Assignment: due July the 15th of 2024, 23:59 hrs

Prepare a report on the following two projects. In your report motivate for each project, the apprach you choose (adress well posedness of the problem, coercivity of your vector field, choice of decreasing vs. fixed gain size), explain your experimental setting (sample size, confidence bounds etc.), and provide a proper output analysis of your SA.

Project 1 [20 Credits]

An investor has total capital of 1 that she wants to invest in n companies. The run time for the investment is t time units. The market value of company i at time t is given by X_i. Let $x_i \in \mathbb{R}, i = 1, \ldots, n$ be given thresholds. Company i will not be able to generate any profit if $X_i < x_i$. In case $X_i \geq x_i$ the return on the investment is given by Y_i. Investing a fraction p_i of the capital yields expected return

$$p_i \mathbb{E}[Y_i 1_{X_i \geq x_i}]$$

for company i, where

$$\sum_{i=1}^{n} p_i = 1 \quad \text{and} \quad 0 \leq p_i \leq 1. \tag{1}$$

Assume that $n = 3$ and Y_i are independent uniformly distributed on $[0, X_i]$, and we let

$$X_i = \frac{\rho V + \sqrt{1-\rho^2}\eta_i}{\max(W, 1)}, \quad 1 \leq i \leq n,$$

with η_i normally distributed with mean 0 and variance i modelling the company's idiosyncratic risk, V standard normally distributed modelling the common factor that affects the economy, and W exponentially distributed with rate $1/0.3$ modelling common market shocks. The variables V, W, and η_i's are all independent. Set the weight factor $\rho = 0.6$, and the thresholds $x_1 = 2, x_2 = 3$, and $x_3 = 1$.

The investor wants to find the optimal investment strategy by maximizing the risk adjusted performance of the investment (also known as Sharpe ratio), which leads to

$$\max_{p_1, p_2, p_3 \geq 0, \sum p_i = 1} \mathbb{E}\left[\frac{\sum_{i=1}^{3} p_i Y_i 1_{X_i \geq x_i}}{std\left(\sum_{i=1}^{3} p_i Y_i 1_{X_i \geq x_i}\right)}\right], \tag{2}$$

where $std(X)$ denotes the standard deviation of random variable X. Find the optimal allocation vector (p_1, p_2, p_3) for solving (2) by applying an appropriate SA algorithm.

Project 2 [25 Credits]

Customers arrive at a service station according to a renewal point process. The inter-arrival times $\{A_n : n \in \mathbb{N}\}$ are iid, with $\mathbb{E}[A_n] < \infty$ and $\mathbb{P}(A_n = 0) = 0$. Customers are served in order of arrival, and consecutive service times are iid random variables $\{S_n(\theta) : n \in \mathbb{N}\}$. Interarrival times and service times are assumed to be mutually independent. The common distribution of the service times depends on parameter $\theta = \mathbb{E}[S_n(\theta)]$. This is what is known as the GI/GI/1 queueing model with FCFS ("first-come first-served") service discipline.

Let $W_n(\theta)$ denote the waiting time of the n-th customer (the time the customer spends in the system until start of service), then we have the following recursion for the waiting times

$$W_{n+1}(\theta) = \max(0, W_n(\theta) + S_n(\theta) - A_{n+1}), \tag{3}$$

for $n \geq 0$, and $W_0(\theta) = S_0(\theta) = 0$; and denote by

$$W^N(\theta) = \frac{1}{N} \sum_{n=1}^{N} W_n(\theta)$$

the *average waiting time over the first N customers.*

Denote the 90 % quantile of the average waiting time over the first N customers by q_θ. For quality control, the system should be operated so that in 90 % of the cases the first N customers do on average not have to wait more than z time units. Without loss of generality, we assume that the cost of operating the server at average speed θ is monotone decreasing in θ. For minimizing costs, we are thus looking for the smallest (and thus least costly) mean service time that achieves the aspired service level. This leads to the following optimization

$$\min_{0<\theta} \mathbb{E}[(q_\theta - z)^2]. \tag{4}$$

Find the optimal mean service time θ by applying an appropriate SA algorithm. For building the estimator, you may use that fact that

$$F_\theta(w) := \mathbb{P}(W^N(\theta) \leq w), \quad w \geq 0,$$

is monotone decreasing in θ for fixed w (longer service times lead to longer average waiting times), and refer to Example 5.6 in the lecture notes for an example on how to track quantiles by means of an SA.

(2.1) (15 Credits) For your experiment, let the interarrival times by exponential with mean value 5, $N = 10$, and let $z = 8$. As detailed in Example 5.5 (pages 137-139), estimating the quantile can be done via SA also. Use this approach for designing an overall SA algorithm for solving the problem in (4).

(2.2) (10 Credits) Run the algorithm where you perform m updates of the SA generating $\{\theta_n\}$ (and thus computing θ_m), and k updates for the sub-problem for tracking the quantile $W^N(\theta_n)$ at θ_n. Keeping your budget mk fixed, what is a good allocation of m and k?