String Theory and Mathematics: an Ongoing Dialogue

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Question

How do physics and mathematics talk to each other?

String Theory	Algebraic Geometry
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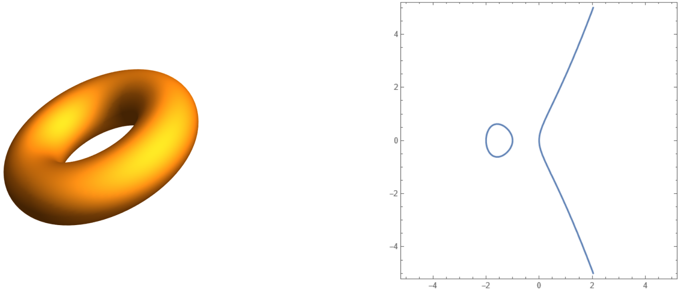
Outline

The Math

A Word on $\mathbb{C}$

Warning

The dimension of a space over $\mathbb{C}$ is half that over $\mathbb{R}$ (e.g. $\mathbb{C}$ is one-dimensional as a complex vector space, but two-dimensional over $\mathbb{R}$). A "curve" is two-dimensional over $\mathbb{R}$, a "surface" is four-dimensional, etc.



(a) Complex solutions to the equation $y^2 = x(x+1)(x+2)$

(b) The real solutions are one-dimensional

Figure 1: Both these images are "curves" according to complex geometers

Algebraic Varieties: Quadrics

Definition

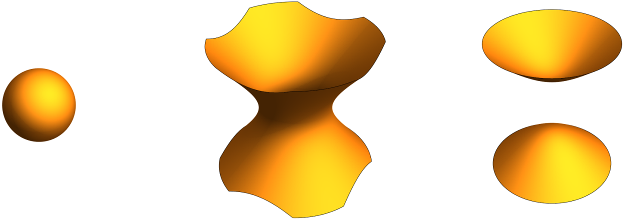
An *algebraic variety* is a geometric shape that can be defined by polynomial equations.

Example (smooth quadrics in $\mathbb{R}^3$)

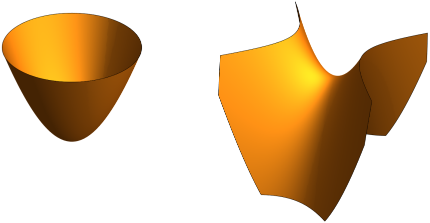
The real zeroes of a real quadratic polynomial in three variables form a surface known as a *real quadric surface*, and generally come in five flavors. A general equation for a quadric surface looks like

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Jz + K = 0$$

Algebraic Varieties: Quadrics



(a) $x^2 + y^2 + z^2 = 1$ (b) $x^2 + y^2 - z^2 = 1$ (c) $x^2 + y^2 - z^2 = -1$



(d) $x^2 + y^2 - z = 1$ (e) $x^2 - y^2 - z = 1$

Figure 2: Real nondegenerate quadrics. Over $\mathbb{C}$, these are all the same surface.

Lines on Quadrics

Theorem

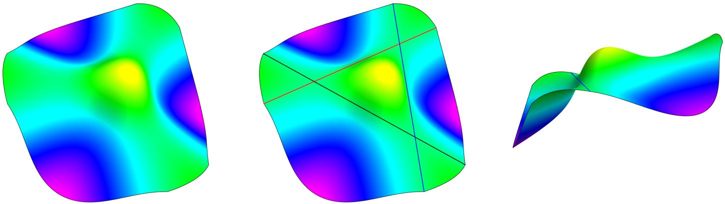
Nondegenerate quadrics have infinitely many lines. These lines come in two one-dimensional families.

Figure 3: The lines on this quadric are clearly visible in two families.

Algebraic Varieties: Cubics

Example (smooth cubics in $\mathbb{C}^3$)

The zeroes of a cubic polynomial in three variables form a surface known as a *cubic surface*.



(a) The real points (color indicates height)

(b) Three Lines

(c) Head-on view showing the lines are straight

Figure 4: The Fermat cubic $x^3 + y^3 + z^3 = 1$

Algebraic Varieties: Cubics

Example (smooth cubics in $\mathbb{C}^3$)

The zeroes of a cubic polynomial in three variables form a surface known as a *cubic surface*.

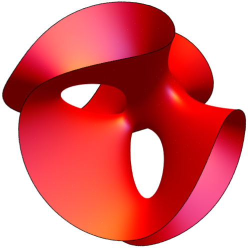


Figure 5: The Clebsch cubic $x^3 + y^3 + z^3 + 1 = (x + y + z + 1)^3$

Lines on Cubics

Theorem

Every smooth cubic surface has ______ exactly 27 lines.

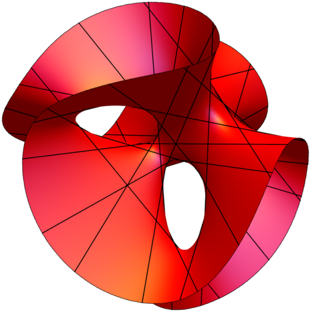
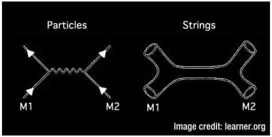


Figure 6: The Clebsch Cubic allows for all 27 lines to be seen in just the real points. Lots of interesting combinatorics here!

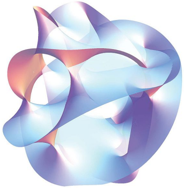
The Physics

String Theory and Algebraic Varieties

- Central hypothesis of string theory: fundamental physical object is a one-dimensional loop.
- Major problem: theory is inconsistent if the dimension of the universe is not 10.
- Solution (compactification): spacetime is a product $\mathbb{R}^4 \times X$ where X is six-dimensional with finite volume
- Consistency: X must be *Calabi-Yau*, in particular an algebraic variety.



(a) If strings are small, they look like particles



(b) A Calabi-Yau manifold

The Quintic Threefold

Theorem (Candelas et. al. [a])

[a] P. Candelas et al. "A Pair of Calabi-Yau manifolds as an exactly soluble superconformal theory". *Nucl. Phys. B* 359 (1991). Ed. by S.-T. Yau, pp. 21–74

The q-expansion of the Yukawa coupling on a family of smooth quintic hypersurfaces in $\mathbb{C}^4$ is given by

$$5 + 2875q + (2^3 \cdot 609250 + 2875)q^2 + \dots$$

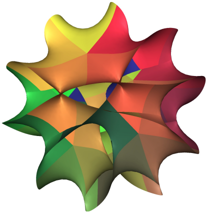


Figure 7: A slice of the Fermat quintic $x^5 + y^5 + z^5 + w^5 = 1$ that Candelas, De La Ossa, Green, and Parkes ran string theory on.

The Quintic Threefold: Enumerative Geometry

Theorem (Candelas et. al)

The q-expansion of the Yukawa coupling on the family of smooth quintic hypersurfaces in $\mathbb{C}^4$ is given by

$$5 + 2875q + (2^3 \cdot 609250 + 2875)q^2 + \dots$$

What makes this result particularly surprising is the following enumerative results:

- 5 is the degree of the quintic hypersurface
- 2875 is the number of lines on the quintic
- 609250 is the number of degree-two curves (conics) on the quintic

The Quintic Threefold: Predictions

The q-expansion predicts the number of degree-three curves on the quintic to be

$$n_3 = 317206375$$

which was unknown at the time.

This was later computed by Ellingsrud and Stromme[1] and found to be correct!

[1] G. Ellingsrud and S. A. Stromme. "The Number of Twisted Cubic Curves on the General Quintic Threefold.". *Mathematica Scandinavica* 76.1 (1995), pp. 5–34

String Theory and Mathematics I: Enumerative Geometry

String Theory	**Algebraic Geometry**
Yukawa coupling	Curve counting
-	-
-	-

The Math

What is a category?

Definition

A category consists of:

- *Objects*: a collection of primitives.
- *Morphisms*: abstract maps between objects.
- *Composition*: $(A \to B) \circ (B \to C)$ yields $A \to C$.

Examples of Categories

Example (The category of groups)

- Objects are groups.
- Morphisms are group homomorphisms.
- Composition is regular function composition.

Example (The category of natural numbers)

- Objects are natural numbers.
- There is a unique morphism from m to n if and only if $m \leq n$.
- Composition: if $k \leq m$ and $m \leq n$, then their composition is the unique morphism from $k \leq n$.

The Derived Category

The derived category is a construction that yields a complicated category containing "homological information" about the object under study.

Example

Given a ring R, the construction associates to R its *derived category* $D(\text{R-Mod})$, containing information about the representation theory of R.

Example

Given a variety X, the construction associates to X its *bounded derived category of coherent sheaves* $D^b(\text{Coh}(X))$, containing information about vector bundles and subvarieties of X.

Derived categories are not very well understood (e.g. no notion of subobject, morphisms are computed using "composable roofs").

The Physics

Strings and Categories

Definition

Theories with open strings allow for *D-branes*, extended physical objects which the string endpoints attach to.

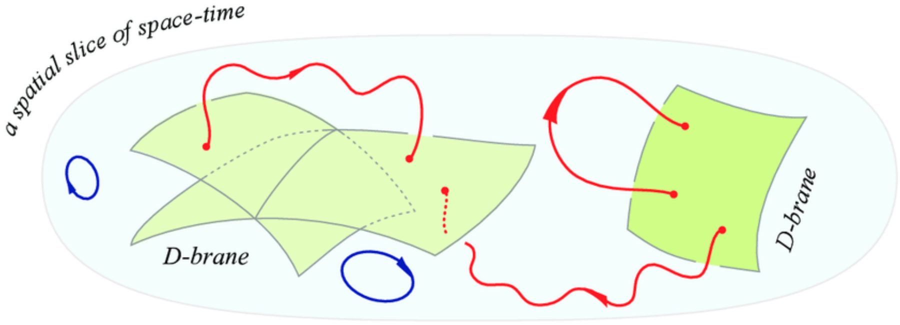


Figure 8: A cartoon of some D-branes with strings attached to them. Image borrowed from arxiv:1406.0929

Key Result

Theorem (Aspinwall and Lawrence [a])

[a] P. S. Aspinwall and A. E. Lawrence. "Derived categories and zero-brane stability". *JHEP* 08 (2001), p. 004. arXiv: hep-th/0104147

The string category associated to a (particular) string theory on a Calabi-Yau algebraic variety X is the derived category of coherent sheaves on X.

String Theory and Mathematics II: Derived Categories

String Theory	**Algebraic Geometry**
Yukawa coupling	Curve counting
Open string states	Derived category
-	-

Insights From Physics

- Michael Douglas [2] observed that strings carry more information than the derived category sees.
- Certain string states can aquire negative mass, indicating the D-branes they are attached to can condense into a new D-brane.
- This process (tachyon condensation) predicts a new structure on the derived category that mathematicians were unaware of.
- Tom Bridgeland formalized this [3] and it has been a central tool in studying derived categories.

[2] M. R. Douglas. "D-branes, categories and N=1 supersymmetry". *J. Math. Phys.* 42 (2001), pp. 2818–2843. arXiv: `hep-th/0011017`

[3] T. Bridgeland. "Stability conditions on triangulated categories". *Ann. of Math. (2)* 166.2 (2007), pp. 317–345

String Theory and Mathematics II: Derived Categories (Bonus)

String Theory	**Algebraic Geometry**
Yukawa coupling	Curve counting
Open string states	Derived category
Tachyon condensation	$D^b(X)$ stability conditions
-	-

Hyperkähler Varieties

Definition

A *hyperkähler variety* is a complex algebraic variety with three independent complex structures IJK satisfying the quaternion relations

$$I^2 = J^2 = K^2 = IJK = -1$$

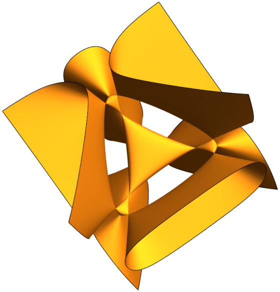


Figure 9: A famous hyperkähler variety known as the K3 surface

Hyperkähler Varieties and Physics

Theorem

An $\mathcal{N} = (4,4)$ supersymmetric string theory can only exist on an algebraic variety that is hyperkähler. Conversely, any string theory on a hyperkähler background can be enhanced to an $\mathcal{N} = (4,4)$ theory.

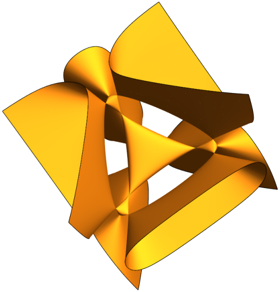


Figure 10: A famous hyperkähler variety known as the K3 surface

Some New Physics Insights?

- The open-string category has yet to be computed in the $\mathcal{N} = (4, 4)$ model.
- The corresponding derived category on the hyperkähler side has yet to be defined properly as well.
- Proper understanding requires the use of *generalized complex geometry*, which is a recent construction not yet well-understood.

String Theory and Mathematics III: New Opportunities

String Theory	**Algebraic Geometry**
Yukawa coupling	Curve counting
Open string states	Derived category
Tachyon condensation	$D^b(X)$ stability conditions
$\mathcal{N} = (4, 4)$ model	???

Thank you for your time!

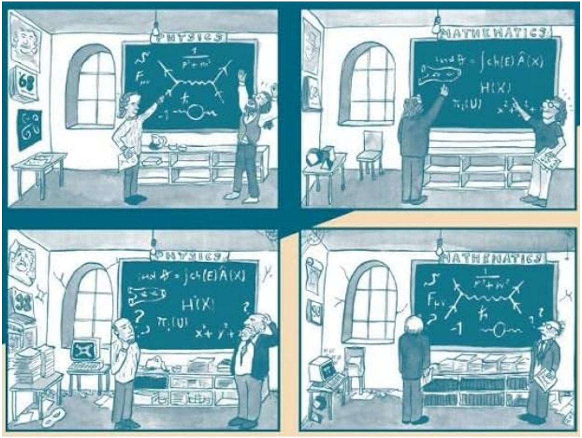


Figure 11: Taken from the cover of "Quantum Fields and Strings: A Course for Mathematicians"