Finding the direction and distance from one point to another in H^2:

Given points $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$,

Let $\mathbf{c}$ be the center of the circle, and r be the radius. We know $c_2 = 0$, $\|\mathbf{x} - \mathbf{c}\| = \|\mathbf{y} - \mathbf{c}\| = r$.

$(x_1 - c_1)^2 + {x_2}^2 = (y_1 - c_1)^2 + {y_2}^2 = r^2$

$= {x_1}^2 - 2x_1c_1 + c_1^2 + {x_2}^2 = {y_1}^2 - 2y_1c_1 + c_1^2 + {y_2}^2$

${x_1}^2 - 2x_1c_1 + {x_2}^2 = {y_1}^2 - 2y_1c_1 + {y_2}^2$

$2(x_1 - y_1)c_1 = {x_1}^2 + {x_2'}^2 - {y_1}^2 - {y_2}^2$

$c_1 = \frac{({x_1}^2 + {x_2}^2) - ({y_1}^2 + {y_2}^2)}{2(x_1 - y_1)}$

$= \frac{\|\mathbf{x}\|^2 - \|\mathbf{y}\|^2}{2(x_1 - y_1)}$

The direction is tangent the circle, which means that its a right angle from the direction to $\mathbf{c}$, which works out to is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (\mathbf{c} - \mathbf{x})$. Setting $x_1 = c_2 = 0$, this becomes $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ -x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ c_1 \end{pmatrix}$. We then normalize this to $\frac{(x_2, c_1)}{\sqrt{{x_2}^2 + c_1^2}}$.

The distance can be calculated using the angle it moves across. Assuming it moves counterclockwise:

$\int_{\theta_1}^{\theta_2} \frac{\sqrt{(\frac{d}{d\theta} r \sin\theta)^2 + (\frac{d}{d\theta} r \cos\theta)^2}}{r \sin\theta} d\theta$

$= \int_{\theta_1}^{\theta_2} \frac{\sqrt{(r\cos\theta)^2 + (-r\sin\theta)^2}}{r\sin\theta} d\theta$

$= \int_{\theta_1}^{\theta_2} \frac{r}{r\sin\theta} d\theta$

$= \int_{\theta_1}^{\theta_2} \csc\theta d\theta$

$= -\ln\left|\frac{\csc\theta_2 + \cot\theta_2}{\csc\theta_1 + \cot\theta_1}\right|$

$= \ln\left|\frac{\csc\theta_1 + \cot\theta_1}{\csc\theta_2 + \cot\theta_2}\right|$

$\csc\theta_1 = \frac{r}{x_2}, \cot\theta_1 = \frac{x_1 - c_1}{x_2}, \csc\theta_2 = \frac{r}{y_2}, \cot\theta_2 = \frac{y_1 - c_1}{y_2}$

$\ln\left|\frac{\frac{r}{x_2} + \frac{x_1 - c_1}{x_2}}{\frac{r}{y_2} + \frac{y_1 - c_1}{y_2}}\right|$

$= \ln\left|\frac{\frac{r + x_1 - c_1}{x_2}}{\frac{r + y_1 - c_1}{y_2}}\right|$

$= \ln\left|\frac{y_2(r + x_1 - c_1)}{x_2(r + y_1 - c_1)}\right|$

Since x_1 and y_1 are within the circle, we know $|r| \geq |x_1 - c_1|, |r| \geq |y_1 - c_1|$, and we already know x_2 and $y_2 > 0$, so $\frac{y_2(r + x_1 - c_1)}{x_2(r + y_1 - c_1)} \geq 0$. Thus, we may remove the absolute value.

$= \ln\frac{y_2(r + x_1 - c_1)}{x_2(r + y_1 - c_1)}$

If it moves clockwise, it comes out to $-\ln\frac{y_2(r + x_1 - c_1)}{x_2(r + y_1 - c_1)} = \ln\frac{x_2(r + y_1 - c_1)}{y_2(r + x_1 - c_1)}$. In general, it's $\left|\ln\frac{y_2(r + x_1 - c_1)}{x_2(r + y_1 - c_1)}\right|$.

Of course, none of this works if the two points share the same first two coordinates. In that case, the problem is easier. The direction is $(0,1)$, and the distance is $\int_{x_3}^{y_3} \frac{1}{t} dt = \ln \frac{y_3}{x_3}$. This gives a vector of $(0, \ln \frac{y_3}{x_3})$.

Finding the point a given distance in a given direction from another:

Given initial point $\mathbf{x} = (x_1, x_2)$ and vector $\mathbf{z} = (z_1, z_2)$,

Since the geodesic is a circle, the center of the circle is on a line perpendicular to the tangent vector.

The tangent line is $\mathbf{x} + t\mathbf{z}$, so the center of the circle is on $\mathbf{x} + t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{z} = (x_1 - tz_2, x_2 + tz_1)$.

This intersects with the x axis when $x_2 + tz_1 = 0$.

$t = -\frac{x_2}{z_1}$

$c_1 = x_1 - tz_2$

$= x_1 + \frac{x_2 z_2}{z_1}$

And as before, $c_2 = 0$.

$r = \|\mathbf{x} - \mathbf{c}\|$

If were letting $x_1 = 0$, that's just $r^2 = c_1^2 + {x_2}^2$.

Now that we have the circle, its just a matter of finding the point at the right distance.

Let $d = \ln \frac{y_2(r+x_1-c_1)}{x_2(r+y_1-c_1)}$

$e^d = \frac{y_2(r+x_1-c_1)}{x_2(r+y_1-c_1)}$

$x_2(r + y_1 - c_1)e^d = y_2(r + x_1 - c_1)$

${x_2}^2(r + y_1 - c_1)^2 e^{2d} = {y_2}^2(r + x_1 - c_1)^2$

$= [r^2 - (y_1 - c_1)^2](r + x_1 - c_1)^2$

$= [r - (y_1 - c_1)][r + (y_1 - c_1)](r + x_1 - c_1)^2$

$= (r - y_1 + c_1)(r + y_1 - c_1)(r + x_1 - c_1)^2$

Since $y_0 > 0$ and $r^2 = |\mathbf{y} - \mathbf{c}|$, it must be the case that $|y_1 - c_1| < r$ so $r + y_1 - c_1 \neq 0$ and can be safely canceled out.

${x_2}^2(r + y_1 - c_1)e^{2d} = (r - y_1 + c_1)(r + x_1 - c_1)^2$

$y_1[{x_2}^2 e^{2d} + (r + x_1 - c_1)^2] = (r + c_1)(r + x_1 - c_1)^2 - {x_2}^2(r - c_1)e^{2d}$

$y_1 = \frac{(r+c_1)(r+x_1-c_1)^2 - {x_2}^2(r-c_1)e^{2d}}{{x_2}^2 e^{2d} + (r+x_1-c_1)^2}$

Now that we know y_1, we can easily find y_2 with $y_2 = \sqrt{r^2 - (y_1 - c_1)^2}$.

We will also need to find the change in orientation.

Let θ_0 be the initial angle and θ_1 be the final angle.

$\sin \theta_0 = \frac{x_2 - c_2}{r}$

$= \frac{x_2}{r}$

$\cos \theta_0 = \frac{x_1 - c_1}{r}$

$= -\frac{c_1}{r}$

Similarly, $\sin \theta_1 = \frac{y_2}{r}, \cos \theta_1 = \frac{y_1 - c_1}{r}$.

We can simply take $\theta_1 - \theta_0$ as the angle, but this will require trigonometry to get the angle, and then trigonometry later to do the rotation with it. It may be more optimal to just calculate $\sin\Delta\theta$ and $\cos\Delta\theta$ using angle sums.

$\sin(\theta_1 - \theta_0) = \sin\theta_1 \cos\theta_0 - \cos\theta_1 \sin\theta_1$

$\cos(\theta_1 - \theta_0) = \cos\theta_0 \cos\theta_1 - \sin\theta_0 \sin\theta_1$

If $z_0 = 0$ so the vector is pointing straight up or down, we have:

$z_2 = \ln \frac{y_2}{x_2}$

$e^{z_2} = \frac{y_2}{x_2}$

$y_2 = x_2 e^{z_2}$

Clearly, y_1 remains constant and there is no rotation.