Model Categories, Topology for the Algebraist

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1 Introduction

The goal of this talk is to construct a homotopy category in a general, yet accessible context. Model categories provide a more universal framework to study anything involving weak equivalences, which includes the classical homotopy theory in topology, but also the general algebraic study of derived functors on Abelian categories. If you lose track of whats going on, keep in mind the model structure on e.g. the category of compactly generated topological spaces, defined in section 3.

2 Definitions and Preliminary Results

(Taken from Hirschhorn's *Model Categories and Their Localizations*)

In an effort to distill the essence of homotopy theory, we define a *Model Structure* on a category $\mathscr{C}$ to be a specification of three classes of morphisms:

- The Cofibrations Cof
- The Fibrations Fib
- The Weak Equivalences W

such that the following five axioms hold:

M1 (Limit Axiom) $\mathscr{C}$ is complete and cocomplete (it contains all small limits and colimits).

M2 (Two-of-Three Axiom) If f, g are composable, and two of the three f, g, fg are in W, then so is the third.

M3 (Retract Axiom) If f is a retract of g (in the category $\mathscr{C}^{\Delta^1}$) and g is in any of the three classes, than so is F (i.e. Fib, Cof, and W are all closed under morphism retracts)

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] \arrow[rr, bend left, "\mathbb{1}"] & X \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\
B \arrow[r] \arrow[rr, bend right, "\mathbb{1}"'] & Y \arrow[r] & B
\end{tikzcd}$$

M4 (Lifting Axiom) If the solid arrow diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

commutes, then the dashed arrow exists if either $i \in \mathrm{Cof}$ and $p \in \mathrm{Fib} \cap W$, or $i \in \mathrm{Cof} \cap W$ and $p \in \mathrm{Fib}$.

More succinctly, we'll say that $(\mathrm{Cof}, \mathrm{Fib} \cap W)$ and $(\mathrm{Cof} \cap W, \mathrm{Fib})$ form *lifting pairs*.

M5 (Factorization Axiom) Every map $g \in \mathrm{Mor}(\mathscr{C})$ can be factorized functorially in two ways:

(a) $g = qi$ with $i \in \mathrm{Cof}$ and $q \in \mathrm{Fib} \cap W$.

(b) $g = pj$ with $j \in \mathrm{Cof} \cap W$ and $p \in \mathrm{Fib}$.

As the axioms suggest, cofibrations behave like good monomorphisms, and fibrations behave like good epimorphisms, where "good" means they have nice lifting properties.

To ease the notation, we'll refer to morphisms in both Fib and W as *trivial fibrations*, and morphisms in both Cof and W as *trivial cofibrations*. Furthermore, if $\mathscr{C}$ has initial and terminal objects, we'll say an object X is *cofibrant* if the map $\star_i \to X$ from the initial object to X is a cofibration, and *fibrant* if the map $X \to \star_t$ from X to the terminal object is a fibration.

As it turns out, a specification of any two of the three classes of morphisms uniquely determines the third. For example, we can show that:

An easy result (from the axioms): a map is a weak equivalence if and only if it can be factored as a trivial cofibration followed by a trivial fibration.

For the other maps:

Theorem 2.1. *A map i is a cofibration if and only if it has the left lifting property with respect to all trivial fibrations.*

Proof. Recall that i is said to have the *left lifting property* with respct to a map p if for every solid-arrow diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

which commutes, the dotted arrow exists.

Suppose first that i is a cofibration. Then, by the lifting axiom we know that i has the left lifting property with respect to p for all trivial fibrations p.

Suppose for the converse that i has the left lifting property against all trivial fibrations. Now, by the factorization axiom we can factor i as $i = pj$ for p a trivial fibration and j a cofibration. Hence, i has the left lifting property with respect to p and the diagram

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"] & X \arrow[d, "p"] \\ Y \arrow[r, equal] \arrow[ur, "q"] & Y \end{tikzcd}$$

exists. Hence, we have the diagram

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "i"] & A \arrow[r, equal] \arrow[d, "j"] & A \arrow[d, "i"] \\
Y \arrow[r, "q"] \arrow[rr, bend right, "\mathbb{1}"] & X \arrow[r, "p"] & Y
\end{tikzcd}$$

so i is a retract of j and since j is a cofibration and the cofibrations are closed under retracts, i is a cofibration as well. ☐

One can prove equivalent statements for trivial cofibrations (tested against fibrations) fibrations (tested against trivial cofibrations) and trivial fibrations (tested against cofibrations).

Hence, cofibrations and trivial cofibrations are closed under coproducts, and fibrations and trivial fibrations are closed against products.

Even stronger, we can show (but I won't) that (trivial) cofibrations are closed under pushouts, and (trivial) fibrations are closed under pullbacks.

3 Early Examples

Model categories started their life in homotopy theory, so we'll review the classical construction now.

3.1 Example: Simplicial Sets

Consider the category Δ whose objects are given by $\{[n] \mid n \in \mathbb{N}\}$ equipped with the natural ordering, and morphisms are given by order-preserving maps of sets (i.e. the skeleton of the category of finite ordered sets). Functors from Δ^{op} to Set determine *Simplicial Sets* and the category $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})$ we will call sSet. If you wanna get fancy, every presheaf on Δ gives rise to a simplicial set, hopefully you can see where this can be generalized.

As expected, simplicial sets can be given a model structure which *models* what we expect the axioms to behave like. This will make precise my comments about heuristically understanding fibrations, cofibrations, and so on.

A quick definition:

Definition 3.1. *A* Kan fibration *is a morphism $X \to Y$ of simplicial sets which has the right lifting property with respect to all horn inclusions*

$$\begin{tikzcd}
\Lambda^k[n] \arrow[r] \arrow[d] & X \arrow[d] \\
\Delta[n] \arrow[r] \arrow[ur, "\exists"] & Y
\end{tikzcd}$$

where $\Lambda^k[n]$ is the boundary of the n-simplex $\Delta[n]$ obtained by omitting the kth face. ($\Delta[n]$ is the simplicial set $\mathrm{Hom}(\bullet, [n])$*)*

Without further ado, here is the model structure on sSet:

Definition 3.2. *Three classes of morphisms can be defined on* sSet*:*

Cof The Cofibrations *are the momomorphisms (which, in* sSet *are degree-wise injections).*

W *The* Weak Equivalences *are the maps which induce weak homotopy equivalences (on the geometric realization of the simplicial sets).*

Fib *The* Fibrations *are the Kan fibrations.*

Theorem 3.1. *The above classes make* sSet *into a model category.*

The proof for this can be found in many places, such as Joyal and Tierney's *An Introduction to Simplicial Homotopy Theory* or similar. The proofs are vaguely combinatorial, and I didn't see them as very enlightening. Maybe it would be better to work with its geometric realization?

Its also provable that this is the smallest model structure with monic cofibrations and homotopy weak equivalences as the weak equivalences, so in a way this example models nicely what we want a model category to look like.

3.2 Pushing Model Structures Around

Under what circumstances can we transport model structures around? Suppose $\mathscr{M}$ is a model category, and $\mathscr{C}$ is any category, and $G : \mathscr{C} \to \mathscr{M}$ is a functor. If G admits a left adjoint $F : \mathscr{M} \to \mathscr{C}$, we can define some classes on $\mathscr{C}$ as:

W For f a morphism in $\mathscr{C}$, if $G(f)$ is a weak equivalence in $\mathscr{M}$, we'll say f is a weak equivalence in $\mathscr{C}$.

Fib For f a morphism in $\mathscr{C}$, we'll say f is a fibration if $G(f)$ is.

Cof The cofibrations in $\mathscr{C}$ are the ones generated by the other two classes. Namely, they are all maps with the left lifting property against the trivial fibrations.

Theorem 3.2. *(Found in Arpon Raksit's notes "Defining the Cotangent Complex") With the situation above, if*

- $\mathscr{C}$ *is complete and cocomplete.*
- $\mathscr{M}$ *is cofibrantly generated (which* sSet *is)*
- G *preserves sequential colimits*
- *A cofibration in* $\mathscr{C}$ *with the left lifting property against all fibrations is trivial.*

then the structure defined above makes $\mathscr{C}$ *into a model category, and* (F, G) *form a Quillen adjunction (see section* 8*)*

3.3 Example: Singular Homology

We can put a model structure on Top (maybe restrict to compactly generated topological spaces) using this result.

Definition 3.3. *Three classes of morphisms are defined:*

Cof *the* Cofibrations *are computed from the other two.*

Fib *the* Fibrations *are the Serre fibrations (they have the right lifting property with respect to all inclusions* $D^n \to D^n \times I$*)*

W the Weak Equivalences *are the weak homotopy equivalences (induces isomorphisms on all homotopy groups)*

To each simplicial object $[n] \in \Delta$ we can associate to it the geometric realization Δ_n as the standard n-simplex. This association is clearly functorial, hence we can compose it with contravariant Hom to get

$$\mathrm{Sing}(X)_\bullet = \mathrm{Hom}_{\mathrm{Top}}(\Delta_\bullet, X)$$

the *singular simplicial set* associated to X. Thus, to each topological space we can associate to it a simplicial set (in a functorial way).

As we'll see, the functor Sing is right-adjoint to the geometric realization functor $|\bullet|$ and this adjunction behaves really well with respect to the model structures (it is a *Quillen Equivalence*). This is not surprising, as we have essentially pulled back the model structure on sSet to Top.

3.4 Example: Dold-Kan Correspondence

Let $\mathscr{A}$ be any Abelian category, with $\mathscr{A}^{\Delta^{\mathrm{op}}}$ the category of simplicial objects in $\mathscr{A}$, and $\mathscr{A}^{\Delta}$ the cosimplicial objects.

Theorem 3.3. *There is an equivalence of categories between $\mathscr{A}^{\Delta^{op}}$ and $\mathrm{Ch}_\bullet^+(\mathscr{A})$ the category of connective (all negative entries are zero) chain complexes in $\mathscr{A}$. This is realized by the functors $N : \mathscr{A}^{\Delta^{op}} \to \mathrm{Ch}_\bullet^+(\mathscr{A})$ the normalized chains complex functor and $\Gamma : \mathrm{Ch}_\bullet^+(\mathscr{A}) \to \mathscr{A}^{\Delta^{op}}$ the realization functor.*

In this case, we can define these explicitly. The geometric realization is given by

$$\Gamma(V_\bullet)_n = \bigoplus_{[n]\to[k]\text{surj.}} V_k$$

with morphisms $\theta : [m] \to [n]$ inducing

$$\theta^* : \bigoplus_{[n]\to[k]\text{surj.}} V_k \to \bigoplus_{[m]\to[r]\text{surj.}} V_r$$

which is given on $\sigma : [n] \to [k]$ by factorizing $\sigma\theta : [m] \to [k]$ as an epi-mono $[m] \to [s] \to [k]$ and mapping

$$V_k \to V_s \to \bigoplus_{[m]\to[r]\text{surj.}} V_r$$

In the other direction, we have the normalized chains complex:

$$N(A_\bullet)$$

which, loosely speaking, is the chain complex from the face maps modulo the degenerate simplices. More specifically, for $A_\bullet$ a simplicial object in $\mathscr{A}$

$$C(A_\bullet)_n = A_n$$

with differential given by

$$\partial_n = \sum_i (-1)^i d_i$$

for d_i induced by $[n-1] \to [n]$ given by skipping the ith spot.

Similarly, define the degenerate complex

$$D(A_\bullet)_n = \langle \cup_i s_i(A_{n-1}) \rangle$$

generated by the degenerate simplices (where s_i is induced by the map $[n] \to [n-1]$ given by contracting the i and $i+1$ position).

Then, we can take $N(A)$ to be $C(A)/D(A)$.

As it turns out, this pair of functors satisfies the hypotheses necessary to carry the model structure from $\mathscr{A}^{\Delta^{op}}$ to $\mathrm{Ch}_\bullet^+(\mathscr{A})$, where we get the model structure on $\mathscr{A}^{\Delta^{op}}$ via the forgetful functor to sSet. This gives us the standard model structure on the connective chain complexes, which is given by:

W the *weak equivalences* are the quasi-isomorphisms

Fib the *fibrations* are the morphisms that are epimorphisms in $\mathscr{A}$ in each positive degree

Cof the *cofibrations* are the degreewise monomorphisms with degreewise projective cokernel.

If we dualize to get cosimplicial objects, we get another model structure on the connective cochain complexes:

W the *weak equivalences* are the quasi-isomorphisms

Fib the *fibrations* are degreewise epimorphisms with injective kernel

Cof the *cofibrations* are the morphisms that are monomorphisms in $\mathscr{A}$ in each positive degree

What are the cofibrant objects in the projective structure?

The projective resolutions!

What are the fibrant objects in the injective structure?

The injective resolutions!

Small caveat to mention: if $\mathscr{A}$ does not have enough projectives, we won't be able to satisfy M5 the factorization axiom for the first structure, and dually if $\mathscr{A}$ does not have enough injectives we can't factorize for the second structure. In particular, M5 (should be) the only axiom that fails generally here. (Are there other model structures that handle this better?)

4 Some Homotopy Theory

Now that we have a model structure, let's play with it a bit. Throughout, we'll be in the context of a general model category $\mathscr{M}$.

Definition 4.1. *A* Cylinder Object *for an object X is a factorization of the map*

$$X \coprod X \to X$$

into a cofibration followed by a weak equivalence. That is, it is an object $\operatorname{Cyl} X$ *along with maps*

$$X \coprod X \xrightarrow{i_0 \coprod i_1} \operatorname{Cyl}(X) \xrightarrow{p} X$$

with $i_0 \coprod i_1$ a cofibration and p a weak equivalence.

Definition 4.2. *Dually, a* Path Object *for an object Y is a factorization of*

$$Y \to Y \prod Y$$

the diagonal map into a weak equivalence followed by a fibration. That is, it is an object $\operatorname{Path}(Y)$ *along with maps*

$$Y \xrightarrow{s} \operatorname{Path}(Y) \xrightarrow{p_0 \times p_1} Y \prod Y$$

with s a weak equivalence and $p_0 \times p_1$ a fibration.

With these, we can start homotopy theory.

Definition 4.3. *A* left homotopy *for maps $f, g : X \to Y$ from f to g, denoted $f \simeq^{\ell} g$ is a cylinder object* $\operatorname{Cyl}(X)$ *for X and a map $H : \operatorname{Cyl}(X) \to Y$ such that $Hi_0 = f$ and $Hi_1 = g$.*

Dually, a right homotopy *for maps $f, g : X \to Y$ from f to g, denoted $f \simeq^{r} g$ is a path object* $\operatorname{Path}(Y)$ *for Y along with a map $H : X \to \operatorname{Path}(Y)$ for which $p_0 H = f$ and $p_1 H = g$.*

If f is both left and right homotopic to g, we say that f is homotopic to g.

Existence of cylinder and path objects follows immediately from M5.

Theorem 4.1. *If $f, g : X \to Y$ are left homotopic and Y is fibrant, then there is a cylinder object of X for which $\operatorname{Cyl}(X) \to X$ is a trivial fibration, and a left homotopy $H : \operatorname{Cyl}(X) \to Y$.*

(and the dual statement)

Proof. Since $f \simeq^{\ell} g$ we have a cylinder object $p' : \operatorname{Cyl}(X)' \to X$ and a map $H' : \operatorname{Cyl}(X)' \to Y$. If we factor p' as a cofibration followed by a trivial fibration, we get a new object $\operatorname{Cyl}(X)$ for which

$$\operatorname{Cyl}(X)' \to \operatorname{Cyl}(X) \to X$$

is a cofibration followed by a trivial fibration. The fact that the middle term is a cylinder object follows from the definitions.

Then, we have the diagram

$$\begin{tikzcd} \operatorname{Cyl}(X)' \arrow[r, "H'"] \arrow[d, "j"] & Y \arrow[d] \\ \operatorname{Cyl}(X) \arrow[r] \arrow[ur, dashed, "H"] & * \end{tikzcd}$$

where j, by two out of three, is a trivial cofibration and hence has the left lifting property against fibrations, so the dashed arrow exists. $\square$

Theorem 4.2. *If* $\mathrm{Cyl}(X)$ *is a cylinder object for* X*, then the injections* $X \coprod X \to \mathrm{Cyl}(X)$ *are weak equivalences. If* X *is cofibrant, then they are trivial cofibrations.*

(and the dual statement)

Proof. Two-of-Three guarantees the injections are weak equivalences. Furthermore, if X is cofibrant, then we have the pushout diagram

$$\begin{tikzcd}
*_i \arrow[r] \arrow[d] & X \arrow[d, "i_1"] \\
X \arrow[r, "i_0"] & X \coprod X
\end{tikzcd}$$

and pushouts of cofibrations are cofibrations. $\square$

As it stands, we don't yet know if homotopy equivalence is an equivalence! Let's get that out of the way now.

Theorem 4.3. *if* X *is cofibrant, then left homotopy is an equivalence relation on the maps* $\mathrm{Hom}(X, Y)$*.*

(and the dual)

Proof. Since X is its own cylinder object, for any $f : X \to Y$ we have $f \simeq^\ell f$.

To show symmetry, observe that for every homotopy $H : \mathrm{Cyl}(X) \to Y$ exhibiting $f \simeq^\ell g$ we can construct an inverse homotopy in the following way: We can construct the inverse cylinder object $\mathrm{Cyl}^{-1}(X)$ by flipping the injections from $X \coprod X$ to $\mathrm{Cyl}(X)$, and $H : \mathrm{Cyl}^{-1}(X) \to Y$ makes $g \simeq^\ell f$.

To show transitivity, observe that if X is cofibrant, we can push out two cylinder objects to get a third. Namely, we have the diagram:

$$\begin{tikzcd}
& & X \arrow[d, "i_0"] & \\
& X \arrow[r, "i_1"] \arrow[d, "i_0'"] & \mathrm{Cyl}(X) \arrow[d] \arrow[dr, bend left, "p"] & \\
X \arrow[r, "i_1'"] & \mathrm{Cyl}(X)' \arrow[r] \arrow[rr, bend right, "p'"'] & \mathrm{Cyl}(X)'' \arrow[r, "p''"] & X
\end{tikzcd}$$

for which two-of-three for p'' gets that it is a weak equivalence (notice that p is a weak equivalence, and i_0' is a trivial cofibration so its pushforward is as well), and the compositions from X to $\mathrm{Cyl}(X)''$ are trivial cofibrations, since they are compositions with inclusions and pushouts of the inclusions, both of which are trivial cofibrations.

Hence, homotopies can be composed. Namely, if $H : \mathrm{Cyl}(X) \to Y$ and $H' : \mathrm{Cyl}(X)' \to Y$ are homotopies exhibiting $f \simeq^\ell g$ and $g \simeq^\ell h$, then we can define their composition $H'' : \mathrm{Cyl}(X)'' \to Y$ from the pushout diagram which exhibits $f \simeq^\ell h$ as desired. $\square$

Finally, as we have seen, cofibrant objects are very nice. Namely,

Theorem 4.4. *If* X *is cofibrant, and* $f, g : X \to Y$ *are left homotopic, then they are also right homotopic.*

(and the dual)

Proof. Let $G : \mathrm{Cyl}(X) \to Y$ be a left homotopy for $f \simeq g$. Then we have the diagram

$$\begin{tikzcd} X \arrow[r, "sf"] \arrow[d, "i_0"] & \mathrm{Path}(Y) \arrow[d, "p_0 \times p_1"] \\ \mathrm{Cyl}(X) \arrow[r, "fp \times G"] \arrow[ur, dotted] & Y \coprod Y \end{tikzcd}$$

where the lifting exists since i_0 is a trivial cofibration and hence has the left lifting property with respect to the fibration $p_0 \times p_1$.

Then, the composition $H = hi_1$ is the desired homotopy. □

Hence, for objects that are both cofibrant and fibrant, both notions of homotopy coincide, and everything nice happens.

5 The Classical Homotopy Category

First, observe that the notion of homotopy equivalence composes nicely. Namely, if $f \simeq g$ as maps from X to Y, and $h : Y \to Z$ is any map, then $h_* : \pi^\ell(X, Y) \to \pi^\ell(X, Z)$ is well-defined, since $hf \simeq^\ell hg$. (similarly for pullbacks)

Hence, for the cofibrant-fibrant objects in $\mathscr{M}$, if $f \simeq g$ from X to Y and $h \simeq k$ from Y to Z, then $hf \simeq kg$. Hence, we have a well-defined composition

$$\circ : \pi(X, Y) \times \pi(Y, Z) \to \pi(X, Z)$$

So, we can define the "classical homotopy category" $\pi\mathscr{M}$ as the category where

1. Objects are the cofibrant-fibrant objects of $\mathscr{M}$
2. Morphisms $\mathrm{Hom}_{\pi\mathscr{M}}(X, Y)$ are the homotopy equivalence classes of maps $\pi(X, Y)$.

Now, if we expect this to look like the derived category, we need to find inverses for all the weak equivalences. As it turns out, this is done easily.

Lemma 5.1. *If A is cofibrant and $p : X \to Y$ is a trivial fibration, then $p_* : \pi^\ell(A, X) \to \pi^\ell(A, Y)$ is an isomorphism.*

(and the dual)

Proof. Suppose $g \in \mathrm{Hom}(A, Y)$. We have the diagram

$$\begin{tikzcd} *_i \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ A \arrow[r, "g"] \arrow[ur, dashed, "f"] & Y \end{tikzcd}$$

where f exists since the left arrow is a cofibration and p is a trivial fibration. Hence $p_* : \pi^\ell(A, X) \to \pi^\ell(A, Y)$ is surjective.

Now, suppse $f, g : A \to X$ are such that $pf \simeq^\ell pg$. Constructing the homotopy $H : \mathrm{Cyl}(A) \to Y$ from pf to pg we get

$$\begin{tikzcd} A \coprod A \arrow[r, "f \coprod g"] \arrow[d] & X \arrow[d, "p"] \\ \mathrm{Cyl}(A) \arrow[r, "F"] \arrow[ur, dashed, "G"] & Y \end{tikzcd}$$

where the dashed arrow exists since the left arrow is a trivial cofibration (A is cofibrant so by 4.2 we get that the injections on the left are trivial cofibrations) and p is by hypothesis a fibration. Hence, G exhibits $f \simeq^{\ell} g$. $\square$

Theorem 5.1 (Whitehead Theorem)**.** *Let $f : X \to Y$ be a weak equivalence in $\mathscr{M}$ between cofibrant-fibrant objects. Then, it is an isomorphism in the classical homotopy category.*

Proof. Begin by factoring f as a cofibration followed by a trivial fibration $f = qp$ for $p : X \to W$ and $q : W \to Y$. W is also cofibrant, and two-of-three tells us that p is a trivial cofibration. Since compositions of invertible morphisms are invertible, we'll show that each p and q has a homotopy inverse.

So, consider $p : X \to W$ the trivial cofibration. We have the diagram

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "p"'] & X \arrow[d] \\ W \arrow[r] \arrow[ur, dashed, "r"] & * \end{tikzcd}$$

where the dashed line exists since p is a trivial cofibration and thus has the left-lifting property against fibrations. Hence $rp = \mathbb{1}_X$.

Now, precomposition with p, as it is a trivial cofibration, induces isomorphisms on the level of $\pi(W, W) \to \pi(X, W)$ (see Lemma 5.1) and so

$$p^*(pr) = prp = p(\mathbb{1}_X) = p = p^*(\mathbb{1}_W)$$

shows that $pr \simeq \mathbb{1}_W$. Hence, p and r are mutual inverses in the classical homotopy category. $\square$

Theorem 5.2. *Let $f : X \to Y$ be an isomorphism in the classical homotopy category. Then, every representative for f in the model category is a weak equivalence.*

A proof for this can be found in the text (Hirschhorn 7.8) but I'll not include it here.

6 The Homotopy Category

For a general category with weak equivalences, we can, in some circumstances, define the localization of $\mathscr{C}$ with respect to W as the unique (up to natural equivalence) category $L_W\mathscr{C}$ such that for any $F : \mathscr{C} \to \mathscr{D}$ taking weak equivalences to isomorphisms, we have

$$\begin{tikzcd} \mathscr{C} \arrow[rr, "F"] \arrow[dr, "\gamma"] & & \mathscr{D} \\ & L_W\mathscr{C} \arrow[ur] & \end{tikzcd}$$

which is 2-coherent (i.e. commutative up to natural isomorphism).

For $\mathscr{M}$ a model category, we can define its *homotopy category* as the localization of $\mathscr{M}$ along the weak equivalences. If $\mathscr{M}$ is small, we can define this explicitly. The objects of $\mathrm{Ho}(\mathscr{M})$ are the objects of $\mathscr{M}$ and the morphisms $\mathrm{Hom}(X, Y)$ are all the morphisms in $\mathscr{M}$ along with formal inverses for the weak equivalences in $\mathscr{M}$.

While this description of the homotopy category is nice, it doesn't lend itself well to much of anything. Here's a better definition:

The objects of $\mathrm{Ho}(\mathscr{M})$ are the objects of $\mathscr{M}$. The Hom-sets can be described as

$$\mathrm{Hom}(X,Y) = \pi(FCX, FCY)$$

for FCX the fibrant-cofibrant approximation to X. We also get a functor $\gamma : \mathscr{M} \to \mathrm{Ho}\,\mathscr{M}$ which sends an object to itself, and does the following to morphisms. For each $f \in \mathrm{Hom}_{\mathscr{M}}(X,Y)$ we have the diagram

$$\begin{tikzcd}
& *_i \arrow[rr] \arrow[d] & & CY \arrow[d] \\
& CX \arrow[r] \arrow[urr, dashed, "C(f)"] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

where the dashed arrow exists because the right arrow is a trivial fibration, and the left arrow is a cofibration.

Next, we have the diagram

$$\begin{tikzcd}
CX \arrow[r, "C(f)"] \arrow[d] & CY \arrow[r] & FCY \arrow[d] \\
FCX \arrow[rr] \arrow[urr, dashed, "FC(f)"] & & *
\end{tikzcd}$$

where the dashed arrow exists because the right arrow is a fibration, and the left arrow is a trivial cofibration. We define $\gamma(f)$ to be $FC(f)$ thought of as a morphism in $\mathrm{Ho}(\mathscr{M})$.

Theorem 6.1. *The classical homotopy category defined in section 5 is the homotopy category (up to natural equivalence)*

Proof. This proof is dumb.

Let $\gamma_{FC} : \pi\mathscr{M} \to \mathrm{Ho}(\mathscr{M})$ be the embedding of $\pi\mathscr{M}$ into $\mathrm{Ho}(\mathscr{M})$ as described above. Consider an inverse $\eta_{FC} : \mathrm{Ho}(\mathscr{M}) \to \pi\mathscr{M}$ as

$$\eta_{FC}(X) = FCX$$
$$\eta_{FC}(f) = f$$

Notice that $\eta_{FC}\gamma_{FC}$ is the identity on $\pi\mathscr{M}$, so all that is left to do is define a natural equivalence from the identity on $\mathrm{Ho}(\mathscr{M})$ to $\gamma_{FC}\eta_{FC}$.

So, consider $\theta_{FC} : \mathbb{1}_{\mathrm{Ho}(\mathscr{M})} \Rightarrow \gamma_{FC}\eta_{FC}$ defined by

$$\theta_X : X \to \gamma_{FC}\eta_{FC}X = FCX$$

being the homotopy class of the identity map of FCX.

Sanity check:

$$\mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})}(X, FCX) = \pi(FCX, FCFCX) = \pi(FCX, FCX)$$

$\square$

7 Derived Functors

Again, $\mathscr{M}$ is our model category, and $\mathscr{C}$ is any category, and $\varphi : \mathscr{M} \to \mathscr{C}$ a functor.

We can define:

Definition 7.1. *The* left derived functor *to φ, denoted $L\varphi$, from $\mathrm{Ho}(\mathscr{M})$ to $\mathscr{C}$ is the right Kan extension of φ along $\gamma : \mathscr{M} \to \mathrm{Ho}(\mathscr{M})$. That is, it is a functor $L\varphi : \mathrm{Ho}(\mathscr{M}) \to \mathscr{C}$ along with a natural transformation $\varepsilon : L\varphi \circ \gamma \to \varphi$ which is "closest to φ from the left"*

(dual definition of the right derived functor)

Theorem 7.1. *If $\varphi : \mathscr{M} \to \mathscr{C}$ takes trivial cofibrations between cofibrant objects to isomorphisms in $\mathscr{C}$, then $L\varphi$ exists.*

(dual for $R\varphi$)

Without proving this result in full, let me tell you what $L\varphi$ is. If we precompose φ with the cofibrant approximation functor C, we get something that sends all weak equivalences to isomorphisms. Thus, by the universality of the homotopy category, we get a functor from $\mathrm{Ho}(\mathscr{M})$ to $\mathscr{C}$ which satisfies all the properties we need.

Just so that we know what they are, here's a cool definition:

Definition 7.2. *The* total left derived functor *of a functor $\varphi : \mathscr{M} \to \mathscr{N}$ of model categories is the left derived functor of the composition $\gamma \circ \varphi : \mathscr{M} \to \mathrm{Ho}(\mathscr{N})$*

(dual statement for right derived)

Theorem 7.2. *If φ takes trivial cofibrations between cofibrant objects in $\mathscr{M}$ into weak equivalences in $\mathscr{N}$ then the total left derived functor exists.*

Proof. Immediate from the previous result. □

8 Bonus content: Quillen Pairs

Suppose (F, G) is an adjunction of functors $F : \mathscr{M} \leftrightarrow \mathscr{N} : G$ between model categories.

Definition 8.1. *If F preserves both cofibrations and trivial cofibrations, or if G preserves both fibrations and trivial fibrations, we say that (F, U) is a Quillen pair.*

Theorem 8.1. *If (F, G) form a Quillen pair, the total left derived functor LF of F exists, the total right derived functor RG exists, and (LF, RG) form an adjoint pair.*

Not gonna prove this here, but it's not too bad, see the book.

Definition 8.2. *If for every cofibrant B in $\mathscr{M}$, every fibrant X in $\mathscr{N}$, and every map $f : B \to GX$ in $\mathscr{M}$, the map f is a weak equivalence in $\mathscr{M}$ if and only if the reflected map $f^\sharp : FB \to X$ is a weak equivalence in $\mathscr{N}$, then (F, G) is a Quillen equivalence.*

Theorem 8.2. *If (F, G) is a Quillen equivalence, then the derived functors (LF, RG) are equivalences of the homotopy categories of $\mathscr{M}$ and $\mathscr{N}$.*