Supersymmetric Topological Sigma Models and Doubling Spaces

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Introduction

Building on the methods of Y. Qin [1], we analyze coherent sheaves on an abelian variety in the framework of the "double field theory" of Hull and Reid-Edwards[2]. The key player is the doubling torus $\mathbb{X}$ which serves as an appropriate ambient geometry. In this setting, line bundles on X become Lagrangian subtori of $\mathbb{X}$, and the intersection theory of these lifts is closely related to the Ext-groups of the line bundles themselves.

This work originated as a Ph.D. thesis at UCSB [3], supervised by Dave Morrison.

The Doubling Torus

- X an abelian variety
- $\hat{X} := \mathrm{Pic}^0(X)$ the dual
- $\mathbb{X} := X \times \hat{X}$ the "doubling torus" of X
- $\mathcal{P}$ the normalized Poincaré bundle on $\mathbb{X}$

Symplectic Structure: The first Chern class of $\mathcal{P}$

$$c_1(\mathcal{P}) = \sum_i dx^i \wedge d\hat{x}^i$$

defines a symplectic form on $\mathbb{X}$, each factor of the product is a Lagrangian subtorus.

Generalized Complex Structures: There is a canonical isomorphism of (real) vector bundles

$$T\mathbb{X} \cong \pi^*(TX \oplus T^*X) \cong \hat{\pi}^*(T\hat{X} \oplus T^*\hat{X})$$

and under this isomorphism, pullbacks of (almost) generalized complex structures on X lift to (almost) complex structures on $\mathbb{X}$.

In particular, if X has complex structure J or symplectic form ω, it induces the complex structures

$$\mathcal{J}_J = \begin{bmatrix} J & 0 \\ 0 & -J \end{bmatrix} \qquad \mathcal{J}_\omega = \begin{bmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{bmatrix}$$

on the doubling torus.

Lifting Line Bundles

- $\mathcal{L}$ a holomorphic line bundle on X
- $E = c_1(\mathcal{L})$ its associated alternating form
- S_{-E} a particular "symmetric" line bundle with $c_1(S_{-E}) = -c_1(\mathcal{L})$

Lifting: Define the lift of $\mathcal{L}$ to be the graph

$$\mathbb{L}(\mathcal{L}) = \Gamma\left(x \mapsto (S_{-E} \otimes t^*_{-x}\mathcal{L})\right)$$

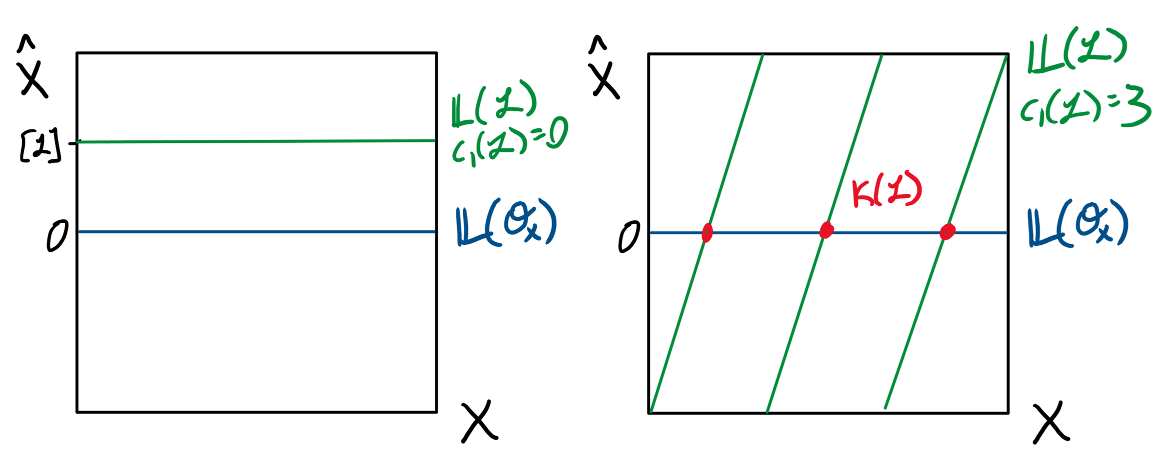
- The lift of $\mathcal{O}_X$ is

$$\mathbb{O} := X \times \{0\}$$

- More generally, if $\mathcal{L} \in \mathrm{Pic}^0(X)$ is trivial,

$$\mathbb{L}(\mathcal{L}) = X \times [\mathcal{L}]$$

- When $c_1(\mathcal{L}) = E$ is nonzero, the lift is a copy of X embedded "with slope E"



An example of lifts on an elliptic curve ($c_1(\mathcal{L}) \in \mathbb{Z}$).
Left: $c_1(\mathcal{L}) = 0$ yields horizontal lifts.
Right: $c_1(\mathcal{L}) \neq 0$ yields lifts with nonzero slope.

Motivation: T-folds

In physics, T-duality identifies theories on tori with dualized factors. Allowing T-duality as part of gluing data defines a "T-fold", a non-geometric background for a nonlinear sigma model. Doubling spaces serve as a geometric "ambient space" for T-folds using local projectors.

Example: (see [2, Section 3]) Let $X \to S^1$ be the T^2-fibration over S^1 with a Dehn twist, and take $\mathbb{X} := X \times_{S^1} \hat{X} \to S^1$ the doubled fibration, with coordinates (x) on the base, and $(y, z, \hat{y}, \hat{z})$ on the fiber.

$$x \mapsto x+1 \implies \begin{array}{ll} y \mapsto y + mz & z \mapsto z \\ \hat{y} \mapsto \hat{y} & \hat{z} \mapsto \hat{z} - m\hat{y} \end{array}$$

To recover the geometric target space from the doubled space, a local projector must be chosen.

Projector:	Project to (y, z)	Project to $(\hat{y}, z)$	Project to $(y, \hat{z})$
Result:	Dehn-twisted T^2-fibration	Trivial T^2-fibration with B-field	not globally defined, T-fold geometry

Key Results

Generalized Submanifolds and Lifts: Generalized complex geometry on X can be pulled back to ordinary complex geometry on $\mathbb{X}$. Using a slight generalization of our lift, we find the doubling torus geometrizes generalized complex geometry:

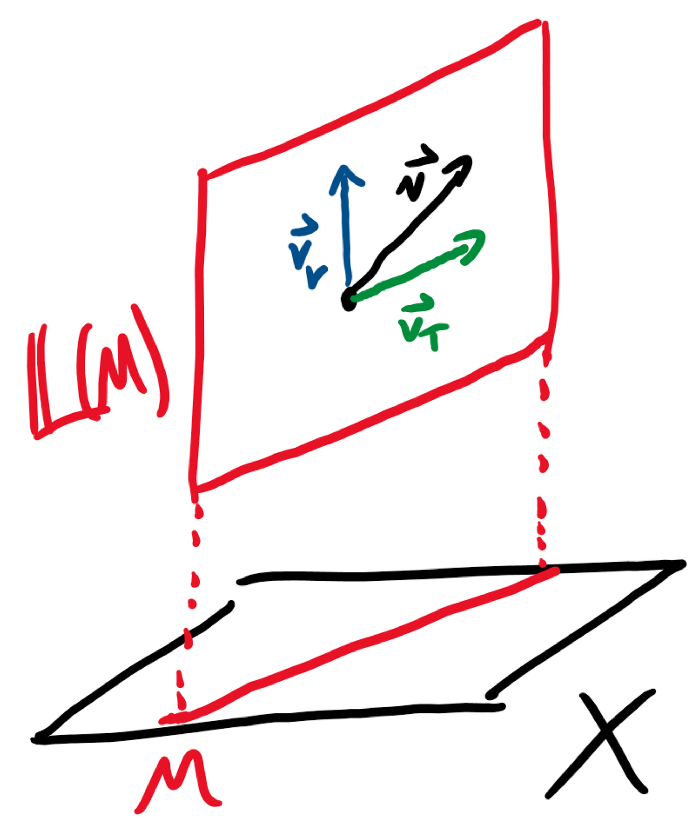
Theorem. *"Lifts are Lagrangian and holomorphic."*
For (M, ∇) a linear subtorus with translation-invariant $U(1)$ connection, the tangent space of the lift is the pullback of the generalized tangent space of (M, ∇). In particular, it is maximal isotropic, and if (M, ∇) is $\mathcal{J}$-holomorphic, so is the lift.

Ext-groups and Intersections: Physics suggests that the intersection theory of the lifts of two line bundles $\mathcal{L}_1, \mathcal{L}_2$ is related to the Ext-groups $\mathrm{Ext}^i(\mathcal{L}_1, \mathcal{L}_2)$.

Theorem. *"$\mathrm{Ext}^*(\mathcal{L}_1, \mathcal{L}_2)$ is encoded in the lifts."*
Let $\mathcal{L}_1, \mathcal{L}_2$ be line bundles, $\mathbb{L}_1, \mathbb{L}_2$ their lifts. If $c_1(\mathcal{L}_1) = c_1(\mathcal{L}_2)$, then

$$\mathrm{HF}^*_{\mathcal{J}_J}(\mathbb{L}_1, \mathbb{L}_2) \cong \mathrm{Ext}^*(\mathcal{L}_1, \mathcal{L}_2)$$

where $\mathrm{HF}_{\mathcal{J}_J}$ is the $\mathcal{J}_J$-holomorphic intersection Floer cohomology of Y. Qin[1].



The lift of (M, ∇) tracks the geometry of M in
$\vec{v}_T \in \pi^*TX$
and the holonomy of ∇ in
$\vec{v}_v \in \hat{\pi}^*T\hat{X} \cong \pi^*T^*X$
via the symplectic form $c_1(\mathcal{P})$.

References

[1] Y. Qin, *Coisotropic branes on tori and homological mirror symmetry*. PhD thesis, University of California, Berkeley, 2020. arXiv:2207.10647 [math.SG].

[2] C. M. Hull and R. A. Reid-Edwards, "Non-geometric backgrounds, doubled geometry and generalised T-duality," *Journal of High Energy Physics* **09** (2009) 014, arXiv:0902.4032 [hep-th].

[3] D. M. Halmrast, *Supersymmetric Topological Sigma Models and Doubling Spaces*. PhD thesis, University of California, Santa Barbara, 2024. arXiv:2507.08181v2 [math.dg].

Future Research

Higher-rank sheaves? Ideas of Mukai suggest that vector bundles can be lifted to submanifolds finite over X. Does the intersection theory of these lifts compute the desired ext-groups? How does lifting work with isogenies?

Hyperkähler geometry: (B,A,A)-branes? The doubling torus lifts both holomorphic and symplectic structures to complex structures on $\mathbb{X}$. Can a (B, A, A)-brane be lifted to a hyperholomorphic object in $\mathbb{X}$? Does this lead to a notion of the derived category of (B, A, A)-branes?

Further Information

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preprint: arXiv:2507.08181 [Math.DG]