THE CIRCLES OF APOLLONIUS: A TOUR OF ENUMERATIVE ALGEBRAIC GEOMETRY

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Abstract. The problem of Apollonius is an ancient problem in geometry: given three circles on the plane, can a circle be constructed which is tangent to all three? For some configurations there are no solutions, while for others there are as much as eight solutions that can be constructed. This problem is an example of an *enumerative problem*, where we are interested in counting certain geometric objects or configurations. We will attack Apollonius' problem with some modern techniques from algebraic geometry, and explore some of the rich geometry hiding therein.

This approach to the problem is taken from Eisenbud and Harris "3264 and all that", chapter 2.3. We will not assume any prior knowledge of algebraic geometry or abstract algebra.

1. Statement of the Problem

The classical problem of Apollonius is stated as follows:

> Given three circles in the plane, construct a circle which lies tangent to all three given circles.

and we will be interested in the refinement:

> Given three circles in the plane, how many solutions to Apollonius' problem are there?

There are some special configurations which admit infinitely many solutions: if all three circles are tangent at the same point p, then *any* circle tangent to any one of the circles at p is automatically tangent to all three, and there are infinitely many such circles.

Conversely, there are special configurations which admit no solutions: if one circle lies completely within another, for example, with the third outside, there can be no solution to the problem.

Is there hope for solving this problem? We will see...

2. Background

The strategy for solving this follows the general principle famously described by Alexander Grothendieck:

> I can illustrate the second approach with the same image of a nut to be opened. The first analogy that came to my mind is of immersing the nut in some softening liquid, and why not simply water? From time to time you rub so the liquid penetrates better, and otherwise you let time pass. The shell becomes more flexible through weeks and months—when the time is ripe, hand pressure is enough, the shell opens like a perfectly ripened avocado!

We will solve this problem by slowly building up general theory, slowly raising the water level, until the problem at hand is seen to be rather trivial.

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2.1. **Quadrics in $\mathbb{P}^2$.** Recall that a circle is a special case of a quadric curve (conic section), a plane curve defined by a polynomial equation in x and y of degree 2. So, let's start with some theory on quadrics in the plane.

Definition 2.1. A *quadric curve* in $\mathbb{R}^2$ is the zero-locus of a polynomial of degree two in two variables. A generic equation for a quadric is

$$f(x,y) = ax^2 + bxy + cy^2 + dx + ey + f = 0$$

One useful tool in the study of such polynomial equations is called *homogenization*. To homogenize a polynomial in x and y, we consider a third variable z and add it to each term in such a way that the total degree of each monomial is the same. For example, our generic quadric homogenizes to

$$F(x,y,z) = ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0$$

and we can then dehomogenize by setting $z = 1$. Notice that if (x_0, y_0, z_0) is a solution, then so is any scalar multiple $\lambda(x_0, y_0, z_0)$. Indeed, multiplying each of x, y, z by λ results in an overall factor of λ^2, which can be safely divided to the other side without consequence.

So, points on our homogenized quadric always come in lines: if $f(p) = 0$, then so does $f(\lambda p)$.

Definition 2.2. *Projective two-space*, denoted $\mathbb{RP}^2$, is the space of all lines in $\mathbb{R}^3$ through the origin. Equivalently, it is the space of all points in $\mathbb{R}^3$ except the origin where two points are identified if they are scalar multiples of each other. The line through the point (x, y, z) is denoted by $[x : y : z]$.

We see now that our solutions are really points in $\mathbb{RP}^2$. Any point (x, y) is identified with $[x : y : 1]$ in $\mathbb{RP}^2$, and thus solutions $f(x, y) = 0$ get lifted to solutions $F(x, y, 1) = 0$. There are also new solutions that don't correspond to points in the xy-plane. These are the ones with $z = 0$, and are called *points at infinity*.

2.2. **Geometry of Real Quadrics.** Consider an arbitrary quadric in $\mathbb{RP}^2$ given by

$$ax^2 + bxy + cy^2 + dxz + eyz + fz^2$$

for real constants $a, ..., f$. We will consider two quadrics "the same" if a linear change of coordinates in x, y, z can relate one to the other.

Theorem 2.3. *In $\mathbb{RP}^2$, there are exactly five conics up to linear changes of coordinates.*

Proof. Notice that the quadric equation can be written as

$$X^T Q X$$

for

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, Q = \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix}$$

and by the spectral theorem for symmetric matrices, Q is (orthogonally) diagonalizable. By scaling x, y, z we can further enforce that these diagonal entries are ± 1 or zero. Thus, we have five cases up to overall scaling by -1:

- $Q = \mathrm{diag}(1, 1, 1)$ yielding the equation

$$x^2 + y^2 + z^2 = 0$$

  which has no real solutions. These are the "complex quadrics"

- $Q = \mathrm{diag}(1,1,-1)$ yielding
$$x^2 + y^2 - z^2 = 0$$
which, when $z = 1$, is a circle.
- $Q = \mathrm{diag}(1,1,0)$ yielding
$$x^2 + y^2 = 0$$
or $(x + iy)(x - iy) = 0$. This is a pair of lines with complex slope whose only real point is their intersection at $(0, 0)$
- $Q = \mathrm{diag}(1,-1,0)$ yielding
$$x^2 - y^2 = 0$$
or $(x + y)(x - y) = 0$. This is a pair of lines on the plane.
- $Q = \mathrm{diag}(1,0,0)$ yielding
$$x^2 = 0$$
which is a "double line" along the y-axis.

Alternatively, notice that completing the square allows us to eliminate all the cross-terms xy, xz, yz, yielding
$$ax^2 + cy^2 + fz^2$$
and by rescaling $x \mapsto x/\sqrt{|a|}$, etc. the only invariant is the sign on x, y, z and whether or not it is zero. Hence the previous cases. $\square$

The first two types of conics are the "nondegenerate", or smooth conics. They are characterized by the equation $\det Q \neq 0$. They are then distinguished by whether or not they admit a real solution. The latter three types of conics are the "degenerate" or singular conics. They are characterized by $\det Q = 0$ and, as the name suggests, are not smooth.

2.3. **The Shape of Quadrics.** The topology of quadrics is actually quite easy to compute. We have seen that we can reduce any arbitrary smooth quadric (which has real points...) to the standard quadric
$$x^2 + y^2 - z^2 = 0$$
hence all smooth quadrics in $\mathbb{P}^2$ are isomorphic, so we just need to find one we understand well to get its topology.

The Veronese map comes to our rescue here. Consider the map
$$\mathbb{P}^1 \to \mathbb{P}^2$$
$$[s : t] \mapsto [s^2 : st : t^2]$$
which is injective and smooth. Away from $t = 0$, it defines a map from the line $\mathbb{A}^1$ to the plane $\mathbb{A}^2$ of the form
$$x := s/t \mapsto (x^2, x)$$
which is a parameterization of the standard parabola. Away from $s = 0$ we get similar behavior. Thus, this map defines an isomorphism between $\mathbb{P}^1$ and the quadric in $\mathbb{P}^2$ defined by
$$xz = y^2$$
This is a smooth quadric, and by the observation that all smooth quadrics are related by a linear change of coordinates, we find that all quadrics look like a $\mathbb{P}^1$ embedded in $\mathbb{P}^2$. The shape of $\mathbb{P}^1$ is quite easy to understand. Over the reals, $\mathbb{RP}^1$ is simply $\mathbb{R} \cup \{\infty\} \cong S^1$ the circle, and over the complex numbers $\mathbb{CP}^1$ is simply $\mathbb{C} \cup \{\infty\} \cong \S^2$ the two-sphere. Hence:

Theorem 2.4. *All smooth quadrics in $\mathbb{RP}^2$, if nonempty, have the topological type of a circle. All smooth quadrics in $\mathbb{CP}^2$ have the topological type of a two-sphere.*

2.4. **The Space of All Quadrics.** Notice that we can think of the coefficients $(a, b, \ldots, f)$ of our standard equation as identifying a quadric curve in $\mathbb{R}^2$. However, scaling all the coefficients by λ has no effect on the resulting solution set. Thus, we see that

Theorem 2.5. *Quadric curves in $\mathbb{RP}^2$ are in one-to-one correspondence with points in $\mathbb{RP}^5$. A quadric curve given by the equation*

$$ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0$$

is represented by the point $[a : b : c : d : e : f]$ in $\mathbb{RP}^5$.

Thus, we can think of $\mathbb{RP}^5$ as the *parameter space*, or *moduli space* of plane quadrics in $\mathbb{RP}^2$. As we have seen, this space decomposes into five parts. Two of them (the smooth conics) form an open set in $\mathbb{RP}^5$ as the complement of $\det Q = 0$, and the other three lie on the hypersurface $\det Q = 0$. There is a further stratification of this hypersurface into the relatively open set of pairs of lines, and the codimension one (in the hypersurface) space of double lines.

2.5. **Intersections of Quadrics.** We now motivate, in part, why we might want to pass to the complex numbers. First, recall the fundamental theorem of algebra:

Theorem 2.6. *A polynomial of degree n over the complex numbers has exactly n roots, counted with multiplicity.*

This of course is not true over the reals, as we have seen with our complex quadrics. Multiplicity here counts how many times the root shows up in the factorization. The canonical example of a root of degree n is $x = 0$ in $f(x) = x^n$.

We can think of this as computing the intersection of the curve defined by the polynomial with the x-axis $y = 0$. A point of intersection with multiplicity ≥ 2 is a point of *tangency* of the intersection, since not only does the curve intersect $y = 0$, but it intersects it tangentially.

This theorem is generalized in the theorem of Bézout:

Theorem 2.7. *The intersection of a curve of degree n with a curve of degree m, if finite, is $m \times n$ points, counted with multiplicity. These may include points at infinity in $\mathbb{CP}^2$.*

And in our case this becomes

Corollary 2.8. *Two quadric curves intersect at four points, counted with multiplicity and allowing for complex intersection points as well as intersections at infinity.*

2.6. **Circles are Quadrics of a Special Type.** How, then, do we identify circles out of all the quadrics? In the xy-plane, a circle is described by an equation of the form

$$(x - x_0)^2 + (y - y_0)^2 - r^2 = 0$$

and, putting this into standard form, we see that there are two things that must hold for a quadric to be a circle. First, the cross-term bxy must vanish, so $b = 0$. Second, the coefficients of x^2 and y^2 must match, so $a = c$. Passing to the complex numbers, this has a very simple description:

Theorem 2.9. *Let $f(x, y, z)$ be a homogeneous polynomial of degree two, describing a quadric curve. Then, $f(x, y, z)$ is a circle if and only if the circle points*

$$\circ_+ = [1 : i : 0]$$
$$\circ_- = [1 : -i : 0]$$

lie on f.

Proof. Substituting the two points into the general equation yield

$$a + ib - c = 0$$
$$a - ib - c = 0$$

from which we determine $b = 0$ and $2a - 2c = 0$, hence the claim. $\square$

Definition 2.10. The *parameter space of real circles* is the linear subspace $\mathbb{RP}^3$ inside $\mathbb{RP}^5$ defined by the equations $a = c$ and $b = 0$. The *parameter space of complex circles* is the linear subspace $\mathbb{CP}^3$ inside $\mathbb{CP}^5$ defined by $a = c$ and $b = 0$.

Notice that this includes a few degenerate cases. For example, the union of a general line with the line through $\circ_+$ and $\circ_-$ is, technically, a circle, but on the plane it appears as a line. Conversely, we could consider the union of two lines L and M with $\circ_+$ in L and $\circ_-$ in M. In the latter case, simple algebra shows that the two lines L and M have imaginary slope, hence they contain only one real point each. We could also consider the double line at infinity, which has no (real or complex) points in the plane. Let's keep this in the back of our heads.

2.7. **Passage to Complex Numbers.** This parameter space is closely related to a bigger parameter space when we allow for complex-valued coefficients. This is called $\mathbb{CP}^3$, or simply $\mathbb{P}^3$. Bézout's theorem suggests that passage to the complex numbers may make our life easier, and we do so now.

Aside: The relationship between $\mathbb{RP}^n$ and $\mathbb{CP}^n$ is a bit subtle. There are two issues at play: first, if V is our $n+1$-dimensional vector space for which $\mathbb{RP}^n = \mathbb{P}(V)$, then $\mathbb{CP}^n$ is naturally a quotient of $V_{\mathbb{C}} = V \otimes \mathbb{C}$ its complexification. However, in the first case $\mathbb{RP}^n$ is a quotient of V by $\mathbb{R}^\times$, whereas $\mathbb{CP}^n$ is a quotient of $V_{\mathbb{C}}$ by the larger group $\mathbb{C}^\times \cong \mathbb{R}^\times \oplus U(1)$. V naturally sits inside $V_{\mathbb{C}}$, but then $\mathbb{RP}^n$ really sits inside $V_{\mathbb{C}}//\mathbb{R}^\times$, which is a circle bundle over $\mathbb{CP}^n$. This circle bundle is the associated frame bundle of the tautological line bundle $\mathcal{O}(1)$ over $\mathbb{CP}^n$, and so $\mathbb{RP}^n$ lives in the (generalized) Hopf fibration over $\mathbb{CP}^n$.

Thus, our circles are secretly complex circles (which are surfaces in the four-dimensional space $\mathbb{P}^2$) and our projective spaces are all complex (and have twice the dimension as their real counterparts). Remarkably, it actually makes more sense to talk about the "complex" dimension of a space, which is half its real dimension. A "curve" in $\mathbb{P}^2$ (read: a surface in the four-dimensional space $\mathbb{P}^2$) is still the zero of a single polynomial, since one complex polynomial is secretly two real polynomials (take the real and imaginary parts).

3. The Space of Tangent Circles

3.1. **Configurations of Intersection Points.** If we take two circles C and D and intersect them, Bézout's theorem guarantees that they intersect at four points, counted with multiplicity. We know two of them already: $\circ_+$ and $\circ_-$. Let's call the other two p and q. There are a few possibilities for where the other two intersection points are:

- p and q are distinct points away from the line at infinity, in which case the two circles intersect at two distinct points in the plane. This is the generic configuration.
- $p = q$ and they are a point away from the line at infinity, in which case the two circles intersect tangentially at $p = q$.
- One of the points, say p, is on the line at infinity and not equal to $\circ_\pm$. Then both C and D contain the three points $\circ_+, \circ_-, p$ on the line at infinity. By Bézout's theorem, this implies that the whole line at infinity is contained in both C and D and C and D are actually lines in the plane. They intersect at q.
- Both points are on the line at infinity, in which case C and D again are lines, but now do not intersect on the plane.

- The circles intersect at $\circ_\pm$ each with multiplicity two. Some basic algebra shows that this is the case of the two circles having the same center. If $C = D$, then they intersect everywhere with multiplicity two. If $C \neq D$, these are concentric circles.
- The circles intersect at one of the circular points, say $\circ_+$, with multiplicity two. Not both C and D can be circles in the plane, then, since they would intersect twice or not at all. Rather, either both are lines in the plane or one is a point and the other a point, line, or circle. They cannot both be plane lines since they must intersect again at $\circ_+$, and any line containing $\circ_+$ has only a point in the real plane.
- The circles intersect at one of the circular points with multiplicity three. These circles don't intersect at all on the plane. One such configuration is given by C being the line at infinity plus a line L_1, and D being the line at infinity plus a line L_2 with L_2 intersecting $\circ_+$. Thus L_2 has a single point on the real plane.

3.2. **The Incidence Correspondence.** Recall that $\mathbb{P}^3$ is the configuration space of circles in the plane. Let us now consider the *incidence correspondence* for a smooth circle D

$$\Phi_D = \{(r, C) \in D \times \mathbb{P}^3 \mid C \text{ intersects } D \text{ tangentially at } r\} \subseteq D \times \mathbb{P}^3$$

and for $r = \circ_\pm$ we require the intersection multiplicity be 3. What sort of structure does this set have? For any r, then, there are two linear conditions on the coefficients of C that must be satisfied (for r not a circle point, C must contain r and also the tangent line to C at r must equal the tangent line to D at r). Thus, for a fixed r the fiber Φ_r is a $\mathbb{P}^1$ inside the parameter space $\mathbb{P}^3$. One may now think of Φ as locally a product $D \times \mathbb{P}^1$, hence Φ is irreducible of dimension 2.

Consider the other projection from Φ onto $\mathbb{P}^3$. Generically, a circle will intersect D tangentially at exactly one point, so for a dense open subset of the image of Φ the map is invertible (such a map is called *birational*). Hence, the image of Φ is also irreducible and of dimension 2. Let us call this

$$Z_D = \pi_{\mathbb{P}^3} \Phi_D$$

which is the surface of circles which lie tangent to D at some point. If we understand the geometry of Z_D, we will understand the problem.

3.3. **The Geometry of Z_D.** The variety Z_D is a surface in $\mathbb{P}^3$, and the most useful thing we can know about it is its degree.

Definition 3.1. The *degree* of a (hyper)surface in $\mathbb{P}^3$ ($\mathbb{P}^n$) is the degree of the homogeneous polynomial defining it.

Equivalently, by Bézout's theorem, the degree of a hypersurface is the number of points of intersection (with multiplicity) of the intersection of the surface with a general line. With this definition at hand, we seek to prove

Theorem 3.2. *For any smooth circle D, the variety Z_D is a quadric (degree two) surface in $\mathbb{P}^3$.*

Proof. To prove this, we intersect Z_D with a general line and count the number of intersection points.

So, let us consider a line $L \subseteq \mathbb{P}^3$ corresponding to a collection of circles $\{C_t\}$ for $t \in \mathbb{P}^1$. Let f and g be the defining polynomials for C_0 and C_∞, so that C_t is defined by

$$F_t = uf + sg$$

with $s/u = t$ (understanding that the case $u = 0$ yields C_∞). The intersection of L with Z_D occurs at the circles who lie tangent to D.

Now, consider the function f/g restricted to D. Since f and g are degree two, this function has two zeroes and two poles (a zero and a pole may occur at the same point and "cancel out", which breaks this argument, but generally this does not happen). This function defines a map

$$f/g : D \to \mathbb{P}^1$$

which is generically two-to-one.

Let us examine this map in more detail. Recall that $F_t = uf + sg$ defines the circle C_t, so the zeroes of F_t correspond to points of intersection of C_t with D. However, the zeroes of F_t occur at values where

$$\begin{aligned} 0 &= uf + sg \\ -sg &= uf \\ -s/u &= f/g \end{aligned}$$

i.e. where $f/g = -t$.

So, the map f/g sends a point d in D to the t-value of the circle in the line L that intersects D at d. This explains why the map is two-to-one generally: most circles intersect D twice!

There are points, however, where the map f/g may fail to be two-to-one and instead be one-to-one. These are called *ramification points* and their t-values correspond to circles which meet D tangentially. We can count these on purely topological grounds using the famous Riemann-Hurwitz formula.

The idea is as follows: the Euler characteristic of a surface behaves very nicely under covering maps. If $p : X \to Y$ is a n-fold covering, then

$$\chi(X) = n\chi(Y)$$

However, if there are ramification points, these contribute negatively to the Euler characteristic. Specifically, each ramfication point subtracts $n-1$ from the Euler characteristic (n points collapse to 1 point). Thus,

$$\chi(X) = n\chi(Y) - \sum_{p \text{ ramified}} (n-1)$$

Putting it all together, we have the following: $\mathbb{P}^1$ and D have Euler characteristic $\chi(\mathbb{P}^1) = \chi(D) = 2$ and so the Riemann-Hurwitz formula with $n = 2$ reads

$$2 = 4 - \sum_{p \text{ ramified}} (1)$$

from whence we find there are two ramified points on this map.

The result is now immediate: the general line L intersects Z_D at exactly two points. Hence, Z_D is a quadric. □

We can actually say a little bit more about Z_D. Consider a pencil of circles passing through D and some other circle $C \in Z_D$. Each element of this pencil intersects D tangentially, hence the whole line is contained in Z_D. This works for any circle $C \in Z_D$, and so Z_D is actually the (projective) cone over a quadric curve with vertex D.

4. Conclusion of the Argument

We now approach the classical problem of Apollonius with these tools under our belt. Let D_1, D_2, D_3 be circles, and consider now the incidence correspondence

$$\Psi = \{(D_1, D_2, D_3, C) \in \mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3 \mid C \text{ lies tangent to each } D_i\}$$

which tracks solutions to the problem. If we examine the projection to the last factor, we see that the fiber over some C is $Z_C^3 \subseteq (\mathbb{P}^3)^3$ which is of dimension six. Thus, the dimension of Ψ is nine (six fiber dimensions, three transverse dimensions from the $\mathbb{P}^3$ we projected onto).

Now consider projection $\pi : \Psi \to (\mathbb{P}^3)^3$ to the first three factors. By Chevalley, since this map is proper the fiber dimension is upper-semicontinuous. That is, the set of fibers of dimension 1 or greater is closed in the image of π. However, I can exhibit a configuration (D_1, D_2, D_3) which has a finite number of solutions, hence there is a nonzero (=dense) open set for which the fiber of π is of dimension zero. Thus, the fiber over (D_1, D_2, D_3) is generically a finite collection of points (the solutions C). How many are there?

Well, a solution to a configuration (D_1, D_2, D_3) corresponds to an intersection point in $Z_{D_1} \cap Z_{D_2} \cap Z_{D_3}$ since it must lie tangent to all three of these quadrics. However, Bézout's theorem guarantees that these three quadrics intersect at exactly $2^3 = 8$ points with multiplicity. Are all eight points distinct, or are they generically separate?

One can compute the tangent spaces of the Z_{D_i} directly and show that they do, in fact, intersect transversally generally. The argument is essentially that the tangent space to Z_D at some C is the plane of all circles containing the point $C \cap D$, and for a general configuration these planes will all be in general position.

Finally, we notice the following surprising property of π. Notice that, as a map between two projective varieties, this map is projective. Hence, it is a closed map and the image $\pi(\Psi)$ of Ψ is closed in $(\mathbb{P}^3)^3$. However, we just saw that the generic fiber of this map is of dimension zero. Thus, the image of π is a closed set in $(\mathbb{P}^3)^3$ which contains a dense open set. It therefore must be all of $(\mathbb{P}^3)^3$, and hence π is surjective! Thus, all problems of Apollonius admit solutions.

5. Real Solutions?

We find that generically we have eight complex solutions to the problem of Apollonius. How many of these are real? This question is generally hard to answer, but here is a sketch:

5.1. **Warm-Up: Real Solutions to Quadratic Equations.**

Consider a general quadratic equation in one variable:

$$Ax^2 + Bx + C = 0$$

The solutions to this follow the quadratic equation, and are described by

$$x = \frac{-B \pm \sqrt{\Delta}}{2A}$$

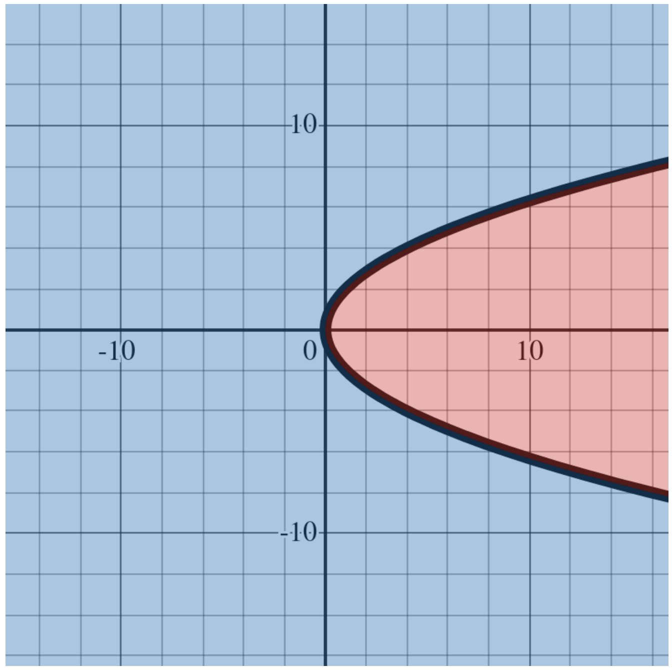
where

$$\Delta = B^2 - 4AC$$

is the *discriminant* of the quadratic. In the parameter space $\mathbb{P}^2$ of all quadrics, this is a quadraic curve. Dehomogenizing by setting $A = 1$ yields the parabola

$$b^2 - 4c$$

in this space.



The discriminant locus on the bc-plane. Blue is $\Delta > 0$ and Red is $\Delta < 0$

When $\Delta > 0$, two distinct real solutions emerge. When $\Delta = 0$, the two solutions coincide to one real double solution. Finally, when $\Delta < 0$ the two solutions become complex conjugate pairs. Thus, the curve $\Delta = 0$ separates the bc plane into two regions which track how many real solutions the polynomial has.

In higher degrees, the discriminant is still defined but is less explicit. If $f(x)$ is a monic polynomial of degree d with roots $r_1, \ldots, r_d$ (possibly repeated), then the discriminant of f is

$$\Delta(f) = \prod_{i<j}(r_i - r_j)^2$$

which is (by some Galois theory) a polynomial function of the coefficients of f. This is zero precisely when f has a repeated root. The zero set $\Delta = 0$ describes a hypersurface in the parameter space $\mathbb{P}^d$ of degree d polynomials, and divides $\mathbb{P}^d \setminus \Delta$ into connected components.

Lemma 5.1. *The number of real solutions is constant on each connected component of* $\mathbb{P}^d \setminus \Delta$.

Proof. The coefficients of f are polynomial functions in the roots of f, hence the correspondence sending a set of roots to the set of coefficients defining them is continuous. Now consider a point in some connected component of $\mathbb{P}^d \setminus \Delta$ corresponding to k real roots $r_1, \ldots, r_k$ and $d-k$ complex roots (necessarily an even number).

Consider a path in $\mathbb{P}^d$ which leaves $k-2$ roots real and turns 2 real roots, say r_1 and r_2 into complex conjugate roots. Restricting the deformation to the roots r_1 and r_2 this path has the following property: if r_1 and r_2 are ever complex, then $r_1 = \bar{r_2}$. Thus, since r_1 and r_2 move continuously in $\mathbb{C}$, we see that necessarily for them to leave the real line they must have the same real part. Hence, this deformation passes through a point where $r_1 = r_2$. Here, $\Delta = 0$. $\square$

Finally, we arrive at a conjectural following description of the number of real solutions to Apollonius' problem:

Conjecture 5.2. *The number of real points in the intersection* $Z_{D_1} \cap Z_{D_2} \cap Z_{D_3}$ *has the following geometric description: There is a wall-and-chamber decomposition of* $(\mathbb{P}^3)^3$ *the configuration space of three circles for which the number of real solutions to Apollonius' problem is constant on each* *chamber. In particular, the number of real solutions is constant on a (Euclidean) open set in* $(\mathbb{P}^3)^3$ *around a given generic configuration.*

Proof. Actually, this is not quite clear to me. Here is a sketch of the proof. We will say that a point $[a_0, \ldots, a_n]$ in $\mathbb{P}^n$ is *real* if there is some $\lambda \in \mathbb{C}^\times$ for which the points $\lambda a_0, \ldots, \lambda a_n$ are real ($a_i = \bar{a}_i$). Real points in $\mathbb{P}^3$ correspond to equations for real circles on the real plane $\mathbb{R}^2$. Thus, our problem is to determine how many of the eight points in the intersection of the three quadrics Z_{D_i} satisfy this reality condition.

The first thing to prove is that since these eight points lie in the intersection of three real quadrics, the set of points is invariant under conjugation. The problem is that conjugation is not well-defined on $\mathbb{P}^n$ since

$$[\lambda \bar{a_0}, \ldots, \lambda \bar{a_n}] \neq \left[\bar{\lambda} \bar{a_0}, \ldots, \bar{\lambda} \bar{a_n}\right]$$

□