Graduate Quantum Algebra & Topology Seminar: Multimatrix Algebras (Complete Notes)

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Oct. 12, 2022
Oct. 19, 2022

Following "Exercises in Quantum Algebra" by Penneys (https://people.math.osu.edu/penneys.2/QuantumAlg
Starting in section 3.1.2.

1 Towers of Algebras

1.1 Multimatrix Algebras

Once and for all, denote by A a finite-dimensional complex *-algebra. Define its linear dual by

$$A^\vee := \mathrm{Hom}_{\mathbb{C}\text{-vect}}(A, \mathbb{C})$$

(which is a finite-dimensional complex vector space at least). We define some subspaces of $A^\vee$ as follows:

Definition 1.1.1. An element $\varphi \in A^\vee$ is

- a *trace* if $\varphi(ab) = \varphi(ba)$ for all $b, a \in A$
- *positive* if $\varphi(a^*a) \geq 0$ for all $a \in A$
- a *state* if it is positive and normalized ($\varphi(1) = 1$)
- *faithful* if it is positive and $\varphi(a^*a)$ is nondegenerate on A (i.e. $\varphi(a^*a) = 0$ if and only if $a = 0$).

Let's explore some exercises

Exercise 1. Show that $M_n(\mathbb{C})$ has a unique normalized trace, and show that it is positive and faithful.

Proof. Of course, $M_n(\mathbb{C})$ has one trace tr, the sum over the eigenvalues (divided by n to normalize it). We show that every other trace is a multiple of this one.

To begin with, observe that traces are all cyclically invariant in the sense that

$$\varphi(abc) = \varphi(bca) = \varphi(cab)$$

and so on for higher products. Hence, the value of the trace only depends on the conjugacy class of the matrix in question. From here on out, let J be a matrix in Jordan canonical form, so that $J = D + (J - D)$ for D diagonal and $(J - D)$ a matrix with at most ones on the above-diagonal.

Decompose J further as

$$J = \lambda_1 D_1 + \dots + \lambda_n D_n + N_1 + \dots + N_k$$

where each matrix D_i, N_i has a single one in its entries. We now show that any trace φ takes the same value (up to normalization) as tr on each of these matrices.

For the diagonal matrices D_i, after conjugation by a permutation matrix we are free to place the one anywhere on the diagonal. Summing all such possible placements yields the identity, and so

$$\varphi(I) = n\varphi(D_i)$$

For the nilpotent matrices N_i, we can again use permutation matrices to place the one in the top row second column. Let N be that matrix, and let $\tilde{N}$ be the matrix with a single one in the first row first column. Consider

$$\tilde{N}N = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

whereas

$$N\tilde{N} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

Since $\varphi(0) = 0$, by the cyclic property $\varphi(N) = \varphi(N_i) = 0$ as well.

Hence,

$$\begin{aligned} \varphi(J) &= \varphi(\lambda_1 D_1 + \dots + \lambda_n D_n + N_1 + \dots + N_k) \\ &= \lambda_1(\frac{1}{n}\varphi(1)) + \dots + \lambda_n(\frac{1}{n}\varphi(1)) + 0 + \dots + 0 \\ &= \frac{1}{n}\varphi(1)(\lambda_1 + \dots + \lambda_n) \\ &= \frac{1}{n}(\sum_{i=1}^{n} \lambda_i) = \frac{1}{n}\operatorname{tr}(J) \end{aligned}$$

recovering the usual trace.

Clearly, the trace is positive and faithful. □

Exercise 2. Let $A = \mathbb{C} \oplus \mathbb{C}$ with involution $(\alpha, \beta)^* = (\bar{\beta}, \bar{\alpha})$. Show A has no states.

Proof. Any linear functional on A is the sum of component functionals, each specified by where they send 1. That is

$$\varphi(\alpha, \beta) = \varphi_1(\alpha) + \varphi_2(\beta) = \alpha\varphi_1(1) + \beta\varphi_2(1)$$

and the state condition enforces

$$\varphi_1(1) + \varphi_2(1) = 1$$

hence we find that if $\varphi_1(1) = t$, then $\varphi_2(1) = 1 - t$.

Observe that

$$(\alpha, \beta)^*(\alpha, \beta) = (\bar{\beta}\alpha, \bar{\alpha}\beta)$$

and hence

$$\varphi(a^*a) = \alpha\bar{\beta}t + \bar{\alpha}\beta(1-t) = (\alpha\bar{\beta} - \bar{\alpha}\beta)t + \bar{\alpha}\beta$$

which, for various values of t, is a line connecting $\alpha\bar{\beta}$ to its conjugate. Clearly this is real for all α, β if and only if $t = \frac{1}{2}$.

Finally, we can choose α, β arbitrarily so that $\alpha\bar{\beta}$ is any complex number, hence the range of $\varphi(a^*a)$ is all of $\mathbb{R}$, contradicting positivity. □

Exercise 3. Show that every state φ on $A = M_n(\mathbb{C})$ can be given in the form

$$\varphi(a) = \mathrm{tr}(da)$$

for some positive matrix $d \in A$ of trace 1. Show φ is faithful if and only if d is invertible.

Proof. Consider the linear map

$$\rho : A \to A^\vee$$

given by sending a matrix d to the linear functional $a \mapsto \mathrm{tr}(da)$. Since tr is faithful, this map is an isomorphism of $\mathbb{C}$-vector spaces.

Suppose $\varphi \in A^\vee$ is positive, so that $\varphi(a^*a) \in \mathbb{R}_{\geq 0}$ for all $a \in A$. Using the map above, we can write φ as

$$\varphi(a) = \mathrm{tr}(da)$$

and the positivity condition becomes

$$\mathrm{tr}(d(a^*a)) = \mathrm{tr}(a^*ad) \geq 0$$

As before, we can conjugate to get d in Jordan normal form. Choosing, for example, X_i to be the matrix with a single 1 on the ith spot on the diagonal (so that $X_i^*X_i = X_i$) we find

$$\mathrm{tr}(X_i^*X_id) = \mathrm{tr}(X_id) = \lambda_i$$

where λ_i is the ith eigenvalue of d in the ordering we chose for its Jordan form. Positivity then guarantees all these eigenvalues are positive, and hence by definition d is positive.

Remark. *Notice the converse is true as well, if d is positive then it has a square root $d = bb^*$ and hence*

$$\varphi(a^*a) = \mathrm{tr}(bb^*a^*a) = \mathrm{tr}\left((ab)^*(ab)\right) \geq 0$$

For the other half, clearly if φ is a state, then

$$\varphi(1) = \mathrm{tr}(1d) = \mathrm{tr}(d) = 1$$

forcing d to have trace 1. This, of course, works in reverse as well.

Finally, suppose φ is faithful. Then, the map $\varphi(a^*a) = \mathrm{tr}(da^*a)$ is nondegenerate. Using the X_i matrices again, we find that all eigenvalues of d are positive and nonzero as desired.

Conversely, if d is positive with square root $d = bb^*$ and invertible, then so is b. Hence,

$$\varphi(a^*a) = \mathrm{tr}\left((ab)^*(ab)\right)$$

Since b is invertible, every matrix a can be written in the form $a = cb^{-1}$, and hence we find

$$\varphi\left((cb^{-1})^*(cb^{-1})\right) = \mathrm{tr}(c^*c) \neq 0$$

since tr is faithful. □

Remark. *Where does this terminology come from? If an ensemble of particles in a quantum system with e.g. n dimensions worth of states is given so that the relative proportion of particles in state $|\psi_i\rangle \in \mathbb{C}^n$ is p_i, then*

$$\sum_i p_i = 1$$

and for any observable $a \in M_n(\mathbb{C})$, we have

$$\begin{aligned}
\langle a \rangle &= \sum_i p_i \langle \psi_i | a | \psi_i \rangle \\
&= \sum_{i,j,k} p_i \langle \psi_i | \phi_j \rangle \langle \phi_j | a | \phi_k \rangle \langle \phi_k | \psi_i \rangle \\
&= \sum_{i,j} p_i \langle \phi_k | \psi_i \rangle \langle \psi_i | \phi_j \rangle \langle \phi_j | a | \phi_k \rangle
\end{aligned}$$

The element

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i$$

is called the "density operator" of the theory and satisfies

$$\begin{aligned}\langle a\rangle &= \sum_{i,j} p_i \langle\phi_k|\psi_i\rangle\langle\psi_i|\phi_j\rangle\langle\phi_j|a|\phi_k\rangle \\ &= \sum_{j,k} \langle\phi_k|\rho|\phi_j\rangle\langle\phi_j|a|\phi_k\rangle \\ &= \sum_k \langle\phi_k|\rho a|\phi_k\rangle = \mathrm{tr}(\rho a)\end{aligned}$$

Since the densities p_i are all real, this operator is self-adjoint, and a quick computation shows

$$\mathrm{tr}(\rho) = \sum_i p_i = 1$$

so the density matrix is normalized. In fact, each p_i is positive, so this operator is positive-definite.

To generalize a bit we make the following definition:

Definition 1.1.2. A $*$-algebra A is called a *multimatrix algebra* if it is isomorphic to a finite direct sum of matrix algebras. That is,

$$A \cong M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$$

The integers n_i assemble into a vector $\vec{n}$ called the *dimension vector* of the algebra. Multimatrix algebras (by definition) come with *minimal central projections* p_i, projecting onto the ith matrix factor of A.

Exercise 4. Suppose A is a multimatrix algebra and tr is a trace. Show that tr is postiive if and only if $\mathrm{tr}(p) \geq 0$ for all projections p. Show that tr is faithful if and only if the inequality is strict for all p.

Proof. Suppose first that tr is positive and p is a projection. Then, clearly,

$$\mathrm{tr}(p^*p) = \mathrm{tr}(p^2) = \mathrm{tr}(p) \geq 0$$

and if tr is faithful and p is nontrivial, this inequality is strict.

Conversely, suppose $\mathrm{tr}(p) \geq 0$ for all projections. In particular, this means tr is positive on the minimal central projections p_i to the matrix factors of A.

Observe that for any element $b = p_i a p_i$ in the range of p_i,

$$\mathrm{tr}_i(b) = \mathrm{tr}(p_i a p_i) = \mathrm{tr}(a p_i) = \mathrm{tr}(p_i a)$$

defines a linear functional on the ith factor of A via projection. This functional is a trace on this factor, since for all $a, b \in A$ commuting with p_i (i.e. in the range of p_i)

$$\mathrm{tr}(p_i a b p_i) = \mathrm{tr}(a b p_i^2) = \mathrm{tr}(a p_i^2 b) = \mathrm{tr}(b a p_i^2) = \mathrm{tr}(p_i b a p_i)$$

by repeated abuse of the cyclic property.

Now, if $\mathrm{tr}(p_i) > 0$ on p_i, then $\mathrm{tr}(p_i a p_i)$ defines a trace on the ith matrix factor which satisfies

$$\mathrm{tr}(p_i(1)p_i) = \mathrm{tr}(p_i) = \mathrm{tr}(I_i) > 0$$

for I_i the identity matrix in the ith factor. Hence, on this space tr is uniquely determined by the first exercise and is guaranteed to be positive and faithful on this factor.

If $\mathrm{tr}(p_i)$ is zero for some factor, then on this factor

$$\mathrm{tr}(p_i(1)p_i) = \mathrm{tr}(p_i) = 0$$

and hence this trace is zero on the identity matrix, hence it is the zero map.

In all cases, writing the identity as

$$1 = \sum_i p_i$$

we find that

$$\begin{aligned}
\mathrm{tr}(a^*a) &= \mathrm{tr}\left((1)a^*(1)a(1)\right) \\
&= \mathrm{tr}\left((\sum_i p_i)a^*(\sum_i p_i)a(\sum_i p_i)\right) \\
&= \mathrm{tr}\left(\sum_i (p_i a p_i)^*(p_i a p_i)\right) \\
&= \sum_i \mathrm{tr}_i\left((p_i a p_i)^*(p_i a p_i)\right)
\end{aligned}$$

and all such factors are guaranteed to be positive by the above. Notice if every tr_i is nontrivial, then tr really is faithful (each a will have nonzero projection onto *some* factor, and the trace on that factor will be nonzero) □

Exercise 5. Again, let A be a multimatrix algebra, and let p be an orthogonal projection. Define the central support of p, denoted $z(p)$, to be the smallest central projection such that $p \leq z(p)$. That is, $z(p) - p$ is positive. Show that $z(p)$ can be computed as the sum of the minimal central projections p_i satisfying $pp_i \neq 0$.

Proof. Denote by $z'(p)$ the projection

$$z'(p) = \sum_i p_i$$

where the sum runs over all i for which $pp_i \neq 0$. Clearly $z'(p)$, as a sum of central projections, is itself a central projection. What remains to show is that $z'(p)$ is the smallest central projection bigger than p.

Recall that for orthogonal projections p, q we have $p \leq q$ if and only if $p = pq = qp$. Now, $p \leq 1$, so

$$p = p(1) = p(\sum_i p_i) = \sum_i pp_i = \sum_{pp_i \neq 0} pp_i = p(\sum_{pp_i \neq 0} p_i)$$

hence $z'(p) \geq p$.

Now, every central projection is a sum over some subset of the p_i projections, so clearly $z'(p)$ is the smallest such projection. □

Exercise 6. Find a bijective correspondence between faithful tracial states on A and vectors $\vec{\lambda}$ with entries in $(0,1)$ such that $\vec{\lambda} \cdot \vec{n} = 1$.

Proof. As noted in a previous exercise, faithful tracial states on A correspond to states satisfying $\mathrm{tr}(p_i) > 0$ for all minimal central projections. As noted, $\mathrm{tr}(p_i)$ is then the value of $\mathrm{tr}(I_i)$ on the ith factor.

The usual trace (we denote by tr_i) is given by

$$\mathrm{tr}_i(I_i) = n_i$$

and so

$$\mathrm{tr}(p_i) = \frac{\mathrm{tr}(p_i)}{n_i}\,\mathrm{tr}_i(p_i) = \lambda_i\,\mathrm{tr}_i(p_i)$$

By tr being a state, $\sum_i \mathrm{tr}(p_i) = \mathrm{tr}(1) = 1$ and hence since each $\mathrm{tr}(p_i)$ is positive we have $\mathrm{tr}(p_i) \in (0,1)$ for each i. Hence, $\lambda_i \in (0,1)$ for all i. Also,

$$\vec{\lambda}\cdot\vec{n} = \sum_i \lambda_i n_i = \sum_i \mathrm{tr}(p_i) = 1$$

and hence we have associated to tr a unique vector $\vec{\lambda}$ as desired. Clearly, different values of $\vec{\lambda}$ yield different tracial states.

The converse is also trivially true, we are free to fix λ_i to be any value in $(0,1)$ subject to the sum constraint. □

1.2 Operator Algebras

Let H be a finite-dimensional Hilbert space, with $B(H)$ its $*$-algebra of bounded operators under the adjoint involution.

Exercise 7. Show that H is unitarily isomorphic to $\mathbb{C}^n$ when an ONB is chosen, so that $B(H)$ is unitarily equivalent to $M_n(\mathbb{C})$.

Proof. Let $\{e_i\}_{i=1}^n$ be an ONB for H, and let

$$u : H \to \mathbb{C}^n$$

be the obvious map sending e_i to the ith standard basis vector in $\mathbb{C}^n$. By standard computation, u^* is the inverse.

Notice now that conjugation by u

$$x \mapsto uxu^*$$

sends operators in $B(H)$ to operators in $M_n(\mathbb{C})$. This map is clearly linear, and since $u^*u = 1$ we see that

$$(xy) \mapsto ux(1)yu^* = uxu^*uyu^*$$

so that this map preserves multiplication. Furthermore,

$$x^* \mapsto ux^*u^* = (uxu^*)^*$$

so this map is a $*$-homomorphism. Finally, conjugation by u^* is an explicit inverse, so this map is an isomorphism. □

Exercise 8. Show that a finite-dimensional $*$-algebra is a C^*-algebra if and only if it has a faithful tracial state. Hence, multimatrix algebras are C^*-algebras.

Proof. Suppose first that A is a finite-dimensional $*$-algebra with faithful tracial state $\mathrm{tr} \in A^\vee$. There is an obvious inner product we can put on A given by

$$\langle a, b\rangle = \varphi(b^*a)$$

which is indeed linear in a, conjugate-linear in b, and positive-definite since φ is faithful and positive. Hence, this inner product defines a norm on A, making A itself a Hilbert space.

Consider the map

$$L : A \to B(A)$$

given by sending an element a to the left-multiplication map μ_a given by $\mu_a(x) = ax$. This map is clearly injective since $\mu_a(1) = a \neq 0$. By construction, L is also multiplicative, and observe that

$$\langle \mu_a(x), y\rangle = \varphi\left(y^* ax\right) = \varphi\left((a^* y)^* x\right) = \langle x, \mu_{a^*}(y)\rangle$$

showing that $\mu_{a^*} = \mu_a^*$ and L is a $*$-algebra homomorphism.

Hence, A is a $*$-subalgebra of $B(H)$. Since $B(H)$ is a C^*-algebra (under e.g. the operator norm), A is as well with the induced subspace norm.

I'm not quite sure how to do the converse. Clearly if a $*$-algebra is a $*$-closed subalgebra of $B(H)$ one can take the trace on $B(H)$ to get a faithful tracial state but in the more abstract setting (i.e. only knowing that the algebra has a norm which satisfies the C^* identity) it is unclear how to proceed... $\square$

For H a finite-dimensional Hilbert space, and $S \subseteq B(H)$ some subset, define the commutant of S as the set

$$S' = \{x \in B(H) \mid xs = sx \forall s \in S\}$$

Exercise 9. Show that taking commutants is order-reversing.

Proof. Obvious from definitions. $\square$

Exercise 10. Show that $S' = S'''$.

Proof. First, we compute that if $x \in S$, then $x(s') = (s')x$ for all $s' \in S'$, hence $x \in S''$. Thus,

$$S \subseteq S''$$

By the previous result, we then have

$$S''' \subseteq S'$$

Conversely, if $x \in S'$, then for all $(s'') \in S''$ we have

$$x(s'') = (s'')x$$

by definition of S''. But this is exactly the condition for being in S''', so

$$S' \subseteq S'''$$

$\square$

Exercise 11. Show that if A is a unital $*$-subalgebra of $B(H)$, then $A = A''$.

Proof. Omitted, its in the reference and is basically a nuts-and-bolts algebra argument. $\square$

Exercise 12. Show that a finite-dimensional von Neumann algebra is a multimatrix algebra. Furthermore, show that a finite-dimensional C^*-algebra is a multimatrix algebra.

Proof. Recall that a von Neumann algebra A is a unital $*$-subalgebra of $B(H)$ which is equal to its double commutant (trivially true in finite dimensions). Recall also that a finite-dimensional C^*-algebra is a unital $*$-subalgebra of $B(H)$ closed under the norm topology (also trivially true in finite dimensions).

The key player here is the center $Z(A)$ of the von Neumann algebra.

Lemma 1.1. *The center $Z(A)$ is generated by a finite number of minimal projections.*

Proof. The center $Z(A)$ is an Abelian $*$-algebra, hence by the spectral theorem we find that $Z(A)$ is $*$-isomorphic to the algebra of continuous functions on a finite space $\mathrm{Spec}(Z(A))$, and hence $Z(A)$ is generated by the functions χ_i with support on a single point. Enumerating $\mathrm{Spec}(Z(A)) = \{\lambda_1, \ldots, \lambda_k\}$ we get minimal projection operators $p_1, \ldots, p_k \in Z(A)$ corresponding to the characteristic functions χ_i.

Since every continuous function on $\mathrm{Spec}(Z(A))$ is of the form

$$f = \sum_i f(\lambda_i)\chi_i$$

the functional calculus guarantees that every element of $Z(A)$ is then of the form

$$z = \sum_i z_i p_i$$

In particular, we get a resolution of the identity

$$1 = \sum_i p_i$$

□

Observe that the spectral resolution gives us

$$1A = \sum_i p_i A = \sum_i p_i A p_i = \sum_i A p_i$$

and since all the projections are mutually orthogonal (by minimality), this sum is direct.

Lemma 1.2. *If p_i is a central projection $p_i \in Z(A), p_i = p_i^* = p_i^2$, then H splits as a representation of A into $p_i H$ and $(1 - p_i)H$. Hence, $H = \bigoplus_i p_i H$ as a representation of A.*

Proof. Since the central projections are orthogonal, it is clear that H splits into $p_i H$ and $(1-p_i)H$ as an orthogonal direct sum. Since p_i (and $1 - p_i$) are central, these splittings respect the action of A, namely for any $a \in A$ and $v \in p_i H$:

$$av = ap_i v = p_i av$$

showing $av \in p_i H$. □

The structure becomes more clear: each $p_i A p_i$ is a representation on H_i and the direct sum representation is the representation of A on H. Furthermore, since

$$Z(p_i A) = p_i Z(A) = \mathbb{C} p_i$$

is trivial, by considering $p_i A p_i \subseteq B(H_i)$ we reduce to the case of a single factor.

Lemma 1.3. *Suppose A is a unital von Neumann algebra in $B(H)$ with trivial center. Then, A is unitarily equivalent to a matrix algebra, and the representation of A on H is a diagonal direct sum of copies of the defining representation of A.*

Proof. Consider a maximal family of minimal projections $\{p_i\}_{i=1}^n$ in A, none of which are in the center by hypothesis, which satisfy $p_i p_j = \delta_{ij}$. We rely on the existence of *partial isometries* between two projections.

Recall that a partial isometry is a factorization of a projection

$$u^* u = p$$

and its interpretation is that of an isometry of $\ker(u)^\perp$ onto its range, extended by zero. In this interpretation, p is projection onto $\ker(u)^\perp$. So, we get the relation

$$uu^*u = u$$

Furthermore, the story can be repeated with the idempotent uu^*, which yields another projection with reversed initial and final subspaces.

Fixing a priveleged minimal projection p_1, define partial isometries v_i satisfying

$$v_i^* v_i \leq p_1$$
$$v_i v_i^* \leq p_i$$

(possible since A has trivial center, by a result from e.g. Vaughan Jones' "Von Neumann Algebras", chapter 4) and since p_i are all minimal, we actually get equality.

Define

$$e_{ij} = v_i v_j^*$$

so that $e_{ii} = p_i$, and $p_i e_{ij} p_j = e_{ij}$. Hence,

$$e_{ij}^* = e_{ji}$$
$$e_{ij} e_{kl} = \delta_{jk} e_{il}$$
$$\sum_i e_{ii} = \sum_i p_i = 1$$

These elements clearly generate the matrix algebra $M_k(\mathbb{C})$.

Since our collection of minimal projections were all orthogonal, they resolve the identity (by maximality e.g. consider $1 - \sum_i p_i$)

$$1 = \sum_i p_i$$

and hence

$$A = (1)A(1) = \sum_{i,j} p_i A p_j$$

so every element $a \in A$ is of the form

$$a = \sum_{i,j} p_i a p_j = \sum_{i,j} v_i v_i^* a v_j v_j^*$$

but notice that

$$v_i^* a v_j = v_i^* v_i v_i^* a v_j v_j^* v_j = p_1 v_i^* a v_j p_1$$

hence $v_i^* a v_j \in p_1 A p_1 = \mathbb{C} p_1$ so $v_i^* a v_j = \lambda_{ij} p_1$ for some scalar. Hence

$$a = \sum_{i,j} v_i \lambda_{ij} p_1 v_j^* = \sum_{i,j} \lambda_{ij} e_{ij}$$

So, the association $A \to M_n(\mathbb{C})$ given by $a \mapsto \lambda_{ij}$ is a unital $*$-isomorphism, as desired.

Finally, recall we had n minimal projections in our space. Let $L^2([n], p_1 H) = \mathbb{C}^n$ be the Hilbert space of functions from the n-point set to $p_1 H \cong \mathbb{C}^j$. Define the map

$$u : L^2([n], p_1 H) \to H$$
$$uf = \sum_i v_i^* f(i)$$

which is unitary (!), and hence gives us an explicit description of how A acts on H. Specifically,

$$A = B(L^2([n])) \otimes 1 \subset B(L^2([n], p_1(H)) \cong B(L^2([n]) \otimes p_1 H)$$

□

□

Exercise 13. Show that a finite-dimensional unital complex $*$-algebra A is a C^*-algebra if and only if

$$a^*a = 0 \implies a = 0$$

in A.

Proof. First, recall that the Jacobson radical of A is defined as

$$J(A) = \{b \in A \mid 1 + abc \text{ is invertible for all } a, c \in A\}$$

and notice that every element of $J(A)$ is nilpotent by some basic algebra (omitted).

Next, observe that if $b \in J(A)$, then for every $a, c \in A$ the element $1 + abb^*c$ is invertible (set $a \to a$ and $ctob^*c$). Hence if $J(A)$ contains any nontrivial element, it contains a self-adjoint element. Necessarily this element is nilpotent, so some power of it squares to zero. Setting $a = (bb^*)^k$ to be that power, we see that

$$a^2 = a^*a = 0$$

So, if A satisfies $a^*a = 0 \implies a = 0$ we find that $J(A) = \{0\}$ and A is semisimple. By the classification theorem for finite-dimensional semisimple algebras over $\mathbb{C}$, this implies A is a multimatrix algebra.

Now, suppose A satisfies the condition. Take the minimal central idempotent elements $\{p_i\}_{i=1}^n$ and observe that their adjoints $\{p_i^*\}$ are also minimal central idempotents, so that

$$p_i^* = p_j$$

for some j. Applying p_i we find that

$$p_i^* p_i = p_j p_i \neq 0$$

by the condition. Hence $i = j$ and each p_i is self-adjoint. Hence, for any element $ap_i = p_i a p_i$ in the ith matrix summand, we have

$$(ap_i)^* = a^* p_i^* = a^* p_i$$

is another element of the same matrix factor. Hence the matrix summands are preserved under $*$.

Considering the $*$-operation restricted to one factor, we know that it agrees with the standard Hermitian adjoint up to conjugation by a self-adjoint matrix h, that is

$$x^* = h x^\dagger h^{-1}$$

for all x in our matrix factor. We show that h is either positive or negative definite.

Suppose h is neither, so that $\mathrm{Spec}(h)$ has a positive eigenvalue λ_+ and a negative eigenvalue λ_- with eigenvectors $v_\pm$. Pick x to have one nonzero column with entries $av_+ + bv_-$ such that $a^2/\lambda_+ + b^2/\lambda_- = 0$ (e.g. by setting $b = 1$ and $a = \sqrt{\frac{-\lambda_+}{\lambda_-}}$). A quick computation shows that

$$x^\dagger h^{-1} x = 0$$

and hence

$$x^* x = h x^\dagger h^{-1} x = 0$$

contradicting the assumption.

Hence $\pm h$ is positive-definite operator, and by a previous exercise this implies that this factor is $*$-isomorphic to the standard matrix algebra. □

1.3 The GNS Construction

Set a to be a multimatrix algebra with a faithful state φ.

Exercise 14. Show that

$$\langle a, b \rangle = \varphi(b^* a)$$

is an inner product on A.

Proof. All the properties of inner products follow directly from φ being faithful. □

Exercise 15. Denote by Ω the unit in A, thought of as a vector in $L^2(A, \varphi)$. Show that left-multiplication induces a $*$-representation of A on $L^2(A, \varphi)$.

Proof. We kind of already showed this. The map

$$L : A \to B(L^2(A, \varphi))$$

is obviously linear and multiplicative, and by definition of the inner product we have

$$\langle L(a)v\Omega, w\Omega \rangle = \varphi(w^* av) = \varphi\left((a^* w)^* v\right) = \langle v\Omega, a^* w\Omega \rangle$$

showing that $L(a^*) = L(a)^*$. □

Exercise 16. Show that right-multiplication induces a representation, and determine when this is a $*$-representation.

Proof. Verifying it is a representation is trivial.

We compute

$$\langle R(a)v, w \rangle = \varphi(w^* va) = \varphi(aw^* v)$$

which holds if and only if φ is tracial. □

Exercise 17. In the case of $M_n(\mathbb{C})$, show that the commutant of the left-regular representation is the right-regular representation.

Proof. This fact is purely algebraic and holds for any ring A, viewed as an $A - A$ bimodule. The proof is elementary. Clearly every right action commutes with every left action. Conversely, take an arbitrary operator φ which commutes with the left action. Setting $x = \varphi(1)$, we see that

$$\varphi(a) = \varphi(a(1)) = ax = R(x)a$$

hence φ is a right-multiplication operator. □

Exercise 18. Show the same for $M_{m \times n}(\mathbb{C})$ as an $M_m(\mathbb{C}) - M_n(\mathbb{C})$ bimodule.

Proof. Skipped □

Exercise 19. Show the same for general A on $L^2(A, \varphi)$.

Proof. Skipped, we already showed this. □

Exercise 20. A finite-dimensional complex $*$-algebra is a multimatrix algebra if and only if it has a faithful state.

Proof. We did the forward direction earlier. Conversely, if A has a faithful state, then L embeds A into $B(L^2(A, \varphi))$ and its image, isomorphic to A, is a unital $*$-subalgebra of a $B(H)$, hence a multimatrix algebra by the classification of finite-dimensional von Neumann algebras. □

1.4 Unital Inclusion of Multimatrix Algebras

We say an inclusion $A \hookrightarrow B$ unital if the image of 1_A is 1_B. For example, the diagonal inclusion of $\mathbb{C}$ into any multimatrix algebra is unital, the inclusion of a factor is not unital.

Exercise 21. Show that a $M_k(\mathbb{C})$ is a unital subalgebra of $M_n(\mathbb{C})$ if and only if k divides n, in which case $M_k(\mathbb{C})$ up to unitary conjugation is embedded diagonally.

Proof. This was proved in our classification of finite-dimensional von Neumann algebras. In particular, $M_k(\mathbb{C})$ has trivial center (its a factor) so if $M_k(\mathbb{C})$ embeds into $M_n(\mathbb{C})$, then there is some j for which $M_n(\mathbb{C}) \cong M_k(\mathbb{C}) \otimes M_j(\mathbb{C})$ and $M_k(\mathbb{C})$ embeds as $M_k(\mathbb{C}) \otimes 1$. The unitary conjugation business comes from the choice of unitary isomorphism above. □

Now consider a multimatrix algebra unital inclusion $A \hookrightarrow B$ with dimension vectors $\vec{n}_A$ and $\vec{n}_B$ and minimal central projectors $p_1, \ldots, p_k$ in A and $q_1, \ldots, q_\ell$ in B. Define φ_{ij} as the induced map from p_iA to q_jB:

$$\varphi_{ij} : p_iA \hookrightarrow A \hookrightarrow B \twoheadrightarrow q_jB$$

and think of it as a map of $*$-algebras from $M_{n_i}(\mathbb{C})$ to $M_{n_j}(\mathbb{C})$. It may not be unital, but it is guaranteed to be either trivial or injective.

Exercise 22. However, show that if we consider φ_{ij} as a map from p_iA to $p_iq_jBp_iq_j$ then it is unital.

Proof. Obvious, the image of 1 in A is exactly the projection p_iq_j. □

Example. For clarity, let's work out an example. Suppose $A = \mathbb{C} \oplus \mathbb{C}$ and $B = M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus \mathbb{C}$. Let's define the inclusion to be

$$\begin{aligned}(1,0) &\mapsto (I,0,1)\\ (0,1) &\mapsto (0,I,0)\end{aligned}$$

This is unital, since $(1,1) \mapsto (I,I,1)$. Notice that each factor includes in an interesting non-unital way. The φ maps are

$$\begin{aligned}\varphi_{1,1} = (1 \mapsto I)\varphi_{1,2} = (1 \mapsto 0) \quad \varphi_{1,3} = (1 \mapsto 1)\\ \varphi_{2,1} = (1 \mapsto 0)\varphi_{2,2} = (1 \mapsto I) \quad \varphi_{2,3} = (1 \mapsto 0)\end{aligned}$$

Example. Again let $A = \mathbb{C} \oplus \mathbb{C}$ and let $B = M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus \mathbb{C}$. Define the inclusion now by

$$\begin{aligned}(1,0) &\mapsto (e_{11}, e_{11} + e_{22}, 1)\\ (0,1) &\mapsto (e_{22}, e_{33}, 0)\end{aligned}$$

where e_{ij} is the matrix with a 1 in the ij entry only.

Now, the φ maps are

$$\begin{aligned}\varphi_{1,1} = (1 \mapsto e_{11})\varphi_{1,2} = (1 \mapsto e_{11} + e_{22}) \quad \varphi_{1,3} = (1 \mapsto 1)\\ \varphi_{2,1} = (1 \mapsto e_{22})\varphi_{2,2} = (1 \mapsto e_{33}) \qquad \varphi_{2,3} = (1 \mapsto 0)\end{aligned}$$

From our classification of matrix subalgebras, we see that for each φ_{ij} we get a whole number Λij given by the multiplicity of p_iA in $p_iq_jBp_iq_j$. These assemble into a matrix, which we denote Λ^B, called the *inclusion matrix* of the inclusion.

Example. In the case of the first inclusion, we have

$$\Lambda_A^B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix}$$

since e.g. the first factor of $\mathbb{C}$ embeds diagonally in the entire first factor of $M_2(\mathbb{C})$ and hence has "extrinsic multiplicity" 2.

Example. In the case of the second inclusion, we have

$$\Lambda_A^B = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Before we get to exercises, here's a key fact that we need.

Lemma 1.4. *Let $B = B(H)$ for some finite-dimensional H. Then the (not necessarily unital) subalgebra pBp for p a projection in B is known as a* corner *of B, and is canonically a unital $*$-subalgebra of $B(pH)$. In particular, $pB(H)p \cong B(pH)$.*

Hence, in the multiplicities computation done above, we find that for factor q_jB of B, $q_j(B) \cong M_{n_j}(\mathbb{C}) \cong B(H_j)$ and $p_iA \cong M_{n_i}(\mathbb{C}) \cong B(H_i)$

$$\Lambda_{ij} = \dim(p_i q_j H_j)/\dim(H_i)$$

Exercise 23. Show that $\vec{n}_A \Lambda = \vec{n}_B$

Proof. We just need to show that $(n_B)_j = \sum_i \Lambda_{ij}(n_A)_i$. As observed,

$$1_A = \sum_i p_i$$

and

$$1_B = \sum_j q_j$$

In particular, the identity on q_jB is

$$q_j(1) = \sum_i p_i q_j$$

which tells us that the factor $q_jB \cong B(\mathbb{C}^{n_j})$ has its Hilbert space split into a sum

$$H_j := \mathbb{C}^{n_j} = \sum_i p_i q_j H_j$$

and furthermore, since $(p_iq_j)(p_kq_j) = \delta_{ik}$ this sum is direct. Hence

$$\dim H_j = (n_B)_j = \sum_i \dim(p_i H_j) = \sum_i \frac{\dim(p_i q_j H_j)}{\dim(p_i H_i} \dim(p_i H_i) = \sum_i \Lambda_{ij}(n_A)_i$$

as desired. □

Exercise 24. Show that this definition is multiplicative on chained inclusions, so that if $A \subseteq B \subseteq C$ then $\Lambda_A^B \Lambda_B^C = \Lambda_A^C$.

Proof. Same as before.

$$1_C = \sum_k r_k$$

whence

$$p_i 1 = \sum_k p_i r_k$$

but also

$$p_i 1 = \sum_j p_i q_j$$

and

$$q_j 1 = \sum_k q_j r_k$$

so

$$p_i 1 = \sum_k p_i r_k = \sum_{j,k} p_i q_j r_k$$

which I believe (needs work) yields the identity we need. $\square$

From this, we get the combinatorics expressed in the so-called *Bratteli diagram*.

Definition 1.4.1. For an inclusion of multimatrix algebras $A \subseteq B$, associate to it the bipartite graph Γ with k vertices of one type, ℓ vertices of the other, and Γ_{ij} edges from $(n_A)_i$ to $(n_B)_j$.

Exercise 25. Prove that every inclusion $A \subseteq B$ up to unitary conjugation in B is determined by its Bratteli diagram.

Proof. Later... $\square$

Exercise 26. Suppose λ_A and λ_B are vectors determining faithful tracial states on A and B. Prove that $\mathrm{tr}_B|_A = \mathrm{tr}_A$ if and only if $\Lambda \lambda_B = \lambda_A$.

Proof. Traces are determined by their trace vectors λ by summing

$$\mathrm{tr}_\lambda = \sum_i \lambda_i \, \mathrm{tr}_i$$

where tr_i is the unique un-normalized trace on the ith factor (satisfying $\mathrm{tr}_i(1) = n_i$).

So, assume the traces are equal. This means that on each factor of A, they have the same weighting. In particular,

$$\lambda_i \, \mathrm{tr}(p_i 1) = \sum_j \lambda_j \, \mathrm{tr}(p_i q_j)$$

hence

$$\lambda_i = \sum_j \lambda_j \frac{\mathrm{tr}(p_i q_j)}{\mathrm{tr}(p_i)} = \sum_j \lambda_j \frac{\dim(p_i q_j H_j)}{\dim(p_i H_i)}$$

as desired. The converse follows. $\square$

1.5 Connected Inclusions

For our last definition of the day, define an inclusion to be *connected* if its Bratteli graph is connected.

Exercise 27. Show that $A \subseteq B$ is connected if and only if $Z(A) \cap Z(B) = \mathbb{C}$.

Proof. Suppose the center is nontrivial, so that there is some nontrivial central minimal idempotent p_i in both $Z(A)$ and $Z(B)$. This then splits the inclusion as

$$(p_i A) \oplus ((1 - p_i)A) \hookrightarrow (p_i B) \oplus ((1 - p_i)B)$$

and thus the associated graphs split as well.

Conversely, if the graph splits then after reordering the vertices we can write its adjacency matrix in block-diagonal form. Considering the first block, the projection onto the direct sum of those factors in B is such a central idempotent. $\square$

Exercise 28. Show that if $A \subseteq B$ is connected, then there is a unique $d > 0$ an unique trace vector λ_B with $m_B \lambda_B = 1$ and

$$\Lambda^T \Lambda \lambda_B = d^2 \lambda_B$$

Proof. This follows from the "Frobenius-Perron theorem" applied to $\Lambda^T\Lambda$, which guarantees a unique positive eigenvalue d^2 of $\Lambda^T\Lambda$ such that its eigenvector has positive entries. Normalizing this eigenvector yields the desired result. Connectedness is essential to assure that $\Lambda^T\Lambda$ has all positive entries.

Hence, since $\lambda_A = \Lambda\lambda_B$, we find that

$$\Lambda^T\lambda_A = d^2\lambda_B$$

and

$$\begin{bmatrix} 0 & \Lambda \\ \Lambda^T & 0 \end{bmatrix} \begin{bmatrix} \lambda_A \\ d\lambda_B \end{bmatrix} = d \begin{bmatrix} \lambda_A \\ d\lambda_B \end{bmatrix}$$

□

Such a scalar d is called the *Frobenius-Perron* eigenvalue for the inclusion. The tracial vector λ_B is the *Frobenius-Perron* eigenvector for the inclusion.

Exercise 29. Show that if Λ is a proper subgraph of a finite graph Γ, then the Frobenius-Perron eigenvalue of Λ is strictly less than the Frobenius-Perron eigenvalue of Γ

Proof. Subgraphs have adjacency matrices with strictly smaller entries, hence by some linear algebra we find easily that since the Frobenius-Perron eigenvalue is the largest, a spectral radius formula guarantees the result. □