Stability Conditions on Topological String Theories

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August 13, 2020

Overview

Summary of the Talk

We start by considering string theory in the twisted nonlinear sigma model with target space a complex projective Calabi-Yau threefold.

The boundary conditions for this theory form a category which is equivalent to $D^b(\mathrm{Coh}(X))$, independent of Kähler form.

Douglas' Π-stability reintroduces the Kähler form dependence by giving a criterion for identifying when certain unstable objects decay into others (via distinguished triangles) as a function on the moduli space of Kähler forms.

In particular, one expects the set of all stable objects at a particular point to form an Abelian category functioning as the heart of our triangulated category.

Bridgeland stability generalizes this to a mathematically rigorous statement concerned with finding the "moduli space" of hearts of our derived category.

What Do We Mean by a Physical Theory?

The definition of a *physical theory* is hard to pin down mathematically, and we do not aim at all to be rigorous here, but for our purposes:

- A *physical theory* is an algebra of operators (describing the "states" of the physical theory) subject to various consistency constraints, along with n-point functions which compute observables as functions of n operators.
- Two physical theories will be considered isomorphic if their observables agree (without getting precise). We'll see an example of this later.
- A *model* is a physical method of generating the operator algebra and the n-point functions. Every physical theory we will review here comes from a model.

The Nonlinear Sigma Model

Our starting point is the nonlinear sigma model, which is a physical theory describing how strings propagate in a Calabi-Yau manifold X (usually of complex dimension 3).

The theory also needs the data of a closed, real two-form B which pairs with the Kähler form to make a *complexified Kähler form* $B + iJ$.

Generally B enters into the computation only modulo integer shifts.

Topological Twisting

One of the striking features of the nonlinear sigma model is the fact that it can be *topologically twisted* in two different ways.

- The *A-model twisting* eliminates the complex structure dependence, and the theory only depends on the topology of X and the complexified Kähler form $B + iJ$ (in fact, only its cohomology class).
- The *B-model twisting* eliminates the dependence on $B + iJ$, and depends only on the complex structure of X.

Mirror Symmetry

The phenomenon of *mirror symmetry* shows itself (albeit incompletely) in these theories in the following way:

Definition

A pair of Calabi-Yau manifolds (X, Y) is called a *mirror pair* if the topological A-model on X is isomorphic as a theory to the topological B-model on Y.

If we restrict our attention of the mirror symmetry phenomenon to the moduli spaces, we get:

Theorem (Expected)

For a mirror pair (X, Y), there exists a local isomorphism (the mirror map *isomorphism) between the moduli space of complexified Kähler forms on X and the moduli space of complex structures on Y, preserving mirror symmetry.*

The Operator Algebra Associated to the B-model

The operator algebra in the B-model is physically obtained by the process of *BRST quantization* on an original "classical" theory, which mathematically amounts to finding a nilpotent operator Q with $Q^2 = 0$ satisfying certain properties, and taking cohomology with respect to Q.

In the case of the B-model on a Calabi-Yau manifold X, the operators start their life in $\Omega_X^{(-p,q)}$, and the operator Q is the $\bar{\partial}$ operator.

Hence, the BRST quantized objects are in $H^{(-p,q)}X = H^q(X, \bigwedge^p T_X)$

Including New Data

There is a variant of the previous construction, which takes in the additional data of a holomorphic vector bundle E on X. (We might require E originally to have a holomorphic connection, but the twisted operator algebra is independent of the connection chosen).

The operators lose their $-p$-form dependence, and take values in $\mathrm{End}(E)$, hence can all be expressed as (cohomology classes of) $(0, q)$ forms with values in $\mathrm{End}(E)$ i.e. $H^{(0,q)}(X, \mathrm{End}(E))$

Physical Interpretation of the Vector Bundle

The modification above can take place at the level of the untwisted (non-topological) nonlinear sigma model, and the conclusion we've stated is the result of twisting and BRST quantizing this variant.

The vector bundle E enters into the calculation as a *boundary term*, and has the effect of adding a $U(r)$-bundle to the boundary of the string for r the rank of E.

Physicists refer to this setup as a "stack" of r Dn-branes, for n the (real) dimension of X, thinking of a rank r vector bundle as a "stack" of rank 1 vector bundles.

Adding the Grading

Physics tells us these branes come with a grading, so we'll include as part of our data a decomposition

$$E = \bigoplus_{n \in \mathbb{Z}} E^n$$

This decomposes our space of operators into

$$\text{BRST local operators} = \bigoplus_{q,m,n} H^{(0,q)}(X, (E^m)^\vee \otimes E^n)$$

and we define the *ghost number* of a definite-graded operator to be $q - m + n$

Generalizing I

We generalize more by allowing E to be a *locally free sheaf* $\mathscr{E}$ of finite rank over X.

Thus $\mathscr{E}$ decomposes as

$$\bigoplus_{n\in\mathbb{Z}} \mathscr{E}^n$$

with all but finitely many $\mathscr{E}^n$ zero.

Operators now live in

$$\text{BRST local operators} = \bigoplus_{q,m,n} \mathrm{Ext}^q(\mathscr{E}^m, \mathscr{E}^n)$$

Generalizing II

The data we have now assembles into a category (denoted $\mathbf{T}_{\mathscr{E}}(X)$) in a nice way.

- The *objects* are the definite-grade parts of $\mathscr{E}$, namely $\{\mathscr{E}^n\}$ for all n for which $\mathscr{E}^n$ is nontrivial (the Dn-branes of our theory)
- The *morphisms* from $\mathscr{E}^m$ to $\mathscr{E}^n$ are given by the higher Ext-groups

$$\mathrm{Hom}_{\mathbf{T}_{\mathscr{E}}(X)}(\mathscr{E}^m, \mathscr{E}^n) = \bigoplus_q \mathrm{Ext}^q(\mathscr{E}^m, \mathscr{E}^n)$$

  (the operators of our theory)
- Composition is given by the Yoneda pairing

$$\mathrm{Ext}^p(\mathscr{E}^\ell, \mathscr{E}^m) \otimes \mathrm{Ext}^q(\mathscr{E}^m, \mathscr{E}^n) \to \mathrm{Ext}^{p+q}(\mathscr{E}^\ell, \mathscr{E}^n)$$

Candidate Category of All Branes

Next, let's try and assemble a category of all possible such theories.

If we vary $\mathscr{E}$ over all possible graded locally free sheaves of finite rank, we get a large category containing a lot of the branes.

We could even allow sheaves supported on submanifolds, yielding the category $\mathrm{Coh}(X)_{\mathbb{Z}}$ of $\mathbb{Z}$-graded coherent sheaves on X, enlarged by allowing the hom-sets to be the total Ext group $\bigoplus_q \mathrm{Ext}^q(-,-)$.

These are referred to as Dp-branes, for p the dimension of the support of the coherent sheaf.

Physical Interpretation of Dp-branes

Since the branes enter in to the calculation as boundary terms, if the brane $\mathscr{E}$ does not have $\mathrm{Supp}(\mathscr{E}) = X$, we must force the boundary of the string to be restricted to $\mathrm{Supp}(\mathscr{E})$.

This amounts to imposing Dirichlet boundary conditions on the endpoints of the string, hence the name D-brane, or Dirichlet-brane.

Assembling the Category of All Branes

We've seen that the category of all branes considered so far has

- **Objects:** Coherent sheaves on X, graded by $\mathbb{Z}$.
- **Morphisms:** Higher Ext-groups

$$\mathrm{Hom}(\mathscr{E}^m, \mathscr{F}^n) = \bigoplus_q \mathrm{Ext}^q(\mathscr{E}^m, \mathscr{F}^n)$$

where $q - m + n$ is the ghost number of the map.

Question

Does this category include all possible D-branes?

To answer this question, we need to look at deformations of our theory.

The Deformation of Q

Let's return to the setup of a single coherent sheaf $\mathscr{E}$ with grading

$$\mathscr{E} = \bigoplus_n \mathscr{E}^n$$

We can deform the theory by deforming the BRST operator Q.

The allowed changes are of the form $Q = Q_0 + \delta Q$ with Q_0 the original BRST symmetry, and

$$\delta Q = t\varphi := d$$

for φ some element of the operator algebra.

The physics demands that Q have definite ghost number 1, so we restrict to φ with ghost number 1. Such operators are called *marginal*.

Candidates for Deformation

The candidate ghost number 1 operators live in

$$\begin{aligned} \mathrm{Ext}^0(\mathscr{E}^n, \mathscr{E}^{n+1}) \\ \mathrm{Ext}^1(\mathscr{E}^n, \mathscr{E}^n) \\ \vdots \\ \mathrm{Ext}^k(\mathscr{E}^n, \mathscr{E}^{n+1-k}) \end{aligned}$$

The operators in $\mathrm{Ext}^1(\mathscr{E}^n, \mathscr{E}^n)$ yield first-order deformations of $\mathscr{E}^n$, and add nothing new.

The operators in $\mathrm{Ext}^0(\mathscr{E}^n, \mathscr{E}^{n+1})$ are worth examining in more detail.

The Ext^0 Deformations

Deforming by an element of $\mathrm{Ext}^0(\mathscr{E}^n, \mathscr{E}^{n+1})$ yields a new theory with branes specified by

- a finite collection of coherent sheaves $\mathscr{E}^n$ of finite rank ($n \in \mathbb{Z}$)
- maps $d_n \in \mathrm{Ext}^0(\mathscr{E}^n, \mathscr{E}^{n+1})$

Consequences

The new BRST symmetry is now

$$Q = Q_0 + d$$
$$d = \bigoplus_n d_n$$
$$d_n : \mathscr{E}^n \to \mathscr{E}^{n+1}$$

In order for this to be a valid deformation, we need $Q^2 = 0$

$$\begin{aligned} Q^2 &= \{Q, Q\} \\ &= Q_0^2 + \{Q_0, d\} + \{d, Q_0\} + d^2 \\ &= 0 + 2\{Q_0, d\} + d^2 \end{aligned}$$

Since $d = t\varphi$, and φ is Q_0-closed, $\{Q_0, d\} = 0$.

Hence $d^2 = 0$ for the theory to be consistent.

The Category of Bounded Complexes

An object in our category of branes now corresponds to a *bounded complex of coherent sheaves* and the category looks more like $\mathrm{Ch}^b(\mathrm{Coh}(X))$.

Deforming Q has the effect of deforming the BRST cohomology, which can be recomputed using a spectral sequence argument. As it turns out, the ghost number p operators between $\mathscr{E}^\bullet$ and $\mathscr{F}^\bullet$ are given by the *hyperext*

$$\mathrm{Hom}_{\mathbf{T}(X)}(\mathscr{E}^\bullet, \mathscr{F}^\bullet) = \bigoplus_p \mathrm{Ext}^p(\mathscr{E}^\bullet, \mathscr{F}^\bullet)$$

Computing Hyperext

The hyperext groups are typically computed using an injective resolution $\mathscr{F} \to \mathscr{I}$, and taking cohomology of the internal hom complex

$$\mathrm{Ext}^p(\mathscr{E}^\bullet, \mathscr{F}^\bullet) = H^p([\mathscr{E}^\bullet, \mathscr{I}^\bullet])$$

where

$$[\mathscr{E}^\bullet, \mathscr{F}^\bullet]^p = \bigoplus_n \mathrm{Hom}(\mathscr{E}^n, \mathscr{I}^{n+p})$$

The New Category

Combining all this data yields a category $\widetilde{K^b}(\mathrm{Coh}(X))$ with

- Objects are bounded complexes of coherent sheaves, now denoted as e.g. $\mathscr{E}$ with the gradings dropped
- Morphisms are given by the total hyperext groups

$$\mathrm{Hom}_{\widetilde{K^b}(\mathrm{Coh}(X))}(\mathscr{E},\mathscr{F}) = \bigoplus_p \mathrm{Ext}^p(\mathscr{E},\mathscr{F})$$

When Are Two Objects the Same?

Two object (branes) $\mathscr{E}_1$ and $\mathscr{E}_2$ are considered *physically identical* if the ext groups can't tell them apart. That is, if and only if

$$\begin{aligned} &\mathrm{Ext}^p(\mathscr{E}_1, \mathscr{F}) \cong \mathrm{Ext}^p(\mathscr{E}_2, \mathscr{F}) \text{ and} \\ &\mathrm{Ext}^p(\mathscr{F}, \mathscr{E}_1) \cong \mathrm{Ext}^p(\mathscr{F}, \mathscr{E}_2) \end{aligned}$$

This condition is necessary and sufficient to guarantee the n-point functions for operators involving $\mathscr{E}_1$, when replaced with $\mathscr{E}_2$, do not change.

The resulting category obtained from $\widetilde{K^b}(\mathrm{Coh}(X))$ by modding out by this equivalence will be denoted $\mathbf{T}(X)$ (the category of *topological field theories* on X).

Consequences

Consider $f : \mathscr{E}_1 \to \mathscr{E}_2$ a quasi-isomorphism (induces isomorphisms on cohomology).

Its mapping cone complex $\mathrm{Cone}(f)$ is therefore acyclic (cohomology zero), and from the spectral sequence, we have for arbitrary $\mathscr{F}$

$$\mathrm{Ext}^p(\mathrm{Cone}(f), \mathscr{F}) \cong \mathrm{Ext}^p(\mathscr{F}, \mathrm{Cone}(f)) \cong 0$$

Hence

$$\mathrm{Ext}^p(\mathscr{E}_1, \mathscr{F}) \cong \mathrm{Ext}^p(\mathscr{E}_2, \mathscr{F})$$

(and vice versa) from the long exact sequence of cohomology.

Mapping From the Derived Category

We have a functor $G : \mathrm{Ch}^b(\mathrm{Coh}(X)) \to \mathbf{T}(X)$ sending an object to itself, and a morphism to its homotopy class modulo physical equivalence in $\mathbf{T}(X)$.

This sends quasi-isomorphisms to isomorphisms in $\mathbf{T}(X)$, so it factors through the derived category $D^b(\mathrm{Coh}(X))$ (bounded, since it maps from bounded chain complexes) and we get a map

$$G : D^b(\mathrm{Coh}(X)) \to \mathbf{T}(X)$$

Locating the Derived Category

Consider the subcategory $\mathbf{T}_0(X)$ with the same objects as $\mathbf{T}(X)$, but morphisms in degree zero only.

It turns out $G : D^b(\mathrm{Coh}(X)) \to \mathbf{T}_0(X)$ is an equivalence of categories.

Summary

The total topological field theory category on X, $\mathbf{T}(X)$, contains morphisms of all degrees with

$$\mathrm{Hom}_{\mathbf{T}(X)}(\mathscr{E},\mathscr{F}) = \bigoplus_p \mathrm{Ext}^p(\mathscr{E},\mathscr{F})$$

modulo physical equivalence.

We can flatten this category to $\mathbf{T}_0(X)$ by considering only the degree-zero morphisms, with

$$\mathrm{Hom}_{\mathbf{T}_0(X)}(\mathscr{E},\mathscr{F}) = \mathrm{Ext}^0(\mathscr{E},\mathscr{F})$$

modulo physical equivalence.

This category is equivalent to $D^b(\mathrm{Coh}(X))$ the bounded derived category of coherent sheaves on X.

Many Physical Theories, One Topological Theory

The physical theory depends critically on the complexified Kähler form $B + iJ$, but BRST quantization and twisting eliminated this dependence.

The critical question:

Question

Can we identify the objects in $\boldsymbol{T}_0(X)$ that correspond to physical branes in a particular physical theory?

Here, a physical brane is, roughly, an admissible boundary term that one can add to the untwisted nonlinear sigma model. Recall that the image of such a term after twisting and taking BRST cohomology lies in $\mathbf{T}_0(X)$.

BPS States and Central Charge

Our motivating example: E a holomorphic vector bundle over X. In effective physics where X is integrated out, E produces a BPS particle which can be studied.

If we consider the effective physics, E has a quantity, the *central charge* associated to it, computed as

$$Z(E) = Q_i(E)\Pi^i$$

where $Q_i(E)$ is a charge vector for E and Π^i depends on the geometry of X.

The Central Charge

In our case, this yields (near the large-volume limit)

$$Z(E) = \int_X \exp(-(B + iJ))\,\mathrm{ch}(E)\sqrt{\mathrm{td}(T_X)} + \text{Quantum Corrections}$$

More generally, we expect there to exist a group homomorphism

$$Z : K(X) \to \mathbb{C}$$

from the Grothendieck group of $D^b(\mathrm{Coh}(X))$ to $\mathbb{C}$

Adding in the $\mathbb{R}$-grading

Fixing $B + iJ$ near the large-volume limit, we can now associate a real number to each vector bundle over X defined as

$$\xi(E) = \frac{1}{\pi} \log(Z(E))$$

This is the $\mathbb{R}$-grade of the vector bundle E, and in general is only defined mod 2.

Physical Interpretation

We can interpret the complex number $Z(E)$ as encoding two quantities:

- The mass of the BPS brane is $\|Z(E)\|$
- The particular $\mathscr{N} = 1 \subset \mathscr{N} = 2$ supersymmetry preserved by E is encoded in $\arg(Z(E))$.

Extending the $\mathbb{R}$-grading

We'll demand that $\xi(E)$ vary continuously with $B + iJ$, so long as E lies in a certain class of objects called the Π-*stable objects* of the theory.

Hence, we expect stability to be a function of points in the path space for the moduli space for $B + iJ$.

We'll also extend the $\mathbb{R}$-grading to the deformed boundary conditions (complexes of locally free sheaves) by

$$Z(\mathscr{E}^\bullet) = \sum_n (-1)^n \mathscr{E}^n$$

By analysis of the A-model picture, we also demand

$$\xi(\mathscr{E}[n]) = \xi(\mathscr{E}) + n$$

What Are the Stable Objects?

We "define" the Π-stable objects at a point $B + iJ$ in the complexified Kähler moduli space to be as:

Definition (Circular!)

Suppose B is an object of $\mathbf{T}_0(X)$, and we've picked a point $B + iJ$ in the complexified Kähler moduli space. Then, B is said to be Π-*stable* at $B + iJ$ if, for all distinguished triangles of the form

$$\begin{tikzcd} & C \arrow[dl, "{[1]}"'] & \\ A \arrow[rr] & & B \arrow[ul] \end{tikzcd}$$

with A and C Π-stable, we have

$$\xi(A) < \xi(B) < \xi(C)$$

What Are the Stable Objects?

By shifting triangles around, we can rephrase this definition:

For a distinguished triangle

$$\begin{tikzcd} & C \arrow[dl, "{[1]}"'] & \\ A \arrow[rr] & & B \arrow[ul] \end{tikzcd}$$

we have:

- C is *stable* against decay to $A[1]$ and B if $\xi(B) < \xi(A) + 1$
- C is *marginally stable* against decay to $A[1]$ and B if $\xi(B) = \xi(A) + 1$

Some Clarification on Notation

Summary of stability conditions:

- **Decay Product Notation:** For the triangle

$$
\begin{tikzcd}
& C \arrow[dl, "{[1]}"'] & \\
A \arrow[rr] & & B \arrow[ul]
\end{tikzcd}
$$

  B is stable against decay to A and B if

$$\xi(A) < \xi(B) < \xi(C)$$

  (think of B as an extension of C by A)

Some Clarification on Notation

Summary of stability conditions:

- **Mapping Cone Notation:** For the triangle

$$\begin{tikzcd} & C \arrow[dl, "{[1]}"'] & \\ A \arrow[rr, "\varphi"] & & B \arrow[ul] \end{tikzcd}$$

C is stable against decay to B and $A[1]$ if

$$\xi(B) - \xi(A) < 1$$

i.e. φ has degree between 0 and 1 (think of C as the mapping cone of φ, and when φ has degree between 0 and 1 it becomes tachyonic)

Physical Interpretation: Tachyon Condensation

Suppose C is a bound state of $A[1]$ and B. This means that $C = \mathrm{Cone}(\varphi : A \to B)$.

Physically, this corresponds to φ being a "tachyonic" state (a state with negative mass) which means the theory with φ as a state will flow to one in which φ binds $A[1]$ and B together.

Walls of Marginal Stability

Suppose we start at a point where C is marginally stable against decay to $A[1]$ and B. We'll say that this point lies on the *ms-wall* for C.

If we move around, one of three things could happen:

- We could stay on the ms-wall, and the gradings of A, B, and C will not change
- We could leave the ms-wall in the direction of C becoming stable, in which case the map $\phi : A \to B$ becomes tachyonic and condenses $A[1]$ and B into C
- We could leave the ms-wall in the other direction, making $\xi(B) - \xi(A) - 1 > 0$ which forces the grade shift on the other two maps of the triangle to go negative, resulting in the CFT breaking and C no longer representing a stable object

Multiple Decay Channels

Setup: $C = \mathrm{Cone}(f_1 : A \to B)$ has a potential decay into $A[1]$ and B, and $B = \mathrm{Cone}(f_2 : E \to F)$ has a potential decay into $E[1]$ and F.

$$\begin{tikzcd} & C \arrow[dl, "{[1]}"'] & \\ A \arrow[rr, "f_1"] & & B \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[dl, "{[1]}"'] & \\ E \arrow[rr, "f_2"] & & F \arrow[ul] \end{tikzcd}$$

Multiple Decay Channels

Start at a point P_0 where both C and B are stable, with P_1 a point where neither are stable.

Consider two paths γ_1 and γ_2 from P_0 to P_1, with γ_1 crossing the ms-wall for C first, followed by the ms-wall for B, and vice versa for γ_2

Moving Around

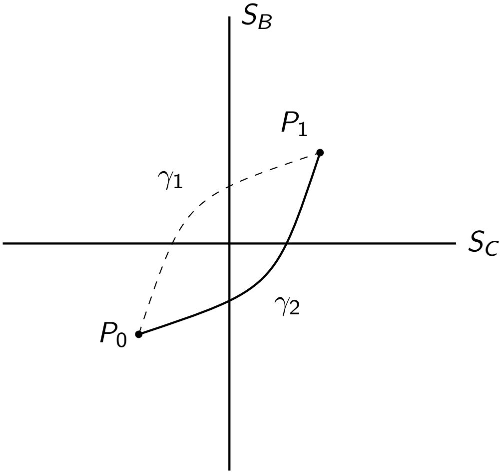

Figure: Two possible paths around a point where two ms-walls cross. If the theory only depends on the homotopy class of paths, C should destabilize in both cases

Multiple Decay Channels

At P_0 we have the following triangles

$$
\begin{tikzcd}
 & C \arrow[dl, "{[1]}"'] & \\
A \arrow[rr] & & B \arrow[ul]
\end{tikzcd}
$$

- $\xi(B) - \xi(A) \leq 1$
- $\xi(C) = \xi(B)$

with $\xi(B) - \xi(A) = 1$ on the C ms-wall

$$
\begin{tikzcd}
 & B \arrow[dl, "{[1]}"'] & \\
E \arrow[rr] & & F \arrow[ul]
\end{tikzcd}
$$

- $\xi(F) - \xi(E) \leq 1$
- $\xi(B) = \xi(F)$

with $\xi(F) - \xi(E) = 1$ on the B ms-wall.

An Apparent Contradiction

If we follow the path γ_1 across the C ms-wall, we conclude that C is unstable at P_1 along the path γ_1.

Suppose instead we take the γ_2 path, where B destabilizes first. Since B is no longer stable, we can't draw any conclusions from the ABC triangle.

Hence, C could remain stable along γ_2, which isn't great.

The Octahedral Axiom for the Derived Category

It turns out that this setup satisfies the hypotheses for the octahedral axiom for triangulated categories.

Axiom

For any two distinguished triangles sharing the same vertex

$$\begin{tikzcd} & Z \arrow[dl, "{[1]}"'] & \\ X \arrow[rr] & & Y \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd} & X' \arrow[dl, "{[1]}"'] & \\ Y \arrow[rr] & & Z' \arrow[ul] \end{tikzcd}$$

there exists an object W and two more distinguished triangles

$$\begin{tikzcd} & X' \arrow[dl, "{[1]}"'] & \\ Z \arrow[rr] & & W \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd} & W \arrow[dl, "{[1]}"'] & \\ X \arrow[rr] & & Z' \arrow[ul] \end{tikzcd}$$

Making the "octahedral diagram" commute.

The Octahedral Diagram

$$\begin{tikzcd}
 & Y \arrow[ddl] \arrow[dr] & \\
X' \arrow[ur, "{[1]}"] \arrow[d, dotted, "{[1]}"'] & & Z' \arrow[ll] \arrow[ddl] \\
Z \arrow[rr, "{[1]}"] \arrow[dr] & & X \arrow[uul] \arrow[u, dotted] \\
 & W \arrow[uul] \arrow[ur, "{[1]}"'] &
\end{tikzcd}$$

Our Example

In our example of A, B, C, E, and F we have

$$
\begin{tikzcd}
 & & B \arrow[ddll] \arrow[drr] & & \\
F[1] \arrow[urr, "{[1]}"] \arrow[d, dotted, "{[1]}"'] & & & & E[1] \arrow[llll] \arrow[ddll] \\
C \arrow[rrrr, "{[1]}"] \arrow[drr] & & & & A \arrow[uull] \arrow[u, dotted] \\
 & & Y \arrow[uull] \arrow[urr, "{[1]}"'] & &
\end{tikzcd}
$$

for Y the new object.

Tracking the Gradings

Suppose we start with everything marginally stable, and $\xi(A) = \varphi$. By the definition of marginally stable, we can compute the stability angles of everything else:

Object	Angle
A	φ
B	$\varphi + 1$
C	$\varphi + 1$
E	φ
$E[1]$	$\varphi + 1$
F	$\varphi + 1$
$F[1]$	$\varphi + 2$
Y	$\varphi + 1$

$$\begin{tikzcd} & C \arrow[dl, "{[1]}"'] & \\ A \arrow[rr] & & B \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd} & F[1] \arrow[dl, "{[1]}"'] & \\ C \arrow[rr] & & Y \arrow[ul] \end{tikzcd}$$

$$\begin{tikzcd} & B \arrow[dl, "{[1]}"'] & \\ E \arrow[rr] & & F \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd} & Y \arrow[dl, "{[1]}"'] & \\ A \arrow[rr] & & E[1] \arrow[ul] \end{tikzcd}$$

Letting B Decay

Let's vary the Kähler form to allow B to decay. This means $\xi(F) - \xi(E) > 1$ and since the grading of B doesn't change (its set by A) we get (for $\alpha, \beta, \gamma > 0$)

Object	Angle
A	φ
B	$\varphi + 1$
C	$\varphi + 1$
E	$\varphi - \alpha$
$E[1]$	$\varphi + 1 - \alpha$
F	$\varphi + 1 + \beta$
$F[1]$	$\varphi + 2 + \beta$
Y	$\varphi + 1 - \gamma$

$$\begin{tikzcd} & C \arrow[dl, "{[1]}"'] & \\ A \arrow[rr] & & B \arrow[ul] \end{tikzcd}
\qquad
\begin{tikzcd} & F[1] \arrow[dl, "{[1]}"'] & \\ C \arrow[rr] & & Y \arrow[ul] \end{tikzcd}$$

$$\begin{tikzcd} & B \arrow[dl, "{[1]}"'] & \\ E \arrow[rr] & & F \arrow[ul] \end{tikzcd}
\qquad
\begin{tikzcd} & Y \arrow[dl, "{[1]}"'] & \\ A \arrow[rr] & & E[1] \arrow[ul] \end{tikzcd}$$

Examining the New Decay Channels

The first triangle has

$$\xi(F[1]) - \xi(Y) = 1 + \beta + \gamma > 1$$

which destabilizes C, as desired!

$$\begin{tikzcd} & F[1] \arrow[dl, "{[1]}"'] & \\ C \arrow[rr] & & Y \arrow[ul] \end{tikzcd}$$

The second triangle has

$$\xi(E[1]) - \xi(A) = 1 - \alpha < 1$$

which stabilizes Y in this decay channel.

$$\begin{tikzcd} & Y \arrow[dl, "{[1]}"'] & \\ A \arrow[rr] & & E[1] \arrow[ul] \end{tikzcd}$$

Hence, we expect the set of stable branes to be dependent only on the homotopy class of path from a known basepoint, and thus well-defined on the universal cover of the moduli space for $B + iJ$.

Describing the Decay

Finally, observe that in the space where both B and C are unstable, we have unstable triangles

$$
\begin{tikzcd}
 & A[1] \arrow[dl, "{[1]}"'] & \\
B \arrow[rr] & & C \arrow[ul]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & E[1] \arrow[dl, "{[1]}"'] & \\
F \arrow[rr] & & B \arrow[ul]
\end{tikzcd}
$$

describing C decaying into B and $A[1]$, and B decaying into F and $E[1]$

$$\xi(B) > \xi(C) > \xi(A[1]) \qquad\qquad \xi(F) > \xi(B) > \xi(E[1])$$

The Filtration

This is summarized in a *generalized filtration* of C as

$$\begin{tikzcd}
0 \arrow[r] & F \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\
& F \arrow[ul, "{[1]}"] & E[1] \arrow[ul, "{[1]}"] & A[1] \arrow[ul, "{[1]}"]
\end{tikzcd}$$

describing how C decays. If the angles are tracked, we get

$$\xi(F) > \xi(E[1]) > \xi(A[1])$$

A main conjecture is that the Π-stable objects at any point in the moduli space are rich enough to allow such a filtration for any object.

Summary

- Π stability of a particular triangle amounts to knowing that two objects are already stable, and knowing the $\mathbb{R}$-gradings of the objects.
- The $\mathbb{R}$-grade of a stable object is given by the argument of its value under the central charge function.
- Π stability is really defined locally around a point where the stable objects are already known, and is expected to be well-defined on points in the universal cover.
- Π stability suggests the existence of filtrations of any object in the derived category into extensions of stable objects.

The Antibrane

Definition

For any brane $\mathscr{E} \in T_0(X)$, its *antibrane* is its image under the left shift functor. That is, $\mathscr{E}[1]$.

Branes and anti-branes tend to annihilate:

$$\begin{tikzcd} & \mathcal{O}_X(-D)[1] \arrow[dl, "{[1]}"'] & \\ \mathcal{O}_X \arrow[rr] & & \mathcal{O}_D \arrow[ul] \end{tikzcd}$$

describes $\mathcal{O}_X$ and $\mathcal{O}_X(-D)$ forming the bound state $\mathcal{O}_D$.

In the large volume limit this triangle is stable in the sense that

$$\xi(\mathcal{O}_X) < \xi(\mathcal{O}_D) < \xi(\mathcal{O}_X(-D)[1])$$

An Explicit Mirror Pair: The Quintic Threefold

Consider a quintic hypersurface $X \hookrightarrow \mathbb{P}^4_{\mathbb{C}}$. It has a mirror pair Y given by a $(\mathbb{Z}_5)^3$ orbifold of a quintic with general equation

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5\psi x_0 x_1 x_2 x_3 x_4$$

The parameter ψ encodes the complex structure of Y, whose moduli space is one-dimensional. We'll set $z = (5\psi)^{-5}$ as a more natural coordinate.

The mirror map then sets $B + iJ := t = \frac{1}{2\pi i} \log(z) + O(z)$ expressing the invariance of B under an integer shift.

The Moduli Space of Y

Some points in the moduli space to consider:

- The point $z = 0$ is mirror to the large-radius limit of X
- The point $z = \infty$ is mirror to the "Gepner point" of X
- The point $z = 1$ is mirror to the "conifold point" of X

The Central Charge

In this case near the large-radius limit we can compute the stability angles of all the line bundles on X coming from $\mathbb{P}^4$.

$$\begin{aligned}\xi(\mathcal{O}_X(N)) &= \frac{1}{\pi}\arg(5(N-B-iJ)^3)\\ &= \frac{3}{\pi}\theta_N - 3\end{aligned}$$

for θ_N the angle between the positive real axis and $B+iJ-(N+i0)$

One Decay Channel

We have the potential decay channel

$$\begin{tikzcd} & \mathcal{O}_X(-N)[1] \arrow[dl, "{[1]}"'] & \\ \mathcal{O}_X \arrow[rr] & & \mathcal{O}_D \arrow[ul] \end{tikzcd}$$

for any map $f : \mathcal{O}_X(-N) \to \mathcal{O}_X$, with D the vanishing locus of f.

This D4-brane $\mathcal{O}_D$ is stable against decay into the D6-brane $\mathcal{O}_X$ and the anti-D6-brane $\mathcal{O}_X(-N)[1]$ when

$$\xi(\mathcal{O}_X) - \xi(\mathcal{O}_X(-N)) < 1$$

or when $\theta_0 - \theta_{-N} < \frac{\pi}{3}$

Lines of Marginal Stability

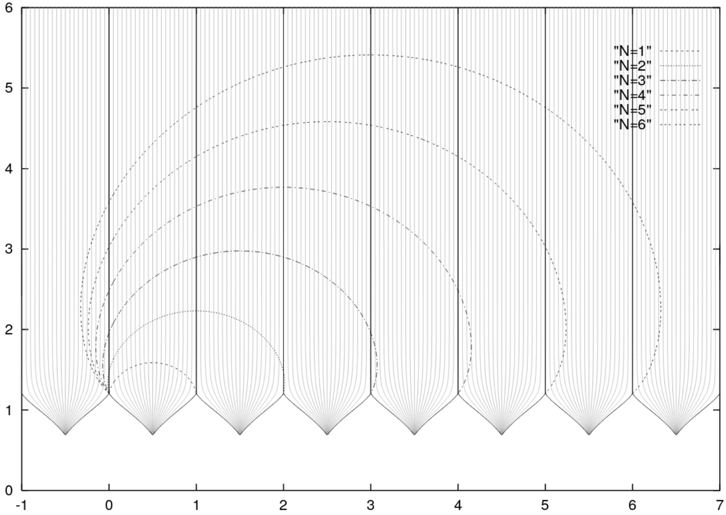

Figure: Borrowed from a paper of Aspinwall and Douglas (hep-th/0110071), the brane $\mathcal{O}_D$ is *stable* above the line in the t-plane.

A New Boundary Condition

Let

$$f \in \mathrm{Hom}(\mathcal{O}_X(-N), \mathcal{O}_X) \cong \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X(N)) \cong H^0(\mathcal{O}_X(N))$$

By Serre duality, we get a map

$$f^\vee \in H^3(\mathcal{O}_X(-N)) \cong \mathrm{Ext}^3(\mathcal{O}_X, \mathcal{O}_X(-N)) \cong \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X(-N)[3])$$

Define $\mathscr{X}_N = \mathrm{Cone}(f^\vee)$

A New Decay Channel?

$$\begin{tikzcd} & \mathscr{X}_N \arrow[dl, "{[1]}"'] & \\ \mathcal{O}_X \arrow[rr] & & \mathcal{O}_X(-N)[3] \arrow[ul] \end{tikzcd}$$

When is $\mathscr{X}_N$ stable? This happens when

$$\xi(\mathcal{O}_X(-N)) + 3 - \xi(\mathcal{O}_X) < 1$$

This is *unstable* in the large-radius limit

A Description of the New Object

By following the mapping cone construction we determine that $\mathscr{X}_N$ is quasi-isomorphic to the complex

$$0 \longrightarrow \mathscr{I}_0 \longrightarrow \mathscr{I}_1 \longrightarrow \mathscr{I}_2 \oplus \mathcal{O}_X \longrightarrow \dots$$

for $\mathscr{I}_\bullet$ an injective resolution of $\mathcal{O}_X(-N)$.

As long as this map is nontrivial, $\mathscr{X}_m$ is not quasi-isomorphic to a single coherent sheaf, or even a direct sum of them!

Formalizing Stability

Bridgeland distills this phenomenon of stability in what is now known as a *Bridgeland stability condition* on a triangulated category $\mathcal{T}$.

Definition

A *Bridgeland stability condition* on a triangulated category $\mathcal{T}$ is the data of a group homomorphism

$$Z : K(\mathcal{T}) \to \mathbb{C}$$

(the *central charge function*), along with a choice of full additive subcategories $\mathcal{P}(\phi)$ for all $\phi \in \mathbb{R}$, such that they satisfy the axioms...

Formalizing Stability

Definition

$\ldots (Z, \mathcal{P})$ satisfies the axioms

- For every $E \in \mathcal{P}(\phi)$, we have $Z(E) = m(E) \exp(i\pi\phi)$ for $m(E) \in \mathbb{R}^+$
- $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$ for all ϕ
- If $\phi_1 > \phi_2$ and $A_j \in \mathcal{P}(\phi_j)$, then

$$\mathrm{Hom}_{\mathcal{T}}(A_1, A_2) = 0$$

- Every object $E \in \mathcal{T}$ admits a *Harder-Narasimhan* filtration with respect to $(Z, \mathcal{P})$

Harder-Narasimhan Filtrations

Definition

A *Harder-Narasimhan Filtration* of an object E with respect to a stability condition $(Z, \mathcal{P})$ is a finite sequence $\phi_1 > \phi_2 > \cdots > \phi_n$ of real numbers, and a filtration

$$\begin{tikzcd} E_0 = 0 \arrow[r] & E_1 \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & \dots \arrow[r] & E_{n-1} \arrow[r] \arrow[d] & E_n = E \arrow[d] \\ & A_1 \arrow[ul, "{[1]}"] & A_2 \arrow[ul, "{[1]}"] & & A_{n-1} \arrow[ul, "{[1]}"] & A_n \arrow[ul, "{[1]}"] \end{tikzcd}$$

with $A_i \in \mathcal{P}(\phi_i)$ and all triangles distinguished.

Understanding Stability

Stability can also be understood in terms of hearts of $\mathcal{T}$.

Definition

A *t-structure* on a triangulated category $\mathcal{T}$ is a choice of two strictly full subcategories $\mathcal{T}_{\geq 0}$ and $\mathcal{T}_{\leq 0}$ such that

- $\mathcal{T}_{\geq 0}$ is closed under suspension, and $\mathcal{T}_{\leq 0}$ is closed under looping
- $\mathrm{Hom}(X, Y[-1]) = 0$ for all $X \in \mathcal{T}_{\geq 0}$ and $Y \in \mathcal{T}_{\leq 0}$
- Every object $X \in \mathcal{T}$ sits in a distinguished triangle

$$
\begin{tikzcd}
 & Z \arrow[dl, "{[1]}"'] & \\
Y \arrow[rr] & & X \arrow[ul]
\end{tikzcd}
$$

with $Y \in \mathcal{T}_{\geq 0}$ and $Z \in \mathcal{T}_{\leq 0}[-1]$

An Example

The standard example of a t-structure on the derived category $D(\mathcal{A})$ is given by

$$D(\mathcal{A})_{\geq 0} = \{A^\bullet \in D(\mathcal{A}) \mid H^i(A^\bullet) = 0 \text{ for all } i < 0\}$$
$$D(\mathcal{A})_{\leq 0} = \{A^\bullet \in D(\mathcal{A}) \mid H^i(A^\bullet) = 0 \text{ for all } i > 0\}$$

Understanding Stability

Given a t-structure on $\mathcal{T}$, we can pick out an Abelian category that generates it.

Definition

The *heart* of a triangulated category $\mathcal{T}$ equipped with a t-structure is the intersection

$$\mathcal{T}^{\heartsuit} := \mathcal{T}_{\geq 0} \cap \mathcal{T}_{\leq 0}$$

Theorem

For $\mathcal{T}$ as above, the heart $\mathcal{T}^{\heartsuit}$ is an Abelian category with $D(\mathcal{T}^{\heartsuit}) = \mathcal{T}$

Understanding Stability

The second type of stability we can demand is on the heart of $\mathcal{T}$

Definition

A *stability function* on an Abelian category $\mathcal{A}$ is the data of a group homomorphism

$$Z : K(\mathcal{A}) \to \mathbb{C}$$

such that for all nonzero objects $E \neq 0$ in $\mathcal{A}$, we have

$$Z(E) \in \mathbb{H}$$

the upper-half plane of $\mathbb{C}$.

This allows us to define the *phase* of an object $E \in \mathcal{A}$ as

$$\phi(E) = \frac{1}{\pi} \arg(Z(E)) \in (0, 1]$$

Understanding Stability

One of the main theorems of Bridgeland's original paper is this:

Theorem

The data of a Bridgeland stability condition on $\mathcal{T}$ is equivalent to the data of a t-structure on $\mathcal{T}$ and a stability function on $\mathcal{T}^\heartsuit$ for which every element of $\mathcal{T}^\heartsuit$ admits a Harder-Narasimhan filtration in $\mathcal{T}^\heartsuit$.

A stability condition can be understood as "picking out a heart" for $\mathcal{T}$ which allows objects in $\mathcal{T}^\heartsuit$ to be decomposed into extensions of stable objects with respect to the central charge.

The Big Result

The set of all stability conditions (satisfying a certain finiteness condition) is denoted $\mathrm{Stab}(\mathcal{T})$, and forms a complex manifold.

Theorem

The space $\mathrm{Stab}(\mathcal{T})$ *admits a natural metric turning it into a topological space*

Theorem

For each connected component $\Sigma \subset \mathrm{Stab}(\mathcal{T})$*, there is a linear subspace* $V(\Sigma) \subset \mathrm{Hom}_{\mathbb{Z}}(K(\mathcal{T}), \mathbb{C})$ *with a natural norm along with a local homeomorphism* $\mathcal{Z} : \Sigma \to V(\Sigma)$ *mapping a stability condition to its central charge.*

Symmetries of the Stability Manifold

The manifold $\mathrm{Stab}(\mathcal{T})$ admits two natural group actions:

- The group $\widetilde{GL_2^+(\mathbb{R})}$ the universal cover of $GL_2^+(\mathbb{R})$ acts on $\mathrm{Stab}(\mathcal{T})$ on the right
- The group $\mathrm{Aut}(\mathcal{T})$ of exact autoequivalences of $\mathcal{T}$ acts on the left by isometries

Furthermore, these actions commute.

Explicit Worked Examples

Bridgeland stability has been studied extensively recently, and certain complex manifolds have been shown to admit Bridgeland stability conditions explicitly (this is definitely not a complete list):

- The elliptic curve
- K3 surfaces
- General smooth projective surfaces
- The total space of $\mathcal{O}_{\mathbb{P}^2}(-3)$
- Fano threefolds
- Abelian threefolds
- Kummer threefolds
- The quintic threefold

Explicit Worked Examples

Π-stability has also been studied in the following cases (again, non-exhaustive):

- The quintic threefold
- The flop transition
- the $\mathbb{C}^3/\mathbb{Z}^3$ orbifold
- Certain quiver gauge theories (McKay correspondence)
- Vanishing del Pezzo surfaces (more general rational surfaces in a CY_3 as well)
- Landau-Ginzburg models
- the elliptic curve (via matrix factorizations)

An Incongruity

Analysis in the special case where $\mathcal{T} = D^b(\mathrm{Coh}(X))$ for X a simply-connected Calabi-Yau threefold shows that the tangent space to the moduli space of complexified Kähler forms on X (which we expect to parameterize stability conditions) is $H^{(1,1)}(X)$

However, the tangent space to $\mathrm{Stab}(X)$ at a point is

$$T_{\mathrm{Stab}(X)} = \bigoplus_p H^{(p,p)}(X)$$

Hence, extra data is needed to find the Kähler form moduli space inside $\mathrm{Stab}(X)$.

Homological Mirror Symmetry

A conjecture of Konstevich suggests that mirror symmetry extends to an equivalence of triangulated categories between the A-model category of branes, and the B-model category of branes on the other side.

The A-model category is the *Fukaya category*, which carries an A_∞ structure, and the conjecture implies that a nontrivial A_∞ structure on $D^b(\mathrm{Coh}(X))$ may play a role in the story. It has been shown that the A_∞ deformations of $D^b(\mathrm{Coh}(X))$ for X a complex projective variety has a tangent space identified with

$$\bigoplus_{p,q} H^p\left(X, \bigwedge^q T_X\right)$$

One might expect for mirror pairs (X, Y) the A_∞ deformations of $D^b(\mathrm{Coh}(X))$ correspond to deformations of stability conditions on $D^b(\mathrm{Coh}(Y))$.

Higher Categories

Recently, the theory of ∞-categories has been formalized, which is a more natural setting for derived categories to be in.

Many of the constructions performed in building the category $\mathbf{T}(X)$ have natural ∞-category analogues, and in particular A_∞ categories are naturally phrased in these terms.

Moreover, every derived category arises as the flattening of a stable ∞-category which arises canonically from the dg-category of chain complexes.

Potential Directions

A possible short-term goal would be to recast this talk in the language of ∞-categories, and examine A_∞ deformations.

I would also like to examine known stability manifolds and compare them to complexified Kähler moduli spaces

Finally, I would like to further explore the "wall-crossing" phenomenon of the flop transition. One expects the hearts of $\mathscr{T}$ to tile the moduli space/stability manifold, and such phenomena are seen in e.g. the gauged linear sigma model

Thank you for your time!

Notation

Throughout this part of the talk, I'll use the following notation (subscripts omitted when context makes it clear):

Symbol	Meaning
$\Omega^k_{X,\mathbb{R}}$	Sheaf of real k-forms on X
$\Omega^k_{X,\mathbb{C}}$	$\Omega_{X,\mathbb{R}} \otimes \mathbb{C}$ the sheaf of complex k-forms on X
$\Omega_X^{(p,q)}$	The (p,q) part of the Hodge decomposition of $\Omega^k_{X,\mathbb{C}}$
T_X	The complexified tangent bundle of X
$T_X^{(1,0)}$, $T_X^{(0,1)}$	The (anti)holomorphic tangent bundle of X
$\Omega_X^{(-p,q)}$	$\bigwedge^p T_X^{(1,0)} \otimes \Omega_X^{(0,q)}$ as sheaves
K_X	The canonical bundle on X

Appendix A

The Nonlinear Sigma Model

What is a Physical Theory?

When we refer to a *field theory*, we mean the following data:

- A manifold M (the *spacetime*) on which the fields lie (usually taken to be $\mathbb{R}^{3,1}$ for Minkowski space, or $\mathbb{R}^1$ for a classical particle)
- A collection of *fields*, which are specified by choosing vector bundles over M with connection. A field is either a section of (a power of) the tangent bundle, or a connection on a principal bundle, or a section of an associated bundle.
- An *action* functional, which takes in a configuration of fields and spits out a number.

What is a Physical Theory?

In classical mechanics, the goal is to find the configuration of fields that extremizes the action.

Example

Take $M = \mathbb{R}^{3,1}$ with a single scalar field φ, and action of the form

$$S[\varphi] = \int_M \left(\frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi \right) dx^4$$

This is extremized at $\left(\partial^2 - m^2\right)\varphi = 0$, which is the equation for a relativistic free particle of mass m.

What is a Physical Theory?

In quantum mechanics, the story changes a bit, and the fields get a new interpretation as elements of some C^* algebra acting on the *Hilbert space of states*

This algebra gets extra structure in the form of the *n*-point functions

$$\langle 0|\varphi(x_n), \ldots, \varphi(x_1)|0\rangle = N \int [D\varphi]\, \varphi(x_n) \ldots \varphi(x_1) \exp(iS[\varphi])$$

where the integral is the *path integral* over the configuration space for φ. There's some non-commutative subtlety going on here that we'll skip over.

What is a Physical Theory?

So, a specification of the spacetime M, the fields φ, and the action S completely determines the physical theory (the C^* algebra of fields and the n-point functions)

Note that this association is typically not injective, as many different actions could lead to the same physical theory. Adding in a field whose equations of motion are constantly zero is a trivial example.

The Physical Nonlinear Sigma Model

The nonlinear Σ model is the starting point. The data of a nonlinear Σ model is:

- The worldsheet Σ, a Riemann surface functioning as spacetime (possibly with boundary)
- The *target space* X, a Calabi-Yau manifold (usually of $\mathbb{C}$-dimension 3) which is smooth over $\mathbb{C}$.
- A *Complexified Kähler form* $B + iJ$ on X, which is the data of a Kähler form J and a closed, real-valued two form B.
- A choice of a square root of the canonical bundle K on Σ, denoted $\sqrt{K}$.
- a smooth map $f : \Sigma \to X$

Actually, when computing, we really want to integrate over all Σ, all $B + iJ$, and all f due to quantization

The Physical Nonlinear Sigma Model

This data assembles to a physical theory by specifying the following fermionic fields:

Field	Bundle
ψ_+^i	$\sqrt{K} \otimes f^*(T_X^{(1,0)})$
$\psi_+^{\bar{i}}$	$\sqrt{K} \otimes f^*(T_X^{(0,1)})$
ψ_-^i	$\overline{\sqrt{K}} \otimes f^*(T_X^{(1,0)})$
$\psi_-^{\bar{i}}$	$\overline{\sqrt{K}} \otimes f^*(T_X^{(0,1)})$

including as well the bosonic fields $x^i, x^{\bar{i}}$ given by the coordinates of f and the pullback of the complexified Kähler form on X.

The Physical Nonlinear Sigma Model

The action is the last bit of information, and it is given by

$$S\left[\Sigma, f, \psi\right] = \int_{\Sigma} \Big(\frac{1}{2}(g_{IJ} + iB_{IJ})\partial_z x^I \partial_z x^J + \frac{\sqrt{-1}}{2} g_{i\bar{i}} \psi_-^{\bar{i}} D_z \psi_-^{i} + \frac{\sqrt{-1}}{2} g_{i\bar{i}} \psi_+^{\bar{i}} D_z \psi_+^{i} + \left(R_{i\bar{i}j\bar{j}} \psi_+^{i} \psi_+^{\bar{i}} \psi_-^{j} \psi_-^{\bar{j}}\right)\Big) d^2 z$$

where I, J run across all coordinates, i, j runs across holomorphic coordinates, and $\bar{i}, \bar{j}$ runs across antiholomorphic coordinates

Topologically Twisting the Sigma Model

The *B-model topologically twisted theory* tensors the fermionic fields with $\sqrt{K}$, its conjugate, and their duals.

The new fields take values in

Field	**Bundle**
ψ_+^i	$K \otimes f^*(T_X^{(1,0)})$
$\psi_+^{\bar{i}}$	$f^*(T_X^{(0,1)})$
ψ_-^i	$\overline{K} \otimes f^*(T_X^{(1,0)})$
$\psi_-^{\bar{i}}$	$f^*(T_X^{(0,1)})$

From this, we get the summed fields that enter explicitly into the action:

$$\eta^{\bar{i}} = \psi_+^{\bar{i}} + \psi_-^{\bar{i}} \in f^* T_X^{(0,1)}$$

$$\rho^i = \psi_+^i + \psi_-^i \in \Gamma(\Omega^1_{\Sigma,\mathbb{C}} \otimes f^*(T_x^{(1,0)}))$$

$$\theta_i = g_{\bar{i}i}\left(\psi_+^{\bar{i}} - \psi_-^{\bar{i}}\right) \in f^*\Omega_X^{(0,1)}$$

The Modified Action

The new action with this twisting is given by

$$
\begin{aligned}
S[\Sigma, f, \eta, \rho, \theta] = \int_\Sigma \Big(& g_{IJ}\partial x^I \bar{\partial} x^J + i\eta^{\bar{i}}\left(D_z\rho^i_{\bar{z}} + D_{\bar{z}}\rho^i_z\right) \\
& + i\theta_i\left(D_{\bar{z}}\rho^i_z - D_z\rho^i_{\bar{z}}\right) \\
& + R_{i\bar{i}j\bar{j}}\rho^i_z\rho^j_{\bar{z}}\eta^{\bar{i}}\theta_k g^{k\bar{j}}\Big) d^2z
\end{aligned}
$$

The Supersymmetry Generator

This action in this form is invariant under an infinitesimal *supersymmetry* generated by

$$
\begin{aligned}
\delta x^{i} &= 0 \\
\delta x^{\bar{i}} &= i\alpha\eta^{\bar{i}} \\
\delta\psi_{+}^{i} &= -\alpha\partial x^{i} \\
\delta\psi_{+}^{\bar{i}} &= -i\alpha\psi_{-}^{\bar{j}}\Gamma_{\bar{j}\bar{k}}^{\bar{i}}\psi_{+}^{\bar{k}} \\
\delta\psi_{-}^{i} &= -\alpha\bar{\partial}x^{i} \\
\delta\psi_{-}^{\bar{i}} &= -i\alpha\psi_{+}^{\bar{j}}\Gamma_{\bar{j}\bar{k}}^{\bar{i}}\psi_{-}^{\bar{k}}
\end{aligned}
$$

where α is an infinitesimal section of $\sqrt{K}^{*}$.

The Supersymmetry Generator

The *supersymmetry operator* Q is defined by

$$\{Q(\alpha), W\} = \delta W$$

for any field W.

The action now becomes

$$\begin{aligned} S[\Sigma, f, \eta, \rho, \theta] &= i \int_\Sigma \{Q, \mathscr{D}\} + U \\ \mathscr{D} &= g_{j\bar{k}} \left(\rho_z^j \bar{\partial} x^{\bar{k}} \right) \\ U &= \int_\Sigma \left(-\theta_j D\rho^j - \frac{i}{2} R_{j\bar{j}k\bar{k}} \rho^j \rho^k \eta^{\bar{j}} \theta^{\bar{k}} \right) \end{aligned}$$

This action is invariant under deformation of the Kähler form, up to a Q-exact term.

Operators in Q-Cohomology

Local operators can be built up from our fields via products. Starting with

$$A = d\bar{z}^{\bar{j}_1} \dots d\bar{z}^{\bar{j}_q} A^{\bar{i}_1 \dots \bar{i}_p}_{\bar{j}_1 \dots \bar{j}_q} \partial_{\bar{i}_1} \dots \partial_{\bar{i}_p}$$

a $(0,q)$-form with values in $\bigwedge^p T_X^{(1,0)}$

We have:

$$W[A] = \eta^{\bar{j}_1} \dots \eta^{\bar{j}_q} A^{\bar{i}_1 \dots \bar{i}_p}_{\bar{j}_1 \dots \bar{j}_q} \theta_{\bar{i}_1 \dots \bar{i}_p}$$

an operator.

If we assemble all these W maps together, we get a big map

$$\bigoplus W : \bigoplus_{p,q} \Gamma(\Omega_X^{(-p,q)}) \to (\text{local operators in the CFT})$$

Q-Cohomology on Operators

The effect of Q on $W[A]$ is

$$\{Q, W[A]\} = W[\bar{\partial} A]$$

Passing to Q-cohomology (BRST quantization) amounts to considering operators that are Q-closed modulo those that are Q-exact.

This allows us to factor W as

$$\begin{tikzcd} \bigoplus_{p,q} \Gamma(\Omega^{(-p,q)}) \arrow[r, "W"] \arrow[dr] & \text{Local Operators} \arrow[r, "\text{BRST}"] & \text{BRST Cohomology} \\ & \bigoplus_{p,q} H^{(-p,q)}(X) \arrow[ur, "\widetilde{W}"] & \end{tikzcd}$$

Upshot: $\widetilde{W}$ is an isomorphism!

A New Action Term

New data we can add: a vector bundle E on X with connection 1-form A and field strength (curvature) F.

For each connected boundary C_k of Σ we can add a new term to the action:

$$S_{C_k} = \oint_{C_k} \left(f^*(A) - \sqrt{-1}\eta^{\bar{i}} F_{\bar{i}j} \rho^j \right)$$

For this to remain supersymmetric (Q-invariant) we need A to be holomorphic, so that F is purely of type $(1,1)$

$$F \in \Gamma(\Omega_X^{(1,1)} \otimes \mathrm{End}(E))$$

Hence, E must be a *holomorphic* vector bundle with connection.

New Operators

The new operators are now $(-p, q)$-forms on X with values in $\mathrm{End}(E)$. Furthermore, we have the relation

$$\theta_j = g_{j\bar{k}} \left(\psi_+^{\bar{k}} - \psi_-^{\bar{k}} \right) = F_{j\bar{k}} \eta^{\bar{k}}$$

and hence we can move all terms in the local operator to η terms, so the operators are of the form

$$\text{BRST local operators} = \bigoplus_q H^{(0,q)}(X, \mathrm{End}(E))$$

The number q associated to an operator is called the *ghost number* of the operator

Summary

Our theory now has the following data:

- A target space X, assumed to be Calabi-Yau, with a complexified Kähler form $B + iJ$
- A Riemann surface Σ with boundary components $\partial\Sigma = \sum_k C_k$
- A holomorphic vector bundle E with connection
- A ring of local BRST operators $H^{(0,q)}(X, \mathrm{End}(E))$

This is the data of a *topological sigma model* with a *Dn-brane* for n the (real) dimension of X

More generally, Dp-branes are vector bundles supported on complex submanifolds of (real) dimension p along with the reasonable generalization of the above picture

Appendix B

Computation of BRST Quantization of Complexes

Computing the New BRST Operators

Let $(\mathscr{E}^\bullet, d^E)$ and $(\mathscr{F}^\bullet, d^F)$ be objects in $\mathrm{Ch}^b(\mathrm{Coh}(X))$ i.e. bounded complexes of coherent sheaves.

The sheaf Hom gives us a double complex

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \big\downarrow{\scriptstyle d_1^{E*}} & & \big\downarrow{\scriptstyle d_1^{E*}} & & \\
\dots & \xrightarrow{d^F_{-1*}} & \mathscr{H}om(\mathscr{E}^1, \mathscr{F}^0) & \xrightarrow{d^F_{0*}} & \mathscr{H}om(\mathscr{E}^1, \mathscr{F}^1) & \xrightarrow{d^F_{1*}} & \dots \\
 & & \big\downarrow{\scriptstyle d_0^{E*}} & & \big\downarrow{\scriptstyle d_0^{E*}} & & \\
\dots & \xrightarrow{d^F_{-1*}} & \mathscr{H}om(\mathscr{E}^0, \mathscr{F}^0) & \xrightarrow{d^F_{0*}} & \mathscr{H}om(\mathscr{E}^0, \mathscr{F}^1) & \xrightarrow{d^F_{1*}} & \dots \\
 & & \big\downarrow{\scriptstyle d_{-1}^{E*}} & & \big\downarrow{\scriptstyle d_{-1}^{E*}} & & \\
 & & \vdots & & \vdots & &
\end{array}$$

Computing the new BRST Operators

We form the total complex by summing across antidiagonals

$$\dots \xrightarrow{d_{-1}} \mathscr{H}om^0(\mathscr{E}^\bullet, \mathscr{F}^\bullet) \xrightarrow{d_0} \mathscr{H}om^1(\mathscr{E}^\bullet, \mathscr{F}^\bullet) \xrightarrow{d_1} \dots$$

where

$$\mathscr{H}om^p(\mathscr{E}^\bullet, \mathscr{F}^\bullet) = \bigoplus_n \mathscr{H}om(\mathscr{E}^n, \mathscr{F}^{n+p})$$

is the grade-p part of the sheaf internal hom for $\mathrm{Ch}(\mathrm{Coh}(X))$ and

$$d_n = d_n^E + d_n^F$$

with d^E and d^F anticommuting.

Computing the Cohomology

This is the sequence on which d operates, and we need to tensor this with the sequence on which Q_0 operates and take the resulting cohomology.

The relevant double complex is:

$$
\begin{tikzcd}
 & \vdots & & \vdots & \\
\dots \arrow[r, "d"] & \Gamma(\Omega^{(0,1)}) \otimes \mathscr{H}om^0(\mathscr{E}^\bullet, \mathscr{F}^\bullet) \arrow[u, "Q_0"] \arrow[r, "d"] & \Gamma(\Omega^{(0,1)}) \otimes \mathscr{H}om^1(\mathscr{E}^\bullet, \mathscr{F}^\bullet) \arrow[u, "Q_0"] \arrow[r, "d"] & \dots \\
\dots \arrow[r, "d"] & \Gamma(\Omega^{(0,0)}) \otimes \mathscr{H}om^0(\mathscr{E}^\bullet, \mathscr{F}^\bullet) \arrow[u, "Q_0"] \arrow[r, "d"] & \Gamma(\Omega^{(0,0)}) \otimes \mathscr{H}om^1(\mathscr{E}^\bullet, \mathscr{F}^\bullet) \arrow[u, "Q_0"] \arrow[r, "d"] & \dots \\
 & \vdots \arrow[u, "Q_0"] & \vdots \arrow[u, "Q_0"] &
\end{tikzcd}
$$

Computing the Cohomology

This double complex is handled explicitly by a spectral sequence described by

$$E_2^{p,q} = H^p(X, H^q(\mathscr{H}\!om^\bullet(\mathscr{E}^\bullet, \mathscr{F}^\bullet))) \implies H_Q^{p+q}$$

which yields our desired Q-cohomology.

This spectral sequence abuts to the *hyperext* of the complexes

$$H_Q^n(\mathscr{E}^\bullet, \mathscr{F}^\bullet) = Ext^n(\mathscr{E}^\bullet, \mathscr{F}^\bullet)$$

In particular, $\mathrm{Ext}^0(\mathscr{E}^\bullet, \mathscr{F}^\bullet)$ are homotopy classes of maps between $\mathscr{E}^\bullet$ and $\mathscr{F}^\bullet$ of degree zero.