Neural Circuit Modelling and Neural Computation

ECE 441

October 19

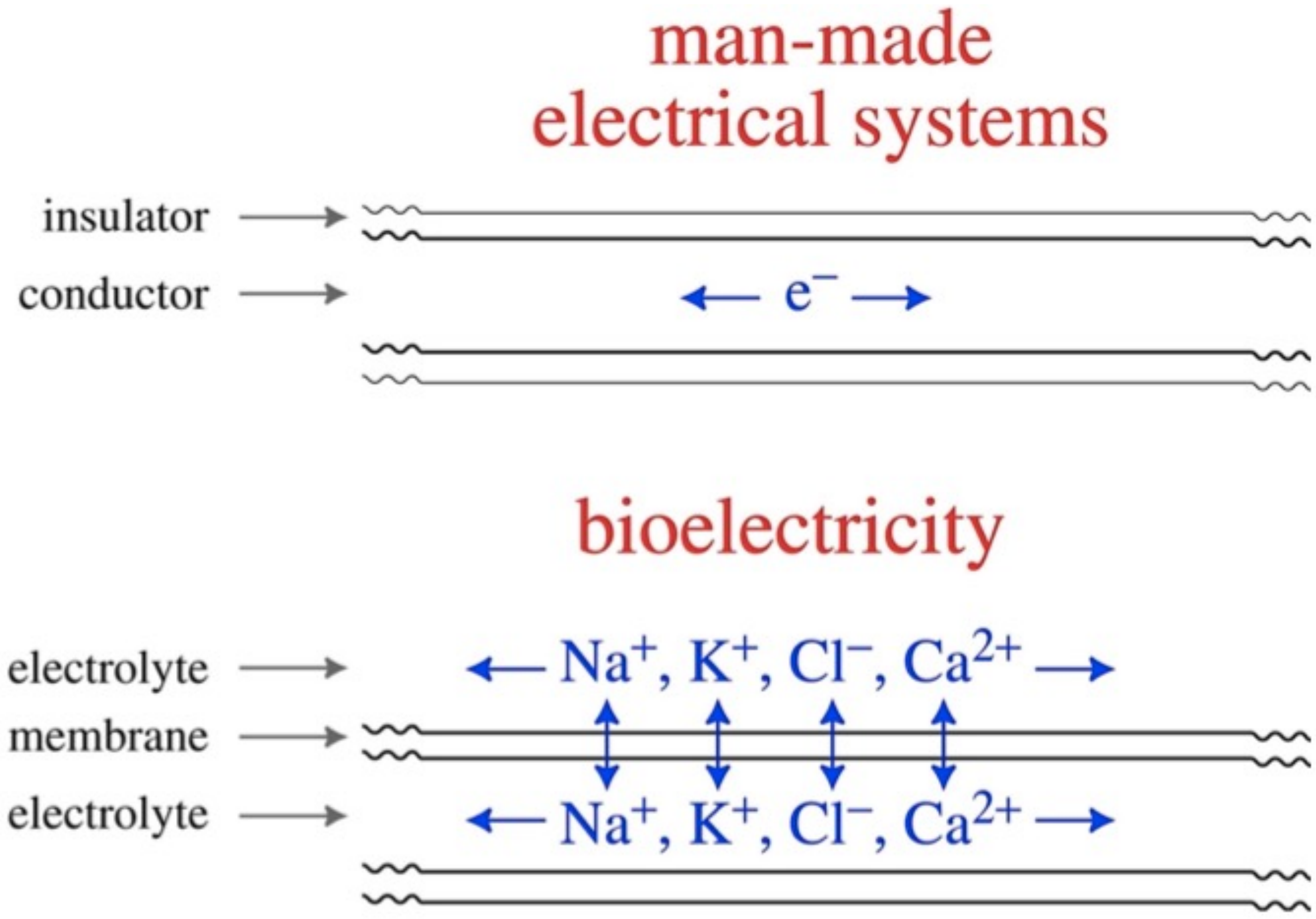
Neural Computation

- Use of mathematical modelling and techniques to analyze neural data
- Requires fundamental understanding of neural circuits and computational principles

Electrical System vs. Bioelectricity

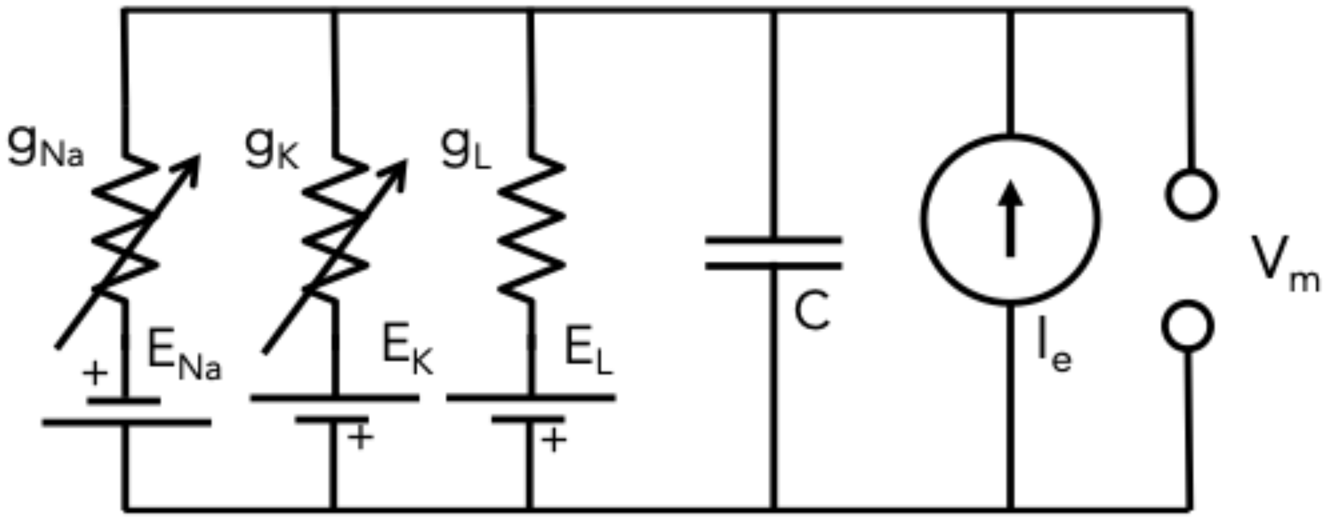
Man-made electrical system	Charge carriers are electrons within a conductor	Current flow within (insulated) conductors
Bioelectricity	Charge carriers are ions within an electrolyte	Current flow inside *and outside* of (partially- insulated) cell membranes

Electrical System vs. Bioelectricity

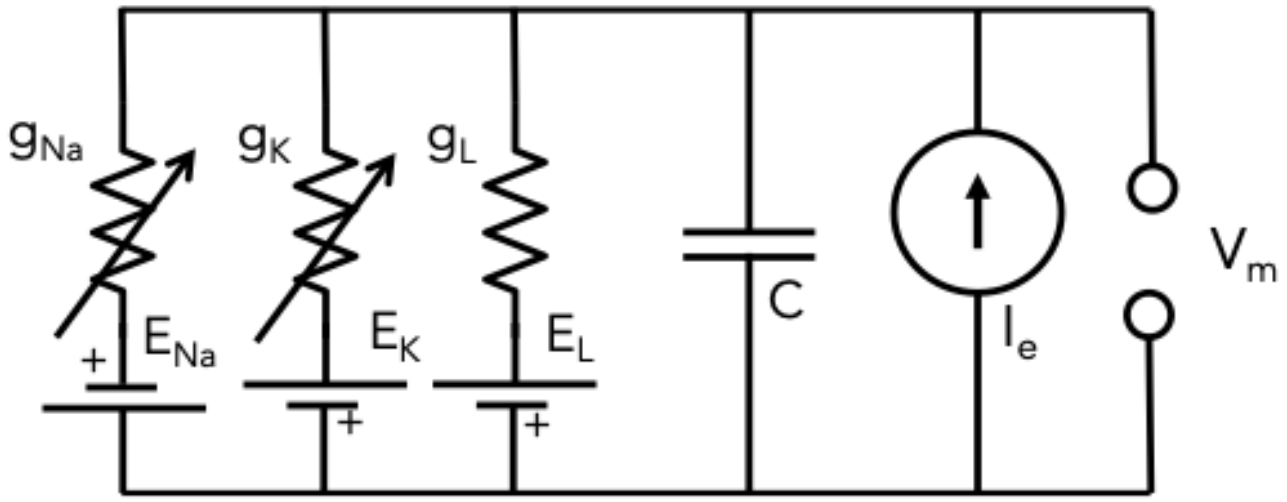


Mathematical Model of Neuron

- Equivalent circuit model



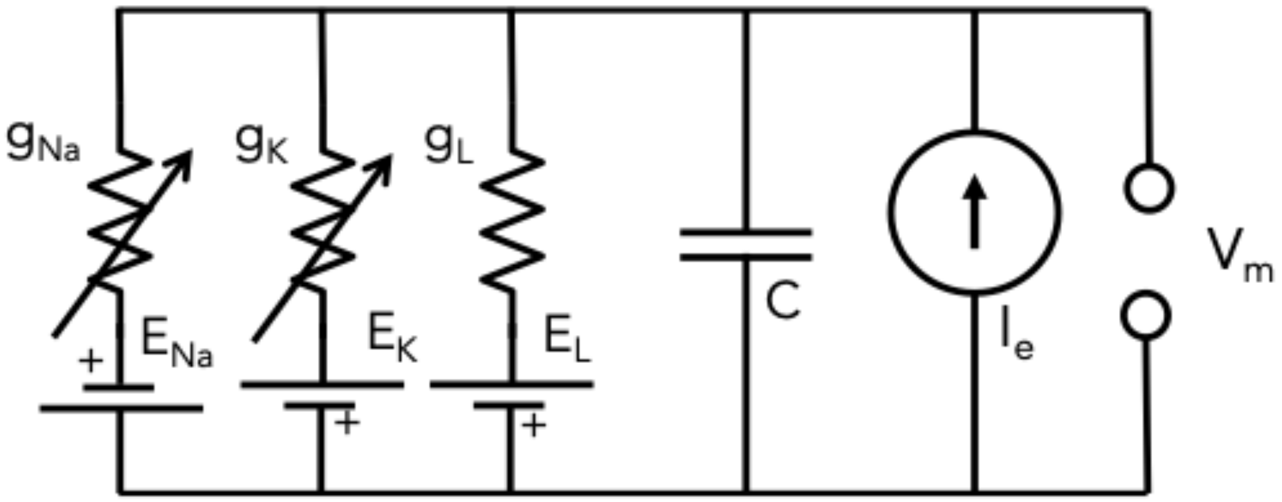
Circuit Model of Neuron



Components:

- Parallel conductance model for ionic channels
- g – conductance
- E – equilibrium potential
- C – capacitance
- I – current
- Vm – resting potential

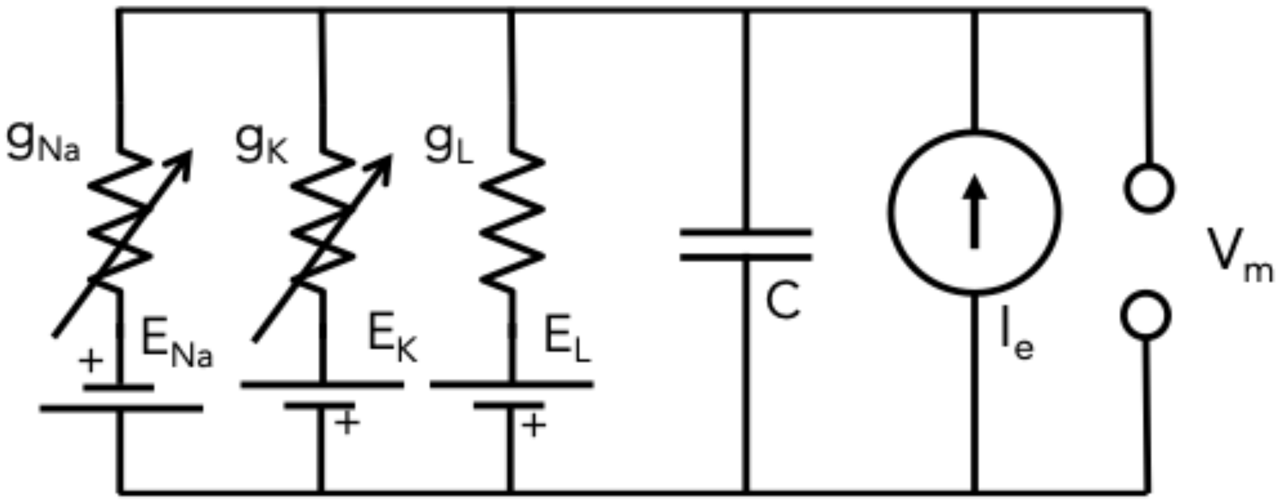
Circuit Model of Neuron



Role of this circuit in bioelectricity:

- Power supplies
- Integrator of past inputs
- Temporal filter to smooth inputs in time
- Spike generator
- oscillator

Circuit Model of Neuron



- In bioelectricity, current flows result from ionic movements

Nernst Planck Equation and Equilibrium Potential

Nernst Planck Equation

- Demonstrates the effects of spatial differences on ionic flow in two types of gradients:
  - Concentration
  - Electric potential

- Effects of a concentration gradient is described by “Fick’s law of diffusion”
- Effects of an electric potential gradient is described by “Ohm’s law of drift”

Fick’s Law of Diffusion

$\bar{J_D} = -D\nabla C$

Where:

$\bar{J_D}$: flux due to diffusion

D: diffusion coefficient (determined empirically, different for each ion)

C: concentration **as a function of position**

∇: Del operator, $\nabla = \frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}$

Ohm’s Law of Drift

$$\bar{J}_e = -u_p \frac{Z_p}{|Z_p|} C_p \nabla\Phi$$

Where:

$\bar{J}_e$: ionic flux due to electric field

$-\nabla\Phi$: electric field (Φ is the potential)

u_p: mobility of pth ion

$\frac{Z_p}{|Z_p|}$: sign of the valence of pth ion (e.g. Cl- : $\frac{-1}{|-1|} = -1$)

C_p: concentration of pth ion

Nernst Planck Equation (Cont'd)

$$\bar{J}_p = \bar{J}_d + \bar{J}_e$$

$$= -D_p\left(\nabla C_{\mathrm{p}} + \frac{Z_p C_p u_p}{|Z_p| D_p}\nabla\Phi\right)$$

According to Einstein's equation:

$$D_p = \frac{u_p RT}{|Z_p|F}$$

R: gas constant

T: absolute temperature

F: Faraday's constant

Nernst Planck Equation (Cont’d)

Substituting u_p into the equation:

$$\bar{J}_p = -D_p\left(\nabla C_\mathrm{p} + \frac{Z_p C_p F}{RT}\nabla\Phi\right)$$

- This is the Nernst Planck Equation

Nernst Equilibrium

- At equilibrium, $\bar{J}_d + \bar{J}_e = \bar{J}_p = 0$,

$$-D_p\left(\nabla C_{\mathrm{p}} + \frac{Z_p C_p F}{RT}\nabla\Phi\right) = 0$$

$$\nabla C_{\mathrm{p}} = -\frac{Z_p C_p F}{RT}\nabla\Phi$$

Assuming one direction only:

$$\frac{dC_p}{dx} = -\frac{Z_p C_p F}{RT}\frac{d\Phi}{dx}$$

$$\frac{dC_p}{C_p} = \frac{Z_p C_p F}{RT}d\Phi$$

Nernst Equilibrium (Cont'd)

- Taking the integral from extracellular to intracellular space across the membrane:

$$\int_e^i \frac{dC_p}{C_p} = -\frac{Z_p F}{RT}\int_e^i d\Phi$$

- Nernst equilibrium potential is then:

$$V_m^{eq} = -\frac{RT}{Z_p F}\ln\left(\frac{\left[C_p\right]_i}{\left[C_p\right]_e}\right)$$

Nernst Equilibrium Example 1

Calculate the equilibrium potential for potassium (K+) at 27 $°C$, when the extracellular ion concentration is 20 mM and the intracellular ion concentration is 400 mM.

Nernst Equilibrium Example 1

$$E_K = -\frac{RT}{Z_K F} \ln\left(\frac{[C_K]_i}{[C_K]_e}\right)$$

$$= -\frac{(8.314)(273+27)}{(+1)(96487)} \ln\left(\frac{400}{20}\right)$$

$$= -77\ mV$$

Nernst Equilibrium Example 2

Calculate the equilibrium potential for chlorine (Cl-) at 27 $°C$, when the extracellular ion concentration is 77.5 mM and the intracellular ion concentration is 1.5 mM.

Nernst Equilibrium Example 1

$$E_{Cl} = -\frac{RT}{Z_{Cl}F}\ln\left(\frac{[C_{Cl}]_i}{[C_{Cl}]_e}\right)$$

$$= -\frac{(8.314)(273+27)}{(-1)(96487)}\ln\left(\frac{1.5}{77.5}\right)$$

$$= 102\ mV$$

Response to Injected Current

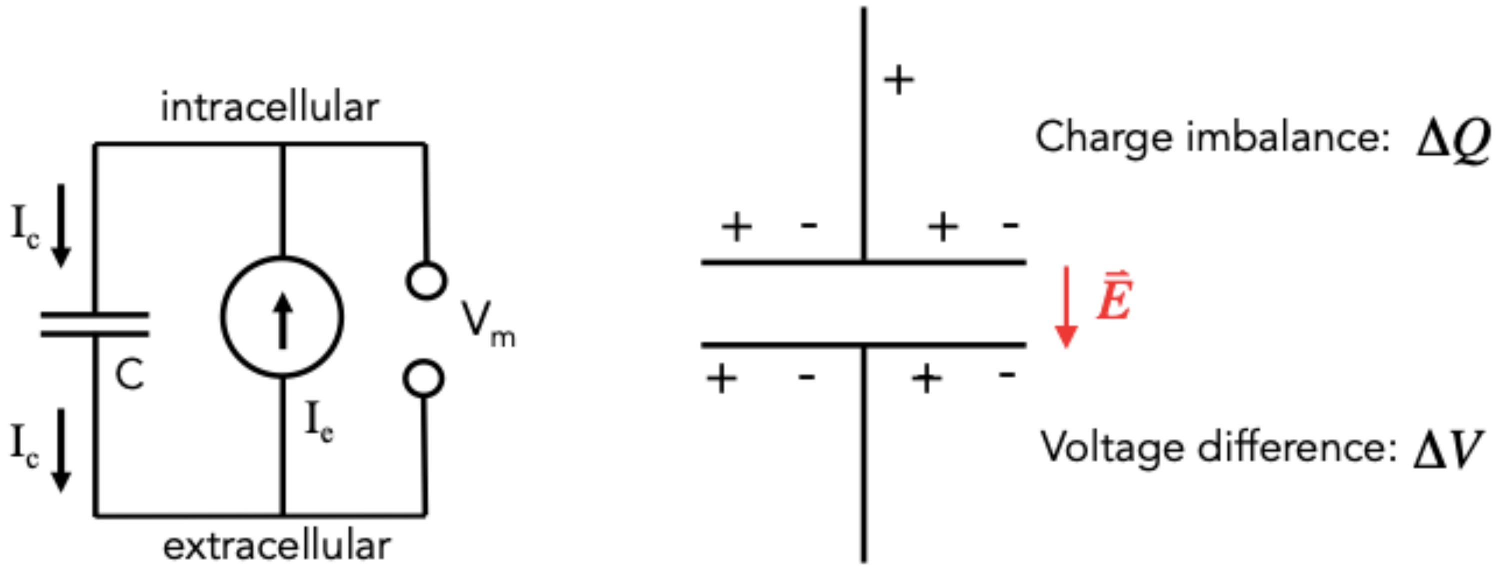
Neuron and Injected Current

- In the brain:
  - ➢Neurons inject current into other neurons through synapses
  - ➢Current can also be injected into neurons as a result of sensory stimuli

- A neuron could be modelled as a capacitor

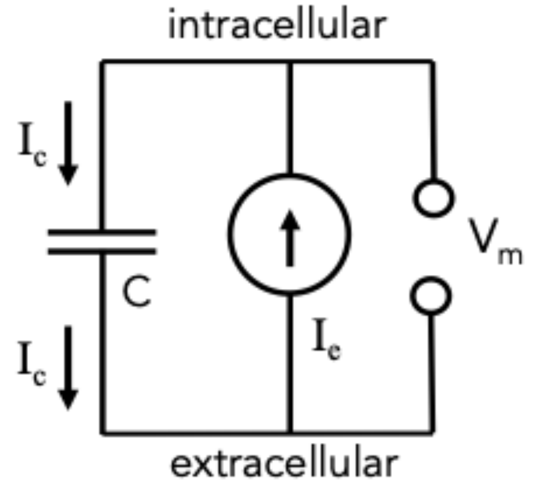
Neuron and Injected Current

- Equivalent circuit and electrical system representation:



- Equation for capacitance: $\Delta Q = C\Delta V$

Neuron and Injected Current



- Equation for capacitive current:

$$I_C(t) = \frac{dQ}{dt} = C\frac{dV_m}{dt}$$

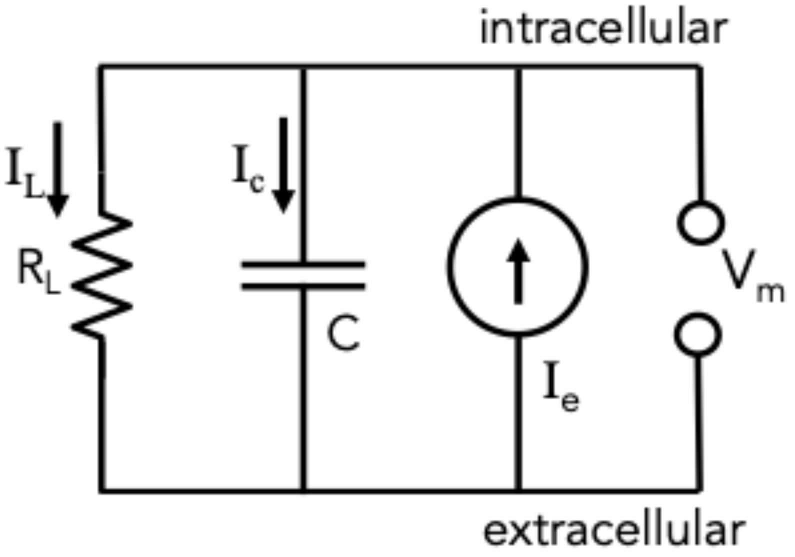
- According to Kirchoff's law:

$$-I_C + I_e = 0$$

- Therefore:

$$I_e(t) = I_C(t) = C\frac{dV_m}{dt}$$

Neuron and Injected Current



- A more accurate way of modelling:
  - ➢Treat the neuron as a **leaky** capacitor
- According to Kirchoff's law:

$$I_L + I_C = I_e$$

$$I_L + C\frac{dV_m}{dt} = I_e$$

- In this case, the current I_L could be simply computed using Ohm's law:

$$I_L = \frac{V_m}{R_L}$$

Neuron and Injected Current

- Putting everything together:

$$\frac{V_m}{R_L} + C\frac{dV_m}{dt} = I_e$$

$$V_m + R_L C\frac{dV_m}{dt} = R_L I_e$$

Neuron and Injected Current

- What is the steady state of this equation?
  - Set $\frac{dV_m}{dt} = 0$
  - $V_\infty = R_L I_e$
- Rewriting our equation:

$$V_m + R_L C \frac{dV_m}{dt} = V_\infty$$

- Therefore:

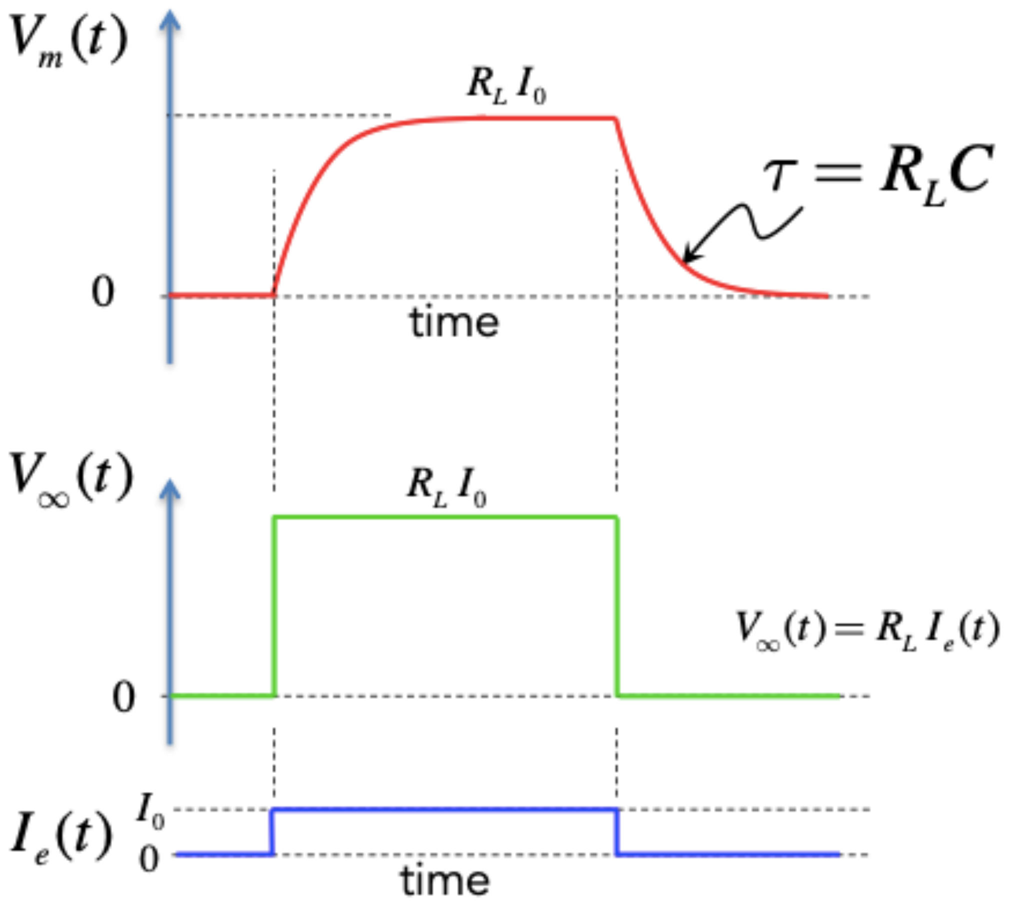
$$\frac{dV}{dt} = \frac{1}{R_L C}(V_\infty - V)$$

Neuron and Injected Current

- Under the condition that both I_e and V_∞ are constant:

$$V(t) = (V_0 - V_\infty)e^{-\frac{t}{R_L C}} + V_\infty$$

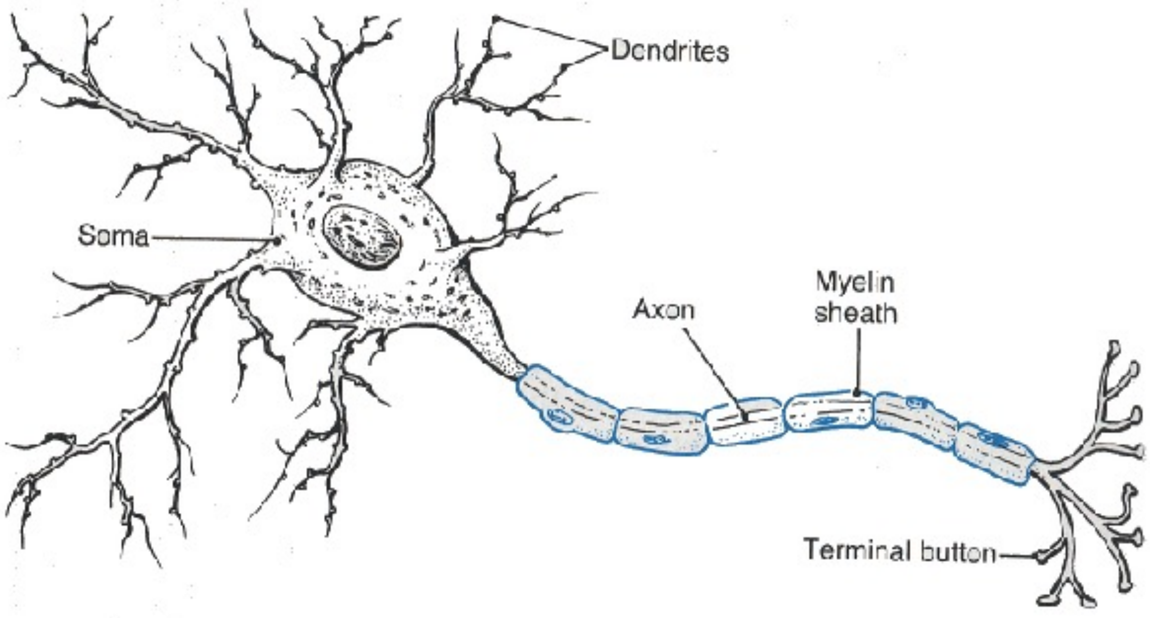
- Graphical representation on the right



Dendrites

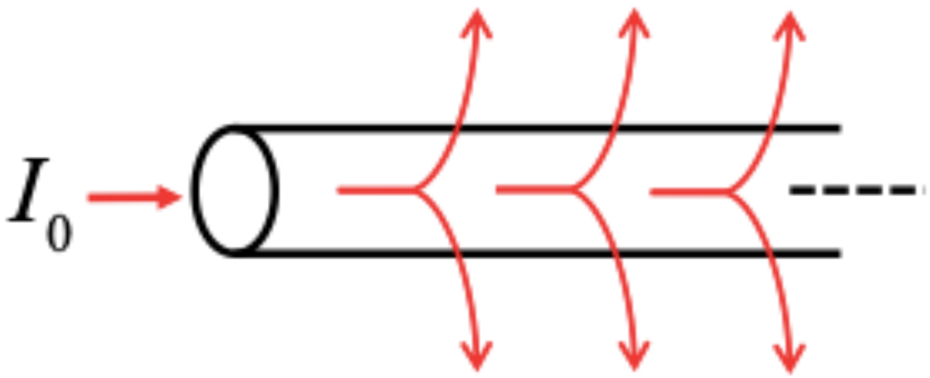
Dendrites

- In our previous models, we assumed that soma is the only part of the neuron
- However, relatively few inputs are made onto the soma
- Inputs first arrive at the dendrites and travel to the soma



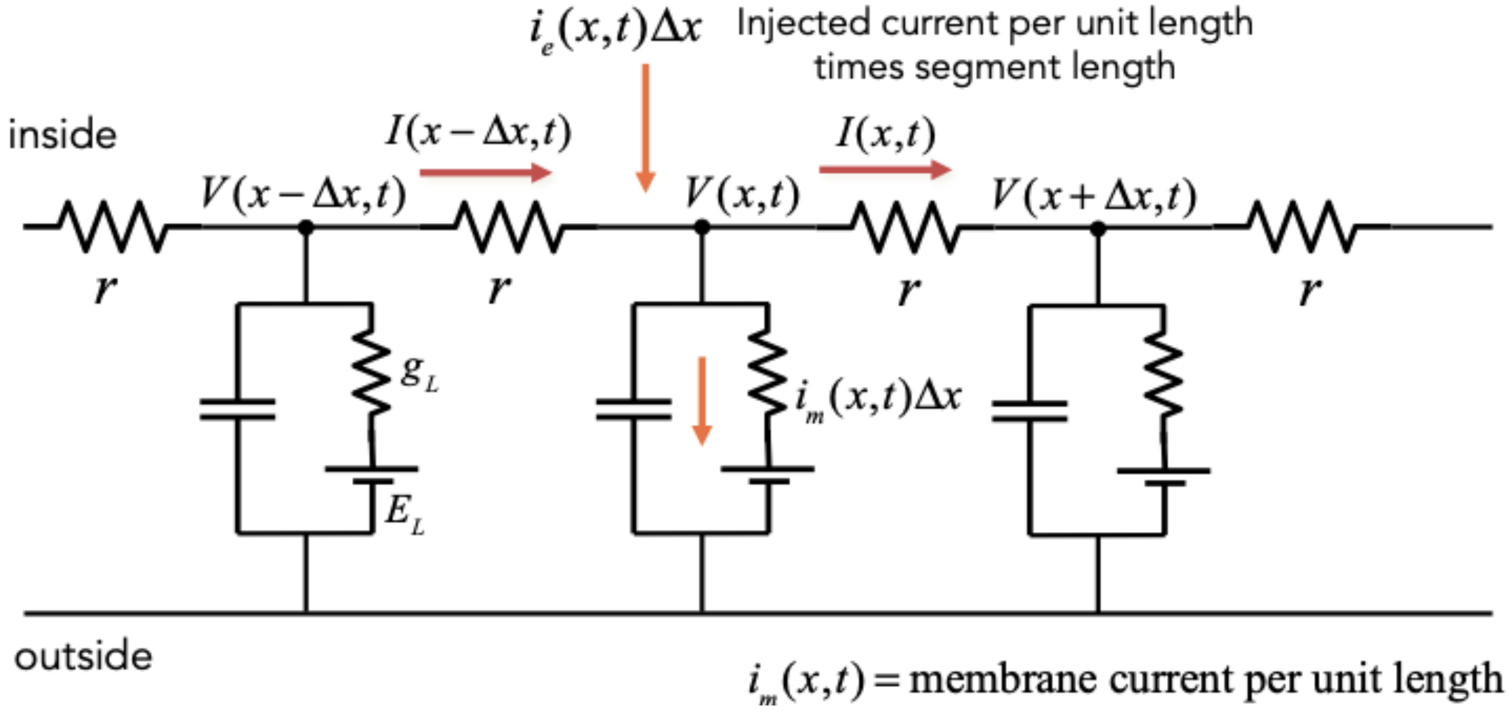
Dendrites

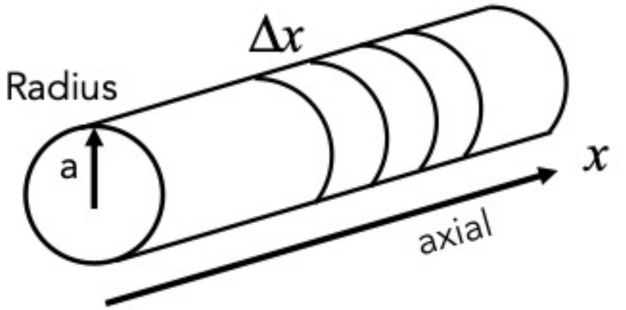
- Dendrites could be modelled as a “leaky garden hose”
- Current represents the water flow
- Voltage represents the pressure



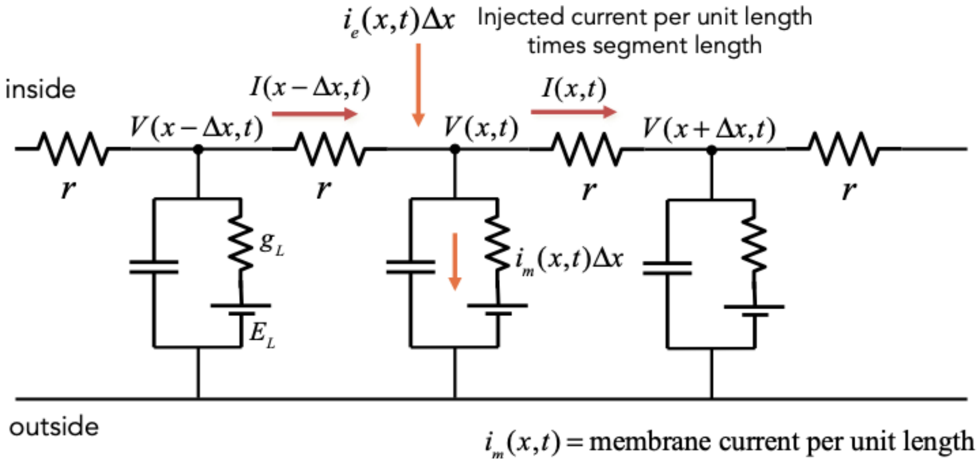
Dendrites Circuit Diagram

- Circuit representation of current flow within dendrites:





Dendrites Circuit Diagram



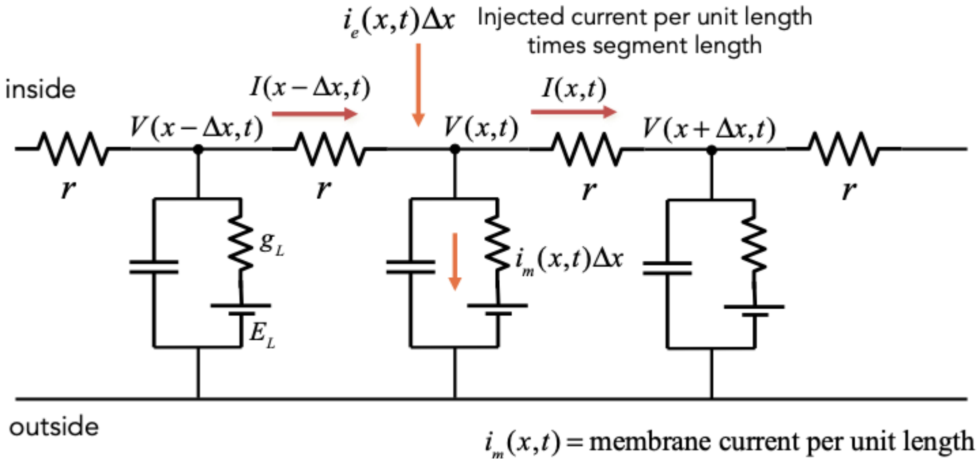
- Kirchoff's law:

$$i_m(x,t)\Delta x - i_e(x,t)\Delta x + I(x,t) - I(x-\Delta x,t) = 0$$

$$i_m(x,t) - i_e(x,t) = -\frac{1}{\Delta x}[I(x,t) - I(x-\Delta x,t)]$$

$$i_m - i_e = -\frac{\partial I}{\partial x}(x,t)$$

Dendrites Circuit Diagram



- According to the diagram:

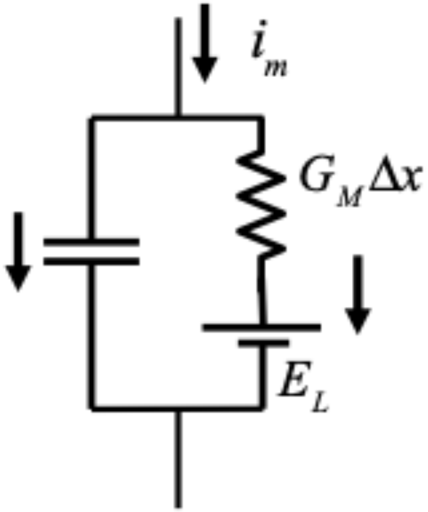
$$-\frac{\partial V}{\partial x} = R_a I(x,t)$$

$$\frac{\partial^2 V}{\partial x^2} = -R_a \frac{\partial I}{\partial x}(x,t)$$

- Therefore

$$i_m - i_e = \frac{1}{R_a}\frac{\partial^2 V}{\partial x^2}(x,t)$$

Dendrites Circuit Diagram

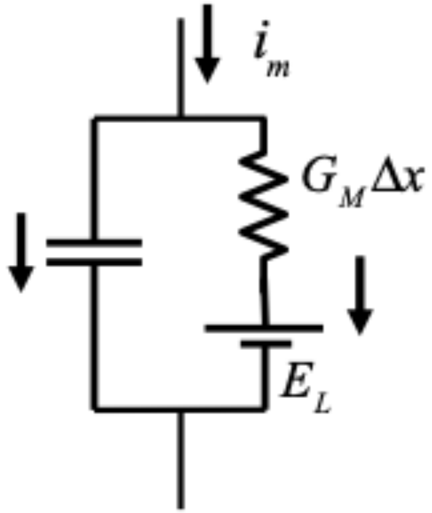


- For each element of length Δx in the cable:

$$i_m(x,t)\Delta x = C_m \Delta x \frac{dV}{dt}(x,t) + G_m \Delta x (V - \cancel{E_L})$$

Where E_L could be approximated as a constant offset

Dendrites Circuit Diagram



- Substituting into $i_m - i_e = \frac{1}{R_a}\frac{\partial^2 V}{\partial x^2}(x,t)$

$$\frac{1}{R_a}\frac{\partial^2 V}{\partial x^2}(x,t) = C_m \frac{\partial V}{\partial x}(x,t) + G_m(V-0) - i_e(x,t)$$

- Dividing both sides by G_m

$$\lambda^2 \frac{\partial^2 V}{\partial x^2}(x,t) = \tau_m \frac{\partial V}{\partial x}(x,t) + V(x,t) - \frac{1}{G_m} i_e(x,t)$$

This is the cable equation!

Where

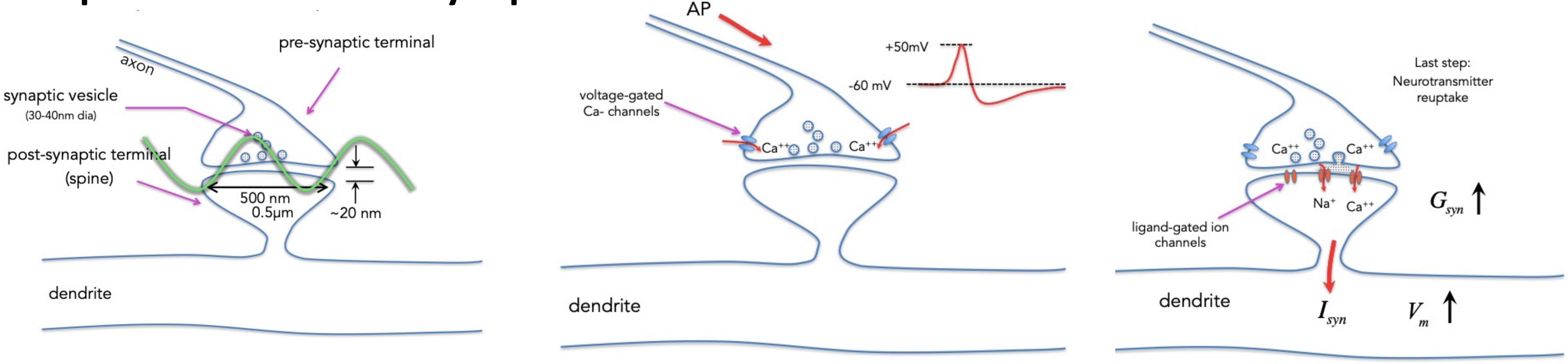
$$\lambda^2 = \frac{1}{G_m R_a} \qquad \tau_m = \frac{C_m}{G_m}$$

Synapse

Structure of Synapse

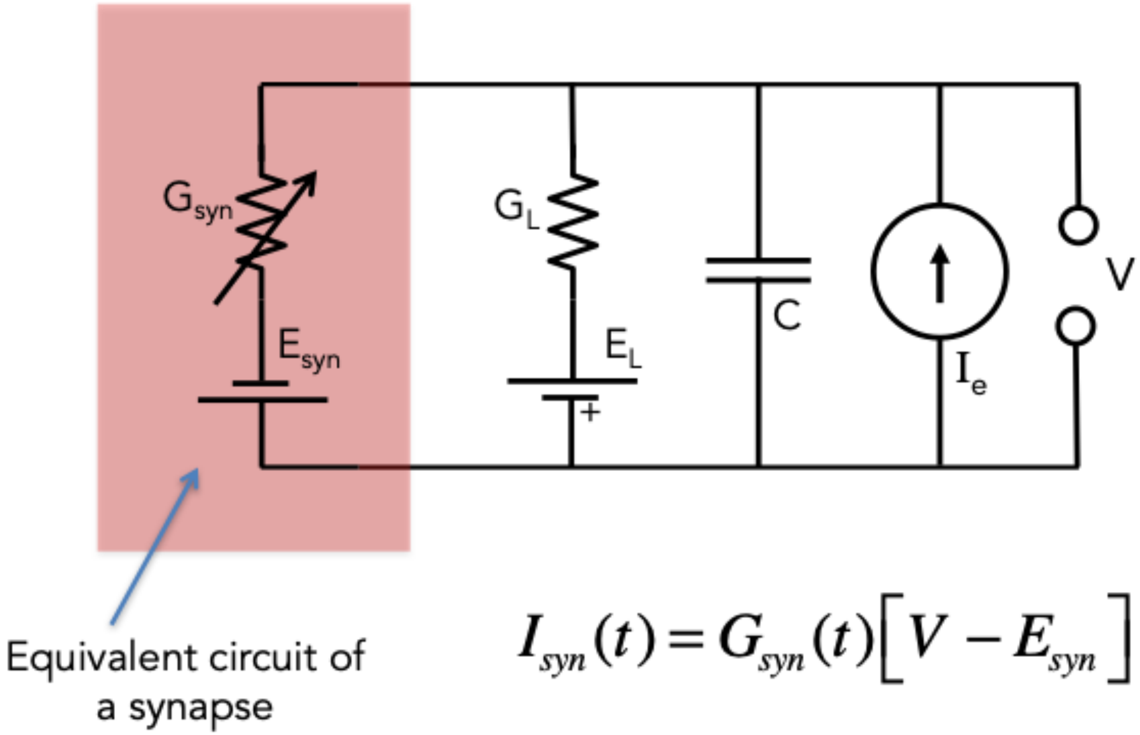
- Very small – contact area is ~0.5 um
- Highly packed ~10^9 synapses/mm^3
- A single cell receives ~10000 synapses

Sequence of Events in Synaptic Transmission:



Circuit Model for Synapse

- Synaptic conductance changes with a current flow through synapse

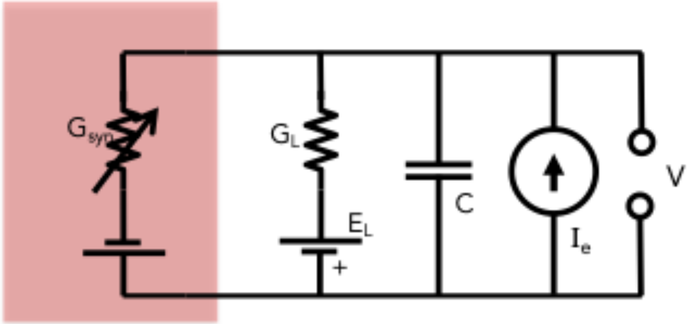


$$I_{syn}(t) = G_{syn}(t)\left[V - E_{syn}\right]$$

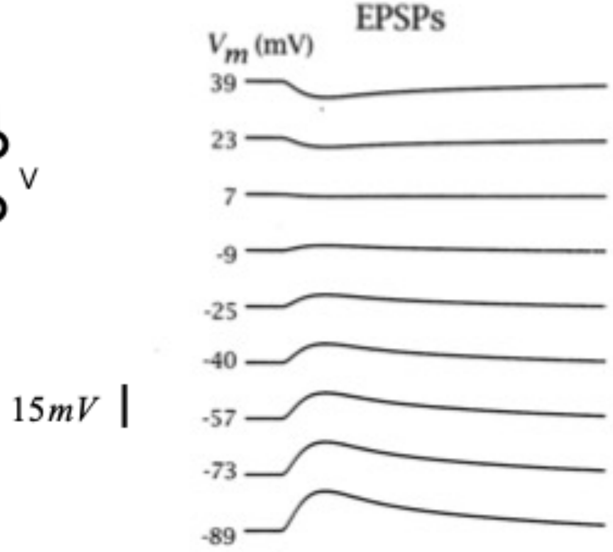
Excitatory Synapse

- Membrane potential (V_m) approaches reversal potential for the synapse

*Injection of current



$E_{syn} = 0\,mV$

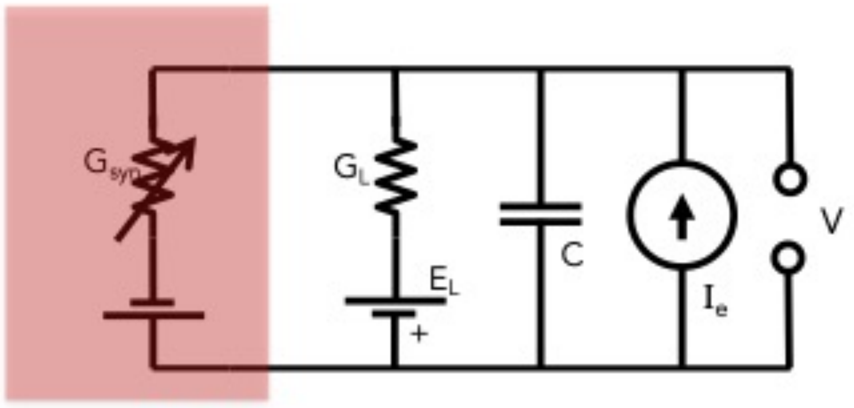


Excitatory synapse if

$$E_{syn} > V_{th}$$

Excitatory postsynaptic potential (EPSP)

Inhibitory Synapse



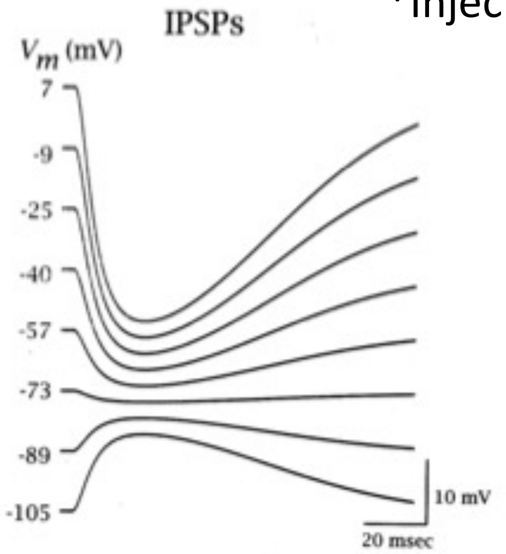
$$E_{syn} = -75\,mV$$

Inhibitory synapse if

$$E_{syn} < V_{th}$$

GABAergic synapse

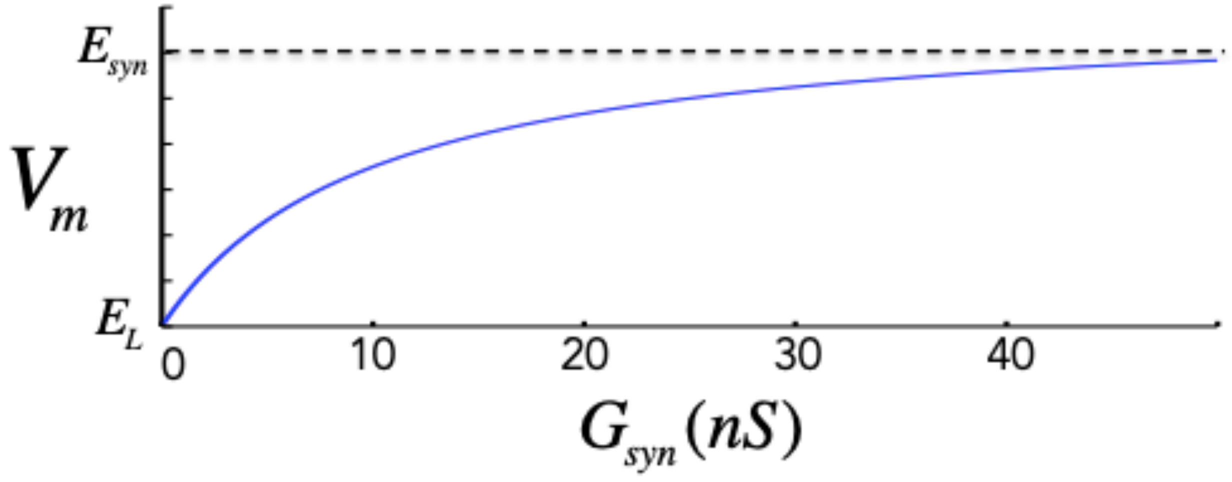
*Injection of current

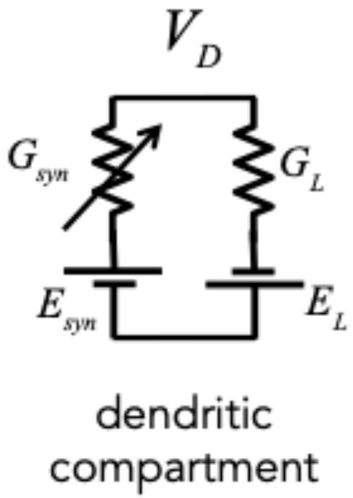


Inhibitory postsynaptic potential (IPSP)

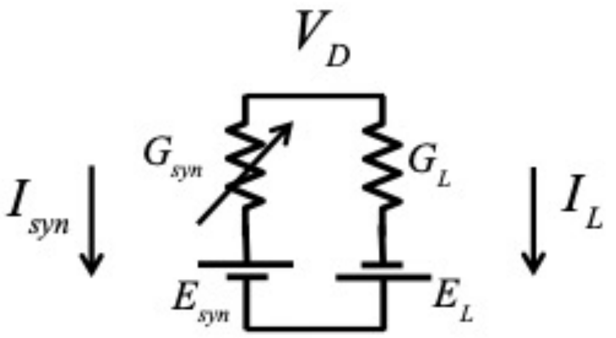
Synaptic Saturation

- How does the potential in dendrite change as a function of amount of excitatory conductance?





Synaptic Saturation



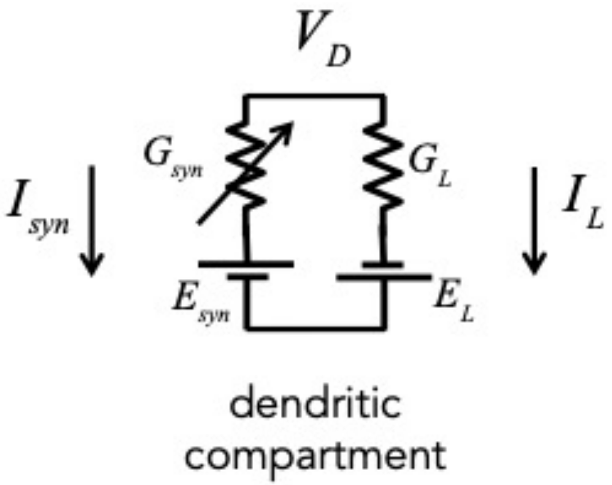
- Kirchoff's law:

$$I_{syn} + I_L = 0$$

$$G_{syn}\left[V - E_{syn}\right] + G_L[V - E_L] = 0$$
$$G_{syn}V - G_{syn}E_{syn} + G_L V - G_L E_L = 0$$
$$V\left(G_{syn} + G_L\right) - \left(G_{syn}E_{syn} + G_L E_L\right) = 0$$

$$V = \frac{G_L E_L + G_{syn}E_{syn}}{G_L + G_{syn}}$$

Synaptic Saturation



$$V = \frac{G_L E_L + G_{syn} E_{syn}}{G_L + G_{syn}}$$

- For $G_L \gg G_{syn}$:

$$V \approx E_L + \frac{G_{syn} E_{syn}}{G_L}$$

- For $G_{syn} \gg G_L$:

$$V \rightarrow E_{syn}$$