# THESIS

## 1. Details on Specific Spaces

1.1. **Surface Of Revolution.** Consider a point $x$ on a 3-surface $U$ that is a surface of revolution and a vector $v$ in the tangent space $U_x$.

Consider the component $v_0$ of $v$ along $S^2$.

Consider the great circle made by extending $x$ to a geodesic in the direction of $v_0$.

We can reflect across this great circle on all the spherical cross-sections of $U$.

By symmetry, the geodesic made by extending $x$ in the $v$ direction must stay on this slice of $U$.

This reduces finding the geodesic on a 3-surface of revolution to finding one on a 2-surface.

1.1.1. *Finding the vectors from a point:* Rather than mapping a sphere to $\mathbb{R}^2$, we can embed it in $\mathbb{R}^3$.

Given points $\mathbf{x}, \mathbf{y} \in S^2 \times \mathbb{R}$,

Let $\mathbf{v} =$ the normalization of the projection of $\mathbf{y}_1$ perpendicular to $\mathbf{x}_1$.

Let $\theta =$ the angle between $\mathbf{x}$ and $\mathbf{y} = \arccos \langle \mathbf{x}, \mathbf{y} \rangle$

Now we take the points $\mathbf{x}' = (0, x_4)$ and $\mathbf{y}' = (\theta, y_4)$ in the two-dimensional version.

Let $\mathbf{z}'$ be a vector between them.

Let the $S^2$ component of $\mathbf{z}$ be $z_0'\mathbf{v}$ and the $\mathbf{R}$ component be $z_1'$.

$\mathbf{z}$ is a vector between the two points.

1.1.2. *Finding a point from a vector:* Given point $\mathbf{x}$ and vector $\mathbf{z}$,

Let $\mathbf{x}' = (0, x_2), \mathbf{z}' = (\|\mathbf{z}_1\|, z_2)$.

Let $\mathbf{v} = \|\mathbf{z}_1\|$.

Find $\mathbf{y}'$ with the two-dimensional version.

$\mathbf{y} = (\mathbf{x}_1 \cos y_1' + \mathbf{v} \sin y_1', y_2')$.

In order to find the rotation, we must work in a more relevant basis. We can do this by commuting with the appropriate matrix.

$\mathbf{e}_1 \mapsto \mathbf{e}_1, \mathbf{v} \mapsto \mathbf{e}_2, \mathbf{x}_1 \mapsto \mathbf{e}_3, \mathbf{x}_1 \times \mathbf{v} \mapsto \mathbf{e}_4$

$(\mathbf{e}_1, \mathbf{v}, \mathbf{x}_1, (\mathbf{x}_1 \times \mathbf{v}))^{-1}$

$= (\mathbf{e}_1, \mathbf{v}, \mathbf{x}_1, (\mathbf{x}_1 \times \mathbf{v}))^T$, since it's a rotation matrix.

First, we use the rotation of the two-dimensional version to find the rotation between $\mathbf{e}_1$ and $\mathbf{v}$.

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Next, we have to deal with the fact that the $S^2$ component rotates.

Let $\phi = y'_2 - x'_2$.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Putting this all together, we get:

$$(\mathbf{e}_1, \mathbf{z}, \mathbf{x}, (\mathbf{x}\times\mathbf{z})) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} (\mathbf{e}_1, \mathbf{z}, \mathbf{x}, (\mathbf{x}\times\mathbf{z}))^T$$

1.1.3. *Portals:* The intersection tells me the orientation as embedded in $\mathbb{R}^3$. I need to convert that to $\mathbb{R} \times S^2 \subseteq \mathbb{R}^4$.

First, I just expand the matrix $M$ to $\begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}$ to put it in $\mathbb{R}^3 \subseteq \mathbb{R}^4$.

Now I need to reflect it twice.

First, I will reflect along the $t$-axis, to get $\begin{pmatrix} M & 0 \\ 0 & -1 \end{pmatrix}$. This doesn't change anything, since that vector was orthogonal to reality, but it does guarantee that the final orientation will have the same sign.

Now I need to reflect between $(1, 0, 0, 0)$ and $(0, v)$ where $v \in S^2$ is the position of the vector.

Let $w$ be a vector I am moving with this, and $R$ be the reflection matrix.

$Rw = w - ((v, 0) - (0, 0, 0, 1)) \langle (v, 0) - (0, 0, 0, 1), w \rangle$

$= w - (v, -1) \langle (v, -1), w \rangle$

$= Iw - (v, -1)(v, -1)^T w$

$= (I - (v, -1)(v, -1)^T)w$

$R = I - (v, -1)(v, -1)^T$

This means that the final orientation is $R \begin{pmatrix} M & 0 \\ 0 & -1 \end{pmatrix}$.