Geometric invariant theory and derived categories of coherent sheaves

by

Daniel Scott Halpern-Leistner

A dissertation submitted in partial satisfaction of the
requirements for the degree of
Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Constantin Teleman, Chair
Professor Martin Olsson
Professor Birgitta Whaley

Spring 2013

Geometric invariant theory and derived categories of coherent sheaves

Abstract

Geometric invariant theory and derived categories of coherent sheaves

by

Daniel Scott Halpern-Leistner

Doctor of Philosophy in Mathematics

University of California, Berkeley

Professor Constantin Teleman, Chair

Given a quasiprojective algebraic variety with a reductive group action, we describe a relationship between its equivariant derived category and the derived category of its geometric invariant theory quotient. This generalizes classical descriptions of the category of coherent sheaves on projective space and categorifies several results in the theory of Hamiltonian group actions on projective manifolds.

This perspective generalizes and provides new insight into examples of derived equivalences between birational varieties. We provide a criterion under which two different GIT quotients are derived equivalent, and apply it to prove that any two generic GIT quotients of an equivariantly Calabi-Yau projective-over-affine variety by a torus are derived equivalent.

We also use these techniques to study autoequivalences of the derived category of coherent sheaves of a variety arising from a variation of GIT quotient. We show that these autoequivalences are generalized spherical twists, and describe how they result from mutations of semiorthogonal decompositions. Beyond the GIT setting, we show that all generalized spherical twist autoequivalences of a dg-category can be obtained from mutation in this manner.

Motivated by a prediction from mirror symmetry, we refine the main theorem describing the derived category of a GIT quotient. We produce additional derived autoequivalences of a GIT quotient and propose an interpretation in terms of monodromy of the quantum connection. We generalize this observation by proving a criterion under which a spherical twist autoequivalence factors into a composition of other spherical twists.

Finally, our technique for studying the derived category of a GIT quotient relies on a special stratification of the unstable locus in GIT. In the final chapter we establish a new modular description of this stratification using the mapping stack $\underline{\mathrm{Hom}}(\Theta, X/G)$, where $\Theta = \mathbb{A}^1/\mathbb{G}_m$. This is the first foundational step in extending the methods of GIT beyond global quotient stacks X/G to other stacks arising in algebraic geometry. We describe a method of constructing such stratifications for arbitrary algebraic stacks and show that it reproduces the GIT stratification as well as the classical stratification of the moduli stack of vector bundles on a smooth curve.

I dedicate this dissertation to my family. My parents, Jay and Marcia, have always encouraged me to pursue my interests. They have nurtured me through some difficult times and cheered for me during good times. My brothers, Adam and Jordan, have not only been great friends, they have kept me grounded and given me a different perspective on the world. Finally, my grandparents' love has shaped me in many ways – especially my grandfather Ernie, who fostered my interest in science from a young age.

Also to Katie, whose love, kindness, and support have sustained me through the preparation of this dissertation.

"I am not sure that I exist, actually. I am all the writers that I have read, all the people that I have met, all the women that I have loved; all the cities I have visited."

-Jorge Luis Borges

Contents

List of Figures

List of Tables

Acknowledgments

I would like to thank my adviser Constantin Teleman for introducing me to his work [43] and many of the other classic papers which were the starting point for this research. His support and mathematical comments were very helpful throughout my studies at Berkeley.

I would also like to thank my dear friends Daniel Pomerleano and Daniel Cristofaro-Gardiner. They have been an inspiration to me and have taught me many things, and they have been the source of many years of interesting conversation.

I appreciate the support of Hideo Mabuchi, who supported me as a visitor in the Stanford applied physics department in 2008, and Peter Teichner and the Max Planck Institute for Mathematics, who hosted me during my studies in the Fall of 2010.

For the material in Chapter 2: I would like to thank Anatoly Preygel for useful conversations about derived algebraic geometry and for carefully reviewing the later sections. I would like to thank Yujiro Kawamata for suggesting that I apply my methods to hyperkähler reduction and flops of Grassmannian bundles, and Kentaro Hori for carefully reviewing my work and discovering some mistakes in the first version of this paper. I would also like to thank Matthew Ballard, David Favero, and Ludmil Katzarkov for many interesting conversations following the completion of my work.

For the material in Chapter 3: I would like to thank the attendees of the School on Algebraic Geometry and Theoretical Physics at the University of Warwick, July 2012 for many stimulating mathematical conversations, with special thanks to Will Donovan, Ed Segal, and Timothy Logvinenko for explaining their work. I also benefited greatly from conversations with Kentaro Hori, Paul Horja, Hiroshi Iritani, Lev Borisov, and Alexander Kuznetsov.

For the material in Chapter 4: I would like to thank Ian Morrison for encouraging me to revisit Kempf's early work on instability in invariant theory. I would also like to thank Martin Olsson and Jacob Lurie for interesting conversations.

Chapter 1

Introduction

Let k be an algebraically closed field of characteristic 0. Let X be a quasiprojective variety over k and consider the action of a reductive group G on X. A classical problem in algebraic geometry is to make sense of the orbit space "X/G."

Simple examples show that finding an algebraic variety which parameterizes orbits is not always possible. For instance the categorical quotient of $\mathbb{A}^n$ by the action of $\mathbb{G}_m$ with weight -1 (we denote this by $\mathbb{A}^n(1)$) is $\operatorname{Spec} k$. Even if G acts freely, such as for the action of $\mathbb{G}_m$ on $\mathbb{A}^1(-1) \times \mathbb{A}^1(1) - \{0\}$, the quotient will not be a separated scheme.

Grothendieck's solution was to generalize the notion of a scheme to that of an algebraic stack. In this more general setting, the quotient of X by G is well defined as a stack. Most of the geometric notions defined for algebraic varieties are also naturally defined for algebraic stacks. For example on can study vector bundles on X/G, which are equivariant vector bundles on X, and if $k = \mathbb{C}$ then one can study the topological cohomology of X/G, which agrees with the equivariant cohomology $H_G^*(X)$. In this thesis, we will denote the quotient stack by X/G.

Mumford's geometric invariant theory [33] offers a different solution. One chooses some additional geometric data, a G-linearized ample line bundle L, and uses this to define an open G-invariant subvariety $X^{ss} \subset X$. In good situations, there will be a variety which parameterizes G orbits in X^{ss}, called a geometric quotient of X^{ss} by G, or alternatively a coarse moduli space for the stack X^{ss}/G. In this thesis, we will use the term "GIT quotient" to refer to the quotient stack X^{ss}/G and not its coarse moduli space.

When comparing the geometry of X^{ss}/G and X/G, one must consider the geometry of the unstable locus $X^{us} = X - X^{ss}$. It turns out that this subvariety admits a special stratification which we call a *Kempf-Ness (KN)* stratification. Classically, this stratification was used by Kirwan and others to describe very precise relationships between the cohomology of X/G and the cohomology of X^{ss}/G. The natural restriction map $H^*(X/G) \to H^*(X^{ss}/G)$ is surjective, and one can describe the kernel of this homomorphism fairly explicitly.

We develop a categorification of these ideas. In Chapter 2 we establish a relationship between the derived category of equivariant coherent sheaves on X, i.e. $\mathrm{D}^b(X/G)$, and the derived category $\mathrm{D}^b(X^{ss}/G)$ which is analogous to Kirwan's results on equivariant cohomol-

ogy. In Chapter 3 we apply these results to study autoequivalences of the derived category predicted by homological mirror symmetry. Finally, in Chapter 4, we revisit the foundations of the subject and discuss how the KN stratifications used in Chapters 2 and 3 can be constructed for stacks which are not global quotient stacks. In future work, we hope to apply these methods to moduli problems in algebraic geometry.

1.1 Background

1-parameter subgroups and parabolic subgroups

Let G be a reductive group over an algebraically closed field k of characteristic 0. A one parameter subgroup is a group homomorphism $\lambda : \mathbf{G}_m \to G$. Given such a λ, we define subgroups of G:

$$L_\lambda = \text{ the centralizer of } \lambda$$
$$P_\lambda = \{p \in G | \lim_{t \to 0} \lambda(t)p\lambda(t)^{-1} \text{ exists}\}$$
$$U_\lambda = \{u \in G | \lim_{t \to 0} \lambda(t)p\lambda(t)^{-1} = 1\}$$

Then $L_\lambda \subset P_\lambda$ is a Levi factor and we have the semidirect product sequence

$$1 \longrightarrow U_\lambda \longrightarrow P_\lambda \overset{\longleftarrow}{\longrightarrow} L_\lambda \longrightarrow 1 \tag{1.1}$$

where $U_\lambda \subset P_\lambda$ is the unipotent radical. The projection $\pi : P_\lambda \to L_\lambda$ maps

$$p \mapsto \lim_{t \to 0} \lambda(t)p\lambda(t)^{-1}.$$

The Hilbert-Mumford numerical criterion

Let $X \subset \mathbb{P}^n \times \mathbb{A}^m$ be a closed subvariety invariant with respect to the action of a reductive group G. We sometimes refer to a closed subvariety of $\mathbb{P}^n \times \mathbb{A}^m$ as projective-over-affine. This condition is equivalent to the canonical morphism $X \to \operatorname{Spec} \Gamma(X, \mathcal{O}_X)$ being projective.

Let $\mathcal{L} := \mathcal{O}_X(1)$ be a choice of G-linearized ample line bundle on X. Then the semistable locus is defined to be the G-equivariant open subvariety

$$X^{ss} := \bigcup_{s \in \Gamma(X, \mathcal{L}^n)^G} \{x \in X | s(x) \neq 0\}$$

The Hilbert-Mumford criterion provides a computationally effective way to determine if a point $x \in X$ lies in X^{ss}.

If $\lambda : \mathbb{G}_m \to G$ is a one parameter subgroup $y = \lim_{t \to 0} \lambda(t)x$ exists, then y is fixed by $\mathbb{G}_m$ under λ, so the fiber $\mathcal{L}_y$ is a one dimensional representation of $\mathbb{G}_m$. We let $\mathrm{weight}_\lambda(\mathcal{L}_y)$ denote the weight (i.e. the integer corresponding to the character) of this representation.

Theorem 1.1.1 (Hilbert-Mumford numerical criterion)**.** *Let a reductive group G act on a projective-over-affine variety X. Let $x \in X$ and $\mathcal{L}$ a G-ample line bundle on X. Then $x \in X^{ss}$ if and only if* $\mathrm{weight}_\lambda \mathcal{L}_y \geq 0$ *for all λ for which $y = \lim_{t \to 0} \lambda(t)x$ exists.*

In the following section, we will show how this numerical criterion can be refined, leading to a stratification of the unstable locus by the "degree of instability."

Stratifications of the unstable locus in GIT

Now in addition to a G-linearized ample line bundle, we choose an inner product on the cocharacter lattice of G which is invariant under the Weyl group action. This allows us to define the norm $|\lambda| > 0$ for all nontrivial one-parameter subgroups. If G is a complex group and $K \subset G$ a maximal compact subgroup, then this is equivalent to specifying a K-invariant Hermitian inner product on $\mathfrak{g}$ which takes integer values on the cocharacters.

We will describe the construction of a statification of the unstable locus $X - X^{ss}$, but first we must recall a general theorem due to Hesselink. We will need the following notion

Definition 1.1.2. If G is a linear algebraic group over k acting on a scheme X over k, we say that the action is *locally affine* if for any 1PS $\lambda : \mathbb{G}_m \to G$, there is an open cover of X by $\mathbb{G}_m$-invariant affine schemes.

This is a fairly mild hypothesis. First, it suffices to find an invariant affine cover for 1PS's in a fixed maximal torus $T \subset G$. Furthermore

Lemma 1.1.3. *If X is a normal k scheme with a G action and $Y \subset X$ is a G-equivariant closed subscheme, then the action of G on Y is locally affine*

Proof. As a consequence of Sumihiro's theorem, any group action on a normal k scheme is locally affine. Furthermore, for any G-quivariant closed immersion $Y \subset X$, if and the action on X is locally affine then the action on Y is locally affine as well. $\qquad\square$

We let X be a k-scheme with a locally affine action of a linear group G, and let $\lambda : \mathbb{G}_m \to G$ be a one-parameter subgroup (1PS). λ induces a $\mathbb{G}_m$ action on X, and we let $X^{\mathbb{G}_m}$ denote the fixed subscheme. When we wish to emphasize the dependence on λ, we will denote the fixed subscheme by X^λ. Hesselink's theorem states[1]

Theorem 1.1.4 ([23])**.** *Let X be a k-scheme admitting a locally finite action by $\mathbb{G}_m$. Then the functor*

$$\Phi_X(T) = \left\{ \mathbb{G}_m\text{-equivariant maps } \mathbb{A}^1 \times T \to X \right\}$$

is representable by a scheme Y. Restriction of a map $\mathbb{A}^1 \times T \to X$ to $\{1\} \times T \subset \mathbb{A}^1 \times T$ defines a morphism $j : Y \to X$ which is a local immersion. Restriction to $\{0\} \times T \subset \mathbb{A}^1 \times T$ defines a morphism $\pi : Y \to X^{\mathbb{G}_m}$ which is affine.

---

[1]Theorem 1.1.4 is the special case of the main theorem of section 4 of [23] for which the "center" is $C = X$ and the "speed" is $m = 1$.

It is also straightforward to verify that $\pi : Y \to X^{\mathbb{G}_m}$ has connected geometric fibers. We note the following alternate characterization of $\Phi_X(T)$

Lemma 1.1.5. *Restriction of an equivariant map $\mathbb{A}^1 \times T \to X$ to $\{1\} \times T$ identifies $\Phi_X(T)$ with the subfunctor*

$$\{f : T \to X | \mathbb{G}_m \times T \xrightarrow{t \cdot f(x)} X \text{ extends to } \mathbb{A}^1 \times T\} \subset \mathrm{Hom}(T, X)$$

Proof. Restriction to $\{1\} \times T$ identifies the set of equivariant maps $\mathbb{G}_m \times T \to X$ with $\mathrm{Hom}(T, X)$. If the corresponding map extends to $\mathbb{A}^1 \times T$ it will be unique because X is separated. Likewise the uniqueness of the extension of $\mathbb{G}_m \times \mathbb{G}_m \times T \to X$ to $\mathbb{G}_m \times \mathbb{A}^1 \times T \to X$ guarantees the $\mathbb{G}_m$ equivariance of the extension $\mathbb{A}^1 \times T \to X$. $\qquad\square$

We will also often make use of the following strengthened version of the Biaynicki-Birula theorem.

Proposition 1.1.6. *If X is a k scheme with a locally affine $\mathbb{G}_m$ action and $X \to S$ is a smooth morphism which is $\mathbb{G}_m$ invariant, then both $X_m^{\mathbb{G}}$ and Y are smooth over S.*
If S is smooth over k, then $Y \to X^{\mathbb{G}_m}$ is a Zariski-locally trivial bundle of affine spaces with linear $\mathbb{G}_m$ action on the fibers.

The first statement of the proposition can be verified by checking that the map of functors $\Phi_X \to h_S$ is formally smooth. The second statement is proved in Section 5 of [23].

Theorem 1.1.4 allows us to define the *blade* corresponding to a connected component $Z \subset X^\lambda$

$$Y_{Z,\lambda} := \pi^{-1}(Z) = \left\{ x \in X | \lim_{t \to 0} \lambda(t) \cdot x \in Z \right\} \tag{1.2}$$

Note that $Y_{Z,\lambda}$ is connected and $\pi : Y_{Z,\lambda} \to Z$ is affine. When X is smooth then both $Y_{Z,\lambda}$ and Z are smooth, and $Y_{Z,\lambda}$ is a fiber bundle of affine spaces over Z by Proposition 1.1.6.

We define the subgroup

$$P_{Z,\lambda} := \{p \in P_\lambda | l(Z) \subset Z, \text{ where } l = \pi(p)\}$$

$P_{Z,\lambda} \subset P_\lambda$ has finite index – it consists of the preimage of those connected components of L_λ which stabilize Z. $Y_{Z,\lambda}$ is closed under the action of $P_{Z,\lambda}$, because

$$\lim_{t \to 0} \lambda(t)px = \lim_{t \to 0} \lambda(t)p\lambda(t)^{-1}\lambda(t)x = l \cdot \lim_{t \to 0} \lambda(t)x.$$

G acts on the set of such pairs (Z, λ) by $g \cdot (Z, \lambda) = (gZ, g\lambda g^{-1})$, and $g \cdot Y_{Z,\lambda} = Y_{gZ,g\lambda g^{-1}}$. Up to this action we can assume that λ lies in a fixed choice of maximal torus of G, and the set of Z appearing in such a pair is finite.

We are now ready to describe the stratification of the unstable locus in GIT. For each pair (Z, λ) we define the numerical invariant

$$\mu(\lambda, Z) = \frac{-1}{|\lambda|} \mathrm{weight}_\lambda \mathcal{L}|_Z \in \mathbb{R} \tag{1.3}$$

One constructs the KN stratification iteratively by selecting a pair $(Z_\alpha, \lambda_\alpha)$ which maximizes μ among those (Z, λ) for which Z is not contained in the previously defined strata. One defines the open subset $Z_\alpha^\circ \subset Z_\alpha$ not intersecting any higher strata, and the attracting set $Y_\alpha^\circ := \pi^{-1}(Z_\alpha^\circ) \subset Y_{Z_\alpha, \lambda_\alpha}$. One also defines $P_\alpha = P_{Z_\alpha, \lambda_\alpha}$ and the new strata is defined to be $S_\alpha = G \cdot Y_\alpha$.

The strata are ordered by the value of the numerical invariant μ. It is a non-trivial fact that $\bar{S}_\alpha \subset S_\alpha \cup \bigcup_{\mu_\beta > \mu_\alpha} S_\beta$, so the Hilbert-Mumford criterion leads to an ascending sequence of G-equivariant open subvarieties $X^{ss} = X_0 \subset X_1 \subset \cdots \subset X$ where each $X_i \setminus X_{i-1}$ is a stratum. It is evident that the stratification of $\mathbb{P}^n \times \mathbb{A}^m$ induces the stratification of X via the embedding $X \subset \mathbb{P}^n \times \mathbb{A}^m$.

In this thesis we will use some special properties of the locally closed subvariety S_α (see [30], [16] and the references therein):

(S1) By construction $Y_\alpha^\circ = \pi^{-1}(Z_\alpha^\circ)$ is an open subvariety of the blade corresponding to Z_α and λ_α. The variety Z_α° is L_α equivariant, and Y_α° is P_α equivariant. The map $\pi : x \mapsto \lim_{t \to 0} \lambda_\alpha(t) \cdot x$ is algebraic and affine, and it is P_α-equivariant if we let P_α act on Z_α° via the quotient map $P_\alpha \to L_\alpha$. Thus $Y_\alpha^\circ = \underline{\mathrm{Spec}}_{Z_\alpha^\circ}(\mathcal{A})$ where $\mathcal{A} = \mathcal{O}_{Z_\alpha^\circ} \oplus \bigoplus_{i<0} \mathcal{A}_i$ is a coherently generated P_α-equivariant $\mathcal{O}_{Z_\alpha^\circ}$ algebra, nonpositively graded with respect to the weights of λ_α.

(S2) $Y_\alpha^\circ \subset X$ is invariant under P_α and the canonical map $G \times_{P_\alpha} Y_\alpha^\circ \to G \cdot Y_\alpha^\circ =: S_\alpha$ is an isomorphism.

(S3) Property (S1) implies that the conormal sheaf $\mathfrak{N}_{S_\alpha/X} = \mathcal{I}_{S_\alpha}/\mathcal{I}_{S_\alpha}^2$ restricted to Z_α° has positive weights with respect to λ_α

Note that properties (S1) and (S3) hold for any subvariety which is the attracting set of some $Z \subset X^\lambda$, so (S2) is the only property essential to the strata arising in GIT. Note also that when G is abelian, $P_\alpha = G$ and $Y_\alpha^\circ = S_\alpha$ for all α, which simplifies the description of the stratification.

Due to the iterative construction of the KN stratification, it will suffice for many of our arguments to analyze a single *closed* stratum $S \subset X$.

Definition 1.1.7 (KN stratification). Let X be a quasiprojective variety with the action of a reductive group G. A closed *Kempf-Ness (KN) stratum* is a closed subvariety $S \subset X$ such that there is a λ and an open-and-closed subvariety $Z \subset X^\lambda$ satisfying properties (S1)-(S3). We will introduce standard names for the morphisms

$$Z \underset{\pi}{\overset{\sigma}{\rightleftarrows}} Y \subset S \overset{j}{\longrightarrow} X \tag{1.4}$$

If X is not smooth along Z, we make the technical hypothesis that there is a G-equivariant closed immersion $X \subset X'$ and a KN stratum $S' \subset X'$ such that X' is smooth in a neighborhood of Z' and S is a union of connected components of $S' \cap X$.

Let $X^u \subset X$ be a closed equivariant subvariety. A stratification $X^u = \bigcup_\alpha \mathfrak{S}_\alpha$ indexed by a partially ordered set I will be called a *KN stratification* if $S_\beta \subset X - \bigcup_{\alpha > \beta} S_\alpha$ is a KN stratum for all β.

Remark 1.1.8. The technical hypothesis is only used for the construction of Koszul systems in Section 2.1. It is automatically satisfied for the GIT stratification of a projective-over-affine variety.

We denote the open complement $V = X - S$. We will use the notation $\mathfrak{X}$, $\mathfrak{S}$, and $\mathfrak{V}$ to denote the stack quotient of these schemes by G. Property (S2) implies that as stacks the natural map $Y/P \to S/G$ is an equivalence, and we will identify the category of G-equivariant quasicoherent sheaves on S with the category of P-equivariant quasicoherent sheaves on Y under the restriction functor. We will also use j to denote $Y/P \to X/G$.

A KN stratum has a particularly nice structure when X is smooth along Z. In this case Z must also be smooth, and Y is a locally trivial bundle of affine spaces over Z. By (S2), S is smooth and hence $S \subset X$ is a regular embedding. In this case $\det(\mathfrak{N}_{S/X})$ is an equivariant line bundle and its restriction to Z is concentrated in a single nonnegative weight with respect to λ (it is 0 iff $\mathfrak{N}_{\mathfrak{S}/\mathfrak{X}} = 0$). For each stratum in a smooth KN stratification we define

$$\eta_\alpha = \text{weight}_{\lambda_\alpha} \left(\det \mathfrak{N}_{S_\alpha/X}|_{Z_\alpha} \right) \tag{1.5}$$

These numbers will be important in stating our main theorem 1.2.1.

1.2 Introduction to Chapter 2

In Chapter 2 we describe a relationship between the derived category of equivariant coherent sheaves on a smooth projective-over-affine variety, X, with an action of a reductive group, G, and the derived category of coherent sheaves on a GIT quotient of X with respect to G. The main theorem connects three classical circles of ideas:

- Serre's description of quasicoherent sheaves on a projective variety in terms of graded modules over its homogeneous coordinate ring,

- Kirwan's theorem that the canonical map $H_G^*(X) \to H^*(X//G)$ is surjective,[30] and

- the "quantization commutes with reduction" theorem from geometric quantization theory equating $h^0(X, \mathcal{L})^G$ with $h^0(X//G, \mathcal{L})$ when the linearization $\mathcal{L}$ descends to the GIT quotient[43].

We denote the quotient stacks $\mathfrak{X} = X/G$ and $\mathfrak{X}^{ss} = X^{ss}/G$ and the bounded derived category of coherent sheaves on $\mathfrak{X}$ by $\mathrm{D}^b(\mathfrak{X})$, and likewise for $\mathfrak{X}^{ss}$.[2] Restriction gives an exact dg-functor $i^* : \mathrm{D}^b(X/G) \to \mathrm{D}^b(\mathfrak{X}^{ss}/G)$, and in fact any bounded complex of equivariant coherent sheaves on X^{ss} can be extended equivariantly to X. The main result of this chapter is the construction of a *functorial* splitting of i^*.

Theorem 1.2.1 (derived Kirwan surjectivity, preliminary statement)**.** *Let X be a smooth projective-over-affine variety with a linearized action of a reductive group G, and let $\mathfrak{X} = X/G$. Specify an integer w_i for each KN stratum of the unstable locus $\mathfrak{X} \setminus \mathfrak{X}^{ss}$. Define the full subcategory of $\mathrm{D}^b(\mathfrak{X})$*

$$\mathbf{G}_w := \left\{ F^\bullet \in \mathrm{D}^b(\mathfrak{X}) \,|\, \mathcal{H}^*(L\sigma_i^* F^\bullet) \text{ supported in weights } [w_i, w_i + \eta_i) \right\}$$

Then the restriction functor $i^* : \mathbf{G}_w \to \mathrm{D}^b(\mathfrak{X}^{ss})$ is an equivalence of categories.

Remark 1.2.2. The full version of the result proved in Chapter 2 is more general than Theorem 1.2.1 in two ways. First, it applies to the situation where X is singular provided the KN strata satisfy two additional properties (L+) and (A). Second, it describes the kernel of the restriction map $\mathrm{D}^b(X/G) \to \mathrm{D}^b(X^{ss}/G)$ explicitly by identifying $\mathbf{G}_w$ as piece of a semiorthogonal decomposition of $\mathrm{D}^b(\mathfrak{X})$, where the remaining semiorthogonal factors generate the kernel of the restriction.

The simplest example of Theorem 1.2.1 is familiar to many mathematicians: projective space $\mathbb{P}(V)$ can be thought of as a GIT quotient of $V/\mathbb{C}^*$. Theorem 1.2.1 identifies $\mathrm{D}^b(\mathbb{P}(V))$ with the full triangulated subcategory of the derived category of equivariant sheaves on V generated by $\mathcal{O}_V(q), \cdots, \mathcal{O}_V(q+\dim V-1)$. In particular the semiorthogonal decompositions described in Section 2.1 refine and provide an alternative proof of Beilinson's theorem that the line bundles $\mathcal{O}_{\mathbb{P}(V)}(1), \ldots, \mathcal{O}_{\mathbb{P}(V)}(\dim V)$ generate $\mathrm{D}^b(\mathbb{P}(V))$.

Serre's theorem deals with the situation in which $G = \mathbb{C}^*$, X is an affine cone, and the unstable locus consists only of the cone point – in other words one is studying a connected, positively graded k-algebra A. The category of quasicoherent sheaves on $\mathrm{Proj}(A)$ can be identified with the quotient of the full subcategory of the category of graded A-modules graded in degree $\geq q$ for any fixed q by the subcategory of modules supported on the cone point. This classical result has been generalized to noncommutative A by M. Artin [4]. D. Orlov studied the derived categories and the category of singularities of such algebras in great detail in [36], and much of the technique of the proof of Theorem 1.2.1 derives from that paper. In fact our Theorem 2.0.3 gives a more refined version of this result, identifying a full subcategory of the category of graded A-modules which gets identified with the derived category of $\mathrm{Proj}(A)$ under the quotient map.

---

[2]On a technical note, all of the categories in this paper will be pre-triangulated dg-categories, so $\mathrm{D}^b(\mathfrak{X})$ denotes a dg-enhancement of the triangulated category usually denoted $\mathrm{D}^b(\mathfrak{X})$. However, all of the results will be statements that can be verified on the level of homotopy categories, such as semiorthogonal decompositions and equivalences of categories, so I will often write proofs on the level of the underlying triangulated category.

In the context of equivariant Kähler geometry, Theorem 1.2.1 is a categorification of Kirwan surjectivity, which states that the restriction map on equivariant cohomology $H_G^*(X) \to H_G^*(X^{ss})$ is surjective.[3] Kirwan's proof procedes inductively, showing that the restriction map $H_G^*(X^{ss} \cup S_0 \cdot \cup S_n) \to H_G^*(X^{ss} \cup S_0 \cup \cdots \cup S_{n-1})$ is surjective for each n. Our proof of Theorem 1.2.1 follows an analogous pattern, although the techniques are different.

One can recover the De Rham cohomology of a smooth stack as the periodic-cyclic homology its derived category[46, 28], so the classical Kirwan surjectivity theorem follows from the existence of a splitting of i^*. Kirwan surjectivity applies to topological K-theory as well[21], and one immediate corollary of Theorem 1.2.1 is an analogous statement for algebraic K-theory

Corollary 1.2.3. *The restriction map on algebraic K-theory $K_i(\mathfrak{X}) \to K_i(\mathfrak{X}^{ss})$ is surjective.*

The fully faithful embedding $\mathrm{D}^b(\mathfrak{X}^{ss}) \subset \mathrm{D}^b(\mathfrak{X})$ of Theorem 1.2.1 and the more precise semiorthogonal decomposition of Theorem 2.0.3 correspond, via Orlov's analogy between derived categories and motives[35], to the claim that the motive $\mathfrak{X}^{ss}$ is a summand of $\mathfrak{X}$. Via this analogy, the results of this paper bear a strong formal resemblance to the motivic direct sum decompositions of homogeneous spaces arising from Białynicki-Birula decompositions[12]. However, the precise analogue of Theorem 1.2.1 would pertain to the equivariant motive X/G, whereas the results of [12] pertain to the nonequivariant motive X.

The "quantization commutes with reduction" theorem from geometric quantization theory relates to the fully-faithfulness of the functor i^*. The original conjecture of Guillemin and Sternberg, that $\dim H^0(X/G, \mathcal{L}^k) = \dim H^0(X^{ss}/G, \mathcal{L}^k)$, has been proven by several authors, but the most general version was proven by Teleman in [43]. He shows that the canonical restriction map induces an isomorphism $R\Gamma(X/G, \mathcal{V}) \to R\Gamma(X^{ss}/G, \mathcal{V})$ for any equivariant vector bundle such that $\mathcal{V}|_{Z_\alpha}$ is supported in weight $> -\eta_\alpha$. If $\mathcal{V}_1$ and $\mathcal{V}_2$ are two vector bundles in the grade restriction windows of Theorem 1.2.1, then the fact that $R\operatorname{Hom}_{\mathfrak{X}}^\bullet(\mathcal{V}_1, \mathcal{V}_2) \to R\operatorname{Hom}_{\mathfrak{X}^{ss}}^\bullet(\mathcal{V}_1|_{\mathfrak{X}^{ss}}, \mathcal{V}_2|_{\mathfrak{X}^{ss}})$ is an isomorphism is precisely Teleman's quantization theorem applied to $\mathcal{V}_2 \otimes \mathcal{V}_1^\vee \simeq R\underline{\operatorname{Hom}}(\mathcal{V}_1, \mathcal{V}_2)$.

In Section 2.2, we apply Theorem 1.2.1 to construct new examples of derived equivalences and embeddings resulting from birational transformations, as conjectured by Bondal & Orlov[8]. The G-ample cone in $NS_G^1(X)$ has a decomposition into convex conical chambers[16] within which the GIT quotient $\mathfrak{X}^{ss}(\mathcal{L})$ does not change, and $\mathfrak{X}^{ss}(\mathcal{L})$ undergoes a birational transformation as $[\mathcal{L}]$ crosses a wall between chambers. Derived Kirwan surjectivity provides a general approach to constructing derived equivalences between the quotients on either side of the wall: in some cases both quotients can be identified by Theorem 1.2.1 with the *same* subcategory of $\mathrm{D}^b(X/G)$. This principle is summarized in Ansatz 2.2.11.

For a certain class of wall crossings, *balanced wall crossings*, there is a simple criterion for when one gets an equivalence or an embedding in terms of the weights of $\omega_X|_{Z_i}$. When $G = T$ is abelian, all codimension-1 wall crossings are balanced, in particular we are able

---

[3]When X^{ss}/G is a DM stack, the equivariant cohomology $H_G^*(X^{ss})$ agrees with the cohomology of the coarse moduli space

to prove that any two generic torus quotients of an equivariantly Calabi-Yau variety are derived equivalent. For nonabelian G, we consider a slightly larger class of *almost balanced wall crossings*. We produce derived equivalences for flops which excise a Grassmannian bundle over a smooth variety and replace it with the dual Grassmannian bundle, recovering recent work of Will Donovan and Ed Segal[18, 17].

Finally, in Section 2.3 we investigate applications of Theorem 2.0.3 beyond smooth quotients X/G. We identify a criterion under which Property (L+) holds for a KN stratification, and apply it to hyperkähler reductions. We also explain how Morita theory[7] recovers derived Kirawn surjectivity for certain complete intersections and derived categories of singularities (equivalently categories of matrix factorizations) "for free" from the smooth case.

The inspiration for Theorem 1.2.1 were the grade restriction rules for the category of boundary conditions for B-branes of Landau-Ginzburg models studied by Hori, Herbst, and Page [22], as interpreted mathematically by Ed Segal [39]. The essential idea of splitting was present in that paper, but the analysis was only carried out for a linear action of $\mathbb{C}^*$, and the category $\mathbf{G}_w$ was identified in an ad-hoc way. The main contribution of this paper is showing that the splitting can be globalized and applies to arbitrary X/G as a categorification of Kirwan surjectivity, and that the categories $\mathbf{G}_w$ arise naturally via the semiorthogonal decompositions to be described in the next section.

1.3 Introduction to Chapter 3

This chapter is joint work with Ian Shipman.

Homological mirror symmetry predicts, in certain cases, that the bounded derived category of coherent sheaves on an algebraic variety should admit *twist autoequivalences* corresponding to a spherical object [40]. The autoequivalences predicted by mirror symmetry have been widely studied, and the notion of a spherical object has been generalized to the notion of a spherical functor [2] (See Definition 3.2.10). In Chapter 3 we apply the techniques of Chapter 2 to the construction of autoequivalences of derived categories, and our investigation leads to general connections between the theory of spherical functors and the theory of semiorthogonal decompositions and mutations.

We consider an algebraic stack which arises as a GIT quotient of a smooth quasiprojective variety X by a reductive group G. By varying the G-ample line bundle used to define the semistable locus, one gets a birational transformation $X_-^{ss}/G \dashrightarrow X_+^{ss}/G$ called a variation of GIT quotient (VGIT). We study a simple type of VGIT, which we call a *balanced wall crossing* (See Section 3.2).

Under a hypothesis on ω_X, a balanced wall crossing gives rise to an equivalance $\psi_w : \mathrm{D}^b(X_-^{ss}/G) \to \mathrm{D}^b(X_+^{ss}/G)$ which depends on a choice of $w \in \mathbb{Z}$, and the composition $\Phi_w := \psi_{w+1}^{-1}\psi_w$ defines an autoequivalence of $\mathrm{D}^b(X_-^{ss}/G)$. Autoequivalences of this kind have been studied recently under the name window-shifts [17, 39]. We generalize the observations of those papers in showing that Φ_w is always a spherical twist.

Recall that if B is an object in a dg-category, then we can define the twist functor

$$T_B : F \mapsto \mathrm{Cone}(\mathrm{Hom}^\bullet(B, F) \otimes_{\mathbb{C}} B \to F)$$

If B is a spherical object, then T_B is by definition the spherical twist autoequivalence defined by B. More generally, if $S : \mathcal{A} \to \mathcal{B}$ is a spherical functor (Definition 3.2.10), then one can define a twist autoequivalence $T_S := \mathrm{Cone}(S \circ S^R \to \mathrm{id}_{\mathcal{B}})$ of $\mathcal{B}$, where S^R denotes the right adjoint. Throughout this paper we refer to a twist autoequivalence corresponding to a spherical functor simply as a "spherical twist." A spherical object corresponds to the case where $\mathcal{A} = \mathrm{D}^b(k - \mathrm{vect})$.

It was noticed immediately [40] that if B were instead an exceptional object, then T_B is the formula for the left mutation equivalence ${}^\perp B \to B^\perp$ coming from a pair of semiorthogonal decompositions $\langle B^\perp, B \rangle = \langle B, {}^\perp B \rangle$.[4] In fact, we will show that there is more than a formal relationship between spherical functors and mutations. If $\mathcal{C}$ is a pre-triangulated dg category, then the braid group on n-strands acts by left and right mutation on the set of length n semiorthogonal decompositions $\mathcal{C} = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle$ with each $\mathcal{A}_i$ admissible. Mutating by a braid gives equivalences $\mathcal{A}_i \to \mathcal{A}'_{\sigma(i)}$, where σ is the permutation that the braid induces on end points. In particular if one of the semiorthogonal factors is the same subcategory before and after the mutation, one gets an autoequivalence $\mathcal{A}_i \to \mathcal{A}_i$.

Summary Theorem 1.3.1 (spherical twist=mutation=window shifts)**.** *If $\mathcal{C}$ is a pre triangulated dg category admitting a semiorthogonal decomposition $\mathcal{C} = \langle \mathcal{A}, \mathcal{G} \rangle$ which is fixed by the braid (acting by mutations)*
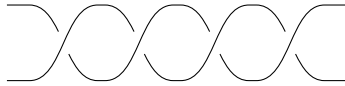
then the autoequivalence of $\mathcal{G}$ induced by mutation is the twist T_S corresponding to a spherical functor $S : \mathcal{A} \to \mathcal{G}$ (Theorem 3.2.11). Conversely, if $S : \mathcal{A} \to \mathcal{B}$ is a spherical functor, then there is a larger category $\mathcal{C}$ admitting a semiorthogonal decomposition fixed by this braid which recovers S and T_S (Theorem 3.2.15).

In the context of a balanced GIT wall crossing, the category $\mathcal{C}$ arises naturally as a subcategory of the equivariant category $\mathrm{D}^b(X/G)$, defined in terms of "grade restriction rules" (Section 3.1). The resulting autoequivalence agrees with the window shift Φ_w (Proposition 3.2.4) and corresponds to a spherical functor $f_w : \mathrm{D}^b(Z/L)_w \to \mathrm{D}^b(X^{ss}_-/G)$, where Z/L is the "critical locus" of the VGIT, which is unstable in both quotients (Section 3.2).

In the second half of the paper we revisit the prediction of derived autoequivalences from mirror symmetry. Spherical twist autoequivalences of $\mathrm{D}^b(V)$ for a Calabi-Yau V correspond to loops in the moduli space of complex structures on the mirror Calabi-Yau $V^\vee$, and flops correspond, under the mirror map, to certain paths in that complex moduli space. We

---

[4]Such semiorthogonal decompositions exist when $\mathrm{Hom}^\bullet(F^\bullet, B)$ has finite dimensional cohomology for all $F^\bullet$.

review these predictions, first studied in [24] for toric varieties, and formulate corresponding predictions for flops coming from VGIT in which an explicit mirror may not be known.

By studying toric flops between toric Calabi-Yau varieties of Picard rank 2 (Section 3.3), we find that mirror symmetry predicts more autoequivalences than constructed in Theorem 1.3.1. The expected number of autoequivalences agrees with the length of a full exceptional collection on the critical locus Z/L of the VGIT. Motivated by this observation, we introduce a notion of "fractional grade restriction windows" given the data of a semiorthogonal decomposition on the critical locus. This leads to

Summary Theorem 1.3.2 (Factoring spherical twists)**.** *Given a full exceptional collection* $\mathrm{D}^b(Z/L)_w = \langle E_0, \ldots, E_N \rangle$, *the objects* $S_i := f_w(E_i) \in \mathrm{D}^b(X_-^{ss}/G)$ *are spherical, and (Corollary 3.3.12)*

$$\Phi_w = T_{S_0} \circ \cdots T_{S_N}.$$

More generally, let $S = \mathcal{E} \to \mathcal{G}$ *be a spherical functor of dg-categories and let* $\mathcal{E} = \langle \mathcal{A}, \mathcal{B} \rangle$ *be a semiorthogonal decomposition such that there is also a semiorthogonal decomposition* $\mathcal{E} = \langle F_S(\mathcal{B}), \mathcal{A} \rangle$, *where* F_S *is the cotwist autoequivalence of* $\mathcal{E}$ *induced by* S. *Then the restrictions* $S_{\mathcal{A}} : \mathcal{A} \to \mathcal{G}$ *and* $S_{\mathcal{B}} : \mathcal{B} \to \mathcal{G}$ *are spherical as well, and* $T_S \simeq T_{S_{\mathcal{A}}} \circ T_{S_{\mathcal{B}}}$ *(Theorem 3.3.13).*

We propose an interpretation of this factorization theorem in terms of monodromy of the quantum connection in a neighborhood of a partial large volume limit (Section 3.3).

1.4 Introduction to Chapter 4

In Chapter 4 we revisit the foundational problem of constructing the stratification of the unstable locus in geometric invariant theory. We are motivated by two questions.

Question 1.4.1. *Theorem 2.0.3 requires that a KN stratification have special properties, Properties (L+) and (A), which hold automatically when* X *is smooth, but can fail even for mildly singular* X. *Is there nevertheless a version of Theorem 2.0.3 which applies in this setting?*

The second question comes from examples of KN stratifications beyond the setting of GIT. For example if G is a reductive group, then the moduli stack of G-bundles on a smooth curve C admits a KN stratification of the unstable locus due to Harder-Narasimhan and Shatz [41]. In the setting of differential geometry, the moduli space of semistable bundles can be constructed as a GIT quotient of an infinite dimensional space by an infinite dimensional "gauge group." The Shatz stratification agrees with the KN stratification for this infinite dimensional quotient, but this description does not carry over to the setting of algebraic geometry.

Question 1.4.2. *Many moduli problems in algebraic geometry which come with a notion of "stability" also come with canonical stratifications of their unstable loci. Is there a way*

to intrinsically construct a stratification of a stack $\mathfrak{X}$ which simultaneously generalizes the KN stratification in GIT (without relying on a global quotient presentation) and the Shatz stratification?

Although these two questions seem unrelated, the key to answering both lies in a modular interpretation for the strata that we establish in this chapter. We consider the quotient stack $\Theta := \mathbb{A}^1/\mathbb{G}_m$. If $\mathfrak{X} = X/G$ is a global quotient, then we show that a morphism of stacks $\Theta \to \mathfrak{X}$ is described uniquely up to 2-isomorphism by specifying a point $x \in X$ and a 1PS $\lambda : \mathbb{G}_m \to G$ under which $\lim_{t\to 0} \lambda(t)x$ exists. Thus for a point $x \in \mathfrak{X}$, the test data in the Hilbert-Mumford numerical criterion are exactly morphisms $f : \Theta \to \mathfrak{X}$ such that $f(1) \simeq x$.

When $\mathfrak{X}$ is of finite type over $\mathbb{C}$, we can define the cohomology of $\mathfrak{X}$ to be the cohomology of the geometric realization of its underlying topological stack [34]. We show that one can define the numerical invariant $\mu(x, \lambda)$ intrinsically in terms of the corresponding morphism $f : \Theta \to \mathfrak{X}$ given the cohomology class $l = c_1(\mathcal{L}) \in H^2(\mathfrak{X}; \mathbb{Q})$ and a class $b \in H^4(\mathfrak{X}; \mathbb{Q})$. While it is well-known that one needs a class $l \in H^2(\mathfrak{X}; \mathbb{Q})$ to define semistability, the importance of a class in $H^4(\mathfrak{X}; \mathbb{Q})$ for defining a stratification of the unstable locus is a new observation. The class b corresponds to the choice of a K-invariant inner product on $\mathfrak{g}$, and it is necessary to define the KN stratification.

Furthermore, the strata S_α/G arising in geometric invariant theory admit a modular interpretation as open substacks of the *mapping stack*

$$\underline{\mathfrak{X}}(\Theta) := \underline{\mathrm{Hom}}(\Theta, \mathfrak{X})$$

We show that this stack, which classifies maps $S \times \Theta \to \mathfrak{X}$ for any test scheme S, is an algebraic stack, and we describe it explicitly using a global quotient presentation of $\mathfrak{X}$. Even when $\mathfrak{X}$ is connected, the stack $\underline{\mathfrak{X}}(\Theta)$ will have infinitely many connected components, and the numerical invariant will define a locally constant real valued function on $\underline{\mathfrak{X}}(\Theta)$.

Note that restricting a morphism $S \times \Theta \to \mathfrak{X}$ to the subscheme $S \times \{1\} \subset S \times \Theta$ defines a morphism $r_1 : \underline{\mathfrak{X}}(\Theta) \to \mathfrak{X}$. We show that the Hilbert-Mumford criterion identifies a sequence of connected components of $\underline{\mathfrak{X}}(\Theta)$ for which r_1 is a closed immersion away from the image of previous connected components (with higher numerical invariant). In other words the morphism r_1 identifies the strata with open substacks of the connected components of $\underline{\mathfrak{X}}(\Theta)$ selected by the Hilbert-Mumford criterion.

Now recall that Property (L+) referred to in Question 1.4.1 states that the relative cotangent complex $\mathbf{L}^\bullet_{S_\alpha/X}$ must have positive weights along Z_α w.r.t. λ_α. The modular interpretation of S_α/G means that every strata fits into a universal evaluation diagram

$$\Theta \times S_\alpha/G \xrightarrow{\;ev\;} \mathfrak{X}$$
$$\downarrow{\scriptstyle \pi}$$
$$S_\alpha/G$$

Let $\mathbf{L}^\bullet_{\mathfrak{X}}$ denote the cotangent complex of $\mathfrak{X}$, and consider the object

$$E^\bullet := \pi_*\left(ev^*\mathbf{L}^\bullet_{\mathfrak{X}}[1](1)\right) \in D^b(S_\alpha/G)$$

One can show that this object automatically has positive weights along Z_α with respect to λ_α. If $\mathfrak{X} = X/G$ is smooth, then $E^\bullet \simeq \mathbf{L}^\bullet_{S_\alpha/X} \simeq \mathfrak{N}_{S_\alpha} X[1]$ is the relative cotangent complex. When X is not smooth, then $E^\bullet$ only represents the cotangent complex of the inclusion $S_\alpha/G \hookrightarrow X/G$ if we equip S_α with its canonical *derived* structure as an open substack of the derived mapping stack $\underline{\mathfrak{X}}(\Theta)$.

Thus using the modular interpretation one can equip S_α/G with a derived structure such that Property (L+) holds automatically. In future work, we will address the consequences of this observation for the extension of Theorem 2.0.3. In this chapter, we take the first key step of establishing the modular interpretation for the strata in the classical (i.e. non-derived setting).

In Chapter 4 we also use the modular interpretation to provide a preliminary answer to Question 1.4.2. In Shatz's stratification of the moduli of unstable vector bundles over a curve with rank R and degree D, the strata are indexed by sequences of points in the plane $(R_0, D_0), \ldots, (R_p, D_p) = (R, D)$ with $0 < R_1 < \cdots < R_p$ and such that the region below the piecewise linear path connecting these points is convex. An unstable vector bundle $\mathcal{E}$ has a unique Harder-Narasimhan filtration, and the sequence of ranks and degrees of the vector bundles in that filtration determines the stratum on which $\mathcal{E}$ lies. The same applies for principal SL_R bundles.

For $G = \mathrm{GL}_R$ or SL_R, we identify classes in H^2 and H^4 of the moduli stack of G-bundles on Σ, $\mathrm{Bun}_G(\Sigma)$, for which the intrinsic Hilbert-Mumford procedure reproduces the Shatz stratification. A map $\Theta \to \mathrm{Bun}_G(\Sigma)$ is equivalent to the data of a locally free sheaf on Σ together with a descending filtration whose associated graded is also locally free. The map $f : \Theta \to \mathrm{Bun}_G(\Sigma)$ which optimizes our numerical invariant subject to the constraint $f(1) \simeq \mathcal{E}$ corresponds to the Harder-Narasimhan filtration. For this f the numerical invariant takes the value

$$\sqrt{\sum (\nu_j)^2 r_j - \nu^2 R}$$

where ν_j denotes slope $(\deg / \mathrm{rank})$ of the jth piece of the graded bundle associated to the Harder-Narasimhan filtration, and r_j denotes its rank. This quantity is strictly monotone increasing with respect to inclusion of Shatz polytopes.

We say that this is a partial answer to question 1.4.2 because both in this example and for the modular interpretation of the KN strata in GIT, we use our prior knowledge of the existence of the stratification and verify that it can be "rediscovered" via our intrinsic description. The key inputs that we use are the existence and uniqueness (up to conjugacy) of a maximally destabilizing one-parameter subgroup for a point in X/G and the existence and uniqueness of a Harder-Narasimhan filtration of an unstable vector bundle on a curve.

There are many moduli problems, such as the moduli stack of all polarized projective varieties, where several different notions of stability have been introduced but where the notion corresponding to the Harder-Narasimhan filtration of an unstable object has not been investigated. Therefore, in order to use our general formulation of the notion of stability to discover new examples of KN stratifications, we must revisit the classical proofs of the existence and uniqueness of Harder-Narasimhan filtrations from an intrinsic perspective.

In the final sections of Chapter 4 we do just that. We introduce a new combinatorial object which we call a *fan*. For an arbitrary algebraic stack $\mathfrak{X}$ and a point $x \in \mathfrak{X}$, we introduce a fan $\mathbf{D}(\mathfrak{X}, x)_\bullet$ which parameterizes all of the one parameter degenerations $f : \Theta \to \mathfrak{X}$ with $f(1) \simeq x$. This fan generalizes the data of the fan of a toric variety X, which can be thought of as describing the various limit points of a generic point of X under the 1PS's of the torus acting on X.

Furthermore, an abstract fan has a geometric realization $|F_\bullet|$, a topological space which is homeomorphic the union of cones in $\mathbb{R}^N$ in many cases. A fan also admits a projective realization $\mathbb{P}(F_\bullet)$ which in good cases is homeomorphic to the intersection of $|F_\bullet|$ with the unit sphere $S^{N-1} \subset \mathbb{R}^N$.

We show that the numerical invariant determined by a class in $H^2(\mathfrak{X}; \mathbb{Q})$ and $H^4(\mathfrak{X}; \mathbb{Q})$ (see Example 4.3.6) defines a continuous function on the topological space $\mathbb{P}(\mathbf{D}(\mathfrak{X}, x)_\bullet)$ which is locally convex in a suitable sense. Furthermore we show that for a global quotient of an affine variety by a reductive group, $\mathfrak{X} = V/G$, the subset of $\mathbb{P}(\mathbf{D}(\mathfrak{X}, x)_\bullet)$ on which the numerical invariant is positive is also convex in a suitable sense. In this context, Kempf's original argument [29] for the existence and uniqueness of maximal destabilizing 1PS's can be boiled down to a simple observation: a convex function on a convex set has a unique maximizer.

In future work, we hope to apply this technique to establish KN stratifications for many other moduli problems in algebraic geometry. This would lead to a notion of "Harder-Narasimhan filtration" for objects other than vector bundles and coherent sheaves.

Chapter 2

Derived Kirwan surjectivity

As discussed in the introduction, we will consider a quasiprojective variety X with the action of a reductive group G. We let $X^{ss} \subset X$ be an open subvariety whose complement admits a KN-stratification (Definition 1.1.7). We will use the symbol $\mathfrak{X}$ to denote the quotient stack X/G, and likewise for $\mathfrak{X}^{ss}$. As the statement of Theorem 1.2.1 indicates, we will construct a splitting of $D^b(\mathfrak{X}) \to D^b(\mathfrak{X}^{ss})$ by identifying a subcategory $\mathbf{G}_w \subset D^b(\mathfrak{X})$ that is mapped isomorphically onto $D^b(\mathfrak{X}^{ss})$. In fact we will identify $\mathbf{G}_w$ as the middle factor in a large semiorthogonal decomposition of $D^b(\mathfrak{X})$.

We denote a *semiorthogonal decomposition* of a triangulated category $\mathcal{D}$ by full triangulated subcategories $\mathcal{A}_i$ as $\mathcal{D} = \langle \mathcal{A}_n, \ldots, \mathcal{A}_1 \rangle$ [9]. This means that all morphisms from objects in $\mathcal{A}_i$ to objects in $\mathcal{A}_j$ are zero for $i < j$, and for any object of $E \in \mathcal{D}$ there is a sequence $0 = E_0 \to E_1 \to \cdots E_n = E$ with $\mathrm{Cone}(E_{i-1} \to E_i) \in \mathcal{A}_i$, which is necessarily unique and thus functorial.[1] In our applications $\mathcal{D}$ will always be a pre-triangulated dg-category, in which case if $\mathcal{A}_i \subset \mathcal{D}$ are full pre-triangulated dg-categories then we will abuse the notation $\mathcal{D} = \langle \mathcal{A}_n, \ldots, \mathcal{A}_1 \rangle$ to mean that there is a semiorthogonal decomposition of homotopy categories, in which case $\mathcal{D}$ is uniquely identified with the gluing of the $\mathcal{A}_i$.

A *baric decomposition* is simply a filtration of a triangulated category $\mathcal{D}$ by right-admissible triangulated subcategories, i.e. a family of semiorthogonal decompositions $\mathcal{D} = \langle \mathcal{D}_{<w}, \mathcal{D}_{\geq w} \rangle$ such that $\mathcal{D}_{\geq w} \supset \mathcal{D}_{\geq w+1}$, and thus $\mathcal{D}_{<w} \subset \mathcal{D}_{<w+1}$, for all w. This notion was introduced and used construct 'staggered' t-structures on equivariant derived categories of coherent sheaves [1].

Although the connection with GIT was not explored in the original development of the theory, baric decompositions seem to be the natural structure arising on the derived category of the unstable locus in geometric invariant theory. The key to our proof will be to consider a single closed KN stratum $\mathfrak{S} \subset \mathfrak{X}$ and construct baric decompositions of $D^b(\mathfrak{S})$ in Proposition 2.1.14 and of $D^b_{\mathfrak{S}}(\mathfrak{X})$, the bounded derived category of complexes of coherent sheaves on $\mathfrak{X}$

---

[1]There are two additional equivalent ways to characterize a semiorthogonal decomposition: 1) the inclusion of the full subcategory $\mathcal{A}_i \subset \langle \mathcal{A}_i, \mathcal{A}_{i-1}, \ldots, \mathcal{A}_1 \rangle$ admits a left adjoint $\forall i$, or 2) the subcategory $\mathcal{A}_i \subset \langle \mathcal{A}_n, \ldots, \mathcal{A}_i \rangle$ is right admissible $\forall i$. In some contexts one also requires that each $\mathcal{A}_i$ be admissible in $\mathcal{D}$, but we will not require this here. See [9] for further discussion.

whose homology is supported on a KN stratum $\mathfrak{S}$, in Proposition 2.1.21. We will postpone a detailed analysis of the homological structure of a single KN stratum to Section 2.1 – here we apply the results of that section iteratively to a stratification with multiple KN strata.

In order to state our main theorem in the setting where X is singular, we will introduce two additional properties on the KN strata. When X is smooth, Z, Y, and S will all be smooth as well and these properties will hold automatically.

(A) $\pi : Y \to Z$ is a locally trivial bundle of affine spaces

(L+) The derived restriction of the relative cotangent complex $L\sigma^* \mathbf{L}_{S/X}^{\bullet}$ along the closed immersion $\sigma : Z \hookrightarrow S$ has nonnegative weights w.r.t. λ.

We will use the construction of the cotangent complex in characteristic 0 as discussed in [31]. Note that when X is smooth along Z, $\mathbf{L}_{S/X}^{\bullet} \simeq \mathfrak{N}_{S/X}[1]$ is locally free on S, so Property (L+) follows from (S3).

For each inclusion $\sigma_i : Z_i \hookrightarrow S_i$ and $j_i : S_i \hookrightarrow X$, we define the shriek pullback functor $\sigma_i^! : \mathrm{D}^b(\mathfrak{X}) \to \mathrm{D}^b(\mathfrak{Z}_i)$ as the composition $F^{\bullet} \mapsto \sigma_i^* j_i^! F^{\bullet}$, where $j_i^! F^{\bullet} = \underline{\mathrm{Hom}}(\mathcal{O}_{\mathfrak{S}_i}, F^{\bullet})$ regarded as an $\mathcal{O}_{\mathfrak{S}_i}$ module.

Theorem 2.0.3 (derived Kirwan surjectivity). *Let $\mathfrak{X} = X/G$ be a stack quotient of a quasiprojective variety by a reductive group, let $\mathfrak{X}^{ss} \subset \mathfrak{X}$ be an open substack, and let $\{\mathfrak{S}_{\alpha}\}_{\alpha \in I}$ be a KN stratification (Definition 1.1.7) of $\mathfrak{X}^u = \mathfrak{X} \setminus \mathfrak{X}^{ss}$. Assume that each $\mathfrak{S}_{\alpha}$ satisfies Properties (A) and (L+). Define the integers*

$$a_i := \mathrm{weight}_{\lambda_i} \det(N_{Z_i} Y_i) \tag{2.1}$$

For each KN stratum, choose an integer $w_i \in \mathbb{Z}$. Define the full subcategories of $\mathrm{D}^b(\mathfrak{X})$

$$\mathrm{D}^b_{\mathfrak{X}^u}(\mathfrak{X})_{\geq w} := \{F^{\bullet} \in \mathrm{D}^b_{\mathfrak{X}^u}(\mathfrak{X}) \,|\, \forall i, \lambda_i \text{ weights of } \mathcal{H}^*(\sigma_i^* F^{\bullet}) \text{ are } \geq w_i\}$$

$$\mathrm{D}^b_{\mathfrak{X}^u}(\mathfrak{X})_{<w} := \{F^{\bullet} \in \mathrm{D}^b_{\mathfrak{X}^u}(\mathfrak{X}) \,|\, \forall i, \lambda_i \text{ weights of } \mathcal{H}^*(\sigma_i^! F^{\bullet}) \text{ are } < w_i + a_i\}$$

$$\mathbf{G}_w := \left\{ F^{\bullet} \,\middle|\, \begin{array}{l} \forall i, \mathcal{H}^*(\sigma_i^* F^{\bullet}) \text{ has weights } \geq w_i, \text{ and} \\ \mathcal{H}^*(\sigma_i^! F^{\bullet}) \text{ has weights } < w_i + a_i \end{array} \right\}$$

Then there are semiorthogonal decompositions

$$\mathrm{D}^b_{\mathfrak{X}^u}(\mathfrak{X}) = \langle \mathrm{D}^b_{\mathfrak{X}^u}(\mathfrak{X})_{<w}, \mathrm{D}^b_{\mathfrak{X}^u}(\mathfrak{X})_{\geq w} \rangle \tag{2.2}$$

$$\mathrm{D}^b(\mathfrak{X}) = \langle \mathrm{D}^b_{\mathfrak{X}^u}(\mathfrak{X})_{<w}, \mathbf{G}_w, \mathrm{D}^b_{\mathfrak{X}^u}(\mathfrak{X})_{\geq w} \rangle \tag{2.3}$$

and the restriction functor $i^* : \mathbf{G}_w \to \mathrm{D}^b(\mathfrak{X}^{ss})$ is an equivalence of categories. We have $\mathrm{Perf}_{\mathfrak{X}^u}(\mathfrak{X})_{\geq v} \otimes^L \mathrm{D}^b_{\mathfrak{X}^u}(\mathfrak{X})_{\geq w} \subset \mathrm{D}^b_{\mathfrak{X}^u}(\mathfrak{X})_{\geq v+w}$.

If $\mathfrak{X}$ is smooth in a neighborhood of $\mathfrak{X}^u$, then Properties (A) and (L+) hold automatically, and we define

$$\eta_i := \mathrm{weight}_{\lambda_i} \det(N_{S_i} X^{\vee}) \tag{2.4}$$

$$= \mathrm{weight}_{\lambda_i} \det(N_{Y_i}^{\vee} X) - \mathrm{weight}_{\lambda_i} \det(\mathfrak{g}_{\lambda_i > 0})$$

Then we have alternate characterizations

$$\mathrm{D}^b_{\mathfrak{X}^u}(\mathfrak{X})_{<w} := \{F^\bullet \in \mathrm{D}^b_{\mathfrak{X}^u}(\mathfrak{X}) \,|\, \forall i, \lambda_i \text{ weights of } \mathcal{H}^*(\sigma_i^* F^\bullet) \text{ are } < w_i + \eta_i\}$$
$$\mathbf{G}_w := \{F^\bullet \,|\, \forall i, \mathcal{H}^*(\sigma_i^* F^\bullet) \text{ has weights in } [w_i, \eta_i) \text{ w.r.t. } \lambda_i\}$$

Proof. Choose a total ordering of I, $\alpha_0 > \alpha_1 > \cdots$ such that α_n is maximal in $I \setminus \{\alpha_0, \ldots, \alpha_{n-1}\}$, so that $\mathfrak{S}_{\alpha_n}$ is closed in $\mathfrak{X} \setminus \mathfrak{S}_{\alpha_0} \cup \cdots \cup \mathfrak{S}_{\alpha_{n-1}}$. Introduce the notation $\mathfrak{S}^n = \bigcup_{i<n} \mathfrak{S}_{\alpha_i}$. $\mathfrak{S}^n \subset \mathfrak{X}$ is closed and admits a KN stratification by the n strata $\mathfrak{S}_{\alpha_i}$ for $i < n$, so we will proceed by induction on n. The base case is Theorem 2.1.31.

Assume the theorem holds for $\mathfrak{S}^n \subset \mathfrak{X}$, so $\mathrm{D}^b(\mathfrak{X}) = \langle \mathrm{D}^b_{\mathfrak{S}^n}(\mathfrak{X})_{<q}, \mathbf{G}_q^n, \mathrm{D}^b_{\mathfrak{S}^n}(\mathfrak{X})_{\geq q}\rangle$ and restriction maps $\mathbf{G}_q^n$ isomorphically onto $\mathrm{D}^b(\mathfrak{X} \setminus \mathfrak{S}^n)$. $\mathfrak{S}_{\alpha_n} \subset \mathfrak{X} \setminus \mathfrak{S}^n$ is a closed KN stratum, so Theorem 2.1.31 gives a semiorthogonal decomposition of $\mathbf{G}_q^n \simeq \mathrm{D}^b(\mathfrak{X} \setminus \mathfrak{S}^n)$ which we combine with the previous semiorthogonal decomposition

$$\mathrm{D}^b(\mathfrak{X}) = \langle \mathrm{D}^b_{\mathfrak{S}^n}(\mathfrak{X})_{<q}, \mathrm{D}^b_{\mathfrak{S}_{\alpha_n}}(\mathfrak{X} \setminus \mathfrak{S}^n)_{<q(\alpha)}, \mathbf{G}_q^{n+1}, \mathrm{D}^b_{\mathfrak{S}_{\alpha_n}}(\mathfrak{X} \setminus \mathfrak{S}^n)_{\geq q(\alpha)}, \mathrm{D}^b_{\mathfrak{S}^n}(\mathfrak{X})_{\geq q}\rangle$$

The first two pieces correspond precisely to $\mathrm{D}^b_{\mathfrak{S}^{n+1}}(\mathfrak{X})_{<q}$ and the last two pieces correspond to $\mathrm{D}^b_{\mathfrak{S}^{n+1}}(\mathfrak{X})_{\geq q}$. The theorem follows by induction. $\qquad\square$

Remark 2.0.4. The semiorthogonal decomposition in this theorem can be refined further using ideas of Kawamata[27], and Ballard, Favero, Katzarkov[6] (See Amplification 2.1.23 below for a discussion in this context).

Example 2.0.5. Let $X \subset \mathbb{P}^n$ be a projective variety with homogeneous coordinate ring A. The affine cone $\mathrm{Spec}\, A$ has $\mathbb{G}_m$ action given by the nonnegative grading of A and the unstable locus is $Z = Y = S =$ the cone point. $\mathcal{O}_S$ can be resolved as a semi-free graded dg-algebra over A, $(A[x_1, x_2, \ldots], d) \to \mathcal{O}_S$ with generators of positive weight. Thus $\mathbf{L}^\bullet_{S/Z} = \mathcal{O}_S \otimes \Omega^1_{A[x_1,\ldots]/A}$ has positive weights. The Property (A) is automatic. In this case Theorem 2.0.3 is essentially Serre's theorem on the derived category of a projective variety.

Corollary 2.0.6. *Let Z be a quasiprojective scheme and $\mathcal{A} = \bigoplus_{i\geq 0} \mathcal{A}_i$ a coherently generated sheaf of algebras over Z, with $\mathcal{A}_0 = \mathcal{O}_Z$. Let $j : Z \hookrightarrow \underline{\mathrm{Spec}}(\mathcal{A})$ be the inclusion. There is an infinite semiorthogonal decomposition,*

$$\mathrm{D}^b(\mathrm{gr} - A) = \langle \ldots, \mathrm{D}^b(Z)_{w-1}, \mathbf{G}_w, \mathrm{D}^b(Z)_w, \mathrm{D}^b(Z)_{w+1}, \ldots \rangle$$

where $\mathrm{D}^b(Z)_w$ denotes the subcategory generated by $j_* \mathrm{D}^b(Z) \otimes \mathcal{O}_X(-w)$, and

$$\mathbf{G}_w = \left\{ F^\bullet \in \mathrm{D}^b(X/\mathbb{C}^*) \,\middle|\, \begin{array}{l} \mathcal{H}^*(j^* F^\bullet) \text{ has weights } \geq w, \text{ and} \\ \mathcal{H}^*(j^! F^\bullet) \text{ has weights } < w \end{array} \right\}$$

and the restriction functor $\mathbf{G}_w \to \mathrm{D}^b(\underline{\mathrm{Proj}}\mathcal{A})$ is an equivalence.

Example 2.0.7. Consider the graded polynomial ring $k[x_1, \ldots, x_n, y_1, \ldots, y_m]/(f)$ where the x_i have positive degrees and the y_i have negative degrees and f is a homogeneous polynomial such that $f(0) = 0$. This corresponds to a linear action of $\mathbb{G}_m$ on an equivariant hypersurface X_f in the affine space $\mathbb{A}^n_x \times \mathbb{A}^m_y$. Assume that we have chosen the linearization such that $S = \{0\} \times \mathbb{A}^m_y \cap X_f$. One can compute

$$\mathbf{L}^{\bullet}_{S/X_f} \begin{cases} (\mathcal{O}_S x_1 \oplus \cdots \oplus \mathcal{O}_S x_n)[1], & \text{if } f \notin (x_1, \ldots, x_n) \\ (\mathcal{O}_S f \to \mathcal{O}_S dx_1 \oplus \cdots \oplus \mathcal{O}_S dx_n)[1] & \text{if } f \in (x_1, \ldots, x_n) \end{cases}$$

Thus $S \subset X_f$ satisfies Property (L+) iff either $\deg f \geq 0$, in which case $f \in (x_1, \ldots, x_n)$, or if $\deg f < 0$ but $f \notin (x_1, \ldots, x_n)$. Furthermore, Property (A) amounts to S being an affine space, which happens iff $\deg f \geq 0$ so that $S = \mathbb{A}^m_y$, or $\deg f = -1$ with a nontrivial linear term in the y_i. Note in particular that in order for X_f to satsify these properties with respect to the stratum of the the opposite linearization, then we are left with only two possibilities: either $\deg f = 0$ or $\deg f = \pm 1$ with nontrivial linear terms. This illustrates the non-vacuousness of Properties (A) and (L+).

Explicit constructions of the splitting, and Fourier-Mukai kernels

Given an $F^{\bullet} \in \mathrm{D}^b(\mathfrak{X}^{ss})$, one can extend it uniquely up to weak equivalence to a complex in $\mathbf{G}_q$. Due to the inductive nature of Theorem 2.0.3, the extension can be complicated to construct. We will discuss a procedure for extending over a single stratum at the end of Section 2.1, and one must repeat this for every stratum of $\mathfrak{X}^{us}$.

Fortunately, it suffices to directly construct a single universal extension. Consider the product $\mathfrak{X}^{ss} \times \mathfrak{X} = (X^{ss} \times X)/(G \times G)$, and the open substack $\mathfrak{X}^{ss} \times \mathfrak{X}^{ss}$ whose complement has the KN stratification $\mathfrak{X}^{ss} \times \mathfrak{S}_{\alpha}$. One can uniquely extend the diagonal $\mathcal{O}_{\mathfrak{X}^{ss} \times \mathfrak{X}^{ss}}$ to a sheaf $\tilde{\mathcal{O}}_{\Delta}$ in the subcategory $\mathbf{G}_q$ with respect to this stratification. The Fourier-Mukai transform $\mathrm{D}^b(\mathfrak{X}^{ss}) \to \mathrm{D}^b(\mathfrak{X})$ with kernel $\tilde{\mathcal{O}}_{\Delta}$, has image in the subcategory $\mathbf{G}_q$ and is the identity over $\mathfrak{X}^{ss}$. Thus for any $F^{\bullet} \in \mathrm{D}^b(\mathfrak{X}^{ss})$, $(p_2)_*(\tilde{\mathcal{O}}_{\Delta} \otimes p_1^*(F^{\bullet}))$ is the unique extension of $F^{\bullet}$ to $\mathbf{G}_q$.

2.1 Homological structures on the unstable strata

In this section we will study in detail the homological properties of a single closed KN stratum $\mathfrak{S} := S/G \subset \mathfrak{X}$ as defined in 1.1.7. We establish a multiplicative baric decomposition of $\mathrm{D}^b(\mathfrak{S})$, and when $\mathfrak{S} \subset \mathfrak{X}$ satisfies Property (L+), we extend this to a multiplicative baric decomposition of $\mathrm{D}^b_{\mathfrak{S}}(\mathfrak{X})$, the derived category of complexes of coherent sheaves on $\mathfrak{X}$ whose restriction to $\mathfrak{V} = \mathfrak{X} - \mathfrak{S}$ is acyclic. Then we use these baric decompositions to construct our main semiorthogonal decompositions of $D^b(\mathfrak{X})$.

Recall the structure of a KN stratum (1.4) and the associated parabolic subgroup (1.1). By Property (S1), $\mathfrak{S} := S/G \simeq Y/P$ via the P-equivariant inclusion $Y \subset S$, so we will identify quasicoherent sheaves on $\mathfrak{S}$ with P-equivariant quasicoherent $\mathcal{O}_Y$-modules. Furthermore, we will let P act on Z via the projection $P \to L$. Again by Property (S1), we

have $Y/P = \underline{\mathrm{Spec}}_Z(\mathcal{A})/P$, where $\mathcal{A}$ is a coherently generated $\mathcal{O}_Z$-algebra with $\mathcal{A}_i = 0$ for $i > 0$, and $\mathcal{A}_0 = \mathcal{O}_Z$. Thus we have identified quasicoherent sheaves on $\mathfrak{S}$ with quasicoherent $\mathcal{A}$-modules on $\mathfrak{Z}' := Z/P$.

Remark 2.1.1. The stack $\mathfrak{Z} := Z/L$ is perhaps more natural than the stack $\mathfrak{Z}'$. The projection $\pi : Y \to Z$ intertwines the respective P and L actions via $P \to L$, hence we get a projection $\mathfrak{S} \to \mathfrak{Z} := Z/L$. Unlike the map $\mathfrak{S} \to \mathfrak{Z}'$, this projection admits a section $Z/L \to Y/P$. In other words, the projection $\mathcal{A} \to \mathcal{A}_0 = \mathcal{O}_Z$ is $\lambda(\mathbb{C}^*)$-equivariant, but not P-equivariant. We choose to work with $\mathfrak{Z}'$, however, because the map $\mathfrak{S} \to \mathfrak{Z}$ is not representable, so the description of quasicoherent $\mathfrak{S}$ modules in terms of "$\mathfrak{Z}$-modules with additional structure" is less straightforward.

We will use the phrase $\mathcal{O}_Z$-module to denote a quasicoherent sheaf on the stack $\mathfrak{Z}' = Z/P$, assuming quasicoherence and P-equivariance unless otherwise specified. λ fixes Z, so equivariant $\mathcal{O}_Z$ modules have a natural grading by the weight spaces of λ, and we will use this grading often.

Lemma 2.1.2. *For any $F \in \mathrm{QCoh}(\mathfrak{Z}')$ and any $w \in \mathbb{Z}$, the submodule $F_{\geq w} := \sum_{i \geq w} F_i$ of sections of weight $\geq w$ with respect to λ is P equivariant.*

Proof. $\mathbb{C}^*$ commutes with L, so $F_{\geq w}$ is an equivariant submodule with respect to the L action. Because $U \subset P$ acts trivially on Z, the U-equivariant structure on F is determined by a coaction $a : F \to k[U] \otimes F$ which is equivariant for the $\mathbb{C}^*$ action. We have

$$a(F_{\geq w}) \subset (k[U] \otimes F)_{\geq w} = \bigoplus_{i+j \geq w} k[U]_i \otimes F_j \subset k[U] \otimes F_{\geq w}$$

The last inclusion is due to the fact that $k[U]$ is non-positively graded, and it implies that $F_{\geq w}$ is equivariant with respect to the U action as well. Because we have a semidirect product decomposition $P = UL$, it follows that $F_{\geq p}$ is an equivariant submodule with respect to the P action. $\qquad\square$

Remark 2.1.3. This lemma is a global version of the observation that for any P-module M, the subspace $M_{\geq w}$ with weights $\geq w$ with respect to λ is a P-submodule, which can be seen from the coaction $M \to k[P] \otimes M$ and the fact that $k[P]$ is nonnegatively graded with respect to λ.

It follows that any $F \in \mathrm{QCoh}(\mathfrak{Z}')$ has a functorial factorization $F_{\geq w} \hookrightarrow F \twoheadrightarrow F_{<w}$. Note that as $\mathbb{C}^*$-equivariant instead of P-equivariant $\mathcal{O}_Z$-modules there is a natural isomorphism $F \simeq F_{\geq w} \oplus F_{<w}$. Thus the functors $(\bullet)_{\geq w}$ and $(\bullet)_{<w}$ are exact, and that if F is locally free, then $F_{\geq w}$ and $F_{<w}$ are locally free as well.

We define $\mathrm{QCoh}(\mathfrak{Z}')_{\geq w}$ and $\mathrm{QCoh}(\mathfrak{Z}')_{<w}$ to be the full subcategories of $\mathrm{QCoh}(\mathfrak{Z}')$ consisting of sheaves supported in weight $\geq w$ and weight $< w$ respectively. They are both Serre subcategories, they are orthogonal to one another, $(\bullet)_{\geq w}$ is right adjoint to the inclusion $\mathrm{QCoh}(\mathfrak{Z}')_{\geq w} \subset \mathrm{QCoh}(\mathfrak{Z}')$, and $(\bullet)_{<w}$ is left adjoint to the inclusion $\mathrm{QCoh}(\mathfrak{Z}')_{<w} \subset \mathrm{QCoh}(\mathfrak{Z}')$.

Lemma 2.1.4. *Any $F \in \mathrm{QCoh}(\mathfrak{Z}')_{<w}$ admits an injective resolution $F \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots$ such that $\mathcal{I}^i \in \mathrm{QCoh}(\mathfrak{Z}')_{<w}$. Likewise any $F \in \mathrm{Coh}(\mathfrak{Z}')_{\geq w}$ admits a locally free resolution $\cdots \to E_1 \to E_0 \to F$ such that $E_i \in \mathrm{Coh}(\mathfrak{Z}')_{\geq w}$.*

Proof. First assume $F \in \mathrm{QCoh}(\mathfrak{Z}')_{<w}$, and let $F \to \mathcal{I}^0$ be the injective hull of F.[2] Then $\mathcal{I}^0_{\geq w} \cap F_{<w} = 0$, hence $\mathcal{I}^0_{\geq w} = 0$ because I^0 is an essential extension of F. $\mathrm{QCoh}(\mathfrak{Z}')_{<w}$ is a Serre subcategory, so $\mathcal{I}^0/F \in \mathrm{QCoh}(\mathfrak{Z}')_{<w}$ as well, and we can inductively build an injective resolution with $\mathcal{I}^i \in \mathrm{QCoh}(\mathfrak{Z}')_{<w}$.

Next assume $F \in \mathrm{Coh}(\mathfrak{Z}')_{\geq w}$. Choose a surjection $E \to F$ where E is locally free. Then $E_0 := E_{\geq w}$ is still locally free, and $E_{\geq w} \to F$ is still surjective. Because $\mathrm{Coh}(\mathfrak{Z}')_{\geq w}$ is a Serre subcategory, $\ker(E_0 \to F) \in \mathrm{Coh}(\mathfrak{Z}')_{\geq w}$ as well, so we can inductively build a locally free resolution with $E_i \in \mathrm{Coh}(\mathfrak{Z}')_{\geq w}$. $\square$

We will use this lemma to study the subcategories of $\mathrm{D}^b(\mathfrak{Z}')$ generated by $\mathrm{Coh}(\mathfrak{Z}')_{\geq w}$ and $\mathrm{Coh}(\mathfrak{Z}')_{<w}$. Define the full triangulated subcategories

$$\mathrm{D}^b(\mathfrak{Z}')_{\geq w} = \{F^\bullet \in \mathrm{D}^b(\mathfrak{Z}') | \mathcal{H}^i(F^\bullet) \in \mathrm{Coh}(\mathfrak{Z}')_{\geq w}\}$$
$$\mathrm{D}^b(\mathfrak{Z}')_{<w} = \{F^\bullet \in \mathrm{D}^b(\mathfrak{Z}') | \mathcal{H}^i(F^\bullet) \in \mathrm{Coh}(\mathfrak{Z}')_{<w}\}$$

For any complex $F^\bullet$ we have the canonical short exact sequence

$$0 \to F^\bullet_{\geq w} \to F^\bullet \to F^\bullet_{<w} \to 0 \tag{2.5}$$

If $F^\bullet \in \mathrm{D}^b(\mathfrak{Z}')_{\geq w}$ then the first arrow is a quasi-isomorphism, because $(\bullet)_{\geq w}$ is exact. Likewise for the second arrow if $F^\bullet \in \mathrm{D}^b(\mathfrak{Z}')_{<w}$. Thus $F^\bullet \in \mathrm{D}^b(\mathfrak{Z}')_{\geq w}$ iff it is quasi-isomorphic to a complex of sheaves in $\mathrm{Coh}(\mathfrak{Z}')_{\geq w}$ and likewise for $\mathrm{D}^b(\mathfrak{Z}')_{<w}$.

Proposition 2.1.5. *These subcategories constitute a* baric decomposition

$$\mathrm{D}^b(\mathfrak{Z}') = \langle \mathrm{D}^b(\mathfrak{Z}')_{<w}, \mathrm{D}^b(\mathfrak{Z}')_{\geq w} \rangle$$

This baric decomposition is multiplicative *in the sense that*

$$\mathrm{Perf}(\mathfrak{Z}')_{\geq w} \otimes \mathrm{D}^b(\mathfrak{Z}')_{\geq v} \subset \mathrm{D}^b(\mathfrak{Z}')_{\geq v+w}.$$

It is bounded, *meaning that every object lies in $\mathcal{D}_{\geq w} \cap \mathcal{D}_{<v}$ for some w, v. The baric truncation functors, the adjoints of the inclusions $\mathcal{D}_{\geq w}, \mathcal{D}_{<w} \subset \mathrm{D}^b(\mathfrak{Z}')$, are exact.*

Proof. If $A \in \mathrm{Coh}(\mathfrak{Z}')_{\geq w}$ and $B \in \mathrm{Coh}(\mathfrak{Z}')_{<w}$, then by Lemma 2.1.4 we resolve B by injectives in $\mathrm{QCoh}(\mathfrak{Z}')_{<w}$, and thus $R\mathrm{Hom}(A, B) \simeq 0$. It follows that $\mathrm{D}^b(\mathfrak{Z}')_{\geq w}$ is left orthogonal to $\mathrm{D}^b(\mathfrak{Z}')_{<w}$. $\mathrm{QCoh}(\mathfrak{Z}')_{\geq w}$ and $\mathrm{QCoh}(\mathfrak{Z}')_{\leq w}$ are Serre subcategories, so $F^\bullet_{\geq w} \in \mathrm{D}^b(\mathfrak{Z}')_{\geq w}$ and $F^\bullet_{<w} \in \mathrm{D}^b(\mathfrak{Z}')_{<w}$ for any $F^\bullet \in \mathrm{D}^b(\mathfrak{Z}')$. Thus the natural sequence (2.5) shows that we have a baric decomposition, and that the right and left truncation functors are the exact functors $(\bullet)_{\geq w}$ and $(\bullet)_{<w}$ respectively. Boundedness follows from the fact that coherent equivariant $\mathcal{O}_Z$-modules must be supported in finitely many λ weights. Multiplicativity is also straightforward to verify. $\square$

---

[2]The injective hull exists because $\mathrm{QCoh}(\mathfrak{Z}')$ is cocomplete and taking filtered colimits is exact.

Remark 2.1.6. A completely analogous baric decomposition holds for $\mathfrak{Z}$ as well. In fact, for $\mathfrak{Z}$ the two factors are mutually orthogonal.

Quasicoherent sheaves on $\mathfrak{S}$

The closed immersion $\sigma : Z \hookrightarrow Y$ is L equivariant, hence it defines a map of stacks $\sigma : \mathfrak{Z} \to \mathfrak{S}$. Recall also that because $\pi : \mathfrak{S} \to \mathfrak{Z}'$ is affine, the derived pushforward $R\pi_* = \pi_*$ is just the functor which forgets the $\mathcal{A}$-module structure. Define the thick triangulated subcategories

$$D^b(\mathfrak{S})_{<w} = \{F^\bullet \in D^b(\mathfrak{S}) | \pi_* F^\bullet \in D^b(\mathfrak{Z}')_{<w}\}$$
$$D^b(\mathfrak{S})_{\geq w} = \{F^\bullet \in D^b(\mathfrak{S}) | L\sigma^* F^\bullet \in D^-(\mathfrak{Z})_{\geq w}\}$$

In the rest of this subsection we will analyze these two categories and show that they constitute a multiplicative baric decomposition.

Complexes on $\mathfrak{S}$ of the form $\mathcal{A} \otimes E^\bullet$, where each E^i is a locally free sheaf on $\mathfrak{Z}'$, will be of prime importance. Note that the differential $d^i : \mathcal{A} \otimes E^i \to \mathcal{A} \otimes E^{i+1}$ is not necessarily induced from a differential $E^i \to E^{i+1}$. However we observe

Lemma 2.1.7. *If $E \in \mathrm{QCoh}(\mathfrak{Z})$, then $\mathcal{A} \cdot (\mathcal{A} \otimes E)_{\geq w} = \mathcal{A} \otimes E_{\geq w}$, where the left side denotes the smallest $\mathcal{A}$-submodule containing the $\mathcal{O}_Z$-submodule $(\mathcal{A} \otimes E)_{\geq w}$.*

Proof. By definition the left hand side is the $\mathcal{A}$-submodule generated by $\bigoplus_{i+j \geq w} \mathcal{A}_i \otimes E_j$ and the left hand side is generated by $\bigoplus_{j \geq w} \mathcal{A}_0 \otimes E_j \subset \mathcal{A} \otimes E_{\geq w}$. These $\mathcal{O}_Z$-submodules clearly generate the same $\mathcal{A}$-submodule. $\square$

This guarantees that $\mathrm{im}\, d^i \subset \mathcal{A} \otimes E^{i+1}_{\geq w}$, so $\mathcal{A} \otimes E^\bullet_{\geq w}$ is a subcomplex, and $E_{\geq w}$ is a direct summand as a non-equivariant $\mathcal{O}_Z$-module, so we have a canonical short exact sequence of complexes in $\mathrm{QCoh}(\mathfrak{S})$

$$0 \to \mathcal{A} \otimes E^\bullet_{\geq w} \to \mathcal{A} \otimes E^\bullet \to \mathcal{A} \otimes E^\bullet_{<w} \to 0 \tag{2.6}$$

Proposition 2.1.8. *$F^\bullet \in D^b(\mathfrak{S})_{\geq w}$ iff it is quasi-isomorphic to a right-bounded complex of the sheaves of the form $\mathcal{A} \otimes E^i$ with $E^i \in \mathrm{Coh}(\mathfrak{Z}')_{\geq w}$ locally free.*

First we observe the following extension of Nakayama's lemma to the derived category

Lemma 2.1.9 (Nakayama). *Let $F^\bullet \in D^-(\mathfrak{S})$ with coherent cohomology. If $L\sigma^* F^\bullet \simeq 0$, then $F^\bullet \simeq 0$.*

Proof. The natural extension of Nakayama's lemma to stacks is the statement that the support of a coherent sheaf is closed. In our setting this means that if $G \in \mathrm{Coh}(\mathfrak{S})$ and $G \otimes \mathcal{O}_Z = 0$ then $G = 0$, because $\mathrm{supp}(G) \cap Z = \emptyset$ and every nonempty closed substack of S intersects Z nontrivially.

If $H^r(F^\bullet)$ is the highest nonvanishing cohomology group of a right bounded complex, then $H^r(L\sigma^* F^\bullet) \simeq \sigma^* H^r(F^\bullet)$. By Nakayama's lemma $\sigma^* H^r(F^\bullet) = 0 \Rightarrow H^r(F^\bullet) = 0$, so we must have $\sigma^* H^r(F^\bullet) \neq 0$ as well. $\square$

Remark 2.1.10. Note another consequence of Nakayama's lemma: if $F^\bullet$ is a complex of locally free sheaves on $\mathfrak{S}$ and $\mathcal{H}^i(F^\bullet \otimes \mathcal{O}_Z) = 0$, then $\mathcal{H}^i(F^\bullet) = 0$, because the canonical map on stalks $\mathcal{H}^i(F^\bullet) \otimes k(z) \to \mathcal{H}^i(F^\bullet \otimes k(z))$ is an isomorphism if it surjective. In particular if $E^i \in \mathrm{Coh}(\mathfrak{Z}')$ are locally free and $\sigma^*(\mathcal{A} \otimes E^\bullet) = E^\bullet$ has bounded cohomology, then $\mathcal{A} \otimes E^\bullet$ has bounded cohomology as well.

Proof of Proposition 2.1.8. We assume that $L\sigma^* F^\bullet \in \mathrm{D}^b(\mathfrak{Z})_{\geq w}$. Choose a right bounded presentation by locally frees $\mathcal{A} \otimes E^\bullet \simeq F^\bullet$ and consider the canonical sequence (2.6).

Restricting to $\mathfrak{Z}$ gives a short exact sequence $0 \to E_{\geq w}^\bullet \to E^\bullet \to E_{<w}^\bullet \to 0$. The first and second terms have homology in $\mathrm{Coh}(\mathfrak{Z})_{\geq w}$, and the third has homology in $\mathrm{Coh}(\mathfrak{Z})_{<w}$. These two categories are orthogonal, so it follows from the long exact homology sequence that $E_{<w}^\bullet$ is acyclic. Thus by Nakayama's lemma $\mathcal{A} \otimes E_{<w}^\bullet$ is acyclic and $F^\bullet \simeq \mathcal{A} \otimes E_{\geq w}^\bullet$. $\qquad\square$

Using this characterization of $\mathrm{D}^b(\mathfrak{S})_{\geq w}$ we have semiorthogonality

Lemma 2.1.11. *If $F^\bullet \in \mathrm{D}^-(\mathfrak{S})_{\geq w}$ and $G^\bullet \in \mathrm{D}^+(\mathfrak{S})_{<w}$, then $R\mathrm{Hom}_{\mathfrak{S}}(F^\bullet, G^\bullet) = 0$.*

Proof. By Proposition 2.1.8 if suffices to prove the claim for $F^\bullet = \mathcal{A} \otimes E$ with $E \in \mathrm{Coh}(\mathfrak{Z}')_{\geq w}$ locally free. Then $\mathcal{A} \otimes E \simeq L\pi^* E$, and the derived adjunction gives $R\mathrm{Hom}_{\mathfrak{S}}(L\pi^* E, F^\bullet) \simeq R\mathrm{Hom}_{\mathfrak{Z}'}(E, R\pi_* F^\bullet)$. π is affine, so $R\pi_* F^\bullet \simeq \pi_* F^\bullet \in \mathrm{D}^+(\mathfrak{Z}')_{<w}$. The claim follows from the fact that $\mathrm{QCoh}(\mathfrak{Z}')_{\geq w}$ is left orthogonal to $\mathrm{D}^+(\mathfrak{Z}')_{<w}$. $\qquad\square$

Remark 2.1.12. The category of coherent $\mathfrak{S}$ modules whose weights are $< w$ is a Serre subcategory of $\mathrm{Coh}(\mathfrak{S})$ generating $\mathrm{D}^b(\mathfrak{S})_{<w}$, but there is no analogue for $\mathrm{D}^b(\mathfrak{S})_{\geq w}$. Consider for instance, when G is abelian there is a short exact sequence $0 \to \mathcal{A}_{<0} \to \mathcal{A} \to \mathcal{O}_Z \to 0$. This nontrivial extension shows that $R\mathrm{Hom}\,\mathfrak{S}(\mathcal{O}_Z, \mathcal{A}_{<0}) \neq 0$ even though $\mathcal{O}_Z$ has nonnegative weights.

Every $F \in \mathrm{Coh}(\mathfrak{S})$ has a highest weight submodule as an equivariant $\mathcal{O}_Z$-module $F_{\geq h} \neq 0$ where $F_{\geq w} = 0$ for $w > h$. Furthermore, because $\mathcal{A}_{<0}$ has strictly negative weights the map $(F)_{\geq h} \to (F \otimes \mathcal{O}_Z)_{\geq h}$ is an isomorphism of L-equivariant $\mathcal{O}_Z$-modules. Using the notion of highest weight submodule we prove

Proposition 2.1.13. *If $\mathcal{A} \otimes E^\bullet$ is a right-bounded complex with bounded cohomology, then $E_{\geq w}^\bullet := (\sigma^*(\mathcal{A} \otimes E^\bullet))_{\geq w}$ has bounded cohomology and thus so does $\mathcal{A} \otimes E_{\geq w}^\bullet$ by Remark 2.1.10. If $\mathcal{A} \otimes E^\bullet$ is perfect, then so are $E_{\geq w}^\bullet$ and $\mathcal{A} \otimes E_{\geq w}^\bullet$.*

Proof. We define the subquotient $\mathcal{A} \otimes E_{[a,b)}^\bullet = \mathcal{A} \otimes (E_{\geq a}^\bullet)_{<b}$ for any $\infty \leq a < b \leq \infty$, noting that the functors commute so order doesn't matter. The generalization of the short exact sequence (2.6) for $a < b < c$ is

$$0 \to \mathcal{A} \otimes E_{[b,c)}^\bullet \to \mathcal{A} \otimes E_{[a,c)}^\bullet \to \mathcal{A} \otimes E_{[a,b)}^\bullet \to 0 \tag{2.7}$$

We will use this sequence to prove the claim by descending induction.

First we show that for W sufficiently high, $\mathcal{A} \otimes E^{\bullet}_{\geq W} \simeq 0$. By Nakayama's lemma and the fact that $\mathcal{A} \otimes E^{\bullet}$ has bounded cohomology, it suffices to show $(L\sigma^* F)_{\geq W} \simeq 0$ for any $F \in \mathrm{Coh}(\mathfrak{S})$, and this follows by constructing a resolution of F by vector bundles whose weights are $\leq$ the highest weight of F.

Now assume that the claim is true for $\mathcal{A} \otimes E^{\bullet}_{\geq w+1}$. It follows from the sequence (2.6) that $\mathcal{A} \otimes E^{\bullet}_{<w+1}$ has bounded cohomology. The complex $E^{\bullet}_{[w,w+1)}$ is precisely the highest weight space of $\mathcal{A} \otimes E^{\bullet}_{<w+1}$, and thus has bounded cohomology as well. Applying σ^* to sequence (2.7) gives $0 \to E^{\bullet}_{\geq w+1} \to E^{\bullet}_{\geq w} \to E^{\bullet}_{[w,w+1)} \to 0$, thus $E^{\bullet}_{\geq w}$ has bounded cohomology and the result follows by induction.

The argument for perfect complexes similar to the previous paragraph. By induction $\mathcal{A} \otimes E^{\bullet}_{<w+1}$ is perfect, thus so is $\sigma^*(\mathcal{A} \otimes E^{\bullet}_{<w+1})$ and its highest weight space $E^{\bullet}_{[w,w+1)}$. Because $E^{\bullet}_{[w,w+1)}$ is concentrated in a single weight, the differential on $\mathcal{A} \otimes E^{\bullet}_{[w,w+1)}$ is induced from the differential on $E^{\bullet}_{[w,w+1)}$, i.e. $\mathcal{A} \otimes E^{\bullet}_{[w,w+1)} = L\pi^*(E^{\bullet}_{[w,w+1)})$. It follows that $\mathcal{A} \otimes E^{\bullet}_{[w,w+1)}$ is perfect, and thus so is $\mathcal{A} \otimes E^{\bullet}_{\geq w}$ by the exact sequence (2.7). $\qquad\square$

Proposition 2.1.14. *The categories* $\mathrm{D}^b(\mathfrak{S}) = \langle \mathrm{D}^b(\mathfrak{S})_{<w}, \mathrm{D}^b(\mathfrak{S})_{\geq w} \rangle$ *constitute a multiplicative baric decomposition. This restricts to a multiplicative baric decomposition of* $\mathrm{Perf}(\mathfrak{S})$, *which is bounded. If* $Z \hookrightarrow Y$ *has finite Tor dimension, for instance if Property (A) holds, then the baric decomposition on* $\mathrm{D}^b(\mathfrak{S})$ *is bounded as well.*

Proof. Lemma 2.1.11 implies $\mathrm{D}^b(\mathfrak{S})_{\geq w}$ is left orthogonal to $G^{\bullet} \in \mathrm{D}^b(\mathfrak{S})_{<w}$. In order to obtain left and right truncations for $F^{\bullet} \in \mathrm{D}^b(\mathfrak{S})$ we choose a presentation of the form $\mathcal{A} \otimes E^{\bullet}$ with $E^{\bullet} \in \mathrm{Coh}(\mathfrak{Z})$ locally free. The canonical short exact sequence (2.6) gives an exact triangle $\mathcal{A} \otimes E^{\bullet}_{\geq w} \to F^{\bullet} \to \mathcal{A} \otimes E^{\bullet}_{<w} \dashrightarrow$. By Proposition 2.1.13 all three terms have bounded cohomology, thus our truncations are $\beta_{\geq w} F^{\bullet} = \mathcal{A} \otimes E^{\bullet}_{\geq w}$ and $\beta_{<w} F^{\bullet} = \mathcal{A} \otimes E^{\bullet}_{\geq w}$. If $F^{\bullet} \in \mathrm{Perf}(\mathfrak{S})$, then by Proposition 2.1.13 so are $\beta_{\geq w} F^{\bullet}$ and $\beta_{<w} F^{\bullet}$.

The multiplicativity of $\mathrm{D}^b(\mathfrak{S})_{\geq w}$ follows from the fact that $\mathrm{D}^b(\mathfrak{Z})_{\geq w}$ is multiplicative and the fact that $L\sigma^*$ respects derived tensor products.

Every $M \in \mathrm{Coh}(\mathfrak{S})$ has a highest weight space, so $M \in \mathrm{D}^b(\mathfrak{S})_{<w}$ for some w. This implies that any $F^{\bullet} \in \mathrm{D}^b(\mathfrak{S})$ lies in $\mathrm{D}^b(\mathfrak{S})_{<w}$ for some w. The analogous statement for $\mathrm{D}^b(\mathfrak{S})_{\geq w}$ is false in general, but if $F^{\bullet} \in \mathrm{D}^b(\mathfrak{S})$ is such that $\sigma^* F^{\bullet}$ is cohomologically bounded, then $F^{\bullet} \in \mathrm{D}^b(\mathfrak{S})_{\geq w}$ for some w. The boundedness properties follow from this observation. $\qquad\square$

Amplification 2.1.15. *If Property (A) holds, then* $\beta_{\geq w} F^{\bullet}$ *and* $\beta_{<w} F^{\bullet}$ *can be computed from a presentation* $F^{\bullet} \simeq \mathcal{A} \otimes E^{\bullet}$ *with* $E^i \in \mathrm{Coh}(\mathfrak{Z}')$ *coherent but not necessarily locally free. Furthermore* $L\pi^* = \pi^* : \mathrm{D}^b(\mathfrak{Z}')_w \to \mathrm{D}^b(\mathfrak{S})_w$ *is an equivalence, where* $\mathrm{D}^b(\mathfrak{S})_w := \mathrm{D}^b(\mathfrak{S})_{\geq w} \cap \mathrm{D}^b(\mathfrak{S})_{<w+1}$ *and likewise for* $\mathrm{D}^b(\mathfrak{Z})_w$.

Proof. If $\pi : Y \to Z$ is flat and $E \in \mathrm{Coh}(\mathfrak{Z}')$, then $\mathcal{A} \otimes E \in \mathrm{D}^b(\mathfrak{S})_{\geq w}$ iff $E \in \mathrm{Coh}(\mathfrak{Z}')_{\geq w}$ and likewise for $< w$. Thus

$$\mathcal{A} \otimes E^{\bullet}_{\geq w} \to F^{\bullet} \to \mathcal{A} \otimes E^{\bullet}_{<w} \dashrightarrow$$

is the exact triangle defining the baric truncations of $F^\bullet$. In fact for any coherent $\mathcal{A}$-module M there is a coherent $E \in \mathrm{Coh}(\mathfrak{Z}')$ and a surjection $\mathcal{A} \otimes E \twoheadrightarrow M$ which is an isomorphism on highest weight subsheaves, and one can use this fact to construct a presentation of this form in which $E^i_{\geq w} = 0$ for $i \ll 0$. So in fact $\beta_{\geq w} F^\bullet$ is equivalent to a *finite* complex of the form $\mathcal{A} \otimes E^\bullet_{\geq w}$.

For the second claim, fully faithfulness of $L\pi^*$ follows formally from the fact that π admits the section $\sigma : \mathfrak{Z} = Z/L \hookrightarrow \mathfrak{S}$. Essential surjectivity follows from the first part of the proposition. $\qquad\square$

Our final observation is that the components of the baric decomposition of $\mathrm{D}^b(\mathfrak{S})$ and $\mathrm{D}^b(\mathfrak{Z}')$ can be characterized pointwise over Z. We let $\mathbb{C}^*$ act on Y via λ and consider the flat morphism of stacks $Y/\mathbb{C}^* \to Y/P$. This gives a pullback (forgetful) functor $\mathrm{D}(\mathfrak{S}) \to \mathrm{D}(Y/\mathbb{C}^*)$. Given a point $p : * \hookrightarrow Z$, one can compose this forgetful functor with the pullback and shriek-pullback to get functors $p^* : \mathrm{D}^-(\mathfrak{S}) \to \mathrm{D}^-(*/\mathbb{C}^*)$ and $p^! : \mathrm{D}^+(\mathfrak{S}) \to \mathrm{D}^+(*/\mathbb{C}^*)$. By abuse of notation we denote the analogous functors $p^* : \mathrm{D}^-(\mathfrak{Z}') \to \mathrm{D}^-(*/\mathbb{C}^*)$ and $p^! : \mathrm{D}^+(\mathfrak{Z}) \to \mathrm{D}^+(*/\mathbb{C}^*)$, so that $p^! = p^!\sigma^!$ and $p^* = p^*\sigma^*$.

Lemma 2.1.16. *A complex $F^\bullet \in \mathrm{D}^-(\mathfrak{Z}')$ lies in $\mathrm{D}^-(\mathfrak{Z}')_{\geq, <w}$ iff $p^*F^\bullet \in \mathrm{D}^-(*/\mathbb{C}^*)_{\geq, <w}$ for all $p : * \hookrightarrow Z$. Dually, a complex $F^\bullet \in \mathrm{D}^+(\mathfrak{Z}')$ lies in $\mathrm{D}^+(\mathfrak{Z}')_{\geq, <w}$ iff $p^!F^\bullet \in \mathrm{D}^+(*/\mathbb{C}^*)_{\geq, <w}$ for all p.*

Proof. It suffices to work over $Z/\mathbb{C}^*$. Because every quasicoherent sheaf functorially splits into λ eigensheaves, $p^*(F^\bullet)_{\geq, <w} = (p^*F^\bullet)_{\geq, <w}$ and $p^!(F^\bullet)_{\geq, <w} = (p^!F^\bullet)_{\geq, <w}$. The claim for $F^\bullet \in \mathrm{D}^-(\mathfrak{Z}')$ now follows by applying derived Nakayama's Lemma to $F^\bullet_{<, \geq w}$. Likewise the claim for $F^\bullet \in \mathrm{D}^+(\mathfrak{Z}')$ follows from the Serre dual statement of derived Nakayama's Lemma, namely that $F^\bullet \in \mathrm{D}^+(Z)$ is acyclic iff $p^!F^\bullet = \mathrm{Hom}(p_*\mathbb{C}, F^\bullet)$ is acyclic for all p (note that we only need Nayama's Lemma in the non-equivariant setting). $\qquad\square$

Corollary 2.1.17. *The subcategories $\mathrm{D}^-(\mathfrak{S})_{\geq w}, \mathrm{D}(\mathfrak{S})_{<w} \subset \mathrm{D}(\mathfrak{S})$ are characterized by their images in $\mathrm{D}(Y/\mathbb{C}^*)$. If we consider all points $p : * \hookrightarrow Z$.*

- *$F^\bullet \in \mathrm{D}^-(\mathfrak{S})$ lies in $\mathrm{D}^-(\mathfrak{S})_{\geq w}$ iff $p^*F^\bullet \in \mathrm{D}^-(*/\mathbb{C}^*)_{\geq w}$, $\forall p$*

- *$F^\bullet \in \mathrm{D}^b(\mathfrak{S})$ lies in $F^\bullet \in \mathrm{D}^b(\mathfrak{S})_{<w}$ iff $p^*F^\bullet \in \mathrm{D}^-(*/\mathbb{C}^*)_{<w}$, $\forall p$*

Dually, if $\pi : Y \to Z$ is a bundle of affine spaces with determinant weight a, then $\mathrm{D}^b(\mathfrak{S})_{\geq, <w}$ are characterized by the weights of $\sigma^!F^\bullet$. We have

- *$F^\bullet \in \mathrm{D}^+(\mathfrak{S})$ lies in $\mathrm{D}^+(\mathfrak{S})_{<w}$ iff $p^!F^\bullet \in \mathrm{D}^+(*/\mathbb{C}^*)_{<w+a}$, $\forall p$*

- *$F^\bullet \in \mathrm{D}^b(\mathfrak{S})$ lies in $\mathrm{D}^b(\mathfrak{S})_{\geq w}$ iff $p^!F^\bullet \in \mathrm{D}^+(*/\mathbb{C}^*)_{\geq w+a}$, $\forall p$*

Proof. The first claim is immediate from the definitions of $\mathrm{D}^-(\mathfrak{S})_{\geq w}$ and $\mathrm{D}(\mathfrak{S})_{<w}$, so for the remainder of the proof we can work in the category $\mathrm{D}(Y/\mathbb{C}^*)$. From Proposition 2.1.14 and the discussion preceding it, we know that $\mathrm{D}^-(\mathfrak{S})_{\geq w}$ and $\mathrm{D}^b(\mathfrak{S})_{<w}$ are characterized by $\sigma^*F^\bullet \in \mathrm{D}^-(Z/\mathbb{C}^*)$. The claim now follows from Lemma 2.1.16.

Now assume that $\pi : Y \rightarrow Z$ is a bundle of affine spaces. Locally over Z, $\sigma^! F^\bullet \simeq \sigma^* F^\bullet(-a)[d]$, where d is the fiber dimension of π and a is the weight of λ on $\det(\mathfrak{N}_{Z/Y})^\vee$, so the weights of $\sigma^! F^\bullet$ are simply shifts of the weights of $\sigma^* F^\bullet$. If $F^\bullet \in \mathrm{D}^b(\mathfrak{S})$ the claim again follows from Lemma 2.1.16. By definition an unbounded object $F^\bullet \in \mathrm{D}^+(\mathfrak{S})$ lies in $\mathrm{D}^+(\mathfrak{S})_{<w}$ iff $\mathcal{H}^i(F^\bullet) \in \mathrm{QCoh}(\mathfrak{S})_{<w}$ for all i, i.e. iff $\tau^{\leq n} F^\bullet \in \mathrm{D}^b(\mathfrak{S})_{<w}$ for all n. One can prove the claim for $\mathrm{D}^+(\mathfrak{S})_{<w}$ by writing $F^\bullet = \varinjlim \tau^{\leq n} F^\bullet$ and that each homology sheaf of $\sigma^! F^\bullet = \varinjlim \sigma^! \tau^{\leq n} F^\bullet$ stabilizes after some finite n.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

The cotangent complex and Property (L+)

We review the construction of the cotangent complex and prove the main implication of the positivity Property (L+):

Lemma 2.1.18. *If $\mathfrak{S} \hookrightarrow \mathfrak{X}$ satisfies Property (L+) and $F^\bullet \in \mathrm{D}^b(\mathfrak{S})_{\geq w}$, then $Lj^* j_* F^\bullet \in \mathrm{D}^-(\mathfrak{S})_{\geq w}$ as well.*

We can inductively construct a cofibrant replacement $\mathcal{O}_\mathfrak{S}$ as an $\mathcal{O}_\mathfrak{X}$ module: a surjective weak equivalence $\mathcal{B}^\bullet \twoheadrightarrow \mathcal{O}_\mathfrak{S}$ from a sheaf of dg-$\mathcal{O}_\mathfrak{X}$ algebras with $\mathcal{B}^\bullet \simeq (\mathbb{S}(E^\bullet), d)$, where $\mathbb{S}(E^\bullet)$ is the free graded commutative sheaf of algebras on the graded sheaf of $\mathcal{O}_\mathfrak{X}$-modules $E^\bullet$ with E^i locally free and $E^i = 0$ for $i \geq 0$. Note that the differential is uniquely determined by its restriction to $E^\bullet$, and letting e be a local section of $E^\bullet$ we decompose $d(e) = d_{-1}(e) + d_0(e) + \cdots$ where $d_i(e) \in \mathbb{S}^{i+1}(E^\bullet)$.

The $\mathcal{B}^\bullet$-module of Kähler differentials is

$$\mathcal{B}^\bullet \xrightarrow{\delta} \Omega^1_{\mathcal{B}^\bullet/\mathcal{O}_\mathfrak{X}} = \mathbb{S}(E^\bullet) \otimes_{\mathcal{O}_\mathfrak{X}} E^\bullet$$

with the universal closed degree 0 derivation defined by $\delta(e) = 1 \otimes e$. The differential on $\Omega^1_{\mathcal{B}^\bullet/\mathcal{O}_\mathfrak{X}}$ is uniquely determined by its commutation with δ

$$d(1 \otimes e) = \delta(de) = 1 \otimes d_0(e) + \delta(d_2(e) + d_3(e) + \cdots)$$

By definition

$$\mathbf{L}^\bullet(\mathfrak{S} \hookrightarrow \mathfrak{X}) := \mathcal{O}_\mathfrak{S} \otimes_{\mathcal{B}^\bullet} \Omega^1_{\mathcal{B}^\bullet/\mathcal{O}_\mathfrak{X}} \simeq \mathcal{O}_\mathfrak{S} \otimes E^\bullet$$

where the differential is the restriction of d_0.

Proof of Lemma 2.1.18. First we prove the claim for $\mathcal{O}_\mathfrak{S}$. Note that $\mathcal{B}^\bullet \twoheadrightarrow \mathcal{O}_\mathfrak{S}$, in addition to a cofibrant replacement of dg-$\mathcal{O}_\mathfrak{X}$-algebras, is a left bounded resolution of $\mathcal{O}_\mathfrak{S}$ by locally frees. Thus $Lj^* j_* \mathcal{O}_\mathfrak{S} = \mathcal{O}_\mathfrak{S} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{B}^\bullet = \mathbb{S}_\mathfrak{S}(E^\bullet|_\mathfrak{S})$ with differential $d(e) = d_0(e) + d_1(e) + \cdots$. The term d_{-1} in the differential vanishes when restricting to $\mathfrak{S}$. Restricting further to $\mathfrak{Z}$, we have a deformation of complexes of $\mathcal{O}_\mathfrak{Z}$ modules over $\mathbb{A}^1$

$$F_t^\bullet := (\mathbb{S}(E^\bullet)|_\mathfrak{Z}, d_0 + t d_1 + t^2 d_2 + \cdots)$$

which is trivial over $\mathbb{A}^1 - \{0\}$. The claim of the lemma is that $(F_1^\bullet)_{<0} \sim 0$. Setting $t = 0$, the differential becomes the differential of the cotangent complex, so $F_0^\bullet = \mathbb{S}(\mathbf{L}_{\mathfrak{S}/\mathfrak{X}}^\bullet)|_{\mathfrak{Z}}$. By hypothesis $\mathbf{L}_{\mathfrak{S}/\mathfrak{X}}^\bullet \to (\mathbf{L}_{\mathfrak{S}/\mathfrak{X}}^\bullet)_{\geq 0}$ is a weak equivalence, so $\mathbb{S}_{\mathfrak{Z}}(\mathbf{L}_{\mathfrak{S}/\mathfrak{X}}^\bullet|_{\mathfrak{Z}}) \to \mathbb{S}_{\mathfrak{Z}}((\mathbf{L}_{\mathfrak{S}/\mathfrak{X}}^\bullet|_{\mathfrak{Z}})_{\geq 0})$ is a weak equivalence with a complex of locally frees generated in nonnegative weights. Thus $(F_0^\bullet|_{\mathfrak{Z}})_{<0} \sim 0$. By semicontinuity it follows that $(F_t^\bullet)_{<0} = 0$ for all $t \in \mathbb{A}^1$, and the lemma follows for $\mathcal{O}_{\mathfrak{S}}$.

Now we consider arbitrary $F^\bullet \in \mathrm{D}^b(\mathfrak{S})$. Let $\widetilde{\mathcal{O}}_{\mathfrak{S}} := \mathbb{S}(E^\bullet)|_{\mathfrak{S}} = Lj^* j_* \mathcal{O}_{\mathfrak{S}}$ denote the derived self intersection. $\mathcal{O}_{\mathfrak{S}}$ is a summand of $\widetilde{\mathcal{O}}_{\mathfrak{S}}$ as an $\mathcal{O}_{\mathfrak{S}}$ module, and we have already established that $\widetilde{\mathcal{O}}_{\mathfrak{S}} \in \mathrm{D}^b(\mathfrak{S})_{\geq 0}$, so $E^\bullet \in \mathrm{D}^b(\mathfrak{S})_{\geq w}$ iff $\widetilde{\mathcal{O}}_{\mathfrak{S}} \otimes E^\bullet \in \mathrm{D}^b(\mathfrak{S})_{\geq w}$. The proof of the lemma follows from this and the projection formula

$$Lj^* j_* (\widetilde{\mathcal{O}}_{\mathfrak{S}} \otimes F^\bullet) = Lj^* (j_* \mathcal{O}_{\mathfrak{S}} \otimes^L j_* F^\bullet) = \widetilde{\mathcal{O}}_{\mathfrak{S}} \otimes^L Lj^* j_* F^\bullet$$

$\square$

Koszul systems and cohomology with supports

We recall some properties of the right derived functor of the subsheaf with supports functor $R\underline{\Gamma}_{\mathfrak{S}}(\bullet)$. It can be defined by the exact triangle $R\underline{\Gamma}_{\mathfrak{S}}(F^\bullet) \to F^\bullet \to i_*(F^\bullet|_{\mathfrak{V}}) \dashrightarrow$, and it is the right adjoint of the inclusion $\mathrm{D}_{\mathfrak{S}}(\mathfrak{X}) \subset \mathrm{D}(\mathfrak{X})$. It is evident from this exact triangle that if $F^\bullet \in \mathrm{D}^b(\mathfrak{X})$, then $R\underline{\Gamma}_{\mathfrak{S}}(F^\bullet)$ is still bounded, but no longer has coherent cohomology. On the other hand the formula

$$R\underline{\Gamma}_{\mathfrak{S}}(F^\bullet) = \varinjlim \underline{\mathrm{Hom}}(\mathcal{O}_{\mathfrak{X}}/\mathcal{I}_{\mathfrak{S}}^i, F^\bullet)$$

shows that the subsheaf with supports is a limit of coherent complexes.

We will use a more general method of computing the subsheaf with supports similar to the Koszul complexes which can be used in the affine case.

Lemma 2.1.19. *Let $\mathfrak{X} = X/G$ with X quasiprojective and G reductive, and let $S \subset X$ be a KN stratum. Then there is a direct system $K_0^\bullet \to K_1^\bullet \to \cdots$ in $\mathrm{Perf}(\mathfrak{X})^{[0,N]}$ along with compatible maps $K_i^\bullet \to \mathcal{O}_{\mathfrak{X}}$ such that*

1. *$\mathcal{H}^*(K_i^\bullet)$ is supported on $\mathfrak{S}$*

2. *$\varinjlim(K_i^\bullet \otimes F^\bullet) \to F^\bullet$ induces an isomorphism $\varinjlim(K_i^\bullet \otimes F^\bullet) \simeq R\underline{\Gamma}_{\mathfrak{S}}(F^\bullet)$.*

3. *$\mathrm{Cone}(K_i^\bullet \to K_{i+1}^\bullet)|_{\mathfrak{Z}} \in \mathrm{D}^b(\mathfrak{Z})_{<w_i}$ where $w_i \to -\infty$ as $i \to \infty$.*

We will call such a direct system a Koszul system *for $\mathfrak{S} \subset \mathfrak{X}$*

Proof. First assume $\mathfrak{X}$ is smooth in a neighborhood of $\mathfrak{S}$. Then $\mathcal{O}_{\mathfrak{X}}/\mathcal{I}_{\mathfrak{S}}^i$ is perfect, so the above formula implies that the duals $K_i^\bullet = (\mathcal{O}_{\mathfrak{X}}/\mathcal{I}_{\mathfrak{S}}^i)^\vee$ satisfy properties (1) and (2) with $K_i^\bullet \to \mathcal{O}_{\mathfrak{X}}$ the dual of the map $\mathcal{O}_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}/\mathcal{I}_{\mathfrak{S}}^i$. We compute the mapping cone

$$\mathrm{Cone}(K_i^\bullet \to K_{i+1}^\bullet) = \left(\mathcal{I}_{\mathfrak{S}}^i/\mathcal{I}_{\mathfrak{S}}^{i+1}\right)^\vee = \left(j_*(S^i(\mathfrak{N}_{\mathfrak{S}/\mathfrak{X}}))\right)^\vee$$

Where the last equality uses the smoothness of $\mathfrak{X}$. Because Property (L+) is automatic for smooth $\mathfrak{X}$, it follows from Lemma 2.1.18 that $Lj^*j_*(S^i(\mathfrak{N}_{\mathfrak{S}/\mathfrak{X}})) \in D^b(\mathfrak{S})_{\geq n}$, hence $\mathrm{Cone}(K_i^\bullet \to K_{i+1}^\bullet)$ has weights $\leq -i$, and the third property follows.

If X is not smooth in a neighborhood of S, then by hypothesis we have a G-equivariant closed immersion $\phi : X \hookrightarrow X'$ and closed KN stratum $S' \subset X'$ such that S is a connected component of $S' \cap X$ and X' is smooth in a neighborhood of S'. Then we let $K_i^\bullet \in \mathrm{Perf}(\mathfrak{X})$ be the restriction of $L\phi^*(\mathcal{O}_{\mathfrak{X}'}/\mathcal{I}_{\mathfrak{S}'}^i)^\vee$. These $K_i^\bullet$ still satisfy the third property. Consider the canonical morphism $\varinjlim(K_i^\bullet \otimes F^\bullet) \to R\underline{\Gamma}_{\mathfrak{S}' \cap \mathfrak{X}} F^\bullet$. Its push forward $\varinjlim \phi_*(K_i^\bullet \otimes F^\bullet) \to \phi_* R\underline{\Gamma}_{\mathfrak{S}' \cap \mathfrak{X}}(F^\bullet) = R\underline{\Gamma}_{\mathfrak{S}'} \phi_* \overline{F^\bullet}$ is an isomorphism, hence the $K_i^\bullet$ form a Koszul system for $\mathfrak{S}' \cap \mathfrak{X}$. Because $\mathfrak{S}$ is a connected component of $\mathfrak{S}' \cap \mathfrak{X}$, the complexes $R\underline{\Gamma}_{\mathfrak{S}} K_i^\bullet$ form a Koszul system for $\mathfrak{S}$. $\qquad\square$

We note an alternative definition of a Koszul system, which will be useful below

Lemma 2.1.20. *Property (3) of a Koszul system is equivalent to the property that for all w,*
$$\mathrm{Cone}(K_i^\bullet \to \mathcal{O}_{\mathfrak{X}})|_{\mathfrak{Z}} \in D^b(\mathfrak{Z})_{<w} \text{ for all } i \gg 0$$

Proof. First, by the octahedral axiom we have an exact triangle
$$\mathrm{Cone}(K_i^\bullet \to \mathcal{O}_{\mathfrak{X}})[-1] \to \mathrm{Cone}(K_{i+1}^\bullet \to \mathcal{O}_{\mathfrak{X}})[-1] \to \mathrm{Cone}(K_i^\bullet \to K_{i+1}^\bullet) \dashrightarrow$$

So the property stated in this Lemma implies property (3) of the definition of a Koszul system.

Conversely, let $K_i^\bullet$ be a Koszul system for $\mathfrak{S} \subset \mathfrak{X}$. For any $F^\bullet \in D^+(\mathfrak{X})$, $j^! F^\bullet \simeq j^! \underline{\Gamma}_{\mathfrak{S}} F^\bullet$, so
$$\sigma^! F^\bullet \simeq \sigma^! \underline{\Gamma}_{\mathfrak{S}} F^\bullet \simeq \varinjlim \sigma^!(K_i^\bullet \otimes F^\bullet) \simeq \varinjlim K_i^\bullet|_{\mathfrak{Z}} \otimes \sigma^! F^\bullet$$

where we have used compactness of $\mathcal{O}_{\mathfrak{Z}}$ as an object of $D^+(\mathfrak{X})$ (which follows from the analogous statement for schemes proved in Section 6.3 of [38]) in order to commute $\sigma^!$ with the direct limit computing $\underline{\Gamma}_{\mathfrak{S}} F^\bullet$.

Now let $\omega^\bullet \in D^b(\mathfrak{X})$ be a dualizing complex, which by definition means that ω is a dualizing complex in $D^b(X)$ after forgetting the G action (see [3] for a discussion of dualizing complexes for stacks). Then $\sigma^! \omega^\bullet$ is a dualizing complex on $\mathfrak{Z}$, and its restriction to $Z/\mathbb{C}^*$ is again a dualizing complex. Any dualizing complex on $Z/\mathbb{C}^*$ must be concentrated in a single weight, so $\sigma^! \omega^\bullet \in D^b(\mathfrak{Z})_N$ for some weight $N \in \mathbb{Z}$.

Now the formula above says that $\sigma^! \omega^\bullet = \varinjlim(K_i^\bullet|_{\mathfrak{Z}} \otimes \sigma^! \omega^\bullet)$. If we assume that $\mathrm{Cone}(K_i^\bullet \to K_{i+1}^\bullet)|_{\mathfrak{Z}} \in D^b(\mathfrak{Z})_{<w_i}$ where $w_i \to -\infty$ as $i \to \infty$, then for any v the canonical map $(K_i^\bullet \otimes \sigma^! \omega^\bullet)_{\geq v} \to \varinjlim(K_i^\bullet \otimes \sigma^! \omega^\bullet)_{\geq v}$ is an isomorphism for $i \gg 0$. In particular for any fixed $v < N$ we have $\sigma^! \omega^\bullet = (\sigma^! \omega^\bullet)_{\geq v} \simeq (K_i^\bullet)_{\geq v-w} \otimes \sigma^! \omega^\bullet$ for all $i \gg 0$. Thus the map $(K_i^\bullet|_{\mathfrak{Z}})_{\geq v-N} \to \mathcal{O}_{\mathfrak{Z}} = (\mathcal{O}_{\mathfrak{Z}})_{\geq v-N}$ is an isomorphism for $i \gg 0$, hence $\mathrm{Cone}(K_i^\bullet \to \mathcal{O}_{\mathfrak{X}})|_{\mathfrak{Z}} \in D^b(\mathfrak{Z})_{<v-N}$ for $i \gg 0$. $\qquad\square$

Quasicoherent sheaves with support on $\mathfrak{S}$, and the quantization theorem

We turn to the derived category $D_{\mathfrak{S}}^b(\mathfrak{X})$ of coherent sheaves on $\mathfrak{X}$ with set-theoretic support on $\mathfrak{S}$. We will extend the baric decomposition of $D^b(\mathfrak{S})$ to a baric decomposition of $D_{\mathfrak{S}}^b(\mathfrak{X})$. Using this baric decomposition we will prove a generalization of the quantization commutes with reduction theorem, one of the results which motivated this work.

Proposition 2.1.21. *Let $\mathfrak{S} \subset \mathfrak{X}$ be a KN stratum satisfying Property (L+). There is a unique multiplicative baric decomposition $D_{\mathfrak{S}}^b(\mathfrak{X}) = \langle D_{\mathfrak{S}}^b(\mathfrak{X})_{<w}, D_{\mathfrak{S}}^b(\mathfrak{X})_{\geq w} \rangle$ such that*

$$j_*(D^b(\mathfrak{S})_{\geq w}) \subset D_{\mathfrak{S}}^b(\mathfrak{X})_{\geq w} \text{ and } j_*(D^b(\mathfrak{S})_{<w}) \subset D_{\mathfrak{S}}^b(\mathfrak{X})_{<w}$$

It is described explicitly by

$$D_{\mathfrak{S}}^b(\mathfrak{X})_{<w} = \{F^\bullet \in D_{\mathfrak{S}}^b(\mathfrak{X}) | Rj^! F^\bullet \in D^+(\mathfrak{S})_{<w}\}$$
$$D_{\mathfrak{S}}^b(\mathfrak{X})_{\geq w} = \{F^\bullet \in D_{\mathfrak{S}}^b(\mathfrak{X}) | Lj^* F^\bullet \in D^-(\mathfrak{S})_{\geq w}\}$$

When Property (A) holds, this baric decomposition is bounded.

Proof. Let $D_{\mathfrak{S}}^b(\mathfrak{X})_{\geq w}$ (respectively $D_{\mathfrak{S}}^b(\mathfrak{X})_{<w}$) be the triangulated subcategory generated by $j_*(D^b(\mathfrak{S})_{\geq w})$ (respectively $j_*(D^b(\mathfrak{S})_{<w})$). By Lemma 2.1.18, $Lj^* j_*(D^b(\mathfrak{S})_{\geq w}) \subset D^-(\mathfrak{S})_{\geq w}$, and so $D_{\mathfrak{S}}^b(\mathfrak{X})_{\geq w}$ is right orthogonal to $D_{\mathfrak{S}}^b(\mathfrak{X})_{<w}$ as a consequence of Lemma 2.1.11.

Next we must show $D_{\mathfrak{S}}^b(\mathfrak{X}) = D_{\mathfrak{S}}^b(\mathfrak{X})_{\geq w} \star D_{\mathfrak{S}}^b(\mathfrak{X})_{<w}$, where the $\mathcal{A} \star \mathcal{B}$ denotes the full subcategory consisting of F admitting triangles $A \to F \to B \dashrightarrow$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

If $\mathcal{A}$ and $\mathcal{B}$ are triangulated subcategories, and $\mathcal{B} \subset \mathcal{A}^\perp$, then the subcategory $\mathcal{A} \star \mathcal{B}$ is triangulated as well. Furthermore, for any $F \in D^b(\mathfrak{S})$ we have the exact triangle $j_* \beta_{\geq w} F \to j_* F \to j_* \beta_{<w} F \dashrightarrow$, so $D_{\mathfrak{S}}^b(\mathfrak{X})_{\geq w} \star D_{\mathfrak{S}}^b(\mathfrak{X})_{<w}$ is a triangulated subcategory containing $j_*(D^b(\mathfrak{S}))$, and so $D_{\mathfrak{S}}^b(X) = D_{\mathfrak{S}}^b(\mathfrak{X})_{\geq w} \star D_{\mathfrak{S}}^b(\mathfrak{X})_{<w}$ as desired.

Now that we have shown that $D_{\mathfrak{S}}^b(\mathfrak{X}) = \langle D_{\mathfrak{S}}^b(\mathfrak{X})_{<w}, D_{\mathfrak{S}}^b(\mathfrak{X})_{\geq w} \rangle$, we can characterize each $D_{\mathfrak{S}}^b(\mathfrak{X})_{\geq w, <w}$ as the orthogonal of the other. The adjunctions $Lj^* \dashv j_*$ and $j_* \dashv Rj^!$ give the a posteriori characterizations in the last statement of the proposition.

$\square$

Remark 2.1.22. Every coherent sheaf on $\mathfrak{S}$ has a highest weight space. Because coherent sheaves generate the bounded derived category, we have $D^b(\mathfrak{S}) = \bigcup_w D^b(\mathfrak{S})_{<w}$. Furthermore, $j_* D^b(\mathfrak{S})$ generates $D_{\mathfrak{S}}^b(\mathfrak{X})$, so we have $D_{\mathfrak{S}}^b(\mathfrak{X}) = \bigcup_w D_{\mathfrak{S}}^b(\mathfrak{X})_{<w}$ as well. The analogous statement for $D^b(\mathfrak{S})_{\geq w}$ is false. Note, however, that $\mathrm{Perf}(\mathfrak{X}) \subset \bigcup_w D_{\mathfrak{S}}^b(\mathfrak{X})_{\geq w}$.

The following is an extension to our setting of an observation which appeared in [6], following ideas of Kawamata [27]. There the authors described semiorthogonal factors appearing under VGIT in terms of the quotient Z/L'.

Amplification 2.1.23. *Define $D_{\mathfrak{S}}^b(\mathfrak{X})_w := D_{\mathfrak{S}}^b(\mathfrak{X})_{\geq w} \cap D_{\mathfrak{S}}^b(\mathfrak{X})_{<w+1}$. If the weights of $\mathbf{L}_{S/X}^\bullet$ are strictly positive, then $j_* : D^b(\mathfrak{S})_w \to D_{\mathfrak{S}}^b(\mathfrak{X})_w$ is an equivalence with inverse $\beta_{<w+1} Lj^*(F^\bullet)$.*

Corollary 2.1.24. *If* $\mathbf{L}^\bullet_{S/X}$ *has strictly positive weights, then the baric decomposition of Proposition 2.1.21 can be refined to an infinite semiorthogonal decomposition*

$$\mathrm{D}^b_{\mathfrak{S}}(\mathfrak{X}) = \langle \dots, \mathrm{D}^b(\mathfrak{Z})_w, \mathrm{D}^b(\mathfrak{Z})_{w+1}, \mathrm{D}^b(\mathfrak{Z})_{w+2}, \dots \rangle$$

where factors are the essential images of the fully faithful embeddings $j_*\pi^* : \mathrm{D}^b(\mathfrak{Z})_w \to \mathrm{D}^b_{\mathfrak{S}}(\mathfrak{X})$.

Finally we will use the baric decomposition of Proposition 2.1.21 to generalize a Theorem of Teleman [43], which was one of the motivations for this paper.

Definition 2.1.25. We define the thick triangulated subcategories of $\mathrm{D}^b(\mathfrak{X})$

$$\mathrm{D}^b(\mathfrak{X})_{\geq w} := \{F^\bullet \in \mathrm{D}^b(\mathfrak{X}) | Lj^*F^\bullet \in \mathrm{D}^-(\mathfrak{S})_{\geq w}\}$$
$$\mathrm{D}^b(\mathfrak{X})_{<w} := \{F^\bullet \in \mathrm{D}^b(\mathfrak{X}) | Rj^!F^\bullet \in \mathrm{D}^+(\mathfrak{S})_{<w}\}$$

Theorem 2.1.26 (Quantization Theorem)**.** *Let* $F^\bullet \in \mathrm{D}^b(\mathfrak{X})_{\geq w}$ *and* $G^\bullet \in \mathrm{D}^b(\mathfrak{X})_{<v}$ *with* $w \geq v$, *then the restriction map*

$$R\operatorname{Hom}_{\mathfrak{X}}(F^\bullet, G^\bullet) \to R\operatorname{Hom}_{\mathfrak{V}}(F^\bullet|_{\mathfrak{V}}, G^\bullet|_{\mathfrak{V}})$$

to the open substack $\mathfrak{V} = \mathfrak{X} \setminus \mathfrak{S}$ *is an isomorphism.*

Proof. This is equivalent to the vanishing of $R\Gamma_{\mathfrak{S}}(R\underline{\operatorname{Hom}}_{\mathfrak{X}}(F^\bullet, G^\bullet))$. By the formula

$$Rj^!\underline{\operatorname{Hom}}_{\mathfrak{X}}(F^\bullet, G^\bullet) \simeq \underline{\operatorname{Hom}}_{\mathfrak{S}}(Lj^*F^\bullet, Rj^!G^\bullet)$$

it suffices to prove the case where $F^\bullet = \mathcal{O}_X$, i.e. showing that $R\Gamma_{\mathfrak{S}}(G^\bullet) = 0$ whenever $Rj^!G^\bullet \in \mathrm{D}^+(\mathfrak{S})_{<0}$.

From Property (S3) we have a system $K_1 \to K_2 \to \cdots$ of perfect complexes in $\mathrm{D}^b_{\mathfrak{S}}(\mathfrak{X})_{\leq 0}$ such that $R\Gamma_{\mathfrak{S}}(G^\bullet) = \operatorname{colim} R\Gamma(K_i^\bullet \otimes G^\bullet)$ so it suffices to show the vanishing of each term in the limit. We have $j^!(K_i^\bullet \otimes G^\bullet) = j^*(K_i^\bullet) \otimes j^!G^\bullet$, so $K_i^\bullet \otimes G^\bullet \in \mathrm{D}^b_{\mathfrak{S}}(\mathfrak{X})_{<0}$. The category $\mathrm{D}^b_{\mathfrak{S}}(\mathfrak{X})_{<0}$ is generated by objects of the form j_*F with $F \in \mathrm{D}^b(\mathfrak{S})_{<0}$, and thus $R\Gamma(F^\bullet)$ for all $F^\bullet \in \mathrm{D}^b_{\mathfrak{S}}(\mathfrak{X})_{<0}$. $\square$

Semiorthogonal decomposition of $\mathrm{D}^b(\mathfrak{X})$

In this section we construct the semiorthogonal decomposition of $\mathrm{D}^b(\mathfrak{X})$ used to prove the derived Kirwan surjectivity theorem. When Y is a bundle of affine spaces over Z, we construct right adjoints for each of the inclusions $\mathrm{D}^b_{\mathfrak{S}}(\mathfrak{X})_{\geq w} \subset \mathrm{D}^b(\mathfrak{X})_{\geq w} \subset \mathrm{D}^b(\mathfrak{X})$.

We prove this in two steps. First we define a full subcategory $\mathrm{D}^b(\mathfrak{X})^{\mathrm{fin}} \subset \mathrm{D}^b(\mathfrak{X})$ of complexes whose weights along $\mathfrak{S}$ are bounded and construct a semiorthogonal decomposition of this category. Then we prove that $\mathrm{D}^b(\mathfrak{X})^{\mathrm{fin}} = \mathrm{D}^b(\mathfrak{X})$ when Y is a bundle of affine spaces over Z.

Definition 2.1.27. The categories of objects with *bounded weights* along $\mathfrak{S}$ are the full triangulated subcategories

$$\mathrm{D}^b(\mathfrak{X})^{\mathrm{fin}} := \bigcup_v \left(\mathrm{D}^b(\mathfrak{X})_{\geq v} \cap \mathrm{D}^b(\mathfrak{X})_{<v} \right)$$

$$\mathrm{D}^b(\mathfrak{X})^{\mathrm{fin}}_{\geq w} := \mathrm{D}^b(\mathfrak{X})_{\geq w} \cap \mathrm{D}^b(\mathfrak{X})^{\mathrm{fin}}, \qquad \mathrm{D}^b(\mathfrak{X})^{\mathrm{fin}}_{<w} := \mathrm{D}^b(\mathfrak{X})_{<w} \cap \mathrm{D}^b(\mathfrak{X})^{\mathrm{fin}}$$

By Remark 2.1.22, any $F^{\bullet} \in \mathrm{D}^b_{\mathfrak{S}}(\mathfrak{X})$ lies in $\mathrm{D}^b_{\mathfrak{S}}(\mathfrak{X})_{<w}$ for some w, so $\mathrm{D}^b_{\mathfrak{S}}(\mathfrak{X})_{\geq w} \subset \mathrm{D}^b(\mathfrak{X})^{\mathrm{fin}}$ for all w.

Proposition 2.1.28. *Let $F^{\bullet} \in \mathrm{D}^b(\mathfrak{X})^{fin}$ and let $K^{\bullet}_i$ be a Koszul system for $\mathfrak{S} \subset \mathfrak{X}$. Then for sufficiently large i the canonical map*

$$\beta_{\geq w}(K^{\bullet}_i \otimes F^{\bullet}) \to \beta_{\geq w}(K^{\bullet}_{i+1} \otimes F^{\bullet})$$

is an equivalence. The functor

$$\beta_{\geq w}\underline{\Gamma}_{\mathfrak{S}}(F^{\bullet}) := \varinjlim_i \beta_{\geq w}\left(K^{\bullet}_i \otimes F^{\bullet} \right) \tag{2.8}$$

is well-defined and is a right adjoint to the inclusions $\mathrm{D}^b_{\mathfrak{S}}(\mathfrak{X})_{\geq w} \subset \mathrm{D}^b(\mathfrak{X})^{fin}_{\geq w}$ and $\mathrm{D}^b_{\mathfrak{S}}(\mathfrak{X})_{\geq w} \subset \mathrm{D}^b(\mathfrak{X})^{fin}$.

Proof. By hypothesis the $C^{\bullet}_i := \mathrm{Cone}(K^{\bullet}_i \to K^{\bullet}_{i+1})$ is a perfect complex in $\mathrm{D}^b_{\mathfrak{S}}(\mathfrak{X})_{<w_i}$, where $w_i \to -\infty$ as $i \to \infty$. Because $F^{\bullet} \in \mathrm{D}^b(X)^{\mathrm{fin}}$, we have $F^{\bullet} \in \mathrm{D}^b(\mathfrak{X})_{<N}$ for some N, so if $w_i + N < w$ we have $C^{\bullet}_i \otimes F^{\bullet} \in \mathrm{D}^b_{\mathfrak{S}}(\mathfrak{X})_{<w}$ and

$$\mathrm{Cone}\left(\beta_{\geq w}(K^{\bullet}_i \otimes F^{\bullet}) \to \beta_{\geq w}(K^{\bullet}_{i+1} \otimes F^{\bullet}) \right) = \beta_{\geq w}(C^{\bullet}_i \otimes F^{\bullet}) = 0$$

Thus the direct system $\beta_{\geq w}(K^{\bullet}_i \otimes F^{\bullet})$ stabilizes, and the expression (2.8) defines a functor $\mathrm{D}^b(\mathfrak{X})^{\mathrm{fin}} \to \mathrm{D}^b_{\mathfrak{S}}(\mathfrak{X})_{\geq w}$.

The fact that $\beta_{\geq w}R\underline{\Gamma}$ is the right adjoint of the inclusion follows from the fact that elements of $\mathrm{D}^b_{\mathfrak{S}}(\mathfrak{X})$ are compact in $\mathrm{D}^+_{\mathfrak{S}}(\mathfrak{X})$ [38]. For $G^{\bullet} \in \mathrm{D}^b_{\mathfrak{S}}(\mathfrak{X})_{\geq w}$ we compute

$$R\mathrm{Hom}(G^{\bullet}, \beta_{\geq w}\underline{\Gamma}_{\mathfrak{S}}F^{\bullet}) = \varinjlim_i R\mathrm{Hom}(G^{\bullet}, K^{\bullet}_i \otimes F^{\bullet}) = R\mathrm{Hom}(G^{\bullet}, F^{\bullet})$$

$\square$

The right orthogonal to $\mathrm{D}^b_{\mathfrak{S}}(\mathfrak{X})_{\geq w}$ can be determined a posteriori from the fact that $\mathrm{D}^b_{\mathfrak{S}}(\mathfrak{X})_{\geq w}$ is generated by $j_* \mathrm{D}^b(\mathfrak{S})_{\geq w}$. Proposition 2.1.28 gives semiorthogonal decompositions

$$\mathrm{D}^b(\mathfrak{X})^{\mathrm{fin}}_{\geq w} = \langle \mathbf{G}_w, \mathrm{D}^b_{\mathfrak{S}}(\mathfrak{X})_{\geq w} \rangle \qquad \mathrm{D}^b(\mathfrak{X})^{\mathrm{fin}} = \langle \mathrm{D}^b(\mathfrak{X})^{\mathrm{fin}}_{<w}, \mathrm{D}^b_{\mathfrak{S}}(\mathfrak{X})_{\geq w} \rangle$$

where $\mathbf{G}_w := \mathrm{D}^b(\mathfrak{X})^{\mathrm{fin}}_{\geq w} \cap \mathrm{D}^b(\mathfrak{X})^{\mathrm{fin}}_{<w}$. What remains is to show that $\mathrm{D}^b(\mathfrak{X})^{\mathrm{fin}}_{\geq w} \subset \mathrm{D}^b(\mathfrak{X})^{\mathrm{fin}}$ is right admissible.

Proposition 2.1.29. *The inclusion of the subcategory* $\mathrm{D}^b(\mathfrak{X})^{fin}_{\geq w} \subset \mathrm{D}^b(\mathfrak{X})^{fin}$ *admits a right adjoint* $\beta_{\geq w}(\bullet)$ *defined by the exact triangle*

$$\beta_{\geq w}F^\bullet \to F^\bullet \to \beta_{<w}\left((K_i^\bullet)^\vee \otimes F^\bullet\right) \dashrightarrow \qquad for\ i \gg 0$$

Proof. First note that for $F^\bullet \in \mathrm{D}^b(\mathfrak{X})^{\mathrm{fin}}$ and for $i \gg 0$, $\mathrm{Cone}(K_i^\bullet \to K_{i+1}^\bullet)^\vee \otimes F^\bullet \in \mathrm{D}^b(\mathfrak{X})_{\geq w}$. It follows that the inverse system $(K_i^\bullet)^\vee \otimes F^\bullet$ stabilizes, as in Proposition 2.1.28.

Consider the composition $F^\bullet \to (K_i^\bullet)^\vee \otimes F^\bullet \to \beta_{<w}((K_i^\bullet)^\vee \otimes F^\bullet) \dashrightarrow$ where we define $\beta_{\geq w}F^\bullet$ as above. The octahedral axiom gives a triangle

$$\mathrm{Cone}(F^\bullet \to (K_i^\bullet)^\vee \otimes F^\bullet) \to \beta_{\geq w}F^\bullet[1] \to \beta_{\geq w}((K_i^\bullet)^\vee \otimes F^\bullet)$$

Thus for $i \gg 0$, $\beta_{\geq w}F^\bullet \in \mathrm{D}^b(\mathfrak{X})^{\mathrm{fin}}_{\geq w}$ and is right orthogonal to $\mathrm{D}^b_{\mathfrak{G}}(\mathfrak{X})_{<w}$. It follows that $\beta_{\geq w}F^\bullet$ is functorial in $F^\bullet$ and is right adjoint to the inclusion $\mathrm{D}^b(\mathfrak{X})^{\mathrm{fin}}_{\geq w} \subset \mathrm{D}^b(\mathfrak{X})^{\mathrm{fin}}$. $\square$

Combining Propositions 2.1.28 and 2.1.29, we have a semiorthogonal decomposition

$$\mathrm{D}^b(\mathfrak{X})^{\mathrm{fin}} = \langle \mathrm{D}^b_{\mathfrak{G}}(\mathfrak{X})^{\mathrm{fin}}_{<w}, \mathbf{G}_w, \mathrm{D}^b_{\mathfrak{G}}(\mathfrak{X})_{\geq w} \rangle \tag{2.9}$$

where the restriction functor $\mathbf{G}_w \to \mathrm{D}^b(\mathfrak{V})$ is fully faithful by theorem 2.1.26. Recall that our goal is to use the semiorthogonal decomposition (2.9) as follows: any $F^\bullet \in \mathrm{D}^b(\mathfrak{V})$ extends to $\mathrm{D}^b(\mathfrak{X})$, then using (2.9) one can find an element of $\mathbf{G}_w$ restricting to $F^\bullet$, hence $i^* : \mathbf{G}_w \to \mathrm{D}^b(\mathfrak{V})$ is an equivalence of categories. Unfortunately, in order for this argument to work, we need $\mathrm{D}^b(\mathfrak{X}) = \mathrm{D}^b(\mathfrak{X})^{\mathrm{fin}}$. In the rest of this section we show that when $\pi : Y \to Z$ is a bundle of affine spaces, $\mathrm{D}^b(\mathfrak{X})^{\mathrm{fin}} = \mathrm{D}^b(\mathfrak{X})$.

Lemma 2.1.30. *Suppose that* $\pi : Y \to Z$ *is a bundle of affine spaces, then* $\mathrm{D}^b(\mathfrak{X})^{fin} = \mathrm{D}^b(\mathfrak{X})$.

Proof. First we show that any $F^\bullet \in \mathrm{D}^b(\mathfrak{X})_{\geq w}$ for some w. Let $P^\bullet$ be a perfect complex in $\mathrm{D}^b_{\mathfrak{G}}(\mathfrak{X})$ whose support contains $\mathfrak{Z}$ – for instance any object in the Koszul system constructed in Lemma 2.1.19 will suffice. We know that $P^\bullet \otimes F^\bullet \in \mathrm{D}^b_{\mathfrak{G}}(\mathfrak{X})_{\geq a}$ for some a by Remark 2.1.22, so $\sigma^*(F^\bullet \otimes P^\bullet) = \sigma^* P^\bullet \otimes \sigma^* F^\bullet \in \mathrm{D}^-(\mathfrak{Z})_{\geq a}$.

Because $P^\bullet$ is perfect, $\sigma^* P^\bullet \in \mathrm{D}^b(\mathfrak{Z})_{<q}$ for some q. It suffices to forget the action of L on Z and work in the derived category of $Z/\mathbb{C}^*$. Let $p : * \hookrightarrow Z$ be a point, then $p^*(\sigma^* P^\bullet \otimes \sigma^* F^\bullet) = p^* P^\bullet \otimes_k p^* F^\bullet$ has weight $\geq a$. However $p^* P^\bullet$ is non-zero by hypothesis and is equivalent in $\mathrm{D}^b(*/\mathbb{C}^*)$ to a direct sum of shifts $k(w)[d]$ with $w > -q$, so this implies that $p^* F^\bullet \in \mathrm{D}^b(*/\mathbb{C}^*)_{\geq a-q}$. This holds for every point in Z, so by Lemma 2.1.16, $\sigma^* F^\bullet \in \mathrm{D}^-(\mathfrak{Z})_{\geq a-q}$. Thus $F^\bullet \in \mathrm{D}^b(\mathfrak{X})_{\geq a-q}$.

By Lemma 2.1.16 we have that $F^\bullet \in \mathrm{D}^b(\mathfrak{X})_{<w}$ iff $\sigma^! F^\bullet \in \mathrm{D}^+(\mathfrak{Z})_{<w+a}$. By the same argument above, we can assume $\sigma^!(P^\bullet \otimes F^\bullet) = \sigma^* P^\bullet \otimes \sigma^! F^\bullet \in \mathrm{D}^+(\mathfrak{Z})_{<N}$ for some N. For any $p : * \hookrightarrow Z$, we have $p^!(\sigma^* P^\bullet \otimes \sigma^! F^\bullet) = p^* P^\bullet \otimes_k p^! \sigma^! F^\bullet$, so by Lemma 2.1.16 we have $F^\bullet \in \mathrm{D}^b(\mathfrak{X})_{<N-q}$ where q is the highest weight in $\sigma^* P^\bullet$. $\square$

Now that we have identified $D^b(\mathfrak{X})^{\mathrm{fin}} = D^b(\mathfrak{X})$ in this case, we collect the main results of this section in the following

Theorem 2.1.31. *Let $\mathfrak{S} \subset \mathfrak{X}$ be a closed KN stratum (Definition 1.1.7) satisfying Properties (L+) and (A). Let $\mathbf{G}_w = D^b(\mathfrak{X})_{\geq w} \cap D^b(\mathfrak{X})_{<w}$, then*

$$\mathbf{G}_w = \left\{ F^\bullet \in D^b(\mathfrak{X}) \,\middle|\, \begin{array}{l} \sigma^* F^\bullet \text{ supported in weights } \geq w, \text{ and} \\ \sigma^! F^\bullet \text{ supported in weights } < w + a \end{array} \right\}$$

where a is the weight of $\det(\mathfrak{N}_{Z/Y})^\vee$. There are semiorthogonal decompositions

$$D^b(\mathfrak{X}) = \langle D^b_{\mathfrak{S}}(\mathfrak{X})_{<w}, \mathbf{G}_w, D^b_{\mathfrak{S}}(\mathfrak{X})_{\geq w} \rangle$$

And the restriction functor $i^* : D^b(\mathfrak{X}) \to D^b(\mathfrak{V})$ induces an equivalence $\mathbf{G}_w \simeq D^b(\mathfrak{V})$, where $\mathfrak{V} = \mathfrak{X} - \mathfrak{S}$.

Proof. Because $\pi : Y \to Z$ is a bundle of affine spaces, Lemma 2.1.30 states that $D^b(\mathfrak{X})^{\mathrm{fin}} = D^b(\mathfrak{X})$, and Lemma 2.1.16 implies that $D^b(\mathfrak{X})_{<w} = \{F^\bullet | \sigma^! F^\bullet \in D^+(\mathfrak{Z})_{<w+a}$. As noted above, the existence of the semiorthogonal decomposition follows formally from the adjoint functors constructed in Propositions 2.1.28 and 2.1.29.

The fully faithfulness of $i^* : \mathbf{G}_w \to D^b(\mathfrak{V})$ is Theorem 2.1.26. Any $F^\bullet \in D^b(\mathfrak{V})$ admits a lift to $D^b(\mathfrak{X})$, and the component of this lift lying in $\mathbf{G}_w$ under the semiorthogonal decomposition also restricts to $F^\bullet$, hence i^* essential surjectivity follows. $\square$

Now let X be smooth in a neighborhood of Z. Passing to an open subset containing Z, we can assume that X is smooth. Recall that in this case S, Y, and Z are smooth, and the equivariant canonical bundle $\omega_{\mathfrak{X}} := (\bigwedge^{top} \mathfrak{g}) \otimes (\bigwedge^{top} \Omega^1_X)$ is a dualizing bundle on $\mathfrak{X}$ and defines the Serre duality functor $\mathbb{D}_{\mathfrak{X}}(\bullet) = R\underline{\mathrm{Hom}}(\bullet, \omega_{\mathfrak{X}}[\mathrm{vdim}\,\mathfrak{X}])$, and likewise for $\mathfrak{S}$ and $\mathfrak{Z}$. The canonical bundles are related by $j^! \omega_{\mathfrak{X}} \simeq \omega_{\mathfrak{S}}[-\mathrm{codim}(S, X)]$ and $\sigma^! \omega_{\mathfrak{S}} \simeq \omega_{\mathfrak{Z}}[-\mathrm{codim}(Z, S)]$.

Using the fact that $\omega_{\mathfrak{Z}}$ has weight 0, so $\mathbb{D}_{\mathfrak{Z}}(D^b(\mathfrak{Z})_{\geq w}) = D^b(\mathfrak{Z})_{<w+1}$, and the fact that $\mathbb{D}_{\mathfrak{Z}} \sigma^* F^\bullet \simeq \sigma^! \mathbb{D}_{\mathfrak{X}}$ and likewise for $\mathfrak{S}$, we have

$$\mathbb{D}_{\mathfrak{S}}(D^b(\mathfrak{S})_{\geq w}) = D^b(\mathfrak{S})_{<a+1-w}, \text{ and } \mathbb{D}_{\mathfrak{X}}(D^b(\mathfrak{X})_{\geq w}) = D^b(\mathfrak{X})_{<a+1-w}$$

where a is the weight of λ on $\omega_{\mathfrak{S}}|_{\mathfrak{Z}}$.

Furthermore any $F^\bullet \in D^b(\mathfrak{X})$ is perfect, so $j^! F^\bullet \simeq j^!(\mathcal{O}_X) \otimes j^* F^\bullet \simeq \det(\mathfrak{N}_{\mathfrak{S}/\mathfrak{X}})^\vee \otimes j^* F^\bullet[-\mathrm{codim}(S, X)]$. If we let η denote the weight of λ on $\det \mathfrak{N}_{S/X}|_Z$, then this implies that

$$D^b(\mathfrak{X})_{<w} = \{F^\bullet | \sigma^* F^\bullet \text{ supported in weights } < w + \eta\}$$

Using this we can reformulate Theorem 2.1.31 as

Corollary 2.1.32. *Let $\mathfrak{S} \subset \mathfrak{X}$ be a KN stratum such that $\mathfrak{X}$ is smooth in a neighborhood of $\mathfrak{Z}$. Let $\mathbf{G}_w = D^b(\mathfrak{X})_{\geq w} \cap D^b(\mathfrak{X})_{<w}$, then*

$$\mathbf{G}_w = \{F^\bullet \in D^b(\mathfrak{X}) | \sigma^* F^\bullet \text{ supported in weights } [w, w + \eta)\}$$

where η is the weight of $\det(\Omega_{\mathfrak{G}}\mathfrak{X})$. There are semiorthogonal decompositions

$$\mathrm{D}^b(\mathfrak{X}) = \langle \mathrm{D}^b_{\mathfrak{G}}(\mathfrak{X})_{<w}, \mathbf{G}_w, \mathrm{D}^b_{\mathfrak{G}}(\mathfrak{X})_{\geq w}\rangle$$

And the restriction functor $i^* : \mathrm{D}^b(\mathfrak{X}) \to \mathrm{D}^b(\mathfrak{V})$ induces an equivalence $\mathbf{G}_w \simeq \mathrm{D}^b(\mathfrak{V})$.

One can explicitly define the inverse using the functors $\beta_{\geq w}$ and $\beta_{<w}$ on $\mathrm{D}^b_{\mathfrak{G}}(\mathfrak{X})$. Given $F^{\boldsymbol{\cdot}} \in \mathrm{D}^b(\mathfrak{V})$, choose a complex $\tilde{F}^{\boldsymbol{\cdot}} \in \mathrm{D}^b(\mathfrak{X})$ such that $\tilde{F}^{\boldsymbol{\cdot}}|_{\mathfrak{V}} \simeq F^{\boldsymbol{\cdot}}$. Now for $N \gg 0$ take the mapping cone

$$\beta_{\geq w}R\underline{\mathrm{Hom}}_{\mathfrak{X}}(\mathcal{O}_X/\mathcal{I}^N_{\mathfrak{G}}, \tilde{F}^{\boldsymbol{\cdot}}) = \beta_{\geq w}R\underline{\Gamma}_{\mathfrak{G}}\tilde{F}^{\boldsymbol{\cdot}} \to \tilde{F}^{\boldsymbol{\cdot}} \to G^{\boldsymbol{\cdot}} \dashrightarrow$$

So $G^{\boldsymbol{\cdot}} \in \mathrm{D}^b(\mathfrak{X})_{<w}$. By Serre duality the left adjoint of the inclusion $\mathrm{D}^b_{\mathfrak{G}}(\mathfrak{X})_{<w} \subset \mathrm{D}^b(\mathfrak{X})_{<w}$ is $\mathbb{D}_{\mathfrak{X}}\beta_{\geq \eta+1-w}R\underline{\Gamma}_{\mathfrak{G}}\mathbb{D}_{\mathfrak{X}}$, and this functor can be simplified using Lemma 2.1.28. We form the exact triangle

$$\tilde{G}^{\boldsymbol{\cdot}} \to G^{\boldsymbol{\cdot}} \to \beta_{<w}(G^{\boldsymbol{\cdot}} \otimes^L \mathcal{O}_X/\mathcal{I}^N_{\mathfrak{G}}) \dashrightarrow$$

and $\tilde{G}^{\boldsymbol{\cdot}} \in \mathbf{G}_w$ is the unique object in $\mathbf{G}_w$ mapping to $F^{\boldsymbol{\cdot}}$.

2.2 Derived equivalences and variation of GIT

We apply Theorem 2.0.3 to the derived categories of birational varieties obtained by a variation of GIT quotient. First we study the case where $G = \mathbb{C}^*$, in which the KN stratification is particularly easy to describe. Next we generalize this analysis to arbitrary variations of GIT, one consequence of which is the observation that if a smooth projective-over-affine variety X is equivariantly Calabi-Yau for the action of a torus, then the GIT quotients of any two generic linearizations are derived equivalent.

A normal projective variety X with linearized $\mathbb{C}^*$ action is sometimes referred to as a birational cobordism between $X//_{\mathcal{L}}G$ and $X//_{\mathcal{L}(r)}G$ where $\mathcal{L}(m)$ denotes the twist of $\mathcal{L}$ by the character $t \mapsto t^r$. A priori this seems like a highly restrictive type of VGIT, but by Thaddeus' master space construction[44], any two spaces that are related by a general VGIT are related by a birational cobordism. We also have the weak converse due to Hu & Keel:

Theorem 2.2.1 (Hu & Keel). *Let Y_1 and Y_2 be two birational projective varieties, then there is a birational cobordism $X/\mathbb{C}^*$ between Y_1 and Y_2. If Y_1 and Y_2 are smooth, then by equivariant resolution of singularities X can be chosen to be smooth.*

The GIT stratification for $G = \mathbb{C}^*$ is very simple. If $\mathcal{L}$ is chosen so that the GIT quotient is an orbifold, then the Z_α are the connected components of the fixed locus X^G, and S_α is either the ascending or descending manifold of Z_α, depending on the weight of $\mathcal{L}$ along Z_α.

We will denote the tautological choice of 1PS as λ^+, and we refer to "the weights" of a coherent sheaf at point in X^G as the weights with respect to this 1PS. We define $\mu_\alpha \in \mathbb{Z}$ to
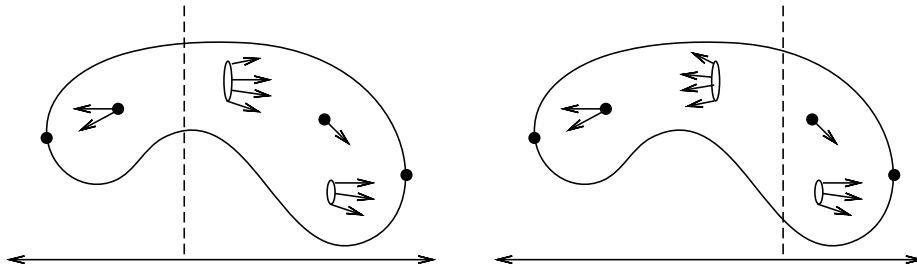
Figure 2.1: Schematic diagram for the fixed loci Z_α. S_α is the ascending or descending manifold of Z_α depending on the sign of μ_α. As the moment fiber varies, the unstable strata S_α flip over the critical sets Z_α.

be the weight of $\mathcal{L}|_{Z_\alpha}$. If $\mu_\alpha > 0$ (respectively $\mu_\alpha < 0$) then the maximal destabilizing 1PS of Z_α is λ^+ (respectively λ^-). Thus we have

$$S_\alpha = \left\{ x \in X \,\middle|\, \begin{array}{l} \lim_{t \to 0} t \cdot x \in Z_\alpha \text{ if } \mu_\alpha > 0 \\ \lim_{t \to 0} t^{-1} \cdot x \in Z_\alpha \text{ if } \mu_\alpha < 0 \end{array} \right\}$$

Next observe the weight decomposition under λ^+

$$\Omega^1_X|_{Z_\alpha} \simeq \Omega^1_{Z_\alpha} \oplus \mathcal{N}^+ \oplus \mathcal{N}^- \tag{2.10}$$

Then $\Omega^1_{S_\alpha}|_{\mathfrak{Z}^\alpha} = \Omega^1_{Z_\alpha} \oplus \mathcal{N}^-$ if $\mu_\alpha > 0$ and $\Omega^1_{S_\alpha}|_{\mathfrak{Z}^\alpha} = \Omega^1_{Z_\alpha} \oplus \mathcal{N}^+$ if $\mu_\alpha < 0$, so we have

$$\eta_\alpha = \begin{cases} \text{weight of } \det \mathcal{N}^+|_{Z_\alpha} \text{ if } \mu_\alpha > 0 \\ -\text{weight of } \det \mathcal{N}^-|_{Z_\alpha} \text{ if } \mu_\alpha < 0 \end{cases} \tag{2.11}$$

There is a parallel interpretation of this in the symplectic category. A sufficiently large power of $\mathcal{L}$ induces a equivariant projective embedding and thus a moment map $\mu : X \to \mathbb{R}$ for the action of $S^1 \subset \mathbb{C}^*$. The semistable locus is the orbit of the zero fiber $X^{ss} = \mathbb{C}^* \cdot \mu^{-1}(0)$. The reason for the collision of notation is that the fixed loci Z_α are precisely the critical points of μ, and the number μ_α is the value of the moment map on the critical set Z_α.

Varying the linearization $\mathcal{L}(r)$ by twisting by the character $t \mapsto t^{-r}$ corresponds to shifting the moment map by $-r$, so the new zero fiber corresponds to what was previously the fiber $\mu^{-1}(r)$. For non-critical moment fibers the GIT quotient will be a DM stack, and the critical values of r are those for which $\mu_\alpha = $ weight of $\mathcal{L}(r)|_{Z_\alpha} = 0$ for some α.

Say that as r increases it crosses a critical value for which $\mu_\alpha = 0$. The maximal destabilizing 1PS λ_α flips from λ^+ to λ^-, and the unstable stratum S_α flips from the ascending manifold of Z_α to the descending manifold of Z_α. In the decomposition (2.10), the normal bundle of S_α changes from $\mathcal{N}^+$ to $\mathcal{N}^-$, so applying det to (2.10) and taking the weight gives

$$\text{weight of } \omega_X|_{Z_\alpha} = \eta_\alpha - \eta'_\alpha \tag{2.12}$$

Thus if ω_X has weight 0 along Z_α, the integer η_α does not change as we cross the wall. The grade restriction window of Theorem 2.0.3 has the same width for the GIT quotient on either side of the wall, and it follows that the two GIT quotients are derived equivalent because they are identified with the *same* subcategory $\mathbf{G}_q$ of the equivariant derived category $\mathrm{D}^b(X/G)$. We summarize this with the following

Proposition 2.2.2. *Let $\mathcal{L}$ be a critical linearization of $X/\mathbb{C}^*$, and assume that Z_α is the only critical set for which $\mu_\alpha = 0$. Let a be the weight of $\omega_X|_{Z_\alpha}$, and let $\epsilon > 0$ be a small rational number.*

1. *If $a > 0$, then there is a fully faithful embedding*
$$\mathrm{D}^b(X//_{\mathcal{L}(\epsilon)}G) \subseteq \mathrm{D}^b(X//_{\mathcal{L}(-\epsilon)}G)$$

2. *If $a = 0$, then there is an equivalence*
$$\mathrm{D}^b(X//_{\mathcal{L}(\epsilon)}G) \simeq \mathrm{D}^b(X//_{\mathcal{L}(-\epsilon)}G)$$

3. *If $a < 0$, then there is a fully faithful embedding*
$$\mathrm{D}^b(X//_{\mathcal{L}(-\epsilon)}G) \subseteq \mathrm{D}^b(X//_{\mathcal{L}(\epsilon)}G)$$

The analytic local model for a birational cobordism is the following

Example 2.2.3. Let Z be a smooth variety and let $\mathcal{N} = \bigoplus \mathcal{N}_i$ be a $\mathbb{Z}$-graded locally free sheaf on Z with $\mathcal{N}_0 = 0$. Let X be the total of $\mathcal{N}$ – it has a $\mathbb{C}^*$ action induced by the grading. Because the only fixed locus is Z the underlying line bundle of the linearization is irrelevant, so we take the linearization $\mathcal{O}_X(r)$.

If $r > 0$ then the unstable locus is $\mathcal{N}_- \subset X$ where $\mathcal{N}_-$ is the sum of negative weight spaces of $\mathcal{N}$, and if $r < 0$ then the unstable locus is $\mathcal{N}_+$ (we are abusing notation slightly by using the same notation for the sheaf and its total space). We will borrow the notation of Thaddeus [44] and write $X/\pm = (X \setminus \mathcal{N}_\mp)/\mathbb{C}^*$.

Inside $X/\pm$ we have $\mathcal{N}_\pm/\pm \simeq \mathbb{P}(\mathcal{N}_\pm)$, where we are still working with quotient stacks, so the notation $\mathbb{P}(\mathcal{N}_\pm)$ denotes the weighted projective bundle associated to the graded locally free sheaf $\mathcal{N}_\pm$. If $\pi_\pm : \mathbb{P}(\mathcal{N}_\pm) \to Z$ is the projection, then $X/\pm$ is the total space of the vector bundle $\pi_\pm^* \mathcal{N}_\mp(-1)$. We have the common resolution

$$\mathcal{O}_{\mathbb{P}(\mathcal{N}_-) \times_S \mathbb{P}(\mathcal{N}_+)}(-1,-1)$$

$$\pi_+^* \mathcal{N}_-(-1) \qquad\qquad \pi_-^* \mathcal{N}_+(-1)$$

Let $\pi : X \to Z$ be the projection, then the canonical bundle is $\omega_X = \pi^*(\omega_Z \otimes \det(\mathcal{N}_+)^\vee \otimes \det(\mathcal{N}_-)^\vee)$, so the weight of $\omega_X|_Z$ is $\sum i \operatorname{rank}(\mathcal{N}_i)$. In the special case of a flop, Proposition 2.2.2 says

$$\text{if } \sum i \operatorname{rank}(\mathcal{N}_i) = 0, \text{ then } \mathrm{D}^b(\pi_+^* \mathcal{N}_-(-1)) \simeq \mathrm{D}^b(\pi_-^* \mathcal{N}_+(-1))$$

General variation of GIT quotient

We will generalize the analysis of a birational cobordism to an arbitrary variation of GIT quotient. Until this point we have taken the KN stratification as given, but now we must recall its definition and basic properties as described in [16].

Let $\mathrm{NS}^G(X)_{\mathbb{R}}$ denote the group of equivariant line bundles up to homological equivalence, tensored with $\mathbb{R}$. For any $\mathcal{L} \in \mathrm{NS}^G(X)_{\mathbb{R}}$ one defines a stability function on X

$$M^{\mathcal{L}}(x) := \max \left\{ \left. \frac{\mathrm{weight}_\lambda \mathcal{L}_y}{|\lambda|} \right| \lambda \text{ s.t. } y = \lim_{t \to 0} \lambda(t) \cdot x \text{ exists} \right\}$$

$M^{\mathcal{L}}(\bullet)$ is upper semi-continuous, and $M^{\bullet}(x)$ is lower convex and thus continuous on $\mathrm{NS}^G(X)_{\mathbb{R}}$ for a fixed x. A point $x \in X$ is semistable if $M^{\mathcal{L}}(x) \leq 0$, stable if $M^{\mathcal{L}}(x) < 0$, strictly semistable if $M^{\mathcal{L}}(x) = 0$ and unstable if $M^{\mathcal{L}}(x) > 0$.

The G-ample cone $\mathcal{C}^G(X) \subset \mathrm{NS}^G(X)_{\mathbb{R}}$ has a finite decomposition into convex conical chambers separated by hyperplanes – the interior of a chamber is where $M^{\mathcal{L}}(x) \neq 0$ for all $x \in X$, so $\mathfrak{X}^{ss}(\mathcal{L}) = \mathfrak{X}^s(\mathcal{L})$. We will be focus on a single wall-crossing: $\mathcal{L}_0$ will be a G-ample line bundle lying on a wall such that for ϵ sufficiently small $\mathcal{L}_\pm := \mathcal{L}_0 \pm \epsilon \mathcal{L}'$ both lie in the interior of chambers.

By continuity of the function $M^{\bullet}(x)$ on $\mathrm{NS}^G(X)_{\mathbb{R}}$, all of the stable and unstable points of $\mathfrak{X}^s(\mathcal{L}_0)$ will remain so for $\mathcal{L}_\pm$. Only points in the strictly semistable locus, $\mathfrak{X}^{sss}(\mathcal{L}_0) = \{x \in \mathfrak{X} | M^{\mathcal{L}}(x) = 0\} \subset \mathfrak{X}$, change from being stable to unstable as one crosses the wall.

In fact $\mathfrak{X}^{us}(\mathcal{L}_0)$ is a union of KN strata for $\mathfrak{X}^{us}(\mathcal{L}_+)$, and symmetrically it can be written as a union of KN strata for $\mathfrak{X}^{us}(\mathcal{L}_-)$.[16] Thus we can write $\mathfrak{X}^{ss}(\mathcal{L}_0)$ in two ways

$$\mathfrak{X}^{ss}(\mathcal{L}_0) = \mathfrak{S}_1^{\pm} \cup \cdots \cup \mathfrak{S}_{m_\pm}^{\pm} \cup \mathfrak{X}^{ss}(\mathcal{L}_\pm) \tag{2.13}$$

Where $\mathfrak{S}_i^{\pm}$ are the KN strata of $\mathfrak{X}^{us}(\mathcal{L}_\pm)$ lying in $\mathfrak{X}^{ss}(\mathcal{L}_0)$.

Definition 2.2.4. A wall crossing $\mathcal{L}_\pm = \mathcal{L}_0 \pm \epsilon \mathcal{L}'$ will be called *balanced* if $m_+ = m_-$ and $\mathfrak{Z}_i^+ = \mathfrak{Z}_i^-$ under the decomposition (2.13).

By the construction of the strata outlined above, there is a finite collection of locally closed $Z_i \subset X$ and one parameter subgroups λ_i fixing Z_i such that $G \cdot Z_i / G$ are simultaneously the attractors for the KN strata of both $\mathfrak{X}^{ss}(\mathcal{L}_\pm)$ and such that the $\lambda_i^{\pm 1}$ are the maximal destabilizing 1PS's.

Proposition 2.2.5. *Let a reductive G act on a projective-over-affine variety X. Let $\mathcal{L}_0$ be a G-ample line bundle on a wall, and define $\mathcal{L}_\pm = \mathcal{L}_0 \pm \epsilon \mathcal{L}'$ for some other line bundle $\mathcal{L}'$. Assume that*

- *for ϵ sufficiently small, $\mathfrak{X}^{ss}(\mathcal{L}_\pm) = \mathfrak{X}^s(\mathcal{L}_\pm) \neq \emptyset$,*

- *the wall crossing $\mathcal{L}_\pm$ is balanced, and*

- *for all Z_i in $\mathfrak{X}^{ss}(\mathcal{L}_0)$, $(\omega_{\mathfrak{X}})|_{Z_i}$ has weight 0 with respect to λ_i*

then $\mathrm{D}^b(\mathfrak{X}^{ss}(\mathcal{L}_+)) \simeq \mathrm{D}^b(\mathfrak{X}^{ss}(\mathcal{L}_-))$.

Remark 2.2.6. Full embeddings analogous to those of Proposition 2.2.2 apply when the weights of $(\omega_{\mathfrak{X}})|_{Z_i}$ with respect to λ_i are either all negative or all positive.

Proof. This is an immediate application of Theorem 2.0.3 to the open substack $\mathfrak{X}^s(\mathcal{L}_\pm) \subset \mathfrak{X}^{ss}(\mathcal{L}_0)$ whose complement admits the KN stratification (2.13). Because the wall crossing is balanced, $Z_i^+ = Z_i^-$ and $\lambda_i^-(t) = \lambda_i^+(t^{-1})$, and the condition on $\omega_{\mathfrak{X}}$ implies that $\eta_i^+ = \eta_i^-$. So Theorem 2.0.3 identifies the category $\mathbf{G}_q \subset \mathrm{D}^b(\mathfrak{X}^{ss}(\mathcal{L}_0))$ with both $\mathrm{D}^b(\mathfrak{X}^s(\mathcal{L}_-))$ and $\mathrm{D}^b(\mathfrak{X}^s(\mathcal{L}_+))$. $\qquad\square$

Example 2.2.7. Dolgachev and Hu study wall crossings which they call *truly faithful*, meaning that the identity component of the stabilizer of a point with closed orbit in $\mathfrak{X}^{ss}(\mathcal{L}_0)$ is $\mathbb{C}^*$. They show that every truly faithful wall is balanced.[16, Lemma 4.2.3]

Dolgachev and Hu also show that for the action of a torus T, there are no codimension 0 walls and all codimension 1 walls are truly faithful. Thus any two chambers in $\mathcal{C}^T(X)$ can be connected by a finite sequence of balanced wall crossings, and we have

Corollary 2.2.8. *Let X be a projective-over-affine variety with an action of a torus T. Assume X is equivariantly Calabi-Yau in the sense that $\omega_X \simeq \mathcal{O}_X$ as an equivariant $\mathcal{O}_X$-module. If $\mathcal{L}_0$ and $\mathcal{L}_1$ are G-ample line bundles such that $\mathfrak{X}^s(\mathcal{L}_i) = \mathfrak{X}^{ss}(\mathcal{L}_i)$, then $\mathrm{D}^b(\mathfrak{X}^s(\mathcal{L}_0)) \simeq \mathrm{D}^b(\mathfrak{X}^s(\mathcal{L}_1))$.*

A compact projective manifold with a non-trivial $\mathbb{C}^*$ action is never equivariantly Calabi-Yau, but Corollary 2.2.8 applies to a large class of non compact examples. The simplest are linear representations V of T such that $\det V$ is trivial. More generally we have

Example 2.2.9. Let T act on a smooth projective Fano variety X, and let $\mathcal{E}$ be an equivariant ample locally free sheaf such that $\det \mathcal{E} \simeq \omega_X^\vee$. Then the total space of the dual vector bundle $Y = \mathrm{Spec}_X(S^*\mathcal{E})$ is equivariantly Calabi-Yau and the canonical map $Y \to \mathrm{Spec}(\Gamma(X, S^*\mathcal{E}))$ is projective, so Y is projective over affine and by Corollary 2.2.8 any two generic GIT quotients $Y//T$ are derived equivalent.

When G is non-abelian, the chamber structure of $\mathcal{C}^G(X)$ can be more complicated. There can be walls of codimension 0, meaning open regions in the interior of $\mathcal{C}^G(X)$ where $\mathfrak{X}^s \neq \mathfrak{X}^{ss}$, and not all walls are truly faithful.[16] Still, there are examples where derived Kirwan surjectivity can give derived equivalences under wall crossings which are not balanced.

Definition 2.2.10. A wall crossing $\mathcal{L}_\pm = \mathcal{L}_0 \pm \epsilon \mathcal{L}'$ will be called *almost balanced* if $m_+ = m_-$ and under the decomposition (2.13), one can choose maximal destabilizers such that $\lambda_i^- = (\lambda_i^+)^{-1}$ and $\mathrm{cl}(Z_i^+) = \mathrm{cl}(Z_i^-)$.

In an almost balanced wall crossing for which $\omega_X|_{Z_i}$ has weight 0 for all i, we have the following general principal for establishing a derived equivalence:

Ansatz 2.2.11. *For some w and w', $\mathbf{G}_w^+ = \mathbf{G}_{w'}^-$ as subcategories of $D^b(X^{ss}(\mathcal{L}_0)/G)$, where $\mathbf{G}_\bullet^\pm$ is the category identified with $D^b(X^{ss}(\mathcal{L}_\pm)/G)$ under restriction.*

For instance, one can recover a result of Segal & Donnovan[17]:

Example 2.2.12 (Grassmannian flop). Choose $k < N$ and let V be a k-dimensional vector space. Consider the action of $G = GL(V)$ on $X = T^* \operatorname{Hom}(V, \mathbb{C}^N) = \operatorname{Hom}(V, \mathbb{C}^N) \times \operatorname{Hom}(\mathbb{C}^N, V)$. A 1PS $\lambda : \mathbb{C}^* \to G$ corresponds to a choice of weight decomposition $V \simeq \bigoplus V_\alpha$ under λ. A point (a, b) has a limit under λ iff

$$V_{>0} \subset \ker(a) \quad \text{and} \quad \operatorname{im}(b) \subset V_{\geq 0}$$

in which case the limit (a_0, b_0) is the projection onto $V_0 \subset V$. There are only two nontrivial characters up to rational equivalence, $\det^\pm$. A point (a, b) is semistable iff any 1PS for which $\lambda(t) \cdot (a, b)$ has a limit as $t \to 0$ has nonpositive pairing with the chosen character.

In order to determine the stratification, it suffices to fix a maximal torus of $GL(V)$, i.e. and isomorphism $V \simeq \mathbb{C}^k$, and to consider diagonal one parameter subgroups $(t^{w_1}, \dots, t^{w_k})$ with $w_1 \leq \dots \leq w_k$. If we linearize with respect to $\det$, then the KN stratification is

$$\lambda_i = (0, \dots, 0, 1, \dots, 1) \text{ with } i \text{ zeros}$$

$$Z_i = \left\{ \left(\begin{bmatrix} \square & | & 0 \end{bmatrix}, \begin{bmatrix} * \\ 0 \end{bmatrix} \right), \begin{array}{l} \text{with } * \in M_{i \times N}, \\ \text{and } \square \in M_{N \times i} \text{ full rank} \end{array} \right\}$$

$$Y_i = \left\{ \left(\begin{bmatrix} \square & | & 0 \end{bmatrix}, b \right), \begin{array}{l} \text{with } b \in M_{k \times N} \text{ arbitrary,} \\ \text{and } \square \in M_{N \times i} \text{ full rank} \end{array} \right\}$$

$$S_i = \{ (a, b) | b \text{ arbitrary, } \operatorname{rank} a = i \}$$

So $(a, b) \in X$ is semistable iff a is injective. If instead we linearize with respect to $\det^{-1}$, then (a, b) is semistable iff b is surjective, the λ_i flip, and the critical loci Z_i are the same *except that the role of $\square$ and $*$ reverse.* So this is an almost balanced wall crossing with $\mathcal{L}_0 = \mathcal{O}_X$ and $\mathcal{L}' = \mathcal{O}_X(\det)$.

Let $\mathbb{G}(k, N)$ be the Grassmannian parametrizing k-dimensional subspaces $V \subset \mathbb{C}^N$, and let $0 \to U(k, N) \to \mathcal{O}^N \to Q(k, N) \to 0$ be the tautological sequence of vector bundles on $\mathbb{G}(k, N)$. Then $\mathfrak{X}^{ss}(\det)$ is the total space of $U(k, N)^N$, and $\mathfrak{X}^{ss}(\det^{-1})$ is the total space of $(Q(N - k, N)^\vee)^N$ over $\mathbb{G}(N - k, N)$.

In order to verify that $\mathbf{G}_w^+ = \mathbf{G}_{w'}^-$ for some w', one observes that the representations of GL_k which form the Kapranov exceptional collection[26] lie in the weight windows for $G_0^+ \simeq D^b(\mathfrak{X}^{ss}(\det)) = D^b(U(k, N)^N)$. Because $U(k, N)^N$ is a vector bundle over $\mathbb{G}(k, N)$, these objects generate the derived category. One then verifies that these object lie in the weight windows for $\mathfrak{X}^{ss}(\det^{-1})$ and generate this category for the same reason. Thus by verifying Ansatz 2.2.11 we have established an equivalence of derived categories

$$D^b(U(k, N)^N) \simeq D^b((Q(N - k, N)^\vee)^N)$$

The astute reader will observe that these two varieties are in fact isomorphic, but the derived equivalences we have constructed are natural in the sense that they generalize to families. Specifically, if $\mathcal{E}$ is an N-dimensional vector bundle over a smooth variety Y, then the two GIT quotients of the total space of $\underline{\mathrm{Hom}}(\mathcal{O}_Y \otimes V, \mathcal{E}) \oplus \underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{O}_Y \otimes V)$ by $GL(V)$ will have equivalent derived categories.

The key to verifying Ansatz 2.2.11 in this example was simple geometry of the GIT quotients $\mathfrak{X}^{ss}(\det^{\pm})$ and the fact that we have explicit generators for the derived category of each. With a more detailed analysis, one can verify Ansatz 2.2.11 for many more examples of balanced wall crossings, and we will describe this in a future paper.

Remark 2.2.13. This example is similar to the generalized Mukai flops of [13]. The difference is that we are not restricting to the hyperkähler moment fiber $\{ba = 0\}$. The surjectivity theorem cannot be applied directly to the GIT quotient of this singular variety, but in the next section we will explore some applications to abelian hyperkähler reduction.

2.3 Applications to complete intersections: matrix factorizations and hyperkähler reductions

In the example of a projective variety, where we identified $\mathrm{D}^b(Y)$ with a full subcategory of the derived category of finitely generated graded modules over the homogeneous coordinate ring of Y, the point of the affine cone satisfied Property (L+) "for free." In more complicated examples, the cotangent positivity property (L+) can be difficult to verify.

Here we discuss several techniques for extending derived Kirwan surjectivity for stacks X/G where X is a local complete intersection. First we provide a geometric criterion for Property (L+) to hold, which allows us to apply Theorem 2.0.3 to some hyperkähler quotients. We also discuss two different approaches to derived Kirwan surjectivity for LCI quotients, using morita theory and derived categories of singularities.

A criterion for Property (L+) and non-abelian Hyperkähler reduction

In this section we study a particular setting in which Property (L+) holds for the KN stratification of a singular quotient stack. This will allow us to address some hyperkähler reductions by nonabelian groups.

Let X' be a smooth quasiprojective variety with an action of a reductive G, and let $S' = G \cdot Y' \subset X'$ be a closed KN stratum (Definition 1.1.7). Because X' is smooth, Y' is a P-equivariant bundle of affine spaces over Z'. Let V be a linear representation of G, and $s : X' \to V$ and equivariant map. Alternatively, we think of s as an invariant global section of the locally free sheaf $\mathcal{O}_{\mathfrak{X}'} \otimes V$. We define $X = s^{-1}(0)$ and $S = S' \cap X$, and likewise for Y and Z.

Note that if we decompose $V = V_+ \oplus V_0 \oplus V_-$ under the weights of λ, then $\Gamma(\mathfrak{S}', \mathcal{O}_{\mathfrak{S}'} \otimes V_-) = 0$, so $s|_{\mathfrak{S}'}$ is a section of $\mathcal{O}_{\mathfrak{S}'} \otimes V_0 \oplus V_+$.

Lemma 2.3.1. *If for all $z \in Z \subset Z'$, $(ds)_z : T_z X \to V$ is surjective in positive weights w.r.t. λ, then*

$$(\sigma^* \mathbf{L}_{\mathfrak{S}}^\bullet)_{<0} \simeq [\mathcal{O}_{\mathfrak{Z}} \otimes V_+^\vee \xrightarrow{(ds_+)^\vee} (\Omega_{Y'}|_Z)_{<0}]$$

and is thus a locally free sheaf concentrated in cohomological degree 0.

Proof. First of all note that from the inclusion $\sigma : \mathfrak{Z} \hookrightarrow \mathfrak{S}$ we have

$$(\sigma^* \mathbf{L}_{\mathfrak{S}}^\bullet)_{<0} \to (\mathbf{L}_{\mathfrak{Z}}^\bullet)_{<0} \to (\mathbf{L}_{\mathfrak{Z}/\mathfrak{S}}^\bullet)_{<0} \dashrightarrow$$

The cotangent complex $\mathbf{L}_{\mathfrak{Z}}^\bullet$ is supported in weight 0 because λ acts trivially on Z, so the middle term vanishes, and we get $(\sigma^* \mathbf{L}_{\mathfrak{S}}^\bullet)_{<0} \simeq (\mathbf{L}_{\mathfrak{Z}/\mathfrak{S}}^\bullet)_{<0}[-1]$, so it suffices to consider the later.

By definition Y is the zero fiber of $s : Y' \to V_0 \oplus V_+$. Denote by s_0 the section of V_0 induced by the projection of P-modules $V_+ \oplus V_0 \to V_0$. We consider the intermediate variety $Y \subset Y_0 := s_0^{-1}(0) \subset Y'$. Note that $Y = \pi^{-1}(Z)$, where $\pi : Y' \to Z'$ is the projection.

Note that $Y_0 \to Z$ is a bundle of affine spaces with section σ, so in particular $\mathfrak{Z} \subset \mathfrak{S}_0$ is a regular embedding with conormal bundle $(\Omega_{Y'}^1|_Z)_{<0} = (\Omega_{X'}^1|_Z)_{<0}$. Furthermore, on Y_0 the section s_0 vanishes by construction, so $Y \subset Y_0$, which by definition is the vanishing locus of $s|_{Y_0}$, is actually the vanishing locus of the map $s_+ : Y_0 \to V_+$. The surjectivity of $(ds)_z$ for $z \in Z$ in positive weights implies that $s_+^{-1}(0)$ has expected codimension in every fiber over Z and thus $\mathfrak{S} \subset \mathfrak{S}_0$ is a regular embedding with conormal bundle $\mathcal{O}_{\mathfrak{S}} \otimes V_+^\vee$.

It now follows from the canonical triangle for $\mathfrak{Z} \subset \mathfrak{S} \subset \mathfrak{S}_0$ that

$$\mathbf{L}_{\mathfrak{Z}/\mathfrak{S}}^\bullet \simeq \mathrm{Cone}(\sigma^* \mathbf{L}_{\mathfrak{S}/\mathfrak{S}_0} \to \mathbf{L}_{\mathfrak{Z}/\mathfrak{S}_0}) \simeq [\mathcal{O}_{\mathfrak{Z}} \otimes V_+^\vee \xrightarrow{ds_+} (\Omega_{Y'}^1|_Z)_{<0}]$$

with terms concentrated in cohomological degree -2 and -1. The result follows. $\square$

Proposition 2.3.2. *Let X' be a smooth quasiprojective variety with reductive G action, and let $Z' \subset S' \subset X'$ be a KN stratum. Let $s : X' \to V$ be an equivariant map to a representation of G.*

Define $X = s^{-1}(0)$, $S = S' \cap X$, and $Z = Z' \cap X$, and assume that X has codimension $\dim V$. If for all $z \in Z$, $(ds)_z : T_z X' \to V$ is surjective in positive weights w.r.t. λ, then Property (L+) holds for $S/G \hookrightarrow X/G$.

Proof. We will use Lemma 2.3.1 to compute the relative cotangent complex $(\sigma^* \mathbf{L}_{\mathfrak{S}/\mathfrak{X}}^\bullet)_{<0}$. We consider the canonical diagram

$$
\begin{array}{ccccc}
[\mathcal{O}_Y \otimes V^\vee \to \Omega_{X'}^1|_Y] & \longrightarrow & [\mathcal{O}_Y \otimes (V_{\geq 0})^\vee \to \Omega_{Y'}^1|_Y] & & \\
\downarrow a & & \downarrow b & & \\
j^* \mathbf{L}_{\mathfrak{X}}^\bullet & \longrightarrow & \mathbf{L}_{\mathfrak{S}}^\bullet & \longrightarrow & \mathbf{L}_{\mathfrak{S}/\mathfrak{X}}^\bullet \dashrightarrow
\end{array}
$$

where the bottom row is an exact triangle and we have used the identification $\mathfrak{S}' \simeq Y'/P$ and $\mathfrak{S} \simeq Y/P$. Because $X \subset X'$ has the expected codimension, it is a complete intersection and the morphism a is a quasi-isomorphism. Lemma 2.3.1 implies that b is a quasi-isomorphism after applying the functor $(\sigma^*(\bullet))_{<0}$.

Thus we have a quasi-isomorphism

$$(\sigma^* \mathbf{L}_{\mathfrak{S}/\mathfrak{X}}^{\bullet})_{<0} \simeq \mathrm{cone}\left([\mathcal{O}_Z \otimes V^{\vee} \to \Omega_{X'}^1|_Z] \to [\mathcal{O}_Z \otimes (V_{\geq 0})^{\vee} \to \Omega_{Y'}^1|_Z]\right)_{<0}$$
$$\simeq \mathrm{cone}\left((\Omega_{X'}^1|_Z)_{<0} \to (\Omega_{Y'}^1|_Z)_{<0}\right) \simeq 0$$

The last isomorphism follows because $\Omega_{Y'}^1|_Z$ is the negative weight eigenspace of $\Omega_{X'}^1|_Z$ by construction. $\qquad\square$

Now let (M, ω) be an algebraic symplectic manifold with a Hamiltonian G action, i.e. there is a G-equivariant algebraic map $\mu : M \to \mathfrak{g}^{\vee}$ satisfying $d\langle \xi, \mu \rangle = -\omega(\partial_\xi, \bullet) \in \Gamma(M, \Omega_M^1)$, where ∂_ξ is the vector field corresponding to $\xi \in \mathfrak{g}$.

For any point $x \in M$, we have an exact sequence

$$0 \to \mathrm{Lie}\, G_x \to \mathfrak{g} \xrightarrow{d\mu} T_x^* M \to T_x(G \cdot x)^{\perp} \to 0 \tag{2.14}$$

Showing that $X := \mu^{-1}(0)$ is regular at any point with finite stabilizer groups. Thus if the set such points is dense in X, then $X \subset M$ is a complete intersection cut out by μ. Thus we have

Proposition 2.3.3. *Let (M, ω) be a projective-over-affine algebraic symplectic manifold with a Hamiltonian action of the reductive group G, and let $X = \mu^{-1}(0) \subset M$. If X^s is dense in X, then Property (L+) holds for the GIT stratification of X.*

Example 2.3.4 (stratified Mukai flop)**.** We return to $M := \mathrm{Hom}(V, \mathbb{C}^N) \times \mathrm{Hom}(\mathbb{C}^N, V)$. In Example 2.2.12 we considered the GIT stratification for the action of $GL(V)$, but this group action is also algebraic Hamiltonian with moment map $\mu(a, b) = ba \in \mathfrak{gl}(V)$. The stratification of $X = \mu^{-1}(0)$ is induced by the stratification of M. Thus the Y_i in X consist of

$$Y_i = \left\{ \left([\, a_1 \mid 0 \,], \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right), \begin{array}{l} \text{with } b_1 a_1 = 0, b_2 a_1 = 0, \\ \text{and } a_1 \in M_{N \times i} \text{ full rank} \end{array} \right\}$$

and $Z_i \subset Y_i$ are those points where $b_2 = 0$. Note that over a point in Z_i, the condition $b_2 a_1 = 0$ is linear in the fiber, and so $Y_i \to Z_i$ satisfies Property (A).

The GIT quotient $X^{ss}/GL(V)$ is the cotangent bundle $T^* \mathbb{G}(k, N)$. Property (A) holds in this example, and Property (L+) holds by Proposition 2.3.2, so Theorem 2.0.3 gives a fully faithful embedding $\mathrm{D}^b(T^* \mathbb{G}(k, N)) \subset \mathrm{D}^b(X/GL(V))$ for any choice of integers w_i. The derived category $\mathrm{D}^b(T^* \mathbb{G}(k, N))$ has been intensely studied by Cautis, Kamnitzer, and Licata from the perspective of categorical $\mathfrak{sl}_2$ actions. We will discuss the connection between their results and derived Kirwan surjectivity in future work.

Extending the main theorem using Morita theory

In this section I remark that Theorem 1.2.1 extends to complete intersections in a smooth X/G for purely formal reasons, where by complete intersection I mean one defined by global invariant functions on X/G.

In this section I will use derived Morita theory ([7],[28]), and so I will switch to a notation more common in that subject. $\mathrm{QC}(\mathfrak{X})$ will denote the unbounded derived category of quasicoherent sheaves on a perfect stack $\mathfrak{X}$, and $\mathrm{Perf}(\mathfrak{X})$ will denote the category of perfect complexes, i.e. the compact objects of $\mathrm{QC}(\mathfrak{X})$. All of the stacks we use are global quotients of quasiprojective varieties, so $\mathrm{Perf}(\mathfrak{X})$ are just the objects of $\mathrm{QC}(\mathfrak{X})$ which are equivalent to a complex of vector bundles.

Now let $\mathfrak{X} = X/G$ as in the rest of this paper. Assume we have a map $f : \mathfrak{X} \to B$ where B is a quasiprojective scheme. The restriction $i^* : \mathrm{Perf}(\mathfrak{X}) \to \mathrm{Perf}(\mathfrak{X}^{ss})$ is a dg-$\otimes$ functor, and in particular it is a functor of module categories over the monoidal dg-category $\mathrm{Perf}(B)^{\otimes}$.

The subcategory $\mathbf{G}_q$ used to construct the splitting in Theorem 1.2.1 is defined using conditions on the weights of various 1PS's of the isotropy groups of $\mathfrak{X}$, so tensoring by a vector bundle f^*V from B preserves the subcategory $\mathbf{G}_q$. It follows that the splitting constructed in Theorem 1.2.1 is a splitting as modules over $\mathrm{Perf}(B)$. Thus for any point $b \in B$ we have a split surjection

$$\mathrm{Fun}_{\mathrm{Perf}(B)}\left(\mathrm{Perf}(\{b\}), \mathrm{Perf}(\mathfrak{X})\right) \underset{i^*}{\overset{\longleftarrow}{\longrightarrow}} \mathrm{Fun}_{\mathrm{Perf}(B)}\left(\mathrm{Perf}(\{b\}), \mathrm{Perf}(\mathfrak{X}^{ss})\right)$$

Using Morita theory, both functor categories correspond to full subcategories of $\mathrm{QC}((\bullet)_b)$, where $(\bullet)_b$ denotes the derived fiber $(\bullet) \times_B^L \{b\}$. Explicitly, $\mathrm{Fun}_{\mathrm{Perf}(B)}\left(\mathrm{Perf}(\{b\}), \mathrm{Perf}(\mathfrak{X})\right)$ is equivalent to the full dg-subcategory of $\mathrm{QC}((\mathfrak{X})_b)$ consisting of complexes of sheaves whose pushforward to $\mathfrak{X}$ is perfect. Because $\mathfrak{X}$ is smooth, and $\mathcal{O}_{(\mathfrak{X})_b}$ is coherent over $\mathcal{O}_{\mathfrak{X}}$, this is precisely the derived category of coherent sheaves $\mathrm{D}^b(\mathrm{Coh}((\mathfrak{X})_b))$. The same analysis applied to the tensor product $\mathrm{Perf}(\{b\}) \otimes_{\mathrm{Perf}(B)} \mathrm{Perf}(\mathfrak{X})$ yields a splitting for the category of perfect complexes.

Corollary 2.3.5. *Given a map $f : \mathfrak{X} \to B$ and a point $b \in B$, the splitting of Theorem 1.2.1 induces splittings of the natural restriction functors*

$$\mathrm{D}^b(\mathrm{Coh}((\mathfrak{X})_b)) \underset{i^*}{\overset{\longleftarrow}{\longrightarrow}} \mathrm{D}^b(\mathrm{Coh}((\mathfrak{X}^{ss})_b))$$

$$\mathrm{Perf}((\mathfrak{X})_b) \underset{i^*}{\overset{\longleftarrow}{\longrightarrow}} \mathrm{Perf}((\mathfrak{X}^{ss})_b)$$

In the particular case of a complete intersection one has $B = \mathbb{A}^r$, $b = 0 \in B$, and the derived fiber agrees with the non-derived fiber.

As a special case of Corollary 2.3.5, one obtains equivalences of categories of matrix factorizations in the form of derived categories of singularities. Namely, if $W : \mathfrak{X} \to \mathbb{C}$ is

a function, a "potential" in the language of mirror symmetry, then the category of matrix factorizations corresponding to W is

$$\text{MF}(\mathfrak{X}, W) \simeq \text{D}^b_{sing}(W^{-1}(0)) = \text{D}^b(\text{Coh}(W^{-1}(0)))/\text{Perf}(W^{-1}(0))$$

From Corollary 2.3.5 the restriction functor $\text{MF}(\mathfrak{X}, W) \to \text{MF}(\mathfrak{X}, W)$ splits. In particular, if two GIT quotients $\text{Perf}(\mathfrak{X}^{ss}(\mathcal{L}_1))$ and $\text{Perf}(\mathfrak{X}^{ss}(\mathcal{L}_2))$ can be identified with the same subcategory of $\text{Perf}(\mathfrak{X})$ as in Proposition 2.2.2, then the corresponding subcategories of matrix factorizations are equivalent

$$\text{MF}(\mathfrak{X}^{ss}(\mathcal{L}_1), W|_{\mathfrak{X}^{ss}(\mathcal{L}_1)}) \simeq \text{MF}(\mathfrak{X}^{ss}(\mathcal{L}_2), W|_{\mathfrak{X}^{ss}(\mathcal{L}_2)})$$

Corollary 2.3.5 also applies to the context of hyperkähler reduction. Let T be a torus, or any group whose connected component is a torus, and consider a Hamiltonian action of T on a hyperkähler variety X with algebraic moment map $\mu : X/T \to \mathfrak{t}^\vee$. One forms the hyperkähler quotient by choosing a linearization on X/T and defining $X////T = \mu^{-1}(0) \cap \mathfrak{X}^{ss}$. Thus we are in the setting of Corollary 2.3.5.

Corollary 2.3.6. *Let T be an extension of a finite group by a torus. Let T act on a hyperkähler variety X with algebraic moment map $\mu : X \to \mathfrak{t}^\vee$. Then the restriction functors*

$$\text{D}(\text{Coh}(\mu^{-1}(0)/T)) \to \text{D}(\text{Coh}(\mu^{-1}(0)^{ss}/T))$$
$$\text{Perf}(\text{Coh}(\mu^{-1}(0)/T)) \to \text{Perf}(\text{Coh}(\mu^{-1}(0)^{ss}/T))$$

both split.

This splitting does not give as direct a relationship between $\text{D}^b(X/T)$ and $\text{D}^b(X////T)$ as Theorem 2.0.3 does for the usual GIT quotient, but it is enough for some applications, for instance

Corollary 2.3.7. *Let X be a projective-over-affine hyperkähler variety with a Hamiltonian action of a torus T. Then the hyperkähler quotients with respect to any two generic linearization $\mathcal{L}_1, \mathcal{L}_2$ are derived equivalent.*

Proof. By Corollary 2.2.8 all $\mathfrak{X}^{ss}(\mathcal{L})$ for generic $\mathcal{L}$ will be derived equivalent. In particular there is a finite sequence of wall crossings $\text{Perf}(\mathfrak{X}^{ss}(\mathcal{L}_+)) \to \text{Perf}(\mathfrak{X}^{ss}(\mathcal{L}_0)) \leftarrow \text{Perf}(\mathfrak{X}^{ss}(\mathcal{L}_-))$ identifying each GIT quotient with the same subcategory. By Corollary 2.3.6 these splittings descend to $\mu^{-1}(0)$, giving equivalences of both $\text{D}^b(\text{Coh}(\bullet))$ and $\text{Perf}(\bullet)$ for the hyperkähler reductions. $\square$

Chapter 3

Autoequivalences of derived categories

3.1 Derived Kirwan surjectivity

In this section we fix our notation and recall the theory of derived Kirwan surjectivity developed in Chapter 2. We also introduce the category $\mathcal{C}_w$ and its semiorthogonal decompositions, which will be used throughout this paper.

We consider a smooth projective-over-affine variety X over an algebraically closed field k of characteristic 0, and we consider a reductive group G acting on X. Given a G-ample equivariant line bundle L, geometric invariant theory defines an open semistable locus $X^{ss} \subset X$. After choosing an invariant inner product on the cocharacter lattice of G, the Hilbert-Mumford numerical criterion produces a special stratification of the unstable locus by locally closed G-equivariant subvarieties $X^{us} = \bigcup_i S_i$ called Kirwan-Ness (KN) strata. The indices are ordered so that the closure of S_i lies in $\bigcup_{j \geq i} S_j$.

Each stratum comes with a distinguished one-parameter subgroup $\lambda_i : \mathbb{C}^* \to G$ and S_i fits into the diagram

$$Z_i \underset{\pi_i}{\overset{\sigma_i}{\rightleftarrows}} Y_i \subset S_i := G \cdot Y_i \xrightarrow{j_i} X \ , \tag{3.1}$$

where Z_i is an open subvariety of $X^{\lambda_i \text{ fixed}}$, and

$$Y_i = \left\{ x \in X - \bigcup_{j > i} S_j \ \middle| \ \lim_{t \to 0} \lambda_i(t) \cdot x \in Z_i \right\} .$$

σ_i and j_i are the inclusions and π_i is taking the limit under the flow of λ_i as $t \to 0$. We denote the immersion $Z_i \to X$ by σ_i as well. Throughout this paper, the spaces Z, Y, S and morphisms σ, π, j will refer to diagram 3.1.

In addition, λ_i determines the parabolic subgroup P_i of elements of G which have a limit under conjugation by λ_i, and the centralizer of λ_i, $L_i \subset P_i \subset G$, is a Levi component for

P_i. One key property of the KN stratum is that $S_i = G \times_{P_i} Y_i$, so that G equivariant quasicoherent sheaves on S_i are equivalent to P_i-equivariant quasicoherent sheaves on Y_i. When G is abelian, then $G = P_i = L_i$, and $Y_i = S_i$ is already G invariant, so the story simplifies quite a bit.

Theorem 3.1.1 (derived Kirwan surjectivity). *Let η_i be the weight of $\det(N^\vee_{S_i} X)|_{Z_i}$ with respect to λ_i. Choose an integer w_i for each stratum and define the full subcategory*

$$\mathbf{G}_w := \{F^\bullet \in \mathrm{D}^b(X/G) | \forall i, \sigma_i^* F^\bullet \text{ has weights in } [w_i, w_i + \eta_i) \text{ w.r.t. } \lambda_i\}.$$

Then the restriction functor $r : \mathbf{G}_w \to \mathrm{D}^b(X^{ss}/G)$ is an equivalence of dg-categories.

The weight condition on $\sigma_i^* F^\bullet$ is called the *grade restriction rule* and the interval $[w_i, w_i + \eta_i)$ is the *grade restriction window*. The theorem follows immediately from the corresponding statement for a single closed KN stratum by considering the chain of open subsets $X^{ss} \subset X_n \subset \cdots \subset X_0 \subset X$ where $X_i = X_{i-1} \setminus S_i$.

The full version of the theorem also describes the kernel of the restriction functor $r : \mathrm{D}^b(X/G) \to \mathrm{D}^b(X^{ss}/G)$. For a single stratum S we define the full subcategory

$$\mathcal{A}_w := \left\{ F^\bullet \in \mathrm{D}^b(X/G) \,\middle|\, \begin{array}{c} \mathcal{H}^*(\sigma^* F^\bullet) \text{ has weights in } [w, w + \eta] \text{ w.r.t. } \lambda \\ \mathcal{H}^*(F^\bullet) \text{ supported on } S \end{array} \right\}$$

we have an infinite semiorthogonal decomposition

$$\mathrm{D}^b(X/G) = \langle \ldots, \mathcal{A}_{w-1}, \mathcal{A}_w, \mathbf{G}_w, \mathcal{A}_{w+1}, \ldots \rangle$$

This means that the subcategories are disjoint, semiorthogonal (there are no $R\,\mathrm{Hom}$'s pointing to the left), and that every object has a functorial filtration whose associated graded pieces lie in these subcategories (ordered from right to left).[1] These categories are not obviously disjoint, but it is a consequence of the theory that no non-zero object supported on S can satisfy the grade restriction rule defining $\mathbf{G}_w$.

Let $\mathrm{D}^b(Z/L)_w \subset \mathrm{D}^b(Z/L)$ denote the full subcategory which has weight w with respect to λ, and let $(\bullet)_w$ be the exact functor taking the summand with λ weight w of a coherent sheaf on Z/L.

Lemma 3.1.2 (see Chapter 2). *The functor $\iota_w : \mathrm{D}^b(Z/L)_w \to \mathcal{A}_w$ is an equivalence, and its inverse can be described either as $(\sigma^* F^\bullet)_w$ or as $(\sigma^* F^\bullet)_{w+\eta} \otimes \det(N_S X)$.*

Using the equivalences ι_w and r we can rewrite the main semiorthogonal decomposition

$$\mathrm{D}^b(X/G) = \langle \ldots, \mathrm{D}^b(Z/L)_w, \mathrm{D}^b(X^{ss}/G)_w, \mathrm{D}^b(Z/L)_{w+1}, \ldots \rangle \tag{3.2}$$

---

[1] The early definitions of semiorthogonal decompositions required the left and right factors to be admissible, but this requirement is not relevant to our analysis. The notion we use is sometimes referred to as a *weak semiorthogonal decomposition*

When there are multiple strata, one can inductively construct a nested semiorthogonal decomposition using $D^b(X_{i-1}/G) = \langle \ldots, \mathcal{A}_w^i, D^b(X_i/G), \mathcal{A}_{w+1}^i, \ldots \rangle$.

In this paper, we will consider the full subcategory

$$\mathcal{C}_w := \{F^\bullet \in D^b(X/G) \,|\, \mathcal{H}^*(\sigma^* F^\bullet) \text{ has weights in } [w, w+\eta] \text{ w.r.t. } \lambda\} \subset D^b(X/G)$$

If we instead use the grade restriction window $[w, w+\eta)$, then we get the subcategory $\mathbf{G}_w \subset \mathcal{C}_w$. The main theorem of Chapter 2 implies that we have two semiorthogonal decompositions

$$\mathcal{C}_w = \langle \mathbf{G}_w, \mathcal{A}_w \rangle = \langle \mathcal{A}_w, \mathbf{G}_{w+1} \rangle. \tag{3.3}$$

We regard restriction to X^{ss} as a functor $r : \mathcal{C}_w \to D^b(X^{ss}/G)$. The subcategory $\mathcal{A}_w$ is the kernel of r, but is described more explicitly as the essential image of the fully faithful functor $\iota_w : D^b(Z/L)_w \to \mathcal{C}_w$ as discussed above.

Lemma 3.1.3. *The left and right adjoints of $\iota_w : D^b(Z/L)_w \to \mathcal{C}_w$ are $\iota_w^L(F^\bullet) = (\sigma^* F^\bullet)_w$ and $\iota_w^R(F^\bullet) = (\sigma^* F^\bullet)_{w+\eta} \otimes \det N_S X|_Z$.*

Proof. Letting $G^\bullet \in D^b(Z/L)_w$, we have $\text{Hom}_{X/G}(F^\bullet, \iota_w G^\bullet) \simeq \text{Hom}_{S/G}(j^* F^\bullet, \pi^* F^\bullet)$ and $\pi^* G^\bullet \in D^b(S/G)_w$. In Chapter 2, we show that $D^b(S/G)$ admits a baric decomposition, and using the baric truncation functors

$$\text{Hom}_{S/G}(j^* F^\bullet, \pi^* G^\bullet) \simeq \text{Hom}_{S/G}(\beta_{<w+1} j^* F^\bullet, \pi^* G^\bullet)$$
$$\simeq \text{Hom}_{Z/L}(\sigma^* \beta_{<w+1} j^* F^\bullet, G^\bullet)$$

Where the last equality uses the fact that $\pi^* : D^b(Z/L)_w \to D^b(S/G)_w$ is an equivalence with inverse σ^*. Finally, we have $\sigma^* \beta_{<w+1} j^* F^\bullet = (\sigma^* j^* F^\bullet)_w = (\sigma^* F^\bullet)_w$.

The argument for ι^R is analogous, but it starts with the adjunction for $j^! F^\bullet \simeq j^!(\mathcal{O}_X) \otimes j^* F^\bullet$, $\text{Hom}_{X/G}(\iota_w G^\bullet, F^\bullet) \simeq \text{Hom}_{S/G}(\pi^* G^\bullet, \det(N_S X) \otimes j^* F^\bullet)$. □

Lemma 3.1.4. *The functor $r : \mathcal{C}_w \to D^b(X^{ss}/G)$ has right and left adjoints given respectively by $r^R : D^b(X^{ss}/G) \simeq \mathbf{G}_w \subset \mathcal{C}_w$ and $r^L : D^b(X^{ss}/G) \simeq \mathbf{G}_{w+1} \subset \mathcal{C}_{w+1}$.*

Now because we have two semiorthogonal decompositions in Equation (3.3), there is a left mutation [10] equivalence functor $\mathbb{L}_{\mathcal{A}_w} : \mathbf{G}_{w+1} \to \mathbf{G}_w$ defined by the functorial exact triangle

$$\iota_w \iota_w^R(F^\bullet) \to F^\bullet \to \mathbb{L}_{\mathcal{A}_w} F^\bullet \dashrightarrow \tag{3.4}$$

Note that restricting to X^{ss}/G, this triangle gives an equivalence $r(F^\bullet) \simeq r(\mathbb{L}_{\mathcal{A}_w} F^\bullet)$. Thus this mutation implements the 'window shift' functor

$$\begin{array}{ccc} \mathbf{G}_{w+1} & \xrightarrow{\quad \mathbb{L}_{\mathcal{A}_w} \quad} & \mathbf{G}_w \\ & r \searrow \quad \nearrow r^{-1}=r^R & \\ & D^b(X^{ss}/G) & \end{array} \tag{3.5}$$

meaning that $\mathbb{L}_{\mathcal{A}_w} F^\bullet$ is the unique object of $\mathbf{G}_w$ restricting to the same object as $F^\bullet$ in $D^b(X^{ss}/G)$.

The category $\mathrm{D}^b(Z/L)_w$

We will provide a more geometric description of the subcategory $\mathrm{D}^b(Z/L)_w$. We define the quotient group $L' = L/\lambda(\mathbb{C}^*)$. Because $\lambda(\mathbb{C}^*)$ acts trivially on Z, the group L' acts naturally on Z as well.

Lemma 3.1.5. *The pullback functor gives an equivalence* $\mathrm{D}^b(Z/L') \xrightarrow{\simeq} \mathrm{D}^b(Z/L)_0$.

Proof. This follows from the analogous statement for quasicoherent sheaves, which is a consequence of descent. □

The categories $\mathrm{D}^b(Z/L)_w$ can also be related to $\mathrm{D}^b(Z/L')$. If $\lambda : \mathbb{C}^* \to G$ has the kernel $\mu_n \subset \mathbb{C}^*$, then $\mathrm{D}^b(Z/L)_w = \emptyset$ unless $w \equiv 0 \mod n$. In this case we replace λ with an injective λ' such that $\lambda = (\lambda')^n$ and $\mathrm{D}^b(Z/L)_{[\lambda'=w]} = \mathrm{D}^b(Z/L)_{[\lambda=nw]}$. Thus we will assume that λ is injective.

Lemma 3.1.6. *Let* $\mathcal{L} \in \mathrm{D}^b(Z/L)_w$ *be an invertible sheaf. Then pullback followed by* $\mathcal{L} \otimes \bullet$ *gives an equivalence* $\mathrm{D}^b(Z/L') \xrightarrow{\simeq} \mathrm{D}^b(Z/L)_w$.

For instance, if there is a character $\chi : L \to \mathbb{C}^*$ such that $\chi \circ \lambda$ is the identity on $\mathbb{C}^*$, then χ induces an invertible sheaf on Z/L with weight 1, so Lemma 3.1.6 applies. If G is abelian then such a character always exists.

Remark 3.1.7. This criterion is not always met, for example when $Z/L = */GL_n$ and λ is the central $\mathbb{C}^*$. What is true in general is that $Z/L \to Z/L'$ is a $\mathbb{C}^*$ gerbe, and the category $\mathrm{D}^b(Z/L)_1$ is by definition the derived category of coherent sheaves on Z/L' twisted by that gerbe. The data of an invertible sheaf $\mathcal{L} \in \mathrm{D}^b(Z/L)_1$ is equivalent to a trivialization of this gerbe $Z/L \xrightarrow{\simeq} Z/L' \times */\mathbb{C}^*$.

3.2 Window shift autoequivalences, mutations, and spherical functors

In this paper we study balanced GIT wall crossings. Let L_0 be a G-ample line bundle such that the strictly semistable locus $X^{sss} = X^{ss} - X^s$ is nonempty, and let L' be another G-equivariant line bundle. We assume that $X^{ss} = X^s$ for the linearizations $L_\pm = L_0 \pm \epsilon L'$ for sufficiently small ϵ, and we denote $X_\pm^{ss} = X^{ss}(L_\pm)$. In this case, $X^{ss}(L_0) - X^{ss}(L_\pm)$ is a union of KN strata for the linearization $L_\pm$, and we will say that the wall crossing is *balanced* if the strata S_i^+ and S_i^- lying in $X^{ss}(L_0)$ are indexed by the same set, with $Z_i^+ = Z_i^-$ and $\lambda_i^+ = (\lambda_i^-)^{-1}$. This is slightly more general than the notion of a truly faithful wall crossing in [16]. In particular, if G is abelian and there is some linearization with a stable point, then all codimension one wall crossings are balanced.

In this case we will replace X with $X^{ss}(L_0)$ so that these are the only strata we need to consider. In fact we will mostly consider a balanced wall crossing where only a single stratum

flips – the analysis for multiple strata is analogous. We will drop the superscript from $Z^\pm$, but retain superscripts for the distinct subcategories $\mathcal{A}_w^\pm$. Objects in $\mathcal{A}_w^\pm$ are supported on $S^\pm$, which are distinct because S^+ consist of orbits of points flowing to Z under λ^+, whereas S^- consists of orbits of points flowing to Z under λ^-. When there is ambiguity as to which $\lambda^\pm$ we are referring to, we will include it in the notation, i.e. $\mathrm{D}^b(Z/L)_{[\lambda^+=w]}$.

Observation 3.2.1. *If $\omega_X|_Z$ has weight 0 with respect to $\lambda^\pm$, then $\eta^+ = \eta^-$ (see Chapter 2). This implies that $\mathcal{C}_w^+ = \mathcal{C}_{w'}^-$, $\mathbf{G}_w^+ = \mathbf{G}_{w'+1}^-$, and $\mathbf{G}_{w+1}^+ = \mathbf{G}_{w'}^-$, where $w' = -\eta - w$.*

This observation, combined with derived Kirwan surjectivity, implies that the restriction functors $r_\pm : \mathbf{G}_w^- \to \mathrm{D}^b(X_\pm^{ss}/G)$ are both equivalences. In particular $\psi_w := r_+ r_-^{-1} : \mathrm{D}^b(X_-^{ss}/G) \to \mathrm{D}^b(X_+^{ss}/G)$ is a derived equivalence between the two GIT quotients. Due to the dependence on the choice of w, we can define the *window shift autoequivalence* $\Phi_w := \psi_{w+1}^{-1}\psi_w$ of $\mathrm{D}^b(X_-^{ss}/G)$.

Lemma 3.2.2. *If there is an invertible sheaf $\mathcal{L} \in \mathrm{D}^b(X/G)$ such that $\mathcal{L}|_Z$ has weight w w.r.t. λ^+, then $\Phi_w = (\mathcal{L}^\vee \otimes)\Phi_0(\mathcal{L}\otimes)$. In particular, if $\mathcal{L}$ has weight 1, then $\psi_v^{-1}\psi_w$ lies in the subgroup of $\mathrm{Aut}\,\mathrm{D}^b(X_-^{ss}/G)$ generated by Φ_0 and $\mathcal{L}\otimes$.*

Proof. The commutativity of the following diagram implies that $(\mathcal{L}^\vee\otimes)\psi_k(\mathcal{L}\otimes) = \psi_{k+w}$

$$
\begin{array}{ccccc}
\mathrm{D}^b(X_-^{ss}/G) & \longleftarrow & \mathbf{G}_{k+w}^- & \longrightarrow & \mathrm{D}^b(X_+^{ss}/G) \\
\downarrow\otimes\mathcal{L} & & \downarrow\otimes\mathcal{L} & & \downarrow\otimes\mathcal{L} \\
\mathrm{D}^b(X_-^{ss}/G) & \longleftarrow & \mathbf{G}_k^- & \longrightarrow & \mathrm{D}^b(X_+^{ss}/G)
\end{array}
$$

$\square$

Here we are able to give a fairly explicit description of Φ_w from the perspective of mutation. Note that because $\mathbf{G}_{w+1}^+ = \mathbf{G}_{w'}^-$, the inverse of the restriction $\mathbf{G}_{w+1}^+ \to \mathrm{D}^b(X_-^{ss}/G)$ is the right adjoint r^R, whereas the inverse of the restriction $\mathbf{G}_{w'+1}^- \to \mathrm{D}^b(X_-^{ss}/G)$ was the left adjoint r_-^L by Lemma 3.1.4.

Proposition 3.2.3. *The autoequivalence Φ_w of $\mathrm{D}^b(X_-^{ss}/G)$ described by the following non-commuting diagram, i.e. $\Phi_w = r_- \circ \mathbb{L}_{\mathcal{A}_w^+} \circ r_-^R$.*

$$
\begin{array}{ccc}
\mathbf{G}_{w+1}^+ & \xrightarrow{\;\mathbb{L}_{\mathcal{A}_w^+}\;} & \mathbf{G}_w^+ \\
& & \\
{\scriptstyle r_-^R=r_-^{-1}}\nwarrow & & \swarrow{\scriptstyle r_-} \\
& \mathrm{D}^b(X_-^{ss}/G) &
\end{array}
$$

Proof. This is essentially rewriting Diagram (3.5) using Observation 3.2.1 and its consequences. $\square$

Window shifts are spherical twists

Next we show that the window shift autoequivalence Φ_w is a twist corresponding to a spherical functor.

This generalizes a spherical object [40], which is equivalent to a spherical functor $D^b(k - vect) \to \mathcal{B}$. By describing window shifts both in terms of mutations and as spherical twists, we show why these two operations have the "same formula" in this setting. In fact, in the next section we show that spherical twists can always be described by mutations.

Let $E := Y^+ \cap X_-^{ss}$, it is P_+-equivariant, and let $\tilde{E} = S^+ \cap X_-^{ss} = G \cdot E$. Then we consider the diagram

$$E/P_+ = \tilde{E}/G \xrightarrow{\ j\ } X_-^{ss}/G \qquad (3.6)$$
$$\downarrow \pi$$
$$Z/L$$

This is a stacky form of the EZ-diagram used to construct autoequivalences in [24]. We define the transgression along this diagram $f_w = j_* \pi^* : D^b(Z/L)_w \to D^b(X_-^{ss}/G)$. Note that we have used the same letters π and j for the restriction of these maps to the open substack $E/P_+ \subset Y^+/P_+$, but we denote this transgression f_w to avoid confusion.

Proposition 3.2.4. *The window shift functor Φ_w is defined for $F^\bullet \in D^b(X_-^{ss}/G)$ by the functorial mapping cone*

$$f_w f_w^R(F^\bullet) \to F^\bullet \to \Phi_w(F^\bullet) \dashrightarrow$$

Proof. This essentially follows from abstract nonsense. By the definition of left mutation (3.4), and by the fact that $r_- r_-^R = \mathrm{id}_{D^b(X_-^{ss}/G)}$, it follows that the window shift autoequivalence is defined by the cone

$$r_- \iota_w^+ (\iota_w^+)^R r_-^R(F^\bullet) \to F^\bullet \to \Phi_w(F^\bullet) \dashrightarrow$$

Furthermore, by construction we have $f_w = r_- \iota_w^+$, so $f_w^R \simeq (\iota_w^+)^R r_-^R$. The claim follows. $\square$

Consider the case where $D^b(Z/L)_w$ is generated by a single exceptional object E. The object $E^+ := \iota_w^+ E \in \mathcal{A}_w^+$ is exceptional, and the left mutation functor (3.4) acts on $F^\bullet \in \mathbf{G}_{w+1}^+$ by

$$\mathrm{Hom}_{X/G}(E^+, F^\bullet) \otimes E^+ \to F^\bullet \to \mathbb{L}_{\mathcal{A}_w^+}(F^\bullet) \dashrightarrow$$

To emphasize the dependence on E^+ we write $\mathbb{L}_{E^+} := \mathbb{L}_{\mathcal{A}_w^+}$. As we have shown, $\mathbb{L}_{E^+}(F^\bullet)|_{X_-^{ss}}$ is the window shift autoequivalence $\Phi_w(F^\bullet|_{X_-^{ss}})$. If we restrict the defining exact triangle for $\mathbb{L}_{E^+}(F^\bullet)$ to X_-^{ss} we get

$$\mathrm{Hom}_{X/G}(E^+, F^\bullet) \otimes E^+|_{X_-^{ss}} \to F^\bullet|_{X_-^{ss}} \to \Phi_w(F^\bullet|_{X_-^{ss}}) \dashrightarrow$$

Define the object $S = E^+|_{X_-^{ss}} \in D^b(X_-^{ss}/G)$. The content of Proposition 3.2.4 is that the canonical map $\mathrm{Hom}_{X/G}(E^+, F^\bullet) \to \mathrm{Hom}_{X_-^{ss}/G}(S, F^\bullet|_{X_-^{ss}})$ is an isomorphism, so that

$\Phi_w = \mathbb{L}_{E^+}|_{X^{ss}_-}$ is the spherical twist T_S by the object S. This can be verified more directly using the following

Lemma 3.2.5. *For $F^\bullet, G^\bullet \in \mathcal{C}^-_{w'}$,*

$$R\Gamma_{S^-}\underline{\mathrm{Hom}}_{X/G}(F^\bullet, G^\bullet) \simeq \mathrm{Hom}_{Z/L}((\sigma^* F^\bullet)_{w'}, (\sigma^* G^\bullet \otimes \kappa_-)_{w'}).$$

Equivalently we have an exact triangle

$$\mathrm{Hom}_{Z/L}((\sigma^* F^\bullet)_{w'}, (\sigma^* G^\bullet \otimes \kappa_-)_{w'}) \to \mathrm{Hom}_{X/G}(F^\bullet, G^\bullet) \to \mathrm{Hom}_{X^{ss}_-/G}(F^\bullet|_{X^{ss}_-}, G^\bullet|_{X^{ss}_-}) \dashrightarrow$$

Proof. Let $\mathcal{C} = \langle \mathcal{A}, \mathcal{B} \rangle$ be a semiorthogonal decomposition of a pretriangulated dg-category, and let $\iota_\mathcal{A}$ and $\iota_\mathcal{B}$ be the inclusions. Applying $\mathrm{Hom}(F, \bullet)$ to the canonical exact triangle $\iota_\mathcal{B}\iota_\mathcal{B}^R G \to G \to \iota_\mathcal{A}\iota_\mathcal{A}^L G \dashrightarrow$ gives the exact triangle

$$\mathrm{Hom}_\mathcal{B}(\iota_\mathcal{B}^L F, \iota_\mathcal{B}^R G) \to \mathrm{Hom}_\mathcal{C}(F, G) \to \mathrm{Hom}_\mathcal{A}(\iota_\mathcal{A}^L F, \iota_\mathcal{A}^L G) \dashrightarrow$$

assuming $\mathcal{B}$ is left admissible. The lemma is just a special case of this fact for the semiorthogonal decomposition $\mathcal{C}^-_{w'} = \langle \mathbf{G}^-_{w'}, \mathcal{A}^-_w \rangle$, using the description of the adjoint functors in Lemma 3.1.3. $\square$

In summary, we have given a geometric explanation for the identical formulas for L_{E^+} and T_S: the spherical twist is the restriction to the GIT quotient of a left mutation in the equivariant derived category.

Example 3.2.6. Let X be the crepant resolution of the A_n singularity. It is the 2 dimensional toric variety whose fan in $\mathbb{Z}^2$ has rays spanned by $(1, i)$, for $i = 0, \ldots, n+1$, and which has a 2-cone for each pair of adjacent rays. Removing one of the interior rays corresponds to blowing down a rational curve $\mathbb{P}^1 \subset X$ to an orbifold point with $\mathbb{Z}/2\mathbb{Z}$ stabilizer. This birational transformation can be described by a VGIT in which $Z/L' \simeq *$. The spherical objects corresponding to the window shift autoequivalences are $\mathcal{O}_{\mathbb{P}^1}(m)$.

Remark 3.2.7. Horja [24] introduced the notion of an EZ-spherical object $F^\bullet \in \mathrm{D}^b(E/P_+)$ for a diagram $Z/L' \xleftarrow{q} E/P_+ \xrightarrow{j} X^{ss}_-/G$ – his notion is equivalent to the functor $j_*(F^\bullet \otimes q^*(\bullet))$ being spherical 3.2.10. Proposition 3.2.4 amounts to the fact that $\mathcal{O}_{E/P_+}$ is an EZ-spherical object for this diagram. By the projection formula $q^*\mathcal{L}$ is EZ-spherical for any invertible sheaf $\mathcal{L}$ on Z. The twist functors corresponding to different choices of $\mathcal{L}$ are equivalent.

Remark 3.2.8. Our results also extend results in [39, 17]. The first work formally introduced grade restriction windows to the mathematics literature and showed that window shift equivalences are given by spherical functors in the context of gauged Landau-Ginzburg models. (See subsection 3.3.) In the second work, the authors study window shift autoequivalences associated to Grassmannian flops, using representation theory of $GL(n)$ to compute with homogeneous bundles.

All spherical twists are mutations

We have shown that the window shift Φ_w is a twist $\mathrm{Cone}(f_w f_w^R \to \mathrm{id})$ corresponding to a functor $f_w : \mathrm{D}^b(Z/L)_w \to \mathrm{D}^b(X_-^{ss}/G)$. Now we show that this f_w is *spherical* [2], and in fact any autoequivalence of a dg-category arising from mutations as Φ_w does is a twist by a spherical functor. Conversely, any spherical functor between dg-categories with a compact generator arises from mutations.

Using the equalities of Observation 3.2.1, we have the following semiorthogonal decompositions of $\mathcal{C}_w^+ = \mathcal{C}_{w'}^-$, all coming from (3.3):

$$\langle \mathcal{A}_w^+, \mathbf{G}_{w+1}^+ \rangle \xrightarrow{L_{\mathcal{A}_w^+}} \langle \mathbf{G}_w^+, \mathcal{A}_w^+ \rangle \xrightarrow{L_{\mathbf{G}_w^+}} \langle \mathcal{A}_{w'}^-, \mathbf{G}_w^+ \rangle \xrightarrow{L_{\mathcal{A}_{w'}^-}} \langle \mathbf{G}_{w+1}^+, \mathcal{A}_{w'}^- \rangle \qquad (3.7)$$

$$\underbrace{\phantom{\langle \mathcal{A}_w^+, \mathbf{G}_{w+1}^+ \rangle \xrightarrow{L_{\mathcal{A}_w^+}} \langle \mathbf{G}_w^+, \mathcal{A}_w^+ \rangle \xrightarrow{L_{\mathbf{G}_w^+}} \langle \mathcal{A}_{w'}^-, \mathbf{G}_w^+ \rangle \xrightarrow{L_{\mathcal{A}_{w'}^-}} \langle \mathbf{G}_{w+1}^+, \mathcal{A}_{w'}^- \rangle}}_{L_{\mathbf{G}_{w+1}^+}}$$

where we conclude a fortiori that each semiorthogonal decomposition arises from the previous one by left mutation. Each mutation gives an equivalence between the corresponding factors in each semiorthogonal decomposition, and the autoequivalence Φ_w, interpreted as an autoequivalence of $\mathbf{G}_w^+$, is obtained by following the sequence of mutations.

Remark 3.2.9. The braid group B_n on n strands acts by mutations on the set of semiorthogonal decompositions of length n (with admissible factors). The fact that the first and last semiorthogonal decompositions in 3.7 are equal means that this semiorthogonal decomposition has a nontrivial stabilizer in B_2 under its action on length two semiorthogonal decompositions of $\mathcal{C}_w$. We would like to point out that this may be a way to produce interesting autoequivalences more generally. Let $\mathcal{G}_n$ be the groupoid whose objects are strong semiorthogonal decompositions (i.e. all factors are admissible subcategories) of length n and whose morphisms are braids that take one to another by mutation. Let $e = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle$ be a semiorthogonal decomposition in the category of interest. Then $\mathrm{Aut}_{\mathcal{G}_n}(e)$ is a subgroup of B_n and for each i there is a representation

$$\mathrm{Aut}(e) \to \mathrm{Aut}(\mathcal{A}_i),$$

the group of exact autoequivalences of $\mathcal{A}_i$ up to isomorphism of functors. By construction the autoequivalences in the image of this representation are compositions of mutations. In the situation above $B_2 = \mathbb{Z}$ and $\mathrm{Aut}(\langle \mathbf{G}_w^+, \mathcal{A}_w^+ \rangle) \subset B_2$ is the index four subgroup.

Let us recall the definition of spherical functor.

Definition 3.2.10 ([2]). A dg-functor $S : \mathcal{A} \to \mathcal{B}$ of pre-triangulated dg-categories is *spherical* if it admits right and left adjoints R and L such that

1. the cone F_S of $\mathrm{id} \to RS$ is an autoequivalence of $\mathcal{A}$, and

2. the natural morphism $R \to F_S L$ induced by $R \to RSL$ is an isomorphism of functors.

If S is spherical, the cone T_S on the morphism $SR \to \mathrm{id}$ is an autoequivalence called the *twist* corresponding to S.

Suppose that $\mathcal{C}$ is a pre-triangulated dg category admitting semiorthogonal decompositions

$$\mathcal{C} = \langle \mathcal{A}, \mathcal{B} \rangle = \langle \mathcal{B}, \mathcal{A}' \rangle = \langle \mathcal{A}', \mathcal{B}' \rangle = \langle \mathcal{B}', \mathcal{A} \rangle.$$

Denote by $i_\bullet$ the inclusion functors. Since $\mathcal{A}, \mathcal{B}, \mathcal{A}', \mathcal{B}'$ are admissible, $i_\bullet$ admits right and left adjoints $i_\bullet^R$ and $i_\bullet^L$, respectively. We can use these functors to describe the mutations

$$\mathbb{L}_\mathcal{A} = i_{\mathcal{B}'}^L i_\mathcal{B} : \mathcal{B} \to \mathcal{B}', \quad \mathbb{R}_\mathcal{A} = i_\mathcal{B}^R i_{\mathcal{B}'} : \mathcal{B}' \to \mathcal{B},$$

with analogous formulae for the other mutations.

Theorem 3.2.11. *The functor $S : \mathcal{A} \to \mathcal{B}$ given by $S = i_\mathcal{B}^L i_\mathcal{A}$ is spherical. Moreover, the spherical twist $T_S : \mathcal{B} \to \mathcal{B}$ is obtained as the mutation*

$$T_S \cong \mathbb{L}_{\mathcal{A}'} \circ \mathbb{L}_\mathcal{A}.$$

Proof. We must produce left and right adjoints for S, then check the two parts of the definition. Clearly the right adjoint to S is $R = i_\mathcal{A}^R i_\mathcal{B}$. In order to compute the left adjoint, we first apply $i_\mathcal{B}^L$ to the triangle

$$i_{\mathcal{B}'} i_{\mathcal{B}'}^R \to \mathrm{id}_\mathcal{C} \to i_{\mathcal{A}'} i_{\mathcal{A}'}^L \dashrightarrow .$$

Since $i_\mathcal{B}^L i_{\mathcal{A}'} = 0$ we see that the map

$$\mathbb{L}_{\mathcal{A}'} i_{\mathcal{B}'}^R = i_\mathcal{B}^L i_{\mathcal{B}'} i_\mathcal{B}^R \to i_\mathcal{B}^L$$

is an isomorphism. Using the fact that $\mathbb{L}_{\mathcal{A}'}$ and $\mathbb{R}_{\mathcal{A}'}$ are biadjoint, it follows that $L = i_\mathcal{A}^L i_{\mathcal{B}'} \mathbb{R}_{\mathcal{A}'}$.

To establish (1), we will express F_S in terms of mutations. Begin with the triangle,

$$\mathrm{id}_\mathcal{C} \to i_\mathcal{B} i_\mathcal{B}^L \to i_{\mathcal{A}'} i_{\mathcal{A}'}^R[1] \dashrightarrow . \tag{3.8}$$

Then apply $i_\mathcal{A}^R$ on the right and $i_\mathcal{A}$ on the left to get a triangle

$$\mathrm{id}_\mathcal{A} \to i_\mathcal{A}^R i_\mathcal{B} i_\mathcal{B}^L i_\mathcal{A} = RS \to i_\mathcal{A}^R i_{\mathcal{A}'} i_{\mathcal{A}'}^R i_\mathcal{A}[1] = \mathbb{R}_{\mathcal{B}'} \mathbb{R}_\mathcal{B}[1] \dashrightarrow .$$

Since $i_\mathcal{A}$ is fully faithful, the first map is the unit of the adjunction between S and R so we see that $F_S \cong \mathbb{R}_{\mathcal{B}'} \mathbb{R}_\mathcal{B}[1]$. Hence it is an equivalence. A very similar computation shows that $T_S \cong \mathbb{L}_{\mathcal{A}'} \mathbb{L}_\mathcal{A}$.

We now verify (2), that the composition $R \to RSL \to F_S L$ is an isomorphism. The map $R \to F_S L$ is the composite

$$R = i_\mathcal{A}^R i_\mathcal{B} \to (i_\mathcal{A}^R i_\mathcal{B})(i_\mathcal{B}^L i_{\mathcal{B}'} i_{\mathcal{B}'}^R i_\mathcal{A})(i_\mathcal{A}^L i_{\mathcal{B}'} i_{\mathcal{B}'}^R i_\mathcal{B}) \to$$
$$RSL = (i_\mathcal{A}^R i_\mathcal{B})(i_\mathcal{B}^L i_\mathcal{A})(i_\mathcal{A}^L i_{\mathcal{B}'} i_{\mathcal{B}'}^R i_\mathcal{B}) \to F_S L = i_\mathcal{A}^R i_{\mathcal{A}'} i_{\mathcal{A}'}^R i_\mathcal{A} i_\mathcal{A}^L i_{\mathcal{B}'} i_{\mathcal{B}'}^R i_\mathcal{B}$$

where the middle map comes from the isomorphism $i_{\mathcal{B}}^L i_{\mathcal{B}'} i_{\mathcal{B}'}^R i_{\mathcal{A}} \to S = i_{\mathcal{B}}^L i_{\mathcal{A}}$ that we discussed in the preceding paragraphs. The first map is obtained by applying $R\mathbb{L}_{\mathcal{A}'}$ and $\mathbb{R}_{\mathcal{A}'}$ to the left and right, respectively of the unit morphism $\mathrm{id}_{\mathcal{B}'} \to (i_{\mathcal{B}'}^R i_{\mathcal{A}})(i_{\mathcal{A}}^L i_{\mathcal{B}'})$. To get the last map one applies $i_{\mathcal{A}}^R$ and $i_{\mathcal{A}} i_{\mathcal{A}}^L i_{\mathcal{B}'} i_{\mathcal{B}'}^R i_{\mathcal{B}}$ to the left and right, respectively of the map $i_{\mathcal{B}} i_{\mathcal{B}}^L \to i_{\mathcal{A}'} i_{\mathcal{A}'}^R[1]$ from the triangle (3.8).

In order to understand the morphism $R \to RSL$, consider the commutative diagram

$$i_{\mathcal{B}}^L i_{\mathcal{B}'} \xrightarrow{\mathrm{id}\circ\epsilon} i_{\mathcal{B}}^L i_{\mathcal{B}'} i_{\mathcal{B}'}^R i_{\mathcal{B}'} \xrightarrow{\mathrm{id}\circ\epsilon\circ\mathrm{id}} i_{\mathcal{B}}^L i_{\mathcal{B}'} i_{\mathcal{B}'}^R i_{\mathcal{A}} i_{\mathcal{A}}^L i_{\mathcal{B}'}$$

$$= \qquad \downarrow \mathrm{id}\circ\eta\circ\mathrm{id} \qquad \downarrow \mathrm{id}\circ\eta\circ\mathrm{id}$$

$$i_{\mathcal{B}}^L i_{\mathcal{B}'} \xrightarrow{\mathrm{id}\circ\epsilon\circ\mathrm{id}} i_{\mathcal{B}}^L i_{\mathcal{A}} i_{\mathcal{A}}^L i_{\mathcal{B}'}$$

In this diagram, units and counits of adjunctions are denoted ϵ and η, respectively. The map $R \to RSL$ is obtained by applying $i_{\mathcal{A}}^R i_{\mathcal{B}}$ and $i_{\mathcal{B}'}^R i_{\mathcal{B}}$ on the left and right, respectively, to the clockwise composition from the upper left to the lower right. On the other hand the counterclockwise composite from the upper left to the lower right comes from the unit morphism $\mathrm{id}_{\mathcal{C}} \to i_{\mathcal{A}} i_{\mathcal{A}}^L$ by applying $i_{\mathcal{B}}^L$ and $i_{\mathcal{B}'}$ on the right and left, respectively. Therefore we get $R \to RSL$ by applying $i_{\mathcal{A}}^R i_{\mathcal{B}} i_{\mathcal{B}}^L$ and $i_{\mathcal{B}'} i_{\mathcal{B}'}^R i_{\mathcal{B}}$ to the left and right of this unit morphism, respectively.

Next, consider the commutative diagram

$$i_{\mathcal{B}} i_{\mathcal{B}}^L \xrightarrow{\mathrm{id}\circ\epsilon} i_{\mathcal{B}} i_{\mathcal{B}}^L i_{\mathcal{A}} i_{\mathcal{A}}^L$$

$$\downarrow \qquad\qquad \downarrow$$

$$i_{\mathcal{A}'} i_{\mathcal{A}'}^R[1] \xrightarrow{\mathrm{id}\circ\epsilon} i_{\mathcal{A}'} i_{\mathcal{A}'}^R i_{\mathcal{A}} i_{\mathcal{A}}^L[1]$$

We have established now that the map $R \to F_S L$ is obtained from the clockwise composition in this diagram by applying $i_{\mathcal{A}}^R$ and $i_{\mathcal{B}'} i_{\mathcal{B}'}^R i_{\mathcal{B}}$ on the left and right, respectively. Let us examine what happens when we apply these functors to the whole commutative diagram. From the triangle (3.8) and the fact that $i_{\mathcal{A}}^R i_{\mathcal{B}'} = 0$ we see that the left vertical map becomes an isomorphism. Moreover, the unit map fits into the triangle

$$i_{\mathcal{B}} i_{\mathcal{B}}^R \to \mathrm{id}_{\mathcal{C}} \to i_{\mathcal{A}} i_{\mathcal{A}}^L \dashrightarrow$$

and since $i_{\mathcal{A}'}^R i_{\mathcal{B}} = 0$ it follows that the bottom horizontal map becomes an isomorphism as well. So $R \to F_S L$ is an isomorphism. $\qquad\square$

Remark 3.2.12. There are other functors arising from the sequence of semiorthogonal decompositions in the statement of Theorem 3.2.11, such as $i_{\mathcal{B}'}^R i_{\mathcal{A}} : \mathcal{A} \to \mathcal{B}'$, which are spherical because they are obtained from S by composing with a suitable mutation. The corresponding spherical twist autoequivalences can also be described by mutation in $\mathcal{C}$.

We can also obtain a converse to this statement. Suppose that $A^\bullet$ and $B^\bullet$ are dg-algebras over k. Write $\mathrm{D}(\bullet)$ for the derived category of right dg modules over $\bullet$. We

begin with a folklore construction. Let $F^\bullet$ be an $A^\bullet - B^\bullet$ bimodule defining a dg functor $F : D(A^\bullet) \to D(B^\bullet)$ given by $F(M^\bullet) = M^\bullet \otimes_A F^\bullet$. Define a new dg algebra

$$C_F = \begin{pmatrix} A^\bullet & F^\bullet \\ 0 & B^\bullet \end{pmatrix}.$$

More precisely, as a complex $C_F = A^\bullet \oplus F^\bullet \oplus B^\bullet$ and the multiplication is given by

$$(a, f, b)(a', f', b') = (aa', af' + fb', bb').$$

By construction $C_F^\bullet$ has a pair of orthogonal idempotents $e_A = (1, 0, 0)$ and $e_B = (0, 0, 1)$. Every module splits as a complex $M^\bullet = M_A^\bullet \oplus M_B^\bullet$, where $M_A^\bullet := M^\bullet e_A$ is an $A^\bullet$ module and $M_B^\bullet := M^\bullet e_B$ is a $B^\bullet$ module. In fact the category of right $C_F^\bullet$ modules is equivalent to the category of triples consisting of $M_A^\bullet \in D(A^\bullet)$, $M_B^\bullet \in D(B^\bullet)$, and a structure homomorphism of $B^\bullet$ modules $M_A \otimes_A F^\bullet \to M_B^\bullet$, with intertwiners as morphisms. In order to abbreviate notation, we will denote the data of a module over $C_F^\bullet$ by its structure homomorphism $[F(M_A^\bullet) \to M_B^\bullet]$

Let $\mathcal{A}, \mathcal{B} \subset D(C^\bullet)$ be the full subcategories of modules of the form $[F(M_A^\bullet) \to 0]$ and $[F(0) \to M_B^\bullet]$ respectively. Then $\mathcal{A} \simeq D(A^\bullet)$, $\mathcal{B} \simeq D(B^\bullet)$, and the projection $D(C_F^\bullet) \to \mathcal{A}$ (resp. $\mathcal{B}$) given by $[F(M_A^\bullet) \to M_B^\bullet] \mapsto M_A^\bullet$ (resp. $M_B^\bullet$) is the left (resp. right) adjoint of the inclusion. We have semiorthogonality $\mathcal{B} \perp \mathcal{A}$ and a canonical short exact sequence

$$[F(0) \to M_B^\bullet] \to [F(M_A^\bullet) \to M_B^\bullet] \to [F(M_A^\bullet) \to 0] \dashrightarrow$$

and therefore $\mathrm{D}(C_F^\bullet) = \langle \mathcal{A}, \mathcal{B} \rangle$.

Lemma 3.2.13. *Suppose that $G^\bullet$ is a $B^\bullet - A^\bullet$ bimodule such that $\otimes G^\bullet$ is right adjoint to $\otimes F^\bullet$. Then there is an equivalence $\Phi : \mathrm{D}(C_F^\bullet) \to \mathrm{D}(C_G^\bullet)$ such that Φ restricts to the identity functor between the subcategories of $D(C_F^\bullet)$ and $D(C_G^\bullet)$ which are canonically identified with $D(A^\bullet)$.*

Proof. Note that the adjunction allows us to identify a module over $C_F^\bullet$ by a structure homomorphism $M_A^\bullet \to G(M_B^\bullet)$ rather than a homomorphism $F(M_A^\bullet) \to M_B^\bullet$. Letting $M = [M_A^\bullet \to G(M_B^\bullet)] \in D(C_F^\bullet)$, we define

$$\Phi(M)_A = \mathrm{Cone}(M_A^\bullet \to G(M_B^\bullet))[-1] \qquad \Phi(M)_B = M_B[-1]$$

with the canonical structure homomorphism $G(M_B^\bullet) \to \Phi(M)_A$ defining an object in $D(C_G^\bullet)$. This construction is functorial.

For $N = [G(N_B) \to N_A] \in D(C_G^\bullet)$, the inverse functor assigns

$$\Phi^{-1}(N)_A = \mathrm{Cone}(G(N_B^\bullet) \to N_A^\bullet) \qquad \Phi^{-1}(N)_B = N_B^\bullet[1]$$

with the canonical structure homomorphism $\Phi^{-1}(N)_A \to G(N_B^\bullet[1])$ defining an object of $D(C_F^\bullet)$ (again using the adjunction between F and G). $\qquad \square$

Remark 3.2.14. If $F^\bullet$ and $G^\bullet$ are perfect bimodules, then $\mathrm{D}(\bullet)$ can be replaced with $\mathfrak{Perf}(\bullet)$ in the above lemma.

Consider a functor $S : \mathrm{D}(A^\bullet) \to \mathrm{D}(B^\bullet)$, given by a $A^\bullet - B^\bullet$ bimodule $S^\bullet$, with right and left adjoints R, L given by bimodules $R^\bullet$ and $L^\bullet$ respectively. Fix morphisms $A^\bullet \to S^\bullet \otimes_B \cdot R^\bullet$, etc., representing the units and co-units of the adjunctions. Note that from these choices we can produce a bimodule $F_S^\bullet$ representing F_S and a quasi-isomorphism $R^\bullet \to L^\bullet \otimes_A \cdot F_S^\bullet$.

Theorem 3.2.15. *If S is spherical then there is a pre-triangulated dg category $\mathcal{C}$ which admits semiorthogonal decompositions*

$$\mathcal{C} = \langle \mathcal{A}, \mathcal{B} \rangle = \langle \mathcal{B}, \mathcal{A}' \rangle = \langle \mathcal{A}', \mathcal{B}' \rangle = \langle \mathcal{B}', \mathcal{A} \rangle.$$

such that S is the inclusion of $\mathcal{A}$ followed by the projection onto $\mathcal{B}$ with kernel $\mathcal{A}'$, as in the previous theorem.

Proof. Let $S' = S[-1]$, $R' = R[1]$, and $L' = L[1]$ and note that the same adjunctions hold. We will show that $\mathcal{C} = \mathrm{D}(C_{S'}^\bullet)$ admits the desired semiorthogonal decomposition. The reason for introducing $S', R',$ and L' is the following. We observe that $S \simeq i_{\mathcal{B}}^L \circ i_{\mathcal{A}}$. Indeed, $\mathcal{A}'$ is the full subcategory of $\mathrm{D}(C_{S[-1]}^\bullet)$ of objects $[S[-1](M_A^\bullet) \to M_B^\bullet]$ where the structural morphism is an isomorphism. So we see that for any $M_A^\bullet$ there is a triangle

$$[S[-1](M_A^\bullet) \to S[-1](M_A^\bullet)] \to [S[-1](M_A^\bullet) \to 0] \to [S'(0) \to S(M_A^\bullet)] \dashrightarrow$$

Hence including $\mathcal{A}$ and projecting to $\mathcal{B}$ away from $\mathcal{A}'$ gives S.

It follows from Lemma 3.2.13 that there are equivalences

$$\mathrm{D}(C_{R'}^\bullet) \underset{\Phi_1}{\overset{\Psi_1}{\rightleftarrows}} \mathrm{D}(C_{S'}^\bullet) \underset{\Phi_2}{\overset{\Psi_2}{\rightleftarrows}} \mathrm{D}(C_{L'}^\bullet)$$

By construction, $\mathrm{D}(C_{S'}^\bullet)$ admits a semi-orthogonal decomposition $\langle \mathcal{A}, \mathcal{B} \rangle$. We define two more full subcategories using the above equivalences. Let $\mathcal{A}' = \Phi_2 \, \mathrm{D}(A^\bullet)$ and $\mathcal{B}' = \Psi_1 \, \mathrm{D}(B^\bullet)$. Then we have the semiorthogonal decompositions

$$\mathrm{D}(C_{S'}^\bullet) = \langle \mathcal{B}', \mathcal{A} \rangle = \langle \mathcal{A}, \mathcal{B} \rangle = \langle \mathcal{B}, \mathcal{A}' \rangle.$$

All that remains is to show that we have a semiorthogonal decomposition $\mathrm{D}(C_{S'}^\bullet) = \langle \mathcal{A}', \mathcal{B}' \rangle$ as well. We will produce an autoequivalence of $\mathrm{D}(C_{S'}^\bullet)$ which carries $\mathcal{A}$ to $\mathcal{A}'$ and $\mathcal{B}$ to $\mathcal{B}'$, establishing the existence of the remaining semiorthogonal decomposition.

The equivalence F_S gives rise to another equivalence, $X : \mathrm{D}(C_{L'}^\bullet) \to \mathrm{D}(C_{R'}^\bullet)$. Let $P^\bullet$ be a $C_{L'}^\bullet$-module. We define

$$X(P^\bullet)_A = F_S(P_A^\bullet) = P_A^\bullet \otimes_A \cdot F_S^\bullet \quad \text{and} \quad X(P^\bullet)_B = P_B^\bullet.$$

Starting with the structural morphism $P_B^\bullet \otimes_{B^\bullet} L^\bullet[1] \to P_A^\bullet$ we produce the structural morphism

$$R'(P_B^\bullet) \xrightarrow{\cong} F_S(L'(P_B^\bullet)) \to F_S(P_A^\bullet).$$

This is invertible because F_S is an equivalence and we have an isomorphism $F_S^{-1} R' \to L'$.

Consider the autoequivalence $\Psi_1 X \Psi_2$ of $\mathrm{D}(C_{S'}^\bullet)$. We observe by a straightforward computation that

$$\Psi_1 X \Psi_2(\mathcal{B}) = \mathcal{B}'.$$

Now, we compute $\Psi_1 X \Psi_2(\mathcal{A})$. First, $\Psi_2(\mathcal{A}) \subset \mathrm{D}(C_{L'}^\bullet)$ is the full subcategory of objects isomorphic to objects of the form $[L'(S'(M_A^\bullet)) \to M_A^\bullet]$, where the structure morphism is the counit of adjunction. Next we compute that

$$X[L'(S'(M_A^\bullet)) \to M_A^\bullet] = [R'(S'(M_A^\bullet)) \to F_S(M_A^\bullet)]$$

where the structure morphism is the composition of the map $R'(S'(M_A^\bullet)) \to F_S L'(S'(M_A^\bullet))$ with the map $F_S(L'S'(M_A^\bullet)) \to F_S(M_A^\bullet)$ induced by the counit morphism. This map is just the map coming from the triangle

$$R'S' = RS \to F_S \to \mathrm{id}[1] \dashrightarrow$$

defining F_S. Therefore, after applying Ψ_1 we get

$$[S'(M_A^\bullet)[1] \to S'(M_A^\bullet)[1]]$$

where the structure morphism is the identity. This is exactly the condition defining the category $\mathcal{A}' = \Phi_2(\mathrm{D}(A^\bullet))$. Thus $\mathrm{D}(C_{S'}^\bullet)$ admits the fourth semi-orthogonal decomposition

$$\mathrm{D}(C_{S'}^\bullet) = \langle \mathcal{A}', \mathcal{B}' \rangle.$$

$\square$

Remark 3.2.16. If $S^\bullet, R^\bullet$, and $L^\bullet$ are perfect bimodules then we may replace $\mathrm{D}(\bullet)$ with $\mathfrak{Perf}(\bullet)$ in the above theorem. If $A^\bullet$ and $B^\bullet$ are smooth and proper, then all cocontinuous functors between $\mathfrak{Perf}(A^\bullet)$ and $\mathfrak{Perf}(B^\bullet)$ are represesented by perfect bimodules.

Remark 3.2.17. There is an alternate formula for the twist. Suppose that we have

$$\mathcal{C} = \langle \mathcal{A}, \mathcal{B} \rangle = \langle \mathcal{B}, \mathcal{A}' \rangle = \langle \mathcal{A}', \mathcal{B}' \rangle = \langle \mathcal{B}', \mathcal{A} \rangle$$

as above. Then $T_S = i_{\mathcal{B}}^L \circ \mathbf{L}_{\mathcal{A}}$. (Compare with 3.2.3, where r plays the role of the quotient functor $i_{\mathcal{B}}^L$.)

3.3 Monodromy of the quantum connection and fractional grade restriction rules

In the remainder of this paper, we will refine the above construction of autoequivalences of $D^b(X^{ss}/G)$ from a variation of GIT quotient. We generalize the grade restriction rules of Theorem 3.1.1 in order to produce additional derived autoequivalences (see Corollary 3.3.12). Our motivation is to explain additional autoequivalences predicted by homological mirror symmetry (HMS). We first review how HMS leads to autoequivalences, as studied in [40, 24, 25], then we frame these predictions in the context of variation of GIT quotient. We would like to emphasize that the following discussion of mirror symmetry is not meant to introduce new ideas of the authors – we only hope to frame existing ideas regarding HMS in the context of GIT.

For simplicity we consider a smooth projective Calabi-Yau (CY) variety V of complex dimension n. HMS predicts the existence of a mirror CY manifold $\hat{V}$ such that $D^b(V) \simeq D^b \operatorname{Fuk}(\hat{V}, \beta)$, where β represents a complexified Kähler class and $D^b \operatorname{Fuk}(\hat{V}, \beta)$ is the graded Fukaya category. The category $D^b \operatorname{Fuk}$ does not depend on the complex structure of $\hat{V}$. Thus if $\hat{V}$ is one fiber in a family of compact CY manifolds $\hat{V}_t$ over a base $\mathcal{M}$, the monodromy representation $\pi_1(\mathcal{M}) \to \pi_0(\operatorname{Symp}^{gr}(\hat{V}, \beta))$ acting by symplectic parallel transport leads to an action $\pi_1(\mathcal{M}) \to \operatorname{Aut} D^b \operatorname{Fuk}(\hat{V}, \beta)$. Via HMS this gives an action $\pi_1(\mathcal{M}) \to \operatorname{Aut} D^b(V)$ (see [40] for a full discussion).

Hodge theoretic mirror symmetry predicts the existence of a normal crossings compactification $\overline{\mathcal{M}}(\hat{V})$ of the moduli space of complex structures on $\hat{V}$ along with a mirror map $\overline{\mathcal{M}}(\hat{V}) \to \overline{\mathcal{K}}(V)$ to a compactification of the "complexified Kähler moduli space" of V. Different regions of $\mathcal{K}(V)$ correspond to different birational models of V, but locally $\mathcal{K}(V)$ looks like the open subset of $H^2(V; \mathbb{C})/2\pi i H^2(V; \mathbb{Z})$ whose real part is a Kähler class on V.[2] Mirror symmetry predicts that the mirror map identifies the *B-model variation of Hodge structure* $H^n(\hat{V}_t)$ over $\mathcal{M}$ with the *A*-model variation of Hodge structure, which is locally given by the quantum connection on the trivial bundle $\bigoplus H^{p,p}(V) \times \mathcal{K}(V) \to \mathcal{K}(V)$ (See Chapter 6 of [14] for details).

Finally, one can combine Hodge theoretic mirror symmetry and HMS: Let $\gamma : S^1 \to \mathcal{K}(V)$ be the image of a loop $\gamma' : S^1 \to \mathcal{M}(\hat{V})$ under the mirror map. Symplectic parallel transport around γ' of a Lagrangian $L \subset \hat{V}_t$ corresponds to parallel transport of its fundamental class in the *B*-model variation of Hodge structure $H^n(\hat{V}_t)$. Thus mirror symmetry predicts that the automorphism $T_\gamma \in \operatorname{Aut}(D^b(V))$ corresponds to the the monodromy of quantum connection around γ under the twisted Chern character $\operatorname{ch}^{2\pi i}$ defined in [25].

From the above discussion, one can formulate concrete predictions in the context of geometric invariant theory without an explicit mirror construction. For now we ignore the requirement that V be compact (we will revisit compact CY's in Section 3.3), and we restrict our focus to a small subvariety of the Kähler moduli space in the neighborhood of a "partial

---

[2]Technically, the complexified Kähler moduli space is locally $\mathcal{K}(V)/\operatorname{Aut}(V)$, but this distinction is not relevant to our discussion.

large volume limit." Assume that $V = X^{ss}_-/G$ is a GIT quotient of a smooth quasiprojective X and that $X^{ss}_-/G \dashrightarrow X^{ss}_+/G$ is a balanced GIT wall crossing with a single stratum and $\omega_X|_Z$ has weight 0, as we studied in Section 3.2.

The VGIT is determined by a 1-parameter family of G-ample bundles $\mathcal{L}_0 + r\mathcal{L}'$, where $r \in (-\epsilon, \epsilon)$. In fact we consider the two parameter space

$$U := \{\tau_0 c_1(\mathcal{L}_0) + \tau' c_1(\mathcal{L}')|\Re(\tau_0) > 0 \text{ and } \Re(\tau')/\Re(\tau_0) \in (-\epsilon, \epsilon)\}$$

This is a subspace of $H^2(X^{ss}_-/G; \mathbb{C})/2\pi i H^2(X^{ss}_-/G; \mathbb{Z})$ obtained by gluing $\mathcal{K}(X^{ss}_-/G)$ to $\mathcal{K}(X^{ss}_+/G)$ along the boundary where $\Re(\tau') = 0$. Because we are working modulo $2\pi i \mathbb{Z}$, it is convenient to introduce the exponential coordinates $q_0 = e^{-\tau_0}$ and $q' = e^{-\tau'}$. In these coordinates, we consider the partial compactification $\bar{U}$ as well as the annular slice U_{q_0}:

$$\begin{aligned} \bar{U} &:= \{(q_0, q') \in \mathbb{C} \times \mathbb{C}^* \, ||q_0| < 1 \text{ and } |q'| \in (|q_0|^\epsilon, |q_0|^{-\epsilon})\} \\ U_{q_0} &:= \{q_0\} \times \mathbb{C}^* \cap \bar{U}. \end{aligned} \tag{3.9}$$

In this setting, mirror symmetry predicts that the quantum connection on U converges to a meromorphic connection on some neighborhood of $U_0 = \{0\} \times \mathbb{C}^* \subset \bar{U}$ which is singular along U_0 as well as a hypersurface $\nabla \subset \bar{U}$. To a path in $U \setminus \nabla$ connecting a point in the region $|q'| < 1$ with the region $|q'| > 1$, there should be an equivalence $\mathrm{D}^b(X^{ss}_-/G) \simeq \mathrm{D}^b(X^{ss}_+/G)$ coming from parallel transport in the mirror family.

Restricting to U_{q_0}, one expects an autoequivalence of $\mathrm{D}^b(X^{ss}_-/G)$ for every element of $\pi_1(U_{q_0} \setminus \nabla)$, which is freely generated by loops around the points $\nabla \cap \bar{U}_{q_0}$ and the loop around the origin. We will refer to the intersection multiplicity of ∇ with the line $\{0\} \times \mathbb{C}^*$ as the *expected number of autoequivalences produced by the wall crossing*. For a generic q_0 very close to 0, this represents the number of points in $\nabla \cap U_{q_0}$ which remain bounded as $q_0 \to 0$.

For the example of toric CY manifolds, the compactification of the Kähler moduli space and the hypersurface ∇ have been studied extensively. In Section 3.3, we compute these intersection multiplicities, which will ultimately inspire the construction of new autoequivalences of $\mathrm{D}^b(X^{ss}_-/G)$ in section 3.3.

Remark 3.3.1 (Normalization)**.** In the discussion above, making the replacements $a\mathcal{L}_0$ and $b\mathcal{L}'$ for positive integers a, b, and reducing ϵ if necessary, does not effect the geometry of the VGIT at all, but it replaces U with the covering corresponding to the map $q_0 \mapsto q_0^a$, $q' \mapsto (q')^b$. The covering in the q_0 direction has no effect on the expected number of autoequivalences defined above, but the covering $q' \mapsto (q')^b$ would multiply the expected number of autoequivalences by b. Fortunately, the VGIT comes with a canonical normalization: When possible we will assume that $\mathcal{L}'|_Z \in \mathrm{D}^b(Z/L)_1$, and in general we will choose $\mathcal{L}'$ which minimizes the magnitude of the weight of $\mathcal{L}'|_Z$ with respect to λ. Multiplying $\mathcal{L}_0$ if necessary, we can define the VGIT with $\epsilon = 1$.

Remark 3.3.2. To simplify the exposition, we have ignored the fact that X^{ss}_-/G is not compact in many examples of interest. To fix this, one specifies a function $W : X^{ss}_-/G \to \mathbb{C}$

whose critical locus is a compact CY V, and the predictions above apply to the quantum connection of V on the image of $\bar{U}$ under the map $H^2(\mathfrak{X}^{ss}_-/G) \to H^2(V)$. We will discuss how autoequivalences of $\mathrm{D}^b(X^{ss}_-/G)$ lead to autoequivalences of $\mathrm{D}^b(V)$ in Section 3.3.

Remark 3.3.3. The region U connects two large volume limits $q_0, q' \to 0$ and $q_0, (q')^{-1} \to 0$. It is possible to reparameterize U in terms of the more traditional large volume limit coordinates around either point ([14], Chapter 6).

The toric case: Kähler moduli space and discriminant in rank 2

A Calabi-Yau (CY) toric variety can be presented as a GIT quotient for a linear action of a torus $T \to \mathrm{SL}(V)$ on a vector space V [15]. Write $X^*(T)$ and $X_*(T)$ for the groups of characters and cocharacters of T, respectively. The GIT wall and chamber decomposition on $X^*(T)_\mathbb{R} = X^*(T) \otimes \mathbb{R}$ can be viewed as a fan known as the GKZ fan. The toric variety defined by this fan provides a natural compactification $\overline{\mathcal{K}}$ of the complexified Kähler space $X^*(T) \otimes \mathbb{C}^*$. A codimension-one wall in $X^*(T)_\mathbb{R}$, which corresponds to a balanced GIT wall crossing, determines an equivariant curve $C \simeq \mathbb{P}^1$ in $\overline{K}$ connecting the two large volume limit points determined by the chambers on either side of the wall. The curve $\bar{U}_0$ corresponding to this VGIT (3.9) is exactly the complement of the two torus fixed points in C.

Such a CY toric variety arises in mirror symmetry as the total space of a toric vector bundle for which a generic section defines a compact CY complete intersection (See Section 3.3, and [14] for a full discussion). In this case, the toric variety defined by the GKZ fan also provides a natural compactification of the complex moduli space of the mirror $\overline{\mathcal{M}}$. Although the mirror map $\overline{\mathcal{M}} \to \overline{\mathcal{K}}$ is nontrivial, it is the identity on the toric fixed points (corresponding to chambers in the GKZ fan) and maps a boundary curve connecting two fixed points to itself. It follows that our analysis of the expected number of autoequivalences coming from the VGIT can be computed in $\overline{\mathcal{M}}$.

The boundary of $\overline{\mathcal{M}}$, corresponding to singular complex degenerations of the mirror, has several components. In addition to the toric boundary, there is a particular hypersurface called the reduced discriminantal hypersurface ∇ in M (see [20]), which we simply call the *discriminant*. It is the singular locus of the GKZ hypergeometric system. For simplicity we will analyze the case when T is rank 2. We will compute the expected number of autoequivalences as the intersection number between C and the normalization of the discriminant. It turns out that this intersection number is equal to the length of a full exceptional collection on the Z/L' appearing in the GIT wall crossing.

Let $V = \mathbb{C}^m$ and $(\mathbb{C}^*)^2 \cong T \subset (\mathbb{C}^*)^m$ be a rank two subtorus of the standard torus acting on V. We can describe T by a matrix of weights,

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_m \\ b_1 & b_2 & \cdots & b_m \end{pmatrix},$$

representing the embedding $(t, s) \mapsto (t^{a_1} s^{b_1}, \ldots, t^{a_m} s^{b_m})$. We assume that all columns are non-zero. The CY condition means that we have

$$\sum_{i=1}^{m} a_i = \sum_{i=1}^{m} b_i = 0.$$

Now, up to an automorphism of V we may assume that the matrix of weights has the following form

$$\binom{a_i}{b_i} = \begin{pmatrix} d_1^1 \chi_1 & \cdots & d_{n_1}^1 \chi_1 & \cdots & d_1^r \chi_r & \cdots & d_{n_r}^r \chi_r \end{pmatrix}$$

where $\chi_j = \binom{\alpha_j}{\beta_j}$ and $\chi_1, \ldots \chi_r$ are ordered counterclockwise by the rays they generate in the plane. Using the fact that a wall between GIT chambers occurs when there exists a strictly semistable point, one can determine that the rays of the GKZ fan are spanned by $-\chi_j$. The GIT chambers, the maximal cones of the GKZ fan, are the cones $\sigma_i = cone(-\chi_i, -\chi_{i+1})$, $i < r$, and $\sigma_r = (-\chi_r, -\chi_1)$. The discriminant admits a rational parameterization, called the Horn uniformization, $f : \mathbb{P}(X_*(T)_{\mathbb{C}}) \dashrightarrow \nabla$ of the following form. Set $d^i = \sum_{j=1}^{n_i} d_j^i$. For a Laurent monomial $x^\lambda \in \mathbb{C}[T]$ we have

$$f^*(x^\lambda) = \prod_{i,j} (d_j^i \chi_i)^{-d_j^i(\chi_i, \lambda)} = d_\lambda \prod_i \chi_i^{-d^i(\chi_i, \lambda)}, \quad d_\lambda := \prod_{i,j} (d_j^i)^{-d_j^i(\chi_i, \lambda)}$$

where we view $X^*(T)$ as a set of linear functions on $X_*(T)_{\mathbb{C}}$. It follows from the CY condition that $f^*(x^\lambda)$ has degree zero as a rational function on $X_*(T)_{\mathbb{C}}$ and that $\mathcal{M}$ is proper. Therefore, f actually defines a regular map $\mathbb{P}(X_*(T)_{\mathbb{C}}) \cong \mathbb{P}^1 \to \mathcal{M}$. We define C_i to be equivariant curve in $\overline{\mathcal{M}}$ defined by the codimension one wall $\mathbb{R}_{\geq 0} \cdot (-\chi_i)$.

Proposition 3.3.4. *If $-\chi_i$ is not among the χ_j, then the length of $\mathbb{P}(X_*(T)_{\mathbb{C}}) \times_{\mathcal{M}} C_i$ is d^i.*

Proof. C_i is covered by the open sets corresponding to σ_{i-1} and σ_i. Let U_i be the chart corresponding to σ_i. Recall that the coordinate ring of U_i is

$$\mathbb{C}[\sigma_i^\vee] = \mathbb{C}\{x^\lambda : \forall \chi \in \sigma_i, (\chi, \lambda) \geq 0\} \subset \mathbb{C}[X_*(T)].$$

Observe that $(\chi, \lambda) \geq 0$ for all $\chi \in \sigma_i$ if and only if $(\chi_i, \lambda), (\chi_{i+1}, \lambda) \leq 0$. Next, we must compute the ideal of C_i in the charts U_i and U_{i+1}. Since $-\chi_i$ spans the wall under consideration the ideal of C_i will be

$$I_i = \mathbb{C}\{x^\lambda : (\chi_i, \lambda) < 0\} \cap \mathbb{C}[U_\bullet].$$

Let $p_j = \{\chi_j = 0\} \in \mathbb{P}(X_*(T)_{\mathbb{C}})$. Then $f(p_j) \in U_i$ if and only if for all λ such that $(\chi_i, \lambda), (\chi_{i+1}, \lambda) \leq 0$ we have $(\chi_j, \lambda) \leq 0$. So if $\chi_j \neq \chi_i, \chi_{i+1}$ then $f(p_j) \notin U_i$ and $f^{-1}(U_i \cap \nabla)$ is supported on $\{p_i, p_{i+1}\}$.

Then there clearly exists a λ such that $(\chi_{i+1}, \lambda) = 0$ but $(\chi_i, \lambda) < 0$. This means that in fact $f^{-1}(U_i \cap \nabla)$ is supported on p_i. So we can compute the length of $f^{-1}(U_i \cap \nabla)$ after

restricting to $\mathbb{P}(X_*(T)_{\mathbb{C}}) \setminus \{p_j\}_{j \neq i}$ where its ideal is generated by $\{\chi_i^{-d_i(\chi_i, \lambda)}\}_{\lambda \in \sigma_i^\vee}$. Finally, we note that

$$\min\{(\chi_i, \lambda) : \lambda \in \sigma_i^\vee\} = 1$$

and therefore the length of $f^{-1}(\nabla \cap U_i)$ is d^i. By an analogous argument we see that $f^{-1}(\nabla \cap U_i) = f^{-1}(\nabla \cap U_{i-1})$. □

Remark 3.3.5. Observe that the image of f avoids the torus fixed points. Indeed, the torus fixed point in U_i lies on $C_i \setminus U_{i-1}$, but $\nabla \cap C_i \subset U_i \cap U_{i-1}$.

Codimension one wall crossings are always balanced [16], but we include the analysis of the Hilbert-Mumford numerical criterion in order to explicitly identify the Z/L' when we cross the wall spanned by $-\chi_i$ where $-\chi_i$ is not also a weight of T acting on V. For any character the KN stratification is determined by data $\{(Z_j, \lambda_j)\}_{j=0}^r$ (see Section 3.1).

Proposition 3.3.6. *Let $\{(Z_j^R, \lambda_j^R)\}_{j=0}^l$ and $\{(Z_j^L, \lambda_j^L)\}_{j=0}^s$ be the data of stratifications immediately to the right and left of the wall spanned by $-\chi_i$, respectively. Then*

1. $\lambda_0^R = -\lambda_0^L$ and $(\chi_i, \lambda_0^\bullet) = 0$,

2. $Z_0^R = Z_0^L = V^{\lambda_0} \setminus 0$, and

3. $\bigcup_{j>0} S_j^R = \bigcup_{j>0} S_j^L$.

Proof sketch. (See [16] for details.) Let χ be a character near $-\chi_i$ (as rays), $\|\cdot\|$ be a norm on $X_*(T)_{\mathbb{R}}$, and $\mu^\chi(\lambda) = \frac{(\chi, \lambda)}{\|\lambda\|}$. In this situation the KN stratification is defined inductively. First, there is a primitive cocharacter λ_{max} which maximizes μ^χ. The most unstable stratum has core $Z_{max} = V^{\lambda_{max}} = 0$ and $S_{max} = \oplus_{i,(\chi_i, \lambda_{max}) \geq 0} \oplus_j V_{i,j}$. The linearization χ determines a choice of generator for the line perpendicular to χ_j. For each j we let λ_j be the primitive cocharacter satisfying (i) $(\chi_j, \lambda_j) = 0$, and (ii) $\mu^\chi(\lambda) \geq 0$. We arrange these in decreasing order according to the value of $\mu^\chi(\lambda_\bullet)$: $\lambda_{j_1}, \ldots, \lambda_{j_n}$. If $V^{\lambda_{j_k}}$ is not entirely contained in $S_{<k} = S_{max} \cup \bigcup_{i<k} S_i$ then we put $Z_k = V^{\lambda_{j_k}} \setminus S_{<k}$ and $S_k = (\oplus_{i,(\chi_i, \lambda_{j_k}) \geq 0} \oplus_j V_{i,j}) \setminus S_{<k}$. Clearly then, the KN stratification only depends on the sequence of $\lambda_\bullet$. Now, as χ varies across the wall, λ_{max} varies, but Z_{max} and S_{max} remain unchanged. Furthermore, $\mu^\chi(\lambda_j)$ remains positive unless $j = i$ and moreover the ordering on λ_j for $j \neq i$ does not change. On the other hand $\mu^\chi(\lambda_i)$ changes sign so that $-\lambda_i$ replaces λ_i as the cocharacter attached to the least unstable stratum. The proposition follows. □

Note that $V^{\lambda_0^\pm} = \oplus_j V_{i,j}$. The action of T on V^{λ_0} factors through χ_i and the weights are simply $d_1^i, \ldots, d_{n_i}^i$ which are all positive. Therefore the stack $Z_0/\mathbb{C}^*$ is a weighted projective space. Its derived category is understood thanks to the following.

Theorem (Theorem 2.12 of [5]). *$D^b(\mathbb{P}(d_1^i, \ldots, d_{n_i}^i))$ has a full exceptional collection of d^i line bundles. In particular $K_0(\mathbb{P}(d_1^i, \ldots, d_{n_i}^i))$ is free of rank d^i.*

In conclusion, we see that the length of a full exceptional collection on Z_0/L_0 associated to a wall i is equal to the intersection multiplicity of $f : \mathbb{P}^1 \to \nabla$ with the curve C_i.

Example 3.3.7. Consider the $T = (\mathbb{C}^*)^2$ action on $\mathbb{A}^8$ given by

$$(t, s) \mapsto (t, t, t, s, s, s, t^{-2}, t^{-1}s^{-3}).$$

The wall and chamber decomposition of $\mathbb{R}^2$ associated to this action is given in the following diagram.
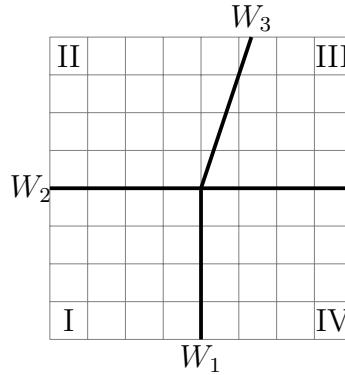
Chamber I corresponds to the total space of $\mathcal{O}(-2,0) \oplus \mathcal{O}(-1,-3)$ over $\mathbb{P}^2 \times \mathbb{P}^2$, and for this reason we will return to this example in subsection 3.3. By Horn uniformization, the discriminant is parameterized by

$$[u : v] \mapsto \left(-4\frac{u + 3v}{u}, -\frac{(u + 3v)^3}{v^3} \right).$$

We will compute the intersection number at wall W_3. This corresponds to the character $(-1, -3)$. No other characters are a rational multiple of this one. Therefore, we should get intersection number 1. We compute the dual cones to chambers II and III, and indicate the ideal of C_3 in the diagram below. The nested grey regions correspond to the monomials in the dual cones and in the ideal of C_3. The red, orange, and blue lines divide the plane into regions corresponding to monomials where u,v, and $(u + 3v)$ respectively appear with positive or negative exponents. It is clear that only $(u + 3v)$ always appears with a positive exponent. Clearly, it appears in x with exponent 1 and therefore the intersection number $\ell(C_3 \cap \nabla)$ is one.

Fractional grade restriction rules

In order to construct additional derived equivalences, we introduce *fractional grade restriction rules* given a semiorthogonal decomposition $\mathrm{D}^b(Z/L)_w = \langle \mathcal{E}_0, \mathcal{E}_1 \rangle$, the data of which we will be denoted e. This will be of particular interest when $\mathrm{D}^b(Z/L)_w$ has a full exceptional collection.
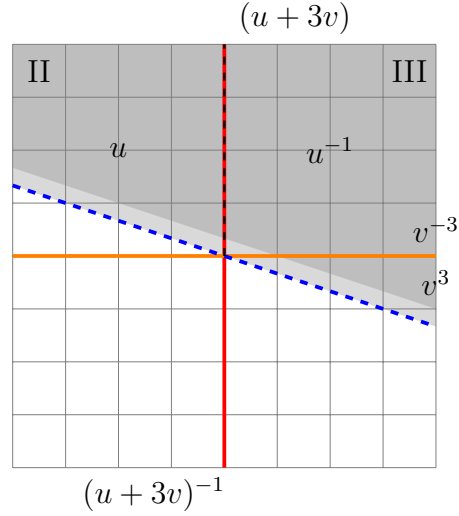
Figure 3.1: The dual cones to chambers II and III and the ideal defining C_3.

The equivalence of Lemma 3.1.2 gives a semiorthogonal decomposition $\mathcal{A}_w^+ = \langle \mathcal{E}_0^+, \mathcal{E}_1^+ \rangle$, where $\mathcal{E}_i^+ = \iota_w(\mathcal{E}_i)$. We can refine the semiorthogonal decompositions (3.3)

$$\mathcal{C}_w^+ = \langle \mathcal{E}_0^+, \mathcal{E}_1^+, \mathbf{G}_{w+1}^+ \rangle = \langle \mathbf{G}_w^+, \mathcal{E}_0^+, \mathcal{E}_1^+ \rangle$$

Because $\mathcal{E}_0^+$ and $\mathcal{E}_1^+$ are left and right admissible in $\mathcal{C}_w^+$ respectively, we can make the following

Definition 3.3.8. Given the semiorthogonal decomposition e, we define the full subcategory $\mathbf{G}_e^+ = (\mathcal{E}_1^+)^\perp \cap {}^\perp(\mathcal{E}_0^+) \subset \mathcal{C}_w^+$. In other words, it is defined by the semiorthogonal decomposition

$$\mathcal{C}_w^+ = \langle \mathcal{E}_0^+, \mathbf{G}_e^+, \mathcal{E}_1^+ \rangle$$

Because $\mathcal{E}_0^+$, and $\mathcal{E}_1^+$ generate the kernel of the restriction functor r_+, it follows formally that $r_+ : \mathbf{G}_e^+ \to \mathrm{D}^b(X_+^{ss}/G)$ is an equivalence of dg-categories.

The mutation equivalence functor factors

$$\mathbf{G}_{w+1}^+ \underset{\mathbb{L}_{\mathcal{E}_1^+}}{\overset{\mathbb{L}_{\mathcal{A}_w^+}}{\rightrightarrows}} \mathbf{G}_e^+ \overset{\mathbb{L}_{\mathcal{E}_0^+}}{\Rightarrow} \mathbf{G}_w^+$$

In order for these intermediate mutations to induce autoequivalences of $\mathrm{D}^b(X_-^{ss}/G)$, we must show that $\mathbf{G}_e^+$ is also mapped isomorphically onto $\mathrm{D}^b(X_-^{ss}/G)$ by restriction. We let $\kappa_\pm$ denote the equivariant line bundle $\det(N_{S^\pm}X)|_Z = (j^\pm)^! \mathcal{O}_X|_Z$.

Lemma 3.3.9. *Let $F^\bullet \in \mathcal{C}_w^+$, then whether $F^\bullet \in \mathbf{G}_e^+$ is determined by the "fractional grade restriction rule":*

$$(\sigma^* F^\bullet)_w \in {}^\perp(\mathcal{E}_0) \quad and \quad (\sigma^* F^\bullet \otimes \kappa_+)_w \in (\mathcal{E}_1)^\perp \tag{3.10}$$

Proof. By definition $F^\bullet \in \mathbf{G}_e^+$ if and only if $\text{Hom}(F^\bullet, \iota_w(\mathcal{E}_0)) = 0$ and $\text{Hom}(\iota_w(\mathcal{E}_1), F^\bullet) = 0$. By Lemma 3.1.3, the left and right adjoint of ι_w can be expressed in terms of $\sigma^* F^\bullet$. We use that $(\sigma^* F^\bullet)_{w+\eta} \otimes \kappa_+ = (\sigma^* F^\bullet \otimes \kappa_+)_w$. $\qquad\square$

One can think of $\mathbf{G}_e^+$ as a refined version of the usual category $\mathbf{G}_w^+$. Previously, we had an infinite semiorthogonal decomposition $\text{D}^b(Z/L) = \langle \ldots, \text{D}^b(Z/L)_w, \text{D}^b(Z/L)_{w+1}, \ldots \rangle$, and the grade restriction rule amounted to choosing a point at which to split this semiorthogonal decomposition, then requiring $\sigma^* F^\bullet$ to lie in the right factor and $\sigma^* F^\bullet \otimes \kappa_+$ to lie in the left factor. Lemma 3.3.9 says the same thing but now we use the splitting

$$\text{D}^b(Z/L) = \langle \langle \ldots, \text{D}^b(Z/L)_{w-1}, \mathcal{E}_0 \rangle, \langle \mathcal{E}_1, \text{D}^b(Z/L)_{w+1}, \ldots \rangle \rangle.$$

The canonical bundle for a quotient stack Z/L is $\omega_{Z/L} = \omega_Z \otimes \det \mathfrak{l}^\vee$. [3] We say that *Serre duality holds for Z/L* if the category $\text{D}^b(Z/L)$ is Hom-finite and $\otimes \omega_{Z/L}[n]$ is a Serre functor for some n, i.e. $\text{Hom}^\bullet_{Z/L}(F^\bullet, G^\bullet \otimes \omega_{Z/L}[n]) \simeq \text{Hom}^\bullet_{Z/L}(G^\bullet, F^\bullet)^\vee$. Because all objects and homomorphism split into direct sums of weights spaces for λ, and $\omega_{Z/L} \in \text{D}^b(Z/L)_0$, this is equivalent to Serre duality holding in the subcategory $\text{D}^b(Z/L)_0 \simeq \text{D}^b(Z/L')$. Thus whenever Z/L' is a compact DM stack, Serre duality holds for Z/L.

Proposition 3.3.10. *Let $\omega_{X/G}|_Z \simeq \mathcal{O}_Z$, and assume that Serre duality holds for Z/L, then $r_- : \mathbf{G}_e^+ \to \text{D}^b(X_-^{ss}/G)$ is an equivalence of dg-categories. More precisely $\mathbf{G}_e^+ = \mathbf{G}_{e'}^-$, where e' denotes the data of the semiorthogonal decomposition*

$$\text{D}^b(Z/L)_{[\lambda^- = w']} = \langle \mathcal{E}_1 \otimes \omega_{Z/L} \otimes \kappa_+^\vee, \mathcal{E}_0 \otimes \kappa_+^\vee \rangle$$

Proof. First note that e' is actually a semiorthogonal decomposition by Serre duality: it is the left mutation of e tensored with $\kappa_+^\vee$.

Applying Serre duality to the characterizaton of $\mathbf{G}_e^+$ in Lemma 3.3.9, and using the fact that $(\bullet)_{[\lambda^+ = w]} = (\bullet)_{[\lambda^- = w'+\eta]}$, it follows that $F^\bullet \in \mathbf{G}_e^+$ if and only if

$$(\sigma^* F^\bullet \otimes \kappa_-)_{[\lambda^- = w']} \in (\mathcal{E}_0 \otimes \omega_{Z/L}^\vee \otimes \kappa_-)^\perp \quad \text{and} \quad (\sigma^* F^\bullet)_{[\lambda^- = w']} \in {}^\perp(\mathcal{E}_1 \otimes \omega_{Z/L} \otimes \kappa_+^\vee)$$

This is exactly the characterization of $\mathbf{G}_{e'}^-$, provided that $\kappa_- \otimes \omega_{Z/L}^\vee \simeq \kappa_+^\vee$.

Consider the weight decomposition with respect to λ^+, $\Omega_X^1|_Z = (\Omega_X^1)_+ \oplus (\Omega_X^1)_0 \oplus (\Omega_X^1)_-$. Then $\omega_{Z/L} \simeq \det((\Omega_X^1)_0) \otimes \det(\mathfrak{g}_0)^\vee$, and $\kappa_\pm^\vee \simeq \det((\Omega_X^1)_\pm) \otimes \det(\mathfrak{g}_\pm)^\vee$, where $\mathfrak{g}_\pm$ denotes the subspace of $\mathfrak{g}$ with positive or negative weights under the adjoint action of λ^+. Hence $\omega_{X/G}|_Z \simeq \kappa_+^\vee \otimes \omega_{Z/L} \otimes \kappa_-^\vee$, so when $\omega_{X/G}|_Z \simeq \mathcal{O}_Z$ we have $\kappa_+^\vee \simeq \kappa_- \otimes \omega_Z^\vee$ as needed. $\qquad\square$

Corollary 3.3.11. *Let $\omega_{X/G}|_Z \simeq \mathcal{O}_Z$ equivariantly, let Serre duality hold for Z/L, and assume we have a semiorthogonal decomposition $\text{D}^b(Z/L)_w = \langle \mathcal{E}_0, \ldots, \mathcal{E}_N \rangle$. If we define $\mathcal{H}_i^+$ as the mutation $\mathcal{C}_w = \langle \mathcal{E}_0^+, \ldots, \mathcal{H}_i^+, \mathcal{E}_i^+, \ldots, \mathcal{E}_N^+ \rangle$, then $r_- : \mathcal{H}_i^+ \to \text{D}^b(X_-^{ss}/G)$ is an equivalence.*

---

[3]This is the same as ω_Z if L is connected.

Proof. Apply Proposition 3.3.10 to the two term semiorthogonal decomposition $\langle \mathcal{A}_0, \mathcal{A}_1 \rangle$, where $\mathcal{A}_0 = \langle \mathcal{E}_0, \dots, \mathcal{E}_{i-1} \rangle$ and $\mathcal{A}_1 = \langle \mathcal{E}_i, \dots, \mathcal{E}_N \rangle$. $\qquad\square$

As a consequence of Corollary 3.3.11 and the results of Section 3.2, one can factor the window shift Φ_w as a composition of spherical twists, one for each semiorthogonal factor $\mathcal{E}_i$. For concreteness, we narrow our focus to the situation where $D^b(Z/L)_w$ admits a full exceptional collection $\langle E_0, \dots, E_N \rangle$. In this case the $\mathcal{E}_i^+$ of Corollary 3.3.11 are generated by the exceptional objects $E_i^+ := j_*^+ (\pi^+)^* E_i$. The category $\mathcal{H}_i^+$ is characterized by the fractional grade restriction rule

$$\begin{aligned} \mathrm{Hom}_{Z/L}((\sigma^* F^\bullet)_{[\lambda^+ = w]}, E_j) &= 0, \text{ for } j < i, \text{ and} \\ \mathrm{Hom}_{Z/L}\left(E_j, (\sigma^* F^\bullet \otimes \kappa_+)_{[\lambda^+ = w]}\right) &= 0, \text{ for } j \geq i \end{aligned} \tag{3.11}$$

Corollary 3.3.12. *Let $\omega_{X/G}|_Z \simeq \mathcal{O}_Z$ and let $D^b(Z/L)_w = \langle E_0, \dots, E_N \rangle$ have a full exceptional collection. Then the objects $S_i := f_w(E_i) = j_*^+ (\pi^+)^* E_i|_{X_-^{ss}} \in D^b(X_-^{ss}/G)$ are spherical, and $\Phi_w = T_{S_0} \circ \cdots \circ T_{S_N}$.*

As noted, this follows for purely formal reasons from Corollary 3.3.11 and the results of subsection 3.3, but for the purposes of illustration we take a more direct approach.

Proof. We use Lemma 3.2.5 and the fact that $(\sigma^* E_i^+)_{[\lambda^- = w']} = (\sigma^* E_i^+)_{[\lambda^+ = w+\eta]} = E_i \otimes \kappa_+^\vee$ to compute

$$\begin{aligned} R\Gamma_S\text{-}\underline{\mathrm{Hom}}(E_i^+, F^\bullet) &\simeq \mathrm{Hom}_{Z/L}(E_i, \sigma^*(F^\bullet)_{w'+\eta} \otimes \kappa_- \otimes \kappa_+) \\ &\simeq \mathrm{Hom}_{Z/L}(E_i, \sigma^*(F^\bullet)_{w'+\eta} \otimes \omega_{Z/L} \otimes \omega_{X/G}^{-1}) \end{aligned}$$

Now let $\omega_{X/G} \simeq \mathcal{O}_Z$. Serre duality implies that

$$\mathrm{Hom}_{Z/L}(E_i, \sigma^*(F^\bullet)_{w'+\eta} \otimes \omega_Z) = \mathrm{Hom}_{Z/L}(\sigma^*(F^\bullet)_{w'+\eta}, E_i)^\vee.$$

Thus by (3.11), the canonical map $\mathrm{Hom}_{X/G}(E_i^+, F^\bullet) \to \mathrm{Hom}_{X_-^{ss}/G}(S_i, F^\bullet|_{X_-^{ss}})$ is an isomorphism for $F^\bullet \in \mathcal{H}_{i+1}^+$. This implies the commutative diagram

$$\begin{CD} \mathbf{G}_{w+1}^+ @>{L_{E_N^+}}>> \mathcal{H}_N^+ @>{L_{E_{N-1}^+}}>> \cdots @>{L_{E_1^+}}>> \mathcal{H}_1^+ @>{L_{E_0^+}}>> \mathbf{G}_w^+ \\ @V{r^-}VV @V{r^-}VV @. @V{r^-}VV @V{r^-}VV \\ D^b(X_-^{ss}/G) @>{T_{S_N}}>> D^b(X_-^{ss}/G) @>{T_{S_{N-1}}}>> \cdots @>{T_{S_1}}>> D^b(X_-^{ss}/G) @>{T_{S_0}}>> D^b(X_-^{ss}/G) \end{CD}$$

Where T_{S_i} is the twist functor $\mathrm{Cone}(\mathrm{Hom}(S_i, F^\bullet) \otimes S_i \to F^\bullet)$. By 3.3.10, the functors r^- are equivalences, and therefore so are T_{S_i}. $\qquad\square$

Corollary 3.3.12, suggests a natural interpretation in terms of monodromy as discussed in the beginning of this section. Let U_{q_0} be the annulus (3.9), with $|q_0|$ small, and let $p_0, \dots, p_N$ be the points of $U_{q_0} \setminus \nabla$ which remain bounded as $q_0 \to 0$. Consider an ordered set of elements $[\gamma_0], \dots, [\gamma_N]$ of $\pi_1(U_{q_0} \setminus \nabla)$ such that

1. γ_i lie in a simply connected domain in U_{q_0} containing $p_0, \ldots, p_N$, and

2. there is a permutation σ_i such that the winding number of γ_i around p_j is δ_{j,σ_i}.

It is natural to guess that the monodromy representation $\pi_1(U_{q_0} \setminus \nabla) \to \operatorname{Aut} \mathrm{D}^b(X^{ss}_-/G)$ predicted by mirror symmetry assigns T_{S_i} to $[\gamma_i]$. In particular, it would be interesting to compare the monodromy of the quantum connection with the action of T_{S_i} under the twisted Chern character.
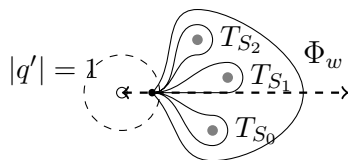
Figure 3.2: Loops in $U_{q_0} \setminus \nabla$ corresponding to monodromy of the quantum connection of X^{ss}_-/G, giving a pictorial interpretation of Corollary 3.3.12.

Evidence for this interpretation of Corollary 3.3.12 is admittedly circumstantial. In 3.3, we verified that the number of autoequivalences predicted by mirror symmetry is the same as the length of a full exceptional collection on $\mathrm{D}^b(Z/L')$ for toric flops of CY toric varieties of Picard rank 2. Letting $q_0 \to 0$, the points $p_0, \ldots, p_n$ converge to $1 \in U_0$. Horja [24] studied the monodromy of the quantum connection and the corresponding autoequivalences for the boundary curve U_0, and his work can be used to verify our interpretation of the loop corresponding to Φ_w.

Furthermore, if we fix a simply connected domain $D \subset U_{q_0}$ containing $p_0, \ldots, p_N$ and let $\operatorname{Diff}(D \setminus \{p_0, \ldots, p_N\}, \partial D)$ denote the topological group of diffeomorphisms which restrict to the identity on the boundary, then $B_{N+1} \simeq \pi_0 \operatorname{Diff}(D \setminus \{p_0, \ldots, p_N\}, \partial D)$ is a braid group which acts naturally on ordered subsets of $\pi_1(U_{q_0})$ satisfying (3.3). The braid group also acts formally by left and right mutations on the set of full exceptional collections $\mathrm{D}^b(Z/L)_w = \langle E_0, \ldots, E_N \rangle$, and these two actions are compatible (See Figure 3.3).
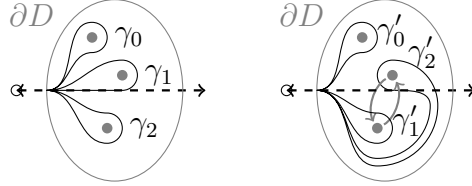
Factoring spherical twists

The arguments used to establish fractional window shift autoequivalences extend to the general setting of Section 3.2. Suppose that $S : \mathcal{E} \to \mathbf{G}$ is a spherical dg functor between pre-triangulated dg categories. Assume that $\mathcal{E}$ and $\mathbf{G}$ have generators and that S and its adjoints are representable by bimodules. Recall that since S is a spherical functor, the functor

$$F_S = \operatorname{Cone}\big(\operatorname{id} \to RS\big)$$

is an equivalence.

Figure 3.3: Dictionary between action of B_3 on loops in $D \setminus \{p_0, p_1, p_2\}$ and on full exceptional collections of $\mathrm{D}^b(Z/L)_w$ – Loops $(\gamma_0, \gamma_1, \gamma_2)$ correspond to full exceptional collection $\langle E_0, E_1, E_2 \rangle$. After acting by a generator of B_3, $\gamma_1' = \gamma_2$. The corresponding full exceptional collection is the right mutation $\langle E_0, E_2, R_{E_2}E_1 \rangle$. Note that $[\gamma_0 \circ \gamma_1 \circ \gamma_2] = [\gamma_0' \circ \gamma_1' \circ \gamma_2']$, consistent with the fact that the twists T_{E_i} for any full exceptional collection compose to Φ_w.

We will now discuss a sufficient condition for a spherical twist to factor into a composition of other spherical twists. In the following, angle brackets will be used to denote the category generated by a pair (tuple) of semiorthogonal subcategories of the ambient category as well as to assert that a given category admits a semiorthogonal decomposition.

Theorem 3.3.13. *Suppose that $\mathcal{E} = \langle \mathcal{A}, \mathcal{B} \rangle$ and assume that the cotwist functor $F_S : \mathcal{E} \to \mathcal{E}$ has the property that there is a semiorthogonal decomposition*

$$\mathcal{E} = \langle F_S(\mathcal{B}), \mathcal{A} \rangle.$$

Then the restrictions $S_{\mathcal{A}} = S|_{\mathcal{A}}$ and $S_{\mathcal{B}} = S|_{\mathcal{B}}$ are spherical and

$$T_S = T_{S_{\mathcal{A}}} \circ T_{S_{\mathcal{B}}}.$$

By Theorem 3.2.15 there exists a dg category $\mathcal{C}$ such that

$$\mathcal{C} = \langle \mathcal{E}, \mathbf{G} \rangle = \langle \mathbf{G}, \mathcal{E}' \rangle = \langle \mathcal{E}', \mathbf{G}' \rangle = \langle \mathbf{G}', \mathcal{E} \rangle$$

where S, the spherical functor, is the composite $i_{\mathbf{G}}^L i_{\mathcal{E}}$. We use the two mutation equivalences $\mathbb{R}_{\mathbf{G}}, \mathbb{L}_{\mathbf{G}'} : \mathcal{E} \to \mathcal{E}'$ to induce decompositions $\mathcal{E}' = \langle \mathcal{A}_R', \mathcal{B}_R' \rangle = \langle \mathbb{R}_{\mathbf{G}}(\mathcal{A}), \mathbb{R}_{\mathbf{G}}(\mathcal{B}) \rangle$ and $\mathcal{E}' = \langle \mathcal{A}_L', \mathcal{B}_L' \rangle := \langle \mathbb{L}_{\mathbf{G}'}(\mathcal{A}), \mathbb{L}_{\mathbf{G}'}(\mathcal{B}) \rangle$ respectively. Then due to the identity $F_S \simeq \mathbb{R}_{\mathbf{G}'} \mathbb{R}_{\mathbf{G}}[1]$, the hypothesis in the statement of Theorem 3.3.13 is equivalent to the existence of a semiorthogonal decomposition

$$\mathcal{E}' = \langle \mathcal{B}_R', \mathcal{A}_L' \rangle \tag{3.12}$$

We will need the following

Lemma 3.3.14. *Under the hypothesis of Theorem 3.3.13, $(\mathcal{A}_L')^{\perp} = {}^{\perp}\mathcal{A}$ and ${}^{\perp}(\mathcal{B}_R') = \mathcal{B}^{\perp}$ as subcategories of $\mathcal{C}$.*

Proof. We deduce that $(\mathcal{A}_L')^{\perp} = {}^{\perp}\mathcal{A}$ from the following sequence of mutations

$$\mathcal{C} = \langle \mathcal{A}, \mathcal{B}, \mathbf{G} \rangle = \langle \mathbf{G}, \mathcal{A}_R', \mathcal{B}_R' \rangle = \langle \mathbf{G}, \mathcal{B}_R', \mathcal{A}_L' \rangle = \langle \mathcal{B}, \mathcal{G}, \mathcal{A}_L' \rangle. \tag{3.13}$$

where the appearance of $\mathcal{A}'_L$ follows from (3.12). Similarly for $^\perp(\mathcal{B}'_R) = \mathcal{B}^\perp$ we consider

$$\mathcal{C} = \langle \mathbf{G}', \mathcal{A}, \mathcal{B} \rangle = \langle \mathcal{A}'_L, \mathcal{B}'_L, \mathbf{G}' \rangle = \langle \mathcal{B}'_R, \mathcal{A}'_L, \mathbf{G}' \rangle = \langle \mathcal{B}'_R, \mathcal{G}', \mathcal{A} \rangle. \qquad (3.14)$$

$\square$

Proof of Theorem 3.3.13. By assumption we have the semiorthogonal decomposition 3.12, which implies that $\mathcal{B}'_R$ is left admissible and $\mathcal{A}'_L$ is right admissible in $\mathcal{C}$. Furthermore Lemma 3.3.14 implies that $(\mathcal{A}'_L)^\perp \cap {}^\perp\mathcal{B}'_R = {}^\perp\mathcal{A} \cap \mathcal{B}^\perp$, and we call this category $\mathcal{G}_e$. Thus we have semiorthogonal decompositions

$$\mathcal{C} = \langle \mathcal{B}'_R, \mathbf{G}_e, \mathcal{A}'_L \rangle = \langle \mathcal{A}, \mathbf{G}_e, \mathcal{B} \rangle.$$

In particular we have a semiorthogonal decomposition $\mathcal{B}^\perp = {}^\perp\mathcal{B}'_R = \langle \mathbf{G}_e, \mathcal{A}'_L \rangle$

Combining this with the semiorthogonal decompositions (3.13) and (3.14) we obtain

$$^\perp\mathcal{A} = \langle \mathcal{B}, \mathcal{G} \rangle = \langle \mathcal{G}_e, \mathcal{B} \rangle = \langle \mathcal{B}'_R, \mathcal{G}_e \rangle = \langle \mathcal{G}, \mathcal{B}'_R \rangle, \text{ and}$$
$$\mathcal{B}^\perp = \langle \mathcal{A}, \mathbf{G}_e \rangle = \langle \mathbf{G}', \mathcal{A} \rangle = \langle \mathcal{A}'_L, \mathbf{G}' \rangle = \langle \mathcal{G}_e, \mathcal{A}'_L \rangle.$$

An analogous analysis of $^\perp\mathcal{A} = \langle \mathcal{B}, \mathbf{G} \rangle$ gives the sequence of semiorthogonal decompositions.

Thus Theorem 3.2.11 implies that the functors $S_B := i_{\mathcal{G}}^L i_{\mathcal{B}} : \mathcal{B} \to \mathcal{G}$ and $\tilde{S}_{\mathcal{A}} := i_{\mathcal{G}_e}^L i_{\mathcal{A}} : \mathcal{A} \to \mathcal{G}_e$ are spherical. Note that the left adjoints $i_{\mathcal{G}}^L$ to the inclusions $i_{\mathcal{G}} : \mathcal{G} \to \langle \mathcal{B}, \mathcal{G} \rangle$ and to $i_{\mathcal{G}_e} : \mathcal{C}_{\mathcal{G}_e} \to \langle \mathcal{A}, \mathcal{G}_e \rangle$ are the restrictions of the corresponding adjoints for the inclusions into $\mathcal{C}$, so there is no ambiguity in writing $i_{\mathcal{G}}^L$ and $i_{\mathcal{G}_e}^L$ without further specification.

$$\mathcal{G} \underset{\substack{\longrightarrow \\ \mathbb{L}_{\mathcal{B}'_R} = i_{\mathcal{G}}^L i_{\mathcal{G}_e}}}{\overset{\substack{\mathbb{L}_{\mathcal{B}} = i_{\mathcal{G}_e}^L i_{\mathcal{G}} \\ \circlearrowleft T_{S_{\mathcal{B}}} \longrightarrow}}{\rightleftharpoons}} \mathcal{G}_e \underset{\substack{\longrightarrow \\ \mathbb{L}_{\mathcal{A}'_L} = i_{\mathcal{G}_e}^L i_{\mathcal{G}'}}}{\overset{\substack{\mathbb{L}_{\mathcal{A}} = i_{\mathcal{G}'}^L i_{\mathcal{G}_e} \\ \circlearrowleft T_{\tilde{S}_{\mathcal{A}}} \longrightarrow}}{\rightleftharpoons}} \mathcal{G}' \qquad (3.15)$$

Let $\phi : \mathbf{G}_e \to \mathbf{G}$ denote the isomorphism $i_{\mathcal{G}}^L i_{\mathcal{G}_e} = \mathbb{L}_{\mathcal{B}'_R}$ whose inverse is $\phi^{-1} = i_{\mathcal{G}_e}^R i_{\mathcal{G}}$. One checks that $S_{\mathcal{A}} := i_{\mathcal{G}}^L i_{\mathcal{A}}$ is equivalent to $\phi \circ \tilde{S}_{\mathcal{A}} : \mathcal{A} \to \mathcal{G}$ and is thus spherical, and $T_{S_{\mathcal{A}}} \simeq \phi \circ T_{\tilde{S}_{\mathcal{A}}} \circ \phi^{-1}$. Following the various isomorphisms in the diagram (3.15) shows that

$$T_{S_{\mathcal{A}}} \circ T_{S_{\mathcal{B}}} = \phi \circ T_{\tilde{S}_{\mathcal{A}}} \circ \phi^{-1} \circ T_{S_{\mathcal{B}}} = \mathbb{L}_{\mathcal{B}'_R} \mathbb{L}_{\mathcal{A}'_L} \mathbb{L}_{\mathcal{A}} \mathbb{L}_{\mathcal{B}} = T_S$$

$\square$

Example 3.3.15. Let X be a smooth projective variety, and $j : Y \hookrightarrow X$ a smooth divisor. Then the restriction functor $S = j^* : \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ has a right adjoint $R = j_*$ and a left adjoint $L = j_*(\bullet \otimes \mathcal{O}_Y(Y)[-1])$. The cotwist $F_S = \mathrm{Cone}(\mathrm{id} \to j_*j^*) \simeq \bullet \otimes \mathcal{O}_X(-Y)[1]$ is an equivalence, and $F_S L \simeq R$ by the projection formula. The corresponding spherical twist autoequivalence of $\mathrm{D}^b(Y)$ is

$$T_S(F^\bullet) = \mathrm{Cone}(j^*(j_*F^\bullet)) \otimes \mathcal{O}_Y(Y)[-1] \to F^\bullet) \simeq F^\bullet \otimes \mathcal{O}_Y(Y)$$

In the special case where Y is an anticanonical divisor, so that $F_S \simeq \bullet \otimes \omega_X[1]$. Then for any semiorthogonal decomposition $\mathrm{D}^b(X) = \langle \mathcal{A}, \mathcal{B} \rangle$ we have $\mathrm{D}^b(X) = \langle F_S(\mathcal{B}), \mathcal{A} \rangle$ by Serre duality, so Theorem 3.3.13 applies.

Example 3.3.16. An example studied in [2] is that of a hypersurface $j : Y \hookrightarrow X$ where $\pi : Y \simeq \mathbb{P}(E) \to M$ is a projective bundle of rank $r \geq 1$ over a smooth projective variety M. Then $j_*\pi^* : \mathrm{D}^b(M) \to \mathrm{D}^b(X)$ is spherical iff $\mathcal{O}_Y(Y) \simeq \pi^*L \otimes \mathcal{O}_\pi(-r)$. In this case the cotwist is tensoring by a shift of L, so if $L \simeq \omega_M$, then Theorem 3.3.13 applies to any semiorthogonal decomposition $\mathrm{D}^b(M) = \langle \mathcal{A}, \mathcal{B} \rangle$

Autoequivalences of complete intersections

Suppose $X_s \subset X$ is defined by the vanishing of a regular section s of a vector bundle $\mathcal{V}^\vee$. In this section, we will use a standard construction to produce autoequivalences of $\mathrm{D}(X_s)$ from variations of GIT for the total space of $\mathcal{V}$. This forms a counterpart to [6, Sections 4,5], where equivalences between different complete intersections are considered.

We are interested in the case where the total space of $\mathcal{V}$ is Calabi-Yau. If $X = \mathbb{P}^n$ and $\mathcal{V}$ is completely decomposable, this is equivalent to X_s being Calabi-Yau. Since X_s is defined by a regular section, the Koszul complex $(\wedge^\bullet \mathcal{V}, d_s)$ is a resolution of $\mathcal{O}_{X_s}$. The key ingredient in this discussion is an equivalence of categories between $\mathrm{D}(X_s)$ and a category of generalized graded matrix factorizations associated to the pair $(\mathcal{V}, s)$.

We call the data $(\mathfrak{X}, W)$ where $\mathfrak{X}$ is a stack equipped with a $\mathbb{C}^*$ action factoring through the squaring map and W is a regular function of weight 2 a *Landau-Ginzburg (LG) pair*. Let $\pi : \mathcal{V} \to X$ be the vector bundle structure map. There is an obvious action of $\mathbb{C}^*$ on $\mathcal{V}$ by scaling along the fibers of π. We equip $\mathcal{V}$ instead with the square of this action, so that λ acts as scaling by λ^2. Since s is a section of $\mathcal{V}^\vee$, it defines a regular function W on $\mathcal{V}$ that is linear along the fibers of π. By construction it has weight 2 for the $\mathbb{C}^*$ action. The total space of $\mathcal{V}|_{X_s}$ is $\mathbb{C}^*$-invariant and when we equip X_s with the trivial $\mathbb{C}^*$ action we obtain a diagram

$$\begin{array}{ccc} \mathcal{V}|_{X_s} & \xrightarrow{\ i\ } & \mathcal{V} \\ \downarrow{\scriptstyle \pi} & & \\ X_s & & \end{array}$$

of LG pairs where the potentials on $\mathcal{V}|_{X_s}$ and X_s are zero.

The category of curved coherent sheaves on an LG pair $\mathrm{D}(\mathfrak{X}, W)$ is the category whose objects are $\mathbb{C}^*$-equivariant coherent sheaves $\mathcal{F}$ equipped with an endomorphism d of weight 1 such that $d^2 = W \cdot \mathrm{id}$; and whose morphisms are obtained by a certain localization procedure. (For details, see [42].) The maps in the above diagram induce functors

$$\mathrm{D}(X_s) = \mathrm{D}(X_s, 0) \xrightarrow{\ \pi^*\ } \mathrm{D}(\mathcal{V}|_{X_s}, 0) \xrightarrow{\ i_*\ } \mathrm{D}(\mathcal{V}, W)$$

whose composite $i_*\pi^*$ is an equivalence.

Suppose that V is a smooth quasiprojective variety with an action of a reductive algebraic group $G \times \mathbb{C}^*$, where $\mathbb{C}^*$ acts through the squaring map. Let W be a regular function on V which is G invariant and has weight 2 for $\mathbb{C}^*$. Suppose that $\mathcal{L}$ is a $G \times \mathbb{C}^*$ equivariant line bundle so that $(\mathcal{V}, W) \cong (V^{ss}(\mathcal{L})/G, W)$ equivariantly for the $\mathbb{C}^*$ action. For simplicity

assume that $V^u(\mathcal{L})$ consists of a single KN stratum S with 1 PSG λ. Let Z be the fixed set for λ on this stratum and Y its blade. Write $\sigma : Z \to V$ for the inclusion. As above we define full subcategories of $D(V/G, W)$. Let $\mathbf{G}_w \subset D(V/G, W)$ be the full subcategory of objects isomorphic to objects of the form $(\mathcal{E}, d)$ where $\sigma^*\mathcal{E}$ has λ-weights in $[w, w + \eta)$. We also define the larger subcategory $\mathcal{C}_w$ where the weights lie in $[w, w + \eta]$. The analysis for the derived category can be adapted to the category of curved coherent sheaves [6] and we see that $\mathbf{G}_w$ is admissible in $\mathcal{C}_w$. The maps $i : Y \to V$ and $p : Y \to Z$ induce functors $p^* : D(Z/L, W|_Z) \to D(Y/P, W|_Y)$ and $i_* : D(Y/P, W|_Y) \to D(V/G, W)$. Let $D(Z/L, W|_Z)_w$ be the full subcategory of curved coherent sheaves concentrated in λ-weight w. Then $i_*p^* : D(Z/L, W|_Z)_w \to D(V/G, W)$ is fully faithful and has image $\mathcal{A}_w$.

We now consider a balanced wall crossing which exchanges $\lambda = \lambda^+$ with $\lambda^{-1} = \lambda^-$ and $S = S^+$ for S^-. Then we obtain wall crossing equivalences. Since $\mathcal{C}_w$ and $\mathbf{G}_w$ are defined by weight conditions, as above we see that $\mathcal{C}_w^+ = \mathcal{C}_{-w-\eta}^-$ and $\mathbf{G}_w^-$ is the left orthogonal to $\mathcal{A}_w$. Therefore, the window shift autoequivalence in this context is still realized by a mutation.

Example 3.3.17. We consider a K3 surface X_s obtained as a complete intersection of type $(2, 0), (1, 3)$ in $X = \mathbb{P}^2 \times \mathbb{P}^2$. It is well known that line bundles on a K3 surface are spherical. We will see that the window shift automorphisms of $D(X_s)$ coming from VGIT as above are the compositions of spherical twists around $\mathcal{O}_{X_s}(i, 0)$ then $\mathcal{O}_{X_s}(i + 1, 0)$ or around $\mathcal{O}_{X_s}(0, i), \mathcal{O}_{X_s}(0, i + 1)$, and $\mathcal{O}_{X_s}(0, i + 2)$.

Let $\mathcal{V} = \mathcal{O}(-2, 0) \oplus \mathcal{O}(-1, -3)$. Recall that the total space of $\mathcal{V}$ is a toric variety which can be obtained as a GIT quotient of $\mathbb{A}^8$ by $(\mathbb{C}^*)^2$ under the action

$$(t, s) \mapsto (t, t, t, s, s, s, t^{-2}, t^{-1}s^{-3}).$$

We also recall that the wall and chamber decomposition of $\mathbb{R}^2$ associated to this action is given in the following diagram.
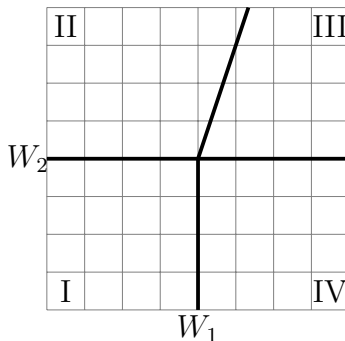
Chamber I corresponds to tot $\mathcal{V}$ and we will analyze the autoequivalences of X_s that come from the walls W_1 and W_2. The window shift autoequivalences of $\mathbf{D}^b(\text{tot } \mathcal{V})$ coming from W_1 do not factor because the associated Z/L is not compact. However, in the presence of a potential, Z/L becomes compact. In fact, the associated Landau-Ginzburg model actually admits a full exceptional collection. To proceed we must compute the KN stratifications

near the walls. Write $V_\bullet$ for the locus defined by the vanishing of the variables occurring in $\bullet$. (So V_x is the locus where all of the x_i are zero.) We obtain the table below.

Near W_1

	Chamber I			Chamber IV	
λ_0	0	V_{xy}	λ_0	0	V_{yp}
$(0,-1)$	$V_{yq} \setminus V_x$	$V_y \setminus V_x$	$(0,-1)$	$V_{yq} \setminus V_x$	$V_y \setminus V_x$
$(-1,0)$	$V_{xpq} \setminus V_{xy}$	$V_x \setminus V_{xy}$	$(1,0)$	$V_{xpq} \setminus V_{xy}$	$V_{pq} \setminus V_y$

Near W_2

	Chamber I			Chamber II	
λ_0	0	V_{xy}	λ_0	0	V_x
$(-1,0)$	$V_{xpq} \setminus V_{xy}$	$V_x \setminus V_{xy}$	$(-1,0)$	$V_{xpq} \setminus V_{xy}$	$V_x \setminus V_{xy}$
$(0,-1)$	$V_{yq} \setminus V_x$	$V_y \setminus V_x$	$(0,1)$	$V_{yq} \setminus V_x$	$V_q \setminus V_x$

Table 3.1: The Kirwan-Ness stratification for T acting on $\mathbb{A}^8$

Consider the potential $W = pf + gq \in \mathbb{C}[x_i, y_i, p, q]_{i=0}^2$, where $f \in \mathbb{C}[x_i]$ is homogeneous of degree 2 and $g \in \mathbb{C}[x_i, y_i]$ is homogeneous of degree $(1,3)$. In order to define an LG pair, we must also specify a second grading on $\mathbb{C}[x_i, y_i, p, q]$. We define the LG weights of p and q to be 2. Assume that f defines a smooth rational curve in $\mathbb{P}^2$. In order to proceed, we need to introduce a particular type of curved coherent sheaf. Consider a line bundle $\mathcal{L}$ on an LG pair which is equivariant for the $\mathbb{C}^*$ action. Given sections $a \in \Gamma(\mathcal{L})$ and $b \in \Gamma(\mathcal{L}^\vee)$ of weight 1, we form a curved coherent sheaf for the potential $b(a)$:

$$\mathcal{O} \underset{b}{\overset{a}{\rightleftarrows}} \mathcal{L}, \quad \text{i.e.} \quad d = \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix},$$

and denote it by $\{a, b\}$. We also write $\mathcal{O}_{triv} = \{1, W\}$ (where $1, W$ are weight 1 section and co-section of $\mathcal{O}(-1)_{LG}$). This object is isomorphic to zero in the category of curved coherent sheaves.

Let us analyze what happens near W_1. First, we have computed that for the least unstable stratum

$$Z_1/L_1 = (V_{xpq} \setminus V_{xy})/T \cong \mathbb{P}^2/\mathbb{C}^*.$$

Next, we notice that $W|_{Z_1} = 0$ and that Z_1 is contained in the fixed set for the LG $\mathbb{C}^*$ action. Therefore the category $\mathrm{D}(Z_1/L_1, W|_{Z_1}) \cong \mathrm{D}(\mathbb{P}^2/\mathbb{C}^*)$ and for any w we have $\mathrm{D}(\mathbb{P}^2/\mathbb{C}^*)_w \cong \mathrm{D}(\mathbb{P}^2)$. It is well known that $\mathrm{D}(\mathbb{P}^2)$ admits a full exceptional collection of length 3. For

example $D(\mathbb{P}^2) = \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle$. By the curved analog of Proposition 3.2.4, we compute the spherical object associated to $\mathcal{O}(i)$ on $\mathbb{P}^2$ by pulling it back to $V_{pq} \backslash V_y$, pushing it forward to $V \backslash V_y$, then restricting it to $\mathcal{V} = (V \backslash V_x \cup V_y)/T$. The locus V_{pq} restricts to the zero section of $\mathcal{V}$, which we also denote by X. The object corresponding to $\mathcal{O}(i)$ on Z_1/L_1 is the line bundle $\mathcal{O}_X(0, i)$, viewed as a curved coherent sheaf supported on the zero section. This object corresponds to an object of $D(X_s)$. To compute this object we observe that there are short exact sequences

$$0 \longrightarrow \mathcal{O}_{triv} \otimes \{q, g\} \longrightarrow \{p, f\} \otimes \{q, g\} \longrightarrow \mathcal{O}_{p=0} \otimes \{q, g\} \longrightarrow 0$$

$$0 \longrightarrow (\mathcal{O}_{p=0})_{triv} \longrightarrow \mathcal{O}_{p=0} \otimes \{q, g\} \longrightarrow \mathcal{O}_S \longrightarrow 0$$

This implies that $\mathcal{O}_X(0, i)$ is equivalent to $\{p, f\} \otimes \{q, g\} \otimes \mathcal{O}(0, i)$ in $D(\mathcal{V}, W)$. Using the analogous short exact sequences for f and g we see that it is also equivalent to $\mathcal{O}_{\mathcal{V}|_Y}(0, i)$. However, this is the image of $\mathcal{O}_{X_s}(0, i)$ under the equivalence $D(X_s) \cong D(\mathcal{V}, W)$.

Next, we consider the wall W_2. In this case, we have

$$Z_2/L_2 = (V_{yq} \backslash V_x)/T \cong (\text{tot } \mathcal{O}_{\mathcal{P}^2}(-2))/\mathbb{C}^*.$$

Moreover $W|_{Z_2} = pf$. So we have $D(Z_2/L_2, W|_{Z_2}) \cong D(C/\mathbb{C}^*)$, where $C \subset \mathbb{P}^2$ is the rational curve defined by f. This means that for any fixed w, $D(Z_2/L_2, W|_{Z_2})_w \cong D(\mathbb{P}^1)$. Of course, we have $D(\mathbb{P}^1) = \langle \mathcal{O}, \mathcal{O}(1) \rangle$. We play a similar game to compute the objects in $D(Y)$ corresponding to these line bundles. First, $\mathcal{O}_C(i)$ corresponds to the curved coherent sheaf $\mathcal{O}_{\text{tot } \mathcal{O}_C(-2)}(i)$ on Z_2/L_2. We push this forward and restrict to $\mathcal{V}$ to get the line bundle $\mathcal{O}(i, 0)$ on the locus $\{q = f = 0\}$. By considering short exact sequences as in the previous case, we see that these objects correspond to the objects $\mathcal{O}_{X_s}(i, 0)$ in $D(X_s)$.

Chapter 4

Stratifications of algebraic stacks

In this chapter we investigate extensions of the KN stratification of the unstable locus in geometric invariant theory. Our goal is to give the KN stratification an "intrinsic", or modular, interpretation. We show that the strata are open substacks of connected components of the mapping stack $\mathfrak{X}(\Theta) = \underline{\mathrm{Hom}}(\Theta, \mathfrak{X})$, where $\mathfrak{X} = X/G$.

We begin by studying the stack Θ itself, and in particular we establish a general classification of principal G bundles and families of principal G bundles on Θ. We use this to show that if $\mathfrak{X}$ is locally a quotient of a finite type k scheme by a locally affine action of a linear group, then the mapping stack $\underline{\mathfrak{X}}(\Theta)$ is also algebraic, and in fact is locally a quotient stack as well.

Next we describe a method of constructing stratifications of an arbitrary local quotient stack which mimic the stratifications in GIT in that the strata are identified with open substacks of connected components of $\underline{\mathfrak{X}}(\Theta)$. The role that the Hilbert-Mumford numerical invariant plays in stratifying the unstable locus in GIT can be generalized by a numerical invariant which depends on a choice of cohomology classes $l \in H^2(\mathfrak{X}; \mathbb{Q})$ and $b \in H^4(\mathfrak{X}; \mathbb{Q})$.

We show how this concept can be applied to the moduli of vector bundles over a curve to recover the Shatz stratification of the moduli of unstable bundles. We also show how to reformulate Kempf's original proof of the existence of the KN stratification intrinsically, which leads to criteria under which the classes l and b define a stratification.

4.1 The stack Θ

In this section we observe some basic properties of $\Theta := \mathbb{A}^1/\mathbb{G}_m$. First of all, Θ has two geometric points, the generic point $1 \in \mathbb{A}^1$ and the special point $0 \in \mathbb{A}^1$. We will discuss quasicoherent sheaves over Θ as well as its cohomology when $k = \mathbb{C}$. Our most important result is the description of the moduli of principal G-bundles over Θ, which plays a key role in Section 4.2 where we establish that $\underline{\mathfrak{X}}(\Theta)$ is an algebraic stack.

Quasicoherent sheaves on Θ

We review the properties of the category of quasicoherent sheaves on Θ. It is equivalent to the category of equivariant quasicoherent sheaves on $\mathbb{A}^1$, which in turn is equivalent to the category of graded modules over the graded ring $k[t]$, where t has degree -1.[1]

Proposition 4.1.1. *The category of quasicoherent sheaves on Θ is equivalent to the category of diagrams of vector spaces of the form*

$$\cdots \to V_i \to V_{i-1} \to \cdots$$

The equivalence assigns a vector space with a decreasing filtration to the module $\bigoplus V_i$ with V_i in degree i, and multiplication by t^k acts by the inclusion $V_i \subset V_{i-k}$. Coherent sheaves correspond to filtered vector spaces such that V_i stabilizes for $i \ll 0$ and $V_i = 0$ for $i \gg 0$.

Remark 4.1.2. The proposition implies that the corresponding derived categories are equivalent as well. The maps $V_i \to V_{i-1}$ need not be injective – injectivity corresponds to the quasicoherent sheaf being torsion free. Note that any object of the derived category $D(\Theta)$ can be represented by a complex of torsion free sheaves, i.e. an honest filtered complex.

Quasicoherent sheaves over $*/\mathbb{G}_m$ are just graded vector spaces. The restriction to the origin of a quasicoherent sheaf on Θ corresponds to taking the associated graded of the filtered vector space. Likewise, restriction to the open substack $\Theta - \{0\} \simeq \mathrm{Spec}(k)$ corresponds to taking the colimit $\varinjlim(V_i)$.

Θ is a quotient of an affine variety by a reductive group, so the push forward to $\mathrm{Spec}(k)$ is exact. Thus if $\cdots V_i^\bullet \to V_{i-1}^\bullet \to \cdots$ is a complex of filtered vector spaces, the derived global section functor on Θ corresponds to taking the complex $V_0^\bullet$.

Lemma 4.1.3. *Let $\pi : \Theta \to \mathrm{Spec}(k)$, then π^* has a left adjoint $\pi_!$, and π_* has a right adjoint $\pi^!$. These functors extend to adjoint functors for the respective functors on the derived category $D(\Theta)$.*

Proof. It is straightforward to verify that

$$\pi_!(V_\bullet) = \mathrm{coker}(V_1 \to \varinjlim V_i), \text{ and } \pi^!(W) = (k[t^\pm]/k[t]) \otimes_k W$$

The functor $\pi^!$ is exact, hence extends to the derived category. The functor $\pi_!$ extends to the derived category via torsion free replacement, and can be expressed as

$$\pi_!(V_\bullet^\bullet) = \mathrm{cone}(V_1^\bullet \to \varinjlim V_i^\bullet)$$

$\square$

---

[1]It is perhaps more natural from an algebraic perspective to consider graded modules over a positively graded ring. Reflecting the gradings $M^i := M_{-i}$ gives an equivalence between the categories of graded modules over the negatively and positively graded polynomial rings. The negative grading is more natural from the geometric perspective, where the $\mathbb{G}_m$ acts by scaling on $\mathbb{A}^1$.

Quasicoherent sheaves on $\Theta \times \mathfrak{X}$

The description of $D(\Theta)$ extends to $D(\Theta \times \mathfrak{X})$ as well, where $\mathfrak{X}$ is an arbitrary stack. Let C be the diagram $\cdots \to \bullet \to \bullet \to \cdots$, then

Proposition 4.1.4. *The category of quasicoherent sheaves on $\Theta \times \mathfrak{X}$ is equivalent to the category of diagrams of quasicoherent sheaves on $\mathfrak{X}$ of the form $\cdots \to V_i \to V_{i-1} \to \cdots$. Likewise we have an equivalence of dg-categories $D(\Theta \times \mathfrak{X}) \simeq \mathrm{Fun}(C, D(\mathfrak{X}))$.*

Proof. This is a general version of the Rees construction. Pulling back to $\mathbb{A}^1 \times \mathfrak{X}$ and pushing forward to $\mathfrak{X}$ identifies quasicoherent sheaves over $\Theta \times \mathfrak{X}$ with the category of graded quasicoherent modules over the graded algebra $\mathcal{O}_{\mathfrak{X}}[T]$ with T in degree -1. The quasicoherent sheaves V_i on $\mathfrak{X}$ correspond to the degree i piece of a graded $\mathcal{O}_{\mathfrak{X}}[T]$ module, and the maps $V_i \to V_{i-1}$ correspond to multiplication by T. The identification $D(\Theta \times \mathfrak{X}) \simeq \mathrm{Fun}(C, D(\mathfrak{X}))$ follows from this description of quasicoherent sheaves. One can also realize this as a consequence of the Morita theory of [7] and the description of $D(\Theta)$ as the category of representations of C. $\square$

Topology of Θ

$\mathrm{Pic}(\Theta) \simeq \mathbb{Z}$ consists of line bundles of the form $\mathcal{O}_\Theta(n)$ which correspond to the free $k[t]$ module with generator in degree $-n$. In particular $\Gamma(\Theta, \mathcal{O}_\Theta(n)) = 0$ for $n > 0$ and $\Gamma(\Theta, \mathcal{O}_\Theta(n)) \simeq k$ for $n \leq 0$. Note that unlike $\mathrm{Pic}(*/\mathbb{G}_m{}^*)$, the invertible sheaf $\mathcal{O}(1)$ can be taken as a canonical generator. It is distinguished from $\mathcal{O}(-1)$ because it has no nonvanishing global sections.

Assume for the moment that $k = \mathbb{C}$. Then Θ has an underlying topological stack whose weak homotopy type [34] is that of the homotopy quotient $\mathbb{C} \times_{B\mathbb{C}^*} E\mathbb{C}^*$, which deformation retracts onto $B\mathbb{C}^* \simeq \mathbb{CP}^\infty$. In particular we have

$$H^*(\Theta; R) = H^*_{\mathbb{C}^*}(\mathbb{C}; R) \simeq R[[q]]$$

Unlike $*/\mathbb{C}^*$, which has an automorphism acting as multiplication by -1 on H^2, the group $H^2(\Theta; R)$ has a canonical generator $q := c_1(\mathcal{O}_\Theta(1))$, and the K-theory $K_0(\Theta)$ is canonically isomorphic to $\mathbb{Z}[u^\pm]$ where $u = [\mathcal{O}_\Theta(1)]$. The fact that we can canonically identify $H^2(\Theta; \mathbb{Q})$ with $\mathbb{Q}$ will be essential in our construction of stratifications of algebraic stacks in Section 4.3.

Principal G-bundles on Θ

Using Proposition 4.1.4, one can show that a vector bundle on $\Theta \times \mathfrak{X}$ is the same as a descending sequence of vector bundles $\cdots V_{i+1} \subset V_i \subset \cdots$ such that V_i/V_{i+1} is a vector bundle for all i, $V_i = 0$ for $i \gg 0$, and V_i stabilizes for $i \ll 0$. However, here we take another approach to vector bundles on Θ which generalizes to arbitrary principal G-bundles and to families of G-bundles. Throughout this section k will be an algebraically closed field and

G will denote a smooth affine group scheme over k. If S and X are a schemes over k, we will use the notation X_S to denote the S scheme $S \times X$, and we use similar notation for the pullback of stacks over k.

For a scheme S, we use the phrase G-bundle over S, principal G-bundle over S, and G-torsor over S interchangeably to refer to a scheme $E \to S$ along with a right action of G_S (left action of G_S^{op}) such that $E \times_S G_S \to E \times_S E$ is an isomorphism and $E \to S$ admits a section étale locally. We can equivalently think of E as the sheaf of sets which it represents over the étale site of S.[2] By definition a principal G-bundle is a morphism $S \to */G$. Similarly one can define a principal G-bundle over a stack to be a map $\mathfrak{X} \to */G$.

Lemma 4.1.5. *Let S be a k scheme. A principal G-bundle over $\Theta \times S$ is a principal bundle $E \to \mathbb{A}^1 \times S$ with a $\mathbb{G}_m$ action on E which is compatible with the action on $\mathbb{A}^1$ under projection and which commutes with the right action of G on E.*

Proof. This is an straightforward interpretation of the descent property of the stack $*/G$ and will be discussed in general in section 4.2 below (See Diagram 4.2). □

If $E \to \mathbb{A}^1$ is a principal G-bundle with compatible $\mathbb{G}_m$ action, we will often say "E is a G-bundle over Θ" even though more accurately, $E/\mathbb{G}_m$ is a G-bundle over Θ.

Given a one parameter subgroup $\lambda : \mathbb{G}_m \to G$, we define the standard G-bundle $E_\lambda := \mathbb{A}^1 \times G$ where G acts by right multiplication and $t \cdot (z, g) = (tz, \lambda(t)g)$. We will show that every G-bundle over Θ is isomorphic to E_λ for some one parameter subgroup. In fact we will obtain a complete description of the groupoid of principal G-bundles over Θ as a corollary of the following main result

Proposition 4.1.6. *Let S be a connected finite type k-scheme and let E be a G-bundle over $\Theta_S := \Theta \times S$. Let $\lambda : \mathbb{G}_m \to G$ be a 1PS conjugate to the one parameter subgroup $\mathbb{G}_m \to \mathrm{Aut}(E_s) \simeq G$ for some $s \in S(k)$ thought of as the point $(0, s) \in \mathbb{A}^1_S$. Then*

1. *There is a unique reduction of structure group $E' \subset E$ to a P_λ-torsor such that $\mathbb{G}_m \to \mathrm{Aut}(E'_s) \simeq P_\lambda$ is conjugate in P_λ to λ, and*

2. *the restriction of E' to $\{1\} \times S$ is canonically isomorphic to the sheaf on the étale site of S mapping $T/S \mapsto \mathrm{Iso}((E_\lambda)_{\Theta_T}, E|_{\Theta_T})$.*

Proof. $(E_\lambda)_{\Theta_S} = E_\lambda \times S/\mathbb{G}_m$ is a G-bundle over Θ_S, and $\underline{\mathrm{Iso}}((E_\lambda)_{\Theta_S}, E)$ is a sheaf over Θ_S representable by a (relative) scheme over Θ_S. In fact, if we define a twisted action of $\mathbb{G}_m$ on E given by $t \star e := t \cdot e \cdot \lambda(t)^{-1}$, then

$$\underline{\mathrm{Iso}}((E_\lambda)_{\Theta_S}, E) \simeq E/\mathbb{G}_m \text{ w.r.t. the } \star\text{-action} \tag{4.1}$$

---

[2]For general G a sheaf torsor may only be represented by an algebraic space $E \to S$, but E is always a scheme when G is affine, see [32, III-Theorem 4.3].

as sheaves over Θ_S.[3]

The twisted $\mathbb{G}_m$ action on E is compatible with base change. Let $T \to S$ be an S-scheme. From the isomorphism of sheaves (4.1), there is a natural bijection between the set of isomorphisms $(E_\lambda)_{\mathbb{A}^1_T} \xrightarrow{\simeq} E|_{\mathbb{A}^1_T}$ as $\mathbb{G}_m$-equivariant G-bundles and the set of $\mathbb{G}_m$-equivariant sections of $E|_{\mathbb{A}^1_T} \to \mathbb{A}^1_T$ with respect to the twisted $\mathbb{G}_m$ action.

The morphism $E|_{\mathbb{A}^1_T} \to \mathbb{A}^1_T$ is separated, so a twisted equivariant section is uniquely determined by its restriction to $\mathbb{G}_m \times T$, and by equivariance this is uniquely determined by its restriction to $\{1\} \times T$. Thus we can identify $\mathbb{G}_m$-equivariant sections with the set of maps $T \to E$ such that $\lim_{t \to 0} t \star e$ exists and $T \to E \to \mathbb{A}^1_S$ factors as the given morphism $T \to \{1\} \times S \to \mathbb{A}^1_S$.

If we define the subsheaf of E over $\mathbb{A}^1_S$

$$E'(T) := \left\{ e \in E(T) | \mathbb{G}_m \times T \xrightarrow{t \star e(x)} E \text{ extends to } \mathbb{A}^1 \times T \right\} \subset E(T),$$

then we have shown that $E'|_{\{1\} \times S}(T) \simeq \mathrm{Iso}((E_\lambda)_{\Theta_T}, E|_{\Theta_T})$. Next we show in several steps that the subsheaf $E' \subset E$ over $\mathbb{A}^1_S$ is a torsor for the subgroup $P_\lambda \subset G$, so E' is a reduction of structure group to P_λ.

Step 1: *E' is representable:* $E \to S$ is affine and $\mathbb{G}_m$ invariant, so the action of $\mathbb{G}_m$ is locally affine. The functor E' is exactly the functor of Lemma 1.1.5, so Theorem 1.1.4 implies that E' is representable by a disjoint union of $\mathbb{G}_m$ equivariant locally closed subschemes of E.

Step 2: *$P_\lambda \subset G$ acts simply transitively on $E' \subset E$:* Because E is a G-bundle over $\mathbb{A}^1_S$, right multiplication $(e, g) \mapsto (e, e \cdot g)$ defines an isomorphism $E \times G \to E \times_{\mathbb{A}^1_S} E$. The latter has a $\mathbb{G}_m$ action, which we can transfer to $E \times G$ using this isomorphism.

For $g \in G(T)$, $e \in E(T)$, and $t \in \mathbb{G}_m(T)$ we have $t \star (e \cdot g) = (t \star e) \cdot (\lambda(t) g \lambda(t)^{-1})$. This implies that the $\mathbb{G}_m$ action on $E \times G$ corresponding to the diagonal action on $E \times_{\mathbb{A}^1_S} E$ is given by

$$t \cdot (e, g) = (t \star e, \lambda(t) g \lambda(t)^{-1})$$

---

[3]To see this, note that a map $T \to \Theta_S$ corresponds to a $\mathbb{G}_m$-bundle $P \to T$ along with a $\mathbb{G}_m$ equivariant map $f : P \to \mathbb{A}^1 \times S$. Then the restrictions $(E_\lambda \times S)_T$ and $E|_T$ correspond (via descent for G-bundles) to the $\mathbb{G}_m$-equivariant bundles $f^{-1}(E_\lambda \times S)$ and $f^{-1}E$ over P. Forgetting the $\mathbb{G}_m$-equivariant structure, the G-bundle $E_\lambda \times S$ is trivial, so an isomorphism $f^{-1}(E_\lambda \times S) \to f^{-1}E$ as G-bundles corresponds to a section of $f^{-1}E$, or equivalently to a lifting

$$
\begin{array}{ccc}
 & & E \\
 & \overset{\tilde{f}}{\nearrow} & \big\downarrow \\
P & \xrightarrow{\quad f \quad} & \mathbb{A}^1 \times S
\end{array}
$$

to a map $\tilde{f} : P \to E \to \mathbb{A}^1 \times S$. The isomorphism of G-bundles defined by the lifting $\tilde{f}$ descends to an isomorphism of $\mathbb{G}_m$-equivariant G-bundles $f^{-1}(E_\lambda \times S) \to f^{-1}E$ if and only if the lift $\tilde{f}$ is equivariant with respect to the twisted $\mathbb{G}_m$ action on E.

The subfunctor of $E \times G$ corresponding to $E' \times_{\mathbb{A}^1_S} E' \subset E \times_{\mathbb{A}^1_S} E$ consists of those points for which $\lim_{t \to 0} t \cdot (e, g)$ exists. This is exactly the subfunctor represented by $E' \times P_\lambda \subset E \times G$. We have thus shown that E' is equivariant for the action of P_λ, and $E' \times P_\lambda \to E' \times_{\mathbb{A}^1_S} E'$ is an isomorphism of sheaves.

Step 3: $p : E' \to \mathbb{A}^1_S$ *is smooth:* Proposition 1.1.6 implies that E' and $E^{\mathbb{G}_m} \subset E'$ are both smooth over S. The restriction of the tangent bundle $T_{E/S}|_{E^{\mathbb{G}_m}}$ is an equivariant locally free sheaf on a scheme with trivial $\mathbb{G}_m$ action, hence it splits into a direct sum of vector bundles of fixed weight with respect to $\mathbb{G}_m$. The tangent sheaf $T_{E'/S}|_{E^{\mathbb{G}_m}} \subset T_{E/S}|_{E^{\mathbb{G}_m}}$ is precisely the subsheaf with weight ≥ 0. By hypothesis $T_{E/S} \to p^* T_{\mathbb{A}^1_S/S}$ is surjective, and $p^* T_{\mathbb{A}^1_S/S}|_{E^{\mathbb{G}_m}}$ is concentrated in nonnegative weights, therefore the map

$$T_{E'/S}|_{E^{\mathbb{G}_m}} = (T_{E/S}|_{E^{\mathbb{G}_m}})_{\geq 0} \to p^* T_{\mathbb{A}^1_S/S}|_{E^{\mathbb{G}_m}} = (p^* T_{\mathbb{A}^1_S/S}|_{E^{\mathbb{G}_m}}))_{\geq 0}$$

is surjective as well.

Thus we have shown that $T_{E'/S} \to p^* T_{\mathbb{A}^1_S/S}$ is surjective when restricted to $E^{\mathbb{G}_m} \subset E'$, and by Nakayama's Lemma it is also surjective in a Zariski neighborhood of $E^{\mathbb{G}_m}$. On the other hand, the only equivariant open subscheme of E' containing $E^{\mathbb{G}_m}$ is E' itself. It follows that $T_{E'/S} \to p^* T_{\mathbb{A}^1_S/S}$ is surjective, and therefore that the morphism p is smooth.

Step 4: $p : E' \to \mathbb{A}^1_S$ *admits sections étale locally:* We consider the $\mathbb{G}_m$ equivariant G-bundle $E|_{\{0\} \times S}$. After étale base change we can assume that $E|_{\{0\} \times S}$ admits a non-equivariant section, hence the $\mathbb{G}_m$-equivariant structure is given by a homomorphism $(\mathbb{G}_m)_{S'} \to G_{S'}$. Lemma 4.1.8 implies that after further étale base change this homomorphism is conjugate to a constant homomorphism. Thus $E|_{\{0\} \times S'}$ is isomorphic to the trivial equivariant $\mathbb{G}_m$-bundle $(E_\lambda)_{\mathbb{A}^1_S} = \mathbb{A}^1_S \times G \to \mathbb{A}^1_S$ with $\mathbb{G}_m$ acting by left multiplication by $\lambda(t)$.

It follows that $E|_{\{0\} \times S'}$ admits an invariant section with respect to the twisted $\mathbb{G}_m$ action. In other words $(E_{\mathbb{A}^1_{S'}})^{\mathbb{G}_m} \to \{0\} \times S'$ admits a section, and $E^{\mathbb{G}_m} \subset E'$, so we have shown that $E' \to \mathbb{A}^1_{S'}$ admits a section over $\{0\} \times S'$. On the other hand, because $p : E' \to \mathbb{A}^1_{S'}$ is smooth and $\mathbb{G}_m$-equivariant, the locus over which p admits an étale local section is open and $\mathbb{G}_m$-equivariant. It follows that p admits an étale local section over every point of $\mathbb{A}^1_{S'}$.

$\square$

Remark 4.1.7. In fact we have shown something slightly stronger than the existence of étale local sections of $E' \to \mathbb{A}^1_S$ in Step 4. We have shown that there is an étale map $S' \to S$ such that $E'|_{S'} \to \mathbb{A}^1_{S'}$ admits a $\mathbb{G}_m$-equivariant section.

We now prove that families of one parameter subgroups of G are étale locally constant up to conjugation, which was the key fact in Step 4.

Lemma 4.1.8. *Let S be a connected k-scheme of finite type and let $\phi : (\mathbb{G}_m)_S \to G_S$ be a homomorphism of group schemes over S. Let $\lambda : \mathbb{G}_m \to G$ be a 1PS conjugate to ϕ_s for some $s \in S(k)$. Then the subsheaf*

$$F(T) = \left\{ g \in G(T) | \phi_T = g \cdot (\mathrm{id}_T, \lambda) \cdot g^{-1} : (\mathbb{G}_m)_T \to G_T \right\} \subset G_S(T)$$

is an L_λ-torsor. In particular ϕ is étale-locally conjugate to a constant homomorphism.

Proof. Verifying that $F \times L_\lambda \to F \times_S F$ given by $(g, l) \mapsto (g, gl)$ is an isomorphism of sheaves is straightforward. The more important question is whether $F(T) \neq \emptyset$ étale locally.

As in the proof of Proposition 4.1.6 we introduce a twisted $\mathbb{G}_m$ action on $G \times S$ by $t \star (g, s) = \phi_s(t) \cdot g \cdot \lambda(t)^{-1}$. Then $G \times S \to S$ is $\mathbb{G}_m$ invariant, and the functor $F(T)$ is represented by the map of schemes $(G \times S)^{\mathbb{G}_m} \to S$. By Proposition 1.1.6, $(G \times S)^{\mathbb{G}_m} \to S$ is smooth, and in particular it admits a section after étale base change in a neighborhood of a point $s \in S(k)$ for which $(G \times S)^{\mathbb{G}_m}_s = (G \times \{s\})^{\mathbb{G}_m} \neq \emptyset$. By construction this set is nonempty precisely when ϕ_s is conjugate to λ, so by hypothesis it is nonempty for some $s \in S(k)$.

By the same reasoning every point has an étale neighborhood on which ϕ is conjugate to a constant homomorphism determined by some one parameter subgroup. Because S is connected and locally finite type it follows that ϕ must be conjugate to the same 1PS λ in each of these étale neighborhoods. Thus $(G \times S)^{\mathbb{G}_m} \to S$ admits a global section after étale base change. $\square$

One immediate consequence of Proposition 4.1.6 is a classification of principal G-bundles over Θ.

Corollary 4.1.9. *Every G-bundle over Θ is isomorphic to E_λ for some one parameter subgroup $\lambda : \mathbb{G}_m \to G$. In addition, $E_{\lambda_0} \simeq E_{\lambda_1}$ if and only if λ_0 and λ_1 are conjugate, and $\mathrm{Aut}(E_\lambda) \simeq P_\lambda$ as an algebraic group.*

Proof. This is essentially exactly statement (2) of Proposition 4.1.6 applied to the case $S = \mathrm{Spec}\, k$, combined with the observation that $E'|_{\{1\}}$ is trivializable because k is algebraically closed. $\square$

In fact Proposition 4.1.6 induces a stronger version of this correspondence – it identifies the category of G-bundles over Θ_S and the category of P_λ torsors over $1 \times S$ by restriction. Furthermore, this identification holds for all k-schemes S in addition to those of finite type.

Corollary 4.1.10. *As a stack over the étale site of k-schemes, we have*

$$\underline{*/G}(\Theta) \simeq \bigsqcup_{[\lambda]} */P_\lambda$$

where $[\lambda]$ ranges over all conjugacy classes of 1PS $\lambda : \mathbb{G}_m \to G$. The maps $*/P_\lambda \to \underline{*/G}(\Theta)$ classify the G-bundles E_λ over Θ.

Proof. The objects E_λ define a 1-morphism $\bigsqcup_{[\lambda]} */P_\lambda \to */G(\Theta)$ of stacks over the site of all k-schemes, and 4.1.6 implies that this is an equivalence of stacks over the sub-site of k-schemes of finite type. The functor $*/G(\Theta)$ is limit preserving by the formal observation

$$\operatorname{Hom}(\varprojlim_i T_i, */G(\Theta)) \simeq \operatorname{Hom}(\varprojlim_i T_i \times \Theta, */G) \simeq \varprojlim_i \operatorname{Hom}(T_i \times \Theta, */G)$$

where the last equality holds because $*/G$ is an algebraic stack locally of finite presentation. The stack $\bigsqcup_{[\lambda]} */P_\lambda$ is locally of finite presentation and thus limit preserving as well. Every affine scheme over k can be written as a limit of finite type k schemes, so the isomorphism for finite type k-schemes implies the isomorphism for all k-schemes. $\qquad\square$

Scholium 4.1.11. *As a stack over the étale site of k-schemes, we have $*/G(*/\mathbb{G}_m) \simeq \bigsqcup_{[\lambda]} */L_\lambda$ where $[\lambda]$ ranges over all conjugacy classes of 1PS $\lambda : \mathbb{G}_m \to \overline{G}$. The maps $*/L_\lambda \to */G(*/\mathbb{G}_m)$ classify the trivial G-bundles $G \to \operatorname{Spec} k$ with $\mathbb{G}_m$ equivariant structure defined by left multiplication by $\lambda(t)$.*

Proof. For S of finite type over k, the proof of Proposition 4.1.6 carries over unchanged for G-bundles over $(*/\mathbb{G}_m) \times S$, showing that étale locally in S they are isomorphic to $S \times G$ with $\mathbb{G}_m$ acting by left multiplication by $\lambda(t)$ on G. In fact, we had to essentially prove this when we considered the $\mathbb{G}_m$-equivariant bundle $E|_{\{0\} \times S}$ in Step 4 of that proof. The amplification of the statement from finite type k schemes to all k schemes is identical to the proof of Corollary 4.1.10. $\qquad\square$

4.2   The stack $\underline{\mathfrak{X}}(\Theta)$

In this section we introduce the mapping stack $\underline{\mathfrak{X}}(\Theta)$ in the case when $\mathfrak{X} \simeq X/G$ is a global quotient of a k-scheme X by a locally affine action (Definition 1.1.2) of a linear group G. We establish an explicit description of $\underline{\mathfrak{X}}(\Theta)$ as a disjoint union of quotient stacks of locally closed sub-schemes of X by parabolic subgroups of G. We also describe a relationship between $\underline{\mathfrak{X}}(\Theta)$ and the stacks $\underline{\mathfrak{X}}(*/\mathbb{G}_m)$ and $\mathfrak{X}$.

By definition, as a weak functor into groupoids we have

$$\underline{\mathfrak{X}}(\Theta)(T) := \operatorname{Hom}(\Theta \times T, \mathfrak{X})$$

where Hom denotes category of natural transformations of presheaves of groupoids, or equivalently the category of 1-morphisms between stacks.

This definition makes sense for any presheaf of groupoids, but if $\mathfrak{X}$ is a stack, then we can describe $\underline{\mathfrak{X}}(\Theta)(T)$ more explicitly in terms of descent data [45]. We consider the first 3 levels of the simplicial scheme determined by the action of $\mathbb{G}_m$ on $\mathbb{A}^1 \times T$

$$\mathbb{G}_m \times \mathbb{G}_m \times \mathbb{A}^1 \times T \underset{a}{\overset{\mu}{\underset{\sigma}{\rightrightarrows}}} \mathbb{G}_m \times \mathbb{A}^1 \times T \underset{a}{\overset{\sigma}{\rightrightarrows}} \mathbb{A}^1 \times T \tag{4.2}$$

Where μ denotes group multiplication, σ denotes the action of $\mathbb{G}_m$ on $\mathbb{A}^1$, and a forgets the leftmost group element. Then the category $\underline{\mathfrak{X}}(\Theta)(T)$ has

- *objects:* $\eta \in \mathfrak{X}(\mathbb{A}^1 \times T)$ along with a morphism $\phi : a^*\eta \to \sigma^*\eta$ satisfying the cocycle condition $\sigma^*\phi \circ a^*\phi = \mu^*\phi$

- *morphisms:* $f : \eta_1 \to \eta_2$ such that $\phi_2 \circ a^*(f) = \sigma^*(f) \circ \phi_1 : a^*\eta_1 \to \sigma^*\eta_2$

It follows from this description, for instance, that the functor $\mathrm{Hom}(\Theta \times T, \mathfrak{X}) \to \mathrm{Hom}(\mathbb{A}^1 \times T, \mathfrak{X})$ is faithful.

As with any Hom-stack, one has a universal evaluation 1-morphism $ev : \Theta \times \underline{\mathfrak{X}}(\Theta) \to \mathfrak{X}$ as well as the projection $\pi : \Theta \times \underline{\mathfrak{X}}(\Theta) \to \underline{\mathfrak{X}}(\Theta)$. In addition we have morphisms

$$\underline{\mathfrak{X}}(*/\mathbb{G}_m) \underset{\sigma}{\overset{r_0}{\leftrightarrows}} \underline{\mathfrak{X}}(\Theta) \overset{r_1}{\longrightarrow} \mathfrak{X} \tag{4.3}$$

where r_0, r_1 are the restriction of a morphism to the points $0, 1 \in \mathbb{A}^1$ respectively. Note that the restriction to the point $1 \in \mathbb{A}^1$ is actually the restriction of the evaluation morphism to the open substack $\mathfrak{X} \simeq (\mathbb{A}^1 - \{0\})/\mathbb{G}_m \times \underline{\mathfrak{X}}(\Theta) \subset \Theta \times \underline{\mathfrak{X}}(\Theta)$. The morphism σ is induced by the projection $\Theta \to */\mathbb{G}_m$. The composition $*/\mathbb{G}_m \to \Theta \to */\mathbb{G}_m$ is equivalent to the identity morphism, so $r_0 \circ \sigma \simeq \mathrm{id}_{\underline{\mathfrak{X}}(*/\mathbb{G}_m)}$.

We recall some notation established in Section 1.1. If G is a linear group acting on X in a locally affine manner (Definition 1.1.2) and λ is a one parameter subgroup, then X^λ denotes the fixed locus of X with respect to $\lambda(\mathbb{G}_m)$. Also, given a connected component $Z \subset X^\lambda$ we can define the blade $Y_{Z,\lambda}$ consisting of points contracted to Z under the action of λ. We also have the subgroup $L_{Z,\lambda} \subset L_\lambda$ of elements preserving Z and the corresponding finite index subgroup $P_{Z,\lambda} \subset P_\lambda$.

Lemma 4.2.1. *There is a map of stacks $\Theta \times (Y_{Z,\lambda}/P_{Z,\lambda}) \to X/G$ which maps the k-point defined by $(z, x) \in \mathbb{A}^1 \times Y$ to the k-point defined by $\lambda(z) \cdot x \in X$. By definition this defines a morphism $Y_{Z,\lambda}/P_{Z,\lambda} \to \underline{\mathfrak{X}}(\Theta)$. Likewise there is a map $*/\mathbb{G}_m \times Z/L_{Z,\lambda} \to \mathfrak{X}$ defining a map $Z/L_{Z,\lambda} \to \underline{\mathfrak{X}}(*/\mathbb{G}_m)$.*

Proof. We will drop the subscripts Z and λ. A morphism $\Theta \times (Y/P) \to X/G$ is a $\mathbb{G}_m \times P$-equivariant G-bundle over $\mathbb{A}^1 \times Y$ along with a G-equivariant and $\mathbb{G}_m \times P$ invariant map to X.

Consider the trivial G-bundle $\mathbb{A}^1 \times Y \times G$, where G acts by right multiplication on the rightmost factor. This principal bundle acquires a $\mathbb{G}_m \times P$-equivariant structure via the action

$$(t, p) \cdot (z, x, g) = (tz, p \cdot x, \lambda(tz)p\lambda(z)^{-1}g)$$

This expression is only well defined when $z \neq 0$, but it extends to a regular morphism because $\lim_{z \to 0} \lambda(z)p\lambda(z)^{-1} = l$ exists. It is straightforward to check that this defines an action of $\mathbb{G}_m \times P$, that the action commutes with right multiplication by G, and that the map $\mathbb{A}^1 \times Y \times G \to X$ defined by

$$(z, x, g) \mapsto g^{-1}\lambda(z) \cdot x$$

is $\mathbb{G}_m \times P$-invariant.

The morphism $*/\mathbb{G}_m \times Z/L_{Z,\lambda} \to X/G$ is simpler. It is determined by the group homomorphism $\mathbb{G}_m \times L_{Z,\lambda} \to G$ given by $(t, l) \mapsto \lambda(t)l \in G$ which intertwines the inclusion of schemes $Z \subset X$. $\square$

Note that for $g \in G$, the subscheme $g \cdot Z$ is a connected component of $X^{\lambda'}$ where $\lambda'(t) = g\lambda(t)g^{-1}$. Furthermore, $g \cdot Y_{Z,\lambda} = Y_{gZ,\lambda'}$ and we have an equivalence $Y_{Z,\lambda}/P_{Z,\lambda} \to Y_{g \cdot Z,\lambda'}/P_{Z,\lambda'}$ which commutes up to 2-isomorphism with the morphisms to $\mathfrak{X}(\Theta)$ constructed in Lemma 4.2.1.

Theorem 4.2.2. *Let $\mathfrak{X} = X/G$ be a quotient of a k-scheme X by a locally affine action (Definition 1.1.2) of a linear group G. The natural morphism $Y_{Z,\lambda}/P_{Z,\lambda} \to \mathfrak{X}(\Theta)$ from Lemma 4.2.1 identifies $Y_{Z,\lambda}/P_{Z,\lambda}$ with a connected component of $\mathfrak{X}(\Theta)$, and in fact these morphisms induce isomorphisms*

$$\mathfrak{X}(\Theta) \simeq \bigsqcup_{[Z,\lambda]} Y_{Z,\lambda}/P_{Z,\lambda}, \quad and \quad \mathfrak{X}(*/\mathbb{G}_m) \simeq \bigsqcup_{[Z,\lambda]} Z/L_{Z,\lambda}$$

The disjoint unions are taken over equivalence classes of pairs $[Z, \lambda]$ where Z is a connected component of X^λ and the equivalence relation on such pairs is generated by $[Z, \lambda] \sim [g \cdot Z, g\lambda g^{-1}]$.

Proof. The map $\mathfrak{X} \to */G$ induces a functor $\mathfrak{X}(\Theta) \to */G(\Theta)$, and we shall use the description of the latter from Corollary 4.1.10 to prove the theorem. Consider the fiber product $\mathfrak{X}(\Theta) \times_{*/G(\Theta)} \operatorname{Spec} k$ where the morphism $\operatorname{Spec} k \to */G(\Theta)$ is induced by the G-bundle E_λ over Θ.

By definition the groupoid of T points of the fiber product [45] consists of $\mathbb{G}_m$-equivariant G bundles $E \to \mathbb{A}^1 \times T$ along with a G-equivariant and $\mathbb{G}_m$-invariant map $E \to X$ and an isomorphism of equivariant G-bundles $E \simeq (E_\lambda)_T$. Of course the data of the G bundle is redundant once we fix an isomorphism with $(E_\lambda)_T$, so we have

$$\left(\mathfrak{X}(\Theta) \times_{*/G(\Theta)} \operatorname{Spec} k\right)(T) \simeq \{\mathbb{G}_m\text{-equivariant maps } \mathbb{A}^1 \times T \to X\} = \Phi_X(T)$$

where $\mathbb{G}_m$ acts on X via λ, and the functor Φ_X is exactly the functor introduced in Theorem 1.1.4. It follows from that theorem that $\mathfrak{X}(\Theta) \times_{*/G(\Theta)} \operatorname{Spec} k$ is represented by the scheme $Y = \bigsqcup_Z Y_{Z,\lambda}$, where the coproduct ranges over all connected components $Z \subset X^\lambda$.

Note that $\operatorname{Aut}(E_\lambda) = P_\lambda$ acts naturally on this fiber product. The quotient of $\bigsqcup Y_{Z,\lambda}$ by the action of P_λ is the disjoint union of $Y_{Z,\lambda}/P_{Z,\lambda}$ where Z ranges over a choice of representatives for each orbit of the action of P_λ on the set of connected components. This is equivalent to a set of representatives for each orbit of the action of L_λ on the set of connected components $\pi_0(X^\lambda)$.

Corollary 4.1.10 implies that the quotient $(\mathfrak{X}(\Theta) \times_{*/G(\Theta)} \operatorname{Spec} k)/P_\lambda$ is the preimage under the canonical morphism $\mathfrak{X}(\Theta) \to */G(\Theta)$ of the connected component $*/P_\lambda$ corresponding

to the conjugacy class of λ. It follows that $Y_{Z,\lambda}/P_{Z,\lambda}$ are the connected components of $\underline{\mathfrak{X}}(\Theta)$. From this description it follows that pairs $[Z, \lambda]$ and $[Z', \lambda']$ define the same connected component of $\underline{\mathfrak{X}}(\Theta)$ if and only if $\lambda' = g\lambda g^{-1}$ and $Z' = g \cdot Z$. One can check that these inclusions $Y_{Z,\lambda}/P_{Z,\lambda} \to \underline{\mathfrak{X}}(\Theta)$ agree with those of Lemma 4.2.1.

The same argument as above implies the statement for $\underline{\mathfrak{X}}(*/\mathbb{G}_m)$ with little modification. By Scholium 4.1.11 the mapping stack $*/G(*/\mathbb{G}_m)$ is isomorphic to $\bigsqcup */L_\lambda$. The morphism $\underline{\mathfrak{X}}(*/\mathbb{G}_m) \to */G(*/\mathbb{G}_m)$ is representable, and the preimage of the connected component $*/L_\lambda$ is the global quotient X^λ/L_λ, which can be further decomposed into connected components. $\qquad\square$

From this explicit description of the stack $\underline{\mathfrak{X}}(\Theta)$ we obtain explicit descriptions of the morphisms r_0 and r_1 from the diagram (4.3) and deduce some basic properties.

Corollary 4.2.3. *The morphism $r_0 : \underline{\mathfrak{X}}(\Theta) \to \underline{\mathfrak{X}}(*/\mathbb{G}_m)$ is finite type with connected fibers. On each connected component, r_0 corresponds to the projection $Y_{Z,\lambda} \to Z$ mapping $x \mapsto \lim_{t\to 0} \lambda(t) \cdot x$, which intertwines the group homomorphism $P_{Z,\lambda} \to L_{Z,\lambda}$.*

Corollary 4.2.4. *On the connected component of $\underline{\mathfrak{X}}(\Theta)$ corresponding to $[Z, \lambda]$, the restriction morphism $r_1 : \underline{\mathfrak{X}}(\Theta) \to \mathfrak{X}$ is equivalent to the inclusion $Y_{Z,\lambda}$ which intertwines the inclusion of groups $P_{Z,\lambda} \subset G$. In particular it is representable and proper over an open substack of $\mathfrak{X}$.*

Furthermore, we can study the morphism $\underline{\mathfrak{W}}(\Theta) \to \underline{\mathfrak{X}}(\Theta)$ induced by a morphism of stacks $\mathfrak{W} \to \mathfrak{X}$.

Proposition 4.2.5. *Let $\mathfrak{W}$ and $\mathfrak{X}$ be quotients of k-schemes by locally affine group actions, and let $f : \mathfrak{W} \to \mathfrak{X}$ be a morphism. We consider the induced morphism $\tilde{f} : \underline{\mathfrak{W}}(\Theta) \to \underline{\mathfrak{X}}(\Theta)$.*

1. *If f is representable by algebraic spaces (respectively schemes), then so is $\tilde{f}$.*

2. *If f is representable by open immersions, then so is $\tilde{f}$, and $\tilde{f}$ identifies $\underline{\mathfrak{W}}(\Theta)$ with the preimage of $\mathfrak{W} \subset \mathfrak{X}$ under the composition $\underline{\mathfrak{X}}(\Theta) \xrightarrow{r_0} \underline{\mathfrak{X}}(*/\mathbb{G}_m) \to \mathfrak{X}$.*

3. *If f is representable by closed immersions, then so is $\tilde{f}$, and $\tilde{f}$ identifies $\underline{\mathfrak{W}}(\Theta)$ with the closed substack $r_1^{-1}\mathfrak{W} \subset \underline{\mathfrak{X}}(\Theta)$.*

Proof. Let $S \to \underline{\mathfrak{X}}(\Theta)$ be an S-point defined by a morphism $\Theta_S \to \mathfrak{X}$. Then the fiber product $\Theta_S \times_{\mathfrak{X}} \mathfrak{W} \to \Theta_S$ is representable and is thus isomorphic to $E/\mathbb{G}_m$ for some algebraic space E with a $\mathbb{G}_m$-equivariant map $E \to \mathbb{A}^1_S$. The fiber of $\underline{\mathfrak{W}}(\Theta) \to \underline{\mathfrak{X}}(\Theta)$ over the given S-point of $\underline{\mathfrak{X}}(\Theta)$ corresponds to the groupoid of sections of $E/\mathbb{G}_m \to \mathbb{A}^1_S/\mathbb{G}_m$, which form a set. Thus $\underline{\mathfrak{W}}(\Theta)$ is equivalent to a sheaf of sets as a category fibered in groupoids over $\underline{\mathfrak{X}}(\Theta)$. Because $\underline{\mathfrak{W}}(\Theta)$ and $\underline{\mathfrak{X}}(\Theta)$ are algebraic, the morphism $\underline{\mathfrak{W}}(\Theta) \to \underline{\mathfrak{X}}(\Theta)$ is relatively representable by algebraic spaces.

Now if $\mathfrak{X} = X/G$ and f is representable by schemes, it follows that $\mathfrak{W} = W/G$ for some scheme W with a G-equivariant map $W \to X$. Let S be a connected k scheme, then

Theorem 4.2.2 implies that a morphism $S \to \underline{\mathfrak{X}}(\Theta)$ corresponds to a P_λ-bundle $E \to S$ along with a P_λ-equivariant map $E \to Y_\lambda$, where Y_λ is the scheme whose existence is guaranteed by Hesselink's Theorem 1.1.4.

To show that if $\mathfrak{W} \subset \mathfrak{X}$ is an open substack, it is necessary and sufficient to show that for any one parameter subgroup, the functor $\Phi_W(T) \subset \Phi_X(T)$ is an open subfunctor. Let $f : \mathbb{A}^1_T \to X$ be a $\mathbb{G}_m$-equivariant morphism, and consider the open subscheme $S = f^{-1}(W) \cap \{0\} \times T \subset T$. Then the $\mathbb{G}_m$-equivariant morphism $\mathbb{A}^1_S \to \mathbb{A}^1_T \to X$ factors through W because any equivariant open subset of $\mathbb{A}^1_S$ containing $\{0\} \times S$ must be all of $\mathbb{A}^1_S$ itself. On the other hand, it is straightforward to show that if $T' \to T$ is such that $\mathbb{A}^1_{T'} \to \mathbb{A}^1_T \to X$ factors through W, then $T' \to T$ factors through S. Thus $S \subset T$ represents the preimage of $\underline{\mathfrak{W}}(\Theta)$ under $T \to \underline{\mathfrak{X}}(\Theta)$, so we have an open immersion.

The argument for closed immersions is similar – we must show that $\Phi_W(T) \subset \Phi_X(T)$ is a closed subfunctor. Let $f : \mathbb{A}^1_T \to X$ be a $\mathbb{G}_m$-equivariant morphism and define the closed subscheme $S = f^{-1}(W) \cap \{1\} \times T \subset T$. Then the morphism $\mathbb{A}^1_S \to \mathbb{A}^1_T \to X$ factors through W, because $f^{-1}W$ is a closed subscheme of $\mathbb{A}^1_T$ containing $\mathbb{G}_m \times S$, and $\mathbb{A}^1_S$ is the scheme theoretic closure of $\mathbb{G}_m \times S$ in $\mathbb{A}^1_T$. In addition if $T' \to T$ is such that $\mathbb{A}^1_{T'} \to \mathbb{A}^1_T \to X$ factors through W, the $T' \to T$ factors through S. Thus in this case $\Phi_W \subset \Phi_X$ is a closed subfunctor. $\qquad\square$

Corollary 4.2.6. *If $\mathfrak{X}$ is an algebraic stack over k which can be covered by a possibly infinite family of open substacks which are quotients of k-schemes by locally affine actions of linear groups, then $\underline{\mathfrak{X}}(\Theta)$ is an algebraic stack. In fact it is also a union of open substacks which are quotient stacks. If $\mathfrak{X}$ is locally of finite type, then so is $\underline{\mathfrak{X}}(\Theta)$.*

Proof. This is an immediate consequence of Part (2) of Proposition 4.2.5. $\qquad\square$

Modular examples of $\underline{\mathfrak{X}}(\Theta)$

We have shown that $\underline{\mathfrak{X}}(\Theta)$ is an algebraic stack locally of finite type whenever $\mathfrak{X}$ is locally a quotient stack (for a locally affine group action). This allows us to study $\underline{\mathfrak{X}}(\Theta)$ for algebraic stacks $\mathfrak{X}$ representing common moduli problems in algebraic geometry.

For example, let Σ be a connected projective k scheme and G a linear group. The stack $\mathrm{Bun}_G(\Sigma)$ of principal G-bundles on Σ is a weak functor valued in groupoids defined by

$$\mathrm{Bun}_G(\Sigma) : T \mapsto \{G\text{-bundles on } T \times \Sigma\} \tag{4.4}$$

This can alternatively be described as the Hom-stack $\underline{\mathrm{Hom}}(\Sigma, */G)(T) = \mathrm{Hom}(T \times \Sigma, */G)$. When Σ is projective, the stack $\mathrm{Bun}_G(\Sigma)$ is algebraic and locally of finite type. This can be deduced from the fact that $\mathrm{Bun}_{\mathrm{GL}_R}(\Sigma)$ is algebraic and locally finite type, and choosing a faithful representation $G \subset \mathrm{GL}_R$, the morphism $\mathrm{Bun}_G(\Sigma) \to \mathrm{Bun}_{\mathrm{GL}_R}(\Sigma)$ is representable and finite type.

Lemma 4.2.7. *Let $\mathfrak{X} = \mathrm{Bun}_G(\Sigma)$, then a geometric point of $\underline{\mathfrak{X}}(\Theta)$, i.e. a morphism $f : \Theta \to \mathrm{Bun}_G(\Sigma)$ is equivalent to either of the following data*

1. *an equivariant G-bundle on $\mathbb{A}^1 \times \Sigma$, where $\mathbb{G}_m^*$ acts on the first factor*

2. *a 1PS $\lambda : \mathbb{G}_m \to G$, and a principal P_λ-bundle, E over X*

and under the second identification the point $f(1) \in \mathrm{Bun}_G(\Sigma)$ is the extension of structure group from P_λ to G. The second identification works in families as well, hence we have

$$\underline{\mathfrak{X}}(\Theta) = \bigsqcup_{[\lambda]} \mathrm{Bun}_{P_\lambda}(\Sigma)$$

Proof. The first description follows from descent on the action groupoid of $\mathbb{G}_m$ on $\mathbb{A}^1 \times X$ and the fact that the functor 4.4 defines a stack (see Diagram (4.2)). The second description follows from the formal observation

$$\underline{\mathrm{Hom}}(\Theta, \underline{\mathrm{Hom}}(\Sigma, */G)) \simeq \underline{\mathrm{Hom}}(\Sigma \times \Theta, */G) \simeq \underline{\mathrm{Hom}}(\Sigma, \underline{\mathrm{Hom}}(\Theta, */G))$$

and the description of $\mathrm{Hom}(\Theta, */G) = \underline{*/G}(\Theta)$ from Corollary 4.1.10. □

Example 4.2.8. A closely related example is the stack $\mathrm{Coh}(X)$ of coherent sheaves on X. A map from Θ corresponds to a coherent sheaf along with a choice of filtration. This follows from the description of $\mathrm{QCoh}(\Theta \times X)$ in Proposition 4.1.4.

Example 4.2.9. One can consider the stack of polarized projective varieties. Here a map from Θ to the moduli stack corresponds to a test configuration as used by Donaldson to define the notion of K-stability.

4.3 Θ-stratifications

Now that we have described the stack $\underline{\mathfrak{X}}(\Theta)$, we return to the theory of stratifications in geometric invariant theory. We shall present a unified framework for constructing stratifications which generalize the KN stratification of the unstable locus in GIT.

Let X be a projective over affine variety with a reductive group action. Recall from Section 1.1 that after fixing a G-ample line bundle $\mathcal{L}$ and an invariant bilinear form $|\bullet|$ on $\mathfrak{g}$, we have a sequence of open G-equivariant subvarieties $X^{ss} = X_0 \subset X_1 \subset \cdots \subset X_N = X$, where the complement $S_i = X_i \setminus X_{i-1}$ are the KN strata.

Combining the description of $\underline{\mathfrak{X}}(\Theta)$ from Theorem 4.2.2 with the description of the KN strata given in Section 1.1, we see that

Proposition 4.3.1. *The KN stratum S_i/G is isomorphic to a connected component of $\underline{X_i/G}(\Theta)$. The closed immersion of stacks $S_i/G \hookrightarrow X_i/G$ corresponds to the canonical morphism $r_1 : \underline{X_i/G}(\Theta) \to X_i/G$.*

This motivates the following definition

Definition 4.3.2. A Θ-*stratum* $\mathfrak{S} \subset \mathfrak{X}$ is a closed substack of a stack $\mathfrak{X}$ which is identified with a connected component of $\underline{\mathfrak{X}}(\Theta)$ by the canonical morphism $r_1 : \underline{\mathfrak{X}}(\Theta) \to \mathfrak{X}$. Likewise, a Θ-stratification of $\mathfrak{X}$ is a family family of open substacks $\mathfrak{X}_0 \subset \mathfrak{X}_1 \subset \cdots$ such that $\mathfrak{X}_i \setminus \mathfrak{X}_{i-1}$ is a union of Θ-strata and $\mathfrak{X} = \bigcup_i \mathfrak{X}_i$.[4]

In GIT the Θ-stratification is determined by an ordered list of connected components of $\underline{\mathfrak{X}}(\Theta)$. We denote these with a superscript $\underline{\mathfrak{X}}(\Theta)^i = Y_{Z_i,\lambda_i}/P_{Z_i,\lambda_i}$. The morphism $r_1 : \underline{\mathfrak{X}}(\Theta)^i \to \mathfrak{X}$ corresponds to the immersion $Y_{Z_i,\lambda_i} \hookrightarrow X$ which intertwines the inclusion $P_{Z_i,\lambda_i} \subset G$. Under the identification $Y/P \simeq G \times_P Y/G$, the morphism r_1 corresponds the G equivariant map $G \times_P Y \to X$. Thus r_1 is representable, but it need not be an immersion, and it need not be proper.

Example 4.3.3. To see that r_1 need not be an immersion, consider the simplest example $*/G(\Theta) = \bigsqcup */P_\lambda$. The fiber of the morphism $r_1 : \underline{*/G}(\Theta) \to */G$ over the cover $* \to */G$ is $\bigsqcup G/P_\lambda$. Hence r_1 is not an immersion.

In fact, r_1 is a local immersion whenever $\mathfrak{X}$ is a global quotient by an abelian group. However we see that r_1 need not be proper

Example 4.3.4. Let $V = \operatorname{Spec} k[x,y,z]$ be a linear representation of $\mathbb{G}_m$ where x, y, z have weights $-1, 0, 1$ respectively, and let $X = V - \{0\}$ and $\mathfrak{X} = X/\mathbb{G}_m$. The fixed locus is the punctured line $Z = \{x = z = 0\} \cap X$, and the connected component of $\underline{\mathfrak{X}}(\Theta)$ corresponding to the pair $[Z, \lambda(t) = t]$ is the quotient $S/\mathbb{G}_m$ where

$$S = \{(x, y, z) | z = 0 \text{ and } y \neq 0\}$$

$S \subset X$ is not closed. Its closure contains the points where $x \neq 0$ and $y = 0$. These points would have been attracted by λ to the "missing" point $\{0\} \in V$ which has been removed in X. It follows that $S/\mathbb{G}_m \to X/\mathbb{G}_m$ is not proper.

In light of these potential pathologies, the main result on the KN stratification in GIT can be interpreted as the statement that for the connected components $\underline{\mathfrak{X}}(\Theta)^i = Y_{Z_i,\lambda_i}/P_{Z_i,\lambda_i}$ selected by the Hilbert-Mumford criterion, we have

1. The substack $\bigcup_{j>i} r_1(\underline{\mathfrak{X}}(\Theta)^j) \subset \mathfrak{X}$ is closed, and

2. $r_1 : \underline{\mathfrak{X}}(\Theta)^i \to \mathfrak{X}$ is a closed immersion over the open substack $\mathfrak{X} \setminus \bigcup_{j>i} r_1(\underline{\mathfrak{X}}(\Theta)^j)$.

Thus the problem of generalizing the KN stratification in GIT rests on an intrinsic description of a method of selecting connected components of $\underline{\mathfrak{X}}(\Theta)$ so that these properties hold.

One caveat is that the stratum S_i/G does not correspond to a unique connected component of $\underline{\mathfrak{X}}(\Theta)$. Consider the map $\mathbb{A}^1 \to \mathbb{A}^1$ given by $t \mapsto t^n$. This is not equivariant with

---

[4]It is possible to modify this notion for applications in which the natural indexing set of the stratification is some other partially ordered set, rather than the natural numbers.

respect to $\mathbb{G}_m$, but it intertwines the group homomorphism $z \mapsto z^n$ and therefore defines a map $\Theta \xrightarrow{\times n} \Theta$ for every $n > 0$. For any algebraic stack $\mathfrak{X}$, we can let the monoid $\mathbf{N}^\times$ act on $\underline{\mathfrak{X}}(\Theta)$ by pre-composing a morphism $\Theta_S \to \mathfrak{X}$ with the n-fold ramified covering map $\Theta_S \xrightarrow{\times n} \Theta_S$.

The action of $\mathbf{N}^\times$ on $\underline{\mathfrak{X}}(\Theta)$ descends to the set of connected components $\pi_0 \underline{\mathfrak{X}}(\Theta)$ as well. The composition $\underline{\mathfrak{X}}(\Theta) \xrightarrow{\times n} \underline{\mathfrak{X}}(\Theta) \xrightarrow{r_1} \mathfrak{X}$ is naturally isomorphic to r_1, so if $\mathfrak{S} \subset \mathfrak{X}$ is a Θ-stratum identified with the connected component $\underline{\mathfrak{X}}(\Theta)^i$, it is also isomorphic to any connected component of $\underline{\mathfrak{X}}(\Theta)$ in the orbit of $\underline{\mathfrak{X}}(\Theta)^i$ under the action of $\mathbf{N}^\times$. So to be precise, the GIT stratification is determined by a sequence of elements of the set $\pi_0 \underline{\mathfrak{X}}(\Theta)/\mathbf{N}^\times$ satisfying the properties above.

Definition 4.3.5. A *numerical invariant* for the stack $\mathfrak{X}$ is a map $\mu : \pi_0 \underline{\mathfrak{X}}(\Theta)/\mathbf{N}^\times \to \mathbb{R} \cup \{-\infty\}$. We can define the *stability function* $M^\mu : \mathfrak{X}(k) \to \mathbb{R} \cup \{-\infty\}$ as

$$M^\mu(p) = \sup\{\mu(f) \,|\, f : \Theta \to \mathfrak{X} \text{ with } f(1) \simeq p \in \mathfrak{X}(k)\}$$

where we are considering μ to be a locally constant function on $\underline{\mathfrak{X}}(\Theta)$.

In the remainder of this section, we will assume that $k = \mathbb{C}$, because our method for constructing numerical invariants makes use of the cohomology of the topological stack underlying the analytification of $\mathfrak{X}$.

Example 4.3.6. Our quintessential example of a numerical invariant is defined using cohomology classes $l \in H^2(\mathfrak{X}; \mathbb{Q})$ and $b \in H^4(\mathfrak{X}; \mathbb{Q})$. Given a map $f : \Theta \to \mathfrak{X}$ the pullback f^*l and f^*b are cohomology classes in $H^2(\Theta) = \mathbb{Q} \cdot q$ and $H^4(\Theta) = \mathbb{Q} \cdot q^2$ respectively. We assume that b is positive definite in the sense that $f^*b \in \mathbb{Q}_{\geq 0} \cdot q^2$ and strict inequality holds if the group homomorphism $\mathbb{G}_m \to \operatorname{Aut} f(0)$ has finite kernel. We define the numerical invariant

$$\mu(f) = f^*l/\sqrt{f^*b} \in \mathbb{R} \tag{4.5}$$

For points for which the homomorphism $\mathbb{G}_m \to \operatorname{Aut} f(0)$ is trivial, we define $\mu = -\infty$. The value of $\mu(f)$ agrees for any two maps $\Theta \to \mathfrak{X}$ corresponding to points in the same connected component of $\underline{\mathfrak{X}}(\Theta)$.[5]

Given a numerical invariant, one can attempt to define a Θ stratification of $\mathfrak{X}$ indexed by real numbers $r \geq 0$

$$\mathfrak{X}_r = \mathfrak{X} \setminus \bigcup_{\mu(\underline{\mathfrak{X}}(\Theta)^i) > r} r_1(\underline{\mathfrak{X}}(\Theta)^i) \tag{4.6}$$

---

[5]Consider a morphism $f : \Theta \times S \to \mathfrak{X}$, where S is a connected scheme of finite type over $\mathbb{C}$. Then $H^*(\Theta \times S) \simeq H^*(\Theta) \otimes H^*(S)$ under the Künneth decomposition, and for any $\mathbb{C}$-point $s \in S$ the restriction morphism $H^*(\Theta \times S) \to H^*(\Theta \times \{s\}) = H^*(\Theta)$ can be identified with the projection onto $H^*(\Theta) \otimes H^0(S) \simeq H^*(\Theta)$. Thus the restriction of $f_s : \Theta \to H^*(\mathfrak{X})$ induces the same homomorphism $f_s^* : H^*(\mathfrak{X}) \to H^*(\Theta)$ for any point $s \in S$.

In other words, the $\mathbb{C}$-points of $\mathfrak{X}_r$ are precisely those $\mathbb{C}$-points of $\mathfrak{X}$ for which $M^\mu(p) > r$. While this definition has the advantage of being very general, it ignores several possible problems: 1) $\mathfrak{X}_r$ does not need to be an open substack as defined, 2) in between $\mathfrak{X}_{r_0} \subset \mathfrak{X}_{r_1}$ there could be infinitely many values of r at which $\mathfrak{X}_r$ jumps, and 3) if the connected component $\underline{\mathfrak{X}}(\Theta)^i$ has numerical invariant $\mu = r$, the morphism $\underline{\mathfrak{X}}(\Theta)^i \to \mathfrak{X}_r$ need not be in immersion and need not be proper.

Nevertheless the KN stratification in geometric invariant theory arises from a numerical invariant of the form (4.5).

Lemma 4.3.7. *The numerical invariant* (1.3) *used to define the KN stratification can be expressed in terms of a class $l \in H^2(\mathfrak{X}; \mathbb{Q})$ and $b \in H^4(\mathfrak{X}; \mathbb{Q})$ as in* (4.5), *and the KN stratification is defined by* (4.6).

Proof. The Hilbert-Mumford criterion uses the numerical invariant $-$ weight $\mathcal{L}_{f(0)}/|\lambda|$, where $|\bullet|$ denotes an invariant inner product on the lie algebra $\mathfrak{g}$. The numerator can be interpreted as $\frac{1}{q} f^* c_1(\mathcal{L}) \in \mathbb{Q}$.

For the denominator, the G invariant bilinear form $|\bullet|$ on $\mathfrak{g}$ can be interpreted as a class in $H^4(*/G; \mathbb{C})$ under the identification $H^*(*/G; \mathbb{C}) \simeq (\mathrm{Sym}(\mathfrak{g}^\vee))^G$. The class is rational if it takes rational values on elements of $\mathfrak{g}$ corresponding to one-parameter subgroups. We pull this back to a class $b \in H^4(X/G)$ under the map $X/G \to */G$. For a morphism $f : \Theta \to X/G$, the pullback f^*b is the pullback of the class in $H^4(*/G)$ under the composition $\Theta \to X/G \to */G$. We therefore have $f^*b = |\lambda|^2 q^2 \in H^4(\Theta)$.

Thus we have identified the Hilbert-Mumford numerical invariant with an invariant of the form (4.5). It is straightforward to verify that (4.6) agrees with the usual definition of the KN stratification. $\qquad \square$

In light of this reformulation of semistability and the KN stratification in GIT, we have the following

Question 4.3.8. *Which existing notions of semistability for moduli problems in algebraic geometry can be described by a class in H^2 and H^4 of a moduli stack? Given classes in H^2 and H^4, under what conditions does the stability function M^μ define a Θ-stratification on the unstable locus?*

In future work we hope to answer this question more fully.

Vector bundles on a curve

Continuing our investigation of the construction of Θ-stratifications, we now study the stack $\mathrm{Bun}_G(\Sigma)$ of principal G-bundles on Σ, where Σ is a smooth algebraic curve and $G = \mathrm{SL}_R$ or GL_R. We will show how the notion of slope semistability as well as the Shatz stratification [41] of $\mathrm{Bun}_G(\Sigma)$ can be recovered from a choice of class in H^4 and H^2. This example serves as proof-of-concept for constructing Θ-stratifications using the intrinsic formulation of the Hilbert-Mumford criterion.

We shall make use of the natural equivalence between the categories of GL_R-bundles and locally free sheaves of rank R on Σ, and between SL_R-bundles and locally free sheaves with trivial determinant. We define the *slope* of a locally free sheaf $\nu(\mathcal{E}) := \deg(\mathcal{E})/\mathrm{rank}(\mathcal{E})$.[6] A locally free sheaf $\mathcal{E}$ on Σ is called *slope semistable* if, for all locally free subsheaves $\mathcal{F} \subset \mathcal{E}$ with locally free quotient one has $\nu(\mathcal{F}) < \nu(\mathcal{E})$.[7]

Any unstable locally free sheaf has a unique filtration (up to indexing) such that the associated graded sheaves $\mathrm{gr}_i(\mathcal{E}_\bullet) = \mathcal{E}_i/\mathcal{E}_{i+1}$ are semistable, and the slope $\nu(\mathrm{gr}_i(\mathcal{E}_\bullet))$ is strictly increasing in i. This is known as the Harder-Narasimhan (HN) filtration. Typically, the indices of the subsheaves $\mathcal{E}_i$ are not taken as part of the data of the HN filtration, but one consequence of our analysis below is that the indices are canonical up to scaling $\mathcal{E}_i \mapsto \mathcal{E}_{ni}$, so we will see that there is a canonical $f : \Theta \to \mathrm{Bun}_G(\Sigma)$ up to the n-fold coverings $\Theta \to \Theta$ which is "maximally destabilizing" for $\mathcal{E}$. By Proposition 4.1.4, we can interpret this as saying that for every unstable $p \in \mathrm{Bun}_G(\Sigma)$, there is a map $f : \Theta \to \mathrm{Bun}_G(\Sigma)$ with $f(1) \simeq [\mathcal{E}]$ which corresponds to the HN filtration of $\mathcal{E}$.

To any map $\Theta \to \mathrm{Bun}_G(\Sigma)$, corresponding to a filtered vector bundle $\mathcal{E}_\bullet$, we associate the sequence $(R_i, D_i) := (\mathrm{rank}(\mathcal{E}_i), \deg(\mathcal{E}_i)) \in \mathbb{Z}^2$ which is a topological invariant, in the sense that these integers are locally constant on $\underline{\mathfrak{X}}(\Theta)$, where $\mathfrak{X} = \mathrm{Bun}_G(\Sigma)$. We also define $R = \mathrm{rank}\,\mathcal{E}$ and $D = \deg\mathcal{E}$. Because $\mathcal{E}_i = \mathcal{E}_{i+1}$ for all but finitely many values of i, we can encode this data more concisely (but equivalently) as a finite sequence $\alpha = \{(r_j, d_j, w_j) | j = 1, \ldots, p\}$ where w_j ranges over indices for which $\mathrm{gr}_{w_j} \mathcal{E}_\bullet \neq 0$, $r_j = \mathrm{rank}\,\mathrm{gr}_{w_j}\mathcal{E}_\bullet$, and $d_j = \deg\mathrm{gr}_{w_j}\mathcal{E}_\bullet$. Given this data we can reconstruct the sequence (R_i, D_i) with the

$$(R_i, D_i) = \left(\sum_{j | w_j \geq i} r_j, \sum_{j | w_j \geq i} d_j \right)$$

To such a sequence, we associate the *polytope* $\mathrm{Pol}(\alpha)$, which is the convex hull of points $(R_i, D_i) \in \mathbb{R}^2$ for $i \in \mathbb{Z}$. Note that because our filtration is decreasing, the points (R_i, D_i) move from right to left in the (r, d)-plane as i varies from $-\infty$ to ∞.

Shatz showed in [41] that the moduli of unstable locally free sheaves on Σ whose HN filtration has a particular polytope (note that the polytope does not depend on the indexing of the filtration) is a locally closed substack S^α of $\mathrm{Bun}_G(\Sigma)$. Furthermore, the closure of S^α is the union of S^β for all β such that $\mathrm{Pol}(\alpha) \subset \mathrm{Pol}(\beta)$. Our goal is to find a numerical invariant μ on $\mathrm{Bun}_G(\Sigma)$ such that the associated stability function $M^\mu([\mathcal{E}])$ (Definition 4.3.2)

---

[6]The slope is more commonly denoted μ, but we have chosen the letter ν to avoid confusion with the notion of a numerical invariant on a stack $\mathfrak{X}$.

[7]For an arbitrary reductive G, a G-bundle $E \to \Sigma$ is semistable if for any one parameter subgroup λ of G and any reduction of structure group to P_λ, the line bundle on Σ associated to any dominant character of P_λ has $\deg \leq 0$ [37]. In the case of $G = \mathrm{GL}_R$ or SL_R, a reduction of structure group to P_λ corresponds to a decreasing filtration $\cdots \subset \mathcal{E}_{i+1} \subset \mathcal{E}_i \subset \cdots \mathcal{E}$ where $\mathrm{rank}\,\mathrm{gr}_i(\mathcal{E}_\bullet)$ is the dimension of the eigenspace of λ of weight i in the fundamental representation of G. One can show that an unstable bundle can be detected by considering two-step filtrations $0 \subset \mathcal{F} \subset \mathcal{E}$, and that this notion of stability agrees with the notion of slope semistability.

is a function of the polytope of the HN filtration of $\mathcal{E}$, and M^μ should be strictly monotone increasing with respect to inclusion of polytopes.

Cohomology classes on $\mathrm{Bun}_G(\Sigma)$ can be constructed geometrically via "transgression" along the universal diagram

$$
\begin{array}{ccccc}
T \times \Sigma & \longrightarrow & \mathrm{Bun}_G(\Sigma) \times \Sigma & \longrightarrow & */G \\
{\scriptstyle \pi_T}\downarrow & & {\scriptstyle \pi}\downarrow & & \\
T & \xrightarrow{\;\;f\;\;} & \mathrm{Bun}_G(\Sigma) & &
\end{array}
\tag{4.7}
$$

If we choose a coherent sheaf F on Σ and a representation V of G, then we have the cohomology class[8]

$$
\mathrm{ch}\, R\pi_*(F \otimes V_{E_{univ}}) \in H^{even}(\mathrm{Bun}_G(\Sigma); \mathbb{Q})
$$

where ch denotes the Chern character and $V_{E_{univ}} = E_{univ} \times_G V$ is the locally free sheaf on $\mathrm{Bun}_G(\Sigma) \times \Sigma$ associated to the representation V by the universal G-bundle E_{univ}.

Proposition 4.3.9. *Let $\sqrt{K}$ be a square root of the canonical bundle on Σ, and let k_p be the structure sheaf of a point $p \in \Sigma$. Let $G = \mathrm{GL}_R$ or SL_R, and let V be the vector representation $V = \mathbb{C}^R$ and $W := V^{\otimes R} \otimes \det^{-1}(V)$. Consider the numerical invariant μ defined by Equation (4.5) using the cohomology classes*

$$
l := \frac{-1}{R^R}\, \mathrm{ch}_1(R\pi_*(\sqrt{K} \otimes W_{E_{univ}})) \in H^2(\mathrm{Bun}_G(\Sigma); \mathbb{Q}), \text{ and}
$$
$$
b := 2\, \mathrm{ch}_2(R\pi_*(k_p \otimes V_{E_{univ}})) \in H^4(\mathrm{Bun}_G(\Sigma); \mathbb{Q})
$$

Then for a locally free sheaf $\mathcal{E}$, $M^\mu([\mathcal{E}]) > 0$ if and only if $\mathcal{E}$ is slope unstable, and if the Harder-Narasimhan filtration of a locally free sheaf $\mathcal{E}$ has graded pieces with slopes ν_j and ranks r_j, then $M^\mu([\mathcal{E}]) = \sqrt{\sum_j \nu_j^2 r_j - \nu^2 R} > 0$.

Remark 4.3.10. When $G = \mathrm{SL}_R$, the cohomology class $-\mathrm{ch}_1(R\pi_*(\sqrt{K} \otimes V_{E_{univ}}))$ in $H^2(\mathrm{Bun}_G(\Sigma); \mathbb{Q})$ induces the same numerical invariant on $\mathrm{Bun}_G(\Sigma)$ as the class l.

We will prove this Proposition in the next section, but first we will analyze the stability function $M^\mu([\mathcal{E}])$.

If $\mathcal{E}_\bullet$ is a vector bundle with decreasing filtration, the sequence of points (R_i, D_i) can be linearly interpolated in a canonical way to a continuous piecewise linear function $h_{\mathcal{E}_\bullet} : [0, R] \to \mathbb{R}$ such that $D_i = h_{\mathcal{E}_\bullet}(R_i)$. Note that $h_{\mathcal{E}_\bullet}$ does not depend on the indexing of the filtration.

---

[8]Using Grothendieck-Riemann-Roch this can also be expressed as the cohomological pushforward of cohomology classes on $\mathrm{Bun}_G(\Sigma) \times \Sigma$, namely we can write this cohomology class as $[\Sigma] \cap ((1 + \frac{1}{2}c_1(K)) \cdot \mathrm{ch}(F) \cdot \mathrm{ch}(V))$.

Corollary 4.3.11. *Let $\mathcal{E}$ be a locally free sheaf of rank R and degree D. If $h_{\mathcal{E}_\bullet} : [0, R] \to \mathbb{R}$ is the piecewise linear function associated to the Harder-Narasimhan filtration of $\mathcal{E}$, then*

$$M^u([\mathcal{E}]) = \sqrt{\int_0^R (h'_{\mathcal{E}_\bullet}(x))^2 dx - \nu^2 R} \tag{4.8}$$

This stability function is strictly monotone increasing with respect to inclusion of Shatz polytopes. Therefore $M^\mu([\mathcal{E}])$ recovers the Shatz stratification of the unstable locus of $\mathrm{Bun}_G(\Sigma)$ via the formula (4.6).

Proof. The integral in (4.8) is simply a reinterpretation of the sum $\sum \nu_j^2 r_j$ in the expression for $M^\mu([\mathcal{E}])$ – each term $\nu_j^2 r_j$ corresponds to an interval of length r_j on which $h'_{\mathcal{E}_\bullet}(x) = \nu_j$ is constant.

We must show that this expression is monotone increasing with respect to inclusion of polytopes. If $h_1, h_2 : [0, R] \to \mathbb{R}$ are continuous piecewise linear functions with $h'_i(x)$ decreasing and with $h_1(x) \leq h_2(x)$ with equality at the endpoints of the interval, then we must show that $\int_0^R (h'_1(x))^2 < \int_0^R (h'_2(x))^2$. First by suitable approximation with respect to a Sobolev norm it suffices to prove this when h_i are smooth functions with $h'' < 0$.[9] Then we can use integration by parts

$$\int_0^R (h'_2)^2 - (h'_1)^2 dx = \int_0^R (h'_2 + h'_1)(h'_2 - h'_1) dx$$

$$= (h'_2 - h'_1)(h_2 - h_1)|_0^R - \int_0^R (h_2 - h_1)(h''_2 + h''_1) dx$$

The first term vanishes because $h_1 = h_2$ at the endpoints, and the second term is strictly positive unless $h_1 = h_2$. $\qquad \square$

Proof of Proposition 4.3.9

Before proving Proposition 4.3.9, we compute the cohomology classes f^*l and f^*b more explicitly for a morphism $f : \Theta \to \mathrm{Bun}_G(\Sigma)$. Given a coherent sheaf F on Σ, we apply base change on the diagram 4.7 to compute

$$f^* \mathrm{ch}\, R\pi_*(F \otimes V_{E_{univ}}) = \mathrm{ch}\,(R(\pi_\Theta)_*(F \otimes V_E))$$

Where $E = f^*E_{univ}$ is the G-bundle classified by f. Thus we must compute the K-theoretic pushforward to Θ of the classes $[F \boxtimes V_E] \in K_0(\Theta \times \Sigma)$ where $F = \sqrt{K}$ or $F = k_p$.

By Proposition 4.1.6 a G-bundle E on $\Theta \times \Sigma$ corresponds to a one-parameter subgroup λ, and a G-bundle on Σ with a reduction of structure group to P_λ. In fact, the E admits a

---

[9]One can probably prove the inequality without appealing to analysis by using a discrete integration by parts argument.

canonical reduction to a P_λ bundle E' on $\Theta \times S$. The subspaces $V_{\lambda \geq i}$ are subrepresentations of P_λ, hence the associated locally free sheaf $\mathcal{V} := V_E = V_{E'}$ on $\Theta \times \Sigma$ is filtered by the associated locally free sheaves $\mathcal{V}_i := (V_{\lambda \geq i})_{E'}$. One the other hand, $\mathcal{V}$ uniquely corresponds to the data of the restriction $\mathcal{E} = \mathcal{V}|_{\{1\} \times \Sigma}$ along with the decreasing filtration $\mathcal{E}_i = \mathcal{V}_i|_{\{1\} \times \Sigma}$. If a locally free sheaf $\mathcal{V}$ on $\Theta \times \Sigma$ is concentrated in weight w, then we have $\mathcal{V} \simeq \mathcal{E}(-w) = \mathcal{O}_\Theta(-w) \boxtimes \mathcal{E}$ for some w. Thus we have the following identity in $K_0(\Theta \times \Sigma)$

$$[V_E] = \sum_i [\mathcal{O}_\Theta(-i) \boxtimes \mathrm{gr}_i \, \mathcal{E}_\bullet] = \sum_i u^{-i}[\mathrm{gr}_i \, \mathcal{E}_\bullet]$$

where the classes $[\mathrm{gr}_i \, \mathcal{E}_\bullet]$ are pulled back from Σ (and given the trivial $\mathbb{C}^*$ action) and u is the class of the trivial line bundle with $\mathbb{C}^*$ action of weight 1 (equivalently the invertible sheaf whose fiber at $\{0\}$ has weight -1). Thus we have for any coherent sheaf F on Σ,

$$\mathrm{ch}(R\pi_*[F \otimes V_E]) = \mathrm{ch}\left(\sum_i u^{-i} \chi(\Sigma, F \otimes \mathrm{gr}_i \, \mathcal{E}_\bullet)\right) = \sum_i e^{-iq} \chi(\Sigma, F \otimes \mathrm{gr}_i \, \mathcal{E}_\bullet) \qquad (4.9)$$

in $H^*(\Theta) = \mathbb{Q}[[q]]$, where $\mathcal{E}_\bullet$ is the filtered locally free sheaf on Σ corresponding (via Proposition 4.1.4) to the locally free sheaf V_E on $\Theta \times \Sigma$.

Lemma 4.3.12. *Let $G = \mathrm{GL}_R$ or SL_R and let l and b be the cohomology classes on $\mathrm{Bun}_G(\Sigma)$ introduced in Proposition 4.3.9. Let $f : \Theta \to \mathrm{Bun}_G(\Sigma)$ be a morphism with $f(1) \simeq [\mathcal{E}]$, which corresponds to a descending filtration $\mathcal{E}_\bullet$ of $\mathcal{E}$. We let $r_j = \mathrm{rank} \, \mathrm{gr}_{w_j}(\mathcal{E}_\bullet)$ and $d_j = \deg \mathrm{gr}_{w_j}(\mathcal{E}_\bullet)$ as w_j ranges over weights in which $\mathrm{gr}_i(\mathcal{E}_\bullet) \neq 0$. Then the cohomology class b is positive definite, meaning that $f^*b \geq 0$ with equality if and only if the induced homomorphism $\mathbb{G}_m \to \mathrm{Aut} \, f(0)$ is trivial. The numerical invariant is*

$$\mu(f) := \frac{f^*l}{\sqrt{f^*b}} = \frac{\sum_{j=1}^p w_j d_j - \nu \sum_{j=1}^p w_j r_j}{\sqrt{\sum_{j=1}^p w_j^2 r_j}}$$

Proof. We apply Equation (4.9) to the standard representation $V = \mathbb{C}^R$ and $F = k_p$ to compute the class

$$f^*b = \sum_i i^2 q^2 \chi(k_p \otimes \mathrm{gr}_i \, \mathcal{E}) = \sum_i w_j^2 r_j q^2$$

This expression is nonnegative, and it vanishes if and only if the filtration of $\mathcal{E}$ is trivial and concentrated in weight 0, which corresponds to the homomorphism $\mathbb{G}_m \to \mathrm{Aut}(\mathrm{gr} \, \mathcal{E}_\bullet)$ being trivial.

In order to compute f^*l we consider the representation $W = V^{\otimes R} \otimes \det^{-1}(V)$. Now f classifies a principal G-bundle E over $\Theta \times \Sigma$ with $f(1) \simeq [\mathcal{E}]$ and we have

$$[W_E] = u^{\sum i r_i}[\det(\mathcal{E})^{-1}] \cdot \left(\sum u^{-i}[\mathrm{gr}_i \, \mathcal{E}]\right)^R \in K_0(\Theta \times \Sigma)$$

Using Riemann-Roch we see that $\chi(\sqrt{K} \otimes \bullet) = \mathrm{ch}_1(\bullet) \cap [\Sigma]$, which is a derivation over the ring homomorphism $\mathrm{ch}_0 : K_0(\Sigma) \to \mathbb{Z}$. Furthermore, $R(\pi_\Theta)_*(\sqrt{K} \otimes \bullet) : K_0(\Theta \times \Sigma) \to K_0(\Theta)$ is the $\mathbb{Z}[u^{\pm}]$-linear extension of this derivation. This allows us to evaluate

$$R(\pi_\Theta)_*[\sqrt{K} \otimes W_E] = u^{-\sum w_j r_j} \left[(-D)(\sum u^{-w_j} r_j)^R + R(\sum u^{-w_j} r_j)^{R-1}(\sum u^{-w_j} d_j) \right]$$

Taking the Chern character gives

$$\mathrm{ch}\, R(\pi_\Theta)_*[\sqrt{K} \otimes W_E] = e^{\sum w_j r_j q}(\sum e^{-w_j q} r_j)^R (\sum e^{-w_j q} d_j - \nu \sum e^{-w_j q} r_j)$$

The last factor vanishes when $q = 0$, thus the coefficient of q in this power series is

$$f^* l = \frac{-1}{R^R} \mathrm{ch}_1 R\pi_*[\sqrt{K} \otimes W_E] = \frac{1}{R^R} \left(e^{\sum w_j r_j q}(\sum e^{-w_j q} r_j)^R \right)\Big|_{q=0} (\sum w_j d_j - \nu \sum w_j r_j)$$
$$= \sum w_j d_j - \nu \sum w_j r_j$$

$\square$

Remark 4.3.13. If $G = \mathrm{SL}_R$, then a morphism $f : \Theta \to \mathrm{Bun}_G(\Sigma)$ with $f(1) \simeq [\mathcal{E}]$ is a decreasing filtration of $\mathcal{E}$ with indexing such that $\sum_i i\, \mathrm{rank}(\mathrm{gr}_i \mathcal{E}_\bullet) = 0$. The requirement on the ranks of the graded pieces expresses the fact that a 1PS in SL_R is a choice of weight decomposition of $\mathbb{C}^R$ with precisely the same rank constraint. In this case $f^* l = \sum w_j d_j$, and one can compute that $\sum w_j d_j = \sum_i -iq\chi(\sqrt{K} \otimes \mathrm{gr}_i \mathcal{E}_\bullet) = -f^* \mathrm{ch}_1 R\pi_*[\sqrt{K} \otimes V_E]$ as well. Therefore we could have used the cohomology class $-\mathrm{ch}_1 R\pi_*[\sqrt{K} \otimes V_E] \in H^2(\mathrm{Bun}_{\mathrm{SL}_R}(\Sigma); \mathbb{Q})$ instead of the class l.

Remark 4.3.14. Note that given a filtration of $\mathcal{E}$, one can simultaneous shift the indexing $\mathcal{E}_i \mapsto \mathcal{E}_{i+k}$. $f^* l$ is unchanged by this shift as a result of the fact that $Z_0(\mathrm{GL}_R)$ acts trivially on W. If this were not the case, then any bundle could be made unstable by suitably shifting a filtration. This is why the class $-\mathrm{ch}_1 R\pi_*(\sqrt{K} \otimes V_E)$ is not suitable for defining a numerical invariant when $G = \mathrm{GL}_R$.

The numerical criterion for semistability implied by the cohomology class $f^* l$ says that a vector bundle $\mathcal{E}$ on Σ of rank R and degree D is semistable iff for every decreasing filtration of $\mathcal{E}$ we have $\sum i(\deg(\mathrm{gr}_i \mathcal{E}_\bullet) - \nu\, \mathrm{rank}(\mathrm{gr}_i \mathcal{E}_\bullet)) \leq 0$. If we have a single subbundle $\mathcal{F} \subset \mathcal{E}$, we consider this as a filtration where $\mathrm{gr}_b \mathcal{E}_\bullet = \mathcal{F}$ and $\mathrm{gr}_a \mathcal{E}_\bullet = \mathcal{E}/\mathcal{F}$ with $b > a$. The numerical criterion says that if $\mathcal{E}$ is semistable then

$$0 \geq b(\deg \mathcal{F} - \nu\, \mathrm{rank}\, \mathcal{F}) + R^R a(\deg \mathcal{E}/\mathcal{F} - \nu\, \mathrm{rank}\, \mathcal{E}/\mathcal{F})$$
$$= (b - a)\, \mathrm{rank}(\mathcal{F}) \left(\frac{\deg \mathcal{F}}{\mathrm{rank}\, \mathcal{F}} - \nu \right)$$

So a slope unstable bundle is also unstable with respect to the numerical invariant μ. The converse, that a bundle with $M^\mu([\mathcal{E}]) > 0$ is slope unstable, will follow from our explicit

analysis of the filtration which maximizes μ over all $f : \Theta \to \operatorname{Bun}_G(\Sigma)$ with $f(1) \simeq \mathcal{E}$, assuming that there is at least one filtration such that $\mu(f) > 0$, i.e. assuming $\mathcal{E}$ is unstable.

Given a filtration of $\mathcal{E}$ we have some flexibility to re-index, which gives different $f : \Theta \to \operatorname{Bun}_G(\Sigma)$. As above we denote the data of the filtration by the sequence (r_j, d_j, w_j) for $j = 1, \ldots, p$. The different choices of indexing in the filtration correspond precisely to choices of w_j subject to the inequality $w_1 < \cdots < w_p$ (and the constraint $w_1 r_1 + \cdots + w_p r_p = 0$ if $G = \operatorname{SL}_R$). As discussed above this data can be visualized as a piecewise linear path in the (r, d)-plane from the point $(R_0, D_0) = (R, D)$ to the point $(R_p, D_p) = (0, 0)$. The slope of the jth segment is $\nu_j := d_j / r_j$ for $j = 1, \ldots, p$. The path is strictly convex if $\nu_1 < \cdots < \nu_p$.

Lemma 4.3.15. *If $0 \subset \mathcal{E}_{w_p} \subset \cdots \subset \mathcal{E}_{w_1} = \mathcal{E}$ is a decreasing filtration of $\mathcal{E}$ such that $\mu \geq 0$ and $\nu_j \geq \nu_{j+1}$ for some j, then discarding the sub-bundle $\mathcal{E}_{w_{j+1}}$ from our filtration and relabelling*

$$w'_j := \frac{w_j r_j + w_{j+1} r_{j+1}}{r_j + r_{j+1}}$$

does not decrease μ. If necessary, we can scale the w_j without affecting μ so that w'_j is an integer.

Proof. We draw the relevant vertices of the path in the (r, d)-plane corresponding to the filtration $\mathcal{E}_\bullet$.
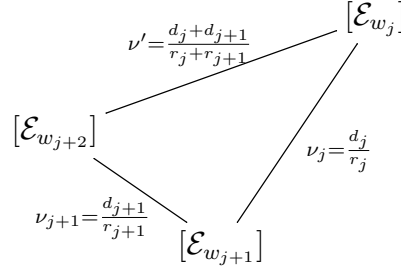
We denote $\mu = L/\sqrt{B}$, then discarding $\mathcal{E}_{w_{j+1}}$ and relabelling w'_j as above, the numerator and denominator change by

$$\begin{aligned}
\Delta L &= w'_j (d_j + d_{j+1}) - w_j d_j - w_{j+1} d_{j+1} \\
&= w_j (\nu' r_j - d_j) + w_{j+1} (\nu' r_{j+1} - d_{j+1}) \\
\Delta B &= (w'_j)^2 (r_{j+1} + r_j) - w_j^2 r_j - w_{j+1}^2 r_{j+1} \\
&= -\frac{r_j r_{j+1}}{r_j + r_{j+1}} (w_j - w_{j+1})^2
\end{aligned}$$

Note that $\Delta B \leq 0$. Also $(\nu' r_j - d_j) + (\nu' r_{j+1} - d_{j+1}) = 0$, so $\Delta L = (w_{j+1} - w_j)(\nu' r_{j+1} - d_{j+1})$. By hypothesis $\nu' r_{j+1} - d_{j+1} \geq 0$, so $\Delta L \geq 0$, and assuming $\mu \geq 0$ to begin with we see that $\mu' \geq \mu$. $\qquad\square$

Thus if we are trying to maximize μ over all destabilizing filtrations of $\mathcal{E}$, it suffices to consider only those flags whose corresponding path in the (r, d)-plane are convex, meaning

$\nu_1 < \cdots < \nu_p$, because we can discard sub-bundles in any destabilizing filtration until it satisfies this property. Next we find the optimal indexing for a given strictly convex filtration

Lemma 4.3.16. *Let $\mathcal{E}_{w_p} \subsetneq \cdots \subsetneq \mathcal{E}_{w_1}$ be a filtration of $\mathcal{E}$ such that $\nu_1 < \cdots < \nu_p$, then $\mu(f)$ is maximized by assigning the indices $w_j \propto \nu_j - \nu$, where $\nu = D/R$. The maximum is thus*

$$\mu = \sqrt{\left(\sum \nu_j d_j\right) - \nu D} = \sqrt{\left(\sum \nu_j^2 r_j\right) - \nu^2 R}$$

Remark 4.3.17. For the group SL_R, we must maximize μ subject to the constraint $\sum w_i r_j = 0$. However, this condition is automatically satisfied by the assignments $w_j \propto \nu_j - \nu$. Therefore, this lemma applies equally to both $G = \mathrm{GL}_R$ and SL_R.

Proof. We can think of the numbers $r_1, \cdots, r_p$ as defining an inner product $\vec{a} \cdot \vec{b} = \sum a_j b_j r_j$. Then given an indexing of the filtration $\vec{w} = (w_1, \cdots, w_p)$, the numerical invariant can be expressed as

$$\mu = \frac{1}{|\vec{w}|}\vec{w} \cdot (\vec{\nu} - \nu\vec{1})$$

where $\vec{\nu} = (\nu_1, \cdots, \nu_p)$ and $\vec{1} = (1, \ldots, 1)$. From linear algebra we know that this quantity is maximized when $\vec{w} \propto \vec{\nu} - \nu\vec{1}$, and the maximum value is $|\vec{\nu} - \nu\vec{1}|$. In the case when $\nu_1 < \cdots < \nu_p$ the assignment $\vec{w} \propto \vec{\nu} - \nu\vec{1}$ satisfies the constraints $w_1 < w_2 < \cdots < w_p$. $\qquad \square$

We have thus completed the proof of Proposition 4.3.9.

4.4 Existence and uniqueness of generalized Harder-Narasimhan filtrations

In Section 4.3, we gave an a posteriori description of the stratification of the unstable locus in GIT intrinsically in terms of classes $l \in H^2(\mathfrak{X}; \mathbb{Q})$ and $b \in H^4(\mathfrak{X}; \mathbb{Q})$. In this section, we study the problem of when two such classes on an arbitrary stack can be used to define a Θ-stratification. We revisit the original construction of the stratification of the unstable locus in V/G where V is an affine variety and G a reductive group.

In this case for any point $p \in \mathfrak{X} = V/G$ there is a unique $f : \Theta \to \mathfrak{X}$ with an isomorphism $f(1) \simeq p$ which maximizes the numerical invariant. In order to investigate this theorem from an intrinsic perspective, we first introduce a combinatorial tool for studying the set of all such maps $f : \Theta \to \mathfrak{X}$.

A combinatorial structure describing degenerations of a point in a stack

Let $p \in \mathfrak{X}(k)$ be a point in an algebraic stack. We have seen that the Hilbert-Mumford numerical criterion can be formulated intrinsically as a maximization of a numerical invariant

$\mu(f)$ over the set of isomorphism classes of maps $f : \Theta \to \mathfrak{X}$ with $f(1) \simeq p$. We denote the fiber product $\underline{\mathfrak{X}}(\Theta)_p = \underline{\mathfrak{X}}(\Theta) \times_{r_1, \mathfrak{X}} \operatorname{Spec} k$, which parameterizes $\Theta \to \mathfrak{X}$ with a choice of isomorphism $f(1) \simeq p$. It is a scheme,[10] but we will sometimes abuse notation and use $\underline{\mathfrak{X}}(\Theta)_p$ to refer to its set of k-points. The numerical invariant is locally constant and manifestly invariant under the action of $\operatorname{Aut}(p)$ on $\underline{\mathfrak{X}}(\Theta)_p$, so instability is equivalent to μ attaining a positive value on the set $\underline{\mathfrak{X}}(\Theta)_p / \operatorname{Aut}(p)$.

Even in simple examples, this set is infinite, so the existence and uniqueness of a maximizer for μ is not immediate. However, we will show that $\underline{\mathfrak{X}}(\Theta)_p / \operatorname{Aut}(p)$ are the "rational" points of a certain topological space and that μ extends to a continuous function on this space, which will allow us to address the problem of maximization. In fact, this space will typically be the geometric realization of a simplicial complex, but we will introduce a different combinatorial structure which is better suited to our application.

Definition 4.4.1. We define a category of *integral simplicial cones* $\mathfrak{C}$ to have

- objects: nonnegative integers $[n]$ with $n \geq 0$,

- morphisms: a morphism $\phi : [k] \to [n]$ is an injective group homomorphism $\mathbb{Z}^k \to \mathbb{Z}^n$ which maps the standard basis of $\mathbb{Z}^k$ to the cone spanned by the standard basis of $\mathbb{Z}^n$.

We define the category of *fans*

$$\operatorname{Fan} := \operatorname{Fun}(\mathfrak{C}^{op}, \operatorname{Set})$$

For $F \in \operatorname{Fan}$ we use the abbreviated notation $F_n = F([n])$. Unless otherwise specified, we assume that all of our fans are connected, which we take to mean $F_0 = \{*\}$.

For any $F \in \operatorname{Fan}$, we can define two notions of geometric realization. First form the comma category $(\mathfrak{C}|F)$ whose objects are elements $\sigma \in F_n$ and morphisms $\xi_1 \to \xi_2$ are given by morphisms $\phi : [n_1] \to [n_2]$ with $\phi^* \xi_2 = \xi_1$. There is a canonical functor $(\mathfrak{C}|F) \to \operatorname{Top}$ assigning $\xi \in F_n$ to the cone $(\mathbb{R}^n)_+$ spanned by the standard basis of $\mathbb{R}^n$. Using this we can define the geometric realization of F

$$|F| := \operatorname*{colim}_{(\mathfrak{C}|F)} (\mathbb{R}^n)_+$$

This is entirely analogous to the geometric realization functor for simplicial sets. We think of an object $F \in \operatorname{Fan}$ as an abstraction of the usual notion of a fan in a vector space.

Given a map $\phi : [k] \to [n]$ in $\mathfrak{C}$, the corresponding linear map $\phi : \mathbb{R}^k \to \mathbb{R}^n$ is injective. Thus ϕ descends to a map $\Delta_{k-1} \to \Delta_{n-1}$, where $\Delta_{n-1} = ((\mathbb{R}^n)_+ - \{0\}) / \mathbb{R}_+^\times$ is the standard $(n-1)$-simplex realized as the space of rays in $(\mathbb{R}^n)_+$. Thus for any $F \in \operatorname{Fan}$ we have a

---

[10]We have only shown that the morphism $r_1 : \underline{\mathfrak{X}}(\Theta) \to \mathfrak{X}$ is representable when $\mathfrak{X}$ is locally a quotient of a k-scheme by a locally affine action of a linear group. However, one can directly check that the groupoid which is the fiber of r_1 over an S-point of $\mathfrak{X}$ is equivalent to a set (i.e. trivial automorphism groups). Thus we can always consider $\underline{\mathfrak{X}}(\Theta)_p$ as a set without making use of the representability results of section 4.2

functor $(\mathfrak{C}|F) \to$ Top assigning $\xi \in F_n$ to Δ_{n-1}. We define the *projective realization* of F to be

$$\mathbb{P}(F) := \operatorname*{colim}_{(\mathfrak{C}|F)} \Delta_{n-1}$$

Construction 4.4.2. *A subset $K \subset \mathbb{R}^N$ which is invariant under multiplication by $\mathbb{R}_+ = \{t \geq 0\}$ is called a* cone *in $\mathbb{R}^N$. Given a set of cones $K_\alpha \subset \mathbb{R}^N$, we define*

$$R_*(\{K_\alpha\})_n := \{injective\ homomorphisms\ \phi : \mathbb{Z}^n \to \mathbb{Z}^N \,|\, \exists \alpha\ s.t.\ \phi(e_i) \subset K_\alpha, \forall i\} \quad (4.10)$$

The sets $R_*(\{K_\alpha\})_n$ naturally define an object of Fan.

Remark 4.4.3. We use the phrase *classical fan* to denote a collection of rational polyhedral cones in $\mathbb{R}^N$ such that a face of any cone is also in the collection, and the intersection of two cones is face of each. We expect that if $K_\alpha \subset \mathbb{R}^n$ are the cones of a classical fan Σ, it is possible to reconstruct Σ from the data of $R_*(\{K_\alpha\})$.

Lemma 4.4.4. *Let $K_\alpha \subset \mathbb{R}^N$ be a finite collection of cones and assume that there is a simplicial classical fan $\{\sigma_i\}$ in $\mathbb{R}^N$ such that each K_α is the union of some collection of σ_i. Then the canonical map $|R_*(\{K_\alpha\})| \to \bigcup K_\alpha$ is a homeomorphism. Furthermore, $\mathbb{P}(F) \simeq S^{N-1} \cap \bigcup_\alpha K_\alpha$ via the evident quotient map $\mathbb{R}^N - \{0\} \to S^{N-1}$.*

Proof. Consider the fans $F = R_*(\{K_\alpha\})$ and $F' = R_*(\{\sigma_i\})$. By hypothesis F' is a subfunctor of $F : \mathfrak{C}^{op} \to$ Set. Hence we have a functor of comma categories $(\mathfrak{C}|F') \to (\mathfrak{C}|F)$ and thus a map of topological spaces $|F'| \to |F|$ which commutes with the map to $\mathbb{R}^N$.

Note that the map $|F'| \to |F|$ is surjective on points because any point on a cone in K_α is contained in a cone σ_i for some σ_i. If the composition $|F'| \to |F| \to \bigcup_\alpha K_\alpha = \bigcup \sigma_i$ were a homeomorphism it would follow that $|F'| \to |F|$ was injective on points as well, and one could use the inverse of $|F'| \to \bigcup_\alpha K_\alpha$ to construct and inverse for $|F| \to \bigcup_\alpha K_\alpha$. Thus it suffices to prove the lemma for a simplicial fan in $\mathbb{R}^N$.

For a single simplicial cone $\sigma \subset \mathbb{R}^N$ of dimension n whose ray generators $v_1, \ldots, v_n$ form a basis for the lattice $\operatorname{span}(v_1, \ldots, v_n) \cap \mathbb{Z}^N$, the fan $R_*(\sigma) \subset R_*(\mathbb{R}^N)$ is equivalent to the representable fan $h_{[n]}([k]) = \operatorname{Hom}_{\mathfrak{C}}([k], [n])$. The category $(\mathfrak{C}|h_{[n]})$ has a terminal object which is the linear map $\mathbb{R}^n \to \mathbb{R}^N$ mapping the standard basis vectors to the ray generators of σ. It follows that $|R_*(\sigma)| \to \sigma$ is a homeomorphism.

By subdividing our rational simplicial fan $\Sigma = \{\sigma_i\}$ in $\mathbb{R}^N$, we can assume that the ray generators of each σ_i form a basis for the lattice generated by $\sigma_i \cap \mathbb{Z}^N$. Let $\sigma_1^{\max}, \ldots, \sigma_r^{\max}$ be the cones of Σ which are maximal with respect to inclusion and let n_i be the dimension of each. Then $\bigsqcup h_{[n_i]} \to F$ is a surjection of functors. In fact if we define $\sigma'_{ij} := \sigma_i^{\max} \cap \sigma_i^{\max}$, then by definition this is a cone of Σ as well, and we let n_{ij} denote its dimension. By construction

$$F = \operatorname{coeq}\left(\bigsqcup_{i,j} h_{n_{ij}} \rightrightarrows \bigsqcup_i h_{n_i}\right)$$

as functors $\mathfrak{C}^{op} \to$ Set. Our geometric realization functor commutes with colimits, so it follows that

$$|F| = \text{coeq}\left(\bigsqcup_{i,j}\sigma'_{ij} \rightrightarrows \bigsqcup_{i}\sigma^{\max}_{ij}\right)$$

Which is homeomorphic to $\bigcup \sigma_i$ under the natural map $|F| \to \mathbb{R}^N$. $\qquad\square$

Example 4.4.5. Objects of Fan describe a wider variety of structures than classical fans. For instance if K_1 and K_2 are two simplicial cones which intersect but do not meet along a common face, then $R_*(K_1, K_2)$ will not be equivalent to $R_*(\{\sigma_i\})$ for any classical fan $\Sigma = \{\sigma_i\}$.

Example 4.4.6. While objects of Fan are more general than classical fans, the definition is broad enough to include some pathological examples. For instance, if $K \subset \mathbb{R}^3$ is the cone over a circle, then $|R_*(K)|$ consists of the rational rays of K equipped with the discrete topology and is not homeomorphic to K. If $K \subset \mathbb{R}^2$ is a convex cone generated by two irrational rays, then $|R_*(K)_\bullet|$ is the interior of that cone along with the origin. There are also examples of fans whose geometric realizations are not Hausdorff, such as multiple copies of the standard cone in $\mathbb{R}^2$ glued to each other along the set of rational rays.

We now return to our application. Let $\mathfrak{X} = X/G$ be the quotient of a k-scheme by a locally affine action of a linear group G. We consider the iterated mapping stacks

$$\underline{\mathfrak{X}}(\Theta^n) := \underline{\text{Hom}}(\Theta^n, \mathfrak{X}) \simeq \underline{\text{Hom}}(\Theta, \underline{\text{Hom}}(\Theta, \cdots, \underline{\mathfrak{X}}(\Theta)))$$

It follows from iterated applications of Theorem 4.2.2 that $\underline{\mathfrak{X}}(\Theta^n)$ is an algebraic stack, and in fact its connected components are quotient stacks of locally closed subschemes of X by subgroups of G. $\Theta^n = \mathbb{A}^n/\mathbb{G}_m^n$, and restricting a morphism to the point $(1, \ldots, 1) \in \mathbb{A}^n$ defines a representable morphism $r_1 : \underline{\mathfrak{X}}(\Theta^n) \to \mathfrak{X}$.

As before we let $\underline{\mathfrak{X}}(\Theta^n)_p$ denote the fiber of the morphism r_1 over $p \in \mathfrak{X}(k)$. It is a scheme, and it is locally of finite type over k if $\mathfrak{X}$ is.

Definition 4.4.7. We define the *degeneration fan* for a point $p \in \mathfrak{X}$ as

$$\mathbf{D}(\mathfrak{X}, p)_n := \{f \in \underline{\mathfrak{X}}(\Theta^n)_p(k) \,|\, \mathbb{G}_m^n \to \text{Aut}(f(0, \ldots, 0)) \text{ has finite kernel}\} \qquad (4.11)$$

Where the homomorphism $\mathbb{G}_m^n \to \text{Aut}(f(0, \ldots, 0))$ is induced by f under the identification $\text{Aut}((0, \ldots, 0)) = \mathbb{G}_m^n$ in Θ^n. Likewise we define the *reduced* degeneration fan consisting of the orbit sets $\tilde{\mathbf{D}}(\mathfrak{X}, p)_n := \mathbf{D}(\mathfrak{X}, p)_n / \text{Aut}(p)$.

Lemma 4.4.8. *The sets* $\mathbf{D}(\mathfrak{X}, p)_n$ *define a functor* $\mathfrak{C}^{op} \to$ Set, *as do the sets* $\tilde{\mathbf{D}}(\mathfrak{X}, p)_n$.

Proof. A morphism $\phi : [k] \to [n]$ in $\mathfrak{C}$ is represented by a matrix of nonnegative integers ϕ_{ij} for $i = 1, \ldots, n$ and $j = 1, \ldots, k$. One has a map of stacks $\Theta^k \to \Theta^n$ defined by the map $\mathbb{A}^k \to \mathbb{A}^n$

$$(z_1, \ldots, z_k) \mapsto (z_1^{\phi_{11}} \cdots z_k^{\phi_{1k}}, \ldots, z_1^{\phi_{n1}} \cdots z_k^{\phi_{nk}})$$

which is intertwined by the group homomorphism $\mathbb{G}_m^k \to \mathbb{G}_m^n$ defined by the same formula.

Pre-composition gives a morphism $\underline{\mathfrak{X}}(\Theta^n) \to \underline{\mathfrak{X}}(\Theta^k)$ which commutes with r_1 up to natural isomorphism. Thus one gets a morphism $\phi^* : \underline{\mathfrak{X}}(\Theta^n)_p \to \underline{\mathfrak{X}}(\Theta^k)_p$. It is straightforward to check that this construction is functorial.

The set $\mathbf{D}(\mathfrak{X}, p)_n$ admits an action by the group $\mathrm{Aut}(p)$ which commutes with all of the pullback maps ϕ^*, so $\tilde{\mathbf{D}}(\mathfrak{X}, p)_n$ defines a fan as well. $\qquad\square$

Remark 4.4.9. One can also consider the fan of connected components $\pi_0 \underline{\mathfrak{X}}(\Theta^n)_p$, but we will not use this notion here.

Example 4.4.10. If $G = T$ is a torus, then $\mathbf{D}(*/T, *)_n$ is the set of all injective homomorphisms $\mathbb{Z}^n \to \mathbb{Z}^r$ where $r = \mathrm{rank}\,T$. This fan is equivalent to $R_*(\mathbb{R}^r)$ where $\mathbb{R}^r \subset \mathbb{R}^r$ is thought of as a single cone. Because this cone admits a simplicial subdivision, Lemma 4.4.4 implies that $|\mathbf{D}(*/T, *)| \simeq \mathbb{R}^r$ and $\mathbb{P}(\mathbf{D}(*/T, *)) \simeq S^{r-1}$.

Proposition 4.4.11. *Let T be a torus acting on a variety X, let $p \in X(k)$, and define $T' = T/\mathrm{Aut}(p)$. Define $Y \subset X$ to be the closure of $T \cdot p$ and $\tilde{Y} \to Y$ its normalization. $\tilde{Y}$ is a toric variety for the torus T' and thus determines a classical fan consisting of cones $\sigma_i \subset N'_{\mathbb{R}}$, where N' is the cocharacter lattice of T'. Let $\pi : N_{\mathbb{R}} \to N'_{\mathbb{R}}$ be the linear map induced by the surjection from the character lattice of T. Then the cones $\pi^{-1}\sigma_i \subset N_{\mathbb{R}}$ define a classical fan, and*

$$\mathbf{D}(X/T, p) \simeq \tilde{\mathbf{D}}(X/T, p) \simeq R_*(\{\pi^{-1}(\sigma_i)\})$$

Proof. The map $Y/T \to X/T$ is a closed immersion, so by Proposition 4.2.5 the map $\mathbf{D}(Y/T, p) \to \mathbf{D}(X/T, p)$ is an isomorphism, so it suffices to consider the case when X is the closure of a single open orbit, i.e. $X = Y$.

A morphism of stacks $f : \mathbb{A}^n/\mathbb{G}_m^n \to Y/T$ along with an isomorphism $f(1) \simeq p$ is determined uniquely up to unique isomorphism by the group homomorphism $\psi : \mathbb{G}_m^n = \mathrm{Aut}((0, \dots, 0)) \to T$. Given such a group homomorphism ψ, the morphism f is determined by the equivariant map

$$f(t_1, \dots, t_k) = \psi(t) \cdot f(1) = \psi(t) \cdot p \in Y$$

This map is defined on the open subset $\mathbb{G}_m^n \subset \mathbb{A}^n$, and if it extends equivariantly to all of $\mathbb{A}^n$ then the extension is unique because Y is separated. We will use the term "equivariant morphism" $f : \mathbb{A}^n \to Y$ to denote the data of the morphism along with a group homomorphism $\psi : \mathbb{G}_m^n \to T$ which intertwines it.

In the language of fans, this observation says that $\mathbf{D}(Y/T, p)$ is a sub-fan of the fan $\mathbf{D}(*/T, *) \simeq R_*(N_{\mathbb{R}})$ discussed in Example 4.4.10. The open orbit $T \cdot p \subset Y$ is smooth, so the map from the normalization $\tilde{Y} \to Y$ is an isomorphism over this open subset. The projection $\tilde{Y} \to Y$ is finite, and $\tilde{Y}$ has a unique T action covering the T action on Y. If an equivariant morphism $\mathbb{A}^n \to Y$ lifts to $\tilde{Y}$, it does so uniquely because $\tilde{Y} \to Y$ is separated.

Now fix an equivariant morphism $f : \mathbb{A}^n \to Y$ with $f(1, \ldots, 1) = p$. Using the T action on Y we can extend this to a morphism

$$T \times \mathbb{A}^n \to T \times Y \to Y$$

which is equivariant with respect to the action of $T \times \mathbb{G}_m^n$ and is dominant. Thus the morphism factors through $\tilde{Y}$ by the universal property of the normalization. We can then restrict this lift to get an equivariant lift $\mathbb{A}^n \times \{1\} \subset \mathbb{A}^n \times T \to \tilde{F}$ of our original $f : \mathbb{A}^n \to Y$. Thus we have shown that the canonical map

$$\mathbf{D}(\tilde{Y}/T, p)_n \to \mathbf{D}(Y/T, p)_n$$

is a bijection. It thus suffices to prove the proposition when Y is normal.

If Y is normal, then it is a toric variety under the action of T', and it is determined by a fan $\Sigma = \{\sigma_i\}$ in $N'_{\mathbb{R}} = N' \otimes_{\mathbb{Z}} \mathbb{R}$. Equivariant maps between toric varieties preserving a marked point in the open orbit are determined by maps of lattices such that the image of any cone in the first lattice is contained in some cone of the second [19]. Applying this to the toric variety $\mathbb{A}^n$ under the torus $\mathbb{G}_m^n$ and to Y under the torus T', equivariant maps from $\mathbb{A}^n$ to Y correspond exactly to homomorphisms $\phi : \mathbb{Z}^n \to N'$ such that the image of the standard cone in $\mathbb{Z}^N$ lies in some cone of Σ.

Because the T action on Y factors through T', a group homomorphism $\mathbb{G}_m^n \to T$ determines a map $\Theta^n \to Y/T$ if and only if the composite $\mathbb{G}_m^n \to T'$ determines a map $\Theta^n \to Y/T'$. Thus $\mathbf{D}(Y/T, p)_n$ consists of injective group homomorphisms $\phi : \mathbb{Z}^n \to N$ such that the image of the standard basis under the composite $\mathbb{Z}^n \to N \to N'$ lies in some cone of Σ. This is exactly $R_*(\{\pi^{-1}\sigma_i\})_n$. $\qquad\square$

Example 4.4.12. Let X be an affine toric variety defined by a rational polyhedral cone $\sigma \subset \mathbb{R}^n$ and let $p \in X$ be generic. Then $\mathbf{D}(X/T, p)_\bullet \simeq R_*(\sigma)$ as defined in (4.10), and $\mathrm{Aut}(p)$ is trivial so $\tilde{\mathbf{D}}(X/T, p)_\bullet \simeq \mathbf{D}(X/T, p)_\bullet$. For instance, $\mathbf{D}(\mathbb{A}^n/\mathbb{G}_m^n, (1, \ldots, 1))_\bullet = h_{[n]}$, the fan represented by the object $[n] \in \mathfrak{C}$.

Example 4.4.13. Let $\mathfrak{X} = */G$ where G is a reductive group, and let p be the unique k point. Then by Proposition 4.1.6, we have $\underline{\mathfrak{X}}(\Theta)_p \simeq \bigsqcup G/P_\lambda$, where λ ranges over all conjugacy classes of one parameter subgroups of G. Thus if k is an uncountable field, the set $\underline{\mathfrak{X}}(\Theta)_p$ is uncountable as well. However, the points of $\tilde{\mathbf{D}}(\mathfrak{X}, p)_1 = \underline{\mathfrak{X}}(\Theta)_p(k)/G(k)$ are exactly the conjugacy classes of nontrivial one-parameter subgroups.

Kempf's optimality argument revisited

Now that we have a combinatorial framework in which to study $\underline{\mathfrak{X}}(\Theta)_p$, we revisit Kempf's construction of the stratification of the nullcone of V/G where V is an affine k-scheme of finite type and G is reductive [29].

For this section we let $k = \mathbb{C}$ so that we may discuss the classical topological stack underlying $\mathfrak{X}$. The underlying topological stack of an algebraic stack $\mathfrak{X}$ locally of finite type

over $\mathbb{C}$ is defined by taking a presentation of $\mathfrak{X}$ by a groupoid in schemes and then taking the analytification, which is groupoid in topological spaces. The cohomology is then defined as the cohomology of the classifying space of this topological stack [34]. For global quotient stacks $\mathfrak{X} = X/G$ this agrees with the equivariant cohomology $H^*_{G^{an}}(X^{an}) = H^*_K(X^{an})$ where $K \subset G$ is a maximal compact subgroup.

Lemma 4.4.14. *Recall that $H^{2l}(\Theta; \mathbb{Q}) = \mathbb{Q} \cdot q^l$. Given a cohomology class $\eta \in H^{2l}(\mathfrak{X})$ we define $\hat{\eta}(f) = \frac{1}{q^l} f^* \eta \in \mathbb{Q}$ for any $f : \Theta \to \mathfrak{X}$. Then $\hat{\eta}$ extends uniquely to a continuous function $\hat{\eta} : |\mathbf{D}(\mathfrak{X}, p)_\bullet| \to \mathbb{R}$ which is homogeneous of degree l with respect to scaling, i.e. $\hat{\eta}(e^t x) = e^{lt} \hat{\eta}(x)$. The function $\hat{\eta}$ is $\mathrm{Aut}(p)$ invariant, and thus descends to a continuous function $\hat{\eta} : |\tilde{\mathbf{D}}(\mathfrak{X}, p)_\bullet| \to \mathbb{R}$ as well.*

Proof. The geometric realization is a colimit, so a continuous function $|F| \to \mathbb{R}$ is defined by a family of continuous functions $(\mathbb{R}^n)_+ \to \mathbb{R}$ for each $\xi \in F_n$ which is compatible with the continuous maps $(\mathbb{R}^k)_+ \to (\mathbb{R}^n)_+$ for each morphism in $(\mathfrak{C}|F)$.

In order to define such a family of functions for $F = \mathbf{D}(\mathfrak{X}, p)_\bullet$ it suffices to show that a cohomology class $\eta \in H^{2l}(\Theta^n; \mathbb{Q})$ defines a unique continuous function on $(\mathbb{R}^n)_+$ which is homogeneous of degree l and takes the value $\hat{\eta}(f)$ on the map $f : \Theta \to \Theta^n$ determined by each integer lattice point in the standard cone of $\mathbb{R}^n$. Furthermore this function should be natural in the sense that if $\phi : \mathbb{Z}^k \to \mathbb{Z}^n$ is injective, $\phi_{\mathbb{R}} : (\mathbb{R}^k)_+ \to (\mathbb{R}^n)_+$ is the corresponding map on cones, and we use $\phi : \Theta^k \to \Theta^n$ to denote the corresponding morphism as well, then $\hat{\eta} \circ \phi_{\mathbb{R}} = \widehat{\phi^* \eta}$.

Such an identification between cohomology classes and homogeneous (polynomial) functions on affine space is accomplished by the Cartan model for the equivariant cohomology of Θ^n. One computes

$$H^{2l}(\Theta^n; \mathbb{R}) \simeq H^{2l}(*/\mathbb{G}_m^n; \mathbb{R}) \simeq \mathrm{Sym}^l(\mathbb{R}^n)^\vee$$

Where $\mathbb{R}^n$ in the final expression is interpreted as the lie algebra of the compact group $(S^1)^n \subset \mathbb{G}_m^n$. Furthermore a homomorphism $\phi : \mathbb{Z}^k \to \mathbb{Z}^n$ induces a morphism $\Theta^k \to \Theta^n$, and the pullback map in cohomology $H^{2l}(\Theta^n; \mathbb{R}) \to H^{2l}(\Theta^k; \mathbb{R})$ agrees with the restriction of degree l polynomials along the linear map $\phi : \mathbb{R}^k \to \mathbb{R}^n$ □

Let $p \in \mathfrak{X}(k)$, and consider two elements of $f, g \in \underline{\mathfrak{X}}(\Theta)_p$. Let $U = \mathbb{A}^1 - \{0\}$. We consider f and g as morphisms $U \times \mathbb{A}^1/\mathbb{G}_m^2 \to \mathfrak{X}$ and $\mathbb{A}^1 \times U/\mathbb{G}_m^2 \to \mathfrak{X}$ respectively with a fixed isomorphism of their restrictions to $U \times U/\mathbb{G}_m^2 \simeq *$, so we can glue them to define

$$f \cup g : \mathbb{A}^2 - \{0\}/\mathbb{G}_m^2 \to \mathfrak{X}$$

This is a morphism from the toric variety defined by the two rays $\mathbb{R} \cdot e_1$ and $\mathbb{R} \cdot e_2$ in $\mathbb{R}^2$. The morphism $f \cup g$ extends over the point $\{0\} \in \mathbb{A}^2$ if and only if the two rays determined by f and g lie on a common cone in $\mathbf{D}(\mathfrak{X}, p)_\bullet$.

Definition 4.4.15. Let $F : \mathfrak{C}^{op} \to$ Set be a fan. An open subset $W \subset \mathbb{P}(F_\bullet)$ is *convex* if any two $f, g \in F_1$ which correspond to points in W lie in a common cone. This is equivalent to the canonical map

$$\underline{\mathfrak{X}}(\Theta^2) \to \underline{\mathfrak{X}}(\Theta) \times \underline{\mathfrak{X}}(\Theta) = \underline{\mathrm{Hom}}(\mathbb{A}^2 - \{0\}/\mathbb{G}_m^2, \mathfrak{X})$$

being surjective onto the subset set corresponding to points in $W \times W$.

Lemma 4.4.16. *Let G be a reductive group acting on an affine k-scheme V of finite type. Let $l \in H^2(V/G)$ be any cohomology class. Then the subset*

$$\{x \in \mathbb{P}(\mathbf{D}(\mathfrak{X}, p)_\bullet) | \hat{l}(x) > 0\}$$

is convex for every $p \in V$.

Proof. First note that V admits an equivariant embedding $V \subset \mathbb{A}^N$, where G acts linearly on $\mathbb{A}^N$, so by Part (3) of Proposition 4.2.5 it suffices to prove the claim for $\mathbb{A}^N$ itself. By Proposition 4.2.2, f and g are given by one parameter subgroups λ_f and λ_g such that $\lim_{t \to 0} \lambda_f(t) \cdot p$ exists, and likewise for λ_g. Note that $P_{\lambda_f} \cap P_{\lambda_g}$ must contain a maximal torus T for G [11]. Therefore we can find $p_f \in P_{\lambda_f}$ such that $p_f \lambda_f(t) p_f^{-1} \in T$, and this new one parameter subgroup defines the same point of $\underline{\mathfrak{X}}(\Theta)_p$. We can likewise choose a representative of g given by a one parameter subgroup of T.

Thus we can assume that λ_f and λ_g commute i.e. that that the point $f \cup g$ is defined by a map $\mathbb{A}^2 - \{0\} \to V$ taking $(1, 1) \mapsto p$ and a group homomorphism $\mathbb{G}_m^2 \to G$ intertwining this map. This homomorphism has finite kernel as long as $\lambda_f \neq \lambda_g^n$ for any $n \in \mathbb{Z}$. Note that if $\lambda_f = \lambda_g^n$, then $\hat{l}(f) = n\hat{l}(g)$. Because $\hat{l}(f), \hat{l}(g) > 0$, it follows that $n \geq 1$ and $[f] = [g] \in \mathbb{P}(\mathbf{D}(\mathfrak{X}, p)_\bullet)$.

The fact that the point $x \in \mathbb{A}^N$ defines a map $\mathbb{A}^2 - \{0\}/\mathbb{G}_m^2 \to \mathbb{A}^N/\mathbb{G}_m^2$ is equivalent the the fact that the point p, as a vector in $\mathbb{A}^N$, lies in the span of $\mathbb{G}_m^2$ eigenspaces which are positive with respect to both copies of $\mathbb{G}_m$. This in turn implies that the map extends to all of $\mathbb{A}^2$, and the fact that $V \subset \mathbb{A}^N$ is closed implies that the map factors through V as well. $\square$

Remark 4.4.17. Note that for $f, g \in \underline{\mathfrak{X}}(\Theta)_p$, the morphism $\mathbb{A}^2/\mathbb{G}_m^2 \to V/G$ extending the morphism $f \cup g$ is actually defined by a map $\mathbb{A}^2 \to V$ and a group homomorphism $\mathbb{G}_m^2 \to G$ intertwining this map.

In this framework, Kempf's argument [29] for the uniqueness of a maximal destabilizing 1PS is quite simple. We choose an $l \in H^2(\mathfrak{X}; \mathbb{Q})$ and $b \in H^4(\mathfrak{X}; \mathbb{Q})$ which is positive definite in the sense that $f^* b \in \mathbb{Q}_{>0} \cdot q^2 \subset H^4(*/\mathbb{G}_m)$ for any map $f : */\mathbb{G}_m \to \mathfrak{X}$ with finite kernel. This implies that $\hat{b} > 0$ everywhere except for the cone point of $|F|$. We define the numerical invariant $\mu = \hat{l}/\sqrt{\hat{b}}$, which is well defined away from the cone point. Both the numerator

and denominator are homogeneous of weight 1 with respect to scalar multiplication, so the function descends to the projective realization

$$\mu = \frac{\hat{l}}{\sqrt{\hat{b}}} : \mathbb{P}(\mathbf{D}(\mathfrak{X}, p)_\bullet) \to \mathbb{R} \tag{4.12}$$

Proposition 4.4.18. *Let $\mathfrak{X} = V/G$ be an affine quotient stack of finite type over $\mathbb{C}$, and let $l \in H^2(\mathfrak{X}; \mathbb{Q})$ and $b \in H^4(\mathfrak{X}; \mathbb{Q})$ be positive definite. Then for each $p \in V$ either $\mu(f) \leq 0$ for all rational points $x \in \mathbb{P}(\mathbf{D}(\mathfrak{X}, p)_\bullet)$, or else there is a unique rational point $x \in \mathbb{P}(\mathbf{D}(\mathfrak{X}, p)_\bullet)$ which maximizes μ. Such a point corresponds to a morphism $f : \Theta \to \mathfrak{X}$ with isomorphism $f(1) \simeq p$ which is uniquely determined up to the identification $f \sim f^n$ for $n \geq 0$.*

Proof. The existence of a maximizer follows from analyzing the fan $\tilde{\mathbf{D}}(\mathfrak{X}, p)_\bullet$ rather than $\mathbf{D}(\mathfrak{X}, p)_\bullet$. We will show in Lemma 4.4.19 that there is a finite collection of cones $\sigma_i \in \tilde{\mathbf{D}}(\mathfrak{X}, p)_{n_i}$ such that the corresponding map

$$\bigsqcup h_{[n_i]}([1]) \xrightarrow{\sqcup \sigma_i} \tilde{\mathbf{D}}(\mathfrak{X}, p)_1$$

is surjective. The function μ restricted to $\mathbb{P}(\bigsqcup h_{n_i}) = \bigsqcup \mathbb{P}(h_{n_i}) \simeq \bigsqcup \Delta_{n_i-1}$ must attain a maximum because it is continuous. Thus μ attains a maximum on $\mathbb{P}(\tilde{\mathbf{D}}(\mathfrak{X}, p)_\bullet)$, and because $\mathbb{P}(\mathbf{D}(\mathfrak{X}, p)_\bullet) \to \mathbb{P}(\tilde{\mathbf{D}}(\mathfrak{X}, p)_\bullet)$ is surjective, the function μ attains a maximum on $\mathbb{P}(\mathbf{D}(\mathfrak{X}, p)_\bullet)$ as well.

Now let $f, g :\in \mathbf{D}(\mathfrak{X}, p)_1$ with $\mu(f), \mu(g) > 0$. Lemma 4.4.16 states that we can find a morphism $e : \mathbb{A}^2/\mathbb{G}_m^2 \to \mathfrak{X}$ such that f and g are the restriction of e to two different rays in $\mathbb{R}^2$. e corresponds to a morphism of fans $e : h_{[2]} \to \mathbf{D}(\mathfrak{X}, p)_\bullet$ and hence a morphism

$$e : \Delta_1 = \mathbb{P}(h_{[2]}) \to \mathbb{P}(\mathbf{D}(\mathfrak{X}, p)_\bullet)$$

The restriction $\mu \circ e$ to Δ_1 is equal to $\widehat{e^*l}/\sqrt{\widehat{e^*b}}$, where $e^*l, e^*b \in H^*(\mathbb{A}^2/\mathbb{G}_m^2; \mathbb{Q})$. Thus $\mu \circ \eta$ is the function induced on Δ_1 by the quotient of a positive, rational, linear function by the square root of a positive definite rational quadratic form on $(\mathbb{R}^2)_+ - \{0\}$. It is an elementary exercise in convex geometry that such a function attains a maximum at a unique rational point. Thus the maximizer for μ on $\mathbb{P}(\mathbf{D}(\mathfrak{X}, p)_\bullet)$ is unique and rational. $\qquad\square$

In order to complete the proof, we prove the following

Lemma 4.4.19. *Let X be a variety with an action of a reductive group G. Then there is a finite collection of cones $\sigma_i \in \mathbf{D}(\mathfrak{X}, p)_{n_i}$ such that the corresponding morphism $\bigsqcup h_{[n_i]} \xrightarrow{\sqcup \sigma_i} \tilde{D}(\mathfrak{X}, p)_\bullet$ is surjective on 1-cones.*

Proof. By an argument exactly parallel to Proposition 4.1.6, one can show that giving a map $\Theta^n \to \mathfrak{X} = X/G$ is equivalent to specifying a group homomorphism $\phi : \mathbb{G}_m^n \to G$ and a point $x \in X$ under which $\lim_{t \to 0}(t^{k_1}, \ldots, t^{k_n}) \cdot x$ exists for all $k_i \geq 0$. The pairs (ϕ, x)

and $(g\phi g^{-1}, gx)$ define 1-morphisms $\Theta^n \to \mathfrak{X}$ which are isomorphic, and they thus define the same element of $\tilde{\mathbf{D}}(X/G, p)_n$.

Let $T \subset G$ be a maximal torus. Every homomorphism $\phi : \mathbb{G}_m^n \to G$ is conjugate to one which factors through T. It follows that $\mathbf{D}(X/T, p)_n \to \tilde{\mathbf{D}}(X/G, p)_n$ is surjective for all n. Thus it suffices to prove the Lemma for $G = T$. Lemma 4.4.11 describes $\mathbf{D}(X/T, p)_\bullet$ explicitly as $R_*(\{\pi^{-1}\sigma_i\})$ where $\pi : N_\mathbb{R} \to N'_\mathbb{R}$ is a linear map and σ_i is a fan of strictly convex rational polyhedral cones in $N'_\mathbb{R}$. Our claim follows from the fact that each cone $\pi^{-1}(\sigma_i)$ can be covered by finitely many simplicial cones. $\qquad\square$

Remark 4.4.20. The existence of a finite collection of cones in $\mathbf{D}(\mathfrak{X}, p)_\bullet$ which generate all of $\mathbf{D}(\mathfrak{X}, p)_1$ is the weakest notion of finiteness that suffices to prove Proposition 4.4.18. It is evident from the proof of Lemma 4.4.19 that $\bigsqcup \mathbb{P}(h_{n_i}) \to \mathbb{P}(\tilde{\mathbf{D}}(\mathfrak{X}, p)_\bullet)$ is surjective as well, but for general fans this is not equivalent.

Note, however, that the strongest notion would be for the map $\bigsqcup_i h_{[n_i]} \to \tilde{\mathbf{D}}(\mathfrak{X}, p)_\bullet$ to be surjective as a natural transformation of functors, but this is not the case. Even the fan $R_*(\sigma)$, where $\sigma \subset \mathbb{R}^N$ is a strictly convex rational polyhedral classical cone, does not admit a surjection from a finite collection of cones unless σ was simplicial.

In future work, we hope to apply this intrinsic reformulation of Kempf's existence and uniqueness argument to prove the existence and uniqueness of Harder-Narasimhan filtrations for moduli problems where the question has not yet been investigated, such as the moduli of polarized varieties.

In these examples, and already in the case of quotients X/G where X is projective rather than affine, the convexity property of Lemma 4.4.16 fails to hold. Nevertheless the basic idea of Kempf's argument in Proposition 4.4.18 can be extended to this setting. For such stacks, classes $l \in H^2(\mathfrak{X}; \mathbb{Q})$ and $b \in H^4(\mathfrak{X}; \mathbb{Q})$ must satisfy an additional convexity property in order for the analogue of Proposition 4.4.18 to hold.

Bibliography

[1]   Pramod N. Achar and David Treumann. "Baric structures on triangulated categories and coherent sheaves". In: *Int. Math. Res. Not. IMRN* 16 (2011), pp. 3688–3743. ISSN: 1073-7928.

[2]   Irina Anno. *Weak representation of tangle categories in algebraic geometry*. Thesis (Ph.D.)–Harvard University. ProQuest LLC, Ann Arbor, MI, 2008, p. 79. ISBN: 978-0549-61433-3.

[3]   Dmitry Arinkin and Roman Bezrukavnikov. "Perverse coherent sheaves". English, with English and Russian summaries. In: *Mosc. Math. J.* 10.1 (2010), pp. 3–29, 271. ISSN: 1609-3321.

[4]   M. Artin and J. J. Zhang. "Noncommutative projective schemes". In: *Adv. Math.* 109.2 (1994), pp. 228–287. ISSN: 0001-8708. DOI: `10.1006/aima.1994.1087`.

[5]   Denis Auroux, Ludmil Katzarkov, and Dmitri Orlov. "Mirror symmetry for weighted projective planes and their noncommutative deformations". In: *Ann. of Math. (2)* 167.3 (2008), pp. 867–943. ISSN: 0003-486X. DOI: `10.4007/annals.2008.167.867`.

[6]   M. Ballard, D. Favero, and L. Katzarkov. "Variation of geometric invariant theory quotients and derived categories". In: *ArXiv e-prints* (Mar. 2012). arXiv: `1203.6643` `[math.AG]`.

[7]   David Ben-Zvi, John Francis, and David Nadler. "Integral transforms and Drinfeld centers in derived algebraic geometry". In: *J. Amer. Math. Soc.* 23.4 (2010), pp. 909–966. ISSN: 0894-0347. DOI: `10.1090/S0894-0347-10-00669-7`.

[8]   A. Bondal and D. Orlov. "Derived categories of coherent sheaves". In: *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*. Beijing: Higher Ed. Press, 2002, pp. 47–56.

[9]   A. Bondal and D. Orlov. "Semiorthogonal decomposition for algebraic varieties". In: *eprint arXiv:alg-geom/9506012* (1995).

[10]   A. I. Bondal. "Representations of associative algebras and coherent sheaves". In: *Izv. Akad. Nauk SSSR Ser. Mat.* 53.1 (1989), pp. 25–44. ISSN: 0373-2436.

[11]   Armand Borel and Jacques Tits. "Groupes réductifs". In: *Inst. Hautes Études Sci. Publ. Math.* 27 (1965), pp. 55–150. ISSN: 0073-8301.

[12] Patrick Brosnan. "On motivic decompositions arising from the method of Białynicki-Birula". In: *Invent. Math.* 161.1 (2005), pp. 91–111. ISSN: 0020-9910. DOI: `10.1007/s00222-004-0419-7`.

[13] Sabin Cautis, Joel Kamnitzer, and Anthony Licata. "Coherent sheaves and categorical *sl*2 actions". In: *Duke Math. J.* 154.1 (2010), pp. 135–179. ISSN: 0012-7094. DOI: `10.1215/00127094-2010-035`.

[14] David A. Cox and Sheldon Katz. *Mirror symmetry and algebraic geometry.* Vol. 68. Mathematical Surveys and Monographs. American Mathematical Society, 1999, pp. xxii+469. ISBN: 0-8218-1059-6.

[15] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties.* Vol. 124. Graduate Studies in Mathematics. Providence, RI: American Mathematical Society, 2011, pp. xxiv+841. ISBN: 978-0-8218-4819-7.

[16] Igor V. Dolgachev and Yi Hu. "Variation of geometric invariant theory quotients". In: *Inst. Hautes Études Sci. Publ. Math.* 87 (1998). With an appendix by Nicolas Ressayre, pp. 5–56. ISSN: 0073-8301.

[17] W. Donovan and E. Segal. "Window shifts, flop equivalences and Grassmannian twists". In: *ArXiv e-prints: http://arxiv.org/abs/1206.0219* (2012).

[18] Will Donovan. "Grassmannian twists on the derived category via spherical functors". In: *eprint arXiv:math.AG/1111.3774* (2011).

[19] William Fulton. *Introduction to toric varieties.* Vol. 131. Annals of Mathematics Studies. The William H. Roever Lectures in Geometry. Princeton, NJ: Princeton University Press, 1993, pp. xii+157. ISBN: 0-691-00049-2.

[20] I. M. Gel′fand, M. M. Kapranov, and A. V. Zelevinsky. *Discriminants, resultants, and multidimensional determinants.* Mathematics: Theory & Applications. Boston, MA: Birkhäuser Boston Inc., 1994, pp. x+523. ISBN: 0-8176-3660-9. DOI: `10.1007/978-0-8176-4771-1`.

[21] Megumi Harada and Gregory D. Landweber. "Surjectivity for Hamiltonian *G*-spaces in *K*-theory". In: *Trans. Amer. Math. Soc.* 359.12 (2007), 6001–6025 (electronic). ISSN: 0002-9947. DOI: `10.1090/S0002-9947-07-04164-5`.

[22] M. Herbst, K. Hori, and D. Page. "Phases Of N=2 Theories In 1+1 Dimensions With Boundary". In: *eprint arXiv:hep-th/0803.2045* (2008).

[23] Wim H. Hesselink. "Concentration under actions of algebraic groups". In: *Paul Dubreil and Marie-Paule Malliavin Algebra Seminar, 33rd Year (Paris, 1980).* Vol. 867. Lecture Notes in Math. Berlin: Springer, 1981, pp. 55–89.

[24] R. Paul Horja. "Derived category automorphisms from mirror symmetry". In: *Duke Math. J.* 127.1 (2005), pp. 1–34. ISSN: 0012-7094. DOI: `10.1215/S0012-7094-04-12711-3`.

[25] Hiroshi Iritani. "An integral structure in quantum cohomology and mirror symmetry for toric orbifolds". In: *Adv. Math.* 222.3 (2009), pp. 1016–1079. ISSN: 0001-8708. DOI: 10.1016/j.aim.2009.05.016.

[26] M. M. Kapranov. "Derived category of coherent sheaves on Grassmann manifolds". In: *Izv. Akad. Nauk SSSR Ser. Mat.* 48.1 (1984), pp. 192–202. ISSN: 0373-2436.

[27] Yujiro Kawamata. "Derived categories of toric varieties". In: *Michigan Math. J.* 54.3 (2006), pp. 517–535. ISSN: 0026-2285. DOI: 10.1307/mmj/1163789913. URL: http://dx.doi.org/10.1307/mmj/1163789913.

[28] Bernhard Keller. "On differential graded categories". In: *International Congress of Mathematicians. Vol. II.* Eur. Math. Soc., Zurich, 2006, pp. 151–190.

[29] George R. Kempf. "Instability in invariant theory". In: *Ann. of Math. (2)* 108.2 (1978), pp. 299–316. ISSN: 0003-486X. DOI: 10.2307/1971168. URL: http://dx.doi.org/10.2307/1971168.

[30] Frances Clare Kirwan. *Cohomology of quotients in symplectic and algebraic geometry.* Vol. 31. Mathematical Notes. Princeton University Press, 1984, pp. i+211. ISBN: 0-691-08370-3.

[31] Marco Manetti. "Differential graded Lie algebras and formal deformation theory". In: *Algebraic geometry—Seattle 2005. Part 2.* Vol. 80. Proc. Sympos. Pure Math. Providence, RI: Amer. Math. Soc., 2009, pp. 785–810.

[32] James S. Milne. *Étale cohomology.* Vol. 33. Princeton Mathematical Series. Princeton, N.J.: Princeton University Press, 1980, pp. xiii+323. ISBN: 0-691-08238-3.

[33] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory.* Third. Vol. 34. Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]. Berlin: Springer-Verlag, 1994, pp. xiv+292. ISBN: 3-540-56963-4. DOI: 10.1007/978-3-642-57916-5. URL: http://dx.doi.org/10.1007/978-3-642-57916-5.

[34] Behrang Noohi. "Homotopy types of topological stacks". In: *Adv. Math.* 230.4-6 (2012), pp. 2014–2047. ISSN: 0001-8708. DOI: 10.1016/j.aim.2012.04.001. URL: http://dx.doi.org/10.1016/j.aim.2012.04.001.

[35] D. O. Orlov. "Derived categories of coherent sheaves, and motives". In: *Uspekhi Mat. Nauk* 60.6(366) (2005), pp. 231–232. ISSN: 0042-1316. DOI: 10.1070/RM2005v060n06ABEH004292.

[36] Dmitri Orlov. "Derived categories of coherent sheaves and triangulated categories of singularities". In: *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II.* Vol. 270. Progr. Math. Boston, MA: Birkhauser Boston Inc., 2009, pp. 503–531. DOI: 10.1007/978-0-8176-4747-6_16.

[37] S. Ramanan and A. Ramanathan. "Some remarks on the instability flag". In: *Tohoku Math. J. (2)* 36.2 (1984), pp. 269–291. ISSN: 0040-8735. DOI: 10.2748/tmj/1178228852. URL: http://dx.doi.org/10.2748/tmj/1178228852.

[38] Raphaël Rouquier. "Dimensions of triangulated categories". In: *J. K-Theory* 1.2 (2008), pp. 193–256. ISSN: 1865-2433. DOI: `10.1017/is007011012jkt010`.

[39] Ed Segal. "Equivalence between GIT quotients of Landau-Ginzburg B-models". In: *Comm. Math. Phys.* 304.2 (2011), pp. 411–432. ISSN: 0010-3616. DOI: `10.1007/s00220-011-1232-y`.

[40] Paul Seidel and Richard Thomas. "Braid group actions on derived categories of coherent sheaves". In: *Duke Math. J.* 108.1 (2001), pp. 37–108. ISSN: 0012-7094. DOI: `10.1215/S0012-7094-01-10812-0`.

[41] Stephen S. Shatz. "The decomposition and specialization of algebraic families of vector bundles". In: *Compositio Math.* 35.2 (1977), pp. 163–187. ISSN: 0010-437X.

[42] Ian Shipman. "A geometric approach to Orlov's theorem". In: *Comp. Math.* 148.5 (2012), pp. 1365–1389. DOI: `10.1112/S0010437X12000255`.

[43] Constantin Teleman. "The quantization conjecture revisited". In: *Ann. of Math. (2)* 152.1 (2000), pp. 1–43. ISSN: 0003-486X. DOI: `10.2307/2661378`.

[44] Michael Thaddeus. "Geometric invariant theory and flips". In: *J. Amer. Math. Soc.* 9.3 (1996), pp. 691–723. ISSN: 0894-0347. DOI: `10.1090/S0894-0347-96-00204-4`.

[45] Angelo Vistoli. "Grothendieck topologies, fibered categories and descent theory". In: *Fundamental algebraic geometry*. Vol. 123. Math. Surveys Monogr. Providence, RI: Amer. Math. Soc., 2005, pp. 1–104.

[46] Charles Weibel. "The Hodge filtration and cyclic homology". In: *K-Theory* 12.2 (1997), pp. 145–164. ISSN: 0920-3036. DOI: `10.1023/A:1007734302948`.