Graduate Quantum Algebra & Topology Seminar: Jones' Basic Construction

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Following "Exercises in Quantum Algebra" by Penneys (https://people.math.osu.edu/penneys.2/QuantumAlg
Starting in section 3.2.

1 Jones' Basic Construction

We'll fix multimatrix algebras A and B along with an inclusion $A \hookrightarrow B$.

1.1 Conditional Expectations

Fix faithful tracial states tr_A on A and tr_B on B

Definition 1.1.1. A *conditional expectation* is a unital morphism of $A - A$-bimodules

$$E \in {}_A\mathrm{Hom}_A(B, A)$$

We say a conditional expectation is *trace-preserving* if

$$\mathrm{tr}_A(E(x)) = \mathrm{tr}_B(x)$$

and *faithful* if

$$E(x^*x) = 0 \iff x = 0$$

Exercise 1. Show that $E|_A = \mathbb{1}_A$ and that E is idempotent ($E^2 = E$). Use this to deduce that if E is trace-preserving, then $\mathrm{tr}_B|_A = \mathrm{tr}_A$.

Proof. The first assertion follows from the fact that E is $A - A$-bilinear. Recall that this means for all $a, b \in A$ and $x \in B$ we have

$$E(axb) = aE(x)b$$

Setting $x = b = 0$ and a arbitrary yields $E(a) = a$.

Since E maps into A, applying E to anything in the image of E is the identity, hence $E^2 = \mathbb{1}_A \circ E = E$.

Clearly, then, if E is trace-preserving we have

$$\mathrm{tr}_B(a) = \mathrm{tr}_A(E(a)) = \mathrm{tr}_A(\mathbb{1}_A(a)) = \mathrm{tr}_A(a)$$

for all $a \in A$. □

Exercise 2. Show that if E, F are two conditional expectations such that for all $a \in A$, $b \in B$

$$\mathrm{tr}_A(aE(b)) = \mathrm{tr}_A(aF(b))$$

then $E = F$. Use this to deduce that trace-preserving conditional expectations, if they exist, are unique.

Proof. Following the hint, consider the induced representations of B on $L^2(A, \mathrm{tr}_A)$ given by E and F. By the condition given, we know that

$$\langle E(b), a\rangle = \mathrm{tr}_A(a^* E(b)) = \mathrm{tr}_A(a^* F(b)) = \langle F(b), a\rangle$$

By e.g. Riesz representation, since $E(a)$ and $F(a)$ induce the same linear functional, they are the same element of A.

If E, F are any two trace-preserving conditional expectations, then

$$\mathrm{tr}_A(aE(b))\,\mathrm{tr}_B(ab) = \mathrm{tr}_A(aF(b))$$

and we recover the conditions necessary for $E = F$. So, a trace-preserving conditional expectation is unique as desired. □

Problem. Suppose E is trace-preserving. Show that $E(x^*) = E(x)^*$ for all $x \in B$.

Proof. Again, we consider the induced representation on $L^2(A, \mathrm{tr}_A)$. We have for all $a, b \in A$

$$\begin{aligned}
\langle E(x)a, b\rangle &= \mathrm{tr}_A(b^* E(x)a) \\
&= \mathrm{tr}_B(b^* x a) \\
&= \overline{\mathrm{tr}_B(a^* x^* b)} \\
&= \overline{\mathrm{tr}_A(a^* E(x^*) b)} \\
&= \mathrm{tr}_A(b^*(E(x^*) * a)) \\
&= \mathrm{tr}_A\left((E(x^*)b)^* a\right) = \langle a, E(x^*)b\rangle
\end{aligned}$$

showing that $E(x^*) = E(x)^*$ as desired. □

By similar reasoning (I'll skip it) we can show $E(x^*x)$ always maps to a positive operator in A.

Exercise 3. Consider $L^2(A, \mathrm{tr}_B|_A)$ as a subspace of $L^2(B, \mathrm{tr}_B)$, and let $e_A \in B(L^2(B, \mathrm{tr}_B))$ be its associated orthogonal projection. Define a map $E : B \to A$ as $E(b) = e_A(b)$. Show that E is a faithful conditional expectation, and is trace-preserving if and only if $\mathrm{tr}_B|_A = \mathrm{tr}_A$.

Proof. First, we show E is unital and $A - A$ bilinear. Since $1 \in A$, $e_A(1) = \mathbb{1}_A(1) = 1$ so E is unital. To show it is $A - A$-bilinear, we use the following argument (I'm not too happy with it, but it's the only one I can think of).

Orthogonal projection can be characterized by the following: For all $a \in A$ and $x \in B$,

$$\langle x, a\rangle = \langle e_A(x), a\rangle$$

derived from the characterization $x - e_A(x) \perp A$. Now, consider for $a, b, c \in A$ and $x \in B$,

$$\langle e_A(axb), c\rangle = \langle axb, c\rangle = \langle x, a^* c b^*\rangle$$

while

$$\langle a e_A(x) b, c\rangle = \langle e_A(x), a^* c b^*\rangle$$

and since $a^*cb^* \in A$, we get equality. Hence by e.g. Riesz representation, $e_A(axb) = ae_A(x)b$ as desired.

Such a map is faithful, since if $e_A(x^*x) = 0$, then in particular

$$0 = \langle e_A(x^*x), 1\rangle = \langle x^*x, 1\rangle = \mathrm{tr}_B(x^*x)$$

but tr_B is faithful, so this implies $x = 0$ as desired.

Finally, observe that

$$\langle e_A(x), 1\rangle = \mathrm{tr}_B(e_A(x))$$
$$\langle e_A(x), 1\rangle = \langle x, 1\rangle = \mathrm{tr}_B(x)$$

so that $\mathrm{tr}_B(x) = \mathrm{tr}_B|_A(e_A(x))$. This is equal to $\mathrm{tr}_A(e_A(x))$ if and only if $\mathrm{tr}_A = \mathrm{tr}_B|_A$ as we wanted. □

Exercise 4. Using the same projection as before, show that for all $b \in B$, $L_{e_A(b)}e_A = e_A L_b e_A$ in $B(L^2(B, \mathrm{tr}_B))$. Then, show that $b \in A$ if and only if $e_A L_b = L_b e_A$.

Proof. To show equality, we show it pointwise. So, let $x \in L^2(B, \mathrm{tr}_B)$. Consider

$$L_{e_A(b)}e_A(x) = e_A(b)e_A(x)$$

and

$$e_A L_b e_A(x) = e_A(be_A(x)) = e_A(b)e_A(x)$$

by A-linearity. So, the two are equal.

Clearly if $b \in A$, then

$$e_A L_b x = e_A(bx) = be_A(x) = L_b e_A x$$

Conversely, if the above holds, then setting $x = 1$ we find

$$e_A(b) = be_A(1) = b$$

which shows that $b \in A$. □

Example. Consider $M_k(\mathbb{C}) \hookrightarrow M_{nk}(\mathbb{C}) \cong M_n(\mathbb{C}) \otimes M_k(\mathbb{C})$ the diagonal embedding, with each given their unique tracial state. Abstractly, the unique trace-preserving conditional expectation on $M_n(\mathbb{C}) \otimes M_k(\mathbb{C})$ is given by the orthogonal projection e_A we constructed in generality.

Lemma 1.1. *The trace on $M_n(\mathbb{C}) \otimes M_k(\mathbb{C})$ is the product of the normalized traces on $M_n(\mathbb{C})$ and $M_k(\mathbb{C})$.*

Proof. Let tr_k and tr_n be the unique normalized traces. Then, consider the map

$$\mathrm{tr}_n \times \mathrm{tr}_k : M_n(\mathbb{C}) \times M_k(\mathbb{C}) \to \mathbb{C}$$

which is clearly bilinear, hence defines a map

$$\mathrm{tr}_n \otimes \mathrm{tr}_k : M_n(\mathbb{C}) \otimes M_k(\mathbb{C})$$

defined on pure tensors $A \otimes B$ as

$$\mathrm{tr}(A \otimes B) = \mathrm{tr}_n(A)\,\mathrm{tr}_k(B)$$

This functional is tracial, since on products of pure tensors

$$\mathrm{tr}\left((A \otimes B)(C \otimes D)\right) = \mathrm{tr}\left(AC \otimes BD\right) = \mathrm{tr}_n(AC)\,\mathrm{tr}_k(BD)$$
$$\mathrm{tr}\left((C \otimes D)(A \otimes B)\right) = \mathrm{tr}\left(CA \otimes DB\right) = \mathrm{tr}_n(CA)\,\mathrm{tr}_k(BD)$$

with equality since both traces are, well, traces.

This functional, finally, is a faithful state since for pure tensors

$$\mathrm{tr}\left((A^* \otimes B^*)(A \otimes B)\right) = \mathrm{tr}_n(A^*A)\,\mathrm{tr}_k(B^*B)$$

and each of tr_n, tr_k is a faithful state. □

Using this, we can construct the projection quite easily. On pure tensors, define

$$E(A \otimes B) = \mathrm{tr}_n(A)B$$

Notice that for $B \in M_k(\mathbb{C})$,

$$E(B) = E(1 \otimes B) = \mathrm{tr}_n(1)B = B$$

and for $A, B \in M_k(\mathbb{C})$ and $X \otimes Y \in M_n(\mathbb{C}) \otimes M_k(\mathbb{C})$ we have

$$E\left((1 \otimes A)(X \otimes Y)(1 \otimes B)\right) = E(X \otimes AYB) = \mathrm{tr}_n(X)AYB = A\,\mathrm{tr}_n(X)YB = AE(X \otimes Y)B$$

showing E is in fact a conditional expectation. By construction, E preserves traces so it really is the unique map we were looking for.

Example. Consider the connected inclusion of $A = M_n(\mathbb{C}) \oplus M_k(\mathbb{C})$ with trace vector $\lambda = (1/(n+k), 1/(n+k))$ into $B = M_{n+k}(\mathbb{C})$ with its unique trace.

The inclusion is given by the obvious block-diagonal embedding. Decomposing $M_{n+k}(\mathbb{C})$ as

$$M_{n+k}(\mathbb{C}) = M_n(\mathbb{C}) \oplus M_{n\times k}(\mathbb{C}) \oplus M_{k\times n}(\mathbb{C}) \oplus M_k(\mathbb{C})$$

with projections π_n, $\pi_{n\times k}$, $\pi_{k\times n}$, π_n we take E to be

$$E = \pi_n \oplus \pi_k$$

Clearly E is unital, and showing it is $A - A$ bilinear is a simple exercise (write each matrix in block form and compute). Notice also that

$$\mathrm{tr}_B(M) = \frac{1}{n+k}\,\mathrm{tr}(M) = \frac{1}{n+k}(\mathrm{tr}_n(\pi_n(M)) + \mathrm{tr}_k(\pi_k(M))) = \mathrm{tr}_A(E(M))$$

Example. Consider the connected inclusion of $A = \mathbb{C} \oplus \mathbb{C}$ to $B = M_2(\mathbb{C}) \oplus \mathbb{C}$ with Bratteli diagram

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

and traces $\lambda_A = (\phi^{-2}, \phi^{-1})$ $\lambda_B = (\phi^{-2}, \phi^{-3})$ with ϕ the golden ratio.

Recall that the Bratteli diagram tells us the inclusion is given by sending the first factor of $\mathbb{C}$ to the e_{11} matrix in $M_2(\mathbb{C})$, and the second factor of $\mathbb{C}$ to $(e_{22}, 1)$ in the sum.

Our conditional expectation E will be defined in the following way. Let π_1 be the projection of $M_2(\mathbb{C})$ onto the e_{11} space, and π_2 onto e_{22}. Then define

$$E(M, z) = (\pi_1(M), (\phi^{-1}\pi_2(M) - \phi^{-2}z))$$

Again, simple computation shows E is $A - A$ bilinear. This also preserves traces, since

$$\mathrm{tr}_B(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, z) = \phi^{-2}(a+d) + \phi^{-3}(z)$$

and

$$\mathrm{tr}_A(E(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, z)) = \mathrm{tr}_A(a, \phi^{-1}d + \phi^{-2}z) = \phi^{-2}a + \phi^{-1}\left(\phi^{-1}d + \phi^{-2}z\right)$$

1.2 The Basic Construction

Suppose now that $\operatorname{tr}_A = \operatorname{tr}_B|_A$, and as before let e_A be orthogonal projection onto $L^2(A \operatorname{tr}_A) \subset L^2(B, \operatorname{tr}_B)$, defining the unique trace-preserving conditional expectation $E : B \to A$.

Definition 1.2.1. The *basic construction* of $A \subseteq B$ is the unital $*$-subalgebra of $B(L^2(B, \operatorname{tr}_B))$ generated by B and e_A, denoted $\langle B, e_A\rangle$.

Exercise 5. Characterize $\langle B, e_A\rangle$ as $B + Be_AB$.

Proof. We can use the relation $e_Abe_A = E(b)e_A$ for $b \in B$ as well as $ae_A = e_Aa$ for $a \in A$ to reduce words. Namely, an element of $\langle B, e_A\rangle$ generically is a word (linear sum of words) in B and e_A, with words necessarily reduced to the form

$$c = b_1e_Ab_2e_A \dots b_ne_Ab_{n+1}$$

or

$$c = b$$

since $e_A^2 = e_A$. By repeated contraction using the above, we can reduce further to

$$c = b_1E(b_2)\dots E(b_n)e_Ab_{n+1} \in Be_AB$$

hence the claim is proved. □

It is noted that we can, with work, show that $\langle B, e_A\rangle = Be_AB$ but I will not do that here.

Definition 1.2.2. The *modular conjugation* is the map $J : L^2(B, \operatorname{tr}_B) \to L^2(B, \operatorname{tr}_B)$ given by $b \mapsto b^*$.

Modular conjugation plays well with the inner product:

$$\langle Ja, b\rangle = \operatorname{tr}(b^*Ja) = \operatorname{tr}(b^*a^*) = \langle Jb, a\rangle$$

and commutes with e_A

$$Je_A = e_AJ$$

Exercise 6. Show that $JA'J = (JAJ)'$

Proof. Suppose first that $x \in A'$. Consider an arbitrary $a \in A$, and observe that

$$JxJ(JaJ) = JxaJ = JaxJ = JaJJxJ$$

so $JxJ \in (JAJ)'$.

Conversely, if $x \in (JAJ)'$, then for all $a \in A$,

$$x(JaJ) = (JaJ)x$$
$$JxJa = aJxJ$$

showing that $JxJ \in A'$, hence $J(JxJ)J = x \in JA'J$ as desired. □

Exercise 7. Show that $\langle B, e_A\rangle = JA'J$.

Proof. Observe that since $B = JB'J \subseteq JA'J$ and $e_A = Je_AJ \in JA'J$ we have inclusion

$$\langle B, e_A\rangle \subseteq JA'J$$

On the other hand,

$$(J\langle B, e_A\rangle J)' \subseteq (JBJ)' \cap (Je_AJ)' = B \cap \{e_A\}'$$

However, elements of B commuting with e_A are exactly the elements of A by our earlier result. Hence,

$$(J\langle B, e_A\rangle J)' \subseteq A$$

Taking commutants and using the bicommutant theorem, we find

$$\begin{aligned} J\langle B, e_A\rangle J &\supseteq A' \\ JA'J &\subseteq \langle B, e_A\rangle \end{aligned}$$

proving our result. ☐

The picture is the following: We have an inclusion $A \subseteq B$ and a conjugation action

$$\begin{aligned} JAJ &= ?? \\ JBJ &= B' \\ J??J &= A' \end{aligned}$$

the missing algebra is $\langle B, e_A\rangle$, filling out the diagram as

$$\begin{aligned} JAJ &= (\langle B, e_A\rangle)' \\ JBJ &= B' \\ J\langle B, e_A\rangle J &= A' \end{aligned}$$

Here, the inclusions go downwards.

1.3 The New Inclusion

Since $\langle B, e_A\rangle$ is also a von Neumann subalgebra of $B(L^2(B, \mathrm{tr}_B))$, it is a multimatrix algebra and the inclusion $B \subseteq \langle B, e_A\rangle$ can be analyzed in the same way we did last week. For ease of notation, we set $A_2 = \langle B, e_A\rangle$.

So, let's assume $A \subseteq B$ is a unital inclusion of multimatrix algebras, with dimension vectors n_A and n_B and inclusion matrix Λ_A^B. Let $\{p_1, \ldots, p_k\} \subseteq Z(A)$ be minimal central projections in A, and $\{q_1, \ldots, q_\ell\} \subseteq Z(B)$ minimal central projections in B. We'll assume $\mathrm{tr}_A = \mathrm{tr}_B|_A$, so that $\lambda^A = \Gamma_A^B \lambda^B$.

Theorem 1.1. *The inclusion matrix for $B \subseteq A_2$ is the transpose of Γ_A^B.*

Proof. Our author presents three proofs, each with less and less abstract machinery. We'll review two of them.

(Proof 1: Morita Equivalence) We'll use the fact that $A_2 = Be_AB$, which we have not proved (it was a double-star problem...).

Lemma 1.2. *The rings A and A_2 are Morita-equivalent by the Morita equivalence $A_2 - A$ bimodule B.*

Proof. This proof follows from the main theorem of Morita equivalence, which says

Theorem 1.2. *For an $R - S$ bimodule P, the following are equivalent*

1. *There is an $S - R$ bimodule Q such that $P \otimes_S Q \cong S$ and $Q \otimes_R P \cong R$.*

2. *Setting $Q_0 = \mathrm{Hom}_R(P, R)$ the canonical morphism*

$$P \otimes_S Q_0 \to R$$

is an isomorphism of $R - R$ bimodules, and $Q_0 \otimes_R P \cong S$ as $S - S$ bimodules.

3. *P is faithfully flat as an R-module of finite presentation and $S \cong \mathrm{End}_R(P)$ as a right S-module.*

4. *P is a faithfully flat right S-module of finite presentation and $R \cong \mathrm{End}_S(P)$ as a left R-module.*

5. *$P \otimes_S -$ is an equivalence of categories from left S-modules to left R-modules.*

6. *$\mathrm{Hom}_R(P, -)$ is an equivalence of categories from left R-modules to left S-modules.*

Moreover, in case 1 Q is isomorphic to $\mathrm{Hom}(P, R)$ as an $S - R$ bimodule.

proof can be found in e.g. Kashiwara Schapira "Categories and Sheaves" §19.5 (I'm sure there are easier references as well. . .).

Using this, we can establish the Morita equivalence. Take $R = A_2$ $S = A$ and $P = B$ thought of as an $R - S$ bimodule. Since $A_2 \cong Be_AB \cong B \otimes_A B$ we see that P satisfies condition 4, hence furnishes a Morita equivalence. □

Since A_2 and A are Morita-equivalent, they have the same center. Take minimal projections $r_1, \ldots, r_k$ in $Z(A_2)$ which are reflections of $p_1, \ldots, p_k$ under the Morita equivalence.

The matrix entry $(\Lambda_A^B)_{ij}$ counts the number of summands of p_iBq_j as an $A - B$ bimodule. Applying the Morita-equivalence yields

$$B \otimes_A p_iBq_j \cong r_i\langle B, e_A\rangle q_j$$

which (since tensoring with B is an equivalence) has the same number of summands. This is then the same number of summands as the conjugate bimodule $q_j\langle B, e_A\rangle r_i$ which is $(\Lambda_A^B)_{ji} = (\Lambda_A^B)^T_{ij}$ as desired.

(Proof 2 using operator algebra theory) Since $A_2 = JA'J$ the map $z \mapsto JzJ$ is an isomorphism of $Z(A)$ and $Z(A_2)$.

Now, endomorphisms of $p_iBq_j = p_iq_jB$ can be identified with

$$\mathrm{End}_{A-B}(p_iq_jB) \cong (p_iq_jA)' \cap (p_iq_jBp_iq_j)$$

and chasing definitions (using the fact that we're dealing with matrix algebras) we find

$$(\Lambda_A^B)_{ij} = \dim(\mathrm{End}_{A-B}\, p_iq_jB)^{1/2}$$

but the map

$$\begin{aligned} \mathrm{Ad}(J) : B &\to B' \\ b &\mapsto Jb^*J \end{aligned}$$

is (details in Penneys' notes) a $*$-algebra anti-isomorphism from $\mathrm{End}_{A-B}(p_iq_jB)$ to $\mathrm{End}_{B-A_2}(q_jJp_iJB)$. Hence

$$(\Lambda_A^B)_{ij} = \dim(\mathrm{End}_{A-B}(p_iq_jB))^{1/2} = \dim(\mathrm{End}_{B-A_2}(q_jJp_iJB))^{1/2} = (\Lambda_B^{A_2})_{ji}$$

as desired.

□

Finally, let's examine compatible traces. Assume the inclusion $A \subseteq B$ is connected, so that $B \subseteq A_2$ is as well. Then

Theorem 1.3. *The following are equivalent for positive d:*

- $\mathrm{tr}_{A_2}(xe_A) = d^{-2}\,\mathrm{tr}_B(x)$ *for all $x \in B$, and*
- $\lambda^{A_2} = d^{-2}\lambda^A$

Hence we make connection with the Frobenius-Perron theorem, since λ^{A_2} was the Frobenius-Perron eigenvalue.