the *local*) is just the materialization of this marvelous convergence property of Brownian motion. It is the same thing and the same form. The form that Brownian motion is, is the same form that the market is, when it is understood that the *form of the market* is the relation between underlying and derivative. Because of this equivalence of form, the instant option price is the instant estimator of volatility. Either the option price exists to translate the form of formal Brownian motion, or it doesn't exist and it jumps into existence because of the martingale representation property, because the contingent payoff gets written as a substitute for the withholding of time. The form of BSM that needs generalizing is the one that says that the options market doesn't exist. Generalizing it amounts to saying that this market exists.

The alternative generalization of BSM takes place through the accidental yet unavoidable time dimension. We generalize BSM by assuming that volatility is stochastic or that there are jumps in the underlying motion. The market becomes incomplete. The underlying market loses its trading force. Time enters into it and non-tradable quantities, such as volatility or the sizes of jumps, become the other states' variables. The underlying price turns into a lottery; it is no longer a market in the sense of force and intensity, the sense of the trading pit out of which only writings emerge, either to materialize the ex-post infinite accounting equation on the spot, or to materialize the contingent payoff in front of the frozen time.

Recall that the force of trading makes probability unnecessary. It lies in the fact that the underlying is, at the same time, generating the random outcomes (it is triggering the lottery) and the ticket to buy the corresponding lottery. The force of trading is the fact that the future random outcomes are the future prices of the underlying combined with the fact that the present value of the corresponding lottery is the present price. This is the distinguishing feature of trading. Price is at the same time the state, or the underlying random variable which admits of no deeper underlying state, and the valuation taken over that state. (If the state were ever needed to compute and support a value, then the price is both the state and the value; it is quicker than a valuation and, for this reason, it ultimately dismisses the state.)

When time enters in the picture and the trading process of the underlying is temporally differentiated into a stochastic volatility process with jumps (when the door is opened to statistics and history), the valuation of lotteries becomes the only perspective and derivatives, recognized as such lotteries, are then valued following non-arbitrage, over the grid of states of the world that now take over. Now, suddenly, the market becomes equivocal. There is equivocation on the name of the market as its name will now be used to argue that there is one name and one market, therefore there is one equivalent martingale measure that will be selected to value the derivatives. Contingent claims are here supposed to exist beforehand; we have not generated their writing in the sense of genesis. The market no longer just means the original forceful market, the one that was driven only by the underlying trading and whose trading strategies synthesized contingent payoffs. The market is now just an empirical and passive reality against which we go and test the family of theoretically equivalent martingale measures in order to determine (by inference, by testing, by calibration) which one was selected.

From values of derivatives that are now functions of states of the world (the underlying and its stochastic parameters) we illegitimately draw a trading force and argue that the market can be completed, that the value processes of the derivatives belong to the (one) market after all and, as such, are trading processes that can serve to hedge other derivatives.

I claim this is not the right way to generate trading processes. There is a difference between starting out with the price processes of assets, even derivatives, as original trading summaries, as initially constituting the force of the market, and using second-hand value processes as price processes. How could the contingent claim not exist in one case (the Brownian case) and admit of a market only through an extraordinary jump in the void – a jump between the first impression of a contingent payoff that has just been synthesized and the second impression that, precisely for the reason that the contingent claim did not exist but only *coincided* through paper with the analytical (redundant) contingent payoff, it is not actually redundant and its market, as a consequence, may then exist; a jump in the void, directly from formalism to the instant reality of the market of the contingent claim, time being suspended – and, in the other case, exist beforehand, lend itself to valuation as an independent lottery, then, through the equivocation and looseness of the concept of the market (when its concept was so stringent, so compressed between the underlying force and the only void in the first case), become a trading process that is used to hedge other contingent claims?

The only way to generate a market for the contingent claims (provided we do not desire the picture of chaos in which underlying and derivatives trade indistinctly; provided, that is, we wish to build the narrative and the order of technology in the genesis of the market of contingent claims; provided, that is, we wish to recount a genesis) is to argue that their market does not exist in BSM *and for this reason will exist*, or, which amounts to the same, that market-makers trade options and make their market using BSM *because* they are certain what volatility is, not uncertain what it is. There is certainty in the formalism, not uncertainty. There is certainty of volatility and certainty that the contingent claim does not exist. This is why the market of contingent claims is the result of the form of BSM, not of BSM as a mathematical model with content (potentially to be interpreted in physical reality).

This is the mode of creation of the market of contingent claims. It is neither available in the same chaos as the underlying, nor brought beside it in a second step as the valuation problem of lotteries written on the underlying (now turned into a mere stochastic process over states of the world that take over the whole picture and take over the market). The market of contingent claims is of a different category than the market of the underlying. It is created and, for this reason, it ‘returns’ and becomes originary.⁸ It completes the force of the underlying market (it completes the meaning of trading) and shows it how to resist the exit in the time dimension. Only by becoming of the same category of the market of the underlying, and by joining its unique sense and unique force, will the market of contingent claims usher the way out of statistics and the time series (usher the way into place).

The generalization of BSM for a market-maker of options is not to generalize the underlying time process but to generate the options market. The result of the form of BSM is implied volatility, we said. Generalizing its form is generalizing implied volatility; that is to say, taking the prices of contingent claims as given, as true trading processes, and recalibrating the model against them.

Implied volatility is recalibration. There is no way the price processes of contingent claims could be given originally and independently. Non-arbitrage always forces us back to writing a generalized *underlying* stochastic process. A mathematically consistent pricing model has to acknowledge that. However, if recalibration is made its main feature, it will be a way of sticking with the force of the surface of the market, the one that doesn't fold back to the background of states but keeps returning from them eternally, by recalibration.

When trading with force, the underlying generates randomness that does need to be overlaid on a grid of states of the world. The state means valuation over that state. The force and compactness of trading of the underlying are such, however, that it is at the same time the state and the valuation; succinctly, it is the price. Price moves because it is the result of an exchange and price is the present value of future random prices because of the same exchange. The underlying price doesn't need probability in either way, neither on the way of forward propagation and motion, because it is intensively and intrinsically random as being the result of an exchange, neither on the way of establishing the states, nor on the way of backward valuation over those states, because, as a price, it is the own present value of its future random prices. With the underlying, the background space (absolute space, the ether) is not needed; we are truly riding the photon. It is absolute randomness.

Now, as history and probability become pressing (because time unavoidably claims a section in absolute randomness), as time insists on making its entry and differentiating the trading process as a stochastic process, the only way to keep probability from hatching (*la tuer dans l'œuf*) and to keep it restrained is to differentiate the process through the writing of the contingent claims, which coincide with the contingent payoffs that get written, at first, as a substitute for the extension of time. The extension of writing, the extension and matter of the paper, its materialization and, hence, its destination to the exchange – because to write is to exchange – are thus a substitute for the extension of time. The force of trading further resists differentiation in time, it further remains in its intensive pit, through the force of the conversion, or the idea (which I had already intuited in *The Blank Swan*) that $\Delta S + B$ is the manufacture of the contingent payoff – hence, later of the contingent claim – before extensive time is even activated.⁹

The force of trading is consubstantial with Brownian motion even before the emergence of probability. (Vovk makes the Wiener measure emerge after trading.)

The force of trading, when it is kept away from the probabilistic exit, translates into the writing of contingent payoffs, which later materialize into contingent claims. Alternatively, the marvel of Brownian motion, which I have called its form, or the fact that instantaneous volatility can be estimated on the spot because of the availability of an infinite time series in the shrinking time interval and the corresponding convergence of the statistical estimation, translates into the local relation between underlying and derivative. The local estimation of volatility materializes into the instant derivative price. Or, again, the ex-post accounting equation, which is represented by the stochastic integral summing up the gains of the hedging strategy, converges on the spot, thus exchanging ex-post and ex-ante, thus lending to the ex-ante concept of instantaneous volatility the ex-post matter of the hedging account, or the matter of the derivative price.

One way or the other, the statistical locality of Brownian motion – this marvel – is translated into the instant derivative price, or market, where ‘instant’ means real; it means that the price is suddenly given (given in a lightning flash). The form of Brownian motion translates into the matter of the market at the pre-quantitative stage, when the quantity of volatility (its numerical value) is still not conceived. Volatility is *only* quantitatively constant in BSM; it could have quantitatively depended on time. What really matters is the form, and the translation of it into the instant option price. Because option prices are instant (market given) and translate the form of Brownian motion without an intermediary passage into physical reality, they can assume any shape and any structure. The implied volatility surface is the direct consequence of the replacement of statistics by the matter of the market. We can always try and explain the implied volatility surface by local volatility (if we are keen on the differentiation in time) and, as a result, volatility becomes instantaneously dependent on time and space; however, this is not physical, this is just the translation of form. It is the consummation of the fact that the matter of price is the translation of the local Brownian form.

The Book of the Market

15.1 Formalism and meta-formalism

There is a whole new science of interpretation, or of the passage between formalism and matter, that is to be learned from how BSM has produced the options market as a new matter – not one that translates the content of the model, but that is layered over its form. BSM never described an options market; however, it was used, as form (almost as empty form), to create and to make the options market.

The options market is inconceivable in its genesis without the history of BSM, without that historical archive (the historical succession of the first beginning and the other beginning). BSM never created the options market in the sense that that market enacted BSM (its content); the market was never in coincidence with the inside of BSM. It grew outside of it. The matter that got translated into the market wasn't inside the formalism (its content); it was the formalism itself.

The matter was the empty shell. The options market grew over layers of empty shells and dead models. BSM created the options markets in the sense that the writing of BSM was needed to support the market, as well as the writing of the quantitative financial papers and books that followed in the steps of BSM. BSM was needed as a historical archive and a book to create the market; it was needed from outside its content. The relation between BSM and the reality of the market is not empirical but historical. BSM historically left its book in the market, its shell, its cover. The market is itself a history book, so it is only normal that its own history should be part of its volume and, even, of its matter.

The options market as industry is not conceivable without the quantitative models, not in the sense of models of physical reality, not by their quantitative content, but by their layering, their written nature, their book. Quantitative papers or treatises, when they are rigorous like Harrison and Pliska's (a paper which I elevate to the status of fetish because of the word 'verbiage'), speak neither of economics nor of econometrics, nor of the market. They contain stochastic processes and pricing formulas. They are pure form. Yet, they generate the matter of the market.

There is a new link between the formalism and its translation, as we said. We even wonder what this literature really means. We should conduct a study into the archaeology of knowledge in order to determine what this literature has really – I mean, really – brought to the field. Models of quantitative finance are not performed or enacted, as sociologists of finance hold; they join the archive, in a somewhat parallel classification to the written payoffs themselves. This kind of literature and layering, this way of building the archive, the volume and, hence, the sediment (the matter) is appropriate to the *genesis* of the market – which is new in its kind, occurring through the jump in the void, precisely because of the peculiar matter of paper.

Something can generally be deduced concerning the metaphysics of books and their writing, given the way in which the books and treatises of quantitative finance are written and layered (and contribute to the matter of the field). The market is necessary to read as a book because the story of its genesis is historical. One necessarily reads stories. For the first time, the reality of something (the market) emerged from a layering that has no equivalent but that of a book. The way the market is supported by the book is a *historical* way. There is a historical layering, not an interpretation or a translation, between formalism and matter. To the extent that ontology is part of the interpretation of a formalism (the formalism is consistent when models are said to exist), history is now part of the ontology of the market. The whole field is still puzzled over the link between model and reality, time and statistics, value and price, because new tools of reasoning about books and history have not been investigated.

15.1.1 The instant of the formalism and the instant of the market

Brownian motion gave us statistics compressed in the spot. By that, we mean statistical convergence, the law of large numbers compressed in the spot. It gave us the notion of instantaneous volatility and, theoretically, the means to infer it statistically. In theory, we could exactly estimate instantaneous volatility because there is meaning in an infinite series of samples all packed in the instantaneous spot together with the convergence of the law of large numbers. Instantaneous volatility is both theoretical (as are all the other parameters of continuous time processes) and theoretically empirically estimable. Drift is theoretically instantaneous; however, it remains *metaphysically* instantaneous. Volatility by contrast is, in the limit, empirically computable and attainable. However, the limit cannot be empirically reached.

Instantaneous volatility has meaning because of the surface of options prices. Because we could write (K, T) continuously (materially, yet not empirically, or in the infinite limit of our empirical capabilities) and because, as a result of our argument for the writing of contingent claims, the surface of prices can be delivered to the market and vary continuously, the underlying is pictured with instantaneous or local volatility. The possibility of estimating volatility is instantaneously equivalent to the instant givenness of the option price and its instant givenness is equivalent to the continuous freedom of the volatility surface. Surely, instantaneous volatility may vary in time as the consequence of its being instantaneous. However, it may sound unrealistic that instantaneous volatility should vary in time and space according to a deterministic schedule when the whole discourse is about randomness and the absence of schedule. Better to imagine a continuous and continuously varying price surface of options.

The matter of the market (its givenness) is the best substitute for the unrealistic instantaneous statistics. The local volatility surface is the formal (not physical) exploitation of BSM. The time in which volatility varies (physical time) is not the same as the instant of time that we need to define instantaneous volatility. There is equivocation on time. It so happens that instantaneous volatility (whose meaning comes from the instant convergence of the estimation on the spot) can also be interpreted as varying in time. But this is a coincidence and an equivocation. It is a mistake to exit from the formal argument into physical time. It is useless to sample Brownian motion discretely because we leave the instant convergence and we then evolve in physical time. Constant volatility is also a false route. The power of BSM lies in the local relation between option price and underlying (or the material estimation of instantaneous volatility), not in constant volatility. The result of the formal property of BSM is the instant givenness of the options market. It is not the dimension of physical time in which statistical frequencies converge in the long run.

We have already remarked that the immanent character of statistics, which is that they are given – the character which is a relief from the ex-ante notion of probability – is independent of time, even though it looks as if it is unavoidably linked to time. The strike of contingency, we wrote, is the same as frequency, when frequency is relieved from the necessity of being thought in chronology. The givenness of the option price is the same as the given character of statistics when it is separated from time (it is a one-time statistics). When matter is thought as immanence and ex-post, the option price and statistics become the same immanence and same matter, except that the ‘degree of matter’ is different in the case of the option price. The degree corresponding to the option price does not require a time series; it is a one-time statistics. For this reason, it is later accidental and almost parasitical that it should vary in physical time. *The instant of reality and givenness of the market – whether it is due to the qualitative argument of the replication of contingent payoffs in which chronological time is withheld, or to the convergence of the estimation of instantaneous volatility, or to the necessity of closing the infinite ex-post accounting equation on the spot – is a different notion of the instant than the one which varies in physical time.*

It all fell together at the same time. Suddenly, we have the concept of instantaneous volatility, which not only makes sense theoretically (as all the other theoretical parameters do), but also empirically, albeit in the infinite limit which is not empirically realized. Suddenly, we can conceive of instantaneous volatility and the beauty of its concept (yes, esthetics) is telling us not to infer it historically, over an extended time series, but locally, or instantaneously; something we feel we could achieve virtually – that is to say, really, but not actually. (Probably the distinction between virtual and actual, with regard to the real, is here at its best and is most relevant.) The temptation is too strong to infer it from the given option price, which is precisely local.

We know these are two different things and that there is probably no relation between instantaneous volatility and the option price (surely, the options market hasn’t computed or inferred instantaneous volatility). But then, to the beauty of the concept of instantaneous volatility, there just corresponds the beauty of the concept of the locality and the givenness of the option price. Because they are given instantly – which is to say locally and on the spot; and let us not forget that the meaning of the market or the greatest marvel of the market is that prices should be given – option prices are just given, and this means that their structure is anything whatsoever and they may as

well vary along the dimensions of strike and expiry date (K, T) . There is a salutary equivocation here between being given on the spot and in the instant, and being empirically given – that is, being anything whatsoever.

It is true that a single option price says nothing of instantaneous volatility (except if the option is instantly expiring); it doesn't get us into the infinitesimal time interval of the differential calculus. The easiest confusion for the single option price and its finite horizon has been the hypothesis of constant volatility. However, the instant implied volatility *surface* (K, T) , in the variety of its shape and its structure, leaves no choice but to consider instantaneous volatility, in the sense that it varies instantly.¹ The concept of instantaneous volatility, as formal possibility that could be *estimated* in theory if, empirically, we could extract an infinite time series in the shrinking interval, is formally translated by the implied volatility surface.

It is true that the question can be posed again of whether the instant option prices physically really correspond to the real instantaneous volatility (has the market effectuated the computation, and so on?); however, as we said, the physical route is a diversion really and a different exit; we don't need to take it; enough if the formal marvel of instantaneous volatility (due to the marvel of Brownian motion) is translated by the *form* that the implied volatility surface is. It is because of this correspondence of forms that we are now saying that the implied volatility surface formally, and not materially, translates the marvel of Brownian instantaneous volatility. This capacity that the market offers – this new matter, which is not the direct material translation of the formalism, but is a form itself, corresponding to the form of the formalism – deserves that we no longer care about convergence in the long run and real correspondence.

We should seriously consider the market as alternative to statistics. Something is pressing here. That the law of large numbers makes frequencies converge in the long run (making it possible to estimate theoretical posits such as probability or volatility, and such, in the long run) was good, but not good enough. Who could wait until the infinite long run? What can be estimated in theory is, here, coincident with an infinite horizon and an infinitely long wait. Now, with the marvel of Brownian motion, the infinite horizon is down to the spot. Surely, we still can't wait indefinitely until the infinite sample is extracted; however, the advantage here is the locality. Now, we don't have to wait in time in order to believe in the reality of volatility; we have to believe in the reality of volatility *before* we get the infinite convergence, because volatility is going to vary instantly anyway. It is instant, in the sense that it will instantly move, and this somehow dispenses us from waiting for the infinite long run on the spot. It presses us to move. It had better be real (convergent in the infinite long run and even virtually empirically computed) because it is going to move!

Instantaneous statistics is the last word in the act of faith of statistics and the reality of the theoretical posits. Surely, all this may still sound formal and theoretical; surely, the marvel of Brownian motion is only possible on paper; but now, we have the formal translation of the market and of options prices! Finally, the formalism has a material correspondent, albeit from outside, that is to say, corresponding to the form that the formalism is, and not directly to the formalism. This is what we meant when we said that we don't really need to go and see whether the option prices really

correspond to instantaneous volatility. *The market is a meta-formal translation of the formalism.* It is not the physical correspondent of the quantitative volatility; it is not a correspondence from inside, but the translation of the possibility that the formalism could be translated. The expression 'theoretical estimation' is meaningless. If something is to be estimated, it has to be empirically estimated. Nobody needs to estimate anything in theory; things are just posited and given in theory. Here, however, we can make sense of the expression, not only because volatility can be estimated in the long run and frequency converges in the long run (as usual with the law of large numbers), but because it will instantly move.

It all fell together nicely, because option prices are given in the variety of their shapes (and, therefore, formally translate instantaneous volatility) and we imply instantaneous volatility from their prices in order to compute their instantaneous hedge, which closes the argument and opens us to the variation of the options prices themselves.

15.1.2 The infinity of the option price and the infinity of matter

That the volatility of Brownian motion could be, in theory, estimated on the spot – that is, while instantaneous volatility doesn't vary in time (the drift, for instance, calls for a finite, possibly very long, time interval in order to be estimated, during which it could vary of course), and while an infinite sample is extracted on the spot and the series converges on the spot – this marvel is so pressing and so eventful that it presses for a reality or, rather, for a matter. It is an event of thought that something that is supposed to depend very much on time, something *statistical* like volatility, should escape the danger of varying in time (on which it so crucially depends) and find, in an instant, the infinite time series that it needs in order to fulfill its statistical character, or the fact that it depends on time and on time series for its very definition. This event of thought *generates* the market (this is our thesis for the reality of the market).

Yes, the argument here is that a reality can be generated because of a captivating or arresting feature in the formalism of Brownian motion (what we called its *form*). To repeat, this reality is different from the reality of probability that is associated with the ex-ante stance. Drift, for instance, doesn't enjoy a comparable feature and this, we wish to argue, is the reason why drift doesn't appear in the quantitative relation between underlying and derivative (or in derivative pricing). That the volatility, and not the drift, should materially exist on the spot (be susceptible to be estimated on the spot) is the reason why it alone appears in derivative pricing.

The market is all about locality. Lotteries are not to be traded. They are evaluated by objective probability and actuarial science (break-even in the long run), or they are evaluated by utility functions on a single run; however, this is not trading. We already have the argument that contingent claims that are written 'elsewhere' as lotteries on the underlying and only imported to the market as an evaluator of lotteries (in a non-arbitrage relation with the underlying) do not enter into trading except falsely, through an illegitimate equivocation on the meaning of the market. Pricing of the underlying is in no need of probability anyway, as we said (not even in need of its concept, it predates its concept) and the manufacture of contingent payoffs, as we saw, is only the full exploitation of the force and meaning of trading (random walk and variable trading size). This is why they are written

and delivered *locally* to the market; that is to say, before time and the probabilistic exit are even envisaged.

Dynamic replication, which is characteristic of finance and trading, ultimately has nothing to do with probability or the long run and is only local; it has to do with volatility. There is something in Brownian motion that goes beyond the mere number that is moving randomly (in this case, the underlying price). The size of the trade has to enter the picture. In a paper on game-theoretic probability, Vovk deduces Brownian motion in the absence of any prior probabilistic assumptions, simply from continuous dynamic trading in fractional size.² Brownian motion as emerging from the exchange of material paper has, as its other face, the variability of the *size* of the trade. (How could the EMH be conceivable without a material asset to be bought and sold, and therefore traded in variable size?) Lotteries don't trade; we needed the full exploitation of the marvel of Brownian motion, we needed the invention of writing and of the written contingent payoffs – which will become contingent claims – in order to complete the idea of the market and have a market. The market, as local, wouldn't have been possible without the local character of volatility (that marvel of Brownian motion) and the genesis of the instant contingent claims – written instantly, thanks to the extension of writing, which replaced the extension of time.

The market *means* that the prices of contingent claims take over the statistics. This is no market, where we only trade for epistemology, in order to guess some fixed random generator. There never is a long run. The local is all there is. We needed to go through the local and invert. Price has everything to do with volatility (price is local) and nothing to do with probability or the long run. Price as trading with force (the pit of the underlying) had to be completed by the prices of contingent claims in order to take over the remaining quantitative aspect of Brownian motion – the value of volatility. Volatility should be estimated on the spot, never in a time series. Infinity can and should be had on the spot, but this can no longer happen as a time series; it happens as the instant option price. There is infinity in the instant option price because there is infinity in matter.

Again, the local convergence of the estimation of volatility is so good that it must materialize into a reality. However, as the infinite series cannot materially be had on the spot, and because it is a misinterpretation and a disruption of the beauty of the thing to go and infer volatility over a time series that is extended in time, the only reality that offers itself both infinitely and on the spot is the option price. The notion of time in which we consider the instant volatility, or the instant convergence of the estimation thanks to the marvel of Brownian motion, or the instant convergence of the ex-post accounting equation which allows us to exchange ex-ante and ex-post on the spot – and how could that be without an unusual conception of time? – is totally different from external physical time, in which we sample Brownian motion discreetly and volatility is supposed to be constant. This is the time in which the market enters and takes over statistics. It is of a different nature.

15.1.3 Formal deduction of matter

I wish to deduce a whole new matter, a whole market, from the formal possibility of Brownian motion. This should impress many a media student. Has a formalism (mathematics) ever been

the medium of emergence of matter? Where is being, where is matter, where is form, where is the technology (which is a material process) in such a question?

The market is metaphysical anyway, hence the originality of its mode of deduction (yes, from a formalism, from a formula). The reality of the market is all the more pressing and all the more real as it is deduced *from a formalism*. It is so real that it is created. This is metaphysics, in which there is no physical causality or agency. Yes, it suffices to think it for it to exist. Brownian motion is formally written, then it is historically thought (BSM), then the market comes into existence.

Where does that leave the philosophical debate on realism vs. antirealism? Does instantaneous volatility really exist; is it real? When statistical parameters were estimated over an extended time series (or a discrete population), one could always argue that the mean or the variance or, for that matter, the probability of any individual outcome, were only theoretical posits that did not exist in reality, and that the only thing that existed were the frequencies, whose ultimate convergence was the expression of the existence of matter (an immanent, ex-post fact) and not the consequence of the existence of a random generator. But now we are talking about instantaneous volatility, what's more, moving in time! How could it not be real? Surely, all this makes sense and exists in the formalism (the instant volatility of Brownian motion is formally defined, and so on) but isn't it also only in the formalism that an infinite time series could be extracted on the spot so as to estimate instant volatility? Isn't the whole Brownian motion purely theoretical?

Instantaneous volatility is a theoretical construct and the whole philosophical question is whether this is reified or not, real or not. When we say it is instantly moving, is it really instantly moving? Even if volatility were not considered to depend on time and spot (which is already stretching its realism too much) but was supposed to be stochastic in order to explain the phenomenon, doesn't the stochastic process that we assume for it presuppose that it is instantly defined, therefore instantly real? An antirealist can always argue that instantaneous volatility or any other instantaneous statistical parameter, whether they are considered instantaneous in order to vary deterministically or stochastically, are just theoretical inexistent posits and that all that exists is the time series of the underlying, whose statistics may vary in time, of course. It is only theory that attempts to explain the variation by positing an ex-ante random generator (thus, a reification) and by assuming it varies in time. Maybe so, but we wish to argue that the prices of contingent claims, in all the variety of their structures and shapes, are real (the market really exists) and that their reality was generated by all the formal-theoretical posits that we are talking about, starting with Brownian motion and ending with the instantaneous variation of any of its parameters (volatility, volatility of volatility, and so on).

There is a very strange game going on here between existence and inexistence, reality and unreality. Contingent claims, let alone their market, do not exist in the formalism and, because we had first thought that they were redundant when, in fact, they did not even exist, *their inexistence in the formalism will be cause that they will exist*.

Someone will argue that we don't care anyway because the formalism doesn't exist in the first place, being a formalism. Yet, I insist that it is the real cause of the existence of the market of contingent claims (what I called a 'historical' trigger, not a physical trigger). So, the market of contingent claims now exists in the infinite variety of their shapes, degrees of complexity, and variety

of prices; and this, we said, is a radical alternative to the probabilistic exit – which means that, after the existence of the market, a quantitative stochastic process should not even be conceived for the underlying. The random generator should no longer exist.

Never before has the reality of the stochastic process (in the sense of the instantaneous existence of its parameters, the total marvel of Brownian motion, and so on) been so pressing, a reality that flies to the face of any antirealist reservation, at least formally, at least momentarily, until, that is, the reality of the market of contingent claims is generated; and never before has the reality of the stochastic process or random generator been so subsequently denied, on grounds that the force of trading is incompatible with the background states of the world and the probability exit. But the reality that is now denied to the probability concepts is different from the one that they were denied after the statistics. The market of contingent claims takes over statistics; it is an alternative reality and it takes over, in the statistics, the way in which the statistics ruled against the reality of the probability concepts and posits, philosophically. The formalism of probability in its finest, Brownian motion, is literally the trigger of the reality of the market of contingent claims; and, as trigger, it disappears and ‘in-exists’, in ways that were not previously conceivable, philosophically speaking (I mean, regarding the philosophical debate between realism and antirealism).

Objection: So what? Just as we had statistics and volatility was only a theoretical posit, now we have prices of contingent claims, which are just as empirical. The option price was generated by supply and demand, purely on the surface; it really had nothing to do with the instant volatility and its infinite extraction, and so on. Why argue for a different *unreality* of the theoretical posits? They never existed to begin with.

The answer is that we needed the historical trigger and the theoretical passage into the pit, into the local. The formalism of probability stands materially between the statistics and the market of contingent claims (which are two different interpretations of the formalism). The market is radically alternative to statistics and it needs the formalism (its historical existence) in order to affirm this radical alterity. To repeat, the market of contingent claims was not conceivable and couldn’t have happened without the formalism in its finest.

15.1.4 A new book for a new reality

Why is a philosophy of the market of contingent claims still missing? We need a philosophy, something both impeccable and necessary that thought has to think about the market. What is happening there, and what are they doing, they who use derivative pricing formulas and formalisms to price and trade derivatives? The industry has become so compact, so well cemented with all the sediments, that it deserves a unifying treatment. Clearly, this is not the philosophy of probability. There is the philosophy and formalism of the trading force, but this produces articles like Harrison and Pliska’s, in which contingent claims do not exist. Clearly, the philosophy can no longer withstand the imperfections of the time line, arguments for informational efficiency³ or its lack; that is, whether or not the market prices of contingent claims will eventually realign themselves with some reality of underlying random generators. A philosophy, both of the link and of the absence of link between the underlying process and the derivatives market is needed.

As we said, the fashion in which the reality of the derivatives is produced severs any link with the statistics and blocks any thought that derivatives prices will converge to their fair values in the long run. The market of contingent claims has become so complete and cemented a book that it has to include the formalisms and ‘papers’ of quantitative finance. To say ‘What they are doing?’ (this overarching philosophical view) is to make the usage of the formulas part of the ontology. There is a *matter* of the market of contingent claims (not to say a subject matter), pressing for its philosophy. If our reasoning from inside is not enough, the view from above is a sufficient motivation. We long to see the book that explains it all, that changes the point of view, to deliver the point of view.

Philosophy (i.e. ontology, what there is and what they are doing, what is real and what is not, the point of view, the lack of naivety, the illuminating and necessary thought) has had a good time with the problem of probability and statistics. Frequencies observably converged in the long run; this was explained by objective probability (with the question of whether the thing that needed explaining wasn’t less mysterious, after all, than the thing that explained it); and we had the classical realist vs. antirealist debate about the reality of probability. But now we have the prices of contingent claims. Are they real or not? And do they not correspond to the reality of volatility, even pushed to the instantaneous limit? This follow-up and addition to the philosophical debate on the reality of probability has not been taken up by philosophers.

To repeat, the philosophy of probability (apart from quantum mechanics) was settled with the mathematical formalism (Popper, resting the case of propensity with Kolmogorov⁴), but little did the philosophers know that the mathematical formalism would engender the market of contingent claims. All this happened mathematically, without philosophical critique or even philosophical conscience. The debate about the reality of probability more or less boiled down to the notion of instant probability or propensity. Well, now, we have the instant option price; we no longer have the statistical series of underlying prices over which the reality of probability could be dissolved (thought to be just a reification, a summary talk). The single option price stands for volatility – a statistical notion!

But isn’t the price of a lottery real too? Doesn’t the insurance claim reflect a reality of actuarial tables, something also reified? We feel that the option price and options market are different, regarding the realist debate. The price of the lottery and the insurance claim are just the summary of an ex-post accounting equation. If anything, they plead in favor of the inexistence of probability (they deconstruct its ex-ante outlook). The market is different because the prices are instantaneous, because there is volatility; reality seems more pressing, here. We all feel that the option price is more real and is liable to reflect real volatility (in the sense of the realist debate) more so than the insurance claim or the price of a lottery. Why? Probably because, at the back of our minds, we feel that the market is a new reality. Because it is new, it is more real and better supports the reality of volatility. At least it gives it a new perspective, a new debate, even if the result is to block that reality.

There is reality in the force of trading. This whole new angle potentially changes the realist debate of probability because, ultimately, this is not probability. The traditional thinking about the reality of options prices is still divided between subjective probability and informational efficiency (convergence in the long run). It hasn’t grasped the jump to the new reality. A new book, therefore a new philosophy, a new point of view, is needed for the market of contingent claims. The industry

has cemented itself. It is neither as if we were talking about general equilibrium (loose concept), nor as if we were talking about stochastic processes. There are tools (calibration and recalibration) and there is a technology. At least let us write the book to formalize that. We have separate pieces which have not yet been unified. Clearly, a book on stochastic processes and arbitrage theory is not enough. Clearly, the whole complete vs. incomplete market debate wouldn't acknowledge the turning to the derivatives market as sole horizon, in which the purpose is to trade derivatives and the inevitable issue is to complete the market.

To repeat, we are looking at a conundrum. We have the underlying process as sole world and sole economy, because we are pricing derivatives. On the other hand, that we are pricing derivatives means that we are looking to build a derivatives market, therefore any incompleteness in the underlying process is potentially fixed. As a matter of fact, all the papers start with the risk-neutral measure; they consider the pricing is solved already.

15.2 The book of genesis

It is the book that I am looking at, before the philosophy (the philosophy will come after). Why a book? Why does the derivatives market call for the shape of the book? Because, precisely, something has to be written about derivatives and their market. Not a descriptive book, not a history book, not a book of sociology, but a *formal* book somehow, yet not a book of equations and stochastic processes.

The market of contingent claims is perfect. It is quantitative, it is precise, it is all about numbers, and therefore we can write a book that is not a loose economy book. On the other hand, this is a market, this is real, and this is matter. So, this may not be only formal: it has history in it; it has the event in it. Perhaps, finally the book about writing, in the sense that writing is after the fact, that writing is material and historical; that it is not reducible to abstract states; yet, in the sense that writing is something precise and formal.

This is the book of the immanent theory of the market. There must be something like the immanent theory of the market, because there wasn't anything like it before. We literally discovered what was crucially needed for the market (precisely because it is a market) – namely an immanent theory; and the discovery was concomitant with the analysis of the derivatives market. As theory, it needs writing, therefore it needs writing a book. A book has a start and a finish, yet, as Borgesian creature, it has no beginning or end; the book is immanent, too. From the pages and sheets of the market to the pages and sheets of the book, without the passage through theory. The book is unique (it is necessary) in this sense.

We feel that the book of the market, this book, is necessary because it is necessary to read it; and it is necessary to read it because only by reading it – that is, by reading a book not a theory – could we understand it. The book doesn't articulate concepts from without, transcendentally and in exchangeable or permutable fashion. The market is a book, too, which is necessary to read, because otherwise it cannot be understood. It cannot be understood from without. There is only one market, not a model of the market. There is only the token. The market is historical; it cannot be repeated or exchanged, abstracted. There is only one way (*sens unique*): reading the formalism in one way

until the derivatives market emerges, and reading the book in one way, without going back. Never before was it written – it is crucially lacking, too. There is matter and there is history in the market, therefore a book is needed. Books describe matter; they narrate histories, whereas theories cannot. If I were writing a theory of stochastic processes, there would have been many book combinations. But this one is unique.

15.2.1 Only the book can write history

The market can only be told in a book, *because the ontology of the market is that of a genesis*. The formal deduction of the market is a meta-formal deduction; it is obtained outside the formalism. In the formalism, there is no market but only contingent payoffs that can be manufactured. In the formalism, there is the theoretical convergence of the estimation of volatility, there is the theoretical matter (the infinite series that is extractible on the spot); but this ‘formal’ existence of matter can generate reality only through a jump outside the formalism through the local, by saying this option price is now local and is now given, too (it is matter): however, there is nothing to guarantee formally its correspondence with the real volatility. As a matter of fact, the reality of the market is a substitute for the reality of volatility. Likewise, it is a jump outside the formalism to argue that the option price is given materially, against which the ex-post accounting equation can be formed and the law of large numbers can produce convergence on the spot.

The deduction is meta-formal and ‘historical’ because the market is made by a historical look at the beauty of the formalism (and a historical misinterpretation: the first impression that contingent claims are redundant, and the second impression that they, in fact, didn’t exist in the formalism); the market historically uses the beauty of the formula; it requires that the formula exists historically, which means that it looks at the formula from outside, considering its form, and this usage of the formula and the embedding, in the ‘formalism’ of the market (or the book of the market to be), of the fact that the formula is a beautiful formula, this usage and embedding is what I call *historical*.

For the reason that it is historical, it cannot be formalized by theory, but only by a book. This is not a history book (in the sense of a contingent narration or story or anecdote) or a book of sociology; it is the formalism of the market that we wish to produce, *only we conclude that it cannot be had in a theory*. The market is not contingent; it is necessary; in this sense, the book of its genesis (or the formalization of the market) is apodictic and ‘theoretical’; however, because of the way that matter was transferred, from its theoretical existence (in the spot where the estimation of volatility converges in theory) to the materially given option price, we talk of a genesis or a meta-formal deduction of the market, and of a book of genesis rather than of theory.

There is a way in which matter follows form and form follows matter, which can only be told in a book. The book itself is such stratification. It is a continual exit from form to matter, every time a page is completed. This is why books have thickness, they have a way of piling up form and matter, in which form and matter don’t interpenetrate but historically succeed to each other. There is historical irreversibility in the piling up of the book, and this is why a book can only be read or only be written. Reading and writing, here, are two material modes of apodictic (not anecdotal) thought that differ from theory. Theory introduces a reciprocity and reversibility that is incompatible with the market.

How much of my deduction or genesis of the market depends on continuous time and the marvel of Brownian motion? Why is continuity crucial in creating the new matter and the new reality? The philosophical discussion of the reality of probability (ontological probability) has been limited to quantum mechanics, because this is physics (whereas the market is nothing) and because of the prestige of physics. Perhaps the market is only metaphysics (it emerges only from a formalism, and it can only be told in a book). Popper conceived of propensity, or ex-ante probability, because of quantum mechanics; however, there is no instant generator in quantum mechanics; there is something else. Randomness and probability are not there, in the world, in the sense of the reification of the random generator. Statistics and the law of large numbers in which nothing converges except the odds plead against the reality of probability. The only matter is the accounting equation. The only thing that exists is ex-post. Brownian motion almost reinstated the reality of probability and the ex-ante stance, because of the virtual estimation of volatility on the spot (what we have called the theoretical existence of matter); however, we turned that ex-post again by the existence of the matter of the market instead, real this time and no longer theoretical.

This passage and transfer of matter is what confers *new reality* to the market (see Chapter 6). This genesis (where the *meaning* of the reality of the market is constituted at the same time as its reality) allows us to wonder about the reality of the market prices in a new way. Doubtless the prices are real, will an outside observer of the marketplace argue. But the new reality we have in mind is reality of the price *insomuch as* it replaces the reality of volatility or probability.

Probability and volatility are only real in the formalism, because of the instant. Volatility almost became 'really' real because of the theoretical existence of matter, but ultimately failed to become actually real. (Prior to this, empiricists will have argued, of course, that the real underlying is not continuously moving and is not continuously sampled anyway, even though we may have proposed that price was ideal for Brownian motion in the sense of the infinite causes of Bachelier, which become form.) However, the reality of the option price took over. This is not physical reality; it is metaphysical because of its mode of deduction, because of its historical succession to a formalism and of its passage through a book. Yet, we wish to argue for this new form of reality, which is the *translation* of the continuous-time stochastic processes.

The continuous-time stochastic process is an event which presses for a reality; it is had metaphysically by the reality of the market. Yes, something *happened* with the advent of continuous-time processes; therefore, something is *real*. I am looking at a whole interplay of form and matter and reality. Vovk deduces probability from the reality of trading.⁵ There is something pressing for a reality since the beginning of probability and its emergence – because it is related to matter and mass, maybe; because statistics is real; because money is real and contingency is real. There is the stroke of matter, maybe a different sense of reality which can only be metaphysically deduced. Can the continuous time process be conceivable without the convergence of the estimation of volatility? With the market, we can now write *continuously* the payoffs and imagine their continuous and differentiable price surfaces.

The 'formal theory' of the market (or its immanent theory) can only be had in a *book*. I call it formal and I call it a theory, because it is supposed to succeed to the accepted theories (of derivative

pricing, of stochastic processes and, even, of probability). There is a unity of subject matter, even the cement of an industry, in the market of contingent claims (or, generally, the market); there is even matter, we can say, of such nature as to command a unifying formalism. We recognize the matter of the market – its matter is pressing – in much the same way that the matter of probability was recognizable and called for formalization through the challenge posed by Hilbert in 1900. I call it ‘formal’ because my account has been a criticism of the formalism of probability theory or derivative valuation theory, and I have been asked for a substitute: ‘What better formalism can you offer?’ Also, because a step back from matter seems to be required, as is customary when thought recedes from the contingency of the material fact or the material shape in order to establish eternal form.

Yet, it seems to me that the immanent theory of the market cannot be detached from its material *history* – not its contingent history, but the peculiar succession of form and matter that has made the market and that only history can witness. The fact that the makers of the market of contingent claims should have their eyes set on the beauty of the formula and of formalisms such as BSM, and that the matter they produce – the market – should have the formula as indissoluble sediment, is history and is material. For this reason, it cannot be recuperated by theory. This succession of form and matter, this irreversible history, can only be recounted in a book.

Books are made for this. For the reason that the BSM formula is historically piled up, for the reason that it is a page in the book of the market, it cannot be dispensed with. The book of the market requires it; it is based on it, not in the sense in which reality corresponds to the contents of a theory, or the sense of a correspondence between model and reality, but in the sense that the BSM formula has become a material and historical layer and is no longer a form.

15.2.2 One book instead of two theories

We have a theory of probability and we have a theory of dynamic replication. The theory of dynamic replication stopped short of engendering the market. It was deviated from its intensive and genetic force by a quantitative argument, which was only due to a quantitative coincidence. From the coincidence – the marvel and the miracle – of Brownian motion, it should have been understood that the turn should never pass to probability and to the probability exit. From the coincidence that the binomial (up and down) or its time-continuous limit, Brownian motion, when it is combined with trading in variable size, ends up replicating perfectly the contingent payoffs without there ever having been a mention of contingent claims or their market, in an argument which exhausts and completes what is meant by the trading of the underlying, the market of contingent claims should have been precisely deduced via the invention of their writing.

The theory of dynamic replication is only coincidentally a quantitative theory, soon to be overtaken by the probabilistic generalizations (thus, quantitative) of Brownian motion, and should have remained qualitative and formal, in the sense that it will *engender* (yes, in a genesis, soon to be recounted by a book) the matter of writing and the matter of the market of contingent claims.

It should have been understood from the fact that the theory of dynamic replication starts out as the completed form of trading of the underlying (completed in the sense of motion and size) yet yields no market (and no mention, even) of the contingent claims that it was an incomplete theory

or a proto-theory, the seed of a theory or, rather, a form, waiting to engender new matter and to be generalized in ways opposite to the probabilistic exit.

As a theory of trading of the underlying, the theory of dynamic replication should have never exited in the direction of probability and of complicating the underlying Brownian process by jumps or stochastic volatility, or any other cause of incompleteness. The reason why Brownian motion has the martingale representation property is that it is a complete theory of trading – as witness, Vovk deduces the Wiener measure from trading⁶ – and it is an improper mixture to go and generalize it in the way of jumps or stochastic volatility. From the way of trading and from the intensity of trading (from which probability is absent), the only proper generalization is the market of contingent claims, or the invention of writing.

As trading, the form admits only of up and down as qualities, and of the variation of the size. The martingale representation property of Brownian motion is just the expression of the completeness of this point of view and of its intensity. The dual of trading the underlying asset at different times t and different prices S is owning it at different prices K and different maturities T . There is duality between (S, t) and (K, T) and this is why a price surface for the vanilla options (K, T) is equivalent to instantaneous volatility and to the corresponding intensive meaning of trading.

The martingale representation property takes us to the doorstep of a market of contingent claims, which is all the more a door as the market doesn't exist and will literally be an *exit*, which is why Brownian motion shouldn't have been generalized, under the register of trading that excludes probability, in any other way than the market of contingent claims.

We don't know what becomes of dynamic replication (which is so beautiful and so powerful, so unique, an argument) when we postulate states of the world (ω_1, ω_2) under the underlying price and generalize Brownian motion through probability. To recoup dynamic replication, we then either have to mix it with probability, as with mean-variance optimal replication or, even worse, with loose talk about the market, nevertheless, and about the martingale measure 'selected' by the market, thanks to which value processes of derivatives become trading processes and can help complete the market again.

On the other hand, we have a theory of probability, in which nothing trades, for the underlying itself is now underlain by probability states. With probability theory, we have actuarial science and break-even in the long run; we have non-arbitrage and risk preferences which provide valuation in the absence of the long run. When probability was no longer needed because the underlying spot price was equal to the present value, probability was used again to value other lotteries, together with the underlying which was now, itself, interpreted as a lottery.

We have two theories. There could be a theory of probability and there could be a theory of underlying trading which stops short of the market of contingent claims. However, there could only be a *book* of the market. If there must be one topic and one argumentative line in my book, it is that the market of contingent claims takes over and replaces the probabilistic exit, right after the qualitative theory of dynamic replication. The whole book is this diverted exit, which consists precisely in saying how the market is made incomplete by the prices of contingent claims and not by varying the underlying stochastic process. The whole book is to exit from theory, into the book, precisely.

15.2.3 Only the book can bind the void

It is not easy to argue for the disappearance of one reality and the genesis of another. This requires a book – a book is this passage or this bridge, this peculiar succession of form and matter. The trading process of the underlying presented all the features of a stochastic process which causally triggered derivatives written on it. It is not easy to argue that, precisely because this is a market, because the underlying is immersed in its market (whose matter is in the making) and because the genesis, and even the definition (the meaning), of this market won't be complete until underlying and derivative trade on the same level, together with the required dependency and the required independence (a contradiction which can only be resolved through a historical record and the fold of a book), it is not easy to argue that, precisely because of all that, the stochastic process of the underlying *is not* what it seems – that it is not a stochastic process.

We may not require the category of probability and of stochastic process to price the underlying (as its price is immediately given) and only require it for valuing derivatives; however, this is precisely what we are now eliminating (rejecting). There truly *is* no way to value derivatives. They look like lotteries, but they are not, because their underlying is trading in a market and, for this reason, is no longer a stochastic process. We think we can value them by suddenly imprinting probabilistic states on the picture; however, this is a ghostly print; it is a mere coincidence. We can value them as lotteries, through this print, however we won't trade them; we will have exited the market. (This includes both valuing them by actuarial science, through the break-even of insurance companies, and valuing them by non-arbitrage overlaid on the states of the world, as lotteries considered on an equal footing with the underlying.) We can only price them in a market, as if they were an underlying or basic asset. For the reason, however, that their underlying asset is trading, we cannot hide it or screen it off as we would usually do the economic factors that typically underlie the trading of a basic asset. The real problem is that the asset underlying the derivatives must, itself, trade in the same market as they do. It is one market and we see all the market.

This leaves as only possibility the genesis of their market through dynamic replication, which makes it look, first, as if they were redundant and makes us realize, second, that they are not, because they don't exist. Nobody knows how to value derivatives, because their underlying is trading and this excludes the theory of probability in favor of the only theory of replication – which, in turn, falls short of the market of contingent claims and requires the book. The void that opens up (nobody knows, nobody can value, and so on) calls for the matter of the market all the better, and for the book as only form to collect it. We have to recognize that nobody knows and that there exists no stochastic process (epistemological and ontological void) before we recognize the jump into the void, which is the genesis of the market, and we recognize the book alone as only bridge and only binder.

In this exclusion there is exclusivity, and in this exclusivity there is irreversibility and univocity (only one way – *sens unique*), leading both to the book and to the genesis of the market. To repeat, the whole argument for the diversion into the market introduces itself by the very fine (very narrow, very exclusive) observation that contingent claims are not included or given by the formalism. It takes a book to argue that dynamic replication, in which they don't exist, is the only way to value contingent claims (as a matter of fact, contingent payoffs), before their price takes over and their

market exists and they exist as contingent claims. This diversion to the market takes place after categories of such metaphysical significance (the void, the writing across it) that only the book can bind them and report them. Only the book can collect them and *think* them, in Heidegger's sense of the term.⁷ The materiality of prices that are to emerge next is dependent on this diversion (or jump) and on the void where it takes place.

It is, then, crucial not to differentiate Brownian motion in time, through probability again. The new reality is material only inasmuch as it takes over the probability exit. (There is a book of the market inasmuch as there is no theory.) History and the book are a necessary layer in the new matter and, as such, they mark a total break with the probability paradigm. There suddenly is a previously unseen void. Contingent claims are written, they are there, and they fall into a void; nobody knows how to value them. The whole probability paradigm which seemed to be well in place suddenly collapses. There is nothing more pressing than the sudden void. A book is required for this (to collect this).

Derivatives have to be tradable at once, because this is the matter of the market that we are making. They have to provide material for calibration and recalibration; we have to jump, after matter was engendered by form historically, into the era of resilience and firmness of the new matter (calibration and recalibration).

There seems to be, unavoidably and irreducibly, this void. Probability is bypassed. No sooner has Brownian motion been recognized to be too unrealistic, than the market of the contingent claims steps in to generalize it and make it incomplete (rather than stochastic generalizations). There is no intermediary step of valuation by non-arbitrage in the martingale measure. *Derivatives are valued at once as prices, or as inputs for recalibration.* This is the trick. The market is virtually total; that is to say, virtually totally incomplete. The market of contingent claims is given (a genesis) faster than valuation. We require a pricing kernel to enforce non-arbitrage; however, we need the capacity to use the calibration inputs as hedges and we need to keep the hedging imperfect in order to allow for recalibration.

Recalibration requires time and space homogeneity of the dynamic model we are recalibrating because the instant in which the market of contingent claims takes over, which is defined intensively, should not be confused with the extensive instant and the extensive time which define the stochastic parameters and potentially open the probability exit. The only reification and interpretation here is via the market of contingent claims (via price) and not through propensity or the generator. Enunciate this loudly.

Why is the problem of form and matter consubstantial to probability? Matter is contingent; matter is absolute. Probability tries to lift the face of matter (the face on which matter lies). Hence, form. It is not a coincidence if van Fraassen, on the one hand, speaks of probability and probability formalism and, on the other hand, introduces me to Coffa. Not a coincidence if Coffa is the first to introduce semantic factualism (second order semantic factualism), akin to Meillassoux.⁸ Why is probability such a deep category (as witness quantum mechanics and the related essential indeterminism)? The formalism of probability by Kolmogorov made its history disappear. There is no book left in this, no archaeology. By contrast, the market re-introduces the event through the totality of the market and, as such, necessitates the book. History is not eliminable.

15.2.4 Only the book can settle the succession

The genesis of the market is historical and, for this reason, it requires a book, not a theory. History is written. Theory tends to eliminate history and to erase the layering. The layering is made of form, from which we successively exit to produce matter, and then matter constitutes the next layer. The genesis of the market is historical because of the way it exits the formalism (thus producing matter and this immixture of matter writes history precisely).

The realization that contingent claims do not materially exist in the formalism, when it follows the impression that they are redundant and, for this reason, engenders their market (for the reason that the realization of non-existence follows the impression of redundancy in that order), is historical. In this historical *succession*, which only a book can capture, the formula gets imprinted and, for this reason, is no longer dispensable.

The formula gets captured by the book and, for this reason, becomes an indispensable part of the cement of the industry. The book provides both the matter and the binder, the press that compacts the matter. The formula is needed because it provides the perfect replication of the contingent payoff and thus writes it, producing the threshold of its market, opening the door to the market. This exit into the genesis of the market is the only way to make the market incomplete; yet, this took place outside probability, using the other door and, for this reason, the formula did not theoretically fail; it historically failed or, rather, it was historically succeeded (successful failure!).

The formula would theoretically fail if its criticism came from the probability door. But it fails because it says there is no paper and no market, when there is one that succeeds to it precisely – precisely, historically and materially. The formula is not immersed in a larger probability space, in which it is resolved that it theoretically fails. Rather, the new space is an unforeseen space; it is the space of the formally inexistent market. For the reason of its eventfulness (which is history, then), the market makes it such that the formula is both exceeded (or, rather, succeeded) and needed.

The new movement of prices of contingent claims in the new space is not probabilistic. It is not an extension of the previous space. It is not an extension even absolutely; it is an intension. Because the market is incomplete in this new sense, the formula of perfect replication can still be used and, as a matter of fact, is needed. Yes, volatility is no longer to be inferred statistically – it can only be inferred on the spot, it is to be inferred from the market price of the derivative (at-the-money option, variance swap), yet the link with the underlying is still required. Because of the historical, rather than theoretical, succession, there is no contradiction between the new motion of prices (due to the formally inexistent market) and the previous perfect replication.

The book of the market is the attempt to ‘solve’ the contradiction between inferring the statistical parameters from the prices of derivatives and applying them in a perfect replication formula involving the underlying. Let us not forget that the formal parameter going into the formula was only valid on paper. It was neither statistical nor material; there was even no question of inferring it. By inferring it from the price of the contingent claim, we both contradict the form of the formalism (‘not to be inferred’) and we exit from it (as we rely on a market that is formally inexistent).

My problem is ontological, not sociological. Statistics doesn’t exist. The stochastic process doesn’t exist. It is a very strange register, the one in which I can thus argue for the inexistence of the

stochastic process. How could it not exist when obviously it does and we do observe it in time? I say it doesn't because of the incompatibility of registers, because of the prohibited probabilistic exit. The probabilistic exit was blocked, together with the statistical temporal register (observing the time series of the underlying), because the market of contingent claims was created as an alternative. There is a creation of matter, or an extraction and transfer of matter from the intensive pit of Brownian motion, either to provide the ex-post accounting equation; or to materialize volatility which could – in theory, yet not physically – be extracted; or by the invention of writing and the market of the contingent claims. This creation and this transfer are metaphysical, not physical.

The genesis of the market takes place through the book and its passage and, for this reason, it is metaphysical. We are not looking at the physical reality of the market from above, in which we perceive the underlying and the derivative trading in concert. We are not observing chaos. Physical ontology applies to chaos. Sociologists note the lack or the contradiction in BSM (namely, that a market for contingent claims is disallowed), so they invoke the sociological dimension in order to supply what is missing. But have they considered the simplest problem of the inference of the volatility that goes into BSM?

Everybody thinks the matter is settled or secondary, when it is not. Volatility only exists formally. There is no physical time in BSM such that statistics or a long run can even be conceived, or informational efficiency be framed, or the convergence of derivative prices to theoretical values even considered. On the other hand, there is no feedback effect. It is not the case that the existence of the derivatives market is interfering with the underlying and disturbing its statistics. Either way, this backward view has to be abandoned altogether. Because this is a market and because the underlying is trading, the completion of its trading and, virtually, its total incompatibility with states of the world and the probability framework are achieved with the invention of writing and the emission of the market of the contingent claim, and this alone is the reason why we can no longer look at the time series.

The market wasn't created from the statistics and the random generator (there is no sense of trading in this) but from the trading of the underlying in its intensive pit and the instant from which the contingent claim was emitted and the door opened to its market, in contradistinction to the probabilistic exit.⁹ The market was created formally. There is pure genesis in this; there is history and no theory; there is the book and, for this reason, we leave physical time and the corresponding ontology. Sociologists seem not to care about the inexistence of the market of contingent claims, therefore about the ontology of the market.

What is the market when the best formula and formalism describing the market (of the underlying) prohibit the way of completing it with the invention of writing? The market of contingent claims was engendered through the formalism, in the void that *succeeds* to it; and for this reason, it is absolutely severed from the time series and statistics. From the moment that the derivative price exists, from the moment this new reality takes place (the reality of the market which obtains only inasmuch as the book replaces the theory and inasmuch as the probability exit is blocked), it becomes prohibited to look at the time series and to infer volatility statistically. Statistics no longer exists.

We cannot formalize the market when the market is conceived – as it should be – as the simultaneous trading of underlying and derivative. It is a market, so it has to be viewed from inside, not from above. The view of the chaos from above is of no help. With states of the world being the underlying, the underlying asset and its derivatives become mere lotteries, so we exit the market. The only way to formalize the market is the way of the book, what I have called its ‘passage’ and its ‘succession’, in which the formula (the formalism) gets imprinted and, for this reason, remains like a sediment (or a ruin), like the page in the book. This way of historical genesis has to be investigated (investigate its application to contemporary art, maybe).

We begin with the trading of the underlying as sole market, and then the manufacture of contingent payoffs makes it pressing to invent the writing of contingent claims, when we recede into the intensive pit, and subsequently to emit their market. The only way to formalize the whole market is this continual exit and jump outside the formalism. It is this exit and this meta-formal thinking of the market (the *thinking* of its matter, a metaphysical materialism at last) which presents the matter, which gives the price of the derivative, against which alone we infer the volatility and completely dispel the time series.

So, it becomes a specialist’s market and a specialist’s book. There is only one reading and one travel. Buy your ticket now; be among the happy few. You may be the only reader of the book, the only traveler of the market – may you take pleasure in this distinction! The reason why the market of contingent claims has no end is the same reason why contemporary art has no end. Doesn’t the label ‘contemporary’ mean that it is always taking place now, as we speak and write, therefore that it has no end? The book has no end either, provided thought never tries to revert to a recollection or to a theory, or to refer the book to fixed and delimited states of the world.

15.2.5 Contemporary art

The book or the market is not related to chronology and neither is contemporary art. Contemporary art is the art of trading the boundaries of art and constantly revisiting the meaning of the work of art, and what it means to be making art while making or presenting this particular work of art. For this reason, the context of the work is part of the work and, therefore, part of the art. Typically, this constant revisiting and constant weaving of matter and expression, of context and objet, is a bottomless pit and a constant singularity; it is a continual event and it is always ‘now’, for this reason. It doesn’t happen in chronology or in time, but in repetition. It happens in place.

Contemporary art is the central nervous system of art (the brain of art). It is no longer necessary that the outside world witnesses the proceedings of the art engine. We are told that the work of contemporary art is often shipped from the artist’s studio (which may no longer be an actual studio; it could be as minimal a place as his brain, in case of conceptual art, or art that takes place and form in no other way than its actual presentation and installation) directly to the museum of contemporary art. It goes down as a work of art without the general public even noticing or judging the event. Enough if the brain of art, or the register of art, or the history of art, registers the event internally.

Nobody watches art externally any longer. That the museum collects the piece has now become an internal episode. The museum is no longer a public place but becomes a register that is private to

art and to the history of art. The work of contemporary art has no other meaningful destination than the museum, which, in turn, has no other mission than to receive the work of art privately. Because of this 'blind' loop, the present museum becomes its own book and its own history. I was told the purpose of a museum of contemporary art is nothing else, any longer, than the presentation of the future museum of contemporary art – presenting, as of today, what future generations will read as the history of contemporary art.

It is not that future history is being written today. Rather, the book starts when all is finished. The museum is no longer collecting the recognized work of art; it is making it, too. What the making materially amounts to is that, when the work is collected (like in a book), it will be thought: 'Yes, this is a museum of contemporary art, this is contemporary art, this is the *history* of contemporary art, this is the collection of contemporary art, this is what contemporary art has passed to be.'

Because of the exclusion and sophistication, nobody except its own brain and own specialists understands contemporary art any longer. Because of the internal episode, the process and the history become endless. The history of contemporary art will always have its event and will always be contemporary. The similarity with the market of contingent claims is that the latter never ends either, and nobody understands it either. The market of contingent claims is no longer accountable to statistics, or to convergence in the long run, or to physics (or even to chaos) altogether. It is only accountable to the next layer of written contingent claims, to the next page of writing. It is only accountable to its endless book.

This is matter; how can anybody understand matter? Matter is written, not thought or conceptualized, and it is left to its only book and only history, to its only travel. The sophistication and the exclusivity of thought, ultimately, are matter. Matter is the exclusion of the thought, for this reason. Matter is endless. The ontology of contemporary art should borrow its book from the market of contingent claims (with regard to chronology and to the relation with representation and external thought or framing). Likewise, the ontological twist that we bring to the market and to finance, by reading them as an endless book and not as a recollection or as a theory, is contemporary. Our gesture is contemporary, as opposed to modern. Inasmuch as we care what we are thinking, now that we deal with the market (what sophisticated endless thought, no longer naïve, very refined), our worry is contemporary and our mood is contemporary.

15.2.6 An ontology made of paper

Recalibration does not mean changing or differentiating (making more complex) the underlying stochastic process. Recalibration doesn't occur in the space of probability and states of the world because of a difference of degree or complexity in that space, or because we think that Brownian motion is unrealistic. Recalibration wouldn't take place if the formula and the formalism didn't precisely engender what was most unexpected and surprising – the market of contingent claims. There wasn't supposed to be a market, or even a mention of contingent claims, in the formalism. Recalibration is recognizing that the prices of contingent claims are prices, not values, and that contingent claims are trading.

Recalibration is admitting the inadmissible; namely, that contingent claims are supposed to trade with force (basically and originally as independent tradables), when there was supposed to be no market except the underlying market, following the argument. The argument started with the trading of the underlying alone, and the argument is precisely what we have called the 'force of trading'; namely, that the underlying price is both producing the randomness and is the price of the endogenous lottery. Probability doesn't exist in the course of that argument and the question is how to open the market, how to add further trading processes, when probability remains disallowed (because we never want to forego the force of trading), and when the only complications must be derivative on the underlying (as it is the sole world and horizon).

We want to add complications to the market (to differentiate it), without parting with its force, but how could this work and how could the compact market open up? How could the closed argument according to which the underlying is its own lottery and its own lottery ticket possibly differentiate without opening up in time and, therefore, to probability? We want to add derivatives as already trading, as prices already and not as values that were computed with probability. This could only be achieved through the impossible and the inadmissible, through the void, precisely through the layering which provides that contingent claims do not exist, then suddenly trade and exist because they would be redundant and would not trade if they existed and, precisely, they don't exist.

The market is the pit; it is the void; it is the book and the metaphysical ontology. We could have contented ourselves with the underlying trading alone, and we wonder what ever inspired us to write additional complications. Only this wouldn't have produced matter and wouldn't have opened up to the real, to the event. We want to separate ourselves from the compact trading of the underlying, yet remain in the intensity of the pit. We don't want to resort to states of the world and to the probabilistic exit. The only way is to produce continuous matter instead of fixed states; it is to invent writing – writing *invented* in the sense that the contingent claims are written and their market emitted just when this is impossible, thus triggering recalibration, which, as we know, is virtually limitless and is incompatible with any total set of states of the world. Yes, matter is produced out of a formula, the market is deduced out of a formalism, and more radically so than we may have imagined.

The market of contingent claims is not the mathematical consequence of BSM but, worse and even more incalculably, the consequence of the succession of matter and form, which we have identified with the book. The ontology of the market is made of paper; that is to say, it is both material and formally deduced. It follows from the invention of writing which precisely takes place in the void outside the formalism, thus finding matter precisely outside the formalism and precisely succeeding to it. An ontology made of paper (thus, the book) has a peculiar relation with reality; we infer volatility from the prices of contingent claims, from the surface of the paper, and real volatility no longer has any meaning (in the sense of convergence or the long run).

If we have the underlying trading with force and we have it as sole universe and market, we cannot add complications to it except the ones that are written on it (derivatives). Yet, we don't have extra paper and we don't know what writing means. Writing has to be invented and the *invention of writing* is precisely what takes place in the intensive instant, when we look for the intensive equivalent of the manufactured contingent payoffs and we wonder what the capacity to trade the underlying at

different prices and in variable sizes means intensively, before extensive time is opened. As writing is invented, the market of contingent claims is emitted. Their prices are produced directly, thanks to the surprising market, and the trading process of the underlying is differentiated in this way and not through the probabilistic exit.

To repeat, we wish to differentiate the market, to make it occupy place and connect with the event, yet we do not wish to part with its trading force and its pit. Recalibration is the channel.

We have the underlying trading with force in its own market, both producing the lottery (the randomness) and the lottery ticket (the price), and we wonder how additional paper – namely, derivatives written on that underlying (for there doesn't exist any other asset) – can be imagined and introduced. A sociologist would have no problem opening up the trading pit of the underlying to its derivatives. To him, the market is not just a formal circle and it is not sociologically closed. It hides many sociological motivations and mechanisms, and one of them can emerge and join the surface of the market; namely the wish of the society of traders of the underlying to additionally exchange derivatives, for examples contracts that pay out in case certain triggers are met, or simply that delay the purchase or the sale of the underlying at certain predetermined prices which may turn out to be advantageous.

The sociologist may thus imagine that such derivatives are thrown into the trading arena from the beginning, together with the underlying. However, no such opening presents itself from the formal point of view of the market, the one in which the underlying exists alone. In the formalism, even though the natural language (the interpretation of the formalism) may lead us to believe in the existence of derivatives and even to mention them (for we feel that if the underlying S is interpreted as a trading process and as a market, then nothing would stop us from imagining the derivative $V(S)$ and from wondering what its value or its market may be), there is a strict order and therefore a strict exclusivity that must be followed. In the formalism, we are not giving ourselves an underlying asset S , with all that it may mean economically and sociologically, such that its market may then be imagined (with all that it may mean economically and sociologically) and such that derivatives, which may be imagined for the economical or sociological reasons we have mentioned, may then join that market.

In the formalism, a certain stochastic process is given, which is *interpreted* as the trading process of the underlying. The underlying is given as trading from the beginning, we may say, and trading with force. This is the first given and the first entry of the formalism. Surely, another asset may be given as well, as trading. The entry level is such that two trading processes (or any number of them) can be imagined. Anything that may be added has to enter right at that level and no derivative on the first underlying may be added and included in the formalism unless its trading process is specified from the beginning.

If the formal point of view is one of trading with force, then the derivative may only enter it as trading originally and with force. To the two independently given trading processes, the payout condition, linking underlying and derivative at maturity (or before) must then be added. Since the underlying is the sole determinant of the derivative value at its expiry (or at its knockout barrier), we know that inter-temporal arbitrage will force non-arbitrage relations between the two price processes

prior to expiry, and we know that this can only be achieved by robbing the derivative of its trading force and its originality, and by proposing that the real state variables underlying the derivative are the underlying price and perhaps, additionally, parameters of the underlying stochastic process, such as its instantaneous volatility, or the size of its jumps, and so on. If one wishes to be strict about the order of the formalism and to respect its entry level, it seems one cannot consider the addition of derivatives, for their consideration will only rob them of the trading force which we said was a requirement for entry.

We have already expanded the thought that making instantaneous volatility stochastic, or adding jumps to the underlying process, will certainly account for the variability of value of the derivative for a given fixed underlying price, but that such an expedient was precisely no trading. The so-called ‘martingale measure’ that the market is supposed to select in that case is actually the consummation of the thought that this is a valuation process and not a trading process. (The market *doesn’t* select a martingale measure.) The probabilistic exit leads to no market, we said, but only to the valuations of lotteries whose underlying is now identified with a vulgar random trigger. Strange as it may seem, respecting the formalism and the order of its thought, respecting its entry level, seems to block any entry and any addition of the derivative from the beginning. One cannot, from the start, look at the market as an arena in which underlying and derivative are initially trading in concert. This has to be engendered (as in a genesis).

The argument from the trading force is the qualitative argument according to which the underlying is complete in its trading pit and does not need probability. This, as we recall, has as a consequence that contingent payoffs are manufactured while contingent claims are not even mentioned. This, as we recall, too, is precisely the reason why the market of contingent claims can then be emitted and the reason why, because of the void and the total departure from any valuation and any probabilistic framework (any grid of states of the world), they will then trade with force. In sum, to respect the order of thought of the formalism, which is the trading force and the entry only to that extent, the only way to have derivatives formally and to have them trade with force is to create their market out of the void (the genesis). This is how they can be formally added, through a subtraction really. Formally: because the *form of the market* is the trading force.

15.3 The trading force

The market either trades originally and with force or it doesn’t. You select the martingale measure and guarantee non-arbitrage in your valuation, but this is no market. It is quite indescribable in the formalism how you transfer your choice to the market. Are you a market-maker? How is this described formally? You select the measure, there is no arbitrage, this is a lottery and this is no market; this is the probabilistic exit. Closing the market on the underlying should have dismissed probability altogether and adopted the force of trading instead, where mention is not made of the contingent claim. Markets may be incomplete and we may stop at the multiplicity of martingale measures and at the unavailability of a price for the contingent claim when we take the probabilistic outlook, but this is a dead end.

The market is incomplete when there isn't a sufficient number of trading assets. But the market was never supposed to be incomplete through complications of the stochastic process of one underlying asset. This is the only way we have found to make the market incomplete once we had closed it on the sole underlying. In this, we are contradicting ourselves, because the purpose of the exercise is to trade derivatives. We have imported the old incompleteness from general states of the world and the economy to the states (prices) of the underlying and to states of the parameters of the stochastic process. Hence, the tension and the confusion – the talk about the market selecting its martingale measure.

We say the market is complete when all the contingent claims trade and cover the number of contingencies of the underlying price process. But this is precisely when there is no longer any market! I prefer the opposite view. Contingent claims can only trade; they were written for that purpose and they make the market incomplete by their trading, because they invent new states of the world. The view of fixed states of the world, which is the probabilistic view, forces the exit from the market and forces market incompleteness at a point where the market is left behind and is really in no place to select the martingale measure. The whole language of martingales and incomplete markets comes from the probabilistic framework, in which it is not surprising that the market stops and that some contingent claims no longer trade. This is no market when we begin by counting the contingencies and then the number of trading assets. The whole idea that states of the world may be identified as well as the number of trading instruments is contrary to the market. I prefer to call a market the view in which states of the world are created – a genesis of the market. This literally traverses the formula and the complete market.

Contingent claims came into being and into trading because *they didn't exist* in a formalism which, for this reason, did not say they were redundant. Dynamic replication is a great thing; it is the discovery of the equation between the number of contingencies and the number of trading assets. Contingencies translate in money and the trading of the assets guarantees the manufacture of contingent payoffs. There is this equation. But when the contingencies relate to the only underlying process, how can you increase the number of traded assets? How can you make derivatives originally trade when you know that arbitrage will make them derivative anyway on a previous grid of states of the underlying and its process? The only way that they could trade originally and with force is that they do so outside the grid.

We have evolved to a new view of the market, a new matter even, in which everything trades, the underlying asset and all grades of derivatives written on it. The market is not complete; it is incomplete. Not because contingent claims do not trade (on the contrary, they all do), but because there is no longer a total of states of the world. Any total will make a certain contingent payoff attainable, but this is an argument to say that the corresponding contingent claim is *precisely* not redundant. The only way to have the underlying and the derivative trade together is, strangely, to disallow (to not even mention) the derivative at first and have only the market of the underlying. The market cannot select a martingale measure out of a multiplicity of such measures. Rather, there is no martingale measure to select, there is even no market (of the contingent claim) and there is no

contingent claim. But this is why there will *precisely* be a market. Not by selection, but by force and, even, forcing.

What is the market? It is chaos indeed, in which everything trades. All we are doing is layering it and showing its genesis. The whole complete vs. incomplete market language suggests the wrong layering. It seems to be chronological – showing us how derivatives are priced *after* the underlying. We must have them all simultaneously instead. But how? This can only be done through the other layering, the historical one, through the book.

According to the canon, the market is complete when all contingent claims trade or admit of prices. No matter how complex and differentiated we make the underlying stochastic process, if enough derivatives of a high enough order of complexity trade, the market will be complete. But who says the trading of derivatives should be measured against a backdrop of probability states and transitions, or against the underlying stochastic process? The market is complete when the contingent payoffs are replicable, not the contingent claims. The contingent payoff is replicated in BSM and we know exactly the initial premium we should invest in order to manufacture it. However, this is different from the jump in the void that now occurs because of the first impression that the contingent claim is redundant and replicable and the second impression that it actually never was because it formally didn't exist.

The contingent claim was never written and put to trading in the first formal layer where the contingent payoff is attained. Precisely the invention of its writing and its subsequent trading will make the market incomplete by making implied volatility stochastic (if, that is, we wished to invert the formula and to imply volatility from the *price of the contingent claim*, which is different from the value of the contingent payoff). Then, we feel we should respond by making real volatility stochastic and changing the underlying process, when there never was such a demand. We say that real volatility is now stochastic and this is a new contingency that the trading of the contingent claim will attain, thus making the market complete. But, in reality, it is the other way round. The underlying trades and the contingent claim trades and nobody has ever counted the existing contingencies and closed the probabilistic grid. The two of them now trade in an attempt to replicate a contingent payoff of higher-order complexity and to create the market of the corresponding contingent claim, thus opening the market again.

From the moment we ask the question 'What is a market?' and identify the market with the trading force, we depart from the statistical picture. A market is the medium in which both the underlying and the derivative admit of prices, or that very material which feeds into the pricing tools for calibration and recalibration. The market is the market-maker's stuff. He is always already in the middle of the market. There wasn't such a thing as the underlying market which was followed chronologically by the derivative market. All derivative valuation models (and theory) try (and fail) to narrate such a chronological story. They teach us how to value the derivative given the price of its underlying, but they fail to establish the derivative market. With the statistical paradigm (distinguishing between complete and incomplete markets) we are stuck with valuation, when the market-maker accomplishes no such thing. The market-maker expresses the force of trading. He

uses the BSM formula precisely because the formalism harbors no *valuation* of contingent claims but only of contingent payoffs – precisely because contingent claims do not exist in the formula.

The BSM formula is a trading tool because it is solely based on the trading force of the underlying (no statistics) and expels the market of the contingent claim outside its theoretical domain. From the moment the market of derivatives is considered, the statistical picture is out. We are left with an epistemological and ontological question, as we ask: ‘What is happening then behind the prices? Is there a statistical convergence in view or is the market-maker only forming opinions?’ The answer is to dismiss both the epistemology and the ontology. The derivatives market has no relation, either ontological or epistemological, with the underlying statistics. The BSM formalism has the semblance of statistics in it. It speaks of volatility and this suggests time series. Yes, there is time and there is a theoretical time series of the underlying in the BSM formalism and in the formula, because how could things be otherwise? However, the formula was never intended as a reflection of the real statistics; the formula is certainly not meant to be used to *infer* the statistics of the underlying. (Think what this *really* means, that the formula shouldn’t be used in that direction – what this implies for its relation with the real statistics.) Implying volatility with the use of the formula has nothing to do with the underlying, as we have tirelessly said.

We have already tried to picture the BSM formula in reality, even with the knowledge of volatility, even going in the same direction as the formula, and we have felt how strange this was. Of course, the formula can allow for volatility that changes in time; however, this is not equal to volatility that changes really, in empirical time. The formula is just the quantitative or extensive proof (for the formalism of probability is extensive, to the best of our knowledge) of the equivalence between the force of trading of the underlying and the manufacture of contingent payoffs. This has nothing to do with statistics, or with the empirical time in which statistics is accounted for. The only connection of the formula with reality is a strange connection. The formula implies there is no market for the contingent claims when obviously there is one in reality, then, when it is realized in a second thought that the formula implies no such thing in reality because it contains no mention of the contingent claim, this realization *becomes* the connection of the formalism with the reality of the market. Better, the reality of the market becomes a consequence of the formalism (through the void). It becomes a consequence of the form of the formalism (what we have called a meta-formal deduction, or the matter of the market which is, in a sense, a form), not of its quantitative content, and this is why by implying volatility from the price of the contingent claim we don’t relate to the reality of the underlying but to something else.

Volatility is constant in BSM because it is just a symbol. Even if it formally depends on time, it is constant because it remains a symbol. It is forced on us by the necessity of the quantitative picture, by the unavoidability of time. In reality, it relates to the intensive trading of the underlying and to the qualitative argument which expresses the exchange as a manufacture of contingent payoffs. Now, their writing is invented at a later stage, they become contingent claims, and their market is emitted. Suddenly, their prices admit of structure $C(K, T)$. But this is no reason to imply a volatility which would now depend on time and spot. There is no connection between the reality of the market of

contingent claims (as thus deduced) and the underlying. Ultimately, this reality is connected with the reality of the event, which, as we know, doesn't exist.

The reality of the market is the invention of a new matter which cannot connect with the statistics. It is the new matter the event is made of. But while it is easy not to speak of the event (because it doesn't exist), it is difficult not to speak of the market because there is matter and we feel we have something to say. On the other hand, it is difficult to speak of the market and of this matter (this reality of the market) because, as we said, the derivatives prices are connected neither ontologically nor epistemologically with the underlying process.

There probably is such a thing as a time series of the underlying (at least, in the eyes of he who has waited long enough to watch it unfold), but who cares? As we review it in mind and realize that volatility wasn't constant but stochastic after all, then that the volatility of volatility wasn't constant either and, as a matter of fact, that the path wasn't even continuous but admitted of jumps, we keep upgrading the dynamic replication strategy that was supposed, in the ideal end, finally to replicate a given contingent payoff. When we realize that volatility wasn't constant but has always been stochastic, and later that the volatility of volatility has always been stochastic in its turn, we backtrade and declare that the dynamic replication strategy should have included a further instrument that would ultimately replicate the target contingent payoff, then a further instrument still. If we imagine the time series of the underlying, then we are tempted by the logic of constant correction and revision of what prior random generators we had in mind. Time's most powerful illusion is that it flows and we can read series in time. *It is in the register of this illusion* that we imagine that the market prices of derivatives will ultimately catch up with the fair value that the ultimate replication strategy had in store for them, or, at least, that their buyers and sellers form such expectations. The truth is that their market (their price) preceded their value, even logically.

We are still prisoners, at the back of our minds, of the picture of ultimate convergence and ultimate arbitrage, and of prices being the temporary passage towards value. We keep expanding, backwards in time, what the final picture should have been (volatility has always been stochastic, and so on), convinced that such a picture exists. Now, it suffices to realize that the contingent claims trade for the reason that they weren't supposed to trade or, rather, for the reason that they don't exist in a framework where they were supposed to be redundant or, again, for the reason that their writing was invented and their market was created through this void and jump outside the formalism which are only due to the layering of history and the book, in order to realize that their trading is original and happens with force – in order, that is, to understand that it is totally parasitic upon the original existence of their market that we should wonder whether their prices will converge towards a programmed value, or whether somebody thinks of them that way.

Precisely because of the void and the narrow coincidence, BSM caused derivatives to trade without connection with what formal time (i.e. time as registered in the formalism) had in store. The time of their market is different and is totally *separated from the time of the time series of the underlying*. This is the only way how a seemingly statistical parameter, volatility, can be put to trading. We know what it means to trade the underlying originally and with force, but it somehow amazes us that a

statistical parameter (pertaining to a whole time series) should be traded. We can only imagine its trading against the background of statistical convergence and learning. Conversely, it is difficult to imagine the original trading of derivatives (as the arbitrage constraint pressures us to re-establish the frame of states of the world and ultimate convergence). As a matter of fact, the statistical parameter the trading of which amazes us doesn't really exist. Instantaneous volatility is a mental construct or a reification. We always observe an empirical time series and it is we who postulate a random generator (volatility, jumps, and so on). The time series is always ex-post. Probably, it is for this reason that only written payoffs can be traded as contingent claims and theoretical parameters are never written.

It is a profound thing (and not just a coincidence) that writing the payoff, say, of a variance swap should have the same discrete and empirical nature as the actual ex-post time series from which realized variance can be calculated. The argument from the replication of the payoff requires the theoretical parameter of instantaneous variance and, for the reason of theory, the contingent payoff is redundant. But precisely because it is written discretely and empirically, it will escape in its own market as a *contingent claim*. It is because the variance swap has realized variance inscribed on it and not a theoretical instantaneous variance that trading it is justified and that, owing to the presence of jumps, its price will diverge from the theoretical replication value.

The time series entering in the computation of realized variance is always ex-post; for this reason, it is *matter*, it is statistical, it is written (as I have already said, concerning statistics). Equivalently, it can be written on paper, as the payoff of a variance swap – written on one undivided sheet. Similarly, it is always ex-post that we realize whether the underlying has settled above or below a certain strike level at a certain maturity. This can be written, too, and the writing can provide for the ex-post nature of the thing through the conditional and the ex-post nature of written testaments (if above the strike, then pay this; if below, then pay that). No wonder the condition and the difference it expresses requires writing, and writing immediately makes for a material (the written sheet) which is matter and is, thereby, ex-post.

That the time series should effectively be given in time, in the chronological probabilistic picture, and should constitute a statistics is mediated by writing, in the alternative picture, and by its immediate translation into price. Price is the substitute for time when written statistics (and statistics are always written) are effectively written on paper. The reason for which it is always too late 'theoretically' to 'construct' a time series and for which, given the time series, the theory can only be revised and reconstructed backwards is the same reason why the written contingent claim can always be reconstructed as having not been redundant or replicated. Theory always plays in the background and in a retreat. (As once said, the detour to the market should be taken directly from the immanence and matter of statistics, not after probability.)

The probability tree is backward and the corresponding time is always back trading, trying to reformulate what 'has always been' or 'should have always been'. The way to release the market forwards is precisely through the historical layering which traverses the formulas and formalism. We shouldn't be amazed that statistical parameters are traded, because they are not. Only contingent claims are traded and the impossibility of capturing the ultimate backward tree is made possible in

the forward way by the possibility of trading of all contingent claims – because they are written, because we have the tools to replicate the contingent payoff, and because we cannot go back and revise the tool (due to the writing barrier, which introduces an incompatibility of contexts and, therefore, irreversibility). The total market (the chaos) is possible through the genesis, and the genesis is, in turn, possible through the layering.

Markets are said to be complete when you can buy (that is to say, one sells) contingent claims spanning the contingencies. It rains or it shines tomorrow and the contingent claim paying off \$1 in case of rain and 0 otherwise is available for trading. The market is complete when the riskless bond and the contingent claim span the contingencies. When the contingencies are the up and down price movement of a traded asset, the market is by definition complete because we are talking about price movement in the first place, which means that an asset is tradable, and, surprise, it pays its own price in the two states of the world, thus spanning the contingencies when combined with the riskless bond.

A market which we have assumed to be incomplete cannot be completed. If contingent claims spanning the contingencies are not traded, what is to make them trade? To repeat, if the only contingencies are up and down prices movement of a traded asset, by definition the contingencies are spanned by a traded contingent claim. Derivative pricing theory tempts us into thinking that the incomplete market can be completed. When the market is complete in derivative pricing theory, there is no mention of contingent claims or of their market. Nothing needs to be done. When the market is incomplete in derivative pricing theory (and this can only be achieved through complications of the underlying process), nothing can be done. Even contingent payoffs are not replicable. However, now we cheat. We presuppose that contingent claims exist as independent lotteries and we value them by non-arbitrage by selecting one martingale measure. But this is no market. The market hasn't done anything; it wasn't completed, because the contingent claims that *we* (and not the market) have valued are still not traded.

A forcing is committed nevertheless – or, rather, an equivocation. We save ourselves the trouble of selecting as we pose the derivative pricing problem already in the risk-neutral measure. Then, we imperceptibly slide to the market, because now that there is no choice (we covertly didn't allow for any), we assume the choice has already been made by the market. We imperceptibly act as if the risk-neutral measure was already the pricing measure of the market. Wasn't our purpose the pricing of derivatives? Now that we have 'priced' them, how far are we from considering that their 'price' is their market price?

It all starts with giving the stochastic process of the underlying. In this, we have already committed ourselves to a certain register of trading, which compels us to think that derivatives are ultimately traded. The stochastic process is confusingly very similar to a trading process. There seems to be no time (for the time series) other than the market time, so the equivocation leads to assuming that, since the stochastic process of the derivative will derivatively be given in time, too, then this process will also be a trading process. When contingencies were abstract (rain, shine, general states of the economy), there wasn't this unavowed drive towards trading. Not to mention the other confusion. If the trading of the underlying is considered, if the market is considered, then who was able to exit

from it in the first place in order to contemplate its statistics and to consider stochastic volatility or jumps, or the complications that we have mentioned? This is just a formal facileness. Simply because the process is formally written, we imagine its volatility is formally stochastic. But never any of this was imagined *in reality*. The image of reality is important. We are imagining the market, no?

15.4 Coda

There is a feeling of void and melancholy in our field because of the absence of matter. Ours is not a physical industry but a metaphysical one. The whole void and lack of foundation is inscribed in the apparently unnoticed finding that the contingent payoff is not equivalent to the contingent claim, and that complicating the underlying process through probability (which seems to be the only way, but really is not) leads to an exit from the market, since derivatives will no longer trade with force. I prefer to reserve the term 'contingent claim' for the argument from the void, the one in which contingent claims are precisely not equivalent to contingent payoffs and are synthetic (not analytic) pieces of paper, while keeping the term 'derivative' or 'contingent payoff' for the framework of probability where the underlying process is just differentiated in time. In *The Blank Swan*, I said that the whole book was trying to find the meaning of implied volatility. This is the same void as the blank interval between contingent payoff and contingent claim. The melancholy is the realization that nothing exists in the blank and that matter has to be created anew. Hence, the book.

Denouement: The Theory after the Two Narratives

I

Einstein once built a theory by postulating that the speed of light was the same in all frames of reference. I propose to build a theory of the market in which the price of the underlying, and the price of the derivative, although numerically different, are of the same nature.

But first, why should the derivative be a constituent part of the market? Why should the definition of the market even reside in the fact that the underlying and the derivative trade alongside each other without any distinction of degree or nature? Because it is too late now and we have a derivatives market. Derivatives are definitely what the market is all about. It is time we grew up and no longer satisfied ourselves with the antiquated order of presentation where the underlying asset is the only thing that trades and the derivative is just *valued* off the underlying stochastic process. Valuation is not trading. If the only instance of price is the price of the underlying, and if the derivative is only ever valued, then it is not priced; then the 'price' of the derivative must refer to something else, which is external to our hypothesis. As a consequence, the price of the underlying and the price of the derivative will not be of the same nature.

We live in a post-BSM world. The BSM technology and its ramifications have helped to establish the trading of derivatives. It is true BSM assumes that the underlying process is the only trading floor and the valuation of the derivative is something extra, something above the floor, something almost detached from the market and taking place in a different realm, as it were. But, precisely, a contemporary theory of the market should move past this stage and include this history as constituent part. It should almost have it wired in its axiomatic system that, although the theory of BSM does not account for the derivative trading but only provides for its valuation, the *usage* of BSM is the trigger and the amplifier of the derivative trading. How the material usage of one theory (BSM) could become part of the axiomatic system and formalism of another theory (the contemporary theory) is precisely our problem.

It simply should not be acceptable that, on the one hand, the format for price – and, accordingly, the format for the market – should be one in which the price of the traded asset follows a stochastic process without any deeper explanation of the manifest randomness than mere trading and that, on the other hand, derivatives should be brought in contact with that picture, and should even lend themselves to trading and pricing, yet without respecting the same format.

If the picture is a picture of trading and if to be in that picture means that a stochastic process for the price of the traded asset is the first given, then derivatives will not belong in that picture unless the first given concerning them is a stochastic process summarizing their trading, as well. Some have proposed models for derivative trading in which, initially, two stochastic processes are given: one ruling the underlying asset price, and another ruling the BSM implied volatility of the derivative of interest.¹ Options traders trade implied volatility, so why not model directly the ‘price’ process of implied volatility?

Unfortunately, arbitrage constraints, combined with the fact that the derivative value ultimately derives on the underlying price and is a mathematical function of that price, prevent the picture from ending there. They prevent supply and demand of the derivative (what I have called its ‘mere trading’) from being the ultimate engine.² The arbitrage constraints cannot but slip an underlying cause under the surface phenomenon of the derivative trading. Typically, the instantaneous volatility of the underlying is assumed to be stochastic and this becomes the only reason why the derivatives prices (or rather, values) move in a dimension of their own, for a fixed underlying price.

To repeat, if the format for trading is one in which the price of the traded asset is the state variable, then the derivative seems unable to fit into that format. Derivative prices seem unable to become state variables alongside the underlying price. Arbitrage compels us, systematically, to postulate a deeper state variable, *which pertains to the underlying alone*, in order to explain the variability of prices of the derivative, typically the instantaneous volatility of the underlying, or the size of its jumps, and so on.

II

There can only be a theory of derivative valuation and it doesn’t connect with the market. As soon as we shift our focus to the valuation of derivatives, we exit from the market. Suddenly, the underlying price process is no longer a trading process; it becomes a stochastic process. Suddenly, probability takes over. By that, I mean that the trading action is extinguished and all that remains of the market is a coincidence between underlying prices that ‘used’ to be generated by the exchange and the intensity of the exchange, and numbers that are generated by a random generator. These numbers are quantitatively the same. For this reason, insofar as the underlying prices are used in the derivative valuation problem, for all we care they can be generated by a random generator.

Derivative valuation is equivalent to the valuation of a lottery. It looks as though the prices of the underlying are the underlying states but, in reality, they are not. The derivative and the underlying are now equally lotteries, equally remote from the market, mere devices that pay out $f(S)$ for the one and S for the other. The real underlying is now the probabilistic elementary event ω , also called state

of the world, which is an element of the universe of possibilities Ω . A presentation like Rebonato's makes this very clear.³ It is only a coincidence if the states of the world ω correspond to prices of the underlying S . An omniscient and totally indifferent being has now taken over the task of generating the randomness. It is no longer the case that randomness is the consequence of the market (EMH). Now, there is randomness, first, and the market is just one of its cases.

Non-arbitrage implies that the value of the lotteries is equal to an expectation. It is now only a coincidence if the value of one lottery, namely S , is the current price S itself. Under the schema of probability, the non-arbitrage value of the lotteries can be anything, provided it is expressed as an expectation; however, in the present situation which happens to be a market (as if by coincidence), we just know that the value of something that trades originally and is random on account of this trading is, well, equal to the traded price. Look at the roundabout way of conceiving the market and price that the probabilistic picture forces on us! As a matter of fact, the states of the world may not be coincident with a price. A general derivative valuation problem could be one in which weather derivatives or credit risk derivatives are considered.

III

It seems to me there are two ways of presenting the problem, which are incompatible with each other. The literature has consisted so far in confusing the two – or, rather, it did not see the point of distinguishing them. Some readers may argue the distinction is purely metaphysical – hence, unimportant. I believe it is very important. This is not metaphysics; this is semantics; this is distinguishing the registers of discourse. This is exactly watching over the formalism and what it has to say – what exact interpretations are overlaid on it.

The two incompatible presentations are the following: either states of the world and their probabilities are considered first, or the trading assets and their prices are considered first. Either randomness is considered first – generated God knows how and of which it is only one possible and *later observation* that prices of an underlying trading asset are coincident with its states (and along with this initial randomness general lotteries are considered, whose payoffs are defined in all future states of the world and whose values are studied in the present state) – or the market is the only thing that is considered, the only first and the only last, the only source of randomness and the only horizon. Either the problem is one of valuation of lotteries, or it is one of trading, in which case there should be no valuation but only transactions and prices.

I am saying that when the stochastic process ruling the price of a certain traded asset S is given, as in BSM, you have to decide in which register of discourse you want to situate yourself, never to leave it again, on pains of inconsistency. If S is given as trading, then it is the only thing that exists. No contingent claims written on S are given or known to exist. If you disagree and insist they are given, then – if we may ask again – where is the trading process that is initially given for them? Let us not forget that to be given, in this picture, is to be given as initially trading. If we are immersed in the market of the underlying asset as only picture and only horizon, we cannot, unless we exit from the market, suddenly change our mind and entertain the thought of valuation of derivatives that are

independently given and waiting to be valued. (Of course, nothing stops us from considering the trading of two underlying assets S_1 and S_2 , or more, as initially given.)

As a matter of fact, if the logic of BSM is strictly followed through, in which the asset S and the bond B are the only things that trade, we find that contingent claims written on S , or derivatives, have never been replicated by S , as we have all believed, and consequently valued. In fact, they have never existed! What the dynamic replication strategy guarantees is the manufacturing of *contingent payoffs*, not contingent claims. The distinction is crucial. In investing the initial amount of money, which is known as the ‘option premium’ and is computed by the BSM algorithm, in the dynamic trading strategy involving the underlying asset S and the riskless bond B , you end up, at any expiry date or knock-out barrier of your choice or, indeed, at the end of any path of prices of the underlying S of your choice, with the amount of money that is a predefined function of the price of S at that expiry date or at that barrier or, indeed, a function of the whole path. You end up generating the contingent payoff of your choice. It is a completely different thing to write, in advance, this contingent payoff on a piece of paper and to call it a *contingent claim*, which therefore becomes a contract binding two parties, and is ipso facto open to trading in its own independent market.

In their 1980 article, ‘Martingales and stochastic integrals in the theory of continuous trading’, Harrison and Pliska make it very clear that to consider that parallel market and the independent writing of the contingent claim is *verbiage* relative to what they have to say.⁴ Harrison and Pliska’s is the first article to reconsider the formalism of BSM in a fully consistent manner. The authors explicitly say they are after a ‘self-contained mathematical theory’. As no market and no counterparty are present for the contingent claim, it is no wonder that all that Harrison and Pliska are able to say is that investors ‘manufacture call options *for themselves*’ (my emphasis). The asset S and the bond B are the only existent things, and the market and the world stop at them. When it is reduced to its formal mathematical core, which is the martingale representation theorem of Brownian motion, and consequently formalized by Harrison and Pliska, all that the BSM theory establishes is that the dynamic trading of S and B is equivalent to the manufacture of contingent payoffs. This is a result that has only to do with S and its trading.

IV

Now, to move to the register where probability and randomness are given first and where it is only a coincidence if the prices of a traded underlying asset S are identical with the states of the world; the picture is now open, as we said, to the existence – and, therefore, to the valuation – of any pre-written lottery. If the lottery admits of a monetary outcome, then the outcome may be expressed as a function of the price of the underlying asset, because the latter is expressed in money, too. Non-arbitrage is the only constraint, combined with the circumstance that the present value of one lottery is known (the one that yields the future random price S). Now, it is only a coincidence if the corresponding martingale measure is unique under Brownian motion. No one is *trading* the underlying S and replicating perfectly the contingent payoff as a result (as in the previous case). Here, the random generator just happens to be Brownian, and the value of one lottery just happens

to be the price S , and the equivalence between the martingale measure and the real measure, which ensures non-arbitrage, just happens to preserve the volatility number by Girsanov's theorem and to yield a unique martingale measure as a result. There never was a market. There never was a trading core, but only a series of outside coincidences.

So, what is the real (intuitive) reason why the value of derivatives is uniquely determined under Brownian motion? Is it the argument from perfect replication, or is it the argument from the uniqueness of the equivalent martingale measure? Of course, the arguments are mathematically the same (or, at least, mathematically reducible to each other), since perfect replication is due to the martingale representation theorem of Brownian motion and the uniqueness of the martingale measure is due to Girsanov's theorem applied to Brownian motion, and there is mathematically only one Brownian motion. Probably, one could use this formal-mathematical equivalence of the two arguments to argue for their semantic equivalence.

Indeed, one can consider it as a *definition* of trading that the value of the lottery paying out the future price of S should be equal to the current price of S ; or, to put it differently, one can consider the exceptional circumstance in which a thing, S , is moving randomly and is, at the same time, the ticket to buy the corresponding lottery as the very meaning of the trading of S and of the existence of a market. For this reason, it may look as though the decisive argument for the uniqueness of the martingale measure – namely, the fact that the value of one lottery, S , is determined and is equal to the price of S – is the same argument as the trading of S . It may look as though invoking the uniqueness of the martingale measure under Brownian motion is the same as invoking the dynamic trading of S under Brownian motion and, therefore, the perfect replication of derivatives by S . However, we know that the two arguments are not really the same because they do not fall in the same semantic register. To repeat, there is equivocation about the inclusion of contingent claims in the picture or their exclusion from it. In the register of dynamic replication there are no independently written contingent claims, or lotteries, and it is only the mathematical formalism, in whose eyes there is only one Brownian motion and only non-arbitrage as quantitative constraint, which is confusing the two registers.

The mathematical formalism does not see the difference between contingent payoff and contingent claim. It doesn't see the material sheet of paper on which the contingent claim is, as we said, independently written and *for this reason* subject to its own (parallel) market. All it sees is the content of the paper, the mathematical expression of the payoff which is the same in the case of the contingent payoff and in the case of the contingent claim. Also, the formalism doesn't see the difference between trading process and stochastic process, between value and price, because all it sees is probability and the principle of non-arbitrage.

V

And what is so special about Brownian motion anyway, which, as we saw, is the key factor in confusing the two registers? Perhaps the distinction between the two registers should be made, first, inside the concept of Brownian motion itself. Maybe there isn't one Brownian motion, as we said, but

‘two’ Brownian motions – or, rather, two views of Brownian motion; one I should call the qualitative view, and the other the quantitative view. In the qualitative view, Brownian motion is native to trading. To trade is essentially for the price to go up or down with non-zero probability in either direction at any time scale, and Brownian motion is the continuous-time limit of this binomial walk. Up and down are only directions in the qualitative view. They are qualities, not quantities. We do not specify the magnitude of the up or down movement, therefore we do not specify a quantity known as volatility. ‘Volatility’ is only qualitative for now and is synonymous with trading. To complete the qualitative view of trading, a dimension other than the direction of movement of price (up or down) is required and this is the size of the trade. Let us try and think, for now, that the size of the trade is not quantitative either. Just as the intuitive meaning of trading involved going up or down, it essentially involves the variable size of the trade as a quality, too.

Trading is not just the random walk of a number known as the price. It involves the material exchange of an asset, and this is expressed by the size dimension. Now, in such a qualitative picture, where we only discuss the essence of trading (up or down and the size as qualities), it should be intuitive that the trading of S suffices to span the ‘states of the world’ that are essentially produced by the trading of S ; namely up and down. In other words, this is not a coincidence, due to an observed and therefore accidental equality or inequality between the number of states of the world and the number of traded assets. The binomial tree is qualitatively (and not quantitatively) different from the trinomial tree and the rest. They differ in nature, not in degree or number of states.

It is only when the probabilistic view takes over and fixes everything in states of the world that outside coincidences are *later* observed, and the binomial tree and the trinomial tree are quantitatively compared. By contrast, the trading of S , when it is considered in the full qualitative sense of the up and down movements of the price and the variable size of the trade, is essentially equivalent to the manufacture of payoffs that are contingent on S , and this, before the quantitative introduction of probability and states. What I am trying to do, here, is to understand the martingale representation theorem for Brownian motion qualitatively, not quantitatively. I am trying to interpret the perfect replication of contingent payoffs as the perfect equivalent of the trading of S . What I am trying to say is that, in the qualitative view of trading and of Brownian motion, the market is always complete.

It is only when we move to the quantitative view that the market becomes incomplete. Quantitative measure is introduced and we can now distinguish, for instance, between two up price movements of different magnitudes. Now, the trinomial tree, or any tree with more than two nodes, can be introduced. Now, volatility becomes quantitative and can itself become stochastic and entail a further separation of states of the world. Now, jumps can be introduced – for instance, in a trinomial tree featuring a node that is considerably more distant than the other two. Accordingly, Brownian motion will no longer be native to trading and becomes a stochastic process like any other.

VI

It is my view that incomplete markets are a heritage of the times that preceded BSM and that it is only by the same confusion of registers that we speak of incomplete markets after BSM. Typically, the

pre-BSM literature talks of the abstract states of a thing called the ‘economy’, the ‘contingencies’; then, it considers a thing called the ‘market’ and it compares the number of traded assets with the number of abstract states. BSM changed everything when they considered the market of the underlying – that is, its stochastic price process – as the only economy. An incomplete market could no longer invoke states of the economy whose number exceeds, by fiat, the number of traded assets. All it could do is argue, as we said, that the true underlying motion is not diffusion with constant volatility but a jump-diffusion, or that volatility is stochastic. But since when has the ‘true’ motion of the underlying asset been observed? Do we now have to resort to statistics and to time series in order to find the ‘states’ of our economy? What kind of a market is this, where we can suddenly retreat into an attic and take the time to study time series? Or maybe the times series have just been postulated?

We can see that this whole view is in trouble because it is soon to fall victim to its own contradictions. It is certainly acceptable to do statistics and consider time series of the underlying price. It is also perfectly acceptable to postulate a random generator as a result. When the underlying motion is not Brownian with constant volatility, an infinity of equivalent martingale measures is admissible and all we have to do is select one in order to prevent non-arbitrage in the valuation of lotteries. It is perfectly acceptable to exit from the market in this way. But this isn’t what the literature says. The literature says: The *market* selects the martingale measure when the underlying stochastic process is incomplete.⁵ What market? The market of the underlying asset S ? How could the underlying market suddenly turn into an evaluator of derivatives? Or is it the derivatives markets? And when exactly has such a market been introduced? Once we exit from the market and start viewing it only probabilistically and statistically, it is we who select the martingale measure with which we value lotteries triggered by the corresponding random generator, not the market. In any case, this *deus ex machina* choice of the martingale measure cannot be made part of the initial formalism of the market.

We all know that the pricing of derivatives and the manufacture of pricing tools for the derivatives trader are the real purpose of the quantitative articles that have followed in the steps of BSM. We all know that these articles have all, ultimately, in mind a market for derivatives. Only this is tacit knowledge; it cannot be formalized within the article. It is only tacitly that these articles think of a derivatives market, but they think it so hard that, as a result, we give up and say: they may as well postulate it from the beginning! But to repeat: has the price process of the derivative been given initially, similarly to the underlying?

To add to the confusion, now that the market has supposedly selected the martingale measure, some articles further extend the argument – for what is to stop them now? – by turning this ‘valuation’ of derivatives by the market properly into a pricing, therefore a trading, and they argue that the market can now be completed with the help of the derivatives as additional trading assets. By (improperly) mixing tacit knowledge with formal knowledge, it seems the literature has found the recipe always to end up with a complete market! Doesn’t the majority of articles now begin with the sibylline statement: ‘Let the underlying process be the following in the risk-neutral measure’, thus brushing aside all questions relative to the provenance of this martingale measure, of whether we are in the business of valuation or the business of pricing, and of whether we are inside the market or outside the market? And doesn’t the exercise subsequently consist in calibrating the parameters of

this risk-neutral underlying process from the prices of derivatives, thus confirming that the market of derivatives has always been there?

When the abstract states of the economy used to be the background, one could always argue that these states were so abstract that contingent claims spanning them could not even be written, let alone traded. These states were maybe so abstract that we only knew they existed without knowing what they were and how to represent them. By contrast, when the only background is the explicitly given stochastic process of the underlying – as this should be the case when the valuation of derivatives written on it is at issue – and no matter how complex or incomplete this process, what is to stop us from writing exactly the derivative that would span the exact contingency; that is, the variance swap to span the stochastic volatility, the gap option to span the jump, the option on variance to span the stochastic volatility of volatility, and so on? And once these derivatives are conceived and written, and given that their market is our sole preoccupation, what is to stop us from assuming that they trade?

VII

We know exactly where to draw the limit in order to stop this verbiage and proliferation. We should stick with the self-contained formal theory that Harrison and Pliska underline. We have to keep in mind that any contingent claim that is subsequently written is destined to valuation, or to the ‘parallel market’ that Harrison and Pliska talk about, so it cannot be re-embedded in the original underlying market. The original market and the parallel market cannot fuse into one market unless the word ‘market’ is so loosely understood as to become meaningless. A stringently defined market cannot, at the same time, be incomplete and complete itself; it cannot select an equivalent martingale measure for itself and claim, as a result, that the equivalent martingale measure is unique. Now that the derivatives that had been valued by the market have become traded assets in a sufficiently great number to complete the market and to determine uniquely the martingale measure, how indeed can we make sense of the fact that there were originally infinitely many equivalent martingale measures?

Following strictly the formalism and the distinction we have made between contingent payoffs and contingent claims, we should therefore stop at the conclusion that contingent payoffs are no longer perfectly replicable when the volatility is stochastic, or when there are jumps in the underlying price, and so on. This is all there is to say and this is still a result that has only to do with the underlying. No contingent claims will ever have been in sight. There is no hope of completing the market because no additional assets can be introduced. Derivatives written on the underlying are potentially the only candidates for spanning the new contingencies of the ‘economy’ – namely, the stochastic volatility and the jumps of the underlying – yet, price processes can never be given for them without turning into valuation processes. The derivatives can never be introduced in the formalism as originally trading. They can only be valued derivatively, pending the choice of the martingale measure, of course – a choice that the market is formally never able to make.

Before we ask how to complete the underlying market when it is incomplete, we should worry how to introduce new trading assets in the first place. How, indeed, to get to derivatives prices

before derivatives values get to us? Derivative valuation corrupts everything as it immediately throws us outside the market and turns the latter into a mere random generator. Derivative valuation immediately puts us in the situation where the true underlying state is no longer the underlying price or the underlying market, but the general probabilistic state over which all kinds of lotteries are defined, one of which is particular and is equal to the underlying asset price. Even when the underlying motion is Brownian, to consider derivative valuation is to exit from the market. (By the way, this is precisely what Black and Scholes do in their 1973 original article. They consider the option as independently given and, therefore, consider its valuation, not its market. They postulate that its value is solely a function of S and t , thus confirming from the start that there are no market forces specific to it that can pull it away.)

So, how to remain confined within the formalism of the underlying market (how never to exit from it) yet, at the same time, to open it? I believe the answer is that derivatives, or contingent claims, should never be considered independently for, if they were, they could then only be submitted to valuation. Rather, they should be created from inside. A great virtue of a formalism like Harrison and Pliska's is that contingent claims do not exist in it, but only the underlying and the contingent payoffs that are dynamically attainable by the underlying. And here is the key finding: if contingent claims do not exist, then they can be created. And hopefully their creation from inside a formalism that is shut off to valuation will keep them insulated from valuation.

VIII

The qualitative picture which establishes the equivalence between the trading of S and the replication of contingent payoffs is shut off to quantity, on the one side, and to the contingent claims, on the other side. It is shut off to making volatility quantitative, on the one side, and to materializing the contingent payoff into a contingent claim, on the other side. More generally, we can say that it is shut off to the realm of probability (in which alone volatility can become stochastic and jumps can appear), on the one side, and to the material realm (in which alone the contingent payoff can be written on a material sheet of paper and make a difference as a contingent claim), on the other side. So, here is my proposition. We should exit the qualitative picture from the other side. We should not open the probabilistic door. Instead of throwing the qualitative Brownian motion in time and putting quantitative measure on the diffusion, eventually making its volatility stochastic or adding jumps to it, and so on, we should use the essential equivalence between the trading of S and the replication of contingent payoffs and try to open the other door.

It is only when volatility is quantified that one is able to compute the premium initially to invest in the dynamic trading strategy in order to manufacture the contingent payoff, but alternatively one can consider another quantity, which is also absent from the original qualitative picture and which is the price of the corresponding contingent claim!

Our discovery is as simple as it is astonishing: Contingent claims have never been valued inside the formalism of Harrison and Pliska; only contingent payoffs have been manufactured and the premium to do so has been computed. Now, if the contingent payoff is interpreted as a contingent

claim *at the same time as* the premium to manufacture the first is interpreted as the market price of the second, then we would have extended our formalism with prices of the contingent claims without ever considering their valuation.

Obviously, we are forcing the formalism here; we are opening it. An interpretation of the formalism is always a step outside the formalism. It is an equal forcing of the qualitative picture to interpret the qualitative volatility in time, therefore to start becoming aware of time series (stochastic volatility, jumps), or to interpret (or force) the contingent payoff on paper, as a contingent claim. How the contingent payoff becomes a contingent claim whose price is available directly without there having been the intermediary step of valuation, is by inverting the BSM formula. This, we think, is the meaning of implied volatility. Option traders use the BSM formula to trade options, not to value them. To trade options is to imply volatility from their prices.

IX

Let me try to formulate this again. Our purpose is to formalize the market of contingent claims, to make it formal and not just empirical that the derivative price exists and is of the same *traded* nature as the underlying price. Our purpose is to find a theory of derivative pricing that does not collapse into a theory of derivative valuation. Of course, the derivatives market exists empirically, and the derivative and the underlying trade empirically alongside each other; however, the formalism cannot start by initially giving the two corresponding price processes. As we said, non-arbitrage would then impose that the derivative price process is recuperated as a valuation process and is read out from a deeper stochastic process that has nothing to do with the derivative but only with the underlying, such as stochastic volatility, and so on. As a matter of fact, when the derivative is so much as considered to be written in advance, any variation of its value that we may wish to represent in order to model its market can only consist in shaking the model we've had until then of the underlying process and in saying: 'Wait a minute, volatility is stochastic and not constant; there are jumps and not only a diffusion; the market is incomplete, not complete.' The derivative can never break free into its own market; it is always swallowed back in an ever-enlarged picture of the underlying process, in which new states of the world are added – stochastic volatility, jumps, and so on – only to offer further valuations of the derivative and never further prices.

The only picture in which there is not a hint of valuation but only trading is the one at the beginning of BSM (as formalized by Harrison and Pliska), in which the underlying price process is the only process. This picture is our preferred formalism. And now it appears as an opportunity, rather than a restraint, that contingent claims should not exist in this formalism. Writing should be invented from within the formalism, not given in advance. It is not volatility that we need to alter and make uncertain or stochastic in order to open the underlying market to contingent claims; it is the contingent payoff, whose manufacture, we said, is equivalent to the underlying market, that we need to write as a contingent claim. What should be varied is not a quantity (volatility) but a quality – the quality of a thing as it shifts from being a contingent payoff to being a contingent claim.

Counterintuitive as it may be, the idea should really start to make its way into our minds that the equivalence between the trading of S and the manufacture of contingent payoffs occurs at a

pre-quantitative/pre-probabilistic stage, a stage which precedes the notion of stochastic processes altogether. Two distinct and mutually incompatible roads present themselves from there: either to differentiate this 'embryo' into the full organism of stochastic processes of the underlying S and their successive complications (stochastic volatility, jumps, stochastic jumps, stochastic volatility of volatility, and so on), or to differentiate it into the full organism of derivatives written on S in the successive complications of their payoff structures and their price structures.

Trading options with the BSM formula does not mean that volatility has to become stochastic; it means that the contingent payoff has become a contingent claim. As we said, the usage of BSM should become part of the formalism of the market of contingent claims. Traders do not use BSM because it replicates perfectly the contingent claim – for, if it did, what could be the *raison d'être* of its market? – they use it because it replicates perfectly the *contingent payoff*. They use BSM because volatility is *known and constant* in the formalism of replication of contingent payoffs, not because volatility is uncertain or stochastic in the market of contingent claims. However, the practice of the usage of BSM is to invert its formula and to compute implied volatility.

Implied volatility belongs in a realm that is separate from the formalism. It has no connection whatsoever with the formal volatility that we read in BSM and should not be identified with it. Implied volatility can be stochastic – as this is what a market of contingent claims means – without entailing that we should revise the model and make its volatility stochastic. When they imply it, traders recognize the alternative road – the one in which the contingent payoff becomes coincident with the contingent claim and in which there are no stochastic processes, no probability, no valuation and, consequently, no principle of non-arbitrage. Non-arbitrage pertains to the tool, not to the matter to which the tool is applied.

Creating this matter, or extending the market of the underlying into a market for contingent claims written on it, or, again, formalizing the market in the full sense of the term – which is that the underlying price and the derivative price are of the same nature – therefore does not proceed along the line of underlying stochastic processes of increasing complexity but along the line of recalibrations to contingent claims of increasing complexity. Although in empirical reality the price of the underlying and the price of the derivative are of the same nature and mingle perfectly on the trading floor, in the formalism we need on the contrary to separate the second from the first by a wall or, rather, by a jump outside the formula. The jump occurs when we force the contingent payoff, which had only to do with the trading of the underlying, into being an independently written contingent claim, or derivative.

We must avoid at all costs that the price of the derivative and the price of the underlying should be immersed in the same space of states of the world, for otherwise the derivative would only be valued off the underlying process and its price would reduce to a value. Forcing the contingent payoff into a contingent claim precisely breaks that space. Crucially, forcing the contingent payoff into a contingent claim does not amount to adding a new range of states of the world to the existing one, as when volatility from constant becomes stochastic or when jumps are added. It amounts to changing the whole world and leaving the whole space.

A formalism of the market of contingent claims therefore requires recalibration processes, not stochastic processes. It is a meta-formalism vis-à-vis the formalism of stochastic processes because

what is being recalibrated is a stochastic process. This is how the *usage* of BSM (and of its cognates, such as stochastic volatility models or jump-diffusion models, and so on) can become part of a superior formalism. Implying volatility in BSM is the first instance of recalibration. It is time we understood that implying volatility from the market price of a contingent claim is not information that benefits to the knowledge of the underlying stochastic process – inferring its parameters somehow. On the contrary, it is moving to the plane where there no longer exists such a process. By contrast, if somebody could tell us how much it costs to replicate a *contingent payoff*, then this would be actionable in the study of the underlying price process and we could invert the BSM formula to infer the volatility of the underlying.

X

In conclusion, we tried very hard to introduce the price of the contingent claim in the formalism without it becoming a valuation and we couldn't. We found that the only way to do so was by forcing it into existence, and this meant introducing it through the pass where it had to be invented and to abolish, in the process of its invention, the whole quantitative idea of states of the world and probability attaching to the underlying. The usage of BSM literally *makes* the market of the contingent claim, but this creation of matter literally blocks any communication with the domain of extension of the stochastic process.

The BSM model is the perfect tool to replicate contingent payoffs which look perfectly like contingent claims, except that the latter do not exist and the whole operation takes place in a domain where the trading of the underlying is the only concern and the only horizon. And when the model is 'readjusted' to account for the contingent claim and its market, this takes place in total antagonism with the stochastic process and its domain of extension. We need the abolition of states of the world and probability every time the price of the contingent claim is introduced, for this is what ensures that it will never collapse into a valuation. For there to be a price for the contingent claim, and for this price not to be equal to a value, the price of the underlying has to be suspended from being a random number and from following a stochastic process. Price, ultimately, is not a number.

Conclusion

This book is ambitious because it aims to redefine the market. Not to define it from previous or first principles, but immanently and implicitly, from the matter that the market has already produced (derivatives). The underlying trades *and* the derivative trades. From recognizing what they have in common – namely, trading – as one single principle, rather than a hierarchy, we implicitly define the market. If we may summarize our results:

1. What we propose in this book is not a theory; it is a technology. There can be a theory of probability, a theory of Brownian motion, or a theory of valuation of derivatives, but there cannot be a theory of the market of ‘contingent claims’. In our excursion from form into matter, derivatives are renamed contingent claims and their valuation theory is renamed their ‘market’, or their ‘pricing technology’. To repeat, the market of contingent claims is a technology.
2. Derivative valuation theory, when it is complemented with the crucial process of recalibration, can no longer be a theory; it becomes a technology (i.e. theory + material procedure). Implied volatility from the derivative market price is the first instance of recalibration. It says that BSM should be adopted and simultaneously exceeded.
3. Another formulation of the recalibration problem is the *fundamental principle* according to which states of the world in the market are prices, all the prices and nothing but the prices (i.e. they are not abstract states of the world). You cannot couch this principle in a probabilistic theoretical framework.
4. The notion of price is incompatible with the notion of underlying state. The market (even, and especially, when ideally thought) is not a theory. Philosophy, or meta-theory, is then faced with the choice either of becoming a philosophy of technology, or of becoming a metaphysics. Either you argue that recalibration exceeds theory and can only extend into a technology, or you dig deep in the categories of thought to understand the metaphysical underpinnings of the notion of state and why price contradicts it. We have attempted both.
5. Any provider of a derivative pricing technology (as opposed to a financial theorist writing a theoretical valuation paper) is supposed to solve the recalibration problem. The technology we think is the most adapted to that purpose is the regime-switching tool.

6. We don't aim to impact the market or change it, but merely to understand what the market is (given the fundamental principle) and to change the way we think of the market. For some, in the field, this (fortunately or unfortunately) means starting to think.
7. In any case, we aim to change the way books on derivatives are written.

Now you, my dear reader, may still object in the last instance that in 99 percent of the markets, 95 percent of the time, stuff just trades because a seller meets a buyer, so why do we need this escalation into derivatives? I would first answer that we are talking of quantitative finance here, not of basic sociological analyses or journalistic summaries of the market. To that end, we need quantitative models. So, let us forget about derivatives – if they seem to annoy you as too special or pathological – and let us just ponder what general equilibrium theory, for instance, has to offer instead. General equilibrium theory has no framework to go by to this date other than states of the world and probabilities applied to them. In that framework, we compute the prices of assets by solving our favorite equation or our favorite fixed-point problem. However, the big question that this whole methodology leaves unanswered is the following: 'Won't those prices then qualify as states of the world/market?' And was the move legitimate, in the first place, to postulate states of the world/market that were other than prices (all the prices and nothing but the prices)?

Derivative pricing has only sharpened this very severe – and, indeed, foundational – problem because it directly postulates that states of the world/market are prices; namely, the prices of the underlying. It doesn't solve a general-equilibrium fixed-point kind of problem, because it considers it solved as far as the underlying is concerned and it takes as *given* the stochastic process that the underlying prices are supposed to follow. The current saying is that this mathematical hypothesis screens off the 'real economy' and leaves derivatives to trade on the surface relatively to each other and to their underlying without any real connection with the real underlying economic factors. Some say this is a bad thing; others think it is a blessing, because it has allowed derivative pricing to become an engineering science (and allowed us all – at least, the derivatives players among us – to be here), when economic theory is still trapped in game theory and other textbook paradigms.

In any case, the big question above remains unanswered. Indeed, won't the derivatives prices that we compute off our theoretical underlying stochastic hypothesis subsequently become additional states of the world/market? And weren't they supposed *to be* states of the world/market in the first place? We recognize in this question the problem of *recalibration* – a word that seems barbaric or too specific to derivative pricing. However, the word 'recalibration' is only the sharpening and the rewording of the one big foundational problem mentioned above to the effect that price – the price of anything and not just of derivatives – may have to be thought as a category totally at variance with the notion of states of the world and the extensionality thereof. There is a paradox here (not a contradiction), a non-closure that is similar, in style not in content, to the measurement problem in quantum mechanics, in which the formalism speaks of probability of measurements (and nothing else) and the measuring devices themselves are quantum systems that interact in their own right and must be governed by the very formalism that they contribute to establish.

The foundational problem I am pointing to isn't comparable with the usual criticism of probability models and probability distributions. It cuts deep into the whole make-up of thought, representation and objectivity. To repeat, we require a 'quantitative' framework (maybe not numerical, maybe not even mathematical in the ordinary sense of the word) to deal with future contingency *without the intermediation of possible states and probability*. This is beyond the reach of the average literature or criticism. I mentioned metaphysics (literally, what lies beyond physics) because this is really the level we are getting at with this depth of questioning, before we rise back to the surface with some formalization or axiomatics of some kind.

Metaphysics does not mean empty or loose talk. To the contrary, when properly conducted, metaphysics is the place of the precision of thought. Of course, everybody knows that contingency is everywhere and that the locality of everything is self-evident; however, the thought process becomes exacting and truly creative when one wonders what exactly can be deduced from that universal fact, if its consequences are radically and totally followed through.

You maintain that hardly anybody assumes a 'random generator'. Well, do they? Apparently, you haven't been talking to quants, or attending quant conferences, or reading any theoretical work in finance (not to mention econometrics) over the last decades.

'Oh, but traders and market practitioners "without the elaborate vocabulary" (Brooklyn boys, as Taleb would call them) do not assume random generators or fictions of that ilk! They just trade the stuff; they don't model it or theorize about it.'

'Fine; but, then, what do you make of the builders, like myself, of a *technology* of derivative pricing? Where do you rank technology? On the side of theory or the side of practice?'

It is truly unfortunate that I shouldn't be able to present a derivative price to the potential user of the technology I build unless that price was backed by the procedure of hedging it. Consequently, it is truly unfortunate that my technology could not but internally rely on stochastic processes and random generators of some kind; or that I could not produce derivative prices based on technical analysis or loose journalistic talk relative to a buyer meeting a seller and to the overlap of their prospects. On the other hand, it is truly unfortunate that the user of the derivative pricing technology could not but *use it* in the actual market and could not be satisfied with the passing observation that the market is an accident or an imperfection that falls beyond the jurisdiction of the corresponding theoretical model or the corresponding quantitative paper. Quants can forever deny recalibration and living market-makers can forever deny models. But the technology sits right in the middle, or right in the knot, and it dictates that recalibration should become a technological process and should be *re-embedded in the technology*. This, the problem of making sense in a systematic way – that is, no longer approximately or vaguely – of the technology that my company is providing, is what prompted the metaphysical systematization or at least systematic critique that I undertake in the book.

To repeat, insofar as you understand the derivative pricing technology as fusing completely with the market, even understand the derivatives market itself as a technology (which is by definition superior to the theoretical framework of states of the world and probability, if only because of recalibration), you have no choice but to build a *system of thought* – that is to say, a metaphysics – in order to account for this new fusion or new medium or new category.

This is not a sideshow having only to do with a tiny fraction of the population and a tiny class of assets. Every derivative payoff is but a summary of a whole trading strategy of the underlying. Derivatives are just ways that the market rewrites itself in synthetic fashion, only to re-embed this rewriting in the exchange. I do not care that in 99 percent of cases and 95 percent of the time the market may seem to you, dear incredulous reader, as a relaxed arena that is totally unconcerned with my pressing and very precise point – what you call the ‘infection in my philosophy’. For this is only how things appear on the surface. I claim that the derivative pricing problem, which sharpens the pricing problem, and the recalibration problem, which sharpens the derivative pricing problem, are where the essence of the market lies. I do not believe in a point of view of the market from above – a kind of balcony from which you watch the overlap of expectations and the meeting of curves of supply and demand. The only valid point of view is the point of view of the *individual market-maker* and user of the market technology. However, this is not anthropology. This becomes a philosophy of science, or better, a logic of science in the sense of Carnap,¹ when we attempt to formalize the market-maker’s point of view, or recognize the genesis of the market of contingent claims as a step outside the formalism that is caused by the formalism.

This step outside is a genuine event. On the other hand, the only place in which the market resides is the bid-and-ask spread that an individual market-maker quotes. Notice that the bid-and-ask spread is precisely an instance of no trading and no overlap. This is how the trade later emerges as an *event*, as a singularity that precisely goes counter to the very idea of pre-computation, programming and algorithmic output.

When BSM implies that contingent claims shouldn’t trade because they are redundant, people usually argue that contingent claims do trade because BSM is wrong; that is, volatility is stochastic, or uncertain, or unknown, or there are jumps, and so on. Yet, if you made volatility stochastic in a model superior to BSM, contingent claims won’t trade either, even though they will no longer be replicable by the underlying, because their values will now be function of two states variables; namely the underlying and volatility. This is still a function and to be the value of a function is not to be a price. Price is not meant to be a function of anything underlying it. So now, again, people will argue that contingent claims, nevertheless, trade in reality because this stochastic volatility model is as much a model as BSM, therefore it is wrong, too.

I am not happy with error or imperfection of the model being the only explanation of the market of contingent claims. Why can’t their market be an impeccable conclusion of the formalism and not something due to chaos and mess? The problem is that we cannot independently postulate a trading process for the contingent claim in the same way as we would for the underlying, or for another underlying. Derivatives prices cannot have their own independent volatility. Their volatility has always to be explained by the volatility of the volatility of the underlying (hence, the stochastic volatility model). No matter what we do with the formalism, derivatives or contingent claims end up always valued, never priced (that is to say, traded).

My latest discovery is that if we scrutinize the mathematical formalism of derivative pricing theory (or the pricing theory of contingent claims), for instance in a theoretical paper such as Harrison and Pliska (1980) and not in a heuristic paper such as the original Black and Scholes (1973), we find

that there is only mention of the market of the underlying asset. With the underlying asset, we can only manufacture or synthesize contingent payoffs, not contingent claims. The distinction is important. When the underlying is given and is trading with Brownian motion in its market, we may buy it or sell it and dynamically readjust our holdings in order to end up with any payoff profile as a function of S , or $f(S)$, at any time horizon, also known as ‘contingent payoff’. This is the major discovery of Black, Scholes and Merton (leading to the Nobel Prize). But the contingent payoff that we have manufactured is no contingent claim. Nobody has given us permission, in the formalism, to assume that contingent claims or derivatives exist independently as written contracts on which such contingent payoffs have been marked, and of which we may subsequently wonder whether we can replicate them with the underlying.

In the formalism, when it is properly read, there is no mention of the contingent claim, or of an independent paper that may admit of an independent price, and of which we later argue that its price is curbed by the replication. In other words, the market of contingent claims doesn’t exist in the formalism and only the market of the underlying does. So, it is not that contingent claims are redundant (as it is traditionally thought) and, for this reason, they won’t admit of their own market. Rather, they simply don’t exist. Because they don’t exist in the formalism, the formalism doesn’t say they are redundant; hence, it may become an impeccable consequence or conclusion of the formalism that they admit of a market. *This is how we impeccably derive their market from the formalism or, rather, outside the formalism.*

The reason why we don’t discard the formalism altogether and why we insist that the market of contingent claims should be its ‘consequence’ is that the formalism is needed by the market-maker in order to hedge the contingent claims. There wouldn’t have been an industry of contingent claims and their market wouldn’t have been cemented without the historical layer of the formalism. Yet, the matter of this market lies literally in the void outside the formalism. We are literally talking about an inceptive medium, properly material and not formal, properly historical, in which alone the market of contingent claims can be thought and given form; that is, ‘formalized’, albeit meta-formally.

The derivative character of contingent claims seems to deprive them of a market, or of a trading force that animates them properly as if they were immersed in their own original market. The way out is truly to believe and ontologize what BSM has produced. Really to create a market for contingent claims, there shouldn’t exist a market before, or even mention of the contingent claim. Statistics muddles the problem. Contingent claims have to be imposed by surprise, by the event, by creation; this is the merit of BSM, of which we have discovered that only the market of the underlying was ever considered and then something happened (the replication of the contingent payoff) which, in our genetic reconstruction, opened a void or interval in which precisely the market of contingent claims could start afresh, without the link with arbitrage and completely outside the comprehensive picture of all-inclusive states of the world.

We have the duty to formalize what happened with BSM, *as this is the market of contingent claims*. The way in which the contingent claim is considered after BSM (by the void and the coincidence, by genesis) is different from considering it as independently written in the framework of statistics or lotteries. To *invent* the writing of the contingent claim is precisely not to exit from the heart

of the market. The layer of implied volatility prevents us from falling back in the comprehensive picture. Ultimately, the capacity of the paper materially to inscribe the event even of its own disappearance and recalibration, or the making of state out of price, are connected with the invention of writing (which is the hinge) and the layering of matter which prevents the recuperation by the comprehensive picture.

The gap in BSM, which consists in noting that only contingent payoffs were manufactured and not contingent claims, is potentially richer and more decisive in the creation (genesis) of the market of contingent claims than any complication of the diffusion process, either by further stochastic processes (jumps, stochastic volatility) or by the uncertainty surrounding its parameters. We think that contingent claims are redundant in BSM and that the only way to explain the existence of their market is to make volatility either stochastic (by adding volatility of volatility, or jumps) or uncertain. In reality, contingent claims do not exist in BSM, and a better and more decisive argument for the existence of their market is to note that they are not redundant because they don't exist.

Noting that there is a perfect coincidence between the contingent payoff and the contingent claim (between the payoff that was manufactured, in the first case, and the payoff that is written, in the second), yet that there is a perfect gap (because the contingent payoff *is not* an existing material claim which can be put to independent trading) and using the coincidence to force and fill the gap is a formal generalization or extension (matter is extended) of BSM that is better than any quantitative extension or *variation* of its parameters, either by stochasticity or by uncertainty.

The road to the market of contingent claims traverses BSM necessarily as well as the narrow passage, or transition, between the coincidence and the gap. We should neither interpret the diffusion process in BSM as an encouragement to its probabilistic and statistical generalization, nor interpret the market of the underlying that is thus summarized as an encouragement to adding to it, from the beginning, the independent trading process of the contingent claim. In the first case, we would have exited the market and it is only through an abuse of language that we could say that the independently written contingent claims are now valued under the martingale measure that the 'market' had selected. It is an abuse of language because an arbitrage-free valuation under a martingale measure that we alone postulate and select for the sake of the argument is no market. In the second case, the comprehensive non-arbitrage picture would have *subsequently* impose itself and to recuperate the trading process of the contingent claim under a valuation, thus exiting the market again.

It is important to realize the register in which BSM is written. It is precisely because it was so written that its formal mathematical consequence was the manufacture of contingent payoffs (not contingent claims) and its later material consequence (by forcing) was the market of contingent claims. The underlying diffusion process in BSM is given in a formal register, in which there is no doubt, uncertainty, knowledge or inference. We may give ourselves a statistical model of the underlying price evolution with a view to physical reality and to inferring its parameters from reality. However, the formal register of BSM is past that stage. It is the premise of a mathematical problem. Let the volatility be σ . It is because of this register, in which there is no doubt, no epistemology and no reality even, that the mathematical problem unfolded in full and the contingent payoff was manufactured.

The manufacture of the contingent payoff (its so-called ‘valuation’) is equivalent to formally giving the diffusion process, with no room allowed for speculation on what volatility may be, because the volatility is formally given. The market of contingent claims was not created because volatility suddenly became stochastic or suddenly became uncertain *in the formalism*, but because of the succession of formal discoveries; first, the discovery that the contingent payoff was coincident with, yet materially different from the contingent claim and, second, the discovery that the contingent claims which were thought to be redundant because of the coincidence actually do not exist and are not redundant because of the difference. This succession was *historical* – we call it historical, we define ‘historical’ so to call it – and it created the industry and the technology (recalibration) in which, alone, the prices of contingent claims could be considered as given states, without an underlying valuation.

The probabilistic framework would have never created a market of contingent claims. It could evaluate them, but what could we make variable in the evaluation (either by further stochasticity, or by further uncertainty) in order to create a market? In the formal gap between contingent payoffs and contingent claims, everything could be made variable because there is no link anyway but only the void. The market of the contingent claims is only ahead. They coincide with the contingent payoffs; they are seemingly redundant on this side; however, they are totally free on the other side. Their market is total and absolute, yet the coincidence explains that the underlying process is still given.

It is important to realize the register we are in when the underlying stochastic process is formally given at the beginning of BSM. We are past any valuation worry because only the net result of trading the underlying is described (what I have called a summary), only the surface consequence which is the random behavior of the price; and we are also past the statistical worry – in itself already a movement past the underlying hidden causes and a preoccupation only with the surface – which would have consisted in proposing a statistical distribution for the price movements of the underlying with a view to inferring its moments from the empirical times series. Any question concerning the real reasons why an exchange issues in a random walk (such as the efficient market hypothesis), or any question whether Brownian motion is a realistic model of the empirical time series, is now behind us. We may have worried about these questions in the past and we may have treated them; however, the register we are in now is purely formal; it looks completely in the other direction as it now formally *lets* an underlying stochastic process *be*, with drift μ and volatility σ – *in the other direction*, which means that there is no longer any question about the knowledge of the moments, or uncertainty about them, or speculation about their value.

Writing the underlying stochastic process in this way is a conclusion and a closure. But we could also say it is a beginning. It is a new way; it is a whole new register, a whole new book – the formal mathematical book, in which unexpected things could be found or derived but in which (empirical) reality is, in any case, no longer expected. It is in the recess of this book, in which the underlying is the only matter and has, moreover, cut off any epistemological relation with reality (such as knowledge, or inference, or uncertainty, or speculation), that it is shown that Brownian motion for the underlying price, when it is understood that the financial market that is hereby formalized is one in which variable quantities of the underlying can be bought and sold continuously at any

price level, is equivalent to the perfect manufacture of contingent payoffs. It is important to see that this unexpected result – the perfect replication of contingent payoffs – is totally on the side of the mathematical book; that is to say, completely disconnected from reality, in order to better see that the later identification of the contingent payoff with the contingent claim is a forcing of reality (and a creation of matter) that will equally lie completely on the other side of the previous reality in which we would have worried about the inference or the knowledge of the parameters.

The way in which this identification (a coincidence combined with a gap) brings back a reality is a revolutionary way. The contingent claim was identified with the contingent payoff through the new formal register in which the underlying was cut off from the previous reality and had become its own conclusion and closure, through the irreversible reading and irreversible institution of that register. The contingent claim was invented, we may say, *after* the formalism and the unexpected consequence of the replication of contingent payoffs, after the end (the conclusion, the closure) of the previous register in which there still existed a correspondence between the underlying stochastic process and a certain reality in which its parameters would be inferred, or would be known to lie only be known to be unknown.

This was unexpected, that the manufacture of contingent payoffs would emerge in the formal register in which the underlying stochastic process was given, and it is only for this reason, I claim – namely, this total unexpectedness combined with the total isolation and irreversibility (to reality) of the formal register – that the most incredible thing could take place: contingent claims could be traded – I mean to say, could find the reality (a revolutionary one) in which they are traded and no longer appear as mere valuations. The reality of the price of the contingent claim was no longer to include in the previous reality in which the underlying stochastic process was *realistic* and real concerns could be voiced about its parameters, such as their inference or their uncertainty. Rather, its reality was acquired *through the industry*, and by that I mean that because the contingent payoff was perfectly replicable in a formal register, completely in isolation from the previous reality and even in irreversibility towards it, it was thought that the price of the contingent claim could now exist, as if originally and absolutely.

The coincidence and the perfection of the formula (or the end product of the formalism) created the pure and perfect price of the contingent claim – it made its market – totally across the void and across its irreversibility, in total isolation from the evaluative (epistemological, statistical) context in which one *simultaneously* worries about the value of the parameters of the stochastic process and values the contingent claims as independently written lotteries (not as created writings). The market of the contingent claims, which is made across the formalism and the corresponding closure and irreversibility, is made totally independently of any concern about the knowledge or the uncertainty of σ ; even better, it is made with the total knowledge that σ is precisely what it is in the formalism – absolutely given, absolutely beyond any doubt.

For the prices of contingent claims to be cut off from any valuation and to be original states, there needed to be the passage through the formalism in which every valuation was forgotten and in which it was then unexpected that contingent payoffs should be manufactured (they were, for the first time, precisely because of the formalism and its attached irreversibility and forgetfulness

of the past knowledge). It was both essential that the trading of the underlying should formally be shown to be equivalent to contingent payoffs (it was essential that the contingent payoff should be synthesized) and that the contingent payoff should be different from (but only coincident with) the contingent claim. Replication by the trading of the underlying was essential, keeping in mind that there would be a gap and a difference, a void across which alone a market could be had (made), when there was supposed to be none.

There is something unique (unexpected) that happened with BSM. The formula was perfect both in replicating the contingent payoff and in separating completely (although invisibly to the naked eye) the contingent payoff and the contingent claim. The mathematical side of the formula, the formal argument for the replication of contingent payoffs, had made it clear that we were past any epistemological or even realistic worry about the underlying process (that there was no doubt, no uncertainty, no inference and no notion of knowledge, even, of its parameters) and, for this reason, the usage of the formula in the reality of the market of contingent claims was an altogether different jump into reality, or different application (interpretation) of the formula and formalism. It was new of its kind; it literally created (genesis) the market of contingent claims, with both the consequences (due to a double separation from the previous reality and realism of the underlying process: the first separation being the overall formal register which it is methodologically very important to separate, as we said, from epistemology and realism; and the second separation being the jump, totally on the other side, totally due to a forcing of the formalism – the invention of writing – and not to a quantitative variation of its content, into the reality of the market of contingent claims) that the prices of the contingent claims had now to be totally separated from any uncertainty or stochasticity or inference of the parameters of the underlying process, or any convergence to them, and that it made possible what we thought was impossible; namely, that contingent claims should be tradable with original force, in their own newly created market without the comprehensive, arbitrage-free picture recuperating this market into a valuation.

The creation of the reality of the market of contingent claims (the invention of their writing) is necessary. One shouldn't consider the contingent claims as pre-existent. If they were, then the temptation would be too strong either to consider them as lotteries whose variety of prices can only be explained by complications (stochastic volatility, jumps) of the underlying process, or to consider their independent trading process, which the comprehensive picture would soon recuperate. There is a new mode of reasoning (of considering the formalism and the exit from it) at play here, which measures how crucial and indispensable BSM is, in the creation of the market of contingent claims. BSM is not a model and, because it is not a model, no model can surpass it.

We have to remain aware of the dissimilarity between the underlying process and a stochastic process. The exit into probability will only result in the valuation of the contingent claims as lotteries. On the other hand, the exceptional circumstance that only Brownian motion of the underlying price produces the perfect replication of contingent payoffs should make us aware of its exception; that is, both of the facts that nothing has been described here other than a phenomenon specific to the underlying (and resulting in no market of other claims), and that Brownian motion shouldn't be quantitatively generalized.

The distinctions we wish to make don't seem to preoccupy a lot of people. Not many people seem to be worried that summarizing the market of the underlying as a stochastic process is in a different category than the probabilistic valuation of derivatives written on that underlying and, as a consequence, shouldn't lend itself to such an valuation. Why are we so keen on such distinctions between categories? The underlying process seems to change its meaning completely when it becomes mere probabilistic states underlying the valuation of derivatives. By preserving the meaning and insisting that the contingent claims should enter the picture under the sole category of market, what other gains of thought are we hoping to achieve; what new frames of thought are we pursuing? For instance, we say that otherwise a whole new matter, even a whole new industry, would be missed.

Appendix A

Regime-Switching Model

A.1 A meta-contextual pricing tool

We must deliver a pricing tool, not a valuation theory, to the derivatives trader.¹ The tool must be delivered to him in his capacity of a vanilla options trader and, as such, in the full knowledge that he will trade those options; or it must be delivered to him in his capacity of a barrier options trader and in the full knowledge that, as such, he will trade those barrier options. It must be thus embedded in the scope of the pricing *technology* (as opposed to valuation theory) that derivative prices are bound to overstep the valuation theory and to become traded market prices. The technology of derivative pricing strictly exceeds the theory. It must be embedded in the pricing tool that the daily recalibration of the stochastic process that is supposed to be given in a specific context will entail the ‘stochasticization’ of that context.

The states of the world have to be identified and totalized in order that valuation theory may operate. In order to value an option written on a certain underlying, we need to determine, first, whether the underlying price is the sole stochastic process, say Brownian motion with constant volatility, or whether volatility is stochastic, too, and follows an independent process. The fixity of the context is mandatory, for only in this way can the dynamic replication of the derivative instrument take place and its valuation be completed. Yet, the trading of the derivative instrument will trigger a step outside its valuation context, for its value now becomes a price; that is to say, a new state of the world which changes the previous context. Worse, it is only in the new context, which is in reality incompatible with the previous one, that the contingent payoff that has just been replicated is written as a *contingent claim* which can now be traded independently.

Changes of context, or recalibration, is precisely where the market lies. Consequently, the pricing tool we are looking for should not be limited to a given context. It is a *meta-contextual* pricing tool, in the sense that it will not be determined in which context it lies – whether volatility is constant or stochastic, whether the volatility of volatility is constant or stochastic, and so on – until the scope of the calibration inputs is determined and calibration is performed. Recalibration will

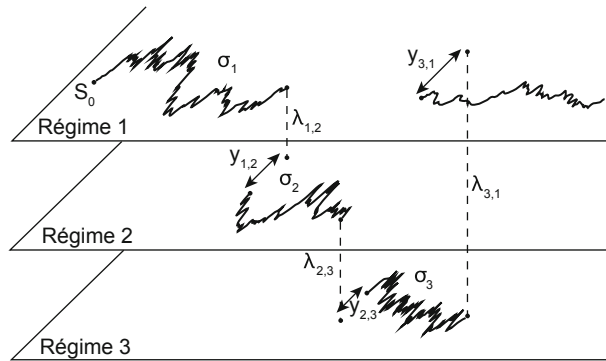


Figure A.1 In the regime-switching model (whose number of regimes we have limited to three in the graphic), the underlying share price follows a *jump-diffusion* process whose parameters – namely, the Brownian volatility σ , the Poisson intensity of the jump process λ , and the size of the jump y – are themselves stochastic and admit of different values depending on the regime we are in. The share starts in regime 1 in which it diffuses with volatility σ_1 , but it can, with a probability measured by the Poisson intensity $\lambda_{1,2}$ (or $\lambda_{1,3}$, or $\lambda_{1,n}$), jump in regime 2 (or 3, or n), where both its price will undergo a discontinuous jump of size $y_{1,2}$ (or $y_{1,3}$, or $y_{1,n}$) and its Brownian volatility will jump to the value σ_2 (or σ_3 , or σ_n). In regime 2 (or 3, or n), the intensities of jumping in the other regimes and the sizes of the corresponding jumps will be different in turn

keep enlarging this scope and will, consequently, change the context; however, this will be achieved without re-initializing the pricing tool.

How the problem will be solved, and the meta-contextual level properly addressed, is by exhibiting a ‘stochastic process’ that will have the property of being *similar to its stochasticization*. Instead of describing a sequence in which the number of state variables augments every time and their names differ every time (underlying share price, jumps of the share price, volatility of the share price, jumps of volatility, volatility of volatility, jumps of the volatility of volatility, and so on), the new solution consists in considering, *for all times*, a state variable *without a name* which will acquire the appropriate name depending on the context and which will change names in the next context.

Strangely, the proposed model looks simpler than the traditional models. It is a regime-switching model, which can be interpreted, at first blush, as a discretization of the traditional stochastic volatility or jump-diffusion processes. Its ‘variable with no name’ is the regime (see Figure A.1).

The fundamental property of the regime-switching model is that the stochasticization of a regime-switching model is still a regime-switching model (see Figures A.2 and A.3).

A regime can ‘reference’ several state variables at once (volatility, volatility of volatility, jump size, and so on). Thus, the potential richness of the model is unlimited. When a state variable admits the same value in all the regimes, it will not be stochastic and it will play the role of a parameter, rather than of a state variable.

When the model is calibrated to the vanillas only, we can identify it with a simple stochastic volatility model. But as soon as it is calibrated to the barrier options on top of the vanillas, we will recognize that the volatility of volatility is now itself stochastic (or, in other words, that the model is now the stochasticization of the model elaborated in the preceding context) and that *this has*

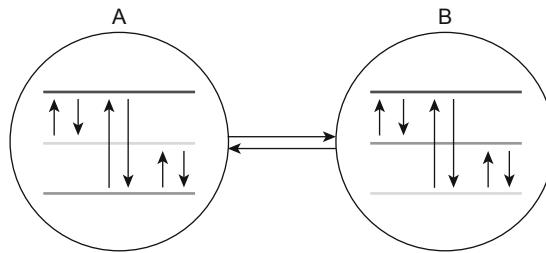


Figure A.2 Let us imagine that the current pricing tool is regime-switching model A . The regimes are symbolized by the horizontal lines of different shades and the transitions between regimes by the arrows of opposite directions. Recalibration of model A to the prices of different derivative instruments changes it into regime-switching model B , whose parameters are different. We may thus entertain the thought that the ‘true’ model is, in fact, a meta-model of regime-switching in which stochastic transitions occur between models A and B , following the recalibration

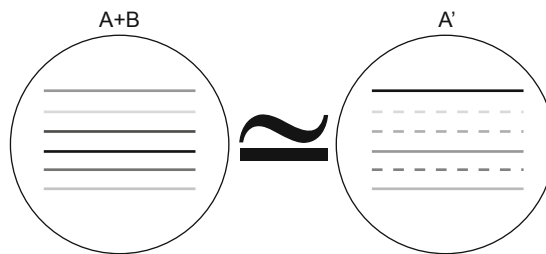


Figure A.3 By virtue of the *associativity* of the stochastic transition between regimes and between regime-switching models, the meta-model is, in fact, itself a regime-switching model, which we have labelled $A + B$. At first sight, this model appears to be a regime-switching model with six regimes. Since regimes do not possess any *metaphysical reality* in our instrumentalist philosophy and their only purpose is to provide a *tool* for calibration, for pricing and for dynamic replication, and since mathematical expectation (not to mention prices) is more important to our eyes than individual states of the world or individual probabilities, the decisive observation is that, insofar as calibration, pricing and dynamic replication are *the only things that matter in practice*, a model which is more parsimonious than $A + B$ – for instance, a regime-switching model with three regimes, A' – can very well be equivalent to $A + B$. In the end, when faced with a regime-switching model with, say, three regimes, one cannot tell whether it is an ‘original’ model like A or B , or whether it is the reduction to three regimes (A' or B') of a meta-model $A + B$. Until one has elucidated the context of utilization, one cannot tell whether the pricing tool is a certain regime-switching model or the meta-model thereof

always been the case as a matter of fact, only we couldn’t see it and we had no way, empirically, of distinguishing it, until we specifically calibrated the model to the barrier options. This is how past history gets reinterpreted under the light of the present; and this is what corresponds, in our situation, to the retro-diction that is characteristic of the meta-contextual level. The seemingly ‘temporal’ process of the market is, thus, constantly folded back into its *unique event*, into a ‘single dice-throw’ which belongs in virtual time, rather than in chronological time. In other words, the market is *returned* to its meaning, which is essentially an *exchange* and a constant rewriting, not a temporal succession. An intension, not an extension.

Thus, the pricing tool makes use of possibility or stochastic processes as one of its working parts, never as its ultimate frame. As pricing tool, it must deal with the *prices* of contingent claims, not

with the *values* of derivatives, insomuch as contingency is always original and never derivative, as it is precisely what exceeds possibility and always upsets it, what can never be replicated and made redundant by the algorithm of possibility.

A.2 Recalibration

Recalibration is the real market process, and it cannot evolve in possibility or be modeled as a stochastic process. It parallels the process of differentiation whereby payoffs of increasing complexity keep being generated, not the process of generation of data by a random generator. Recalibration is the sign of contingency; it is the insurance that the market will never be completed and that the contingent claims will never turn redundant.

Recalibration means that any valuation model will virtually become stochastic and that the market price of any contingent claim, no matter how exotic or complex, will virtually act as a calibration input to the model. This imposes, as pricing tool, a structure that must be at once open and invariant. We believe the regime-switching model is one such.

Recalibration certainly cannot be modeled as a stochastic process, for, then, this process would itself be prone to recalibration and to stochastic change. Recalibration, where the market process really lies, does not take place in chronological extensive time, we said, but in intensive time. Recalibration is the process of differentiation whereby no contingent claim shall ever be redundant.

Any instance of the regime-switching model – for example our three-regime implementation above – is only a stage or a given context in which we momentarily pause in order to perfectly replicate the contingent payoff of a given degree of complexity. Why momentarily? Because, in this given instantiation, the contingent payoff, although replicated, has not yet been written as an independently tradable contingent claim. No matter how many regimes our implementation may include, no matter how incomplete the market that it represents may be, there always comes a stage of the continually creative process of differentiation of payoffs where every subsequent payoff will turn out to be perfectly replicable by a dynamic strategy involving the contingent claims that have been written and put to trading at that stage, in that context. In other words, there always comes a stage where a further contingent claim, which has only been replicated of late as a *contingent payoff*, needs to be recognized as a *contingent claim*; that is to say, written originally and put to trading in its turn.

Another way of pressing the point is to remark that any instance of the regime-switching model is ‘instantaneously’ made stochastic and does not have to wait to be recalibrated over time to that end, because the intention (or intensity) of any pricing tool is to price *and* trade contingent claims, and because to be pricing contingent claims with the intention of trading them is to be assuming, from the beginning, that they will trade at variance with the theoretical premium one is computing for the replication of their (contingent) payoffs. For example, any trader using the BSM model to price vanilla options is ipso facto assuming that implied volatility shall be (as a matter of fact, is already) stochastic. For this reason, regime-switching can appear as the real generalization of BSM. It pursues the *implications* of BSM in the way of its usage in the market. The contemporary theory of the market

must include in its formalism the *usage* of the valuation theory, we said. By contrast, models that have traditionally succeeded to BSM in the literature of derivative valuation are just *complications* of it in the way of possibility, through the probabilistic exit.

Imagine we recalibrate our three-regime switching model in time. Imagine a calibration, followed by a recalibration the next day. Calibration against the surface of vanilla options cannot, in fact, guarantee the unicity of the solution. The minimization problem admits of multiple local minima and the undetermination can only be lifted by adding *other* payoff structures, or exotics, in the calibration procedure. The multiple local minima thus become a single global minimum. The current saying is that different smile models can very well agree on the same volatility smile, yet disagree on the smile dynamics; or, equivalently, that they disagree on the pricing of exotics (such as barrier options, or forward-starting options). Only if we calibrate the smile model to the relevant exotics, simultaneously with the vanillas, are we guaranteed to discriminate between the different smile dynamics.

So, imagine we calibrate our three-regime switching model, not just against the vanillas, but against what it takes of exotic options for the calibration to admit of a unique solution. The price of any other imaginable exotic is now imposed, as we have no further leeway and our information set is complete. As soon as we notice, however, that the calibrated parameters of our regime-switching model do change the following day because of recalibration, we come to the conclusion that the ‘real’ model wasn’t, in fact, a three-regime switching model, but a larger model which switches between the two three-regime instantiations of day 1 and day 2; in other words, a six-regime switching model.

The corollary is that our calibration wasn’t actually uniquely determined from the start, because a six-regime switching model actually needs to be calibrated against exotics in greater number and of even higher degree of complexity than before, in order to admit of a unique set of parameters. This means that the prices of the ‘other imaginable exotics’ weren’t, in fact, imposed. As a matter of fact, if such prices had been available in the market, they would have helped us *reduce* the undetermination of the calibration of the six-regime switching model, so on and so forth.

Now, the whole trick of moving from the actual to the virtual while remaining in the real is to think that the six-regime switching model *is*, in reality, our model from the beginning, only we have ‘diagrammed’ it under three regimes. (Indeed, there is no way to tell whether each of the initial three regimes isn’t, in reality, a superposition of two identical regimes waiting to get differentiated.) And why stop there? As we said, any pricing model virtually becomes stochastic; so not only six, but a twelve-regime switching model is, in reality, our model from the beginning; and why not twenty-four regimes, an infinity of regimes, and so on?

The whole idea is to stop thinking that this infinity of regimes is progressively revealed *over time*, as a consequence of recalibration making any previous model stochastic, and to start thinking that this infinity is *virtually* present from the beginning. It is ‘progressively’ revealed only to the extent that we practically limit our progress at the beginning, and practically limit our calibration to the vanillas and a few exotics.

Recalibration is not a process of recalibration over time, but the instantaneous process of ‘mentally’ defeating any calibration we may have actually performed, and of thinking that we could

have virtually calibrated our pricing tool, instead, against a larger set of exotics of higher degree of complexity, if only their prices had been actually available.

But does it really matter if they're not? For this is the market we are standing in, not a stochastic process, and the market is at once virtual and real. What the market really means is that all the prices of all sorts of contingent claims are virtually present *and* are not redundant. All we need, consequently, in order to solve the problem of recalibration, is a model that we can virtually calibrate against any kind of exotic payoff of any degree of complexity and that we can virtually expand to any number of regimes, even if it means that this expansion will be retro-jected in the past by a backward narrative.

Appendix B

Menard's *Quixote*

Can contingency be independent of time? Pierre Menard does not write the *Quixote* in time, because Cervantes's work pre-exists Menard's, and because, if he did, then his work would be no more than a chronological *rewriting* of the *Quixote* (rather than a writing of it). In time, the writing of the *Quixote* is no longer a possibility, since it is already in the past, and the text is already actual. Menard writes the *Quixote* outside of possibility and replication; he does not repeat the text, he repeats its contingency. The contingency of the *Quixote* survives time and the actualization of the text, since it is always true that the text is contingent, even after its completion and the extinction of its state of possibility.

Menard does not repeat, does not affirm, in regard to the *Quixote*, that the text *is*; he repeats, and affirms, that the text is *written*. Between the underlying state of the *Quixote* and its writing, there is an interval, a distance, into which Menard enters. The writing of the *Quixote* is not a layer superimposed upon its 'state', to imprint it, to engrave it (to duplicate it). It is not the function that comes to attribute to the state of the *Quixote* the value 'existence', and is not the equivalent of that state. It is the layer that comes to *replace* its state so as to say, precisely, that the text could have been different (could have been other). This replacement is all the more difficult to conceive in that the *Quixote* 'is' nothing other than the written text. Neither was it anything for Cervantes before he wrote it. But whereas Cervantes traverses it in the direction of realization and actualization, Menard traverses it in the direction of counteractualization, or virtualization.

Writing is at once that which presents the text, in a movement of thought that goes from the state to value, distinguishing the text and conferring upon it its identity, and certifying (through writing) that to 'this state, here' corresponds 'this text, here'; and, at the same time, writing is a mark and an incision, that which *remains* of the text once the attribution of the value of identity and the positive sense of the attributive function have been interred by the cataclysm we spoke of above, and which may be nothing other than the extreme metaphor of the passage of time. As a remainder or vestige from which the movement of presentation has absented itself, the mark no longer 'presents' anything except its contingent character. Just as writing, when we read it in the sense

that imprints positively the state of the text in it, attests that the text is and *identifies* it, delivers to us what it is, writing, when we reverse our gaze and see it as that which *remains* when the movement of presentation is retracted, says that the text could have been different. Whereas the state imprints the text, the writing of the text, or the printed strike, exchanges it. It no longer refers to time which points to the text like an arrow, makes it be and evaluates it, but towards the spatiality of the marked strike, a relation which the written text maintains with the elsewhere of time, a place where it is exchanged.

This is the privilege (the distinction) of the mark as opposed to the state. Precisely, it *remains*; and to remain is not to be. To remain is to remain unexplained and contingent. That which, in the now forgotten positive movement, was only the effect of the certified imprint of the state, or the mark that it left subsequently and derivatively by the sole effect of its outward saliency, through the affirmation of its identity, becomes, once the necessitating cause is retracted and buried, a contingent and irreversible sign, an 'inexplicable' sign which, from now on, points only towards difference, in the sense that this mark, in being what it is and even *before* being what it is, only is what it is so as to say that it could have been other.

One thus traverses the text with surprise and excitement, in the direction of the mark and no longer that of the state, as if one's eyes were constantly being drawn elsewhere. Now, this elsewhere is no more *identified* than the mark (without state) is. One traverses no other *thing* than the mark, which is to say (through a play on words) that one traverses the mark itself, but one traverses it precisely in a direction contrary to its identity (or independent of it). One traverses the mark 'itself' only insofar as it says that it is *this* and not another thing, that it is contingent and, consequently, that it *could have been* other.

Pierre Menard stepped out of the chronological dimension where the writing of the text is attributed to its state, in a reciprocal play between the possible and the real. Pierre Menard no longer has at his disposal the time, or the possibility, of writing the *Quixote*. However, the work that he 'realizes' seems to demand a certain time and to elicit a certain credibility. How is this possible, and how can Menard's work be credible, if not because it is real? The two crucial points of Borges's novel are precisely that Menard's enterprise is credible, and that it is a matter of writing. Precisely, Menard must write the *Quixote*; it is essential that he writes it, rather than simply reading it or conceiving of it. If Menard's work enjoys a duration that no longer coincides with the time of the positive attribution of the writing of the text to its state, it is because he takes the text and his work of writing elsewhere. If it is in time that identity comes to the state, then difference comes to writing (the thought that the text is written, and that, because it is written, it could have been different) in an elsewhere of time which I call the *place*.

Menard does not write in time, but in place. He writes in a movement of *contraction* towards the paper (as if the function which attributed value to states, which attributed the reality of the *Quixote* to the state of the *Quixote*, were contracted into its mark and its absolute difference, towards its engraving and its matter). This movement of contraction is precisely what is called the *contract*. Menard writes in the contract of the *Quixote* (its writing contracted without its state) which says, at the last moment, that it could have been different.

Certainly, the strike of writing is a copy of the underlying state, but it is above all an incision, an irrevocable difference, something that guides contingency and is capable of saying that the world is not, but *could be* and *could have been* (different) at the moment of the reading of the contract and of the opening of the testament, without the state as a guide. Menard insinuates himself between the state which he does not repeat and the mark which he does repeat and which would come back to the same as the text if it were illuminated in the light of identity. The mark leaves marked the fact that 'it could have been different', without the state.

One might persist in thinking that the text (the mark) is different because its state is different, not wanting to recognize in difference any other trait than its derivative (analogical) character and its underlying cause. But once, in a lightning flash of thought, this link is broken and one realizes that the mark could have been different in an absolute sense, after the ruination and the burying of the underlying state; once one thinks that it could have been different after rebounding from the final actuality of the text, yet without rewinding the path of possibility and of the attribution of difference through the *inheritance* of the state, then the space towards which thought is turned is that of the place. All that then remains of the written formula (of the text) is the materiality of writing disburdened of the infrastructure of the state. Thought can no longer come to it in the chronological order of thought; and, so, it is another order of thought, perhaps even an order other *than* thought, which takes up the text – or rather, returns it, *exchanges* it.

Menard projects himself into the exchange. He drives the *Quixote* towards the place of the exchange. Menard situates himself in the strange dimension of the exchange. What makes his enterprise credible is that there is no difficulty in reinjecting time – that is to say will, labour – into this dimension without time and without possibility (which is the final actuality and the final realization of the *Quixote*), once the latter is illuminated by difference and transported by the exchange. Menard wrote the *Quixote* in *no time*, and this is why we might say that he dreamt it, given that the dream is this regime of the exchange of thought; that is to say, its writing.

When writing is considered as derivative on the underlying state, difference is merely analogical. It is by analogy with the difference of the state that the mark (or writing) of the text, which imprints this state, is in this case different. (To be sure, the state of the text is nothing other than its written state, to be sure there is no deeper text, no deeper state, underlying the text; the state in question lies in thought and nowhere else; precisely, it defines the *order* of thought, the way in which thought orients itself towards the text and comes to it.) But when writing is considered as absolute, as that which remains after the disappearance of the state and is no longer relative to the state, but which, nonetheless, bears the entire reality of the text qua contingent, then difference becomes internal. It no longer follows the attributive order of the state. It can no longer sit by the side of the state, and follow along its temporal order; this is why it changes place, and finds *place* as its only place.

Pierre Menard's activity is credible because we can conceive that it takes time. Now, Menard writes, in the text, that which does not come from the state. Menard's writing is not analogical, and does not repeat attribution. It writes without the state and repeats internal difference. Menard has seized, in the *Quixote*, the strike of writing which says that the text could have been different without this difference being traceable back to the state.

It would be ridiculous, for Menard, to unearth different texts, different variations of the *Quixote*, with the sole aim of coming back to the text and showing in this way that it could have been different. The text has been realized, and its contingency is real, but it would be ridiculous, it would be a misinterpretation (*un contresens*), it would be to *go outside* the text, if one were to retreat into alternatives and into other possible texts in order to recover the text. It would be pretence, a playacting. (Isn't possibility a playacting, a representation, by the way?) Menard *remains* (and does not dwell) within the reality of the *Quixote*, but he spends his time rediscovering it. So, what is it that he makes vary? Certainly something other than the *state* of the *Quixote*. It is very difficult to think that the *Quixote* is contingent and that it could have been different without going back in time and venturing into other possible states. However, the 'It could have been different' cannot go back like this since, then, the text would not exist.

Menard walks in an *impossible* (precisely) space, but one that seems to us credible and real. The difficulty in understanding what Menard does owes to the difficulty of imagining something other than time. To refuse to go back, in order to say that the text could have been different (for then it would not exist), to refuse this fictional facility of time, is an act of the same order as affirming that the future is real. For, in both cases, it is a matter of rejecting possibility. If what Menard achieves is real and not ontologically void, then the future is also real and not possible. This is the fundamental equation of the medium of contingency. The real contingency (not the possible contingency) of a past text is the same thing as the real contingency (also not possible) of a future text. It is the fictive possibility of a future and contingent text that prohibits its being thought real. Its reality is not incompatible with its inexistence, or its inactuality.

It is difficult to admit that Menard's 'process' is not temporal but that, despite this, it has duration. There is something else in the mark – another degree of variation and, even, another nature – than the impact of the state that makes a certified copy of it. As soon as the mark is made and one ventures into the 'It could have been different' without a variation of the state (which is *difficult*), one falls in the place and in the *exchange*. And this is what Menard does, this is what he makes: a *market* all by himself.

It is credible that Menard spends 'time' doing so (it is a process), but one must then draw the full consequences of this credibility – namely, that the words written, the words that Menard forms, do not come to their place through copying or through time (through the choice of words, their identification and their attribution). Exchange as process is accidentally, but unavoidably, immersed in time. In exchanging, one exchanges states, one wanders in the possible and in time. But that is merely accidental. *Exchange* as such (internal exchange) is of the order of thought; it is the turning of thought. However, its vision is completed only in time. Here, we are situated at the threshold of a new metaphysics where the exchange must be redefined and the state must be disconnected from writing, where time is accidental but cannot be circumvented.

To redefine writing, to say what it adds – namely, the mark which is almost always associated with the state but which is, in reality, independent of it – one must know how to step into the exchange. To say that the contingent claim could have been different once it is realized, without varying the state, has no value. It has no value – but it has a *price*.

Between the possible state and its realization, between the state of the *Quixote* and its written matter, there is *still* a distance. This distance is always null and void according to the temporal axis; but this is an optical illusion. Menard manages to spin the axis and give us another perspective. To complete the realization of the possible, to complete reality, requires matter; it thus requires a trace, a mark, an engraving, and this movement of engraving seems indecomposable, and yet...

We will not hesitate to overturn whatever metaphysical categories it will take, provided we can make Menard's story credible, and explain it. To be sure, the *Quixote* is *present* for him; but where is it present? In his mind, like a model to be copied? Or is it present as reality? Certainly, Menard never gets outside of it, for such an exit would be ridiculous. What is it in our mind that immediately makes us think that Menard's enterprise is credible? The *Quixote* is not present in his mind (for then he would only have copied it), and yet he has never left it. This is not a *psychological* problem and it is not a matter of diagnosing some mental disorder on Menard's part, a schizophrenia of sorts.

Let us admit that Menard does not count. For, here, we learn something about texts, or about written contingency, and not about the character. And yet Menard expended energy, he made an effort. Against what? (His energy is negative, the inverse of the probability and of the tree that produced the *Quixote*.) He did not explore the external necessity of the *Quixote*, which says that, once the text exists, it becomes a necessity to itself and the only way of returning the same. Menard repeated the *contingency* of the *Quixote*.

What if Menard's enterprise came down simply to placing writing before the state and the reciprocity of the state? For Menard *does nothing but* write, and we have precisely abolished possibility in his writing, we have abolished the *state*. Thus, for him, the *Quixote* is not 'in state' (*en état*) and is not present. To write without the state and without value, to write without attribution and without inheritance, without the tree, is precisely the exchange! Here is the space of absolute writing, here is *pricing*!

Menard's activity endures outside of time. This is credible, and yet very difficult to imagine. For time occupies thought and thought occupies time. (Perhaps dreaming is the elsewhere of thought and the elsewhere of time: the forgetting of the self, contingency, the variation of thought and not the thought of variation.) We must redefine thought at a high enough level to be able to operate this reversal; we must *generalize* thought.

If one writes without the state, this 'takes time' and takes effort without this adding up to the *same* (without the logic of the result and the copy) and yet with this being the same, since the text is at once departure point and instantaneous arrival. The text is there; it is the van falling from the bridge in the film *Inception*, which imprisons the whole world in its *reality* anyway. It is instantaneously real and there is no way we can overturn this. Which is why we overturn something else and make it vary elsewhere.

If one writes without the state, one places oneself in the logic of price and not of value; one inscribes oneself in this variation of thought, which is not the thought of variation but which is the exchange and the market. This does not take time; it takes place.

We must discover that contingency is outside of time, even if it means that we must invent a time (Menard's time) to fill it up. What makes Menard's work credible is precisely that he produces a

writing. It is incredible that he writes an already-written text (for, in the special case of the text, to be written is to be); and yet it is writing, because of its forgetful character, and as if left to itself and to its contingency (that is to say, unexplained), that makes what Menard does credible. It is through the inexplicability and disconnection of writing (its independence from the state) that Menard becomes credible.

Menard did not rediscover the text inasmuch as it *is* written, but inasmuch as it *remains* written even though it is finished and done with. (What is at play here is the dialogue between being and remaining.) Menard is credible; we believe in what he achieves with a faculty of our mind other than that of recognition, with our mind's capacity of subtraction and exchange, with our mind's own *written version* – what I have called the dream, the rapture, of our thought.

Notes

Prelims

- 1 Jorge Luis Borges, Pierre Menard, Author of the *Quixote*, in *Collected Fictions*, translated by Andrew Hurley (New York: Penguin Books, 1999), pp. 88–95.
- 2 Elie Ayache, *The Blank Swan: The End of Probability* (Chichester: John Wiley & Sons Ltd, 2010).
- 3 Elie Ayache, The medium of contingency, *Pli*, 22, 2011.
- 4 Elie Ayache, In the Middle of the Event, in *The Medium of Contingency*, Robin Mackay (ed.) (Falmouth: Urbanomic, 2011).
- 5 Elie Ayache, The Medium of Contingency, in *Algorithmisiert*, Sylvia Eckermann (Wien: Czernin 2014).
- 6 David Deutsch, *The Beginning of Infinity: Explanations that Transform the World* (New York: Penguin, 2011).
- 7 Andrei Nikolaevich Kolmogorov, *Foundations of the Theory of Probability*, (New York: Chelsea Publishing Company, 1950).

Introduction

- 1 See, for instance, the discussion threads of my previous publication *The Blank Swan* on the quantitative financial forums of Wilmott (<http://wilmott.com/messageview.cfm?catid=11&threadid=76822>) and of Nuclear Phynance (<http://nuclearphynance.com/Show%20Post.aspx?PostIDKey=144145>).
- 2 Fisher Black and Myron Scholes, The pricing of options and corporate liabilities, *Journal of Political Economy*, 81(3), May–June 1973, 637–654.
- 3 See Rama Cont, Jose da Fonseca and Valdo Durrleman, Stochastic models of implied volatility surfaces, *Economic Notes*, 31(2), 2002, 361–377.
- 4 ‘Behind all the problems posed by the kantian a priori, there is nothing, in fact, but semantic confusion: confusion regarding concepts and regarding representations, confusion between what we say and that about which we say it. The science that is destined to eliminate these confusions is what I call “semantics”. The basic assumption common to all members of

that movement [the semantic tradition] was that epistemology was in a state of disarray due primarily to semantic neglect. Epistemology presupposes semantics. A theory of the foundation of our beliefs presupposes that we understand what we say. Semantics was for the members of the semantic tradition what metaphysics used to be for others: the *prima philosophia*. In particular, they thought, the key to the a priori lay in an appreciation of the nature and role of concepts, propositions and senses' (adapted from J. Alberto Coffa, *The Semantic Tradition from Kant to Carnap: To the Vienna Station* (Cambridge: Cambridge University Press, 1991), p. 2, and J. Alberto Coffa, *La Tradition sémantique et l'a priori*, in *Le Cercle de Vienne: doctrines et controverses*, Jan Sebestik and Antonia Soulez (eds) (Paris: Méridiens Klincksieck, 1986), p. 83.

- 5 See J. Michael Harrison and Stanley R. Pliska, *Martingales and stochastic integrals in the theory of continuous trading*, *Stochastic Processes and their Applications*, 11, 1981, 215–260 and our whole discussion around it in Chapter 13.

1 The end of probability

- 1 Part of this section was first published under the title 'The Technology of the Future', in *Realism, Materialism, Art*, ed. Christoph Cox, Jenny Jaskey, Suhail Malik (New York: Center for Curatorial Studies, Bard College; Berlin, Sternberg Press, 2015).
- 2 'By the sole fact of being accomplished, reality casts its shadow behind it into the indefinitely distant past: it thus seems to have been pre-existent to its own realization, in the form of a possible. From this results an error which vitiates our conception of the past; from this arises our claim to anticipate the future on every occasion' (Henri Bergson, *The Creative Mind* (1946), translated by Mabelle Louise Cunningham Anderson (New York: Dover, 2007), p. 11).
- 3 'How can we fail to see that if the event can always be explained afterwards by an arbitrary choice of antecedent events, a completely different event could have been equally well explained in the same circumstances by another choice of antecedent – nay, by the same antecedents otherwise cut out, otherwise distributed, otherwise perceived, – in short, by our *retrospective attention*? Backwards over the course of time a constant remodelling of the past by the present, of the cause by the effect, is being carried out.' Ibid. pp. 84–85.
- 4 A Black Swan is a highly improbable event carrying a great impact (cf. Nassim Taleb, *The Black Swan: The Impact of the Highly Improbable*, (New York: Random House, 2007)).
- 5 Henri Bergson, *Creative Evolution* (New York: Dover Publications, 1998), Nelson Goodman, *Fact, Fiction and Forecast* (Cambridge, Massachusetts: Harvard University Press, 1955), Bas van Fraassen, *Laws and Symmetry* (Oxford: Oxford University Press, 1989).
- 6 Michel Bitbol, *Quantum Mechanics as Generalized Theory of Probabilities*, translated by Robin Mackay, *Collapse*, VIII, 2014.
- 7 Michael Jubien, *Possibility* (Oxford: Oxford University Press, 2009).
- 8 Gilles Deleuze, *The Logic of Sense*, translated by Mark Lester (New York: Columbia University Press, 1990).

- 9 'Almost surely' means with probability equal to 1.
- 10 David Deutsch, *The Beginning of Infinity: Explanations that Transform the World* (New York: Penguin, 2011).

2 The vision ahead

- 1 Elie Ayache, *The Blank Swan: The End of Probability* (Chichester: John Wiley & Sons Ltd, 2010), p. 41.
- 2 David Lewis, *On the Plurality of Worlds* (Oxford, England and Cambridge, Massachusetts: Blackwell Publishers, 1986).
- 3 J. Alberto Coffa, *The Semantic Tradition from Kant to Carnap: To the Vienna Station* (Cambridge: Cambridge University Press, 1991).
- 4 Bruno de Finetti, *Theory of Probability*, vols. 1 and 2, English translation (New York: John Wiley & Sons Ltd, 1974).
- 5 Elie Ayache, *The Blank Swan: The End of Probability* (Chichester: John Wiley & Sons Ltd, 2010).
- 6 Gilles Deleuze, *Nietzsche and Philosophy*, translated by Hugh Tomlinson (Minneapolis, Minnesota: University of Minnesota Press, 1983).
- 7 Pierre Menard is a fictional character created by Jorge Luis Borges. Menard is a twentieth-century Frenchman who has dedicated his life to writing two chapters of *Don Quixote* – not a modern adaptation, but the *Quixote* itself, using exactly the same (Spanish) words Cervantes has used. Menard's book has a completely different meaning than the original *Quixote* and, according to Borges, is, in fact, far more profound (Jorge Luis Borges, Pierre Menard, Author of the *Quixote*, in *Collected Fictions*, translated by Andrew Hurley (New York: Penguin Books, 1999), pp. 88–95).
- 8 François Zourabichvili, *Deleuze: A Philosophy of the Event*, Gregg Lambert and Daniel W. Smith (eds), translated by Kieran Aarons (Edinburgh: Edinburgh University Press, 2012), Alain Badiou, *Being and Event*, translated by Oliver Feltham (London and New York: Verso, 2006), Quentin Meillassoux, *After Finitude: An Essay on the Necessity of Contingency*, translated by Ray Brassier (London: Continuum, 2008).
- 9 Jacques Derrida, *Margins of Philosophy*, translated by Alan Bass (Chicago: University of Chicago Press, 1982).
- 10 Elie Ayache, The medium of contingency, *Pli*, 22, 2011.
- 11 In the last part, in which we tackle the formalization of the market of contingent claims, this whole process is dehumanized and the way to look at the market will be through the eyes, or the channel, of the formalism as it is constantly exceeded by writing and recalibration.
- 12 In the last part, this human episode will be formally re-expressed as the n^{th} episode of manufacture of the $(n+1)^{\text{th}}$ contingent payoff with the help of the n existing contingent claims, which is followed by the writing and trading of the corresponding $(n+1)^{\text{th}}$ contingent claim, whose price provides the next exit and the next occasion to invert the model. Calibration and

recalibration will be formally re-expressed in terms of perpetually checking the limit of the formalism and stepping outside that limit. For now, however, the trader is the exit.

3 Introducing the market

- 1 This chapter was first published, in a slightly different version, under the title ‘The Writing of the Market’, in *Collapse VIII*, the Journal of Philosophical R&D, edited by Robin Mackay (Falmouth: Urbanomic, 2015).
- 2 Nassim Taleb, *Antifragile: How to Live in a World We Don’t Understand* (London and New York: Penguin Books, 2012).
- 3 Jean Baudrillard, *Impossible Exchange*, translated by Chris Turner (London and New York: Verso, 2001).
- 4 Julian Barbour, *The End of Time: The Next Revolution in Physics* (Oxford: Oxford University Press, 1999). In an interview with Edge, Barbour says: ‘My basic idea is that time as such does not exist. There is no invisible river of time. But there are things that you could call instants of time, or “Nows”. As we live, we seem to move through a succession of Nows, and the question is, what are they? They are arrangements of everything in the universe relative to each other in any moment, for example, now. [...] I suggest that our belief in time and a past arises solely because our entire experience comes to us through the medium of static arrangements of matter, in Nows, that create the appearance of time and change’ (<https://edge.org/conversation/the-end-of-time>).
- 5 ‘The problems arising through a misinterpretation of our forms of language have the character of *depth*’ (Ludwig Wittgenstein, *Philosophical Investigations*, translated by G. E. M. Anscombe (Oxford: Basil Blackwell Ltd, 1953), p. 47).
- 6 David Wood, *Thinking After Heidegger* (Cambridge: Polity, 2002).
- 7 Quentin Meillassoux, *After Finitude: An Essay on the Necessity of Contingency*, translated by Ray Brassier, (London: Continuum, 2008).
- 8 François Laruelle, *The Concept of Non-Photography/Le concept de non-photographie*, translated by Robin Mackay (bilingual edition, Urbanomic and Sequence Press: Falmouth and New York, 2011).
- 9 Quentin Meillassoux, *The Number and the Siren*, translated by Robin Mackay (Urbanomic and Sequence Press: Falmouth and New York, 2012).
- 10 Elie Ayache, Why 13 Can Only Succeed to 11, or, The End of Probability, *Wilmott*, July 2006, pp. 30–38.
- 11 Salih N. Neftci, *Principles of Financial Engineering* (San Diego, California: Academic Press, 2004), p. 128.

4 The thought behind

- 1 David Bohm, *Causality and Chance in Modern Physics* (London: Routledge & Kegan Paul, 1957).

- 2 Karl Popper, Quantum Mechanics without ‘The Observer’, in *Quantum Theory and Reality*, Mario Bunge (ed.) (Berlin, Heidelberg and New York: Springer-Verlag, 1967).
- 3 Paulette Destouches-Février, *La Structure des Théories Physiques*, with a foreword by Louis de Broglie (Paris: Presses Universitaires de France, 1951).
- 4 Jan von Plato, *Creating Modern Probability: Its Mathematics, Physics and Philosophy in Historical Perspective* (Cambridge: Cambridge University Press, 1994).
- 5 J. Alberto Coffa, *The Semantic Tradition from Kant to Carnap: To the Vienna Station* (Cambridge: Cambridge University Press, 1991).
- 6 Glenn Shafer and Vladimir Vovk, *Probability and Finance: It Is Only a Game!* (New York: John Wiley & Sons Ltd, 2001).
- 7 The strong law of large numbers is a theorem that was proven by Emile Borel in 1909. It states that the frequency of appearance of a certain event A in a series of independent trials converges to the probability p of that event almost surely, or in other words, with probability equal to 1.
- 8 Cournot’s principle is the principle according to which an event whose probability is very close to 1 will happen. Surprising as it may sound, this principle is not part of probability theory. Kolmogorov adds it separately to the axioms of the theory in a paragraph entitled ‘Relation to Experimental Data’ (Andrei Nikolaevich Kolmogorov, *Foundations of the Theory of Probability* (New York: Chelsea Publishing Company, 1950), p. 4). Indeed, it is an *interpretation* of the formalism of probability theory, or a bridge between the formalism and reality, to specify as a *meaning* of a probability close to 1 that the event to which this probability is assigned by the theory will empirically happen.
- 9 Glenn Shafer and Vladimir Vovk, *Probability and Finance: It Is Only a Game!* (New York: John Wiley & Sons Ltd, 2001), p. 3.
- 10 Glenn Shafer and Vladimir Vovk, *Probability and Finance: It Is Only a Game!* (New York: John Wiley & Sons Ltd, 2001), p. 25.

5 From throwing the dice to grasping brownian motion

- 1 A condensed version of this chapter was originally published, under the title ‘A Formal Deduction of the Market’, in Collapse VIII, the Journal of Philosophical R&D, edited by Robin Mackay (Falmouth: Urbanomic, 2015).
- 2 Andrei Nikolaevich Kolmogorov, *Foundations of the Theory of Probability* (New York: Chelsea Publishing Company, 1950).
- 3 ‘An axiomatized theory of truth may be compared with, say, Kolmogorov’s axiomatization of probability, which puts clear constraints on the concept of probability but leaves open such questions as whether probability is to be further characterized as relative frequency, degree of belief, or something else’ Donald Davidson, *Truth & Predication* (London, England and Cambridge, Massachusetts: Harvard University Press, 2005), p. 32.
- 4 Karl Popper, *Realism and the Aim of Science* (London and New York: Routledge, 1983).

- 5 'The concept of mutual *independence* of two or more experiments holds, in a certain sense, a central position in the theory of probability. [...] Historically, the independence of experiments and random variables represents the very mathematical concept that has given the theory of probability its particular stamp. [...] We thus see, in the concept of independence, at least the germ of the peculiar type of problem in probability theory' (Andrei Nikolaevich Kolmogorov, *Foundations of the Theory of Probability* (New York: Chelsea Publishing Company, 1950), pp. 8-9).
- 6 Richard von Mises, *Probability, Statistics and Truth* (New York: Dover, 1981).
- 7 Andrei Nikolaevich Kolmogorov, *Foundations of the Theory of Probability* (New York: Chelsea Publishing Company, 1950), p. 2.
- 8 Richard von Mises, *Probability, Statistics and Truth* (New York: Dover, 1981), p. 88.
- 9 The probability distribution of a random variable is a positive function on the real set and is defined as the measure of the subset of sample space whose image by the random variable is the set of real numbers that fall below a certain real number. The mean of the random variable is defined as a mathematical integral running over its different values with a weighting equal to the measures of their pre-images by the random variable. Two random variables are said to be independent when their joint probability distribution is equal to the product of the marginal ones.
- 10 Karl Popper, *Realism and the Aim of Science* (London and New York: Routledge, 1983).
- 11 Through the virtual, of course, or an interpretation of reality through the immanence of the gambling strategy that Shafer and Vovk propose in their game-theoretic probability formalism and no longer through its actual realization; immanence because the strategy is ex-post in its attitude, even if it looks ex-ante (it is the prefiguration of the market). The strategy is the occupation of the place in which to gamble and play and only through place, which it occupies and in which the event precisely takes place, can it apparently recede back to a place in time which is situated before the event and can it look ex-ante. In reality, this is immanence and ex-post because we have retired the count, the generator, the state and the ex-ante expectation. As we will see later, it is for the reason that statistics make sense materially through the ex-post accounting equation and the retiring of the ex-ante attitude that the immanence in statistics is the same as the immanence in Shafer and Vovk.
- 12 'For Deleuze, the ontology of mathematics is not reducible to axiomatics, but must be understood much more broadly in terms of the complex tension between axiomatics and what he calls "problematics" [...] The concept of multiplicity, even within mathematics itself, cannot simply be identified with the concept of a set; rather, mathematics is marked by a tension between extensive multiplicities or sets (the axiomatic pole) and virtual or differential multiplicities (the problematic pole), and the incessant translation of the latter into the former [...] It is this *other* mathematics – problematics, as opposed to axiomatics as a "specifically scientific Oedipus" – that Deleuze attempts to uncover and formalize in his work [...] The theory of extensional multiplicities (Cantor's set theory) and its rigorous axiomatization (Zermelo-Fraenkel et al.) is one of the great achievements of modern mathematics [...] For

Deleuze, the task was quite different, since he himself had to construct a hitherto non-existent (philosophical) formalization of differential or virtual multiplicities which are, by his own account, selected against by “royal” mathematics itself [...] The calculus was shot through with dynamic notions such as infinitesimals, fluxions and fluents, thresholds, passages to the limit, continuous variation – all of which presumed a *geometrical* conception of the continuum: in other words, the idea of a process. For most mathematicians, these were considered to be “metaphysical” ideas that lay beyond the realm of mathematical definition [...] It was not until the end of the nineteenth century that the calculus would receive a “rigorous” foundation through the development of the “limit-concept.” “Rigor” meant that the calculus had to be separated from its problematic origins in geometrical conceptions or “intuitions,” and reconceptualized in purely arithmetic terms [...] The aim of Weierstrass’s “discretization” program was to separate the calculus from the geometry of continuity and base it on the concept of number alone [...] Cantor’s set theory, finally, gave a discrete interpretation of the notion of infinity itself, treating infinite sets like finite sets (the power set axiom) – or rather, treating all sets, whether finite or infinite, as mathematical objects (the axiom of infinity) [...] The assumption of Weierstrass’s discretization problem – that only arithmetic is rigorous, and that geometric notions are unsuitable for secure foundations – are now largely identified with the “orthodox” or “major” view of the history of mathematics as a progression toward ever more “well-founded” positions [...] None the less, [Deleuze] insists that the fundamental difference in kind between problematics and axiomatics remains, even in contemporary mathematics [...] One of the aims of Deleuze’s own theory of multiplicities is to assess the status of such notions *as* problematic [...] According to Deleuze, mathematics is constantly producing notions that have an objectively problematic status; the role of axiomatics (or its precursors) is to codify and solidify these problematic notions, providing them with a theorematic ground or rigorous foundation [...] Problematic concepts often (though not always) have their source in what Deleuze terms the “ambulatory” sciences, which includes sciences such as metallurgy, surveying, stonecutting, and perspective [...] Such sciences are linked to notions – such as heterogeneity, dynamism, continuous variation, flows, and so on – that are “barred” or banned from the requirements of axiomatics, and consequently they tend to appear in history as that which was superseded or left behind [...] What is crucial between the two poles [axiomatics and problematics] is thus the processes of translation that take place between them – for instance, in Descartes and Fermat, an algebraic translation of the geometrical; in Weierstrass, a static translation of the dynamic; in Dedekind, a discrete translation of the continuous [...] In general, Deleuze’s work in mathematical “epistemology” tends to focus on the reduction of the problematic to the axiomatic, the intensive to the extensive, the continuous to the discrete, the non-metric to the metric, the non-denumerable to the denumerable, the rhizomatic to the arborescent, the smooth to the striated [...] At times, Deleuze suggests, axiomatics can possess a deliberate will to halt problematics. “State science retains of nomad science only what it can appropriate; it turns the rest into a set of strictly limited formulas without any real scientific status, or else simply

represses and bans it” (Gilles Deleuze and Félix Guattari, *A Thousand Plateaus*, translated by Brian Massumi (Minneapolis: University of Minnesota Press, 1989), p. 362) [...] Deleuze’s project, as we have seen, is to formalize the conception of multiplicities that corresponds to the problematic pole of mathematics. In other words, problematics, no less than axiomatics, is the object of pure mathematics’ (Daniel W. Smith, *Essays on Deleuze* (Edinburgh: Edinburgh University Press Ltd, 2012), pp. 288–300. All of this to say that, in this book, we clearly side with problematics, rather than axiomatics. Our ambulatory science, our metallurgy, is the market and the technology of the derivative price. Due to its intensive nature, price poses a problem to the extensional system of probability. We talk of matter and continuity as opposed to state and arborescence. However, we clearly do not reject axiomatics, as we are ultimately looking for a formalization of price, or for a ‘solution’ to our problem.

- 13 Jacques Bonitzer writes: ‘In substituting probability that is equal to zero in the limit for “very small” probability, does the strong law of large numbers escape the infinite regress that the weak law seemed to carry? [...] When scrutinized, the infinite regress seems indeed to subsist. Surely, in the strong law, the limit of the average of random variables *is equal* to the mean (as in the axiomatics of von Mises), but this equality occurs only with *probability* equal to 1, that is to say, up to an event of measure zero. The limit may very well *logically* differ from the mean, but the frequency of such an event, in an infinite sequence of infinite sequences of random variables, will be zero; and this forms a regress because a frequency that is equal to zero will in turn only be realized with a probability equal to 1. It is this staunch persistence of a germ of infinite regress in the logic of probability that constitutes its specificity. No formal-logical reasoning can get rid of it. Is this not because the logic of probability is, in the end, a *practical* logic?’ (Jacques Bonitzer, *Philosophie du hasard* (Paris: Éditions Sociales, 1984), p. 171, my translation). To which I add that ‘practical’ can be interpreted as ‘material’. It is because it reaches to matter that the formalism of probability experiences infinity.
- 14 Writing without waiting or expecting, writing without hoping, this is the only material process. It seems we are running with time, or writing the event ahead of time, however writing is a ruin, it takes place in the middle of event, in its happening. We are blind to the identity of the event while we write, we are blind to the time of identification – writing is a forgetting – and this is why, although writing takes place incidentally in time, ahead of the event, this temporal separation from the event doesn’t *matter*. The time of writing, or material time, is different from chronological time. Writing is a *market* in the sense that the market is connected to the event materially. Writing is an exchange, for this reason sent to the period that is incidentally prior to the event in time. What matters in writing is the exceptional strategy, the place that we occupy while we write: a place in the train, from which hope or the woman has vanished and in which the outcome no longer matters. Even if incidental time is still around, anyone who has retired hope and expectation from the process qualifies as a writer without hope, always ex-post and always dwelling in the middle of the event, in the sense that he only occupies place from now on.
- 15 Jean Ville, *Étude critique de la notion de collectif* (Paris : Gauthiers-Villars, 1939).

- 16 Originally, a martingale is any strategy for placing bets (not necessarily a strategy with guaranteed positive outcome). Ville formalized the concept by calling 'martingale' the strategy's capital process, or the real valued function that assigns to each sequence of outcomes the capital that you will have accumulated by betting on each outcome and having settled your bets. In a fair game, the value of the bet is supposed to reflect the probability of the outcome and your capital gains are not supposed to grow indefinitely. As a matter of fact, the expected value of your future accumulated gains must be equal to the gains you have already accumulated. If the concept of probability was not available and you only knew of money and of betting, the condition that your capital process should not grow to infinity could be used as an alternative to probability. This is done by Shafer and Vovk: 'We say that an event happens almost surely (that is, with probability one) if there is a nonnegative capital process that diverges to infinity if the event does not happen' (Glenn Shafer and Vladimir Vovk, *Probability and Finance: It Is Only a Game!* (New York: John Wiley & Sons Ltd, 2001), p. 53).
- 17 Elie Ayache, *The Blank Swan: The End of Probability* (Chichester: John Wiley & Sons Ltd, 2010).
- 18 Glenn Shafer and Vladimir Vovk, *Probability and Finance: It Is Only a Game!* (New York: John Wiley & Sons Ltd, 2001), p. 56.
- 19 'Almost surely, [Brownian motion] is *nowhere* differentiable! This is particularly interesting, as it is not easy to construct a continuous, nowhere differential function without the help of randomness' (Peter Mörtens and Yuval Peres, *Brownian Motion* (Cambridge: Cambridge University Press, 2010), p. 3).
- 20 Salih N. Neftci, *Principles of Financial Engineering* (San Diego, California: Academic Press, 2004).
- 21 Fisher Black and Myron Scholes, The pricing of options and corporate liabilities, *Journal of Political Economy*, 81(3), May–June 1973, 637–654.
- 22 David Deutsch, *The Beginning of Infinity: Explanations that Transform the World* (New York: Penguin, 2011).
- 23 The 'smile' phenomenon is the observation that options prices, as empirically given by the options market, do not agree with the Black-Scholes-Merton option pricing model and the assumption it makes that the price process of the underlying is governed by Brownian motion with constant volatility. Instead of reading as outputs of the Black-Scholes-Merton formula, with the same volatility number plugged into it, empirical options prices require a different volatility number to be plugged into the formula, each time we wish to apply it to explain them theoretically. Although written on the same underlying, options with different strike prices will thus imply different volatility numbers. Out-of-the-money options (call options whose strike price is above the underlying spot price and put options whose strike price is below the underlying spot price) will typically imply a volatility number that grows larger and larger as the strike moves away on either side of the spot price, thus exhibiting a curve whose shape resembles a smile. The smile problem consists in finding an alternative model to Black-Scholes-Merton to try to explain this phenomenon.

6 From the marvel of brownian motion to the reality of the market

- 1 Part of this section was first published under the title 'The Technology of the Future', in *Realism, Materialism, Art*, ed. Chirstoph Cox, Jenny Jaskey, Suhail Malik (New York: Center for Curatorial Studies, Bard College; Berlin, Sternberg Press, 2015).
- 2 'The event is forbidden, ontology rejects it' Alain Badiou, *Being and Event*, translated by Oliver Feltham (London: Continuum, 2005), p. 184.
- 3 Gilles Deleuze, *Bergsonism*, translated by Hugh Tomlinson and Barbara Habberjan (New York: Zone, 1988).
- 4 In his presentation of Badiou's philosophy, Peter Hallward writes: 'Where exactly lies the ontological peculiarity of the event? Unlike all normally structured or well-founded multiples, an event belongs to no already existent set. Insofar as it "exists" at all – and remember that to exist means to belong to a set – the event simply belongs to itself. It is, as an occurrence, self-founding, which is to say that it is properly unfounded. [...] In set-theoretic terms, an event is exceptional because it does not comply with the axiom of foundation, that is, the axiom proposed (by Zermelo in 1906) precisely in order to block the paradoxical possibility of sets belonging to themselves. Because it violates the axiom of foundation, the event is forbidden; ontology rejects it' (Peter Hallward, *Badiou: A Subject to Truth* (Minneapolis and London: University of Minnesota Press, 2003), p. 116).
- 5 Henri Bergson, *Matter and Memory*, translated by N. M. Paul and W. S. Palmer (New York: Zone Books, 1990).
- 6 Maurice Blanchot, *The Book to Come*, translated by Charlotte Mandell (Stanford, California: Stanford University Press, 2003).
- 7 Elie Ayache, *The Blank Swan: The End of Probability* (Chichester: John Wiley & Sons Ltd, 2010).

7 The paper and the tree

- 1 Part of this section was first published under the title 'The Medium of Contingency', in *Algorithmisiert* by Sylvia Eckermann (Wien: Czernin 2014).

8 Archaeology of the multiple

- 1 Cf. the difficulty of the notion of realization in the probability formalism.
- 2 Gilles Deleuze, *Bergsonism*, translated by Hugh Tomlinson and Barbara Habberjan (New York: Zone, 1988).
- 3 Gilles Deleuze, *Difference and Repetition*, translated by Paul Patton (New York: Columbia University Press, 1994).
- 4 '[...] negation is not the work of pure mind, I should say of a mind placed before objects and concerned with them alone [...] negation is but the half of an intellectual act, of which the other half is left indeterminate [...] A negative judgment is therefore really one which

indicates a need of substituting for an affirmative judgment another affirmative judgment, the nature of which, however, is not specified, sometimes because it is not known, more often because it fails to offer any actual interest, the attention bearing only on the substance of the first [...] in neither of these two acts is there anything but affirmation [...] It is in vain, then, that we attribute to negation the power of creating ideas *sui generis*, symmetrical with those that affirmation creates, and directed in a contrary sense [...] How comes it, then, that affirmation and negation are so persistently put on the same level and endowed with equal objectivity? [...] The reason is, no doubt, that *both* negation and affirmation are expressed in propositions, and that *any* proposition, being formed of *words*, which symbolize *concepts*, is something relative to social life and to the human intellect [...] From this point of view, which is that of formal logic, to affirm and to deny are indeed two mutually symmetrical acts [...] But how do we fail to see that the symmetry is altogether external and the likeness superficial? Suppose language fallen into disuse, society dissolved, every intellectual initiative, every faculty of self-reflection and of self-judgment atrophied in man [...] The intellect will still affirm, in implicit terms [...] But this passive intelligence, mechanically keeping step with experience, neither anticipating nor following the course of the real, would have no wish to deny' (Henri Bergson, *Creative Evolution* (New York: Dover Publications, 1998), pp. 290–292).

- 5 Jean Baudrillard, *Impossible Exchange*, translated by Chris Turner (London and New York: Verso, 2001).
- 6 The present state which integrates the different and analogical states is quite simply the *random variable* which summarizes them and with which we rightly associate a distribution of probability. But the formal random variable knows nothing of randomness and nothing of physical time. Physical time and randomness are just an *interpretation* of the probability formalism.
- 7 To continue quoting from Bergson: 'Suppose language fallen into disuse, society dissolved, every intellectual initiative, every faculty of self-reflection and of self-judgment atrophied in man [...] The intellect will still affirm, in implicit terms [...] But this passive intelligence, mechanically keeping step with experience, neither anticipating nor following the course of the real, would have no wish to deny [...] For such an intellect to reach the point of denying, it must awake from its torpor, formulate the disappointment of a real or possible expectation, correct an actual or possible error [...] For a mind which should follow purely and simply the thread of experience, there would be no void, no nought, even relative or partial, no possible negation. Such a mind would see facts succeed facts, states succeed states, things succeed things. What it would note at each moment would be things existing, states appearing, events happening. It would live in the actual, and, if it were capable of judging, it would never affirm anything except the existence of the present. Endow this mind with memory, and especially with the desire to dwell on the past; give it the faculty of dissociating and of distinguishing: it will no longer only note the present state of the passing reality; it will represent the passing as a change, and therefore as a contrast between what has been and what is. And as there is

no essential difference between a past that we remember and a past that we imagine, it will quickly rise to the idea of the “possible” in general’ (Henri Bergson, *Creative Evolution* (New York: Dover Publications, 1998), pp. 292–294).

- 8 Elie Ayache, The medium of contingency, *Pli*, 22, 2011.
- 9 In a chapter on the individualistic (as opposed to statistical) interpretation of quantum mechanics, Peter Mittelstaedt writes: ‘[...] a probability proposition P is a proposition the referent of which is a compound quantum system S composed of N originally equally prepared systems S_n [prepared by preparation φ] after a measurement of [observable] B . The proposition P states that the relative frequency of obtaining the result b in the ensemble S approaches the value p for $N \rightarrow \infty$ [...] However, this explanation does not answer the question what the term p could mean for an individual system S with preparation φ . This question is motivated by the fact that quantum probabilities express some genuine indeterminism and cannot be reduced to a distribution of hidden initial conditions [...] In order to attach probabilities p to a single physical system, two often discussed proposals have been made by Heisenberg and by Popper. In the first approach, Heisenberg referred to the philosophy of Aristotle. Being aware that quantum probabilities must not be considered as the quantification of the observer’s subjective only, he tried to understand probabilities also as some kind of physical reality that pertains to the object system in question. In this sense, he argued that quantum probabilities are a quantitative formulation of the concept of *potentia* in Aristotle’s philosophy [...] The second attempt to give some meaning to the probability of a single-event was made by Popper. He introduced the concept of *propensity* in order to provide a single-case interpretation of probability. However, the propensity concept was not primarily conceived as a means for interpreting quantum mechanics. Instead, it was understood by Popper as a new concept that can be used for all kinds of probabilistic problems [...] one has to distinguish the deterministic case, e.g. dice-tossing experiments, and the indeterministic case, which presumably incorporates quantum mechanical situations. Whereas for deterministic situations the propensity concept is often considered to be void, some authors claim that propensities might be particularly useful in quantum mechanical situations, which are considered to be indeterministic. In spite of these intuitively rather convincing arguments, in an actual experimental situation this way of reasoning amounts to saying almost nothing. First, there is no measurable property of a single system that could be interpreted as a probability. Second, and this is more important, a pure state φ provides already maximal information about the system $S(\varphi)$. Indeed, the values of all objective properties can be deduced from the knowledge of φ , and these values will be obtained with certainty by a measuring process [...] These arguments show that for an individual system $S(\varphi)$ there is no need for an additional “potentia” or “propensity”, since from the state φ one can deduce all objective properties, even the statistical ones that turn out to be yes–no properties of a conveniently defined ensemble’ (Peter Mittelstaedt, *The Interpretation of Quantum Mechanics and the Measurement Process* (Cambridge: Cambridge University Press, 1998), pp. 61–63).

- 10 See the note above.
- 11 Elie Ayache, In the Middle of the Event, in *The Medium of Contingency*, Robin Mackay (ed.) (Falmouth: Urbanomic, 2011).

9 Archaeology of the exchange

- 1 Jean Baudrillard, *Impossible Exchange*, translated by Chris Turner (London and New York: Verso, 2001).
- 2 Alain Badiou, *Deleuze: The Clamor of Being*, translated by Louise Burchill (Minneapolis and London: University of Minnesota Press, 2000), p. 42.
- 3 There is, of course, the field of *imprecise probability* as a generalization of classical probability, and it is not foreign to the work of Shafer and Vovk. However, what we are talking about in the present instance is not imprecision in the assignment of probability, which the theories of imprecise probability usually deal with by lower and upper probabilities or intervals of probability, but imprecision of the states of the world themselves, or worse, their inexistence.
- 4 François Zourabichvili, *Deleuze: A Philosophy of the Event*, Gregg Lambert and Daniel W. Smith (eds), translated by Kieran Aarons (Edinburgh: Edinburgh University Press, 2012), p. 100.
- 5 In reference to the reference class problem affecting Popper's propensity theory, Donald Gillies writes: 'A long-run propensity theory is one in which propensities are associated with repeatable conditions, and are regarded as propensities to produce, in a long series of repetitions of these conditions, frequencies which are approximately equal to the probability. A single-case propensity theory is one in which propensities are regarded as propensities to produce a particular result on a specific occasion. [...] [Popper's] characterization of propensities corresponds to our long-run propensities, and yet he wanted these propensities to apply to the single case as well. This position ran into difficulties connected with the reference class problem [...]. My own preference is for long-run propensity theory, and for dealing with the single case by subjective probabilities which may however be objectively based' (Donald Gillies, *Philosophical Theories of Probability* (London and New York: Routledge, 2000), p. 126.). Gillies further writes: 'Finally we can introduce probabilities for single instances of the set S_1 of conditions. These singular probabilities are, however, only validly defined with respect to the given set of repeatable conditions and the given set of possible outcomes under which the single event in question is subsumed' (Donald Gillies, *An Objective Theory of Probability* (London: Methuen & Co. Ltd, 1973), pp. 155–156.)
- 6 To emulate Einstein, who turned simultaneity into a notion more fundamental than time and replaced the metaphysics of absolute time with the material procedure of synchronization of distant clocks.
- 7 Jean Baudrillard, *Impossible Exchange*, translated by Chris Turner (London and New York: Verso, 2001).
- 8 Quentin Meillassoux, *After Finitude: An Essay on the Necessity of Contingency*, translated by Ray Brassier (London: Continuum, 2008).

10 Matter and geometry

- 1 David Bohm, *Causality and Chance in Modern Physics* (London: Routledge & Kegan Paul, 1957), p. 2.

11 Towards a contemporary theory of the market

- 1 Julian Barbour, *The End of Time: The Next Revolution in Physics* (Oxford: Oxford University Press, 1999).
- 2 See Rama Cont, Jose da Fonseca and Valdo Durrleman, Stochastic models of implied volatility surfaces, *Economic Notes*, 31(2), 2002, 361–377.

12 Incomplete markets

- 1 ‘An advantage of game-theoretic probability is that one does not have to start with a full-fledged probability measure from the outset to arrive at interesting conclusions, even in the case of continuous time. [...] The standard qualification “with probability one” is replaced with “unless a specific trading strategy increases manyfold the capital it risks”. [...] This paper makes the next step, showing that the Wiener measure emerges in a natural way in the continuous trading protocol’ (Vladimir Vovk, Continuous-time trading and the emergence of probability, *Finance and Stochastics*, 16(4), October 2012, 561–609).
- 2 ‘In an incomplete market the requirement of no arbitrage is no longer sufficient to determine a unique price for the derivative. [...] When dealing with derivative pricing in an incomplete market we thus have to fix a specific martingale measure Q , or equivalently a market price of risk λ , and the question arises as to how this is to be done. Question: Who chooses the martingale measure? From the discussion above the answer should by now be fairly clear. Answer: The market!’ Tomas Björk, *Arbitrage Theory in Continuous Time* (Oxford: Oxford University Press, 2009), pp. 224–225.
- 3 This is the step taken by Romano and Touzi (1997) and, more recently, by Mark Davis and Jan Obloj (2008). Romano and Touzi first recognize that ‘the determination of a pricing rule for nonattainable contingent claims in incomplete markets is not clearly solved in the existing literature’. By pricing rule, what is understood is a certain rationale for the pricing of contingent claims, which may not generally assume a model for the underlying dynamics. When the latter is considered, however, and volatility is, for instance, assumed to be stochastic, pricing rules can, for example, be utility-based or rely on the quadratic minimization of hedging error. In all cases, they consist in a particular choice of the volatility risk premium. It can be shown that the pricing rule is, in general, induced by an equivalent martingale measure, and the link with the objective underlying process is thus established. When the underlying asset follows Brownian motion with stochastic volatility, Romano and Touzi show that if the price process of a certain contingent claim with convex payoff is given and is a martingale under some equivalent martingale measure, then a dynamic self-financing

strategy involving the underlying asset and this contingent claim eventually replicates the payoff of any other contingent claim. The market is thus completed. It will have selected, of course, the equivalent martingale measure in which the price of the contingent claim we used in the dynamic strategy was initially supposed to be a martingale. As a matter of fact, the market has always been complete – its completeness didn't suddenly emerge with the proof of Romano and Touzi – and the equivalent martingale measure has always been unique, since it is the one the market has uniquely selected. Davis and Obloj, who continue this line of thought, even go as far as declaring that 'the distinction between an "underlying asset" and a "contingent claim" largely disappears in this approach'. As a matter of fact, in the so-called 'market models', 'one specifies directly the price processes of all traded assets, be they underlying assets or derivatives'. The whole puzzle of the equivalent martingale measure that the market selects is thus only an artifact of the order of presentation of the 'martingale models', in which the underlying asset is still distinguished, at least in setting the extent to which the market is first recognized to be incomplete. In truth, the underlying asset must be distinguished, if only because it is the ultimate trigger of the payoff of the contingent claims and the prices of the latter have to verify inter-temporal non-arbitrage, given this ultimate event. Unsurprisingly, Davis and Obloj recognize that while 'the problem with martingale models is that the modelling of asset volatilities is too indirect' (indeed, the absence of arbitrage imposes that the 'direct' volatilities have ultimately to be the volatility of the underlying asset and the volatility of any of the parameters of its process that we may have assumed to be stochastic in order to create market incompleteness), 'the problem with market models is the extremely awkward set of conditions required for absence of arbitrage' (Marc Romano and Nizar Touzi, Contingent claims and market completeness in a stochastic volatility model, *Mathematical Finance*, 7(4), October 1997, pp. 399–410; and Mark Davis and Jan Obloj, Market Completion Using Options, in *Advances in Mathematics of Finance*, L. Stettner (ed.), Banach Center Publications, 83 (Warsaw: Institute of Mathematics, Polish Academy of Sciences, 2008), pp. 49–60.)

- 4 'Is it possible that an observed and incomplete set of prices and payoffs is generated by some complete-markets, contingent-claim economy? The answer is, yes, if there is no arbitrage on the observed prices and payoffs' (John H. Cochrane, *Asset Pricing* (Princeton and Oxford: Princeton University Press, 2001), p. 74
- 5 John H. Cochrane, *Asset Pricing* (Princeton and Oxford: Princeton University Press, 2001).
- 6 For instance, Tomas Björk writes: 'Take a fixed derivative as given. What is a "fair" price for the contract? [...] There exist two natural and mutually contradictory answers. Answer 1: "Using standard principles of operations research, a reasonable price for the derivative is obtained by computing the expected value of the discounted future stochastic payoff." Answer 2: "Using standard economic reasoning, the price of a contingent claim, like the price of any other commodity, will be determined by market forces. In particular it will be determined by the supply and demand curves for the market for derivatives. Supply and demand will in their turn be influenced by such factors as aggregate risk aversion, liquidity preferences, etc., so it

is impossible to say anything concrete about the theoretical price of a derivative.” The reason that there is such a thing as a theory for derivatives lies in the following fact. Main Result: *Both answers above are incorrect! It is possible (given, of course, some assumptions) to talk about the “correct” price of a derivative, and this price is not computed by the method given in Answer 1 above*’ (Tomas Björk, *Arbitrage Theory in Continuous Time* (Oxford: Oxford University Press, 2009), p. 4).

- 7 Michael Magill and Martine Quinzii write: ‘In the late-sixties-early-seventies two papers appeared which were the first to explore equilibrium with “incomplete” financial markets. Diamond’s model (1967) inspired by the finance literature of the 1960’s, focused directly on the “realistic” market structure consisting of the stock and bond markets, while Radner’s model (1972) was inspired by the abstract tradition of Arrow-Debreu theory and explored the consequence of replacing Arrow securities by an incomplete set of contracts for the contingent delivery of commodities. Both papers took the financial structure as given, without discussing why the markets might be incomplete. Indeed Radner made it explicit at the end of his introduction that he took the available securities as given “without any explanation of why some contracts are allowed and others are not”. The assumption that there is an exogenously given financial structure is characteristic of the incomplete markets literature [...]. While the structure of the securities embedded in the model is progressively enriched, the reasons for the incompleteness are typically left out of the discussion and are not present in the model’ (Michael Magill and Martine Quinzii, *Incomplete Markets, Volume 1, Finite Horizon Economies*, International Library of Critical Writings in Economics (Cheltenham: Edward Elgar Publishing Company, 2009); and R. Radner, Existence of equilibrium of plans, prices and price expectations in a sequence of markets, *Econometrica*, 40, 1972, 289–304.
- 8 See Rama Cont and Sara Biagini, Model-free representation of pricing rules as conditional expectations, in: Stochastic processes and applications to mathematical finance, Proceedings of the 6th Ritsumeikan International Symposium, World Scientific (2007), pp. 53–66.
- 9 See Michael Magill and Martine Quinzii, *Incomplete Markets, Volume 1, Finite Horizon Economies*, International Library of Critical Writings in Economics (Cheltenham: Edward Elgar Publishing Company, 2009).
- 10 This leaves Pierre Menard as a writer without underlying states, a writer who moves between contexts or whole ranges of possibilities and not between possible states, a writer who can produce original work even in writing an existent text.

13 The central knot

- 1 To cite only one: the fact that the market should never even remotely be thought to be complete, for that is a contradiction in terms; or at least the market should only be complete as a representation internal to the tool whose usage in trading, thanks to the theoretical completeness, precisely makes the market incomplete in the only way that the market should be thought to be incomplete and in the only way that it should be *thought*, which is that it

is thought to be incomplete without an alternative and without a container inside which its incompleteness is gauged.

- 2 Vladimir Vovk, Continuous-time trading and the emergence of probability, *Finance and Stochastics*, 16(4), October 2012, 561–609.
- 3 Fisher Black and Myron Scholes, The pricing of options and corporate liabilities, *Journal of Political Economy*, 81(3), May–June 1973, 637–654.
- 4 J. Michael Harrison and David M. Kreps, Martingales and arbitrage in multiperiod securities markets, *Journal of Economic Theory*, 20, 1979, 381–408; J. Michael Harrison and Stanley R. Pliska, Martingales and stochastic integrals in the theory of continuous trading, *Stochastic Processes and their Applications*, 11, 1981, 215–260.

14 The hard problem

- 1 ‘The mathematical study of continuous time random processes emerged suddenly in 1929. [...] Why does it come so late? One aspect at least is that it was difficult to imagine a continuously acting chance. Continuous time formed part of determinist classical physics’ (Jan von Plato, *Creating Modern Probability: Its Mathematics, Physics and Philosophy in Historical Perspective* (Cambridge: Cambridge University Press, 1994), p. 135).
- 2 Louis Bachelier, *Théorie de la Spéculation* (Paris: Éditions Jacques Gabay, 1995).
- 3 Op. cit., p. 134.
- 4 See Vladimir Vovk, Continuous-time trading and the emergence of volatility, *Electronic Communications in Probability*, 13, 2008, 319–324.
- 5 Glenn Shafer and Vladimir Vovk, *Probability and Finance: It Is Only a Game!* (New York: John Wiley & Sons Ltd, 2001).
- 6 Riccardo Rebonato, *Volatility and Correlation: The Perfect Hedger and the Fox*, second edition (Chichester: John Wiley & Sons Ltd, 2004).
- 7 Peter Whittle, *Probability via Expectation* (New York: Springer-Verlag, 2000).
- 8 ‘What Nietzsche’s doctrine of eternal return allows Deleuze to say, in a nutshell, is that only differings or creations have being. Creations return, creatures do not – or rather, what returns of a creature is its creating alone’ (Peter Hallward, *Out of this World: Deleuze and the Philosophy of Creation* (London: Verso, 2006), p. 149).
- 9 In *The Blank Swan*, I write: ‘Let us assume that time and possibility are not given yet. All we have now is the *moment* of the conversion, or $\Delta S + B$. All we have is the genesis of the contingent claim: contingency not possibility. [...] Let us not call our picture static either. Indeed, the dynamics I will be investigating are the *dynamics of genesis*: the process of creation of the market as a whole (or at least, its reconstruction), the unfolding of the contingent claim and of the marketplace (as the place of pricing of the contingent claim) in one and the same movement of conversion. I call this picture dynamic because of the process of morphogenesis. By contrast, what we traditionally think is the dynamic picture of a reified market – prices that fluctuate randomly in time for no other reason than a so-called “essence” of the market

as a random generator that makes them fluctuate – is a static essentialist view’ (Elie Ayache, *The Blank Swan: The End of Probability* (Chichester: John Wiley & Sons Ltd, 2010), p. 346).

15 The book of the market

- 1 If we are not happy with volatility as a deterministic and continuous function of time and space, let us consider Dupire’s argument to the effect that local volatility can be conceived as the local expectation of stochastic volatility, conditional on the spot level. As Gatheral writes: ‘It is unlikely that Dupire, Derman and Kani ever thought of local volatility as representing a model of how volatilities actually evolve. Rather, it is likely that they thought of local volatilities as representing some kind of average over all possible instantaneous volatilities in a stochastic volatility world’ (Jim Gatheral, *The Volatility Surface: A Practitioner’s Guide* (Hoboken, New Jersey: John Wiley & Sons, Inc., 2006).
- 2 ‘We consider an idealized financial security with continuous price path, without making any stochastic assumptions. It is shown that typical price paths possess quadratic variation, where “typical” is understood in the following game-theoretic sense: there exists a trading strategy that earns infinite capital without risking more than one monetary unit if the process of quadratic variation does not exist. Replacing time by the quadratic variation process, we show that the price path becomes Brownian motion’ (Vladimir Vovk, Continuous-time trading and the emergence of probability, *Finance and Stochastics*, 16(4), October 2012, 561–609).
- 3 Riccardo Rebonato writes: ‘The fundamental approach [...] is underpinned by one very strong assumption: the trader who chooses and calibrates models [...] is subscribing to the view that the market-created option prices must be fully consistent with the true, but a priori unknown, process for the underlying. The market, in other words, must be a perfect information-processing machine, which absorbs all the relevant information about the unknown process followed by the “stock”, and produces prices consistent with each other (no arbitrage) and with this information set (informational efficiency)’ Riccardo Rebonato, *Volatility and Correlation: The Perfect Hedger and the Fox*, second edition (Chichester: John Wiley & Sons Ltd, 2004), p. 5.
- 4 Popper writes: ‘[...] the transition from the frequency interpretation to the propensity interpretation corresponds to the transition from the *mathematical frequency theory*, developed by von Mises, Copeland, Wald, Church (and myself) to the *neo-classical or measure-theoretical treatment of probability* which, I am satisfied, is superior to the frequency theory, not only from a philosophical but also from a purely mathematical point of view’ (Karl R. Popper, *Realism and the Aim of Science* (London and New York: Routledge, 1983), p. 360.
- 5 Vladimir Vovk, Continuous-time trading and the emergence of probability, *Finance and Stochastics*, 16(4), October 2012, 561–609.
- 6 Vladimir Vovk, Continuous-time trading and the emergence of probability, *Finance and Stochastics*, 16(4), October 2012, 561–609.

- 7 Martin Heidegger, *What Is Called Thinking?*, translated by J. Glenn Gray (New York: Harper & Row, 1968).
- 8 Elie Ayache, The medium of contingency, *Pli*, 22, 2011.
- 9 High frequency trading (HFT) and algorithmic trading have attracted a lot of attention recently. Critical thinkers, artists and activists have had a tendency to confuse the market with algorithms and to think its ultimate secret lay in high frequency trading. They seem fascinated by algorithms and expect with fear the moment when algorithms will dominate the world. It should be clear from our argument, and generally from the thrust of this whole book, how trivial and uninteresting HFT is, and how utterly misleading it can be in pointing the place or the dimension where the market lies. The market as genesis and creation – that is to say, as meaning – does not lie in time series of prices of the underlying asset, no matter whether sampled in nanoseconds, but in the instant and infinite landscape of prices of derivatives written on that underlying.

16 Denouement: The theory after the two narratives

- 1 See Rama Cont, Jose da Fonseca and Valdo Durrleman, Stochastic models of implied volatility surfaces, *Economic Notes*, 31(2), 2002, 361–377.
- 2 When discussing the filtrations generated by the underlying asset price process and by the implied BSM volatility process, Cont and da Fonseca recognize that they need not be coincident in general. They write: ‘This is an important property of the model and corresponds indeed to the real situation encountered on the market: option prices can be affected by factors other than the underlying asset; and, inversely, although market prices of options do give some information on the underlying asset, there is not enough information on the options market to retrieve, in a unique way, the implied dynamics of the underlying asset. This is, by the way, the fundamental reason the problem of calibrating smile models to option prices is ill-posed in general.’ However, they recognize that their perspective is risk-management under the real dynamics of the implied volatility surface and that arbitrage constraints, if they must be considered, might entail a different risk-neutral dynamics for the BSM implied volatility. In other words, Cont and da Fonseca do not extract the pricing kernel which would enforce non-arbitrage and in which the instantaneous volatility of the underlying price would ultimately be the explanatory variable and would make the two filtrations coincide. However, they cite articles by other authors where such a risk-neutral framework is adopted.
- 3 Riccardo Rebonato, *Volatility and Correlation: The Perfect Hedger and the Fox*, second edition (Chichester: John Wiley & Sons Ltd, 2004).
- 4 J. Michael Harrison and Stanley R. Pliska, Martingales and stochastic integrals in the theory of continuous trading, *Stochastic Processes and their Applications*, 11, 1981, 215–260.
- 5 Tomas Björk, *Arbitrage Theory in Continuous Time* (Oxford: Oxford University Press, 2009).

Conclusion

- 1 Alan W. Richardson, *Carnap's Construction of the World: The Aufbau and the Emergence of Logical Empiricism* (Cambridge: Cambridge University Press, 1998).

Appendix A: Regime-switching model

- 1 This description of the regime-switching model was first published, in a slightly different version, in my previous book, *The Blank Swan: The End of Probability* (Chichester: John Wiley & Sons, 2010).

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Index

- absolute contingency, 22, 41–2, 56, 64, 80, 88, 147, 155, 165, 194, 233, 240, 279
- absolute difference, 43, 99, 167, 380
- absolute local, 74–6
- absolute valuation, 193–5
- abstraction, 188–91, 219–22
- accounting equation, 332
see also insurance
- actuality, 146–7
- actuarial valuation, 12, 27
- aggregate of beliefs, 36
- aleatory point, 25, 165
- alternative exit, 289–93
- alternative worlds, *see* possible worlds
- analogical difference, 80, 191–2, 194–9, 234, 236, 244
- antifragility, 55–6, 74, 75, 78
- arbitrage, 5, 12, 14, 26, 37, 86, 251–2, 254–8, 352
 fundamental principle of, 251, 308
 intrinsic, 86, 252
- Arrow-Debreu securities, 38–9, 91–2, 280–1
- assignment, 44–5
- asymmetric payoffs, 74
- averaging, 179
- axiomatic system, of Shafer and Vovk, 92–7
- axiom of randomness, 110, 113–16, 119, 131, 134, 138, 148
- Bachelier, Louis, 300, 301, 332
- backward narrative, 19, 41
- Badiou, Alain, 41
- Baudrillard, Jean, 56
- Bayesian theory, 20, 86
- Bergson, Henri, 41, 42, 108, 147, 198
- binomial tree, 272, 274, 283, 284, 288, 301, 356
- Black-Scholes-Merton (BSM) model, 3–6, 76, 90–3, 142, 147, 279–80, 310, 351, 366, 368, 371
 assumptions of, 32
 as formalism, 316
 implied volatility and, 32–3, 62–3, 265, 318–19
 martingale measure of the market and, 270–2
 options market and, 321
 as trading tool, 346
 use of, 49
 volatility and, 9–11, 313–14, 346
- Black Swan events, 40–1, 50–1, 53–4, 66, 72–3
- Blanchot, Maurice, 63–4
- Bohm, David, 82, 239
- book of the market, 213–14, 314, 321–50
- Born's rule, 198, 201
- break-even, 27–8, 71, 75, 87, 90, 129–31, 140–2, 144–5, 148–9, 151, 154, 155, 179, 180, 231, 304
- Brownian motion, 4, 6, 26–8, 70, 75, 96, 103–45, 147, 151–3, 268, 283–5, 297, 308–10, 316, 319–20, 354–6, 369–70
 with constant volatility, 32
 contingency and, 155
 derivative prices and, 258–60
 dynamic replication and, 333–4
 market concept and, 144–5
 probability and, 142–3, 301–4, 336
 qualitative view of, 356
- BSM, *see* Black-Scholes-Merton (BSM) model
- chaos, 4, 11, 338, 345
- chrono-logical schema of thought, 4, 205, 219, 284
- chrono-logic, 115–17, 194–5, 197–200, 262
- chronology, 57, 65, 95, 115–16, 118, 126–8, 155, 171, 197–200, 205–6
- Coffa, Alberto, 84, 336
- collectives, 110
- complete markets, 267–70, 274–83, 344–5, 349
- conceptual event, 218
- constant volatility, 32, 261, 277, 310, 313, 323–4, 357, 373, 393
- contemporary art, 339–40
- contemporary theory of the market, 87, 251–66
- contingency, 1–2, 21–3, 45, 63, 82, 119
 absolute, 22, 41–2, 56, 64, 80, 83, 88, 147, 155, 165, 194, 196, 233, 240, 279
 exchange of, 214–16
 after the fact, 57
 future, 193–4
 market as medium of, 34

- material, 148, 169, 233, 241
- medium of, 25, 56–66
- possibility and, 83
- reality of, 124–6
- repetition and, 126
- strike of, 119–22, 130, 148, 153–4, 169, 204, 237–8
- of the world, 190–1
- univocity of, 192–4, 204
- writing and exchanging and, 162–7
- contingent claims, 2, 14–15, 24, 62–3, 135–6, 204, 205, 231–2, 291, 348, 362, 367
 - contingent payoff vs., 174–6, 285–6, 355, 368–9
 - as conversion of debt, 38
 - dynamics of, 43–4
 - market of, 297–9, 312–13, 318, 330, 334–6, 340–1, 360–2, 367–71
 - market prices for, 296–7
 - material exchange of, 149–50
 - medium of, 40
 - non-existence of, 298–9
 - non-redundancy of, 92
 - payoff of, 88
 - prices and, 43–5, 61, 92, 214, 345, 370–1
 - probability and, 35–7
 - real world and, 37–8
 - trading of, 344, 348–9
 - value of, 160–1, 172–3, 180, 181
 - writing, 56–7, 86, 147, 162–7, 170, 176–81, 295
- contingent payoff, 73, 174–6, 179, 285–6, 297, 304–11, 355, 359–62, 368–70
- continual event, market as, 48–50
- continuous time, 332
- continuous-time stochastic process, 300, 332
- contracts, 48–9, 161, 202, 245
- conversion of debt, 38, 153
- counterfactuals, 20
- Cournot's principle, 94–5, 106, 119, 125, 127–31, 134, 139, 171, 226
- data generating process (DGP), 12–13, 39
- debt, 153
- de Finetti, Bruno, 35–6, 132, 226
- definite states, 251–4
- Deleuze, Gilles, 41, 63, 127
- derivative prices, 26, 60–1, 75–6, 90, 142, 152, 153, 268–9, 273, 351, 358–9, 366
 - Brownian motion and, 258–60
 - instant, 320
 - market, 309
 - as states, 254–6
- derivative pricing, 2–9, 30, 48–9, 62, 270, 349, 357, 365
- derivatives, 2, 4, 351
 - exchange of, 48–9
 - independent trading of, 264–6
 - market prices of, 71
 - as quantitative formulas, 4–5
 - tradability of, 268–70
- derivatives market, 24, 28–9, 279–80
- derivative valuation theory, 4–9, 48, 167–8, 309–14, 351–62
 - probability and, 65
 - recalibration and, 32–5
- derivative value, 260–2
- Derrida, Jacques, 41, 240–1
- Destouches-Février, Paulette, 82
- determinism, 198, 300
- Deutsch, David, 29
- die, 25, 82, 108, 122, 134, 169
 - throw of the, 63
- difference, 79–80
 - absolute, 43, 99, 167, 380
 - analogical, 80, 191–2, 194–9, 234, 236, 244
 - extensive vs. intensive, 98–100
- diffusion coefficient, 286, 288, 297, 300, 309–10
- discontinuity, 53–4
- dissymmetry, 26, 289
- dynamic replication (hedging), 6, 45, 75–6, 326, 333–4, 335
- efficient market hypothesis (EMH), 3, 260, 280, 369
- Einstein, Albert, 60, 296, 351
- equivocation, 272–4
- estimation, 324–5
- event(s), 104
 - accounting for the, 19, 200–12
 - conceptual, 218
 - future, 223
 - impossible, 146–7
 - money and the, 203–7
 - place and, 209–11, 213
 - price and the, 168–70
 - radically emergent, 19–20, 148
 - singularity of the, 121
 - unpredictability of, 169
 - variation of the, 154–5, 156
 - writing the, 211–12
- ex-ante, 128–30, 135, 137–41, 148–9, 154–5, 170–1, 180, 225–6, 233–5
- exchange, 27, 58, 68–9, 149–50, 153–4, 170–2, 181–4
 - impossible, 214–16
 - matter in the, 228–38
 - necessary, 214–16
 - price and, 80
 - singularity of the, 244–8
 - writing and, 56
- exit
 - alternative, 289–93
 - probabilistic, 286–8
- expectation(s), 9, 20, 86, 117, 197–9, 226, 264, 291, 306
 - exogenous, 281
 - rational, 3, 25
- ex-post, 129, 133, 137–41, 144, 148–9, 155, 170, 171, 224, 228, 244, 304, 348
- extensive difference, 98–100

- fixed-point problem, 25
- formalism, 1, 13–15, 42, 103–7, 110–16, 119–20, 127, 137,
321–30, 342, 358–60, 370–1
of Brownian motion, 76, 151, 325
and BSM, 316, 346, 361
contingent claims and, 286–90, 298
of the market, 269, 273, 291, 299, 313, 331, 343
and matter, 93, 120–7, 324, 341
of probability, 84, 91, 103–16
of quantum mechanics, 21–3
and reality, 312, 315
stepping outside of, 278, 331, 346, 366
timelessness of, 175, 311
as a void of interpretation, 29, 42, 134, 291, 338
- formal reality, 12–15
- fragility, 55–6
- frequency(ies), 35, 51, 54–5, 58, 62, 69, 90, 138, 140, 144,
155, 160, 171
- fundamental principle of the market, 49–50, 69, 363
- future, technology of the, 23, 60–3
- future events, 59–60, 131, 193–4, 223
probability of, 200–12
- future possibilities, 21–2
- future world, 190–1, 193–4
- gambling, 26, 131–2, 133, 139
- generating condition, 82, 105, 108, 225
- geometry, market as, 83–4
- Girsanov theorem, 258, 355
- hedging, dynamic, 75–6
- Henrotte, Philippe, 31
- historical events, 51–2
- historical archive, 321
- history, 23, 149, 165
book, 321
quantitative, 52
succession of form and matter, 333, 337
- identification, 138, 187–8, 189
of the event, 180, 235–7
of the random sequence, 116, 137, 169–71
- identity, 79
- iid variables, 106, 120, 122–5
- immanence, 215, 222–5, 235, 244
- immanent theory of the market, 332–3
- implied volatility, 5, 6, 9–11, 24, 30, 32–4, 61–3, 70, 76, 77, 91,
96, 265, 316, 318–19, 324, 352, 361, 368
- impossible events, 146–7
- impossible exchange, 212–16
- imprecise probability, 396n3
- incomplete markets, 267–82, 317, 349, 356–7
vs. complete markets, 267–70
when the market is all there is, 274–82
- indeterminism, 198
essential, 82, 305, 336
- individual, 233–5
- infinitesimal limit, 143–4, 301–4
- infinity, 108, 110–11, 116, 117, 119, 120, 135, 143, 225
of markets, 25–8
of matter, 325–6
and probability, 117–8
of option price, 325–6
- instantaneous diffusion, 273–4
- instantaneous statistics, 324–5
- instantaneous volatility, 322–3, 343
- insurance
account, 131, 139
and accounting equation, 76, 140
company, 12, 26
paradigm, 301
- integral moment, *see* present state
- integrals, 195–7, 202
- intensive difference, 98–100
- intensive nontemporal price process, 52–3
- intrinsic arbitrage, 86, 252
- intrinsic non-arbitrage relations, 61
- intuition, 108–9, 119, 152
- Jubien, Michael, 23
- Kolmogorov, Andrei Nikolaevich, 90, 134, 138, 199
- Kolmogorovian probability, 21, 23, 82, 83, 90, 103, 104–9,
111–13, 114
- law of large numbers, 94–5, 103, 105–8, 111, 113, 115–16,
119–22, 126, 131, 137, 138, 142, 143, 150–1, 171,
180, 332
- Lewis, David, 35
- limiting frequency, 105, 171
- limit theorems, 121
- locals, 74–6
- lotteries, 305, 329
- lottery value, 256–60, 307, 352–3, 355
- many-valued function, 185–6
- mark, state vs., 84–9
- market, for derivatives, 28–9, 309
- market-makers, 47, 73–4, 77, 89, 145, 266, 345–6, 366
- market prices, 1, 3, 24, 27, 61, 63, 71, 75, 206–7, 247, 259–60,
296–7, 309
- market(s), 1–2, 46–78, 213–17
beyond antifragility, 55–6
book of the, 63–6, 213–14, 314, 321–50
Brownian motion and concept of, 144–5
complete, 274–82, 283, 344, 345, 349
contemporary theory of the, 87, 251–66

-
- of contingent claims, 297–9, 312–13, 318, 330, 334, 335–6, 340–1, 360–2, 367–8, 369, 371
 - as continual event, 48–50
 - as conversion of the image of thought, 43–4
 - definition of, 2, 3–4
 - for derivatives, 24, 279–80
 - dynamics, 73–6
 - form of, 317
 - formal theory of the, 332–3
 - fundamental principle of the, 49–50
 - genesis of the, 2, 66, 337
 - as geometry, 83–4
 - incomplete, 317, 349, 356–7
 - infinity of, 25–8
 - instant of the, 322–35
 - as inverted order of thought, 153–6
 - locality of, 75, 311, 325
 - making of a, 2–3
 - martingale measure of the, 270–2, 273, 284, 334, 343–4, 357
 - matter of the, 4
 - as medium of contingency, 34
 - model of the, 38
 - as opportunity for speculative thought, 42–3
 - options, 311–12, 315–16, 321
 - as place of exchange, 46–56
 - as quantitative history, 50–2
 - reality of the, 149, 150–2
 - semantic theory of the, 314–20
 - statistics as proto-market, 217–28
 - stochastic narrative of the, 251–62
 - surface of the, 69–72
 - technology of the, 146–50
 - time and, 159–72
 - trading narrative of the, 262–6
 - whole, 59, 60, 172–84, 206, 215, 217
 - markets, complete vs. incomplete, 267–70
 - markets, incomplete, 267–82
 - martingale measure of the market, 270–2, 273, 284, 334, 343–4, 357
 - martingales, 132, 133, 264, 266
 - mass, 124–5, 169, 233–6, 245–6
 - inertial and gravitational, 83, 207, 228, 248
 - phenomena, 83
 - massivity, 57–9, 176, 216, 253
 - material hypothesis, 116
 - material mode, 153
 - material time, 226
 - mathematical models, 5
 - matter, 1, 2, 4, 23–5, 48, 109–13, 115, 124–5, 136–7, 150–1, 153, 332, 336
 - arrangement of, 60, 254
 - changing meaning of, 119–26
 - emergence of, 6
 - in the exchange, 228–38
 - formal deduction of, 326–8
 - formalism and, 107, 127
 - hypothesis of, 115
 - infinity of, 325–6
 - in statistics, 225–8
 - measure theory, 95, 105–6, 113, 127–8
 - medium of contingency, 25, 56–66
 - Meillassoux, Quentin, 41, 64–5
 - Menard, Pierre, 40, 196, 206
 - Menard's *Quixote*, 379–84
 - meta-formalism, 321–30
 - metalogical level, 44–5
 - metaphysics, 1, 5, 13, 23, 33, 36–7, 39–41, 47, 53, 87, 88, 159, 223–4, 240, 365
 - meta-probabilistic predictive tool, 21
 - meta-probability, 83
 - money, 68–9, 95, 121, 126–36, 144, 171–2, 202–8, 213–14, 230
 - accountability of, 224, 227
 - event and, 203–7
 - as place, 133–6
 - time and, 131–3
 - multiple, 183–4, 185–212
 - multivalued function, 200–1
 - necessary exchange, 214–16
 - non-arbitrage, 6–12, 11, 14–15, 38, 61–2, 86, 88, 90, 91, 92, 258–61, 272–4, 319, 325, 343, 353
 - non-individual singular, 232–4
 - nonlinearity, 74
 - non-standard analysis, 304
 - nontemporal price process, 52–3
 - number
 - generation of, 81–3
 - law of large numbers, *see* law of large numbers
 - objectivation, 20
 - objective probability, 141, 200, 226–7, 264–5
 - objective semantics, 35
 - objectivist language, 198–9
 - objectivity, 20–1
 - Occam's razor, 37
 - one (1), 191–2
 - one and the many problem, 122–4
 - options prices, 31, 37, 264–5, 274, 311
 - infinity of the, 325–6
 - instant, 320
 - options market, 311–12, 315–16, 321
 - payoff, 97–8, 252
 - perfect replication, 274
 - philosophy, 2, 64, 65, 328–9
 - physical probability, 199
 - physical reality, 12–15, 311–20, 332
 - physics, 4

- place, 75, 152
 event and, 209–11, 213
 time and, 143–5
- Popper, Karl, 69, 82, 95, 105, 108, 109, 112, 224, 225, 396
- population of individuals, 236–7
- possibilities, 19–22, 25, 58–9, 62, 147, 214
 contingency and, 83
 contingent claims and, 43–4
 future, 21–2
 Kolmogorovian probability, 104–9
 vs. real, 20
 suspension of possibility, 188–91
- possible worlds, 2, 23, 35–8, 43, 58, 190–3, 237
- precise state, 96
- prediction, 33–4, 152
- present reality, 192
- present state, 195–6, 198, 219–22, 227
- present value, 183–4
 see also expectation
- prevision, 59, 162–7, 172
- price movements, 46
- price(s)/pricing, 1–2, 24, 45, 103, 135–6, 177, 182–3, 193, 356, 362, 366
 Arrow-Debreu, 38–9, 91–2
 of contingent claims, 45, 345, 370–1
 contingent claims and, 43–4, 61
 derivative, 4, 26, 75–6
 derivative, as states, 254–6
 derivatives, 2–3
 dynamics, 69–70, 73
 event and, 168–70
 exchange and, 80
 at expiry, 156
 format for, 352
 formulas, 73–4
 genesis of, 1, 81–3
 intensive nontemporal price process, 52–3
 market, 1, 3, 24, 27, 61, 63, 75, 206–7, 247, 259–60, 296–7
 mathematics of, 89
 meta-contextual pricing tool, 373–6
 option, 31, 37, 264–5, 274, 311, 320, 325–6
 and paper, 172, 253
 probability and, 38–9, 132, 287
 process, 34, 52–3
 randomness of, 256–60, 283–4, 288, 290–1, 295–6, 305–6
 technology, 6–9, 15
 time and, 167–8
 trace and, 170–2
 of underlying, 31, 270–2, 273–4, 276–7, 279–80, 287
 vs. valuation, 66–78
 value and, 178–9
 virtual, 6, 63, 71
 volatility, 9–11, 24, 161–2
- probabilistic exit, 286–8, 295, 296, 319
- probability, 1, 3, 4, 19–29, 65, 68–9, 80, 116–17, 155, 160, 170–1, 173–4, 178, 180, 301–4, 306–7, 332
 Brownian motion and, 142–3, 301–4, 336
 definition of, 195
 dynamics of, 73
 formalism, 104–5, 112, 120
 of future event, 200–12
 imprecise, 396n3
 infinite, 110–11, 117–18
 as an integral, 195–7
 as internal episode, 89–92
 Kolmogorovian, 21, 23, 82, 83, 90, 103, 111–14
 law of large numbers and, 107
 meaning of, 104–19
 medium of, 61
 as modality, 108–9
 models, 42
 necessity of, 35–8
 objectification of, 196
 objective, 103, 141, 200, 226–7, 264–5
 physical, 199
 price and, 38–9, 132, 287
 risk-neutral, 39, 91, 310
 science, 53, 54
 Shafer and Vovk on, 95–6
 subjective, 132
 theory, 3, 6, 19, 24–5, 29, 91, 106, 132, 134, 147–8, 300, 334
 time and, 202–3, 296
- probability calculus, 191–2
- probability distributions, 25–6, 48–52, 111, 124–5, 165
- problematics, 127
- propensity notion, 82, 109, 112, 224
- quantitative diffusion coefficient, 288
- quantitative formulas, 4–5
- quantitative history, market as, 50–2
- quantum mechanics, 20–1, 22, 40–1, 82
- quiddity, 233
- radically emergent event, 19–20, 148
- random generator, 48, 51, 53, 70–1, 81–3, 96, 111, 133, 137, 147, 149–53, 198, 201, 225, 365
- randomness, 104, 110, 113–15, 148, 295–6, 305–6, 319
 absolute, 312
 axiom of, *see* axiom of randomness
 Brownian motion, 26, 140–4
 Taleb on, 54
 discrete and continuous, 300, 304
 efficient market hypothesis, 72, 257, 272, 353
 of lottery, 256
 and market, 27, 81, 159–70, 301, 306, 312
 and money, 130–4
 of price, 260, 267, 281–7, 290, 341
 quantum mechanics, 82
- random price, 256–60
- random variables, 111–15, 122–4, 134, 138, 198

-
- rational expectations, 25
 - reality, 20–1, 23, 36–7, 94, 113–16, 133, 137, 146, 147, 189, 310, 328–30
 - changing meaning of, 136–45
 - of contingency, 124–6
 - formalism and, 311
 - formal vs. physical, 12–15
 - of the market, 149, 150–2
 - present, 192
 - synthetic, 192
 - reality hypothesis, 116
 - real volatility, 9–11
 - recalibration, 24, 32–5, 44, 77, 87–8, 92, 97, 279, 297, 336, 340–1, 361–2, 364, 365, 376–8
 - as pricing tool, 39
 - process, 92–3
 - of value model, 49
 - regime-switching model, 30–2, 34, 39–41
 - relativity theory, 61, 63
 - repetition, 27, 106–7, 109, 115–16, 121, 125–6, 199
 - replication, 274
 - dynamic, *see* dynamic replication
 - risk-neutral probability, 39, 91, 310

 - semantic theory of the market, 314–20
 - Shafer, Glenn, 94–7, 120, 125, 128, 130–9, 302, 303–4
 - single-case statistics, 234–7
 - singularity, 233
 - of event, 121
 - of the exchange, 244–8
 - of writing, 239–44
 - smile problem, 31, 71–4, 76
 - speculation, 42
 - speculative materialism, 215
 - speculative thought, 42–3
 - state(s), 91
 - identification of, 39, 67, 85, 97, 187–9, 194–7, 232
 - vs. mark, 84–9
 - precise, 96
 - present, 195–6, 198, 219–22, 227
 - of the world, 2, 19–20
 - statistical analysis, 9
 - statistical parameters, 71–2
 - statistical populations, 89–90, 104
 - statistics, 4, 14–15, 25, 75, 103, 130, 136, 147, 151, 152, 162, 217, 283–4, 315, 324–5, 332
 - end of, 28–9
 - immanence of, 222–5
 - matter in, 225–8
 - as proto-market, 217–28
 - single-case, 234–7
 - as writing, 120–1, 127, 130, 229, 247, 348
 - stochastic process, 2, 3, 6, 8, 12–15, 48–9, 90, 149, 261, 263–4, 270, 279–80, 300, 312, 313, 330, 332, 342, 349–50, 362, 369, 371
 - stochastic structure, 268
 - stochastic volatility, 31, 32, 34–5, 276–7, 317, 358
 - subjective probability, 132
 - sub-selections, 110–11, 116, 129, 131
 - succession, 42, 52, 77, 117, 125–7, 185, 196–8, 205, 220–2
 - of form and matter, 332–41
 - historical, 332, 369
 - symmetry, 192, 195–7
 - synthetic reality, 192

 - Taleb, Nassim, 50–6, 74, 75, 78
 - technology, 1, 5–6
 - derivative pricing, 6–9, 270, 365
 - of the future, 23, 60–3
 - of the market, 146–50
 - pricing, 15
 - tense, 116–19
 - theoretical valuation models, 5
 - theory, 1
 - thought
 - image of, 43–4
 - market as inverted order of, 153–6
 - time, 38, 44, 57, 119, 155, 205, 312
 - accident of, 207–9
 - chronological, 57, 65, 95, 205–6
 - continuous, 332
 - market and, 159–72
 - material, 226
 - money and, 131–3
 - place and, 143–5
 - price and, 167–8
 - probability and, 202–3, 296
 - time series, 70, 152, 347, 348
 - topology, 69, 88
 - trace, price and, 170–2
 - trading process, 2, 3, 343–50
 - trading strategy, 95, 132, 143, 292
 - transcendence, 215
 - transition, 43, 187–8, 189
 - true market dynamics, 73–6

 - uncertainty, 15
 - essential, 48
 - of volatility, 272, 313, 368
 - underlying asset, 4, 8–9, 26, 153, 214, 251–2, 276, 311, 319, 351, 366, 367
 - market price of, 259–60
 - price of, 31, 270–4, 276–7, 279–80, 287
 - stochastic process of, 15
 - tradability of, 258–9, 264
 - trading of, 371
 - univocity, 125, 192–6, 231
 - unity,
 - contingent, 194

-
- unity – *continued*
 - of reality, 191
 - and state, 219
 - unpredictability, 50–1, 169
 - valuation, 12, 44, 97–8, 178, 179, 182–3, 186, 233
 - absolute, 193–5
 - actuarial, 12, 27
 - and assignment, 182, 186
 - derivative, 6–9, 352–5
 - pricing vs., 66–78, 179
 - and tree, 172
 - value
 - of contingent claims, 180, 181
 - derivative, 260–2
 - present, 183–4
 - price and, 178–9
 - Ville, Jean, 131–2
 - virtual,
 - Deleuze, 72, 147, 190, 390n11–2
 - and reality, 193, 323, 377
 - volatility, 5, 50, 70, 75, 263, 297–8, 310–11, 326, 332, 366
 - BSM and, 346
 - certainty of, 313–14
 - constant, 32, 261, 277, 310, 313, 323–4, 357, 373, 393
 - implied, 5, 6, 9–11, 24, 30, 32–4, 61–3, 70, 76, 77, 91, 96, 265, 316, 318–19, 324, 352, 361, 368
 - instantaneous, 322–3, 343
 - lack of, 78
 - price, 24, 161–2
 - real, 9–11
 - stochastic, 31, 32, 34–5, 276–7, 317, 358
 - uncertain, 261
 - of underlying, 252
 - von Mises, R., 83, 84, 106–16, 131, 134, 138, 139, 224–5, 228
 - von Plato, Jan, 84
 - Vovk, Vladimir, 94–7, 120, 125, 128, 130–9, 268, 302, 303–4
 - whole market, 59, 60, 172–84, 206, 215, 217
 - Wood, David, 64
 - work, necessity of, 207–9
 - writing, 41, 52, 64, 203, 213, 326
 - content of, 243–4
 - contingent claims, 56–7, 86, 162–7, 170, 176–81, 295
 - exchange and, 56
 - invention of, 176–81, 293–5, 342
 - singularity of, 239–44
 - two sides of, 79–80
 - zero (0), 191–2
 - zero probability, 19