Consider a point x on a 3-surface U of revolution made by rotation a curve along a sphere, and a vector v in the tangent space U_x.

Consider the component v_0 of v along S^2.

Consider the great circle made by extending x to a geodesic in the direction of v_0.

We can reflect across this great circle on all the spherical cross-sections of U.

By symmetry, the geodesic made by extending x in the v direction must stay on this slice of U.

This reduces finding the geodesic on a 3-surface of revolution to finding one on a 2-surface.

Consider the quotient space in $\mathbb{H}^2$ made by identifying two ultraparallel lines so that the intersections with their mutual perpendiculars are identified, and they both go in the same direction.

Then a miracle happend.

The resulting space is a surface of revolution.

We can extend this to a 3-surface of revolution, then use the above method to reduce finding a geodesic to finding one on this space.

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$y_0 = a\sin(x_0 - \theta_0), y_1 = a\sin(x_1 - \theta_0)$

$\frac{y_1}{y_0} = \frac{\sin(x_1-\theta_0)}{\sin(x_0-\theta_0)}$

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Rather than mapping a sphere to $\mathbb{R}^2$, we can embed it in $\mathbb{R}^3$.

Given points $\mathbf{x}, \mathbf{y}$,

First, we take the first thee coordinates, which gives us points in S^2.

We can easily find the great circle between them.

We set up a geodesic along $\mathbb{H}^2$, and place $\mathbf{x}$ at the distance given by x_4 from the bottleneck, where the spherical cross-section is thinnest.

We then move along the spherical cross-section a distance equal to the distance the first three coordinates of $\mathbf{x}$ are along the great circle. We do this simply by scaling $e^{k\theta}$, where θ is the angle along the great circle, and k is some constant that depends on the specific surface we're using.

We do a similar process for $\mathbf{y}$.

Now we can find the geodesic between them along this $\mathbb{H}^2$ slice.

We can then find the vector representing the distance and direction from $\mathbf{x}$ to $\mathbf{y}$.

Now we simply transform this back to a point on the full space.

(v_1, v_2, v_3) is the component of $\mathbf{v}$ along the S^2 slice, which is just the the component along the line perpendicular to the circle on the euclidean map used to represent the hyperbolic slice. Just transform this back onto S^2. Don't worry about the radius of S^2. We didn't actually use it to calculate distance, so it doesn't matter.

v_4 is the component of $\mathbf{v}$ orthogonal to the S^2 slice, which is along the cricle on the euclidean map of the hyperbolic slice.

Now to find the geodesic from the vector.

First, we look at this on the $\mathbb{H}^2$ slice.

We move x onto $\mathbb{H}^2$ by letting the angle on the geodesic represented by a circle about the origin be x_4. The radius doesn't matter for this, since it doesn't matter where we start the section of the quotient space we look at. Thus, we can map it to $\mathbf{x}' = (\sin x_4, \cos x_4)$.

v_4 gets mapped perpendicular to the direction of $\mathbf{x}'$ from the origin on the Euclidean plane. The rest of $\mathbf{v}$ gets pointed along it.

Now that we have a vector in $\mathbb{H}^2$, we can construct the geosic as when we did it in a slice of $\mathbb{H}^3$.

Once we have $\mathbf{y}'$, we let $y_4 = \arctan \frac{y_1'}{y_0'}$.

$e^{k\theta} = \|\mathbf{y}'\|$

$\theta = \frac{1}{k} \ln \|\mathbf{y}'\|$

$= \frac{1}{2k} \ln \|\mathbf{y}'\|^2$

$(y_1, y_2, y_3) = (x_1, x_2, x_3) \cos\theta + (v_1, v_2, v_3) \sin\theta$

Finding the rotation is more difficult.

First, consider the two-dimensional rotation.

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$\sin\theta_0 = \frac{\mathbf{x}' \times (\mathbf{x}' - \mathbf{c})}{|\mathbf{x}'||\mathbf{x}' - \mathbf{c}|}$

$= \frac{x_0'(x_1' - c_1) - x_1'(x_0' - c_0)}{\sqrt{|\mathbf{x}'|^2 |\mathbf{x}' - \mathbf{c}|^2}}$

$= \frac{x_0' x_1' - x_1'(x_0' - c_0)}{\sqrt{|\mathbf{x}'|^2 |\mathbf{x}' - \mathbf{c}|^2}}$

$= \frac{c_0 x_0'}{\sqrt{|\mathbf{x}'|^2 |\mathbf{x}' - \mathbf{c}|^2}}$

$\cos\theta_0 = \frac{\langle \mathbf{x}', \mathbf{x}' - \mathbf{c} \rangle}{|\mathbf{x}'||\mathbf{x}' - \mathbf{c}|}$

$\sin\theta_1 = \frac{\mathbf{y}' \times (\mathbf{y}' - \mathbf{c})}{|\mathbf{y}'||\mathbf{y}' - \mathbf{c}|}$

$= \frac{c_0 y_0'}{\sqrt{|\mathbf{y}'|^2 |\mathbf{y}' - \mathbf{c}|^2}}$

$\cos\theta_1 = \frac{\langle \mathbf{y}', \mathbf{y}' - \mathbf{c} \rangle}{|\mathbf{y}'||\mathbf{y}' - \mathbf{c}|}$

$\begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \cos\theta_0 & \sin\theta_0 \\ -\sin\theta_0 & \cos\theta_0 \end{pmatrix}$

Let $\mathbf{z} =$ the normalization of the component of $\mathbf{y}$ perpendicular to $\mathbf{x}$.

I'm going to need to commute this with a matrix that moves $\mathbf{z}$ to $(0, 1, 0, 0)$. I'm going to make the first digit be the distance along the arc in $\mathbb{H}^2$.

Let's just let it move $\mathbf{x}$ to $(0, 0, 1, 0)$ and $\mathbf{x} \times \mathbf{z}$ to $(0, 0, 0, 1)$. It will preserve $(1, 0, 0, 0)$.

This makes the matrix

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$\mathbf{e}_1 \mapsto \mathbf{e}_1, \mathbf{z} \mapsto \mathbf{e}_2, \mathbf{x} \mapsto \mathbf{e}_3, \mathbf{x} \times \mathbf{z} \mapsto \mathbf{e}_4$
$(\mathbf{e}_1, \mathbf{z}, \mathbf{x}, (\mathbf{x} \times \mathbf{y}'))^{-1}$
$= (\mathbf{e}_1, \mathbf{z}, \mathbf{x}, (\mathbf{x} \times \mathbf{y}'))^T$, since it's a rotation matrix.
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$(\mathbf{e}_1, \mathbf{z}, \mathbf{x}, (\mathbf{x} \times \mathbf{z}))^T$

$$(\mathbf{e}_1, \mathbf{z}, \mathbf{x}, (\mathbf{x} \times \mathbf{z})) \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_0 & \sin\theta_0 & 0 & 0 \\ -\sin\theta_0 & \cos\theta_0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} (\mathbf{e}_1, \mathbf{z}, \mathbf{x}, (\mathbf{x} \times \mathbf{z}))^T$$

This still fails to take into account the rotation around the sphere.

It just rotates between $\mathbf{x}$ and $\mathbf{z}$, so if we do it mid-conjugation, we can just do it with a 2×2 matrix like so:

$$(\mathbf{e}_1, \mathbf{z}, \mathbf{x}, (\mathbf{x} \times \mathbf{z})) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} \cos\theta_0 & \sin\theta_0 & 0 & 0 \\ -\sin\theta_0 & \cos\theta_0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} (\mathbf{e}_1, \mathbf{z}, \mathbf{x}, (\mathbf{x} \times \mathbf{z}))^T$$

Where $\theta = \frac{1}{2k} \ln \|\mathbf{y}'\|^2$