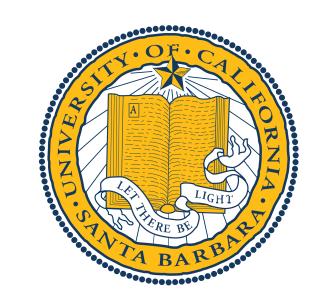
# A REFINED APPROACH TO STABILITY IN THE HYPERKÄHLER SETTING

Daniel Halmrast, Advisor: Dave Morrison

PH.D. EXPECTED 2024

University of California, Santa Barbara



### INTRODUCTION

Bridgeland Stability originated as a description of the low-energy physics of an  $\mathcal{N}=2$  string theory on a Calabi-Yau manifold. When the target has a hyperkähler structure, supersymmetry is enhanced to  $\mathcal{N}=4$ , which has an effect on the allowed objects in the category of boundary conditions.

In the  $\mathcal{N}=2$  case, the category of boundary conditions is the derived category of coherent sheaves [1], but in the hyperkähler case the category is necessarily smaller. Using a doubling construction, these objects can be analyzed using the geometry of the doubled space.

### THE CALABI-YAU CASE

In an open string theory on a Calabi-Yau manifold X, the string ends may lie on a submanifold of Xcalled a "D-brane", which preserves at most half of the total supersymmetry ( $\mathcal{N}=2$  of them in this case). There are two choices for which half to preserve:

	A-model	<b>B-model</b>
Condition	Lagrangian	Holomorphic
Category	$\operatorname{Fuk}(X)$	$D^b(X)$

# HYPERKÄHLER SYMMETRIES

A hyperkähler manifold has three compatible complex structures. D-branes, in this  $\mathcal{N}=4$  case, must be B-type in one complex structure, but can be either B-type or A-type in the other two [2].

In either case, the D-brane is B-type in one complex structure, making it an element of  $D^b(X)$ . Imposing the other two conditions restricts to a subcategory.

### DOUBLING SPACES

doubled manifold is a symplectic manifold  $(M,\omega)$  with a pair of transverse lagrangian foliations  $\mathcal{F}_1, \mathcal{F}_2$ . By the Frobenius theorem, the tangent bundle splits

$$TM \cong T\mathcal{F}_1 \oplus T\mathcal{F}_2$$

into the tangent bundles of the leaves of the two foliations. The symplectic form  $\omega$  then identifies

$$T\mathcal{F}_2 \cong T\mathcal{F}_1^*$$

Then, for a leaf  $S_1 \in \mathcal{F}_1$ 

$$TM|_{S_1} \cong TS_1 \oplus T^*S_1$$

(This is also true at the level of Courant algebroids!)

A doubled product manifold is a doubled manifold M which globally splits as a product  $M = X_1 \times X_2$ whose fibers are leaves of the foliations. Then, the leaf space  $M/\mathcal{F}_2$  is the projection  $\pi: M \to X_1$ .

### THE TORUS CASE

The case of the base a torus has been worked out in detail in [5].

Let  $X = \mathbb{C}^2/\Lambda$  and  $X^{\vee} = \mathbb{C}^2/\Lambda^*$  be dual torii with Kähler structures inherited from  $(g, J, \omega)$  on  $\mathbb{C}^2$ . Their doubling torus is

$$M = X \times X^{\vee}$$

with Kähler structure  $(g \oplus g^{-1}, J \oplus -J^*, \omega \oplus -\omega^{-1})$ , and para-Kähler form  $\sigma_0 = \sum dx \wedge d\hat{x}$ .

The doubling lift of an affine GC submanifold is

$$\mathbb{L} = \{ (x, \hat{x}) \mid x \in L, \hat{x}(\gamma) = (-1)^{s(\gamma)} \operatorname{Hol}_{\nabla}(x + S_{\gamma}^{1}) \}$$

for all  $\gamma \in H_1(L,\mathbb{Z})$ ,  $S^1_{\gamma}$  a linear representative, with s a correction factor. The tangent bundle of  $\mathbb L$  is

$$T\mathbb{L} = \{ X + \xi \mid X \in TL, \xi|_L = \iota_X F \}$$

which is the generalized tangent bundle of L.

Generalized complex submanifolds are submanifolds L with U(1) connections with curvature F such that the generalized tangent bundle

GENERALIZED COMPLEX BRANES

To treat A and B-models uniformly, we use the lan-

guage of generalized complex (GC) geometry, de-

veloped in [3], where both symplectic forms and

complex structures are viewed as endomorphisms of

$$\mathcal{T}L = \{ X + \xi \in TX \oplus T^*X \mid X \in TL, \iota_X F = \xi|_L \}$$

is preserved by the GC structure.

 $TX \oplus T^*X$  squaring to -1.

- B-type: The GC submanifolds are holomorphic submanifolds with holomorphically flat connections.
- A-type The GC submanifolds are coisotropic submanifolds with curvature satisfying the coisotropic brane condition

$$\omega + F\omega^{-1}F = 0$$

# GEOMETRY OF DOUBLING SPACES

Suppose M is a product doubled manifold with base Q. If Q admits a Kähler structure  $(G, J, \omega)$ , then M gets a Kähler structure given by

$$\mathbb{G} = \begin{bmatrix} 0 & g^{-1} \\ g & 0 \end{bmatrix} \mathbb{J} = \begin{bmatrix} J & 0 \\ 0 & -J^* \end{bmatrix} \quad \mathbb{W} = \begin{bmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{bmatrix}$$

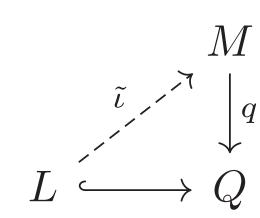
Satisfying  $\mathbb{GJ} = \mathbb{JG} = \mathbb{W}$ . In this case,  $\mathbb{W}$  is another complex structure on M as well. When Q is Hyperkähler with complex structures (I, J, K), M has six complex structures given by  $\mathbb{I}, \mathbb{J}, \mathbb{K}$  and their corresponding Kähler forms  $\mathbb{W}_I, \mathbb{W}_J, \mathbb{W}_K$ .

Geometry of $Q$	Lifted Geometry of ${\cal M}$	
$\mathbb{C}$ -structure $J$	C-structure J	
Symplectic form $\omega$	C-structure W	
Metric G	Metric G	
$HK\left(I,J,K ight)$	$HK\left(\mathbb{I},\mathbb{J},\mathbb{K} ight)$	
Holo. Sym. $(I, \Omega)$	$HK\left(\mathbb{I},\mathbb{W}_{J},\mathbb{W}_{K} ight)$	

- The (B,B,B) hyperkähler structure on M is given by the triple  $(\mathbb{I}, \mathbb{J}, \mathbb{K})$ . This tracks the data of trianalytic submanifolds of Q.
- The (B, A, A) hyperkähler structure on M is given by the triple  $(\mathbb{I}, \mathbb{W}_J, \mathbb{W}_K)$ . This tracks the data of holomorphic coisotropic branes in Q.

# LIFTS OF GC BRANES

A doubling lift of (L, F) is a lift



to the doubled manifold such that  $T(\tilde{\iota}L)$  is isomorphic to the generalized tangent bundle of L. The lift is always maximal isotropic with respect to the para-Kähler form.

In the torus case, the doubling lift of a W-GC brane is holomorphic in the lifted complex structure  $\mathbb{W}$  [5]. As expected, one can show that the doubling lift of a J-GC brane is J-holomorphic as well.

# FUTURE WORK

Every K3 surface is a hyperkähler rotation of an elliptic K3, which is a  $T^2$ -fibration over  $\mathbb{P}^1$ . Although the relative doubling manifold has been explored in detail, a global doubled manifold for the K3 has yet to be constructed in this way. It is expected that doubled geometry implements T-duality, which is an incarnation of mirror symmetry in the K3 case.

Such a doubling space would be a natural setting for the category of  $\mathcal{N}=4$  boundary conditions to be defined in [4], since these boundary conditions come in two flavors which are handled identically in the doubled manifold. However, a full treatment in the sense of [1] requires two additional parts.

- Inclusion of topological deformations: In the  $\mathcal{N}=2$  case this enhances  $\mathrm{Coh}(X)$  to  $D^b(X)$ , but such analysis has not been carried out in the  $\mathcal{N}=4$  case.
- Analysis of enhanced stability: the new category admits stability conditions that enhance Bridgeland stability.

## REFERENCES

- [1] Paul S Aspinwall and Albion Lawrence. Derived categories and zero-brane stability. Journal of High Energy Physics, 2001(08):004, 2001.
- [2] Hirosi Ooguri, Yaron Oz, and Zheng Yin. D-branes on calabi-yau spaces and their mirrors. Nuclear Physics B, 477(2):407–430, 1996.
- [3] Marco Gualtieri. Generalized complex geometry. *Annals of mathematics*, pages 75–123, 2011.
- [4] Anton Kapustin and Yi Li. Open-string brst cohomology for generalized complex branes. *Advances in Theoretical and Mathematical Physics*, 9(4):559–574, 2005.
- [5] Yingdi Qin. Coisotropic branes on tori and Homological mirror symmetry. University of California, Berkeley, 2020.