D stands for Derived: How D-Branes Build the Derived Category

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Although the derived category is well-known to algebraic geometers and commutative algebraists, only recently did it appear in the context of physics. For the next two weeks we will be reviewing Aspinwall and Lawrence's paper "Derived Categories and Zero-Brane Stability" ([AL01]) letting them guide us through the relevant string theory. This will provide motivation for why the derived category is expected to show up in the homological mirror symmetry conjecture, and may hint at additional structure not measured in $D^b(X)$.

We'll pick up where we left off last week, considering a nonlinear sigma model with one (graded sum of) boundary conditions present. Discontented with this, we'll examine deformations of the theory, leading naturally to cochain complexes showing up. Finally, to recover the deformed theory we'll recompute the BRST cohomology, leading to a nice proof of the equivalence between the category of boundary conditions in the B-model and the derived category. Time permitting, we can discuss various ways this equivalence can be enhanced by adding additional structure to $D^b(X)$.

1 Introduction to Deformations

Suppose that our boundary conditions are just a finite collection of vector bundles assembling into a single graded vector bundle

$$\mathscr{E} = \bigoplus_n \mathscr{E}^n$$

As noted, a string of ghost number q lies in

$$\varphi_q \in \bigoplus_{k,m} \operatorname{Ext}^k(\mathscr{E}^m, \mathscr{E}^{m-k+q})$$

Ghost number one operators can be used to deform the theory. For φ a ghost number one operator, adding to the action a term

$$\delta_\varphi S = t \oint_{C_k} \{G, \varphi\}$$

works. The effect of this (as [AL01] argues in §2.4) is that the BRST operator Q gets shifted by

$$Q \mapsto Q + \varphi$$

If $\varphi \in \operatorname{Ext}^1(\mathscr{E}^n, \mathscr{E}^n)$, then φ represents a first-order deformation of the coherent sheaf $\mathscr{E}$. Hence, deformations of this type correspond to "internal" deformations of the boundary conditions inside $\operatorname{Coh}(X)$.

If $\varphi \in \mathrm{Ext}^0(\mathscr{E}^n, \mathscr{E}^{n+1})$ on the other hand, we get something new. Generally, we can take φ to be a sum of such maps

$$\varphi = \sum_n d^n$$

where $d^n \in \mathrm{Hom}(\mathscr{E}^n, \mathscr{E}^{n+1})$. Since Q is nilpotent, consistency of this requires

$$Q^2 = (Q_0 + \delta Q)^2 = 0$$

or

$$Q_0^2 + (\sum_n d^n)^2 + \{Q_0, \varphi\} = 0$$

the first and last terms are zero, hence we require

$$d^{n+1} d^n = 0$$

but this data $(\mathscr{E}^\bullet, d)$ is exactly the data of a *complex* of coherent sheaves! Since we started with a finite collection of D-branes, this is a *bounded* complex.

What remains is to compute the new Q-cohomology operators. Notice that open strings can no longer couple to a single coherent sheaf, since the deformation d "tangles" the D-branes together.

2 The New BRST Cohomology

Let's allow for two distinct D-branes to exist, by allowing our complex of coherent sheaves to be a direct sum of complexes $(E, d) \cong (\mathscr{E}, d_E) \oplus (\mathscr{F}, d_F)$. As before, our open string operators can be built out of elements of $\Omega^{(0,p)} \otimes \mathscr{H}om(\mathscr{E}, \mathscr{F})$. Open strings are not tied to a specific grading, since a general boundary condition is now a complex of locally free sheaves (or coherent sheaves). Since the complexes are, well, complexes, we do get a nice decomposition

$$\mathscr{H}om(\mathscr{E}, \mathscr{F}) = \bigoplus_k \bigoplus_{p+q=k} \mathscr{H}om(\mathscr{E}^p, \mathscr{F}^q) = \bigoplus_{p,q} \mathscr{H}om(\mathscr{E}^p, \mathscr{F}^q$$

and the total differential acts as

$$d = d_E + d_F$$

with $d_E - d_F = 0$. Hence, this is a double complex

$$\begin{tikzcd}
 & \vdots \arrow[d, "d_2^E"] & \vdots \arrow[d, "d_2^E"] & \\
\cdots \arrow[r, "d_F^{-1}"] & \mathscr{H}om(\mathscr{E}^1, \mathscr{F}^0) \arrow[r, "d_F^1"] \arrow[d, "d_1^E"] & \mathscr{H}om(\mathscr{E}^1, \mathscr{F}^1) \arrow[r, "d_F^1"] \arrow[d, "d_1^E"] & \cdots \\
\cdots \arrow[r, "d_F^{-1}"] & \mathscr{H}om(\mathscr{E}^0, \mathscr{F}^0) \arrow[r, "d_F^0"] \arrow[d, "d_0^E"] & \mathscr{H}om(\mathscr{E}^0, \mathscr{F}^1) \arrow[r, "d_F^1"] \arrow[d, "d_0^E"] & \cdots \\
 & \vdots & \vdots &
\end{tikzcd}$$

This has an antidiagonal single complex defined by

$$\mathscr{H}om^n(\mathscr{E}^\bullet, \mathscr{F}^\bullet) = \bigoplus_{p+q=n} \mathscr{H}om(\mathscr{E}^p, \mathscr{F}^q)$$

with differential d. Total cohomology can be computed using the spectral sequence given, and will be given by the result

$$(E_0)^n_m = \mathscr{H}om(\mathscr{E}^m, \mathscr{F}^n) \implies H^{n-m}(\mathscr{H}om^\bullet(\mathscr{E}^\bullet, \mathscr{F}^\bullet)$$

Now consider the total operator algebra. The original BRST operator Q_0 acts as $\bar{\partial}$ and the deformation acts as $d = d^E + d_F$ on the tensor product

$$\Omega^{(0,\cdot)} \otimes \mathscr{H}om^\cdot(\mathscr{E}^\cdot, \mathscr{F}^\cdot)$$

yielding (yet again) a double complex

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & Q_0\uparrow & & Q_0\uparrow & & \\
\cdots & \xrightarrow{d} & \Omega^{(0,1)} \otimes \mathscr{H}om^0(\mathscr{E}^\bullet, \mathscr{F}^\bullet) & \xrightarrow{d} & \Omega^{(0,1)} \otimes \mathscr{H}om^1(\mathscr{E}^\bullet, \mathscr{F}^\bullet) & \xrightarrow{d} & \cdots \\
 & & Q_0\uparrow & & Q_0\uparrow & & \\
\cdots & \xrightarrow{d} & \Omega^{(0,0)} \otimes \mathscr{H}om^0(\mathscr{E}^\bullet, \mathscr{F}^\bullet) & \xrightarrow{d} & \Omega^{(0,0)} \otimes \mathscr{H}om^1(\mathscr{E}^\bullet, \mathscr{F}^\bullet) & \xrightarrow{d} & \cdots \\
 & & Q_0\uparrow & & Q_0\uparrow & & \\
 & & \vdots & & \vdots & &
\end{array}$$

which we strive to take total cohomology of. This is again done by a spectral sequence. Taking cohomology first in the horizontal direction yields the E_1 page

$$\begin{array}{cccc}
 & \vdots & \vdots & \\
 & Q_0\uparrow & Q_0\uparrow & \\
\cdots & \Omega^{(0,1)} \otimes H^0(\mathscr{H}om^\bullet(\mathscr{E}^\bullet, \mathscr{F}^\bullet)) & \Omega^{(0,1)} \otimes H^1(\mathscr{H}om^\bullet(\mathscr{E}^\bullet, \mathscr{F}^\bullet)) & \cdots \\
 & Q_0\uparrow & Q_0\uparrow & \\
\cdots & \Omega^{(0,0)} \otimes H^0(\mathscr{H}om^\bullet(\mathscr{E}^\bullet, \mathscr{F}^\bullet)) & \Omega^{(0,0)} \otimes H^1(\mathscr{H}om^\bullet(\mathscr{E}^\bullet, \mathscr{F}^\bullet)) & \cdots \\
 & Q_0\uparrow & Q_0\uparrow & \\
 & \vdots & \vdots &
\end{array}$$

and taking vertical cohomology yields the E_2 page

$$\begin{array}{cccc}
 & \vdots & \vdots & \\
\cdots & H^1(X, H^0(\mathscr{H}om^\bullet(\mathscr{E}^\bullet, \mathscr{F}^\bullet)) & H^1(X, H^1(\mathscr{H}om^\bullet(\mathscr{E}^\bullet, \mathscr{F}^\bullet)) & \cdots \\
\cdots & H^0(X, H^0(\mathscr{H}om^\bullet(\mathscr{E}^\bullet, \mathscr{F}^\bullet)) & H^0(X, H^1(\mathscr{H}om^\bullet(\mathscr{E}^\bullet, \mathscr{F}^\bullet)) & \cdots \\
 & \vdots & \vdots &
\end{array}$$

So that

$$E_2^{p,q} = H^p(X, H^q(\mathscr{H}om^\bullet(\mathscr{E}^\bullet, \mathscr{F}^\bullet)))$$

This spectral sequence is known as (a version of the) "local-to-global spectral sequence", and is known to abut to

$$E_2^{p,q} \implies \operatorname{Hom}^{p+q}(\mathscr{E}^\bullet, \mathscr{F}^\bullet)$$

where $\operatorname{Hom}^n(\mathscr{E}^\bullet, \mathscr{F}^\bullet)$ is the *hyperext* of the two complexes. In more words than necessary, let's unravel this. The functor

$$\mathscr{H}om^\bullet(\mathscr{E}, -) : K(X) \to K(X)$$

taking a complex $\mathscr{F}$ to its sheaf *hom* complex is right-derivable. Its right-derived functor is

$$R\,\mathscr{H}om(\mathscr{E}, -) : D^b(X) \to D^b(X)$$

Composing this with global sections functor $R\Gamma$ yields

$$R\operatorname{Hom}(\mathscr{E}, -)$$

the *hyperext derived functor* whose cohomology

$$\operatorname{Hom}^i(\mathscr{E}, -) = R^i\operatorname{Hom}(\mathscr{E}, -) = H^i(R\operatorname{Hom}(\mathscr{E}, -))$$

are the *hyperext groups.* By Grothendieck's composition of functors spectral sequence, we can compute

$$R(\Gamma \circ \mathscr{H}om^\bullet(\mathscr{E}, -)) = R(\operatorname{Hom}^\bullet(\mathscr{E}, -))$$

by the above spectral sequence. Hence, our spectral sequence abuts to hyperext.

References

[AL01] Paul S Aspinwall and Albion Lawrence. "Derived categories and zero-brane stability". In: *Journal of High Energy Physics* 2001.08 (2001), p. 004. URL: `https://arxiv.org/abs/hep-th/0104147`.