02156 - Logical Systems and Logic Programming
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DTU - Technical University of Denmark

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Assignment 3

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Problem 1

Question 1.1

The following query:

```
?- tt(p imp q, [(p,t), (q,f)], f).
True .
```

Is showing that, given p is true and q is false then $(p \to q = f)$ is true.

Question 1.2

The boolean/0 program only succeeds if all possible sematics are of two-valued boolean are satisfied. Using the predicate tt in the same vein as in Question 1.1, two-valued boolean logic can be checked. The following query checks that negation works as required:

```
?- tt(neg p, [(p, t)], f), tt(neg p, [(p,f)], t).
```

Question 1.3

The many valued logic is based on the semantics defined in assignment 1. The semantics are as follows:

$$[[\neg P]] = \begin{cases} \text{T} & \text{if } [[P]] = \text{F} \\ \text{F} & \text{if } [[P]] = \text{T} \\ [[P]] & \text{otherwise} \end{cases}$$

$$[[P \wedge Q]] = \begin{cases} [[P]] & \text{if } [[P]] = [[Q]] \\ [[Q]] & \text{if } [[P]] = \text{T} \\ [[P]] & \text{if } [[Q]] = \text{T} \\ \text{F} & \text{otherwise} \end{cases}$$

$$[[P \leftrightarrow Q]] = \begin{cases} \text{T} & \text{if } [[P]] = [[Q]] \\ [[Q]] & \text{if } [[P]] = \text{T} \\ [[P]] & \text{if } [[Q]] = \text{T} \\ [[\neg Q]] & \text{if } [[P]] = \text{F} \\ [[\neg P]] & \text{if } [[Q]] = \text{F} \\ F & \text{otherwise} \end{cases}$$

$$[[P \vee Q]] \equiv \neg(\neg P \wedge \neg Q)$$

$$[[P \to Q]] \equiv P \leftrightarrow (P \wedge Q)$$

Testing

For more extensive testing, see appendix.

```
?- opr(imp, t, x, x).
true.
?- opr(con, x, x, x).
true.
?- opr(dis, f, x, x).
true.
?- opr(eqv, x, x, t).
true.
```

Question 1.4

The semantics of many valued logic still contain the requirements for satisfying two-valued logic, thus boolean/0 still succeed.

Problem 2

Question 2.1

Box $\boxed{1}$ can be infered from (5.)

3. $\vdash \neg p, \neg q, r, \neg(q \to r) \quad \beta \to, 2, 1$

(6.) Introduces $\neg(p \wedge q)$, so box $\boxed{2}$ can be infered to be:

5. $\vdash \neg p, \neg q, r, \neg(p \to (q \to r)) \quad \beta \to, 4, 3$
6. $\vdash \neg(p \wedge v), r, \neg(p \to (q \to r)) \quad \alpha\wedge, 5$

Question 2.2

Box $\boxed{3}$ is an Axiom with the contradiction of r and $\neg r$ present.

Box $\boxed{4}$ can be infered from (16.)

15. $\vdash p \to (q \to r), \neg((p \wedge q) \to r) \quad \alpha \to, 14$

Question 2.3

A is not proved valid with (16.), however with the introduction of the following step it will be proved:

17. $\vdash ((p \wedge q) \to r) \leftrightarrow (p \to (q \to r)) \quad \beta \leftrightarrow, 15, 8$

Problem 3

Question 3.1

Refuting the validity of the proposition, we negate the formular and try to find counter example:

$$\begin{array}{c}
\neg(\exists x\forall y p(x,y) \rightarrow \forall y \exists x p(x,y)) \\
| \\
\exists x\forall y p(x,y), \exists y \forall x \neg p(x,y) \\
| \\
\exists x\forall y p(x,y), \neg\exists x p(x,a) \ \textit{(1)} \\
| \\
\forall y p(b,y), \neg\exists x p(x,a) \\
| \\
\forall y p(b,y), p(b,a), p(b,b), \neg\exists x p(x,a) \\
| \\
\forall y p(b,y), p(b,a), p(b,b), \neg\exists x p(x,a), \neg p(a,a), \neg p(b,a) \\
| \\
\times
\end{array}$$

N.B. duality used at:

- (1): $\exists y \forall x \neg p(x,y) \equiv \neg\forall y \neg\exists x p(x,y)$

which enables the use of delta rule.

The tableau closes as we see the contradiction with: $p(b,a)$ and $\neg p(b,a)$. This concludes a complete closed tableau, thus we can say that the formula is valid. Because we negated the formular, the contradiction becomes a tautology.

Question 3.2

Refuting the validity of the proposition, we negate the formular and try to find counter example:

$$\begin{array}{c}
\neg((\forall x(p(x) \rightarrow \neg\exists y q(y,x))) \rightarrow (p(a) \rightarrow \neg q(a,a))) \\
| \\
\forall x(p(x) \rightarrow \neg\exists y q(y,x)), \neg(p(a) \rightarrow \neg q(a,a)) \\
| \\
\forall x(p(x) \rightarrow \neg\exists y q(y,x)), p(a), q(a,a) \\
| \\
\forall x(p(x) \rightarrow \neg\exists y q(y,x)), p(a) \rightarrow \neg\exists y q(y,a), p(a), q(a,a)
\end{array}$$

$$\begin{array}{cc}
\swarrow & \searrow \\
\forall x(p(x) \rightarrow \neg\exists y q(y,x)), \neg p(a), p(a), q(a,a) & \forall x(p(x) \rightarrow \neg\exists y q(y,x)), \neg\exists y q(y,a), p(a), q(a,a) \\
| & | \\
\times & \forall x(p(x) \rightarrow \neg\exists y q(y,x)), \neg\exists y q(y,a), \neg q(a,a), p(a), q(a,a) \\
 & | \\
 & \times
\end{array}$$

The tableau closes with contradictions $\{..., \neg p(a), p(a)\}$ and $\{..., \neg q(a,a), q(a,a)\}$. This concludes a complete closed tableau, thus we can say that the formula is valid. Because we negated the formular, the contradiction becomes a tautology.

Question 3.3

Refuting the validity of the proposition, we negate the formular and try to find counter example:

$$\neg((\forall x p(x) \wedge \forall x q(x)) \rightarrow \forall x(p(x) \wedge q(x)))$$
$$|$$
$$(\forall x p(x) \wedge \forall x q(x)), \neg\forall x(p(x) \wedge q(x))$$
$$|$$
$$\forall x p(x), \forall x q(x), \neg\forall x(p(x) \wedge q(x))$$
$$|$$
$$\forall x p(x), \forall x q(x), \neg(p(a) \wedge q(a))$$

$$\begin{array}{cc}
\forall x p(x), \forall x q(x), \neg p(a) & \forall x p(x), \forall x q(x), \neg q(a) \\
| & | \\
\forall x p(x), p(a), \forall x q(x), \neg p(a) & \forall x p(x), \forall x q(x), q(a), \neg q(a) \\
| & | \\
\times & \times
\end{array}$$

The tableau closes with contradictions $\{..., \neg p(a), p(a)\}$ and $\{..., \neg q(a), q(a)\}$. This concludes a complete closed tableau, thus we can say that the formula is valid. Because we negated the formular, the contradiction becomes a tautology.

Problem 4

The shortened SeCav is included outcommented in 02156-A3-s201186.pl

Notes on proving $(p \to q) \to p \lor r \to q \lor r$.

- $\Gamma \vdash \triangle$ can be proven if and only if $\vdash \neg\Gamma, \triangle$,
  therefor we start applying rules for: $\neg((p \to q) \to p \lor r \to q \lor r)$
- Used priority: $\alpha > \beta$, utilizing Ext to facilitate the priority.
- The Ext before $\beta\lor$ removed $\neg p \lor r$ and r because a contradiction is found with $\neg q$ and q.
- Basic rule is used to end the proof.

Appendix

Here is a more extensive testing documentation from problem 1.

Implication

```
 ?- opr(imp, t, A, t).
A = t.
?- opr(imp, t, f, A).
A = f.
?- opr(imp, f, t, t).
true.
?- opr(imp, f, f, t).
true.
?- opr(imp, t, x, x).
true.
?- opr(imp, x, t, x).
false.
```

Conjunction

```
 ?- opr(con, x, x, x).
true.
?- opr(con, t, hello, hello).
true.
?- opr(con, f, x, f).
true.
?- opr(con, t, A, t).
A = t.
?- opr(con, A, A, x).
A = x.
?- opr(con, f, anything, f).
true.
?- opr(con, f, anything, t).
false.
```

Disjunction

```
 ?- opr(dis, x, f, x).
true.
?- opr(dis, f, A, t).
A = t.
?- opr(dis, f, f, t).
false.
```

Equivalence

```
 ?- opr(eqv, A, B, t).
A = B.
?- opr(eqv, x, B, t).
B = x.
?- opr(eqv, f, t, t).
false.
?- opr(eqv, x, x, t).
true.
```