THESIS

1. Introduction

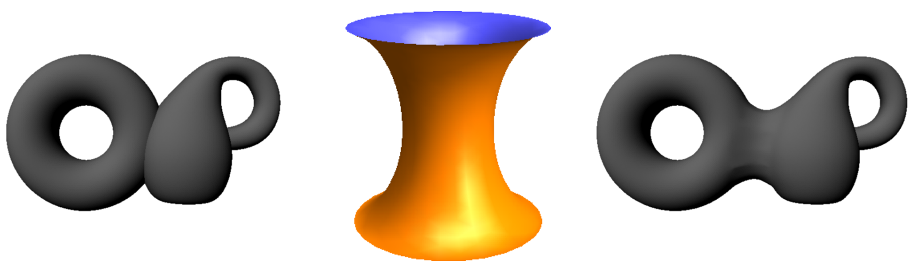
1.1. **Overview.** I am writing a graphics program to visualize three-dimensional non-Euclidean geometry, with emphasis on connected sums of manifolds.

Currently, I have implemented $\mathbb{E}^3$, $\mathbb{H}^3$ and a space homeomorphic to $S^2 \times \mathbb{R}$.

I am in the process of implementing a method to glue multiple spaces together along spherical boundaries.

Outside Euclidean geometry, gluing spaces together like this won't always be smooth. More precisely, since two spheres with the same surface area do not generally have the same curvature, gluing the spaces together by those spheres will result in them having different curvature from each side. A similar problem will occur if the insides or outsides of two spheres are glued together.

In order to address this problem, I have designed and am currently in the process of implementing what I call wormhole.



(a) Two spaces naively connected

(b) A wormhole

(c) Two spaces smoothly connected

I define an (n, m)-surface of revolution as an $(n+m)$-manifold with an automorphism group that has the automorphism group of the n-sphere as a subgroup. An (n, m)-surface of revolution is also an $(n-1, m+1)$-surface of revolution for all $n \geq 1$. In addition, this definition is based entirely upon the intrinsic geometry of the space, and does not require embedding into Euclidean geometry.

I define a 2-wormhole as a quotient space on $\mathbb{H}^2$ made by identifying two ultraparallel lines. This is a $(1, 1)$-surface of revolution.

An n-wormhole is an $(n-1, 1)$-surface of revolution. Generally, this does not have constant curvature, and is not a quotient space on $\mathbb{H}^n$. However, I have found a method to reduce finding a geodesic on an (n, m)-surface of revolution to a $(1, m)$-surface of revolution. In particular, I can find the geodesic on an n-wormhole using the method to find a geodesic on a 2-wormhole.

There are two other spaces that are similar to this wormhole that are of interest. I hope to program these, but they are not part of my main goal. If I run out of time, I will consider the project complete without them. I call one a black hole and the other a cone point space. These spaces, and the wormhole, all are built by extending to the third dimension a quotient space of $\mathbb{H}^2$ in which two lines are identified. The wormhole occurs if the two lines are ultraparallel, the black hole if they're asymptotic, and the cone point space if they intersect.

Black holes and cone spaces can support the connection of two spaces in a similar manner, but they're more interesting just connected to one. An n-black hole is an n-dimensional analogue of a pseudosphere, in which the embedding into $\mathbb{E}^3$ which has spherical cross-sections instead of circular ones. An n-cone point space is notable for containing a cone point and still being easy to connect to $\mathbb{E}^n$ or $\mathbb{H}^n$.

Another way to generalize this is to identify lines on $\mathbb{E}^2$ or S^2 instead of $\mathbb{H}^2$. Again, this is not part of the main project, and is not required for me to consider it complete.

Identifying two parallel lines on $\mathbb{E}^2$ creates a cylinder, which extends to $\mathbb{S}^n \times \mathbb{R}$. Identifying two intersecting lines creates a cone, which extends to a higher dimensional cone. Identifying two lines on a sphere creates something shaped like a lemon, with two cone points on opposite sides. If you extend this to identify two lines that meet at an angle greater than $360°$, which you can construct by gluing multiple spheres together, the resulting space cannot be embedded in $\mathbb{E}^3$.

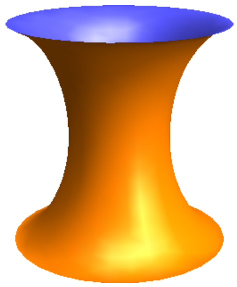
This variation is particularly interesting because it contains a cone point of angle greater than $360°$ and can be smoothly attached to $\mathbb{E}^n$ and S^n. While these cone points can be made easily by generalizing the Cone Point Space and hypercones, they can be smoothly attached only to $\mathbf{H}^3$ and $(n-1,1)$-surfaces of revolution. Furthermore, the only $(n-1,1)$-surface of revolution that I have defined that could be used as an intermediate manifold to attach a Cone Point Space or hypercone to $\mathbf{E}^n$ or S^n is this manifold. In addition, this variation can be used as an intermediate space to glue a ball of $\mathbf{H}^n$ into $\mathbf{E}^n$ or S^n, producing an area that is smaller on the inside.

Figures 1-4 illustrate the 2-dimensional versions of these surfaces. While only the portion of the surfaces that can be embedded into $\mathbb{E}^3$ are shown here, my program will be capable of presenting the entire surface for visualization.

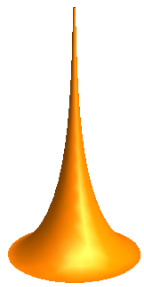
I also hope to implement S^3 and $S^2 \times \mathbb{E}$. I may also implement $\mathbb{H}^2 \times \mathbb{E}$, but it doesn't combine well with the wormhole, so it won't be as useful.

One obstacle will be finding geodesics between two given points when spaces are glued together. In the spaces I have chosen, it is not difficult to construct a geodesic between two given points. However, this does not apply to the compound spaces made by gluing them together.

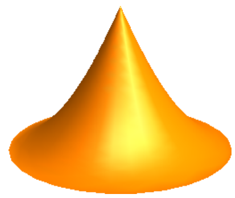
I can easily construct a geodesic from a given point moving in a given direction. By repeatedly constructing such geodesics at set lengths and taking note of the



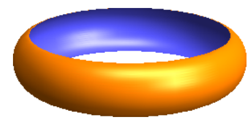
(d) Part of a 2-wormhole



(e) Part of a 2-black hole space



(f) Part of a 2-cone point space



(g) Part of a 2-extended sphere

distance they are from the target point, I can use Newton's method to solve numerically for the geodesic between two given points.

Alternately, I can build a raytracer. Rather than finding the direction each point is in from the camera, I will construct geodesics corresponding to each pixel in the camera, shooting rays to see what lies in that direction. This leads to problems with light sources, since I still need to find the angle and distance of the light source from the given point. I have found three possible solution:

- I can give every point the same lighting, regardless of position. This will likely be useful for debugging, but it results in no shading, which will make it less useful or interesting.
- I can use the camera as the light source. In this case, the geodesic that passes between the camera and a given point is the one I used to find that point to begin with. The shading will be a function only of distance, unless Phong shading is used. It will look better, but it will still be very limited.
- I can simple fire photons in various directions from the light sources, and use them to color objects they strike. Photons can be used to give effects that are otherwise impossible, such as diffuse interreflection, where an object is illuminated by light reflected off of another object, but they generally take a very long time to calculate.

There is another notable detail. In Euclidean geometry, lighting falls of inversely with the square of distance. This does not apply in general. Changing the direction and distance a photon is emitted by $\varepsilon\mathbf{u}$ for some unit vector $\mathbf{u}$ will change where it hits by $M\varepsilon\mathbf{u} + O(\varepsilon^2)$ for some matrix M. The brightness is inversely proportional to $|\det(M)|$.

More accurately, given the list of all vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots$ such that the photons moving in that direction for that distance hit the given point, the brightness at that point is proportional to $\sum \frac{1}{|\det(M_{\mathbf{v}_i})|}$. This is because, in general, two points can be connected by more than one geodesic.

1.2. **Background.** The prime decomposition theorem states that a compact, connected, orientable 3-manifold can be decomposed into a connected sum of prime manifolds (manifolds that cannot be decomposed further, except by trivially removing a sphere) [?], and that this is unique up to insertion or deletion of copies of S^3 [?]. The connected sum of two spaces is made by removing a ball from each and identifying their bounding spheres.

The torus decomposition theorem states that there is a minimal collection of disjointly embedded incompressible tori such that cutting along the edge of each yields components that are each either atoroidal or Seifert-fibered. It also states that this collection is unique up to isomorphism [?] [?] [?] [?].

The geometrization theorem states that any 3-manifold can be decomposed canonically into submanifolds which each is a quotient space of one of the following eight geometries: $S^3, \mathbb{E}^3, \mathbb{H}^3, S^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}, \tilde{SL}(2, \mathbb{R})$, Nil geometry, and Sol geometry. This is done by decomposing along the spheres given by the prime decomposition theorem and tori given by the torus decomposition theorem. The initial work was done by Grisha Perelman [?] [?] [?], and was later completed by other mathematicians [?] [?] [?].

I am working on a program that will allow smooth creation of a connected sum of several geometries as in the geometrization theorem, and visualization of the result. "Smoothness" means that the boundaries that are identified have the same curvatures in each surface. If they are not smoothly identified, then they will appear to have different curvatures from each side. As a result, a geodesic that barely intersects the border can have a very different path from one that barely misses.

For example, if we glue the outside of a sphere in Euclidean geometry to the outside of another such sphere in another copy of Euclidean geometry, the result looks like the portal is a reflective sphere, but with the reflection from the other geometry. This has effects such as blocking anything behind the sphere. This is impossible in a true manifold. Any point is visible from any other. My program will avoid this by using an intermediate geometry of non-constant curvature which contains spheres that can be glued to spaces of any constant curvature.

My program likely will not support torus decomposition, and thus will not be able to show every manifold. However, it is still more general than anything that has been reported previously, as described next.

Previous work has been done to visualize S^3 and $\mathbb{H}^3$ and quotient spaces thereof. In particular, Weeks has written a program that can view compact quotient spaces of $S^3, \mathbb{E}^3$, and $\mathbb{H}^3$ [?]. He also wrote a paper that describes the process in depth [?].

Gunn and Maxwell made a video showing the complementary spaces of knots, most of which are quotient spaces of $\mathbb{H}^3$ [?]. Gunn explains the techniques he uses in [?] [?].

Levy, Munzner and Mark Phillips wrote a geometry viewer called Geomview. Among its features is the ability to render non-Euclidean geometry [?].

The main difference that will distinguish my work is the ability to create connected sums between spaces.

I am unaware of any previous attempts to visualize general connected sums of geometries. However, attempts have been made to visualize a connected sum of two $\mathbb{E}^3$ spaces, with curvature near zero outside of a small wormhole.

Rune Johansen made a video that was designed to convey the idea of a wormhole [?]. This video is primarily artistic in nature, and usees various tricks to make space look curved. There is no geometry that looks precisely like the video.

Corvin Zahn made a computer generated video precisely illustrating a wormhole [?]. He uses a solution to Einstein's field equations that has been found previously [?], and simulated the camera moving through this wormhole. He did not provide specifics about the program used. He detailed what kind of manifold was used, but not how he used it. I have emailed him with questions regarding his implementation, but he has yet to respond.

Zahn's wormhole was made as one continuous geometry with everywhere negative curvature. While this is a much more interesting space than what my program will make, it's much harder to generalize. Adding a second wormhole between the spaces or a second wormhole leading to a third space would require redesigning the entire manifold. In contrast, my program will provide a compact manifold with boundary to connect spaces, so as long as the spheres they replace in Euclidean geometry do not touch, any number can be added easily to the same space.

Apparently, he used a ray tracing method and found the geodesics numerically. I intend to use rasterization and minimal amounts of numerical calculations in order to run the program in real time.

2. Wormhole

In order to allow more interesting manifolds than a few that I hard code in, I plan to implement connected sums of manifolds. Unfortunately, almost none of the well-known manifolds can be smoothly connected without changing their metrics. In order to facilitate this, I have found a class of manifolds that can be used as intermediates to smoothly connect two manifolds of constant curvature, and which allows geodesics to be easily calculated. I call them wormholes.

The wormhole is topologically $S^2 \times \mathbb{R}$, but with a different metric.

Given $p_1, p_2 \in S^2 \times \mathbb{R}$,

$p_1 = (s_1, r_1)$

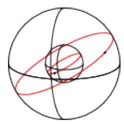
$p_2 = (s_2, r_2)$

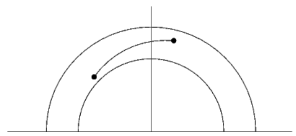
Let S be the great circle containing s_1, s_2.

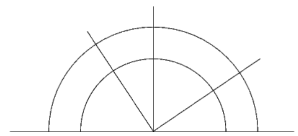
p_1, p_2 are in the slice $S \times \mathbb{R}$ of $S^2 \times \mathbb{R}$.

Let $\theta_1, \theta_2 \in S^1$ be s_1, s_2 under the inverse inclusion map.

$p_1 = (\theta_1, r_1)$
$p_2 = (\theta_2, r_2)$
$S \times \mathbb{R}$ corresponds to a quotient space of the half-plane model of $\mathbb{H}^2$ by $(\theta, r) \mapsto (e^{k\theta} \cos r, e^{k\theta} \sin r)$ where k is a constant depending on the quotient space.
The two semicircles in the picture are identified to form the quotient.


(h) $S^2 \times \mathbb{R}$


(i) Half Plane


(j) Radial Lines

Find a geodesic in $\mathbb{H}^2$ that connects p_1 and p_2, and map it back to $S \times \mathbb{R} \subseteq S^2 \times \mathbb{R}$.

Radial lines are curves of constant curvature. These map to closed loops. The angle of the lines controls how the curvature and diameter relate allowing connection to a space of a specific curvature.

3. Details common to all spaces

3.1. **Orientation.** Much of this program involves orientation. For example, a camera must have an orientation. The orientation is essentially a linear transformation between $\mathbb{R}^3$ and the tangent space of the manifold at that point which preserves magnitude and angles. In the case of $\mathbb{E}^3$, the tangent space at any point has a natural map to $\mathbb{R}^3$, so the orientation can be thought of as a rotation from $\mathbb{R}^3$ to itself.

In order to store the orientation, I use a default orientation for each point. This is commonly the orientation in the map from $\mathbb{R}^3$ to the manifold. For example, in the half-plane model of $\mathbb{H}^3$, the vector $(0, 0, 1)$ maps to the direction of the geodesic that approaches $(0, 0, \infty)$.

This is not necessarily the case. Currently the only exception is SurfaceOfRevolution, which uses a map that is not necessarily conformal, and maps from $S^2 \times \mathbb{R}$ instead of $\mathbb{R}^3$. In this case, $(0, 0, 0, 1)$ maps along the axis, $(x, y, z, 0)$ maps to the natural value, and everything else maps as necessary to make it angle-preserving.

When a point of reference moves along a path, the final orientation is not directly comparable to the initial orientation. However, the final orientation compared to the default is directly comparable to the initial orientation compared to the default. In particular, this difference rotates by a certain amount depending on the path. I generally speak of the point of reference rotating by that amount.

3.2. **Finding the Geodesic Between Two Points Numerically.** In many manifolds, the geodesic between two points can be easily calculated symbolically. However, when two or more simple manifolds are glued together, this method quickly becomes infeasible. You would likely have to find a new equation for every combination, and it's likely that the final equation will quickly get too complicated to be solved easily.

Given a point and a vector, it's still fairly easy to find the geodesic that extends that distance from that point in that direction. Simply find the geodesic in the manifold the point is in, if it intersects with another manifold find where and at what distance, extend the geodesic from there, and repeat until you run out of geodesic.

There is not necessarily only one geodesic between a given pair of points. When drawing a triangle, the necessary geodesics will presumably be close to each other. This can be used to find the geodesic you're looking for.

When drawing a triangle in which the geodesic reaching one of the vertices is known, it can be used for the first iteration to find the other two vertices.

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Given starting point p, ending point q, and vector $\mathbf{v}$ that's near the vector tangent to the geodesic with a magnitude equal to its length,

Let $\psi(\mathbf{u})$ be the endpoint of the geodesic starting at p that is tangent to and of the same magnitude as $\mathbf{u}$. We are attempting to find the value of $\psi^{-1}(q)$ that's close to $\mathbf{v}$.

First, make a continuous injective map, ϕ, from the manifold to the vector field $\mathbb{R}^3$. This set of vectors has little to do with the ones being mapped from in ψ. In order to distinguish them, I will refer to them as $\mathbf{x}, \mathbf{y}$, etc. as opposed to $\mathbf{u}, \mathbf{v}$, etc. ???

S^3 doesn't have such a map. I could get it to work if I remove a point. If I'm gluing it to another space, that isn't a problem because I could just glue it so the point at infinity is in the ball that's removed. If not, I can just stick it in a random spot and hope for the best.

Compound manifolds also don't have such a map. Theoretically, there is no need to put a compound space inside another compound manifold. However, this will simplify programming it. I can bypass this problem by making it so that passing a compound manifold as a submanifold in a compound manifold will result in the outer manifold automatically disassembling the inner manifold and building itself out of its submanifolds.

Now we have $\phi \circ \psi : \mathbb{R}^3 \mapsto \mathbb{R}^3$. This can be inverted with Newton's method. Simply find $(\phi \circ \psi)^{-1} \circ \phi(q) = \psi^{-1} \circ \phi^{-1} \circ \phi(q) = \psi^{-1}(q)$. As long as the triangles are sufficiently small, $\mathbf{v}$ will be sufficiently close to the preferred solution, guaranteeing that it's the solution approached.

There is still the problem that one vertex may have one solution, but a nearby vertex have three. I will deal with this later. I'm pretty sure it won't come up as long as I stick to $\mathbb{E}^3, \mathbb{H}^3$, and Wormhole.

4. Details on Specific Spaces

4.1. **2d Hyperbolic Geometry.**

4.1.1. *Finding the direction and distance from one point to another.* We will be using the half plane model: $\mathbf{H}^2 \mapsto \{(x_1, x_2) : x_2 > 0\}$.

Given points $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$,

The geodesic between them is mapped to a circle on the half-plane with a center on the x-axis. Consider this circle.

Let $\mathbf{c}$ be the center of the circle, and r be the radius. We know $c_2 = 0$, $\|\mathbf{x}-\mathbf{c}\| = \|\mathbf{y} - \mathbf{c}\| = r$.

First, let us solve for c_1.

$(x_1 - c_1)^2 + {x_2}^2 = (y_1 - c_1)^2 + {y_2}^2 = r^2$

$= {x_1}^2 - 2x_1c_1 + c_1^2 + {x_2}^2 = {y_1}^2 - 2y_1c_1 + c_1^2 + {y_2}^2$

${x_1}^2 - 2x_1c_1 + {x_2}^2 = {y_1}^2 - 2y_1c_1 + {y_2}^2$

$2(x_1 - y_1)c_1 = {x_1}^2 + {x_2'}^2 - {y_1}^2 - {y_2}^2$

$c_1 = \frac{({x_1}^2+{x_2}^2)-({y_1}^2+{y_2}^2)}{2(x_1-y_1)}$

$= \frac{\|\mathbf{x}\|^2-\|\mathbf{y}\|^2}{2(x_1-y_1)}$

We can use this value of c_1 to compute r.

The vector at $\mathbf{x}$ pointing to $\mathbf{y}$ is tangent the circle, which means that its a right angle from the direction to $\mathbf{c}$, which works out to be $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (\mathbf{c} - \mathbf{x}) = \begin{pmatrix} x_2 \\ c_1 - x_1 \end{pmatrix}$. We then normalize this to $\frac{(x_2, c_1 - x_1)}{\sqrt{x_2^2+(c_1-x_1)^2}}$.

The distance along the geodesic can be calculated using the angle of the arc between $\mathbf{x}$ and $\mathbf{y}$ on the half plane. Assuming it moves counterclockwise:

$\int_{\theta_1}^{\theta_2} \frac{\sqrt{(\frac{d}{d\theta} r\sin\theta)^2+(\frac{d}{d\theta} r\cos\theta)^2}}{r\sin\theta} d\theta$

$= \int_{\theta_1}^{\theta_2} \frac{\sqrt{(r\cos\theta)^2+(-r\sin\theta)^2}}{r\sin\theta} d\theta$

$= \int_{\theta_1}^{\theta_2} \frac{r}{r\sin\theta} d\theta$

$= \int_{\theta_1}^{\theta_2} \csc\theta d\theta$

$= -\ln\left|\frac{\csc\theta_2+\cot\theta_2}{\csc\theta_1+\cot\theta_1}\right|$

$= \ln\left|\frac{\csc\theta_1+\cot\theta_1}{\csc\theta_2+\cot\theta_2}\right|$

$\csc\theta_1 = \frac{r}{x_2}, \cot\theta_1 = \frac{x_1-c_1}{x_2}, \csc\theta_2 = \frac{r}{y_2}, \cot\theta_2 = \frac{y_1-c_1}{y_2}$

Hence, the distance is $\ln\left|\frac{\frac{r}{x_2}+\frac{x_1-c_1}{x_2}}{\frac{r}{y_2}+\frac{y_1-c_1}{y_2}}\right|$

$= \ln\left|\frac{\frac{r+x_1-c_1}{x_2}}{\frac{r+y_1-c_1}{y_2}}\right|$

$= \ln\left|\frac{y_2(r+x_1-c_1)}{x_2(r+y_1-c_1)}\right|$

Since $\|\mathbf{x} - \mathbf{c}\|, \|\mathbf{y} - \mathbf{c}\| < r$, clearly $|x_1 - c_1|, |y_1 - c_1| < r$, and we already know x_2 and $y_2 > 0$, so $\frac{y_2(r+x_1-c_1)}{x_2(r+y_1-c_1)} \geq 0$. Thus, we may remove the absolute value.

$= \ln\frac{y_2(r+x_1-c_1)}{x_2(r+y_1-c_1)}$

If the path is clockwise instead of counterclockwise, it comes out to $-\ln\frac{y_2(r+x_1-c_1)}{x_2(r+y_1-c_1)} = \ln\frac{x_2(r+y_1-c_1)}{y_2(r+x_1-c_1)}$. In general, the distance is $\left|\ln\frac{y_2(r+x_1-c_1)}{x_2(r+y_1-c_1)}\right|$.

Of course, none of this works if the two points share the same first two coordinates. In that case, the problem is easier. The direction is $(0,1)$, and the distance is $\int_{x_3}^{y_3}\frac{1}{t}dt = \ln\frac{y_3}{x_3}$. This gives a vector of $(0, \ln\frac{y_3}{x_3})$.

4.1.2. *Finding the point a given distance in a given direction from another.* Given initial point $\mathbf{x} = (x_1, x_2)$ and vector $\mathbf{z} = (z_1, z_2)$,

Since the geodesic is a circle, the center of the circle is on a line perpendicular to the tangent vector.

The tangent line is $\mathbf{x}+t\mathbf{z}$, so the center of the circle is on $\mathbf{x}+t\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{z} = (x_1 - tz_2, x_2 + tz_1)$.

This intersects with the x axis when $x_2 + tz_1 = 0$.

$t = -\frac{x_2}{z_1}$

$c_1 = x_1 - tz_2$

$= x_1 + \frac{x_2 z_2}{z_1}$

And as before, $c_2 = 0$.

$r = \|\mathbf{x} - \mathbf{c}\|$

If were letting $x_1 = 0$, that's just $r^2 = c_1^2 + {x_2}^2$.

Now that we have the circle, its just a matter of finding the point at the right distance.

Let $d = \ln\frac{y_2(r+x_1-c_1)}{x_2(r+y_1-c_1)}$

$e^d = \frac{y_2(r+x_1-c_1)}{x_2(r+y_1-c_1)}$

$x_2(r+y_1-c_1)e^d = y_2(r+x_1-c_1)$

${x_2}^2(r+y_1-c_1)^2e^{2d} = {y_2}^2(r+x_1-c_1)^2$

$= [r^2-(y_1-c_1)^2](r+x_1-c_1)^2$

$= [r-(y_1-c_1)][r+(y_1-c_1)](r+x_1-c_1)^2$

$= (r-y_1+c_1)(r+y_1-c_1)(r+x_1-c_1)^2$

Since $y_0 > 0$ and $r^2 = |\mathbf{y}-\mathbf{c}|$, it must be the case that $|y_1 - c_1| < r$ so $r + y_1 - c_1 \neq 0$ and can be safely canceled out.

${x_2}^2(r+y_1-c_1)e^{2d} = (r-y_1+c_1)(r+x_1-c_1)^2$

$y_1[{x_2}^2e^{2d} + (r+x_1-c_1)^2] = (r+c_1)(r+x_1-c_1)^2 - {x_2}^2(r-c_1)e^{2d}$

$y_1 = \frac{(r+c_1)(r+x_1-c_1)^2 - {x_2}^2(r-c_1)e^{2d}}{{x_2}^2e^{2d}+(r+x_1-c_1)^2}$

Now that we know y_1, we can easily find y_2 with $y_2 = \sqrt{r^2-(y_1-c_1)^2}$.

We will also need to find the change in orientation.

Let θ_0 be the initial angle and θ_1 be the final angle.

$\sin\theta_0 = \frac{x_2-c_2}{r}$

$= \frac{x_2}{r}$

$\cos\theta_0 = \frac{x_1-c_1}{r}$

$= -\frac{c_1}{r}$

Similarly, $\sin\theta_1 = \frac{y_2}{r}, \cos\theta_1 = \frac{y_1 - c_1}{r}$.

We can simply take $\theta_1 - \theta_0$ as the angle, but this will require trigonometry to get the angle, and then trigonometry later to do the rotation with it. It may be more optimal to just calculate $\sin\Delta\theta$ and $\cos\Delta\theta$ using angle sums.

$\sin(\theta_1 - \theta_0) = \sin\theta_1\cos\theta_0 - \cos\theta_1\sin\theta_1$

$\cos(\theta_1 - \theta_0) = \cos\theta_0\cos\theta_1 - \sin\theta_0\sin\theta_1$

If $z_0 = 0$ so the vector is pointing straight up or down, we have:

$z_2 = \ln\frac{y_2}{x_2}$

$e^{z_2} = \frac{y_2}{x_2}$

$y_2 = x_2 e^{z_2}$

Clearly, y_1 remains constant and there is no rotation.

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Portals in 3-wormholes will be represented by S^2 slices of $S^2 \times \mathbb{R}$. The intersection with the $S^1 \times \mathbb{R}$ slice, when mapped to the upper half-plane, becomes a radial line. As such, it is necessary to find where geodesics in $\mathbb{H}^2$ intersect with these radial lines.

4.2. **3d Hyperbolic Geometry.**

4.2.1. *Finding the direction and distance from one point to another.* We will use the upper half space model: $\mathbb{H}^3 \mapsto \{(x_1, x_2, x_3) : x_3 > 0\}$.

Given points $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$, the first step is to find the vertical plane that intersects both of them. This way, the problem can be reduced to a problem in $\mathbb{H}^2$. Unless $x_1 = y_1$ and $x_2 = y_2$, we let $u = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$, then set $\mathbf{v} = (v_1, v_2) = (y_1 - x_1, y_2 - x_2)/u$. We can now work with $(x_1', x_2') = (0, x_3)$ and $(y_1', y_2') = (u, y_3)$.

Once we have two points on a plane $\mathbf{x}', \mathbf{y}'$, we can use the two-dimensional solution to find the vector $\mathbf{z}' = (z_1', z_2')$ representing the direction and distance from $\mathbf{x}'$ to $\mathbf{y}'$.

Translating the vector back from the plane is simple. You just use $\mathbf{z} = (z_1, z_2, z_3) = (z_1'v_1, z_1'v_2, z_2')$.

The distance remains the same as it was in the two-dimensional case.

There is a problem if $x_1 = y_1, x_2 = y_2$ because $\mathbf{v}$ is undefined. In this case, $\mathbf{z} = (0, 0, \ln\frac{y_3}{x_3})$

4.2.2. *Finding the point a given distance in a given direction from another.* Given initial point (x_1, x_2, x_3) and vector (z_1, z_2, z_3), we first reduce to the $\mathbb{H}^2$ case as before, unless $z_1 = z_2 = 0$. Let $u = \sqrt{z_1^2 + z_2^2}, (v_1, v_2) = \frac{(z_1, z_2)}{u}$ and solve the two-dimensional case for $\mathbf{y}'$ with $\mathbf{x}' = (x_1', x_2') = (0, x_3)$ and $\mathbf{z}' = (z_1', z_2') = (u, z_3)$.

Now we just need to map $\mathbf{y}'$ back to $\mathbb{H}^3$, which is done via $(y_1, y_2, y_3) = (x_1', x_2', y_2') + y_1'(v_1, v_2, 0) = (x_1' + y_1'v_1, x_2' + y_1'v_2, y_2')$.

We will also need to find the change in orientation.

Our solution in the two-dimensional version gave us an angle which we will call θ. This corresponds to the point of reference being rotated with the rotation matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

You can then expand this to a 3×3 matrix with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix}$ and conjugate it with $\begin{pmatrix} v_1 & v_2 & 0 \\ -v_2 & v_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ to get the 3×3 rotation matrix.

This gives $\begin{pmatrix} v_1 & -v_2 & 0 \\ v_2 & v_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 0 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} v_1 & v_2 & 0 \\ -v_2 & v_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

If $z_1 = z_2 = 0$, then $\mathbf{y} = (x_1, x_2, x_3 e^{z_3})$ and there is no rotation.

4.3. **2d Spherical Geometry.** *This seems to be about 3d spheres*

$y_0 = a\sin(x_0 - \theta_0), y_1 = a\sin(x_1 - \theta_0)$

$\frac{y_1}{y_0} = \frac{\sin(x_1-\theta_0)}{\sin(x_0-\theta_0)}$

???

$(x', y', z') = (\cos y \cos x, \cos y \sin x, \sin y)$

???

Instead of mapping to $\mathbb{R}^3$, as we did for other surfaces, we can simply embed into $\mathbb{R}^4$.

Given points $\mathbf{x}, \mathbf{y}$,

They are almost always not equal or antipodal.

Let $\mathbf{y}'$ be a renormalized form of $\mathbf{y} - \langle \mathbf{x}, \mathbf{y} \rangle$.

The geodesic is $\mathbf{x}\cos\theta + \mathbf{y}'\sin\theta$.

The direction is $\mathbf{y}'$.

The distance is $\arccos\langle \mathbf{x}, \mathbf{y} \rangle$.

The vector is $\arccos\langle \mathbf{x}, \mathbf{y} \rangle \, \mathbf{y}'$.

If we're dealing with a 2-sphere:

We have a rotation of $\arccos\langle \mathbf{x}, \mathbf{y} \rangle$ around the axis $\mathbf{x} \times \mathbf{y}'$.

This gives the quaternion $\langle \mathbf{x}, \mathbf{y} \rangle (\mathbf{x} \times \mathbf{y}')$.

We then convert this to a rotation matrix with:

$$Q = \begin{bmatrix} 1 - 2y^2 - 2z^2 & 2xy - 2zw & 2xz + 2yw \\ 2xy + 2zw & 1 - 2x^2 - 2z^2 & 2yz - 2xw \\ 2xz - 2yw & 2yz + 2xw & 1 - 2x^2 - 2y^2 \end{bmatrix}$$

???

If we're not dealing with a 2-sphere, it gets more complex.

Let $\mathbf{u}, \mathbf{v}$ be unit vectors that are linearly independant with $\mathbf{x}, \mathbf{y}$. You can pick them at random since this will almost certainly be true,

???

Given a point $\mathbf{x}$ and a vector $\mathbf{v}$,

$\mathbf{y} = \mathbf{x} \cos |\mathbf{v}| + \frac{\mathbf{v}}{|\mathbf{v}|} \sin |\mathbf{v}|$

$\mathbf{y}' = \frac{\mathbf{v}}{|\mathbf{v}|}$

The rotation matrix can be found as before.