DTU Course 02156 Logical Systems and Logic Programming (2021)

Week	Date	Main Topics (Prolog Programming in All Lessons)
35 #01	31/8	Course Prerequisites & Tutorial on Logical Systems and Logic Programming
36 #02	7/9	Chapter 1 - Introduction (Prolog Note)
37 #03	14/9	Chapter 2 - Propositional Logic: Formulas, Models, Tableaux
38 #04	21/9	Chapter 3 - Propositional Logic: Deductive Systems
39 #05	28/9	"Isabelle" - Propositional Logic: Sequent Calculus Verifier (SeCaV)
40 #06	5/10	Chapter 4 - Propositional Logic: Resolution
41 #07	12/10	Chapter 7 - First-Order Logic: Formulas, Models, Tableaux
42		(Autumn Vacation)
43 #08	26/10	Chapter 8 - First-Order Logic: Deductive Systems
44 #09	2/11	"Isabelle" - First-Order Logic: Sequent Calculus Verifier (SeCaV)
45 #10	9/11	Chapter 9 - First-Order Logic: Terms and Normal Forms
46 #11	16/11	Chapter 10 - First-Order Logic: Resolution
47 #12	23/11	Chapter 11 - First-Order Logic: Logic Programming
48 #13	30/11	Chapter 12 - First-Order Logic: Undecidability and Model Theory & Course Evaluation

Responsible: Associate Professor Jørgen Villadsen <jovi@dtu.dk>

Assignments & Exam **MUST BE SOLVED INDIVIDUALLY**

Assignment-1 Deadline Sunday 26/9 (Available Wednesday 15/9)

Assignment-2 Deadline Sunday 10/10 (Available Wednesday 29/9)

Assignment-3 Deadline Sunday 31/10 (Available Wednesday 13/10)

Assignment-4 Deadline Sunday 14/11 (Available Wednesday 3/11)

Assignment-5 Deadline Thursday 2/12 (Available Wednesday 17/11)

Written Exam Tuesday 14/12 (2 Hours / No Computer / All Notes Allowed)

The mandatory assignments and the written exam are evaluated as a whole – even if you do well in the mandatory assignments then you still must do decent in the written exam in order to pass the course!

A TEACHER MUST IMMEDIATELY REPORT ANY SUSPICION OF CHEATING TO THE STUDY ADMINISTRATION FOR FURTHER ACTIONS

Queue as two stacks using list reverse (constant amortized complexity)

```
empty(aqueue([], [])).

enqueue(X, aqueue(Xs, Ys), aqueue([X|Xs], Ys)).

dequeue(Y, aqueue(Xs, [Y|Ys]), aqueue(Xs, Ys)).
dequeue(Y, aqueue(Xs, []), aqueue([], Ys)) :- reverse(Xs, [Y|Ys]).
```

```
? empty(Q0), enqueue(N, Q0, Q1), dequeue(N, Q1, Q2).

Q0 = Q2
Q1 = aqueue([N], [])
Q2 = aqueue([], [])

Yes
```

Haskell

```
import Prelude (Maybe(..), print, reverse)

data Queue a = AQueue [a] [a]

empty = AQueue [] []

enqueue x (AQueue xs ys) = AQueue (x : xs) ys

dequeue (AQueue [] []) = (Nothing, AQueue [] [])
dequeue (AQueue xs (y : ys)) = (Just y, AQueue xs ys)
dequeue (AQueue (x : xs) []) =
  (case reverse (x : xs) of y : ys -> (Just y, AQueue [] ys))

main = print (case dequeue (enqueue 0 empty) of (Just x, _) -> x)
```

Isabelle

```
theory Queue imports Main begin

datatype 'a queue = AQueue ‹'a list› ‹'a list›

definition empty :: ‹'a queue› where ‹empty ≡ AQueue [] []›

fun enqueue where
  ‹enqueue x (AQueue xs ys) = AQueue (x # xs) ys›

fun dequeue where
  ‹dequeue (AQueue [] []) = (None, AQueue [] [])› |
  ‹dequeue (AQueue xs (y # ys)) = (Some y, AQueue xs ys)› |
  ‹dequeue (AQueue (x # xs) []) = (case rev (x # xs) of y # ys ⇒ (Some y, AQueue [] ys))›

theorem ‹(case (dequeue (enqueue n empty)) of (Some x, _) ⇒ x) = n›
  unfolding empty_def by simp

export_code empty enqueue dequeue in Haskell

end
```

Info

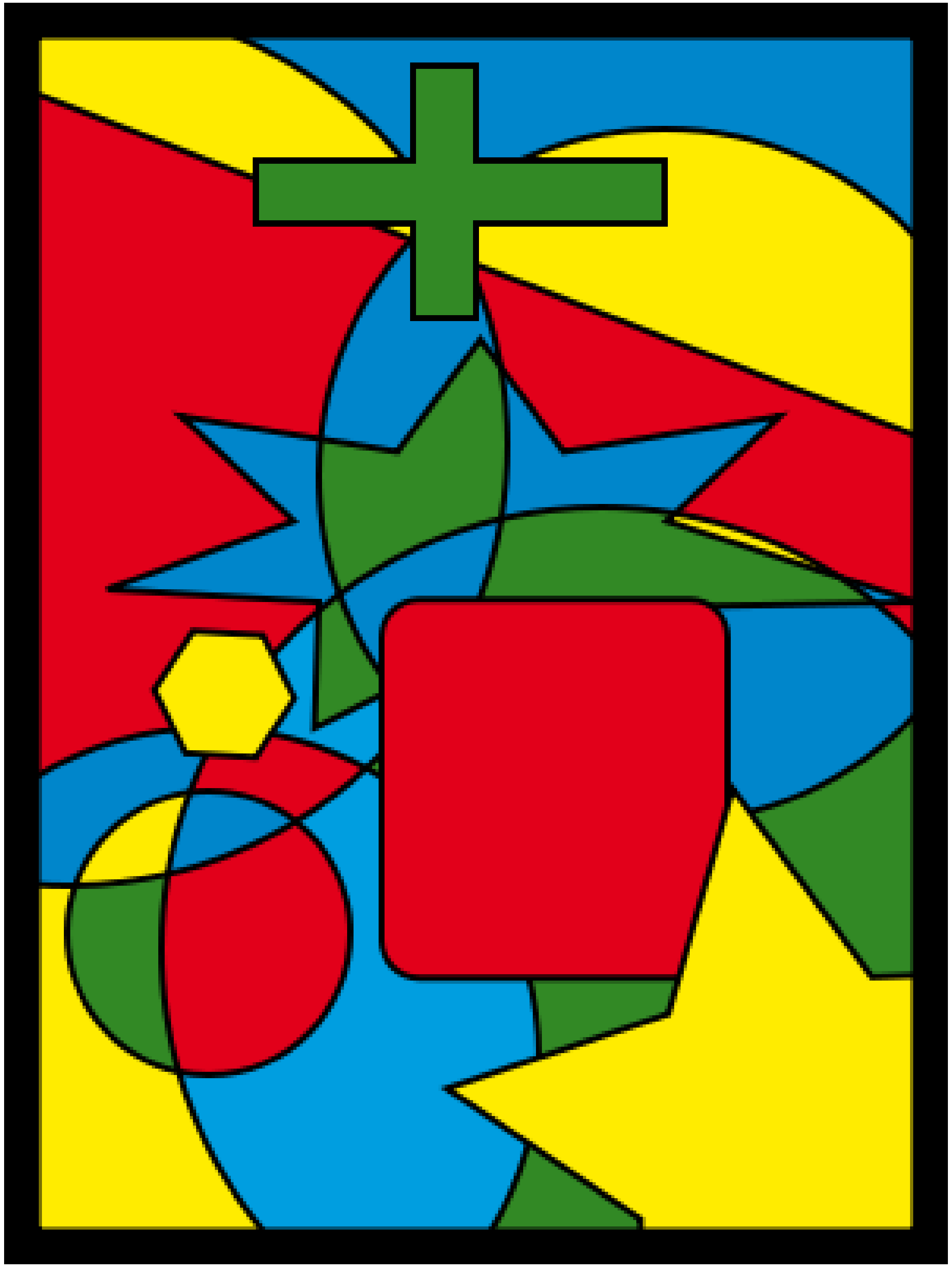
Search: 50%

```
Found termination order: "{}"
```

Info

Search: 50%

```
See theory exports
```



02156 Highlight Weeks 1-5

Important Prolog programs to study:

- Quicksort & Mergesort Algorithms
- Member & Append `basic.pl`
- Truth Table Checker `logic.pl`
- Map Colouring `map.pl`
- Resolution Prover `resolution.pl`

The Four Colour Theorem was the first major theorem to be proven using a computer…

Picture source:

http://commons.wikimedia.org/wiki/Image:Four_Colour_Map_Example.svg

Agenda — Week #4

The four colour theorem — Computer proof — Exercise

The tracer in SWI-Prolog

Prolog note — Cut (pages 10-11)

Propositional logic — Tableaux summary

Gentzen System

Hilbert System

The Four Colour Theorem

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The four colour theorem states that this is always possible.

Today's final exercise is to do map colouring in Prolog... :-)

Map Colouring Exercise — Suggestion 1

Use numbers 0,1,2,3 as colours and show the colours of the neighbours as a list:

```
?- test(X).

X = [ (0, [1, 2, 1], austria),
      (0, [3, 2, 1], belgium),
      (0, [1], denmark),
      (3, [0, 1, 2, 0, 1], france),
      (1, [3, 0, 2, 2, 0, 0], germany),
      (2, [0, 1], holland),
      (1, [3, 0, 2], italy),
      (1, [0], portugal),
      (0, [3, 1], spain),
      (2, [3, 1, 0, 1], switzerland) ]

Yes
```

Map Colouring Exercise — Suggestion 2

In logic programming a variable like X is both input and output which is very different from functional programming!

```
test(X) :- map(X), colouring(X).

map([
  (A,[I,S,G],austria),
  (B,[F,H,G],belgium),
  (D,[G],denmark),
  (F,[E,I,S,B,G],france),
  (G,[F,A,S,H,B,D],germany),
  (H,[B,G],holland),
  (I,[F,A,S],italy),
  (P,[E],portugal),
  (E,[F,P],spain),
  (S,[F,I,A,G],switzerland)
]).
```

Map Colouring Exercise — Suggestion 3

Prolog is elegant but can be a bit hard to master — you must experiment...

Please ask me or the teaching assistants for help if needed!

If SWI-Prolog hides part of the answer behind `...` then enter `w` for `write` in order to have everything written — and enter `p` for `print` in order to have `...` printed again.

Basic Predicates

```
% Prolog file basic.pl

member(H,[H|_]).
member(H,[_|T]) :- member(H,T).

append([],U,U).
append([H|T],U,[H|V]) :- append(T,U,V).
```

Tracer Example

```
member(H,[_|T]) :- member(H,T). % Note that the clauses are swapped
member(H,[H|_]).
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member(H,[_|T]) :- member(H,T). % Note that the clauses are swapped
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?- trace, member(X,[a,b]).

   Call: (1) member(_0,[a,b]) ?
   Call: (2) member(_0,[b]) ?
   Call: (3) member(_0,[]) ?
   Fail: (3) member(_0,[]) ?
   Redo: (2) member(_0,[b]) ?
   Exit: (2) member(b,[b]) ?
   Exit: (1) member(b,[a,b]) ?

X = b ;

   Redo: (1) member(_0,[a,b]) ?
   Exit: (1) member(a,[a,b]) ?

X = a ;

No
```

Tracer I

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See the SWI-Prolog reference manual for the details about the tracer and the debugger.

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Enter `a` to abort the query.

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It is optional to use the tracer for the exercises and for the assignments too.

Key Prolog Programs — Examples

```
?- list([1,2,3]).

Yes
```

```
?- length([1,2,3],_).

Yes
```

```
?- append([1,2,3],_,_).

Yes
```

```
?- member(2,[1,2,3]).

Yes
```

```
?- select(2,[1,2,3],_).

Yes
```

Key Prolog Programs — More Examples

```
?- select(X,[1,2,3],L).

X = 1
L = [2, 3] ;

X = 2
L = [1, 3] ;

X = 3
L = [1, 2] ;

No
```

Key Prolog Programs — Comparison

```
list([]).
list([_|T]) :- list(T).

length([],0).
length([_|T],N1) :- length(T,N), N1 is N+1.

append([],U,U).
append([H|T],U,[H|V]) :- append(T,U,V).

member(H,[H|_]).
member(H,[_|T]) :- member(H,T).

select(X,[X|T],T).
select(X,[Y|T],[Y|U]) :- select(X,T,U).
```

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Original version of the partition program for Quicksort:

```
part(_,[],[],[]).
part(X,[Y|Xs],[Y|Ls],Bs) :- X > Y, part(X,Xs,Ls,Bs).
part(X,[Y|Xs],Ls,[Y|Bs]) :- X =< Y, part(X,Xs,Ls,Bs).
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```

Slightly more efficient version:

```
part(_,[],[],[]).
part(X,[Y|Xs],[Y|Ls],Bs) :- X > Y, !, part(X,Xs,Ls,Bs).
part(X,[Y|Xs],Ls,[Y|Bs]) :- part(X,Xs,Ls,Bs).
```

Cut — Definition

Consider the following schematic clauses for a predicate `p`:

$$
\begin{array}{l}
\texttt{p(}\mathbf{s}_1\texttt{) :- } \mathbf{A}_1. \\
\ldots \\
\texttt{p(}\mathbf{s}_i\texttt{) :- } \mathbf{B}\texttt{,!,}\mathbf{C}. \\
\ldots \\
\texttt{p(}\mathbf{s}_k\texttt{) :- } \mathbf{A}_k.
\end{array}
$$

Suppose that during the execution of a query a call $\texttt{p(}\mathbf{t}\texttt{)}$ is encountered and eventually the *i*-th clause is used and the indicated occurrence of the cut is executed.

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Consider the following schematic clauses for a predicate p:

```
p(s_1)  :-  A_1.
...
p(s_i)  :-  B,!,C.
...
p(s_k)  :-  A_k.
```

Suppose that during the execution of a query a call p(**t**) is encountered and eventually the i-th clause is used and the indicated occurrence of the cut is executed.

Then the indicated occurrence of the cut succeeds immediately, but additionally all alternative ways of computing **B** are discarded, and all computations of p(**t**) using the $i + 1$-th to k-th clause for p are discarded as backtrackable alternatives to the current selection of the i-clause.

Cut — Example — Motivation

Consider the basic `member` predicate:

```
member(H,[H|_]).
member(H,[_|T]) :- member(H,T).
```

Recall that it can be used to enumerate the elements of the list.

Cut — Example — Motivation

Consider the basic `member` predicate:

```
member(H,[H|_]).
member(H,[_|T]) :- member(H,T).
```

Recall that it can be used to enumerate the elements of the list.

It can also succeed more than one time when used to check whether an element is a member of a given list:

```
?- member(a,[a,b,a,c]), write(x), fail.
xx

No
```

The choice point left by the non-deterministic `member` can cause problems in certain situations.

Cut — Example — Implementation

The `membercheck` predicate works as the basic predicate `member`, except that it is deterministic and never leaves a choice point:

```
?- membercheck(a,[a,b,a,c]), write(x), fail.
x

No
```

A cut is used in the first clause in order to commit to the first match and avoid backtracking:

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Use the more efficient `membercheck` in preference to the basic `member`, but only where its restrictions are appropriate.

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```

Use the more efficient `membercheck` in preference to the basic `member`, but only where its restrictions are appropriate.

In SWI-Prolog a built-in predicate `memberchk` is available.

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Similar theorems for tableaux and resolution.

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Soundness and Completeness Theorem: $\models A$ iff $\vdash A$

Similar theorems for tableaux and resolution.

Much more about tableaux, axiomatics and resolution soon for first-order logic. :-)

Different Tableaux 1

The formula (p => q) => ((r \/ p) => (r \/ q)) is valid because the following tableau is closed:

```
~((p => q) => ((r \/ p) => (r \/ q)))
   (p => q),~((r \/ p) => (r \/ q))
---
L R
      ~p,~((r \/ p) => (r \/ q))
         (r \/ p),~(r \/ q),~p
-----
LL LR
            r,~(r \/ q),~p
               ~r,~q,~p,r
X
LR
            p,~(r \/ q),~p
               ~r,~q,~p,p
X
R
      q,~((r \/ p) => (r \/ q))
         (r \/ p),~(r \/ q),q
-----
RL RR
            r,~(r \/ q),q
               ~r,~q,q,r
X
RR
            p,~(r \/ q),q
               ~r,~q,q,p
X
```

Different Tableaux 2

```
The formula (p => q) => ((r \/ p) => (r \/ q)) is valid because the following tableau is closed:

~((p => q) => ((r \/ p) => (r \/ q)))
   ~((r \/ p) => (r \/ q)),(p => q)
      (r \/ p),~(r \/ q),(p => q)
---
L R
         r,~(r \/ q),(p => q)
            ~r,~q,(p => q),r
-----
LL LR
               ~p,r,~r,~q
X
LR
               q,r,~r,~q
X
R
         p,~(r \/ q),(p => q)
            ~r,~q,(p => q),p
-----
RL RR
               ~p,p,~r,~q
X
RR
               q,p,~r,~q
X
```

Different Tableaux 3

```
The formula (p => q) => ((r \/ p) => (r \/ q)) is valid because the following tableau is closed:

~((p => q) => ((r \/ p) => (r \/ q)))
   ~((r \/ p) => (r \/ q)),(p => q)
      ~(r \/ q),(r \/ p),(p => q)
        ~r,~q,(r \/ p),(p => q)
---
L R
            r,(p => q),~r,~q
-----
LL LR
               ~p,~r,~q,r
X
LR
               q,~r,~q,r
X
R
            p,(p => q),~r,~q
-----
RL RR
               ~p,~r,~q,p
X
RR
               q,~r,~q,p
X
```

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For deciding validity of a formula A, start the tableau with $\neg A$.

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A leaf consisting of literals and with a complementary pair of literals $(p, \neg p)$ for some p is marked closed: $\times$

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A leaf consisting of literals and without a complementary pair of literals $(p, \neg p)$ for some p is marked open: $\odot$

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A literal is an atom or a negation of an atom.

A leaf consisting of literals and with a complementary pair of literals $(p, \neg p)$ for some p is marked closed: $\times$

A leaf consisting of literals and without a complementary pair of literals $(p, \neg p)$ for some p is marked open: $\odot$

Two types of rules for tableaux: α-rules create a single child and β-rules create two children.

The Gentzen System

Tableaux turned "tree upside down and signs reversed":

$\neg((p \vee q) \rightarrow (q \vee p))$

$p \vee q, \neg(q \vee p)$

$p \vee q, \neg q, \neg p$

$\overline{\text{L} \qquad \text{R}}$

$p, \neg q, \neg p$

$\times$

R

$q, \neg q, \neg p$

$\times$

1.	$\vdash \neg p, q, p$	Axiom
2.	$\vdash \neg q, q, p$	Axiom
3.	$\vdash \neg(p \vee q), q, p$	$\beta\vee$, 1, 2
4.	$\vdash \neg(p \vee q), (q \vee p)$	$\alpha\vee$, 3
5.	$\vdash (p \vee q) \rightarrow (q \vee p)$	$\alpha \rightarrow$, 4

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$\overline{\text{L} \qquad \text{R}}$

L

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$\times$

R

$q, \neg q, \neg p$
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Hence $\vdash (A \vee B) \to (B \vee A)$

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Soundness and Completeness Theorem: $\models A$ **iff** $\vdash A$

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Hence $\vdash (A \vee B) \rightarrow (B \vee A)$

Soundness and Completeness Theorem: $\models A$ **iff** $\vdash A$

Note: Use implicit set formation for conclusions in the system $\mathcal{G}$

The Hilbert System

One rule MP: $\vdash A, \vdash A \to B \ / \vdash B$

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Axiom 1: $\vdash A \to B \to A$

The Hilbert System

One rule MP: $\vdash A, \vdash A \to B \;/\; \vdash B$

Axiom 1: $\vdash A \to B \to A$

Axiom 2: $\vdash (A \to B \to C) \to (A \to B) \to A \to C$

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One rule MP: $\vdash A, \vdash A \to B \;/\; \vdash B$

Axiom 1: $\vdash A \to B \to A$

Axiom 2: $\vdash (A \to B \to C) \to (A \to B) \to A \to C$

Axiom 3: $\vdash (\neg B \to \neg A) \to A \to B$

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One rule MP: $\vdash A, \vdash A \to B \;/ \vdash B$

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Complicated even for the following tiny example.

1.	$\vdash (A \to (A \to A) \to A) \to (A \to A \to A) \to A \to A$	Axiom 2
2.	$\vdash A \to (A \to A) \to A$	Axiom 1
3.	$\vdash (A \to A \to A) \to A \to A$	MP 1, 2
4.	$\vdash A \to A \to A$	Axiom 1
5.	$\vdash A \to A$	MP 3, 4

The Hilbert System

One rule MP: $\vdash A, \vdash A \to B \;/\; \vdash B$

Axiom 1: $\vdash A \to B \to A$

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Soundness and Completeness Theorem: $\models A$ **iff** $\vdash A$

The Hilbert System

One rule MP: $\vdash A, \vdash A \rightarrow B \;/ \vdash B$

Axiom 1: $\vdash A \rightarrow B \rightarrow A$

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5.	$\vdash A \rightarrow A$	MP 3, 4

Soundness and Completeness Theorem: $\models A$ **iff** $\vdash A$

Note: Use implicit set formation for assumptions in system $\mathcal{H}$

The Deduction Rule

Let U be a set of formulas and let $U \vdash A$ equal $\vdash A$ iff $U = \emptyset$

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Deduction theorem: The deduction rule is a sound derived rule

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Classical logic is explosive — See theorems 3.20 and 3.21.

1.	$\{\neg A\} \vdash \neg A \to \neg B \to \neg A$	Axiom 1
2.	$\{\neg A\} \vdash \neg A$	Assumption
3.	$\{\neg A\} \vdash \neg B \to \neg A$	MP 1, 2
4.	$\{\neg A\} \vdash (\neg B \to \neg A) \to (A \to B)$	Axiom 3
5.	$\{\neg A\} \vdash A \to B$	MP 3, 4
6.	$\vdash \neg A \to A \to B$	Deduction 5

Non-explosive logics are also called paraconsistent logics.

Resolution

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Soundness and completeness must be established in all cases.