D stands for Derived: How D-Branes Build the Derived Category

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Although the derived category is well-known to algebraic geometers and commutative algebraists, only recently did it appear in the context of physics. For the next two weeks we will be reviewing Aspinwall and Lawrence's paper "Derived Categories and Zero-Brane Stability" ([AL01]) letting them guide us through the relevant string theory. This will provide motivation for why the derived category is expected to show up in the homological mirror symmetry conjecture, and may hint at additional structure not measured in $D^b(X)$.

Since no physics knowledge is assumed, we'll spend the first part of the talk reviewing the relevant physics background. Following Aspinwall and Lawrence's wisdom, we then turn our attention to Witten's "topological B-model", which computes Dolbeault cohomology. After introducing open strings, we find that vector bundles (and even coherent sheaves) enter the picture, and the structure of our physics naturally forms that of a category. Time permitting, we'll finish the day studying possible ways to deform the theory, in preparation for next week.

1 Background

1.1 Physical Theories

For our purposes, we adopt the following definition

Definition 1.1.1. A *(quantum) physical theory* is a $\mathbb{C}$-algebra A of operators subject to various consistency constraints, along with a linear form $\langle - \rangle$ which induces a nondegenerate bilinear pairing $\langle a, b \rangle = \langle ab \rangle$. The linear form evaluated on a product of n elements is called the *n-point function* for the theory.

This comes from an alternative (and more modern) approach to quantum mechanics, in which a state $|\psi\rangle$ is given by creation from a vacuum $\psi|0\rangle$ for some operator $\psi \in A$. The n-point functions are expectation values against the vacuum

$$\langle ab \dots d \rangle = \langle 0 | ab \dots d | 0 \rangle$$

Two physical theories A and B are said to be *isomorphic* if there is an isomorphism of algebras $A \to B$ which preserves the n-point functions.

Definition 1.1.2. A *model* of a physical theory is a method of generating the operator algebra and the n-point functions.

One large class of models is given by quantization of Lagrangian field theories, which we review:

Definition 1.1.3. A *Lagrangian field theory* is the data of

- A *spacetime* M
- A collection of *fields* on M, specified by choosing vector bundles over M with connection. Fields will be sections of tensor bundles, or connections on principal bundles, or sections of associated bundles.
- An *action* functional, which takes in a configuration of fields (choices of sections) and returns a number. The action is usually given as the integral over a density $S = \int_M \mathcal{L}$, and the density is the Lagrangian for the theory.

The Lagrangian then completely determines the dynamics of the fields, as the solution, or the equations of motion for the fields is given by minimizing the action.

In quantum, we promote our fields to take values in some operator algebra $B(H)$, and we can define the n-point functions in terms of the action S as

$$\langle \varphi_n(x_n) \dots \varphi_1(x_1) \rangle = N \int D\varphi_1 \dots D\varphi_n \varphi_n(x_n) \dots \varphi_1(x_1) \exp(iS[\varphi_1, \dots, \varphi_n])$$

The key point here is that *the Lagrangian completely determines the n-point functions.* So, proving properties about the observables algebra can reduce to proving things about the Lagrangian.

1.2 Closed vs. Open Strings

In string theory, our field theory takes place on the worldsheet of the string. This theory has two sectors, corresponding to whether or not the string endpoints join up. The sector corresponding to closed strings is the *closed sector* and the other is the *open sector*. One can have a string theory with only a closed sector, but once one has an open sector, the closed sector comes automatically (an open string can dynamically join its endpoints together, forming a closed string, but the opposite cannot happen).

In topological field theories such as the one we are about to examine, the operator algebras for these sectors gain a particularly nice structure. The closed sector is a *commutative Frobenius algebra*, a $\mathbb{C}$-algebra with a faithful state $\theta : A \to \mathbb{C}$ which induces a nondegenerate pairing

$$\theta \circ \mu : A \times A \to \mathbb{C}$$

Examining these in detail leads us down the path of planar algebras.

The open strings, on the other hand, have a rich structure. If we consider the string boundaries to be labeled (interpret this as the endpoints couple to distinct objects, and must stay coupled to a unique object) then the operators corresponding to strings stretching from boundary A to boundary B form a vector space $\mathcal{O}_{AB}$ (not an algebra, we can't join an A boundary to a B boundary!) and for each triple A, B, C we get a pairing

$$\mathcal{O}_{AB} \otimes \mathcal{O}_{BC} \to \mathcal{O}_{AC}$$

which describes two strings with endpoints on B joining. This collection of vector spaces satisfies a bunch of consistency conditions, outlined in [Asp+09] §2.1.

2 Our Main Model: the Nonlinear Sigma Model

One such field theory is the nonlinear sigma model.

2.1 The Physical Model

Suppose X is a Calabi-Yau manifold, smooth over $\mathbb{C}$ (later we may consider singular X, but not now), sometimes assumed to have dimension 3 to make string theory work. We will also include the additional data of the B-field, a real, closed 2-form on X which combines with the Kähler form J to make the complexified Kähler form $B + iJ$.

The nonlinear Σ-model on X is given by

$$\begin{aligned} S\left[\Sigma, f, \psi\right] = \int_{\Sigma} \Big(&\frac{1}{2}(g_{IJ} + iB_{IJ})\partial_z x^I \partial_z x^J \\ &+ \frac{i}{2} g_{i\bar{i}} \psi_-^{\bar{i}} D_z \psi_-^i + \frac{i}{2} g_{i\bar{i}} \psi_+^{\bar{i}} D_z \psi_+^i \\ &+ (R_{i\bar{i}j\bar{j}} \psi_+^i \psi_+^{\bar{i}} \psi_-^j \psi_-^{\bar{j}})(idz \wedge d\bar{z})\Big) \end{aligned}$$

where the fields are taking values in

Field	Bundle
ψ_+^i	$\sqrt{K} \otimes f^*(T_X^{(1,0)})$
$\psi_+^{\bar{i}}$	$\sqrt{K} \otimes f^*(T_X^{(0,1)})$
ψ_-^i	$\overline{\sqrt{K}} \otimes f^*(T_X^{(1,0)})$
$\psi_-^{\bar{i}}$	$\overline{\sqrt{K}} \otimes f^*(T_X^{(0,1)})$

for a square root of the canonical bundle specified. Here f is a map from a Riemann surface Σ in to X with components x^I.

2.2 Topological Twisting

Note that the first term is essentially the integral of the complexified Kähler form pulled back to Σ, and the last two terms ensure conformal invariance and $N = 2$ supersymmetry. This means that there are two linearly independent infinitesimal deformations that mix bosons (x^I) and fermions (ψ), and the action is invariant under conformal transformations on Σ.

Topological twisting changes the bundles the fermions come from.

Field	Untwisted Bundle	A-Twist	B-Twist
ψ_+^i	$\sqrt{K} \otimes f^*T_X^{(1,0)}$	$f^*T_X^{(1,0)}$	$K \otimes f^*T_X^{(1,0)}$
$\psi_+^{\bar{i}}$	$\sqrt{K} \otimes f^*T_X^{(0,1)}$	$K \otimes f^*T_X^{(0,1)}$	$f^*T_X^{(0,1)}$
ψ_-^i	$\overline{\sqrt{K}} \otimes f^*T_X^{(1,0)}$	$\overline{K} \otimes f^*T_X^{(1,0)}$	$\overline{K} \otimes f^*T_X^{(1,0)}$
$\psi_-^{\bar{i}}$	$\sqrt{K} \otimes f^*T_X^{(0,1)}$	$f^*T_X^{(0,1)}$	$f^*T_X^{(0,1)}$

After twisting, the A-model loses its dependence on the complex structure of X, and only depends on the metric and B-field. Conversely, the B-model loses its dependence on the metric and B-field, and only depends on the complex structure of X.

2.3 Operators in the Twisted Theory

We can examine the B model in more detail: the fields

$$\eta^{\bar{i}} = \psi_+^{\bar{i}} + \psi_-^{\bar{i}}$$

and

$$\theta_i = g_{i\bar{i}} \left(\psi_+^{\bar{i}} - \psi_-^{\bar{i}} \right)$$

are worldsheet scalars, and the fields

$$\rho^i = \psi_+^i + \psi_-^i$$

are worldsheet 1-form (with $(1,0)$ and $(0,1)$ components split as above) with values in $f^*T_X^{(1,0)}$. Taking products of these allows us to associate to each $(0,q)$ form (on Σ) with values in $\bigwedge^p T_X^{(1,0)}$ a local operator built from the fields. Namely, for $\theta \in \Omega^{(0,q)}(X, \bigwedge^p T_X^{(1,0)})$ given locally as

$$\theta = h^{j_1 \dots j_p}_{\bar{i}_1 \dots \bar{i}_q} d\bar{z}^{\bar{i}_1} \wedge \cdots \wedge d\bar{z}^{\bar{i}_q} \otimes \partial_{j_1} \wedge \cdots \wedge \partial_{j_p}$$

we can associate the operator $\mathcal{O}_\theta$ defined locally as

$$\mathcal{O}_\theta = h^{j_1 \dots j_p}_{\bar{i}_1 \dots \bar{i}_q} \eta^{\bar{i}_1} \dots \eta^{\bar{i}_q} \theta_{j_1} \dots \theta_{j_p}$$

2.4 BRST Operators

The surviving supersymmetry Q is given by

$$\begin{aligned}
\delta x^i &= 0 \\
\delta x^{\bar{i}} &= i\alpha\eta^{\bar{i}} \\
\delta\psi_+^i &= -\alpha\partial x^i \\
\delta\psi_+^{\bar{i}} &= -i\alpha\psi_-^{\bar{j}}\Gamma^{\bar{i}}_{\bar{j}\bar{k}}\psi_+^{\bar{k}} \\
\delta\psi_-^i &= -\alpha\bar{\partial} x^i \\
\delta\psi_-^{\bar{i}} &= -i\alpha\psi_+^{\bar{j}}\Gamma^{\bar{i}}_{\bar{j}\bar{k}}\psi_-^{\bar{k}}
\end{aligned}$$

where α is some infinitesimal parameter specifying the deformation (an infinitesimal section of $K^{-\frac{1}{2}}$). To say that Q is given by the above means that Q generates the deformation in the sense that

$$\delta W = -i\left\{Q(\alpha), W\right\}$$

for any operator W.

One can easily verify that $Q^2 = 0$ (potentially up to the EOM for the fields i.e. "on-shell"). This Q is known as the *BRST Operator*, and its existence makes computations much easier. It is standard lore that

1. The physical theory is isomorphic to the Q-cohomology $\ker(Q)/\operatorname{Im}(Q)$
2. Variations of the Lagrangian that are Q-exact do not change the theory.

one can then verify that varying the metric, Kähler form, and B-field are all Q-exact variations, and as promised we get a theory that is topological and independent of the symplectic structure.

To compute the Q-cohomology, observe that

$$\delta\mathcal{O}_\theta = \{Q, \mathcal{O}_\theta\} = \mathcal{O}_{\bar{\partial}\theta}$$

so that this sector's operator algebra corresponds to the total Dolbeault cohomology $H^*_{\bar{\partial}}(X, \bigwedge^* T_X^{(1,0)})$ with symmetry operator $\bar{\partial}$.

3 The Open String Theory

3.1 Adding Boundary Conditions

The next step in our journey is to define appropriate *boundary conditions* we can impose on the string. This amounts to adding a term to the action

$$S_{C_k} = \oint_{C_k} \left(f^*(A) - i\eta^{\bar{i}} F_{\bar{i}j}\rho^j\right)$$

where A is the connection 1-form for the vector bundle E associated to a particular boundary condition, and F is its field strength. In this case we are slightly generalizing to allow the boundary of Σ to carry a gauge field associated to E.

In order to make this boundary term invariant under Q, we find that A is a holomorphic connection. Hence, most generally the boundary conditions allowed are holomorphic vector bundles. In fact, we'll see that we can allow these vector bundles to be supported on submanifolds, and we might most generally take coherent sheaves (with connection) to be the boundary conditions.

In this case, the field strength tensor $F = DA = dA + A \wedge A$ can be used to write the θ operators in terms of the η operators

$$\theta_j = F_{j\bar{k}}\eta^{\bar{k}} \in \Omega^{0,q} \otimes \mathrm{End}(E)$$

and so open string operators are given by the Dolbeault cohomology ring $H^{(0,p)}_{\bar{\partial}}(X, \mathrm{End}(E)) = H^p(X, \mathrm{End}(E))$. The index p is called the *ghost number* associated to the open string operator. For those in the know, the ghost number describes the R-charge associated to the string.

3.2 Adding a Grading

In the general theory, D-branes are graded by their central charge (more precisely, the phase of the central charge) and a consistent theory which preserves supersymmetry must have all these charges aligned mod integral shifts. Hence, D-branes are expected to carry an integral grading.

We introduce this artificially by demanding that our vector bundle splits as a direct sum

$$E = \bigoplus_n E^n$$

(one can also arrive at this conclusion using Chan-Paton factors), and that string endpoints couple to a single graded part. This means that open string operators live in

$$H^{(0,p)}_{\bar{\partial}}(X, (E^m)^\vee \otimes E^n)$$

With this new grading, the ghost number gets shifted, so that the new ghost number is $p-m+n$.

3.3 Adding Dirichlet Conditions

Analyzing the conditions necessary for the bosonic fields (the coordinate fields) to be defined in the open sector, we find that in a suitable basis some of the coordinates receive Dirichlet conditions, while others receive Neumann conditions. The directions that receive Dirichlet conditions are interpreted to be fixed, and the result is that the string endpoint is constrained to the submanifold spanned by the directions in which the string still has Neumann conditions.

The D-brane describing the boundary condition now has extra structure: it is a vector bundle *supported over a submanifold*. Holomorphicity constrains this submanifold to be a complex submanifold.

As of yet (as far as I am aware) the jump from "vector bundles supported on holomorphic submanifolds" to coherent sheaves is not entirely justified physically, save the fact that no other natural candidate for objects representing D-branes has appeared. This is discussed for a bit in [Asp+09] §5.3.3.3 but we will not dwell on it further.

3.4 Forming a Category

From now on, we shift to the language of coherent sheaves, and allow our D-branes to be represented by elements of $\mathrm{Coh}(X)$. An open string from boundary condition $\mathscr{E}^m$ to boundary condition $\mathscr{F}^n$ lives in

$$\varphi \in H^{(0,p)}_{\bar{\partial}}\left(X, (\mathscr{E}^m)^\vee \otimes \mathscr{F}^n\right) \cong \mathrm{Ext}^p(\mathscr{E}^m, \mathscr{F}^n)$$

As mentioned before, there should be a well-defined "concatenation map" for strings between triples of objects, and this is given by the Yoneda pairing

$$\mathrm{Ext}^p(\mathscr{E}^m, \mathscr{F}^n) \otimes \mathrm{Ext}^q(\mathscr{F}^n, \mathscr{G}^\ell) \to \mathrm{Ext}^{p+q}(\mathscr{E}^m, \mathscr{G}^\ell)$$

What we have assembled is a *category*, which we will denote as $T_{\mathrm{init}}(X)$. It is defined by

- Objects of $T(X)$ are graded coherent sheaves on X (not necessarily every coherent sheaf, just the ones we use to represent the D-branes present in our model)
- Morphisms from $\mathscr{E}^m$ to $\mathscr{F}^n$ are given by the open string operators, i.e.

$$\mathrm{Hom}_{T_0(X)}(\mathscr{E}^m, \mathscr{F}^n) = \bigoplus_p \mathrm{Ext}^p(\mathscr{E}^m, \mathscr{F}^n)$$

The axioms of a category are easily verified. Since this category completely describes the open string sector of the topological theory, we see that *The category $T_{init}(X)$ completely describes the B-model with prescribed D-branes.*

4 Introduction to Deformations

Suppose that our boundary conditions are just a finite collection of vector bundles assembling into a single graded vector bundle

$$\mathscr{E} = \bigoplus_n \mathscr{E}^n$$

As noted, a string of ghost number q lies in

$$\varphi_q \in \bigoplus_{k,m} \mathrm{Ext}^k(\mathscr{E}^m, \mathscr{E}^{m-k+q})$$

Ghost number one operators can be used to deform the theory. For φ a ghost number one operator, adding to the action a term

$$\delta_\varphi S = t \oint_{C_k} \{G, \varphi\}$$

works. The effect of this (as [AL01] argues in §2.4) is that the BRST operator Q gets shifted by

$$Q \mapsto Q + \varphi$$

If $\varphi \in \mathrm{Ext}^1(\mathscr{E}^n, \mathscr{E}^n)$, then φ represents a first-order deformation of the coherent sheaf $\mathscr{E}$. Hence, deformations of this type correspond to "internal" deformations of the boundary conditions inside $\mathrm{Coh}(X)$.

If $\varphi \in \mathrm{Ext}^0(\mathscr{E}^n, \mathscr{E}^{n+1})$ on the other hand, we get something new. Generally, we can take φ to be a sum of such maps

$$\varphi = \sum_n d^n$$

where $d^n \in \mathrm{Hom}(\mathscr{E}^n, \mathscr{E}^{n+1})$. Since Q is nilpotent, consistency of this requires

$$Q^2 = (Q_0 + \delta Q)^2 = 0$$

or

$$Q_0^2 + (\sum_n d^n)^2 + \{Q_0, \varphi\} = 0$$

the first and last terms are zero, hence we require

$$d^{n+1}d^n = 0$$

but this data $(\mathscr{E}^\bullet, d)$ is exactly the data of a *complex* of coherent sheaves! Since we started with a finite collection of D-branes, this is a *bounded* complex.

What remains is to compute the new Q-cohomology operators. Notice that open strings can no longer couple to a single coherent sheaf, since the deformation d "tangles" the D-branes together.

Appendix A: BRST Symmetries

In this appendix, we show that a supersymmetry generator Q localizes the n-point function computations to Q-cohomology, and that variation of the action by Q-exact terms does not change the theory.

4.1 Localization on Q-cohomology

Suppose Q is a generator of a supersymmetry of our theory with algebra of observables A, and that $Q^2 = 0$. In algebra terms, Q generates a symmetry if

$$\langle \varphi + \delta\varphi \rangle := \langle \varphi + \{Q, \varphi\} \rangle = \langle \varphi \rangle$$

or

$$\langle \{Q, \varphi\} \rangle = 0$$

for all operators φ.

In this case, shifting any field in an n-point function of Q-closed operators by a Q-exact term does not change the n-point function. Shifting $\varphi_a \to \varphi_a + \{Q, \varphi\}$ we get

$$\langle \prod_i \varphi_i \rangle \to \langle \prod_i \varphi_i \rangle + \langle \{Q, \varphi\} \cdot \prod_{i \neq a} \varphi_i \rangle$$

and

$$\langle \{Q, \varphi\} \cdot \prod_{i \neq a} \varphi_i \rangle = \langle \left\{ Q, \varphi \cdot \prod_{i \neq a} \varphi_i \right\} \rangle - \sum_j \langle \varphi \cdot \{Q, \varphi_j\} \cdot \prod_{i \neq a, j} \varphi_i \rangle$$

The first term vanishes since Q is a symmetry, and the second term vanishes since each φ_i was assumed Q-closed.

Some argument needs to be made to ensure that all our physical operators lie in the kernel of Q, I have not found a sufficiently nice treatment of this in any sources I know.

4.2 Localization of the Path Integral

Not only does the operator algebra reduce to a cohomology algebra, but in the presence of Q we can deform our Lagrangian by Q-exact terms without changing the theory. This idea is very powerful, and allows for many computations in topological field theory.

The idea itself, however, is quite simple. Suppose $\mathcal{L}$ is our Lagrangian, and we deform $\mathcal{L}$ by a Q-exact term, i.e.

$$\mathcal{L} \to \mathcal{L}_0 + \{Q, W\}$$

for some operator W. Since Q is fermionic, we require W to be as well so that the Lagrangian is still bosonic. There are additional constraints due to ghost number, but that is perhaps best saved for later.

It is not clear how this shift affects the n-point functions, and so for that we introduce a parameter t (some claim t is related to the temperature of the system) to make our Lagrangian

$$\mathcal{L}(t) = \mathcal{L}_0 + t\{Q, W\}$$

Now, correlation functions are generated from this by

$$\langle \prod_i \varphi_i \rangle = \int (\prod_i D\varphi_i) \exp\left(- \int d^2z (\mathcal{L}_0 + t\{Q, W\} \right) \prod_i \varphi_i$$

If we differentiate with respect to t we get

$$\begin{aligned} \frac{d}{dt} \langle \prod_i \varphi_i \rangle &= \int (\prod_i D\varphi_i) \exp\left(- \int d^2z (\mathcal{L}_0 + t\{Q, W\} \right) (\{Q, W\}) \prod_i \varphi_i \\ &= \langle \{Q, W\} \prod_i \varphi_i \rangle = 0 \end{aligned}$$

which is trivial since a Q-exact operator acts as zero in all n-point functions. Since all n-point functions are independent of t, the whole theory is independent of t.

To see this in action, consider the Lagrangian of the topological B-model. It can be written as

$$\mathcal{L}_B = U + t\{Q, \mathscr{D}\}$$

with

$$\begin{aligned} U &= \int_\Sigma \left(-\theta_j D\rho^j - \frac{i}{2} R_{j\bar{j}k\bar{k}} \rho^j \wedge \rho^k \eta^{\bar{j}} \theta_\ell g^{\ell\bar{k}} \right) \\ \mathscr{D} &= g_{j\bar{k}} \left(\rho^j_z \bar{\partial} \varphi^{\bar{k}} + \rho^j_{\bar{z}} \partial \varphi^{\bar{k}} \right) \end{aligned}$$

Instead of taking the limit $t \to 0$, we can instead take $t \to \infty$. This forces the field configurations to localize on the minima of $\mathscr{D}$. In this model, $\mathscr{D} = 0$ if and only if our bosonic maps φ are constant, showing that *in the B-model, there are no instanton effects.*

References

[AL01] Paul S Aspinwall and Albion Lawrence. "Derived categories and zero-brane stability". In: *Journal of High Energy Physics* 2001.08 (2001), p. 004. URL: `https://arxiv.org/abs/hep-th/0104147`.

[Asp+09] Paul S. Aspinwall et al. *Dirichlet branes and mirror symmetry.* Vol. 4. Clay Mathematics Monographs. Providence, RI: AMS, 2009.