Consider a point x on a 3-surface U that is a surface of revolution and a vector v in the tangent space U_x.

Consider the component v_0 of v along S^2.

Consider the great circle made by extending x to a geodesic in the direction of v_0.

We can reflect across this great circle on all the spherical cross-sections of U.

By symmetry, the geodesic made by extending x in the v direction must stay on this slice of U.

This reduces finding the geodesic on a 3-surface of revolution to finding one on a 2-surface.

Finding the vectors from a point:

Rather than mapping a sphere to $\mathbb{R}^2$, we can embed it in $\mathbb{R}^3$.

Given points $\mathbf{x}, \mathbf{y} \in S^2 \times \mathbb{R}$,

Let $\mathbf{v} =$ the normalization of the projection of $\mathbf{y}_1$ perpendicular to $\mathbf{x}_1$.

Let $\theta =$ the angle between $\mathbf{x}$ and $\mathbf{y} = \arccos \langle \mathbf{x}, \mathbf{y} \rangle$

Now we take the points $\mathbf{x}' = (0, x_4)$ and $\mathbf{y}' = (\theta, y_4)$ in the two-dimensional version.

Let $\mathbf{z}'$ be a vector between them.

Let the S^2 component of $\mathbf{z}$ be $z'_0\mathbf{v}$ and the $\mathbf{R}$ component be z'_1.

$\mathbf{z}$ is a vector between the two points.

Finding a point from a vector:

Given point $\mathbf{x}$ and vector $\mathbf{z}$,

Let $\mathbf{x}' = (0, x_2), \mathbf{z}' = (\|\mathbf{z}_1\|, z_2)$.

Let $\mathbf{v} = \|\mathbf{z}_1\|$.

Find $\mathbf{y}'$ with the two-dimensional version.

$\mathbf{y} = (\mathbf{x}_1 \cos y'_1 + \mathbf{v} \sin y'_1, y'_2)$.

???

In order to create the matrices, we must work in a more relevant basis. We can do this by commuting with the appropriate matrix.

$\mathbf{e}_1 \mapsto \mathbf{e}_1, \mathbf{v} \mapsto \mathbf{e}_2, \mathbf{x}_1 \mapsto \mathbf{e}_3, \mathbf{x}_1 \times \mathbf{v} \mapsto \mathbf{e}_4$

$(\mathbf{e}_1, \mathbf{v}, \mathbf{x}_1, (\mathbf{x}_1 \times \mathbf{v}))^{-1}$

$= (\mathbf{e}_1, \mathbf{v}, \mathbf{x}_1, (\mathbf{x}_1 \times \mathbf{v}))^T$, since it's a rotation matrix.

First, we use the rotation of the two-dimensional version to find the rotation between $\mathbf{e}_1$ and $\mathbf{v}$.

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Next, we have to deal with the fact that the S^2 component rotates.

Let $\phi = y'_2 - x'_2$.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Putting this all together, we get:

$$(\mathbf{e}_1, \mathbf{z}, \mathbf{x}, (\mathbf{x}\times\mathbf{z})) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} (\mathbf{e}_1, \mathbf{z}, \mathbf{x}, (\mathbf{x}\times\mathbf{z}))^T$$

???
$\sin\theta_0 = \frac{\mathbf{x}'\times(\mathbf{x}'-\mathbf{c})}{|\mathbf{x}'||\mathbf{x}'-\mathbf{c}|}$
$= \frac{x_0'(x_1'-c_1)-x_1'(x_0'-c_0)}{\sqrt{|\mathbf{x}'|^2|\mathbf{x}'-\mathbf{c}|^2}}$
$= \frac{x_0'x_1'-x_1'(x_0'-c_0)}{\sqrt{|\mathbf{x}'|^2|\mathbf{x}'-\mathbf{c}|^2}}$
$= \frac{c_0 x_0'}{\sqrt{|\mathbf{x}'|^2|\mathbf{x}'-\mathbf{c}|^2}}$
$\cos\theta_0 = \frac{\langle \mathbf{x}', \mathbf{x}'-\mathbf{c}\rangle}{|\mathbf{x}'||\mathbf{x}'-\mathbf{c}|}$
$\sin\theta_1 = \frac{\mathbf{y}'\times(\mathbf{y}'-\mathbf{c})}{|\mathbf{y}'||\mathbf{y}'-\mathbf{c}|}$
$= \frac{c_0 y_0'}{\sqrt{|\mathbf{y}'|^2|\mathbf{y}'-\mathbf{c}|^2}}$
$\cos\theta_1 = \frac{\langle \mathbf{y}', \mathbf{y}'-\mathbf{c}\rangle}{|\mathbf{y}'||\mathbf{y}'-\mathbf{c}|}$
$\begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \cos\theta_0 & \sin\theta_0 \\ -\sin\theta_0 & \cos\theta_0 \end{pmatrix}$

Let $\mathbf{z}$ = the normalization of the component of $\mathbf{y}$ perpendicular to $\mathbf{x}$.

I'm going to need to commute this with a matrix that moves $\mathbf{z}$ to $(0,1,0,0)$. I'm going to make the first digit be the distance along the arc in $\mathbb{H}^2$.

Let's just let it move $\mathbf{x}$ to $(0,0,1,0)$ and $\mathbf{x}\times\mathbf{z}$ to $(0,0,0,1)$. It will preserve $(1,0,0,0)$.

This makes the matrix
???
$\mathbf{e}_1 \mapsto \mathbf{e}_1, \mathbf{z} \mapsto \mathbf{e}_2, \mathbf{x} \mapsto \mathbf{e}_3, \mathbf{x}\times\mathbf{z} \mapsto \mathbf{e}_4$
$(\mathbf{e}_1, \mathbf{z}, \mathbf{x}, (\mathbf{x}\times\mathbf{y}'))^{-1}$
$= (\mathbf{e}_1, \mathbf{z}, \mathbf{x}, (\mathbf{x}\times\mathbf{y}'))^T$, since it's a rotation matrix.
???
$(\mathbf{e}_1, \mathbf{z}, \mathbf{x}, (\mathbf{x}\times\mathbf{z}))^T$

$$(\mathbf{e}_1, \mathbf{z}, \mathbf{x}, (\mathbf{x}\times\mathbf{z}))\begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta_0 & \sin\theta_0 & 0 & 0 \\ -\sin\theta_0 & \cos\theta_0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}(\mathbf{e}_1, \mathbf{z}, \mathbf{x}, (\mathbf{x}\times\mathbf{z}))^T$$

This still fails to take into account the rotation around the sphere.

It just rotates between $\mathbf{x}$ and $\mathbf{z}$, so if we do it mid-conjugation, we can just do it with a 2×2 matrix like so:

$$(\mathbf{e}_1, \mathbf{z}, \mathbf{x}, (\mathbf{x} \times \mathbf{z}))\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} \cos\theta_0 & \sin\theta_0 & 0 & 0 \\ -\sin\theta_0 & \cos\theta_0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}(\mathbf{e}_1, \mathbf{z}, \mathbf{x}, (\mathbf{x} \times \mathbf{z}))^T$$

Where $\theta = \frac{1}{2k} \ln \|\mathbf{y}'\|^2$