Finding the direction and distance from one point to another in H^2:

We will be using the half plane model: $\mathbf{H}^2 \mapsto \{(x_1, x_2) : x_2 > 0\}$.

Given points $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$,

The geodesic between them is mapped to a circle on the half-plane with a center on the x-axis. Consider this circle.

Let $\mathbf{c}$ be the center of the circle, and r be the radius. We know $c_2 = 0$, $\|\mathbf{x}-\mathbf{c}\| = \|\mathbf{y}-\mathbf{c}\| = r$.

First, let us solve for c_1.

$(x_1 - c_1)^2 + {x_2}^2 = (y_1 - c_1)^2 + {y_2}^2 = r^2$

$= {x_1}^2 - 2x_1c_1 + c_1^2 + {x_2}^2 = {y_1}^2 - 2y_1c_1 + c_1^2 + {y_2}^2$

${x_1}^2 - 2x_1c_1 + {x_2}^2 = {y_1}^2 - 2y_1c_1 + {y_2}^2$

$2(x_1 - y_1)c_1 = {x_1}^2 + {x_2'}^2 - {y_1}^2 - {y_2}^2$

$c_1 = \frac{({x_1}^2+{x_2}^2)-({y_1}^2+{y_2}^2)}{2(x_1-y_1)}$

$= \frac{\|\mathbf{x}\|^2-\|\mathbf{y}\|^2}{2(x_1-y_1)}$

We can use this value of c_1 to compute r.

The vector at $\mathbf{x}$ pointing to $\mathbf{y}$ is tangent the circle, which means that its a right angle from the direction to $\mathbf{c}$, which works out to be $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}(\mathbf{c} - \mathbf{x}) = \begin{pmatrix} x_2 \\ c_1 - x_1 \end{pmatrix}$. We then normalize this to $\frac{(x_2, c_1 - x_1)}{\sqrt{x_2^2+(c_1-x_1)^2}}$.

The distance along the geodesic can be calculated using the angle of the arc between $\mathbf{x}$ and $\mathbf{y}$ on the half plane. Assuming it moves counterclockwise:

$\int_{\theta_1}^{\theta_2} \frac{\sqrt{(\frac{d}{d\theta}r\sin\theta)^2+(\frac{d}{d\theta}r\cos\theta)^2}}{r\sin\theta}d\theta$

$= \int_{\theta_1}^{\theta_2} \frac{\sqrt{(r\cos\theta)^2+(-r\sin\theta)^2}}{r\sin\theta}d\theta$

$= \int_{\theta_1}^{\theta_2} \frac{r}{r\sin\theta}d\theta$

$= \int_{\theta_1}^{\theta_2} \csc\theta d\theta$

$= -\ln\left|\frac{\csc\theta_2+\cot\theta_2}{\csc\theta_1+\cot\theta_1}\right|$

$= \ln\left|\frac{\csc\theta_1+\cot\theta_1}{\csc\theta_2+\cot\theta_2}\right|$

$\csc\theta_1 = \frac{r}{x_2}, \cot\theta_1 = \frac{x_1-c_1}{x_2}, \csc\theta_2 = \frac{r}{y_2}, \cot\theta_2 = \frac{y_1-c_1}{y_2}$

Hence, the distance is $\ln\left|\frac{\frac{r}{x_2}+\frac{x_1-c_1}{x_2}}{\frac{r}{y_2}+\frac{y_1-c_1}{y_2}}\right|$

$= \ln\left|\frac{\frac{r+x_1-c_1}{x_2}}{\frac{r+y_1-c_1}{y_2}}\right|$

$= \ln\left|\frac{y_2(r+x_1-c_1)}{x_2(r+y_1-c_1)}\right|$

Since $\|\mathbf{x}-\mathbf{c}\|, \mathbf{y}-\mathbf{c}\| > r$, clearly $|x_1 - c_1|, |y_1 - c_1| > r$, and we already know x_2 and $y_2 > 0$, so $\frac{y_2(r+x_1-c_1)}{x_2(r+y_1-c_1)} \geq 0$. Thus, we may remove the absolute value.

$= \ln\frac{y_2(r+x_1-c_1)}{x_2(r+y_1-c_1)}$

If the path is clockwise instead of counterclockwise, it comes out to $-\ln\frac{y_2(r+x_1-c_1)}{x_2(r+y_1-c_1)} = \ln\frac{x_2(r+y_1-c_1)}{y_2(r+x_1-c_1)}$. In general, the distance is $\left|\ln\frac{y_2(r+x_1-c_1)}{x_2(r+y_1-c_1)}\right|$.

Of course, none of this works if the two points share the same first two coordinates. In that case, the problem is easier. The direction is $(0,1)$, and the distance is $\int_{x_3}^{y_3}\frac{1}{t}dt = \ln\frac{y_3}{x_3}$. This gives a vector of $(0, \ln\frac{y_3}{x_3})$.

Finding the point a given distance in a given direction from another:

Given initial point $\mathbf{x} = (x_1, x_2)$ and vector $\mathbf{z} = (z_1, z_2)$,

Since the geodesic is a circle, the center of the circle is on a line perpendicular to the tangent vector.

The tangent line is $\mathbf{x} + t\mathbf{z}$, so the center of the circle is on $\mathbf{x} + t\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{z} = (x_1 - tz_2, x_2 + tz_1)$.

This intersects with the x axis when $x_2 + tz_1 = 0$.

$t = -\frac{x_2}{z_1}$

$c_1 = x_1 - tz_2$

$= x_1 + \frac{x_2 z_2}{z_1}$

And as before, $c_2 = 0$.

$r = \|\mathbf{x} - \mathbf{c}\|$

If were letting $x_1 = 0$, that's just $r^2 = c_1^2 + {x_2}^2$.

Now that we have the circle, its just a matter of finding the point at the right distance.

Let $d = \ln\frac{y_2(r+x_1-c_1)}{x_2(r+y_1-c_1)}$

$e^d = \frac{y_2(r+x_1-c_1)}{x_2(r+y_1-c_1)}$

$x_2(r + y_1 - c_1)e^d = y_2(r + x_1 - c_1)$

${x_2}^2(r + y_1 - c_1)^2 e^{2d} = {y_2}^2(r + x_1 - c_1)^2$

$= [r^2 - (y_1 - c_1)^2](r + x_1 - c_1)^2$

$= [r - (y_1 - c_1)][r + (y_1 - c_1)](r + x_1 - c_1)^2$

$= (r - y_1 + c_1)(r + y_1 - c_1)(r + x_1 - c_1)^2$

Since $y_0 > 0$ and $r^2 = |\mathbf{y} - \mathbf{c}|$, it must be the case that $|y_1 - c_1| < r$ so $r + y_1 - c_1 \neq 0$ and can be safely canceled out.

${x_2}^2(r + y_1 - c_1)e^{2d} = (r - y_1 + c_1)(r + x_1 - c_1)^2$

$y_1[{x_2}^2 e^{2d} + (r + x_1 - c_1)^2] = (r + c_1)(r + x_1 - c_1)^2 - {x_2}^2(r - c_1)e^{2d}$

$y_1 = \frac{(r+c_1)(r+x_1-c_1)^2 - {x_2}^2(r-c_1)e^{2d}}{{x_2}^2 e^{2d} + (r+x_1-c_1)^2}$

Now that we know y_1, we can easily find y_2 with $y_2 = \sqrt{r^2 - (y_1 - c_1)^2}$.

We will also need to find the change in orientation.

Let θ_0 be the initial angle and θ_1 be the final angle.

$\sin\theta_0 = \frac{x_2 - c_2}{r}$

$= \frac{x_2}{r}$

$\cos\theta_0 = \frac{x_1 - c_1}{r}$

$= -\frac{c_1}{r}$

Similarly, $\sin\theta_1 = \frac{y_2}{r}, \cos\theta_1 = \frac{y_1 - c_1}{r}$.

We can simply take $\theta_1 - \theta_0$ as the angle, but this will require trigonometry to get the angle, and then trigonometry later to do the rotation with it. It may be more optimal to just calculate $\sin\Delta\theta$ and $\cos\Delta\theta$ using angle sums.

$\sin(\theta_1 - \theta_0) = \sin\theta_1 \cos\theta_0 - \cos\theta_1 \sin\theta_1$

$\cos(\theta_1 - \theta_0) = \cos\theta_0 \cos\theta_1 - \sin\theta_0 \sin\theta_1$

If $z_0 = 0$ so the vector is pointing straight up or down, we have:

$z_2 = \ln\frac{y_2}{x_2}$

$e^{z_2} = \frac{y_2}{x_2}$

$y_2 = x_2 e^{z_2}$

Clearly, y_1 remains constant and there is no rotation.