A Refined Approach to Stability in the Hyperkähler Setting

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Introduction

Bridgeland Stability originated as a description of the low-energy physics of an $\mathcal{N} = 2$ string theory on a Calabi-Yau manifold. When the target has a hyperkähler structure, supersymmetry is enhanced to $\mathcal{N} = 4$, which has an effect on the allowed objects in the category of boundary conditions.

In the $\mathcal{N} = 2$ case, the category of boundary conditions is the derived category of coherent sheaves [1], but in the hyperkähler case the category is necessarily smaller. Using a doubling construction, these objects can be analyzed using the geometry of the doubled space.

The Calabi-Yau Case

In an open string theory on a Calabi-Yau manifold X, the string ends may lie on a submanifold of X called a "D-brane", which preserves at most half of the total supersymmetry ($\mathcal{N} = 2$ of them in this case). There are two choices for which half to preserve:

	A-model	B-model
Condition	Lagrangian	Holomorphic
Category	$\mathrm{Fuk}(X)$	$D^b(X)$

Hyperkähler Symmetries

A hyperkähler manifold has three compatible complex structures. D-branes, in this $\mathcal{N} = 4$ case, must be B-type in one complex structure, but can be either B-type or A-type in the other two [2].

SUSY	(B,B,B)	(B,A,A)
Condition	Trianalytic	Holo. Coisotropic

In either case, the D-brane is B-type in one complex structure, making it an element of $D^b(X)$. Imposing the other two conditions restricts to a subcategory.

Doubling Spaces

A **doubled manifold** is a symplectic manifold (M, ω) with a pair of transverse lagrangian foliations $\mathcal{F}_1, \mathcal{F}_2$. By the Frobenius theorem, the tangent bundle splits

$$TM \cong T\mathcal{F}_1 \oplus T\mathcal{F}_2$$

into the tangent bundles of the leaves of the two foliations. The symplectic form ω then identifies

$$T\mathcal{F}_2 \cong T\mathcal{F}_1^*$$

Then, for a leaf $S_1 \in \mathcal{F}_1$

$$TM|_{S_1} \cong TS_1 \oplus T^*S_1$$

(This is also true at the level of Courant algebroids!)

A **doubled product manifold** is a doubled manifold M which globally splits as a product $M = X_1 \times X_2$ whose fibers are leaves of the foliations. Then, the leaf space $M/\mathcal{F}_2$ is the projection $\pi : M \to X_1$.

The Torus Case

The case of the base a torus has been worked out in detail in [5].
Let $X = \mathbb{C}^2/\Lambda$ and $X^\vee = \mathbb{C}^2/\Lambda^*$ be dual torii with Kähler structures inherited from (g, J, ω) on $\mathbb{C}^2$. Their **doubling torus** is

$$M = X \times X^\vee$$

with Kähler structure $(g \oplus g^{-1}, J \oplus -J^*, \omega \oplus -\omega^{-1})$, and para-Kähler form $\sigma_0 = \sum dx \wedge d\hat{x}$.
The **doubling lift** of an affine GC submanifold is

$$\mathbb{L} = \{(x, \hat{x}) \mid x \in L, \hat{x}(\gamma) = (-1)^{s(\gamma)} \operatorname{Hol}_\nabla(x + S^1_\gamma)\}$$

for all $\gamma \in H_1(L, \mathbb{Z})$, S^1_γ a linear representative, with s a correction factor. The tangent bundle of $\mathbb{L}$ is

$$T\mathbb{L} = \{X + \xi \mid X \in TL, \xi|_L = \iota_X F\}$$

which is the generalized tangent bundle of L.

Generalized Complex Branes

To treat A and B-models uniformly, we use the language of generalized complex (GC) geometry, developed in [3], where both symplectic forms and complex structures are viewed as endomorphisms of $TX \oplus T^*X$ squaring to -1.
Generalized complex submanifolds are submanifolds L with $U(1)$ connections with curvature F such that the generalized tangent bundle

$$\mathcal{T}L = \{X + \xi \in TX \oplus T^*X \mid X \in TL, \iota_X F = \xi|_L\}$$

is preserved by the GC structure.

- **B-type:** The GC submanifolds are holomorphic submanifolds with holomorphically flat connections.
- **A-type** The GC submanifolds are coisotropic submanifolds with curvature satisfying the coisotropic brane condition

$$\omega + F\omega^{-1}F = 0$$

Geometry of Doubling Spaces

Suppose M is a product doubled manifold with base Q. If Q admits a Kähler structure (G, J, ω), then M gets a Kähler structure given by

$$\mathbb{G} = \begin{bmatrix} 0 & g^{-1} \\ g & 0 \end{bmatrix} \mathbb{J} = \begin{bmatrix} J & 0 \\ 0 & -J^* \end{bmatrix} \quad \mathbb{W} = \begin{bmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{bmatrix}$$

Satisfying $\mathbb{G}\mathbb{J} = \mathbb{J}\mathbb{G} = \mathbb{W}$. In this case, $\mathbb{W}$ is another complex structure on M as well. When Q is Hyperkähler with complex structures (I, J, K), M has six complex structures given by $\mathbb{I}, \mathbb{J}, \mathbb{K}$ and their corresponding Kähler forms $\mathbb{W}_I, \mathbb{W}_J, \mathbb{W}_K$.

Geometry of Q	Lifted Geometry of M
$\mathbb{C}$-structure J	$\mathbb{C}$-structure $\mathbb{J}$
Symplectic form ω	$\mathbb{C}$-structure $\mathbb{W}$
Metric G	Metric $\mathbb{G}$
HK (I, J, K)	HK $(\mathbb{I}, \mathbb{J}, \mathbb{K})$
Holo. Sym. (I, Ω)	HK $(\mathbb{I}, \mathbb{W}_J, \mathbb{W}_K)$

- **The** (B, B, B) **hyperkähler structure** on M is given by the triple $(\mathbb{I}, \mathbb{J}, \mathbb{K})$. This tracks the data of trianalytic submanifolds of Q.
- **The** (B, A, A) **hyperkähler structure** on M is given by the triple $(\mathbb{I}, \mathbb{W}_J, \mathbb{W}_K)$. This tracks the data of holomorphic coisotropic branes in Q.

Lifts of GC Branes

A **doubling lift** of (L, F) is a lift

$$\begin{tikzcd} & M \arrow[d, "q"] \\ L \arrow[ur, dashed, "\tilde{\iota}"] \arrow[r, hook] & Q \end{tikzcd}$$

to the doubled manifold such that $T(\tilde{\iota}L)$ is isomorphic to the generalized tangent bundle of L. The lift is always maximal isotropic with respect to the para-Kähler form.
In the torus case, the doubling lift of a $\mathbb{W}$-GC brane is holomorphic in the lifted complex structure $\mathbb{W}$ [5].
As expected, one can show that the doubling lift of a $\mathbb{J}$-GC brane is $\mathbb{J}$-holomorphic as well.

Future Work

Every K3 surface is a hyperkähler rotation of an elliptic K3, which is a T^2-fibration over $\mathbb{P}^1$. Although the relative doubling manifold has been explored in detail, a global doubled manifold for the K3 has yet to be constructed in this way. It is expected that doubled geometry implements T-duality, which is an incarnation of mirror symmetry in the K3 case.

Such a doubling space would be a natural setting for the category of $\mathcal{N} = 4$ boundary conditions to be defined in [4], since these boundary conditions come in two flavors which are handled identically in the doubled manifold. However, a full treatment in the sense of [1] requires two additional parts.

- Inclusion of topological deformations: In the $\mathcal{N} = 2$ case this enhances $\mathrm{Coh}(X)$ to $D^b(X)$, but such analysis has not been carried out in the $\mathcal{N} = 4$ case.
- Analysis of enhanced stability: the new category admits stability conditions that enhance Bridgeland stability.

References

[1] Paul S Aspinwall and Albion Lawrence. Derived categories and zero-brane stability. *Journal of High Energy Physics*, 2001(08):004, 2001.

[2] Hirosi Ooguri, Yaron Oz, and Zheng Yin. D-branes on calabi-yau spaces and their mirrors. *Nuclear Physics B*, 477(2):407–430, 1996.

[3] Marco Gualtieri. Generalized complex geometry. *Annals of mathematics*, pages 75–123, 2011.

[4] Anton Kapustin and Yi Li. Open-string brst cohomology for generalized complex branes. *Advances in Theoretical and Mathematical Physics*, 9(4):559–574, 2005.

[5] Yingdi Qin. *Coisotropic branes on tori and Homological mirror symmetry*. University of California, Berkeley, 2020.