Finding the direction and distance from one point to another in H^3:

We will use the upper half space model: $\mathbb{H}^3 \mapsto \{(x_1, x_2, x_3) : x_3 > 0\}$.

Given points $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$, the first step is to find the vertical plane that intersects both of them. This way, the problem can be reduced to a problem in $\mathbb{H}^2$. Unless $x_1 = y_1$ and $x_2 = y_2$, we let $u = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$, then set $\mathbf{v} = (v_1, v_2) = (y_1 - x_1, y_2 - x_2)/u$. We can now work with $(x_1', x_2') = (0, x_3)$ and $(y_1', y_2') = (u, y_3)$.

Once we have two points on a plane $\mathbf{x}', \mathbf{y}'$, we can use the two-dimensional solution to find the vector $\mathbf{z}' = (z_1', z_2')$ representing the direction and distance from $\mathbf{x}'$ to $\mathbf{y}'$.

Translating the vector back from the plane is simple. You just use $\mathbf{z} = (z_1, z_2, z_3) = (z_1' v_1, z_1' v_2, z_2')$.

The distance remains the same as it was in the two-dimensional case.

There is a problem if $x_1 = y_1, x_2 = y_2$ because $\mathbf{v}$ is undefined. In this case, $\mathbf{z} = (0, 0, \ln \frac{y_3}{x_3})$

Finding the point a given distance in a given direction from another:

Given initial point (x_1, x_2, x_3) and vector (z_1, z_2, z_3), we first reduce to the $\mathbb{H}^2$ case as before, unless $z_1 = z_2 = 0$. Let $u = \sqrt{z_1^2 + z_2^2}, (v_1, v_2) = \frac{(z_1, z_2)}{u}$ and solve the two-dimensional case for $\mathbf{y}'$ with $\mathbf{x}' = (x_1', x_2') = (0, x_3)$ and $\mathbf{z}' = (z_1', z_2') = (u, z_3)$.

Now we just need to map $\mathbf{y}'$ back to $\mathbb{H}^3$, which is done via $(y_1, y_2, y_3) = (x_1', x_2', y_2') + y_1'(v_1, v_2, 0) = (x_1' + y_1' v_1, x_2' + y_1' v_2, y_2')$.

We will also need to find the change in orientation.

Our solution in the two-dimensional version gave us an angle which we will call θ. This corresponds to the point of reference being rotated with the rotation matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

You can then expand this to a 3×3 matrix with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix}$

and conjugate it with $\begin{pmatrix} v_1 & v_2 & 0 \\ -v_2 & v_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ to get the 3×3 rotation matrix.

This gives $\begin{pmatrix} v_1 & -v_2 & 0 \\ v_2 & v_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 0 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} v_1 & v_2 & 0 \\ -v_2 & v_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

If $z_1 = z_2 = 0$, then $\mathbf{y} = (x_1, x_2, x_3 e^{z_3})$ and there is no rotation.