

Astronomy and Astrophysics Olympiad Note

A Comprehensive Note Created by Former Hong Kong Team Members of IOAA



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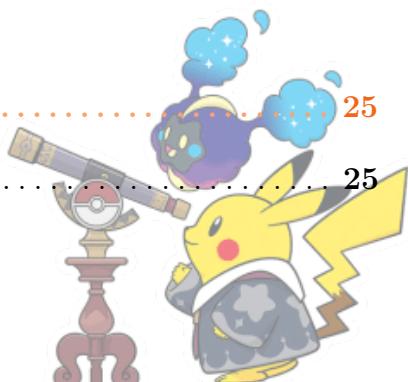
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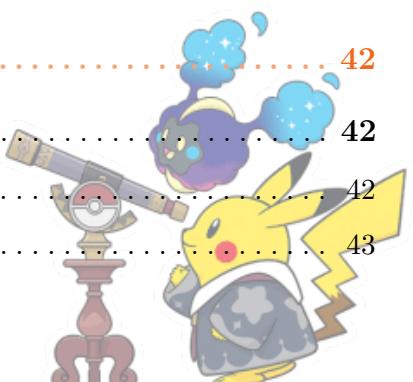
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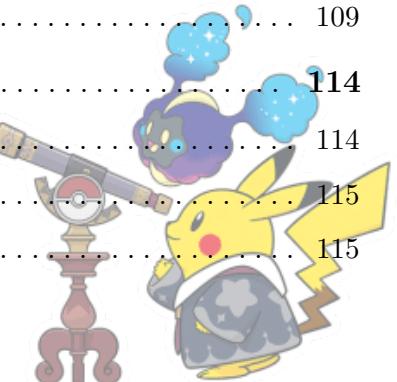
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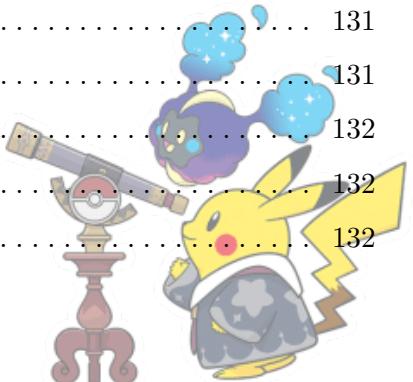
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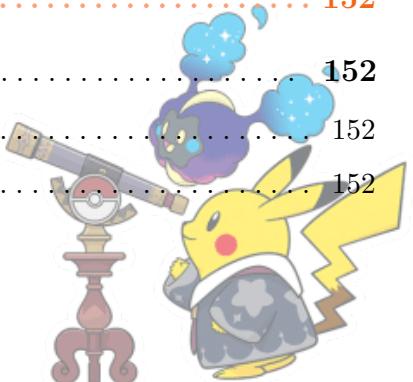
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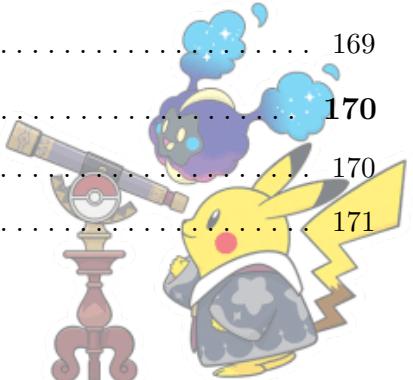
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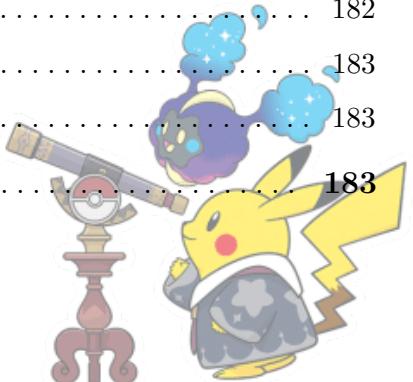
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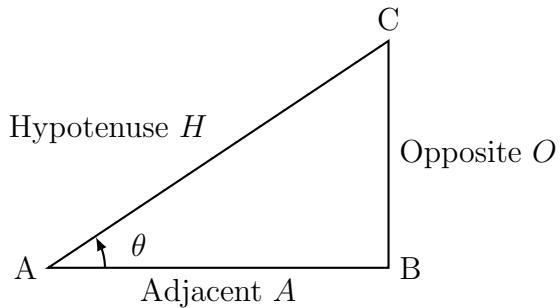


Part I: Theory

1 Mathematics

1.1 Trigonometry

Consider a right triangle with an angle θ , opposite side O , adjacent side A , and hypotenuse H . Recall that



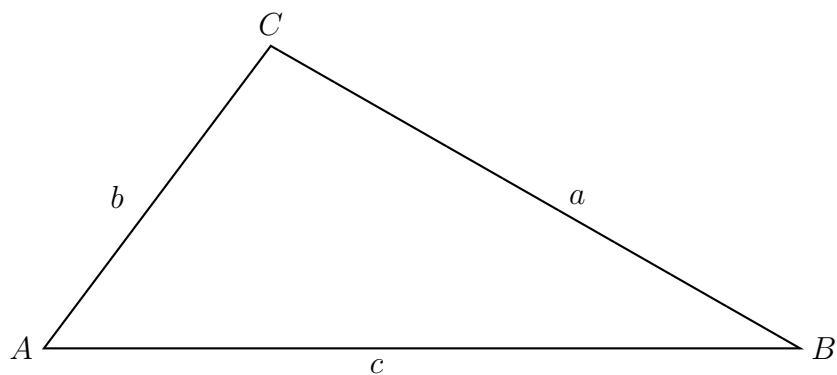
For the angle θ in a right triangle:

$$\sin \theta = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{O}{H}$$

$$\cos \theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{A}{H}$$

$$\tan \theta = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{O}{A}$$

Consider any triangle ABC with sides a , b , and c opposite angles A , B , and C , respectively:



The law of sine states

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The law of cosine states

$$c^2 = a^2 + b^2 - 2ab \cos C$$



1.2 Logarithm

Recall that

$$\log_b x = y \iff b^y = x$$

where

- (1) b is the base ($b > 0, b \neq 1$),
- (2) x is the argument ($x > 0$), and
- (3) y is the logarithm

Remark. \log_{10} can be written as \log .

Definition. 1.1: Natural Number e

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$$

Definition. 1.2: Natural Logarithm

$$y = \ln(x) \iff e^y = x$$

Theorem. 1.1: Natural Logarithm

$$\ln(xy) = \ln x + \ln y$$

$$\ln\left(\frac{x}{y}\right) = \ln x - \ln y$$

$$\ln(x^n) = n \ln x$$

1.3 Summation

The summation symbol \sum is used to denote the sum of a sequence of numbers:

$$\sum_{i=1}^n a_i = a_1 + a_2 + \cdots + a_n$$



Einstein summation notation is a compact and powerful way to write sums over indexed quantities in tensor calculus. Instead of writing explicit summation signs, repeated indices in a term imply a sum over all values of that index. If an index appears twice in a single term (once as an upper index and once as a lower index, or in the same position in Euclidean space), it is implicitly summed over all its possible values:

$$A_i B_i \equiv \sum_{i=1}^n A_i B_i$$

Example. For two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, the dot product can be written using Einstein notation as:

$$\mathbf{u} \cdot \mathbf{v} = u_i v_i = \sum_{i=1}^3 u_i v_i$$

1.4 Dimensional Analysis

Dimensional analysis is used in physics and engineering to:

- check the consistency of equations, and
- predict how different physical quantities relate to one another.

In essence, it is based on the idea that any physical quantity can be expressed in terms of a few fundamental dimensions, such as:

$[M]$ (mass), $[L]$ (length), $[T]$ (time), $[I]$ (electric current), $[\Theta]$ (temperature).

Every valid physical equation must be **dimensionally homogeneous**, meaning all additive terms must have the same dimensions.

Consider Newton's second law:

$$F = ma$$

The dimensions are:

$$[F] = [MLT^{-2}], \quad [m] = [M], \quad [a] = [LT^{-2}]$$

Since $[F] = [m][a]$, the equation is dimensionally consistent.



1.5 Vector and Vector Space

Definition. 1.3: Vector

A vector is a mathematical object that has both a magnitude (size or length) and a direction. This is in contrast to a scalar, which only has magnitude (e.g., temperature, mass).

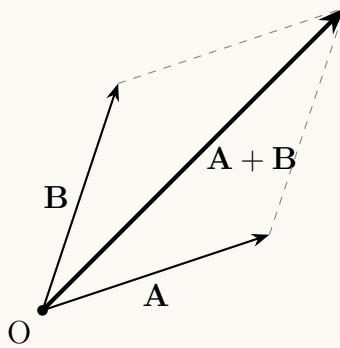
The following are some terminologies:

- A vector is denoted by an arrow above the letter (\vec{A}) or by a boldface letter (\mathbf{A}).
- The magnitude (length) of vector \mathbf{A} is written as $|\mathbf{A}|$ or A .
- A unit vector (\hat{u}) has a magnitude of 1 and defines direction. The standard Cartesian unit vectors are $\mathbf{i}, \mathbf{j}, \mathbf{k}$ along the x, y , and z axes, respectively.
- A vector \mathbf{A} in 3D space can be expressed in terms of its Cartesian components (A_x, A_y, A_z):

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}.$$

Theorem. 1.2: Tail-to-tail Method

The sum of two vectors can be visualized as the diagonal of a parallelogram formed by the two vectors placed tail-to-tail:

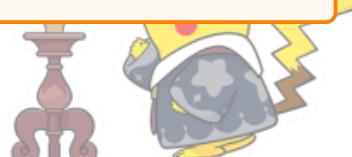


Definition. 1.4: Dot Product and Cross Product

Given that angle between two vectors is θ . If $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$, then

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos(\theta) = A_x B_x + A_y B_y + A_z B_z$$

$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\mathbf{i} - (A_x B_z - A_z B_x)\mathbf{j} + (A_x B_y - A_y B_x)\mathbf{k}$$



Definition. 1.5: Vector Space

A **vector space** V over a field \mathbb{F} (e.g., \mathbb{R} or \mathbb{C}) is a set of vectors equipped with two operations:

- **Vector addition:** $+ : V \times V \rightarrow V$
- **Scalar multiplication:** $\cdot : \mathbb{F} \times V \rightarrow V$

satisfying the following axioms for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and scalars $\alpha, \beta \in \mathbb{F}$:

1. **Associativity of addition:** $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
2. **Commutativity of addition:** $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. **Existence of zero vector:** There exists $\mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$
4. **Existence of additive inverse:** For each $\mathbf{v} \in V$, there exists $-\mathbf{v} \in V$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$
5. **Compatibility of scalar multiplication:** $\alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v}$
6. **Identity element of scalar multiplication:** $1 \cdot \mathbf{v} = \mathbf{v}$
7. **Distributivity over vector addition:** $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$
8. **Distributivity over scalar addition:** $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$

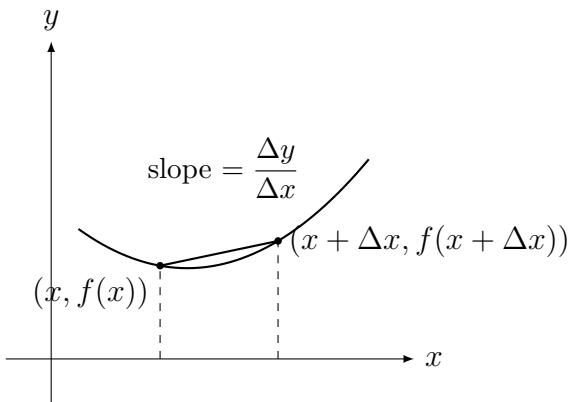
1.6 Differentiation and Partial Differentiation

1.6.1 Introduction

Differentiation is used for functions of a single independent variable, $y = f(x)$. It measures the instantaneous rate of change of y with respect to x . The derivative of a function $f(x)$ with respect to x is defined using a limit and is denoted by $\frac{df}{dx}$ or $f'(x)$.

$$\frac{df}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$



Figure 1: Differentiation is the slope when $\Delta x \rightarrow 0$ **Definition. 1.6: Basic Properties**

If $f(x)$ and $g(x)$ are differentiable functions (the derivative exists), c and n are any real numbers,

- | | |
|---|---|
| 1. $(cf)' = cf'(x)$ | 5. $\frac{d}{dx}(c) = 0$ |
| 2. $(f \pm g)' = f'(x) \pm g'(x)$ | 6. $\frac{d}{dx}(x^n) = nx^{n-1}$ – Power Rule |
| 3. $(fg)' = f'g + fg'$ – Product Rule | 7. $\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$
– Chain Rule |
| 4. $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ – Quotient Rule | |

Definition. 1.7: Formulas

$\frac{d}{dx}(x) = 1$	$\frac{d}{dx}(\csc x) = -\csc x \cot x$	$\frac{d}{dx}(a^x) = a^x \ln(a)$
$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\cot x) = -\csc^2 x$	$\frac{d}{dx}(e^x) = e^x$
$\frac{d}{dx}(\cos x) = -\sin x$	$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\ln(x)) = \frac{1}{x}, x > 0$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\ln x) = \frac{1}{x}, x \neq 0$
$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	$\frac{d}{dx}(\log_a(x)) = \frac{1}{x \ln a}, x > 0$

Partial differentiation is used for functions of **two or more independent variables**, such as $z = f(x, y)$.

A partial derivative measures the rate of change of the function with respect to **one variable**, while **holding all other variables constant**. For example, $\frac{\partial z}{\partial x}$ is the derivative with respect to x while keeping y constant.



1.6.2 Numerical Analysis

Definition. 1.8: Bisection Method

Let the function $f(x)$ be continuous on the interval $[a, b]$, and suppose that $f(a)$ and $f(b)$ have opposite signs, i.e.,

$$f(a) \cdot f(b) < 0.$$

The root r lies between a and b , and we can approximate the root by iteratively halving the interval.

The midpoint m of the interval is given by

$$m = \frac{a + b}{2}.$$

We then check the sign of $f(m)$:

- If $f(m) = 0$, then m is the root.
- If $f(a) \cdot f(m) < 0$, the root lies between a and m , so we set $b = m$.
- If $f(m) \cdot f(b) < 0$, the root lies between m and b , so we set $a = m$.

This process is repeated until the interval becomes sufficiently small.

Definition. 1.9: Newton–Raphson method

Let $f(x)$ be a differentiable function, and let x_0 be an initial guess for the root. The next approximation x_1 is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

The method continues iteratively:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

1.6.3 Taylor's Series

The Taylor Series of a function $f(x)$ about a point a is given by

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f^{(3)}(a)}{3!}(x - a)^3 + \dots$$



1.7 Integration

1.7.1 Introduction

The definite integral of a function $f(x)$ over the interval $[a, b]$ is defined as the limit of a Riemann sum as the partition of the interval becomes infinitely fine. Mathematically, it is expressed as

$$\int_a^b f(x) dx = \lim_{\|\Delta x\| \rightarrow 0} \sum_{i=1}^n f(x_i^*) \Delta x_i$$

where Δx_i is the width of each subinterval, and x_i^* is a point in the i -th subinterval.

Alternatively, the definite integral can be computed using the Fundamental Theorem of Calculus:

$$\int_a^b f(x) dx = F(b) - F(a)$$

where $F(x)$ is any antiderivative of $f(x)$, i.e., $F'(x) = f(x)$. Alternatively, we denote $F(x) = \int f(x) dx$.

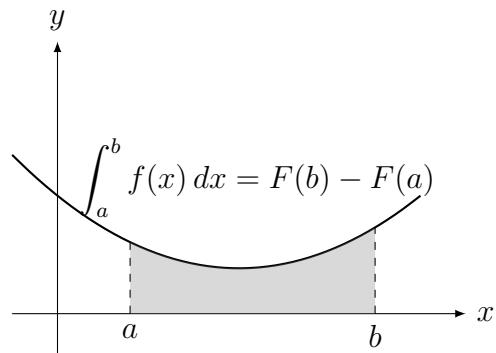


Figure 2: Area under the curve $y = f(x)$ from $x = a$ to $x = b$.

1.7.2 Formula



Definition. 1.10: Formulas (Integration of Constant Omitted)

$\int c \, dx = cx$	$\int \csc^2 x \, dx = -\cot x$	$\int \tan x \, dx = \ln \sec x $
$\int x \, dx = \frac{x^2}{2}$	$\int \sec x \tan x \, dx = \sec x$	$\int \cot x \, dx = \ln \sin x $
$\int cx^n \, dx = \frac{cx^{n+1}}{n+1}, \quad n \neq -1$	$\int \csc x \cot x \, dx = -\csc x$	$\int e^x \, dx = e^x$
$\int \sin x \, dx = -\cos x$	$\int \sec x \, dx = \ln \sec x + \tan x $	$\int \frac{1}{x} \, dx = \ln x $
$\int \cos x \, dx = \sin x$	$\int \csc x \, dx = \ln \csc x - \cot x $	$\int \ln x \, dx = x \ln x - x$
$\int \sec^2 x \, dx = \tan x$		

1.7.3 Integration by Substitution

Suppose we have an integral of the form

$$\int f(g(x))g'(x) \, dx$$

1. Identify an inner function $g(x)$ whose derivative $g'(x)$ appears in the integral.
2. Make the substitution:

$$u = g(x) \implies du = g'(x) \, dx$$

3. Rewrite the integral in terms of u :

$$\int f(g(x))g'(x) \, dx = \int f(u) \, du$$

4. Integrate with respect to u .
5. Substitute back $u = g(x)$ to obtain the result in terms of x .

1.7.4 Integration by Parts

Integration by parts is based on the product rule for differentiation:

$$\frac{d}{dx}[u(x)v(x)] = u'(x)v(x) + u(x)v'(x)$$



Rewriting and integrating both sides gives the formula for integration by parts:

$$\int u(x)v'(x) dx = u(x)v(x) - \int v(x)u'(x) dx$$

where $u(x)$ is chosen to simplify $u'(x)$ and $v'(x)$ is easily integrable.

1.7.5 Double Integration

A double integral is an extension of a single integral to functions of two variables. For a function $f(x, y)$ defined on a region $R \subset \mathbb{R}^2$, the double integral is written as

$$\iint_R f(x, y) dA$$

where dA represents an infinitesimal area element in the xy -plane.

If R is a rectangular region $a \leq x \leq b, c \leq y \leq d$, then

$$\iint_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy.$$

1.7.6 Differential Equation

Introduction An **ordinary differential equation** (ODE) is an equation that involves a function and its derivatives. A general first-order ODE has the form

$$\frac{dy}{dx} = f(x, y)$$

Separable ODE A first-order ODE is **separable** if it can be written as

$$\frac{dy}{dx} = g(x)h(y)$$

Then we solve by separating variables:

$$\frac{1}{h(y)} dy = g(x) dx$$

and integrate both sides.



Integrating Factor A first-order linear ODE has the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

can be solved by introducing

$$\mu(x) = e^{\int P(x)dx}$$

$$y\mu(x) = \int Q(x)\mu(x)dx$$

Initial Condition An **initial condition** is a specification of the value of the solution (and possibly its derivatives) at a particular point, usually denoted t_0 or x_0 . For an n -th order ODE, we typically need n initial conditions to determine a unique solution.

1.8 Conic Section

Introduction A **conic section** is the curve obtained by intersecting a plane with a double-napped cone. Depending on the angle and position of the plane, the intersection can be a

- circle,
- ellipse,
- parabola, and
- hyperbola

The general form of the equation of a conic section in Cartesian coordinates is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Here, A , B , C , D , E , and F are constants that define the shape and orientation of the conic.

Circle

$$(x - h)^2 + (y - k)^2 = r^2$$

- Center: (h, k)
- Radius: r



Ellipse

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

- Center: (h, k)
- Semi-major axis: a , Semi-minor axis: b
- Orientation:
 - If $a > b$, major axis along x -axis
 - If $b > a$, major axis along y -axis

Parabola

$$y - k = a(x - h)^2 \quad (\text{vertical axis})$$

- Vertex: (h, k)

Hyperbola

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \quad (\text{horizontal transverse axis})$$

- Center: (h, k)
- Transverse axis along x -axis
- Conjugate axis along y -axis

1.9 Coordinates System**1.9.1 Polar and Cylindrical Coordinate Systems**

A point in the plane is represented by (r, θ) , where

r is the distance from the origin,

θ is the angle with the positive x -axis.

The conversion to Cartesian coordinates is

$$x = r \cos \theta, \quad y = r \sin \theta$$



Extending polar coordinates to 3D by adding the z -coordinate:

$$(r, \theta, z)$$

The conversion to Cartesian coordinates is

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z$$

1.9.2 Spherical Coordinate Systems

A point in 3D space is represented by (ρ, θ, ϕ) , where

ρ is the distance from the origin,

θ is the polar angle measured from the positive z -axis,

ϕ is the azimuthal angle measured from the positive x -axis in the xy -plane.

The conversion to Cartesian coordinates is

$$x = \rho \sin \theta \cos \phi, \quad y = \rho \sin \theta \sin \phi, \quad z = \rho \cos \theta$$



2 Spherical Trigonometry and Astronomical Coordinate Systems

2.1 Spherical Trigonometry

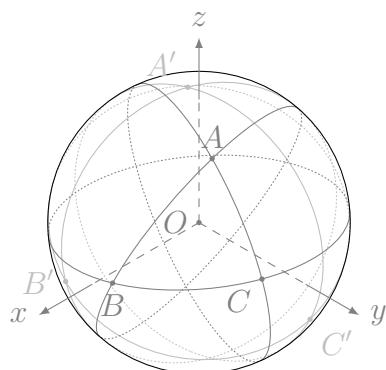
Definition. 2.1: Terminologies

On a sphere of radius R , we define

- a **great circle** is a circle on the sphere whose center is the sphere's center,
- a **spherical triangle** is formed by three arcs of great circles,
- the **sides** of a spherical triangle are arcs of great circles, and
- the **angles** of a spherical triangle are the angles between the tangents to the great circles at the vertices.

Consider a spherical triangle with vertices A , B , and C on a unit sphere (radius = 1). We denote

- Sides: $a = BC$, $b = AC$, $c = AB$ (arc lengths)
- Angles: $A = \angle BAC$, $B = \angle ABC$, $C = \angle ACB$



Definition. 2.2: Excess Angle

The sum of angles in a spherical triangle exceeds π radians (180°). The difference is called the spherical excess E

$$E = A + B + C - \pi$$

Theorem. 2.1: Area of the Spherical Triangle

The area of the spherical triangle on a sphere of radius R is

$$\text{Area} = R^2 \cdot E$$

Theorem. 2.2: Law of Cosine

For a spherical triangle with sides a, b, c and opposite angles A, B, C ,

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

$$\cos b = \cos c \cos a + \sin c \sin a \cos B$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

For the same triangle,

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$

$$\cos B = -\cos C \cos A + \sin C \sin A \cos b$$

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c$$

2.2 Astronomical Coordinate Systems

2.2.1 Horizontal (Alt-Az) Coordinate System

The horizontal coordinate system is based on the observer's location on Earth.

- **Altitude (Alt):** The angle above the horizon (0° at horizon to 90° at zenith).
- **Azimuth (Az):** The angle measured clockwise from the North (0° North, 90° East, 180° South, 270° West).



2.2.2 Equatorial Coordinate System

This system is fixed relative to the stars and independent of the observer's location.

- **Right Ascension (RA):** Analogous to longitude, measured eastward along the celestial equator from the vernal equinox. Usually expressed in hours, minutes, and seconds.
- **Declination (Dec):** Analogous to latitude, measured north or south of the celestial equator (-90° to $+90^\circ$).

2.2.3 Ecliptic Coordinate System

Based on the plane of the Earth's orbit around the Sun.

- **Ecliptic Longitude (λ):** Measured along the ecliptic from the vernal equinox.
- **Ecliptic Latitude (β):** Measured perpendicular to the ecliptic plane.

This system is particularly useful for describing planetary positions and the motion of the Sun, Moon, and planets.

2.2.4 Galactic Coordinate System

Used to map objects relative to the plane of the Milky Way galaxy.

- **Galactic Longitude (l):** Measured in the plane of the Milky Way from the Galactic Center.
- **Galactic Latitude (b):** Measured perpendicular to the Galactic plane.

3 Astronomical Time

3.1 Heliocentric Julian Date (HJD)

The **Heliocentric Julian Date** is the Julian Date (JD) corrected for the light travel time between the Earth and the Sun. This correction accounts for the position of the Earth in its orbit and is essential in high-precision astronomical observations, such as timing of variable stars or exoplanet transits.

$$\text{HJD} = \text{JD} + \frac{\mathbf{r} \cdot \hat{\mathbf{s}}}{c}$$

where



- \mathbf{r} is the vector from the Sun to the Earth,
- $\hat{\mathbf{s}}$ is the unit vector pointing toward the observed object,
- c is the speed of light.

3.2 Time Zones

Time zones are regions of the Earth that share the same standard time. Local civil time is defined relative to the **Greenwich Mean Time (GMT)** or **Coordinated Universal Time (UTC)**:

$$\text{Local Time} = \text{UTC} + \text{Time Zone Offset}$$

3.3 Universal Time (UT)

Universal Time is a time standard based on the rotation of the Earth relative to distant celestial objects. The most commonly used form is **UT1**, which is directly related to the Earth's rotation angle.

3.4 Local Mean Time (LMT)

Local Mean Time is the mean solar time at a particular longitude. It is the local time that would be measured by a clock set to the average solar day rather than the apparent solar day:

$$\text{LMT} = \text{UT} + \frac{\lambda}{15^\circ} \text{ hours}$$

where λ is the longitude east of the Greenwich meridian (positive east, negative west).

3.5 Different Definitions of a Year

There are several astronomical definitions of a year:

- **Sidereal Year:** Time taken for the Earth to complete one orbit relative to the fixed stars. ≈ 365.25636 days.
- **Tropical Year:** Time between successive vernal equinoxes. ≈ 365.24219 days. Used in calendars to keep seasons aligned.
- **Anomalistic Year:** Time between successive perihelion passages. ≈ 365.25964 days.



- **Julian Year:** Exactly 365.25 days. Used in astronomical calculations for simplicity.

3.6 Equation of Time

The **Equation of Time** quantifies the difference between apparent solar time (measured by the Sun's actual position in the sky) and mean solar time (based on a uniform 24-hour day):

$$E(t) = \text{Apparent Solar Time} - \text{Mean Solar Time}$$

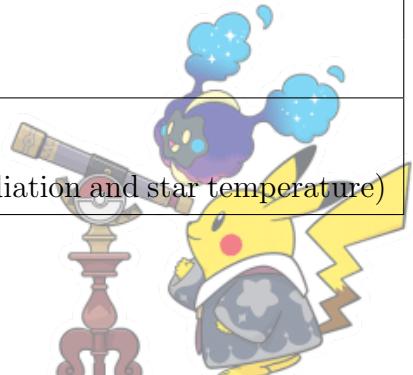
This difference arises due to

- The **ellipticity of Earth's orbit** (Kepler's second law),
- The **tilt of Earth's axis** relative to the orbital plane (obliquity).



4 Measurement in Astrophysics

Quantity	Value / Definition
Distance Units	
Astronomical Unit (AU)	$1 \text{ AU} = 1.496 \times 10^{11} \text{ m}$ (Mean distance between Earth and the Sun)
Parsec (pc)	$1 \text{ pc} = 3.086 \times 10^{16} \text{ m} \approx 3.26 \text{ ly}$ (Distance at which 1 AU subtends 1 arcsecond)
Light-year (ly)	$1 \text{ ly} = 9.461 \times 10^{15} \text{ m}$
Kiloparsec (kpc)	$1 \text{ kpc} = 10^3 \text{ pc}$
Megaparsec (Mpc)	$1 \text{ Mpc} = 10^6 \text{ pc}$
Mass Units	
Solar Mass	$1 M_{\odot} = 1.989 \times 10^{30} \text{ kg}$
Earth Mass	$1 M_{\oplus} = 5.97 \times 10^{24} \text{ kg}$
Radius Units	
Solar Radius	$1 R_{\odot} = 6.96 \times 10^8 \text{ m}$
Earth Radius	$1 R_{\oplus} = 6.37 \times 10^6 \text{ m}$
Time Units	
Julian year	$1 \text{ yr} = 365.25 \text{ days}$
Angle Units	
Degree	$1^\circ = 60' = 3600''$
Radian	$2\pi \text{ rad} = 360^\circ$
Energy Units	
Electron volt	$1 \text{ eV} = 1.602 \times 10^{-19} \text{ J}$
Erg	$1 \text{ erg} = 10^{-7} \text{ J}$
Luminosity and Flux	
Solar luminosity	$1 L_{\odot} = 3.828 \times 10^{26} \text{ W}$
Flux	W m^{-2} or $\text{erg s}^{-1} \text{ cm}^{-2}$
Spectral Units	
Wavelength	$1 \text{ \AA} = 10^{-10} \text{ m}$
Frequency	$\text{Hz} = \text{s}^{-1}$
Temperature	
Kelvin	$K = {}^{\circ}C + 273.15$ (Standard unit for blackbody radiation and star temperature)

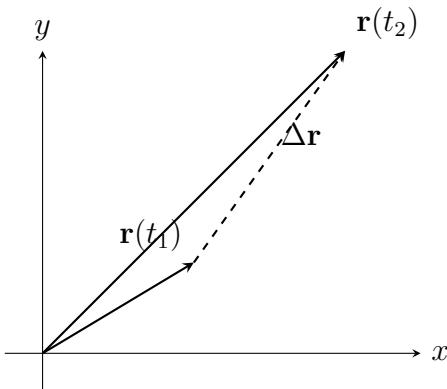


5 Fundamental Mechanics

5.1 Kinematics

5.1.1 Linear Motion

We can use the **position vector** $\mathbf{r}(t)$ to specify the position of a particle in a plane at time t .



The figure above shows the position of a particle at two different times, t_1 and t_2 . The **displacement vector** is given by

$$\Delta \mathbf{r} = \mathbf{r}(t_2) - \mathbf{r}(t_1)$$

The **average velocity** during the time interval $\Delta t = t_2 - t_1$ is:

$$\mathbf{v}_{\text{av}} = \frac{\Delta \mathbf{r}}{\Delta t}$$

The **instantaneous velocity** at time t_0 is obtained by taking the limit as $\Delta t \rightarrow 0$

$$\mathbf{v} = \frac{d \mathbf{r}}{dt}$$

The **instantaneous speed** is the magnitude of the velocity vector

$$v = |\mathbf{v}|$$

The **average acceleration** is the rate of change of velocity

$$\mathbf{a}_{\text{av}} = \frac{\Delta \mathbf{v}}{\Delta t}$$



and the **instantaneous acceleration** is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$$

Suppose a particle moves with a **constant acceleration** \mathbf{a} . At time t_0 , let its velocity and position be \mathbf{v}_0 and \mathbf{x}_0 , respectively. Then, the motion can be described by:

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{v}_0(t - t_0) + \frac{1}{2}\mathbf{a}(t - t_0)^2$$

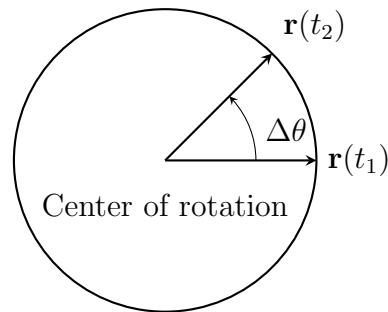
$$\mathbf{v} = \mathbf{v}_0 + \mathbf{a}(t - t_0)$$

$$|\mathbf{v}|^2 - |\mathbf{v}_0|^2 = 2 \mathbf{a} \cdot (\mathbf{x} - \mathbf{x}_0)$$

The time derivative of a vector provides information about the rate of change of the vector's magnitude and direction.

5.1.2 Circular Motion

Introduction Consider an object (for example, a dog on a rope) rotating about a fixed point with a radius r .



- **Angular displacement:** $\Delta\theta$ (in radians)

- **Angular velocity:** $\omega = \frac{d\theta}{dt}$

- **Angular acceleration:** $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$

The rotational kinematic equations (analogous to linear motion) are

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\omega = \omega_0 + \alpha t$$



$$\omega^2 - \omega_0^2 = 2\alpha(\theta - \theta_0)$$

For an object moving in a circle of radius r ,

$$v = r\omega$$

is the **tangential (linear) speed**.

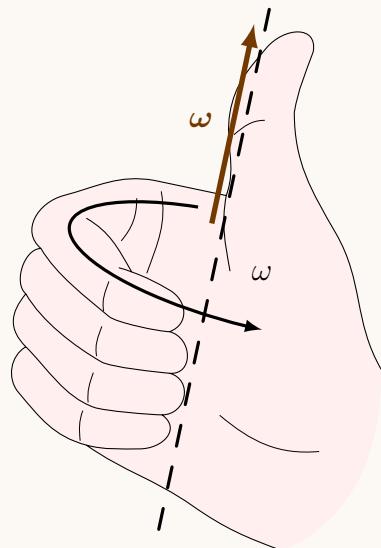
The **tangential acceleration** is

$$a_{\tan} = r\alpha$$

and the **radial (centripetal) acceleration** is

$$a_{\text{rad}} = \frac{v^2}{r} = r\omega^2$$

Theorem. 5.1: Right-hand Rule



Retrograde Motion Retrograde motion is the apparent backward motion of a planet relative to the background stars, caused by the relative orbital motion of Earth and the planet. Earth moves faster in its inner orbit. Earth moves faster in its inner orbit.



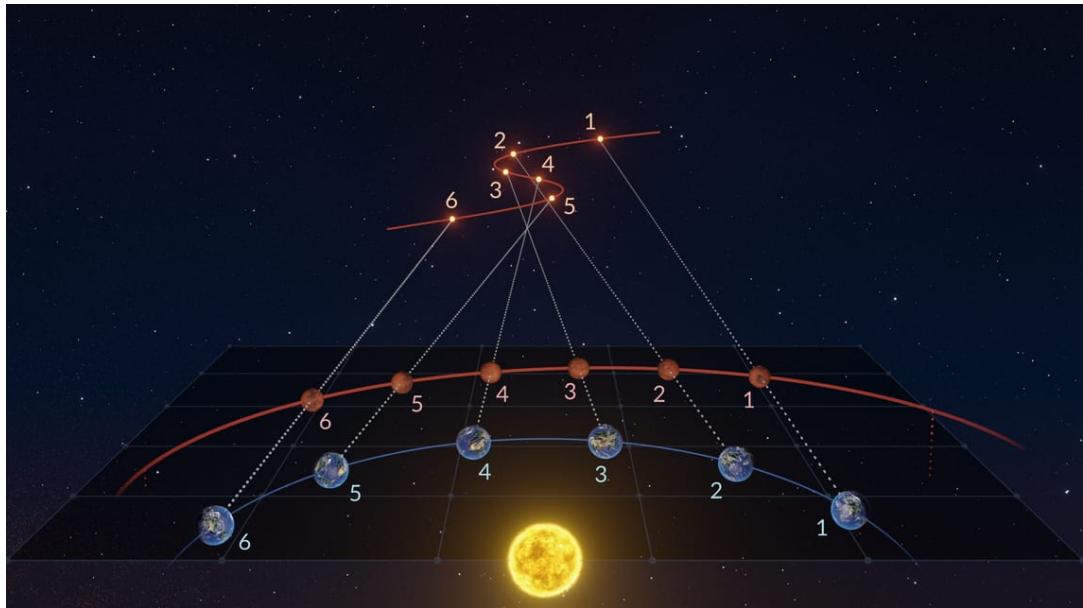


Figure 3: Source: <https://starwalk.space/en/news/what-is-retrograde-motion>

Example. Given:

$$T_E = 365.25, \text{ days}$$

$$T_{Mars} = 686.98, \text{ days}$$

Angular speeds:

$$\omega_E = \frac{360^\circ}{365.25} = 0.9856^\circ/\text{day}$$

$$\omega_{Mars} = \frac{360^\circ}{686.98} = 0.524^\circ/\text{day}$$

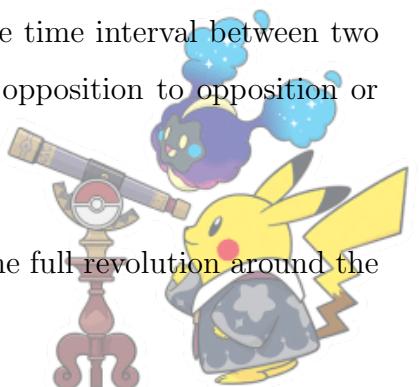
Relative angular speed:

$$\omega_{rel} = \omega_E - \omega_{Mars} = 0.4616^\circ/\text{day}$$

This relative angular motion causes Mars to appear to move backward against the stars during opposition. The duration between successive retrograde motions of a planet is equal to its **synodic period**.

Synodic Period and Sidereal Period The **synodic period** (T_{syn}) is the time interval between two successive identical configurations of a planet as observed from Earth (e.g., opposition to opposition or conjunction to conjunction).

The **sidereal period** (T_s) of a celestial body is the time taken to complete one full revolution around the



Sun with respect to the fixed background stars.

Example. Let

- T_E = sidereal period of Earth
- T_P = sidereal period of the planet

For **inferior planets** (Mercury, Venus):

$$\frac{1}{T_{syn}} = \frac{1}{T_P} - \frac{1}{T_E}$$

For **superior planets** (Mars, Jupiter, Saturn):

$$\frac{1}{T_{syn}} = \frac{1}{T_E} - \frac{1}{T_P}$$

5.2 Newton's Law

First Law (Law of Inertia) An object remains at rest, or moves in a straight line at constant velocity, unless acted upon by an external force.

$$\sum \mathbf{F} = 0 \implies \mathbf{a} = 0$$

Second Law The net force on a particle is equal to the time rate of change of its momentum:

$$\sum \mathbf{F} = \frac{d\mathbf{p}}{dt}$$

For constant mass,

$$\sum \mathbf{F} = m\mathbf{a}$$

This is the most commonly used form of Newton's Second Law.

Third Law For every action, there is an equal and opposite reaction:

$$\mathbf{F}_{AB} = -\mathbf{F}_{BA}$$



5.3 Force and Equilibrium of Body

The force is any interaction that causes a change in an object's motion. It is a vector quantity, having both magnitude and direction. In vector notation:

$$\mathbf{F} = (F_x, F_y, F_z)$$

SI Unit: Newton (N) which is defined as

$$1 \text{ N} = 1 \text{ kg m s}^{-1}$$

Example of Forces

- Gravitational force: $\mathbf{F}_g = mg$
- Normal force: the perpendicular reaction from a surface
- Friction: $\mathbf{f} = \mu N$ (opposes motion)
- Tension: force transmitted through a rope

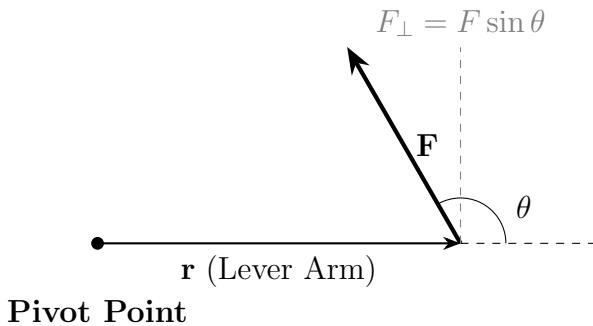
The torque τ produced by a force \mathbf{F} acting at a distance \mathbf{r} from the pivot is the cross product:

$$\tau = \mathbf{r} \times \mathbf{F}$$

The magnitude is given by

$$\tau = rF \sin(\theta)$$

SI Unit: Newton-meter ($\text{N} \cdot \text{m}$)

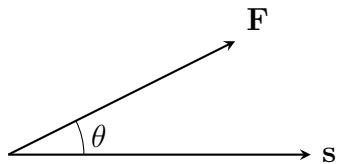


5.4 Energy

When a constant force \mathbf{F} acts on an object and causes a displacement \mathbf{s} , the work done by the force is defined as the dot product

$$W = \mathbf{F} \cdot \mathbf{s} = Fs \cos \theta$$

where θ is the angle between the force and the displacement.



If the force is not constant, then

$$W = \int \mathbf{F} \cdot d\mathbf{s}$$

Remarks.

- Work is a **scalar quantity**.
- $W > 0$ if force and displacement are in the same direction.
- $W < 0$ if force opposes motion (e.g. friction).
- $W = 0$ if $\mathbf{F} \perp \mathbf{s}$ (no work done).

For an object of mass m moving vertically through a height h :

$$W_g = -mgh$$

if the object is lifted upward, and

$$W_g = +mgh$$

if it moves downward. The **kinetic energy** of a body of mass m moving with speed v is

$$K = \frac{1}{2}mv^2$$



5.5 Centripetal Force

Centripetal force is the force that acts on an object moving in a circular path. It is directed towards the center of the circle or axis of rotation. The magnitude of the centripetal force is given by

$$F_c = \frac{mv^2}{r}$$

where

- m is the mass of the object,
- v is the velocity of the object,
- r is the radius of the circular path.

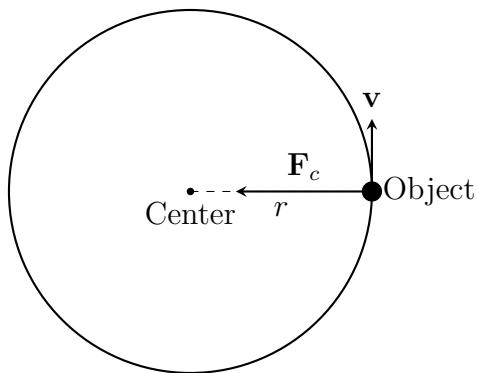


Figure 4: Centripetal force always points toward the center of the circle.

Centrifugal force is a fictitious or pseudo force that appears to act on an object when viewed from a rotating reference frame. It appears to push the object away from the center of rotation. It is equal in magnitude and opposite in direction to the centripetal force in the rotating frame of reference. The magnitude of the centrifugal force is

$$F_{\text{cf}} = \frac{mv^2}{r}$$

The Coriolis force is an apparent force that acts on a mass moving in a rotating system, such as the Earth. It arises due to the rotation of the reference frame. The Coriolis force is given by

$$F_{\text{Coriolis}} = 2m\mathbf{v} \times \boldsymbol{\omega}$$

where

- m is the mass of the object,



- \mathbf{v} is the velocity of the object relative to the rotating frame,
- $\boldsymbol{\omega}$ is the angular velocity vector of the rotating reference frame.

5.6 Angular Momentum

For a particle at position \mathbf{r} (relative to the origin) with linear momentum \mathbf{p} :

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

For a rigid body rotating around a fixed axis with moment of inertia I and angular velocity ω :

$$\mathbf{L} = I\boldsymbol{\omega}$$

Just as force changes linear momentum, torque (τ) changes angular momentum:

$$\tau_{net} = \frac{d\mathbf{L}}{dt}$$

Theorem. 5.2: Conservation of Angular Momentum

If the net external torque on a system is zero ($\tau_{ext} = 0$), the total angular momentum is conserved:

$$L_i = L_f$$

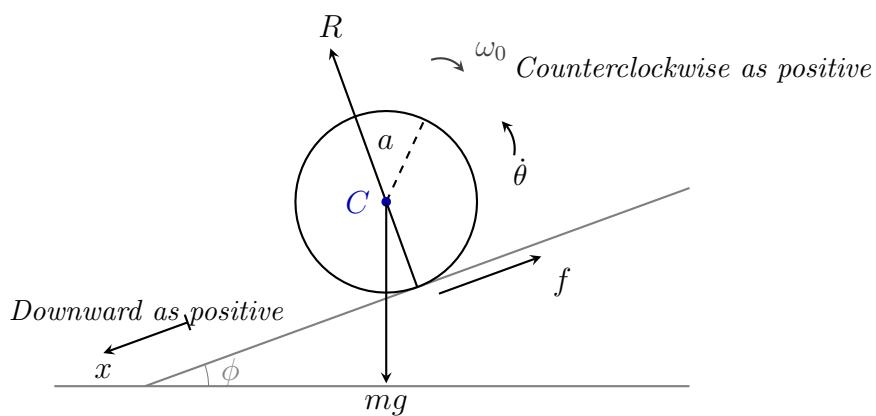


5.7 Case Study

A circular disk rotating about its axis with angular speed ω_0 is placed gently with its axis horizontal on a rough and inclined plane such that the friction acts up the plane. Given that the coefficient of friction is μ and the inclination angle of the plane to the horizontal is ϕ .

- (a) Show that the disk will move upwards if $\mu > \tan \phi$.
- (b) Find the time that elapses before rolling takes place.

Solution. (a)



Equations of Motion: Using Newton's Second Law for translation along the plane and rotation about the center of mass:

$$m\ddot{x} = mg \sin \phi - f \quad (1)$$

$$R - mg \cos \phi = 0 \implies R = mg \cos \phi \quad (2)$$

$$I\ddot{\theta} = \tau \quad (3)$$

Since the disk is slipping, kinetic friction applies:

$$f = \mu R = \mu mg \cos \phi \quad (4)$$

Substituting (4) into (1):

$$m\ddot{x} = mg \sin \phi - \mu mg \cos \phi \implies \ddot{x} = g(\sin \phi - \mu \cos \phi) < 0$$



$$g(\sin \phi - \mu \cos \phi) < 0$$

$$\sin \phi < \mu \cos \phi$$

$$\tan \phi < \mu$$

Hence, the disk will move upwards if $\boxed{\mu > \tan \phi}$.

(b) For pure rolling,

$$v_{\text{contact}} = v + a\omega = 0$$

where $v = \dot{x}$ is the linear velocity and $\omega = \dot{\theta}$ is the angular velocity. Note that

$$v(t) = v_0 + \ddot{x}t = 0 + g(\sin \phi - \mu \cos \phi)t.$$

Also,

$$\tau = -f \cdot a = -(\mu mg \cos \phi)a$$

Using $I = \frac{1}{2}ma^2$ for a disk,

$$\frac{1}{2}ma^2\ddot{\theta} = -\mu m g a \cos \phi \implies \ddot{\theta} = -\frac{2\mu g \cos \phi}{a}$$

Note that

$$\omega(t) = \omega_0 + \ddot{\theta}t = \omega_0 - \frac{2\mu g \cos \phi}{a}t$$

Substitute (5.7) and (5.7) into the rolling condition (5.7):

$$\begin{aligned} v(t) + a\omega(t) &= 0 \\ [g(\sin \phi - \mu \cos \phi)t] + a \left[\omega_0 - \frac{2\mu g \cos \phi}{a}t \right] &= 0 \\ gt(\sin \phi - \mu \cos \phi) + a\omega_0 - 2\mu gt \cos \phi &= 0 \\ gt(\sin \phi - \mu \cos \phi - 2\mu \cos \phi) &= -a\omega_0 \\ gt(\sin \phi - 3\mu \cos \phi) &= -a\omega_0 \end{aligned}$$

$$t = \boxed{\frac{-a\omega_0}{g(\sin \phi - 3\mu \cos \phi)}}$$



6 Orbital Mechanics

6.1 Newton's Law of Universal Gravitation

6.1.1 Introduction

Theorem. 6.1: Newton's Law of Universal Gravitation

The Universal Law of Gravity states that every point mass attracts every other point mass with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between their centers. This is mathematically expressed as

$$F = G \frac{m_1 m_2}{r^2}$$

where

- F is the magnitude of the gravitational force between the two masses,
- G is the gravitational constant,
- m_1 and m_2 are the two masses,
- r is the distance between the centers of the two masses.

Theorem. 6.2: Newton's Law of Universal Gravitation (Vector Form)

The gravitational force between two point masses m_1 and m_2 can be written in vector form as

$$\mathbf{F} = -G \frac{m_1 m_2}{r^2} \hat{r}$$

where

- \mathbf{F} is the gravitational force vector on m_1 due to m_2 ,
- G is the gravitational constant,
- m_1 and m_2 are the masses of the two objects,
- r is the distance between the two masses,
- \hat{r} is the unit vector pointing from m_1 to m_2 , i.e., the direction of the force.



The negative sign indicates that the force is attractive, meaning the force is directed towards the other mass.

Theorem. 6.3: Gauss's Law of Gravity

$$\nabla \cdot \mathbf{g} = -4\pi G\rho$$

where

- $\nabla \cdot \mathbf{g}$ is the divergence of the gravitational field \mathbf{g} ,
- G is the gravitational constant,
- ρ is the mass density of the object.

This law can be integrated over a closed surface to give the integral form of Gauss's Law for gravity:

$$\oint_S \mathbf{g} \cdot d\mathbf{A} = -4\pi GM_{\text{enc}}$$

where

- \oint_S denotes a surface integral over a closed surface S ,
- \mathbf{g} is the gravitational field vector,
- $d\mathbf{A}$ is a differential area element on the surface,
- M_{enc} is the mass enclosed by the surface.

6.1.2 2-body Problem

Consider two point masses, m_1 at position vector \mathbf{r}_1 and m_2 at position vector \mathbf{r}_2 , interacting via a central force:

$$m_1 \ddot{\mathbf{r}}_1 = \mathbf{F}_{21}, \quad m_2 \ddot{\mathbf{r}}_2 = \mathbf{F}_{12} = -\mathbf{F}_{21}$$

The center of mass (barycenter) is located at

$$\mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}$$



and since the internal forces cancel by Newton's third law, the acceleration of the center of mass is

$$\ddot{\mathbf{R}} = 0$$

Hence, the center of mass moves in a straight line with constant velocity. Define the relative position vector

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$

Differentiating twice with respect to time gives:

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = \frac{\mathbf{F}_{21}}{m_1} - \frac{\mathbf{F}_{12}}{m_2} = \frac{\mathbf{F}(\mathbf{r})}{m_1} + \frac{\mathbf{F}(\mathbf{r})}{m_2} = \left(\frac{1}{m_1} + \frac{1}{m_2} \right) \mathbf{F}(\mathbf{r})$$

Define the blueuced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

Then

$$\mu \ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r})$$

which is an effective one-body problem where a particle with mass μ moves under the force $\mathbf{F}(\mathbf{r})$. The angular momentum about the center of mass for the effective one-body problem with blueuced mass μ is

$$\mathbf{L} = \mu \mathbf{r} \times \mathbf{v}$$

For a **central force**

$$\mathbf{F}(\mathbf{r}) = f(r) \hat{\mathbf{r}}$$

the torque is given by

$$\tau = \mathbf{r} \times \mathbf{F}(\mathbf{r}) = \mathbf{r} \times f(r) \hat{\mathbf{r}} = \mathbf{0}$$

Hence,

$$\frac{d\mathbf{L}}{dt} = \tau = \mathbf{0} \quad \Rightarrow \quad \mathbf{L} = \text{constant}$$

The total energy of the system is

$$E = \frac{1}{2} \mu r^2 + \frac{1}{2} \mu r^2 \dot{\theta}^2 + U(r)$$



where $U(r)$ is the potential energy of the central force. Note that

$$L = |\mathbf{L}| = \mu r^2 \dot{\theta}$$

Using

$$\dot{\theta} = \frac{L}{\mu r^2}$$

we have

$$\begin{aligned} \frac{1}{2} \mu r^2 \dot{\theta}^2 &= \frac{L^2}{2\mu r^2} \\ E &= \frac{1}{2} \mu r^2 + \underbrace{\left[U(r) + \frac{L^2}{2\mu r^2} \right]}_{V_{\text{eff}}(r)} \end{aligned}$$

6.2 Kepler's Law

Three Kepler's Laws

Theorem. 6.4: Kepler's First Law

The orbit of every planet is an ellipse with the Sun at one of the two foci. The general equation for an ellipse in polar coordinates (r, θ) with one focus at the origin is

$$r = \frac{a(1 - e^2)}{1 + e \cos \theta}$$

where a is the semi-major axis, and e is the eccentricity ($0 \leq e < 1$). A circular orbit is the special case where $e = 0$.

Proof. Let the system be a planet of mass m orbiting a star of mass M (assumed fixed at the origin).

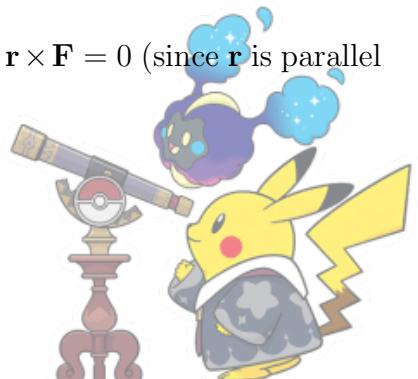
The gravitational force is

$$\mathbf{F} = -G \frac{Mm}{r^2} \hat{r} = m\mathbf{a}$$

where $\mu = GM$ is the gravitational parameter and $\mathbf{r} = r\hat{r}$. The acceleration is $\mathbf{a} = -\frac{\mu}{r^2} \hat{r}$.

- **Angular Momentum L (Kepler's Second Law):** The torque is $\tau = \mathbf{r} \times \mathbf{F} = 0$ (since \mathbf{r} is parallel to \mathbf{F}). Therefore, the angular momentum is constant:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m(\mathbf{r} \times \mathbf{v}) = \text{constant}$$



The orbit is confined to the plane perpendicular to \mathbf{L} .

- **Laplace-Runge-Lenz (LRL) Vector \mathbf{A} :** The LRL vector is a constant of motion only for an inverse-square force. It is defined as

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - m\mu\hat{r}$$

To prove it's conserved, we show $\frac{d\mathbf{A}}{dt} = 0$. Using the product rule,

$$\frac{d\mathbf{A}}{dt} = \frac{d\mathbf{p}}{dt} \times \mathbf{L} + \mathbf{p} \times \frac{d\mathbf{L}}{dt} - m\mu \frac{d\hat{r}}{dt}$$

As $\frac{d\mathbf{p}}{dt} = \mathbf{F}$ and $\frac{d\mathbf{L}}{dt} = \tau = 0$,

$$\frac{d\mathbf{A}}{dt} = \mathbf{F} \times \mathbf{L} - m\mu \frac{d\hat{r}}{dt}$$

By $\mathbf{F} = -\frac{m\mu}{r^2}\hat{r}$ and $\mathbf{L} = m(\mathbf{r} \times \mathbf{v})$:

$$\mathbf{F} \times \mathbf{L} = -\frac{m\mu}{r^2}\hat{r} \times m(\mathbf{r} \times \mathbf{v}) = -\frac{m^2\mu}{r^2} [(\hat{r} \cdot \mathbf{v})\mathbf{r} - (\hat{r} \cdot \mathbf{r})\mathbf{v}]$$

By $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$, $\hat{r} \cdot \mathbf{r} = r$ and $\hat{r} \cdot \mathbf{v} = \dot{r}$,

$$\mathbf{F} \times \mathbf{L} = -\frac{m^2\mu}{r^2} [(\dot{r})\mathbf{r} - (r)\mathbf{v}] = m\mu \left[\frac{r\mathbf{v}}{r^2} - \frac{\dot{r}\mathbf{r}}{r^2} \right]$$

Note that

$$\frac{d\hat{r}}{dt} = \frac{d}{dt} \left(\frac{\mathbf{r}}{r} \right) = \frac{\dot{\mathbf{r}}r - \mathbf{r}\dot{r}}{r^2} = \frac{\mathbf{v}r - \mathbf{r}\dot{r}}{r^2}$$

Hence, $\mathbf{F} \times \mathbf{L} = m\mu \frac{d\hat{r}}{dt}$. Therefore,

$$\frac{d\mathbf{A}}{dt} = m\mu \frac{d\hat{r}}{dt} - m\mu \frac{d\hat{r}}{dt} = 0$$

The LRL vector \mathbf{A} lies in the orbital plane. We find the orbit equation by taking the dot product of \mathbf{A} with the position vector \mathbf{r} :

$$\mathbf{r} \cdot \mathbf{A} = \mathbf{r} \cdot (\mathbf{p} \times \mathbf{L} - m\mu\hat{r}) = \mathbf{r} \cdot (\mathbf{p} \times \mathbf{L}) - m\mu(\mathbf{r} \cdot \hat{r})$$



Using the scalar triple product identity $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$,

$$\mathbf{r} \cdot (\mathbf{p} \times \mathbf{L}) = \mathbf{L} \cdot (\mathbf{r} \times \mathbf{p})$$

Since $\mathbf{r} \times \mathbf{p} = \mathbf{L}$, the identity simplifies to

$$\mathbf{r} \cdot (\mathbf{p} \times \mathbf{L}) = \mathbf{L} \cdot \mathbf{L} = L^2$$

Therefore,

$$\mathbf{r} \cdot \mathbf{A} = L^2 - m\mu r$$

The LRL vector \mathbf{A} is constant. Let $A = |\mathbf{A}|$. We define the angle θ between \mathbf{r} and \mathbf{A} . Then $\mathbf{r} \cdot \mathbf{A} = rA \cos \theta$:

$$rA \cos \theta = L^2 - m\mu r$$

$$r = \frac{L^2}{m\mu} \frac{1}{1 + \frac{A}{m\mu} \cos \theta}$$

Normalizing by $m = 1$ gives

$$r = \frac{\ell}{1 + e \cos \theta}$$

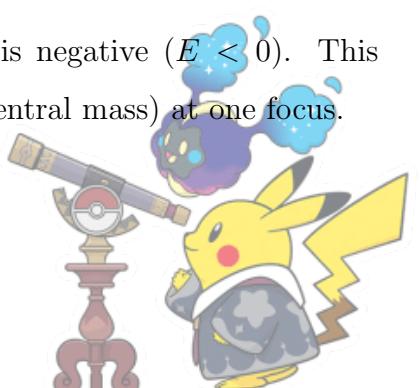
which is the equation of conic section in polar coordinates, where

- The **semi-latus rectum** is $\ell = \frac{L^2}{m\mu}$.
- The **eccentricity** is $e = \frac{A}{m\mu}$.

As A , m , and μ are all positive, the eccentricity e is a non-negative constant.

- If the total energy $E < 0$, then $0 \leq e < 1$, which defines an ellipse.
- If $E = 0$, then $e = 1$, defining a parabola.
- If $E > 0$, then $e > 1$, defining a hyperbola.

Planetary orbits are bound orbit, meaning the total mechanical energy E is negative ($E < 0$). This guarantees $0 \leq e < 1$, proving that the orbit is an ellipse with the Sun (the central mass) at one focus.



Theorem. 6.5: Kepler's Second Law

A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.

Proof.

- The gravitational force \mathbf{F} is always directed along the line connecting the planet and the Sun (it's a **central force**).

$$\mathbf{F} = -G \frac{Mm}{r^2} \hat{r}$$

- The **torque** (τ) on the planet with respect to the Sun (the origin) is given by

$$\tau = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \left(-G \frac{Mm}{r^2} \hat{r} \right)$$

Since \mathbf{r} and \hat{r} are in the same direction, $\mathbf{r} \times \hat{r} = 0$. Hence, the torque is zero: $\tau = 0$.

- By Newton's second law for rotation, the torque is equal to the rate of change of the angular momentum (\mathbf{L}):

$$\tau = \frac{d\mathbf{L}}{dt}$$

Since $\tau = 0$, the angular momentum \mathbf{L} is conserved:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times (m\mathbf{v}) = \text{constant}$$

where \mathbf{p} is the linear momentum.

- The angular momentum can be expressed in terms of the area swept out per unit time. Consider an infinitesimal time dt . The area dA swept by the radius vector \mathbf{r} is approximately the area of a triangle:

$$dA = \frac{1}{2} |\mathbf{r} \times d\mathbf{r}|$$

Since $d\mathbf{r} = \mathbf{v}dt$, we have:

$$dA = \frac{1}{2} |\mathbf{r} \times \mathbf{v}dt| = \frac{1}{2} |\mathbf{r} \times \mathbf{v}| dt$$

- The rate at which area is swept out is $\frac{dA}{dt}$:

$$\frac{dA}{dt} = \frac{1}{2} |\mathbf{r} \times \mathbf{v}| = \frac{1}{2m} |\mathbf{r} \times m\mathbf{v}| = \frac{|\mathbf{L}|}{2m} = \text{constant}$$



Theorem. 6.6: Kepler's Third Law

Kepler's Third Law states that the square of the orbital period T of a planet is directly proportional to the cube of the semi-major axis a of its orbit. Mathematically,

$$T^2 \propto a^3$$

In the case of elliptical orbits, where a is the semi-major axis, the law still holds and it can be expressed as

$$T^2 = \frac{4\pi^2}{GM}a^3$$

where

- T is the orbital period of the planet.
- a is the semi-major axis of the ellipse.
- G is the gravitational constant.
- M is the mass of the central body (e.g., the Sun).

Proof. By Kepler's 2nd law,

$$\frac{\pi ab}{T} = \frac{L}{2m}$$

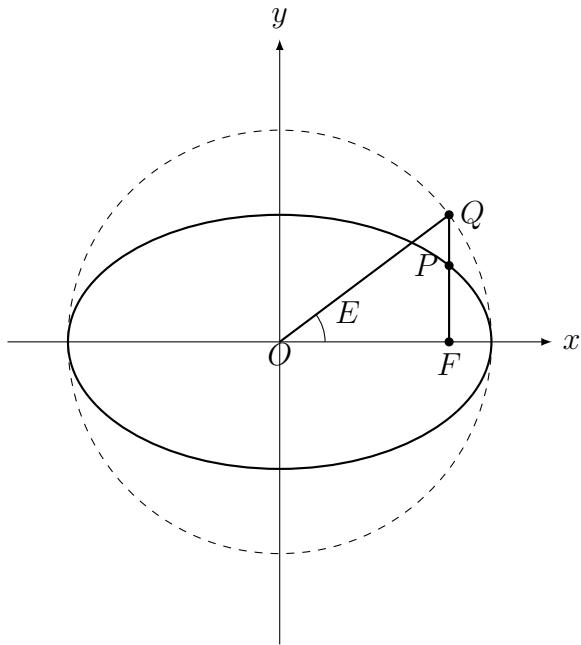
as the area of ellipse is given by πab . By Kepler's 1st law, $r = a(1 - e)$ and

$$r(\theta) = \frac{L^2}{GMm^2(1 + e \cos \theta)}.$$

Then $\frac{L^2}{m^2} = a(1 - e^2)GM$. Substitute the formula involving the area, $T^2 = \frac{4\pi^2}{GM}a^3$.



6.2.1 Kepler's Equation



The eccentric anomaly E is defined using an auxiliary circle:

- Draw a circle of radius a centered at the origin (the auxiliary circle).
- For a point P on the ellipse, draw a vertical line to intersect the auxiliary circle at Q .
- The eccentric anomaly E is the angle $\angle X O Q$.

where O is the center of the ellipse and X is the positive x -axis direction. Although E is not a physical angle measured from the focus, it provides a convenient parametrization of the ellipse. Using the eccentric anomaly E , the position of the orbiting body is

$$\boxed{\begin{aligned}x &= a(\cos E - e) \\y &= a\sqrt{1 - e^2} \sin E\end{aligned}}$$

The **true anomaly** ν is the physical polar angle of the orbiting body measured from periapsis. The relation between ν and E is

$$\tan \frac{\nu}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2}$$

The area swept from periapsis to point P on the ellipse is

$$A = \frac{a^2}{2}(E - e \sin E)$$



The total area of the ellipse is $\pi a^2 \sqrt{1 - e^2}$, but by Kepler's second law, the fraction of area swept equals the fraction of time elapsed:

$$\frac{A}{\pi a^2} = \frac{t - \tau}{T}$$

where T is the orbital period. Using mean anomaly

$$M \equiv n(t - \tau)$$

(where **mean motion** $n = 2\pi/T$ and τ is the time of periapsis passage), we obtain

$$M = E - e \sin E$$

which is the Kepler's equation.

6.2.2 Poisson Bracket

Definition. 6.1: Poisson Bracket

The Poisson bracket of two functions $f(q, p)$ and $g(q, p)$, where q and p represent generalized coordinates and momenta, is defined as

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$$

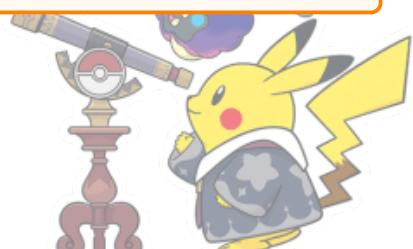
Here, q_i and p_i represent the generalized coordinates and momenta for the i -th degree of freedom.

Definition. 6.2: Levi-Civita Symbol

The Levi-Civita symbol is defined as

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } (i, j, k) \text{ is an even permutation of } (1, 2, 3), \\ -1 & \text{if } (i, j, k) \text{ is an odd permutation of } (1, 2, 3), \\ 0 & \text{if any two indices are equal.} \end{cases}$$

Sometimes, $\sum_{j,k} \epsilon_{ijk} a_j b_k$ is written as $\epsilon_{ijk} a_j b_k$ for simplicity. (**Einstein's Notation**)



For the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, the components of the angular momentum are

$$L_i = \epsilon_{ijk} r_j p_k$$

where ϵ_{ijk} is the Levi-Civita symbol, and we are summing over repeated indices.

$$\{p_i, L_j\} = \{p_i, \epsilon_{jkl} r_k p_l\} = \epsilon_{jkl} (\{p_i, r_k p_l\})$$

As the position r_k and momentum p_l are conjugate variables, we have

$$\{p_i, r_k\} = -\delta_{ik}$$

and

$$\{p_i, p_l\} = 0$$

Note that

$$\{p_i, r_k p_l\} = r_k \{p_i, p_l\} + p_l \{p_i, r_k\} = -p_l \delta_{ik}$$

Finally,

$$\{p_i, L_j\} = \epsilon_{jkl} (-p_l \delta_{ik}) = -\epsilon_{jkl} p_l \delta_{ik} = \epsilon_{ijk} p_k$$

Define $\mathbf{W} = \left\{ \frac{\mathbf{p}}{m}, \mathbf{L} \right\} - \frac{GMm}{r} \mathbf{r}$ which is generalized LRL vector. One can prove that

$$\{W_i, W_j\} = \epsilon_{ijk} \left(\frac{-2E}{m} L_k \right)$$

and show that it forms lie algebra.

Definition. 6.3: Lie Algebra

A **Lie algebra** is a vector space \mathfrak{g} over a field (typically \mathbb{R} or \mathbb{C}) equipped with a binary operation called the **Lie bracket**, denoted by $[\cdot, \cdot]$. The Lie bracket is a bilinear map that satisfies the following two properties:

- **Bilinearity:**

$$[aX+bY, Z] = a[X, Z]+b[Y, Z], \quad [Z, aX+bY] = a[Z, X]+b[Z, Y] \quad \text{for all } X, Y, Z \in \mathfrak{g}, \quad a, b \in \mathbb{F}$$



- **Antisymmetry:**

$$[X, Y] = -[Y, X] \quad \text{for all } X, Y \in \mathfrak{g}$$

- **Jacobi Identity:**

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \quad \text{for all } X, Y, Z \in \mathfrak{g}$$

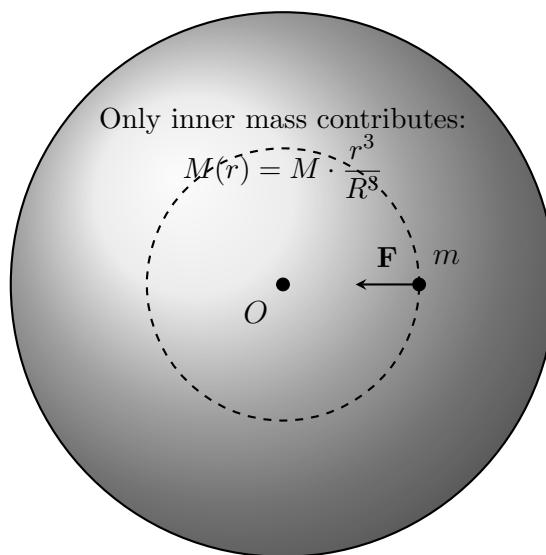
6.3 Shell Theorem

Theorem. 6.7: Shell Theorem

- A spherically symmetric shell of mass exerts **no net gravitational force** on a particle located **inside** the shell.
- A spherically symmetric shell of mass exerts a **gravitational force on an external point mass** as if all the mass were concentrated at the shell's center.

For a point mass located at a distance $r < R$ inside a uniform solid sphere (e.g., a planet), the Shell Theorem implies:

- Only the mass enclosed within radius r contributes to the gravitational force.
- Mass at radii greater than r exerts no net force by First Shell Theorem by considering it as infinitesimal shells.



Assuming uniform density, the mass enclosed within radius r is

$$M(r) = M \cdot \frac{r^3}{R^3}$$

Then, the gravitational force on a test mass m located at distance r from the center is

$$F = \frac{GM(r)m}{r^2} = \frac{GMm}{R^3} \cdot r$$

by second shell theorem.

Example. (Simple Gravity Train) Mass outside the spherical surface of radius r produces *zero net force*. The mass enclosed is

$$M(r) = M \cdot \frac{r^3}{R^3}$$

The gravitational force magnitude is therefore

$$F = -G \frac{M(r)m}{r^2} = -\frac{GMm}{R^3} r = m\ddot{r}$$

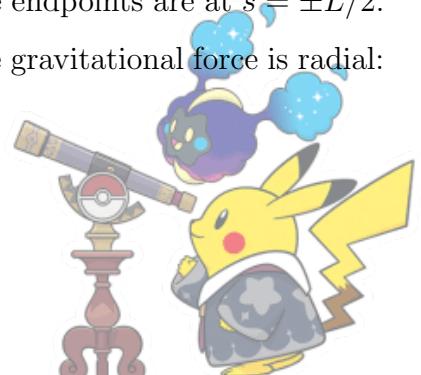
which is in Simple Harmonic Form (SHM) form $a = -\omega^2 x$. Therefore, $\omega = \sqrt{\frac{GM}{R^3}}$ and time of travel $t = \frac{\pi}{\sqrt{\frac{GM}{R^3}}} = \pi \sqrt{\frac{R}{g}}$.

Remark. The gravity train (also known as the Gauss gun, gravity elevator, or vacuum train) is a fascinating thought experiment in classical mechanics. First proposed by Robert Hooke in the 17th century and later analyzed by Isaac Newton, it explores what would happen if a tunnel were drilled through the Earth and an object were dropped through it.

Example. (Generalised Gravity Train) Consider a **straight tunnel** connecting two arbitrary points A and B on Earth's surface, not necessarily antipodal. The tunnel forms a **chord** of Earth's circular cross-section.

Let s be the coordinate along the tunnel, with $s = 0$ at the midpoint M . The endpoints are at $s = \pm L/2$. For a point **inside a homogeneous Earth** at position \mathbf{r} from the center, the gravitational force is radial:

$$\mathbf{F} = -\frac{GM(r)}{r^2} \hat{\mathbf{r}}, \quad \text{with} \quad M(r) = \frac{r^3}{R^3} M$$



Hence,

$$\mathbf{F} = -\frac{GM}{R^3} r \hat{\mathbf{r}}$$

Let the tunnel be at a constant perpendicular distance y from Earth's center. For a point on the tunnel at coordinate s , its radial distance from Earth's center is:

$$r(s) = \sqrt{s^2 + y^2}$$

The **component of gravitational force along the tunnel** (in the $\hat{\mathbf{s}}$ direction) is

$$F_s = \mathbf{F} \cdot \hat{\mathbf{s}} = \left(-\frac{GM}{R^3} r \hat{\mathbf{r}} \right) \cdot \hat{\mathbf{s}}$$

Since $\hat{\mathbf{r}} = \frac{s\hat{\mathbf{s}} + y\hat{\mathbf{y}}}{r}$, we have:

$$F_s = -\frac{GM}{R^3} r \left(\frac{s}{r} \right) = -\frac{GM}{R^3} s = m\ddot{s}$$

which is also in SHM form.

6.4 Gravitational Potential Energy

Theorem. 6.8: Gravitational Potential Energy

The gravitational potential energy (U) of an object in a gravitational field is defined as the work done in moving the object from infinity to a point at distance r from the source of the gravitational field. For two point masses, the gravitational potential energy is given by

$$U = -G \frac{m_1 m_2}{r}$$

where

- U is the gravitational potential energy,
- G is the gravitational constant,
- m_1 and m_2 are the two masses,
- r is the distance between the centers of the two masses.



Proof. The gravitational force is:

$$F = G \frac{Mm}{r^2}$$

Then,

$$\begin{aligned} U &= \int_{\infty}^r F dr \\ &= \int_{\infty}^r G \frac{Mm}{r^2} dr \\ &= GMm \int_{\infty}^r \frac{1}{r^2} dr \\ &= GMm \left[-\frac{1}{r} \right]_{\infty}^r \\ &= -G \frac{Mm}{r} \end{aligned}$$

6.4.1 Vis-viva Equation

Theorem. 6.9: Vis-viva Equation

The Vis-viva equation, also known as the orbital energy conservation equation, relates the velocity of an orbiting body to its position and the geometry of its orbit. It is expressed as

$$v^2 = \mu \left(\frac{2}{r} - \frac{1}{a} \right)$$

where

- v is the orbital speed of the orbiting body.
- r is the distance between the two bodies.
- a is the semi-major axis of the orbit.
- μ is the standard gravitational parameter (GM).

Proof. Let v_1 be the furthest distance from the object to the planet and v_2 be the closest distance from the object to the planet.

By conservation of angular momentum, $m_1 r_1 v_1 = m_2 r_2 v_2 \implies v_2 = \frac{r_1}{r_2} v_1$. By conservation of mechanical energy $\frac{1}{2}mv^2 - \frac{GMm}{r} = \text{constant}$.

$$v_1^2 = \frac{2GMr_2}{r_1(r_1 + r_2)}.$$



By $m_1 r_1 v_1 = m r_2 v_2 \implies v_2 = \frac{r_1}{r_2} v_1$ and $r_1 + r_2 = 2a$,

$$v^2 = \mu \left(\frac{2}{r} - \frac{1}{a} \right).$$

where G is the gravitational constant.

6.4.2 First Cosmic Speed, Second Cosmic Speed and Third Cosmic Speed

Definition. 6.4: First Cosmic Speed

The first cosmic speed is the minimum velocity required for an object to orbit the Earth in a circular trajectory at the lowest possible altitude (negligible height above the surface).

For a satellite to remain in a circular orbit, the gravitational force must provide the required centripetal force.

$$\begin{aligned} F_{grav} &= F_{centripetal} \\ \frac{GMm}{R^2} &= m \frac{v_1^2}{R} \end{aligned}$$

where

- M : Mass of Earth ($\approx 5.97 \times 10^{24}$ kg)
- m : Mass of the satellite
- R : Radius of Earth ($\approx 6.37 \times 10^6$ m)
- G : Gravitational Constant

$$v_1 = \sqrt{\frac{GM}{R}}$$

Definition. 6.5: Second Cosmic Speed

The second cosmic speed is the minimum velocity required for an object to break free from Earth's gravitational attraction, effectively escaping to infinity. It is commonly known as **escape velocity**.

The total energy at Earth's surface must equal the total energy at infinity (where both potential and kinetic energy are zero for a barely escape scenario).



$$K_{surface} + U_{surface} = K_\infty + U_\infty$$

$$\frac{1}{2}mv_2^2 - \frac{GMm}{R} = 0$$

$$v_2 = \sqrt{\frac{2GM}{R}}$$

Note that $v_2 = \sqrt{2}v_1$.

Definition. 6.6: Third Cosmic Speed

The third Cosmic Speed is the launch velocity required from Earth to escape the Solar System.

This requires overcoming Earth's gravity and then having enough residual velocity to escape the Sun's gravity.

- **Hyperbolic Excess Velocity (v_∞):** Utilizing Earth's orbital speed ($v_E \approx 29.8 \text{ km/s}$), the spacecraft needs an excess velocity relative to Earth of

$$v_\infty = v_{esc,\odot} - v_E$$

- **Launch Calculation:** Applying energy conservation from Earth's surface:

$$\frac{1}{2}mv_3^2 - \frac{GMm}{R} = \frac{1}{2}mv_\infty^2$$

$$v_3 = \sqrt{v_\infty^2 + \frac{2GM}{R}}$$

Since $\frac{2GM}{R} = v_2^2$:

$$v_3 = \sqrt{v_\infty^2 + v_2^2}$$

6.4.3 Hohmann Transfer Orbit

The **Hohmann transfer orbit** is the most fuel-efficient two-impulse maneuver for transferring a spacecraft between two coplanar circular orbits around the same central body.



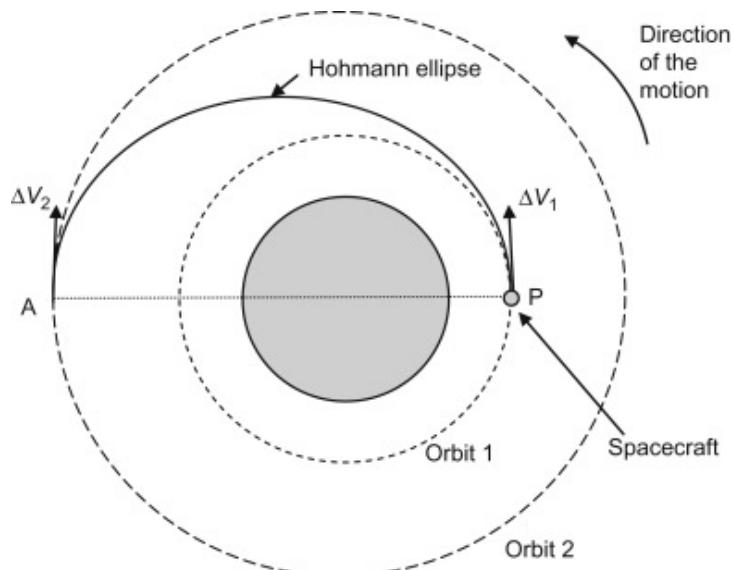


Figure 5: Source: <https://www.sciencedirect.com/topics/engineering/hohmann-transfer>

For a spacecraft in a circular orbit of radius r around a central body of mass M , the orbital speed is

$$v_c = \sqrt{\frac{GM}{r}}$$

For any Keplerian orbit, the speed at distance r is given by the vis-viva equation:

$$v^2 = GM \left(\frac{2}{r} - \frac{1}{a} \right)$$

where a is the semi-major axis of the orbit. Consider two circular orbits of radii r_1 (initial) and r_2 (final), with $r_2 > r_1$. The Hohmann transfer orbit is an ellipse with:

Periapsis at r_1 ,

Apoapsis at r_2 . The semi-major axis of the transfer ellipse is

$$a = \frac{r_1 + r_2}{2}$$

The spacecraft initially moves with circular speed

$$v_1 = \sqrt{\frac{GM}{r_1}}$$



At periapsis of the transfer ellipse, the required speed is obtained from the vis-viva equation:

$$v_p = \sqrt{GM \left(\frac{2}{r_1} - \frac{1}{a} \right)}$$

The first velocity increment is therefore

$$\Delta v_1 = v_p - v_1$$

This burn places the spacecraft onto the elliptical transfer orbit.

At apoapsis of the transfer ellipse ($r = r_2$), the spacecraft speed is

$$v_a = \sqrt{GM \left(\frac{2}{r_2} - \frac{1}{a} \right)}$$

The circular speed in the final orbit is

$$v_2 = \sqrt{\frac{GM}{r_2}}$$

The second velocity increment is

$$\Delta v_2 = v_2 - v_a$$

After this burn, the spacecraft is inserted into the final circular orbit.

The total fuel cost of the maneuver is measured by the total velocity increment:

$$\Delta v_{\text{total}} = \Delta v_1 + \Delta v_2$$

The Hohmann transfer minimizes Δv_{total} among all two-impulse transfers between coplanar circular orbits.

The spacecraft travels half of the elliptical orbit during the transfer. The orbital period of an ellipse is

$$T = 2\pi \sqrt{\frac{a^3}{GM}}$$

Hence, the Hohmann transfer time is

$$t_{\text{transfer}} = \pi \sqrt{\frac{a^3}{GM}}$$

This relatively long transfer time is the main trade-off for fuel efficiency.



6.4.4 Blackhole

The event horizon is a boundary in spacetime beyond which events cannot affect an observer. It is commonly associated with black holes, and marks the point at which the escape velocity exceeds the speed of light. This means that not even light can escape from inside the event horizon, which is why it appears black. For a non-rotating, spherically symmetric black hole, the event horizon is given by the Schwarzschild radius r_s , which is the radius at which the escape velocity equals the speed of light. The Schwarzschild radius is defined as

$$r_s = \frac{2GM}{c^2}$$

where

- G is the gravitational constant,
- M is the mass of the black hole,
- c is the speed of light.

Beyond the event horizon, all paths of particles and light curves back into the black hole, and no signal can escape to the outside universe.

6.4.5 Virial Theorem

For a gravitationally bound system in dynamical equilibrium, such as an isolated galaxy, the virial theorem states that the time-averaged total kinetic energy $\langle T \rangle$ and the time-averaged total gravitational potential energy $\langle U \rangle$ satisfy

$$2\langle T \rangle + \langle U \rangle = 0$$

For a system of particles of total mass M in virial equilibrium, the virial theorem gives

$$2\langle T \rangle + \langle U \rangle = 0$$

The total kinetic energy may be written in terms of the velocity dispersion as

$$\langle T \rangle \sim \frac{1}{2}M\langle v^2 \rangle \sim \frac{3}{2}M\sigma^2$$



where σ is the one-dimensional velocity dispersion.

For a self-gravitating system of characteristic size R , the gravitational potential energy is of order

$$\langle U \rangle \sim -\frac{GM^2}{R}$$

up to a dimensionless factor of order unity that depends on the mass distribution.

Substituting into the virial theorem gives

$$3M\sigma^2 \sim \frac{GM^2}{R}$$

Solving for M yields

$$M \sim \frac{R\sigma^2}{G}$$

which is the virial mass estimate for a galaxy.

6.4.6 The Tully-Fisher Relation

In 1977, R. Brent Tully and J. Richard Fisher discovered an empirical relationship between the luminosity L of spiral galaxies and their maximum rotational velocity v_{\max} :

$$L \propto v_{\max}^\alpha$$

where observations give $\alpha \approx 3.5 - 4$ in the infrared and $\alpha \approx 2.5 - 3$ in the blue band.



6.5 Lagrange Point

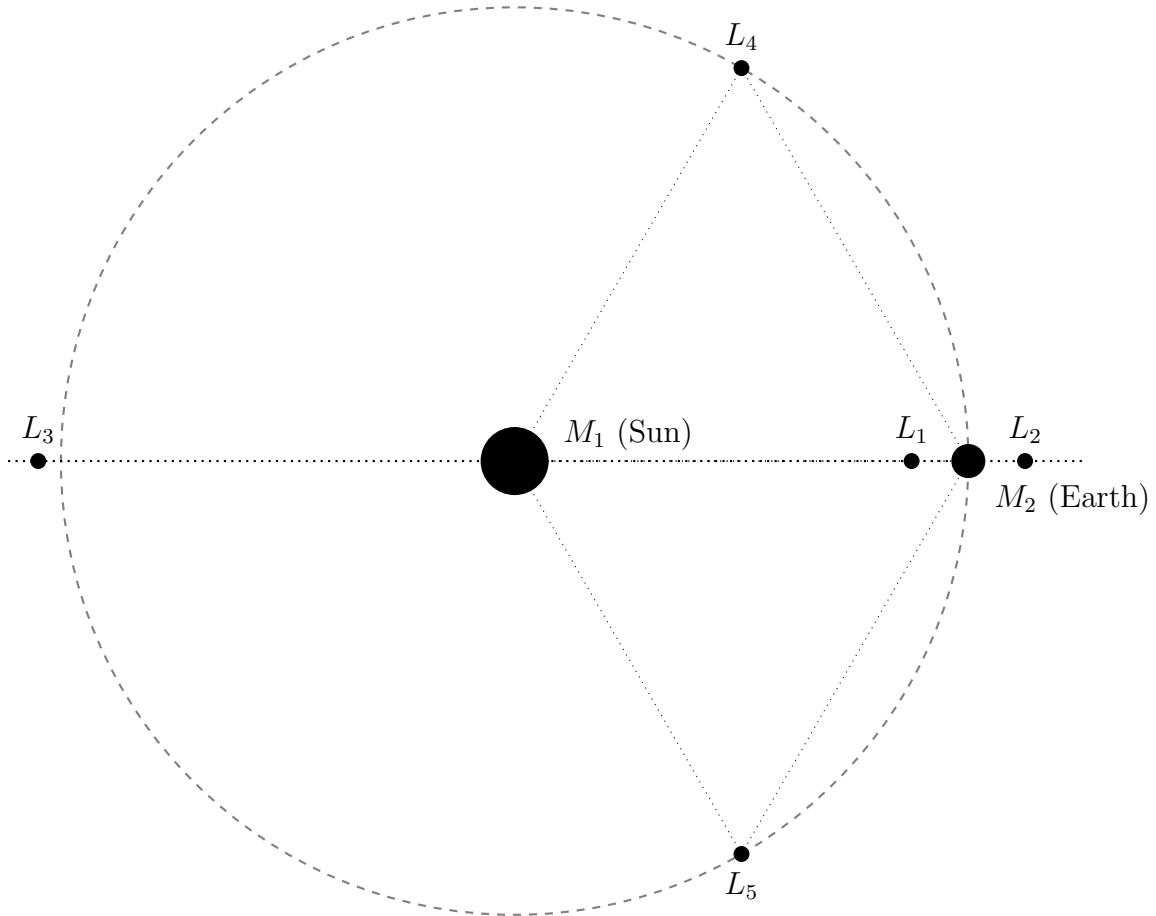


Figure 6: The Five Lagrangian Points (L_1 to L_5) in the Sun-Earth System's Rotating Frame

Two primary masses M_1 and M_2 orbit their common center of mass in circular motion. Let $M_1 > M_2$. Define the blueuced mass

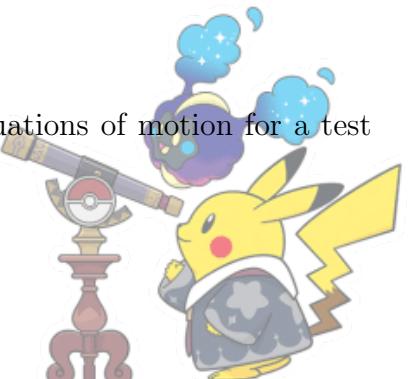
$$\mu = \frac{M_2}{M_1 + M_2}$$

Normalize $M_1 + M_2 = 1$, and the gravitational constant $G = 1$. In the rotating frame centeblue at the center of mass, M_1 is at $(-\mu, 0)$ and M_2 is at $(1 - \mu, 0)$. The angular velocity ω of the rotating frame satisfies (from Kepler's 3rd law for M_1 and M_2)

$$\omega^2 = \frac{G(M_1 + M_2)}{a^3} = 1$$

since $a = 1$, $G = 1$, $M_1 + M_2 = 1$. So $\omega = 1$. In the rotating frame, the equations of motion for a test particle at $\mathbf{r} = (x, y)$ are

$$\ddot{x} - 2\omega\dot{y} = \frac{\partial U}{\partial x}, \quad \ddot{y} + 2\omega\dot{x} = \frac{\partial U}{\partial y}$$



where

$$U(x, y) = \underbrace{\frac{1}{2}\omega^2(x^2 + y^2)}_{\text{Centrifugal Potential}} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$$

with

$$r_1 = \sqrt{(x + \mu)^2 + y^2}, \quad r_2 = \sqrt{(x - 1 + \mu)^2 + y^2}$$

Lagrange points are stationary points of U in the rotating frame (equilibrium points where $\dot{x} = \dot{y} = 0$):

$$\frac{\partial U}{\partial x} = 0, \quad \frac{\partial U}{\partial y} = 0$$

$$\frac{\partial U}{\partial x} = x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3} = 0$$

$$\frac{\partial U}{\partial y} = y \left[1 - \frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3} \right] = 0$$

For $y = 0$, we have $r_1 = |x + \mu|$, $r_2 = |x - 1 + \mu|$. The x -equation becomes

$$x - \frac{1-\mu}{(x+\mu)^2} \operatorname{sgn}(x+\mu) - \frac{\mu}{(x-1+\mu)^2} \operatorname{sgn}(x-1+\mu) = 0$$

- $x > 1 - \mu$ (beyond M_2):

$$x - \frac{1-\mu}{(x+\mu)^2} - \frac{\mu}{(x-1+\mu)^2} = 0$$

- $\mu < x < 1 - \mu$ (between M_1 and M_2):

$$x - \frac{1-\mu}{(x+\mu)^2} + \frac{\mu}{(1-\mu-x)^2} = 0$$

- $x < -\mu$ (opposite side of M_1 from M_2):

$$x + \frac{1-\mu}{(x+\mu)^2} + \frac{\mu}{(x-1+\mu)^2} = 0$$

They have series solution:

$$L_1 : \quad x \approx 1 - \mu - \left(\frac{\mu}{3}\right)^{1/3} + \frac{1}{3} \left(\frac{\mu}{3}\right)^{2/3} + \dots$$

$$L_2 : \quad x \approx 1 - \mu + \left(\frac{\mu}{3}\right)^{1/3} + \frac{1}{3} \left(\frac{\mu}{3}\right)^{2/3} + \dots$$



$$L_3 : \quad x \approx -1 - \frac{5}{12}\mu + \dots$$

From $\frac{\partial U}{\partial y} = 0$,

$$y \left[1 - \frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3} \right] = 0$$

If $y \neq 0$, then

$$\frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3} = 1 \quad (*)$$

From $\frac{\partial U}{\partial x} = 0$,

$$x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3} = 0$$

Multiply (*) by x and subtract from above:

$$-\frac{(1-\mu)\mu}{r_1^3} + \frac{\mu(1-\mu)}{r_2^3} = 0 \quad \Rightarrow \quad \frac{1-\mu}{r_1^3} = \frac{1-\mu}{r_2^3}$$

Hence, for $\mu \neq 1$, we get $r_1 = r_2$. From $r_1 = r_2$, (*) gives

$$\frac{1-\mu}{r_1^3} + \frac{\mu}{r_1^3} = \frac{1}{r_1^3} = 1 \quad \Rightarrow \quad r_1 = 1$$

So $r_1 = r_2 = 1$. Now $(x+\mu)^2 + y^2 = 1$ and $(x-1+\mu)^2 + y^2 = 1$. Solving,

$$x = \frac{1}{2} - \mu, y = \pm \frac{\sqrt{3}}{2}$$

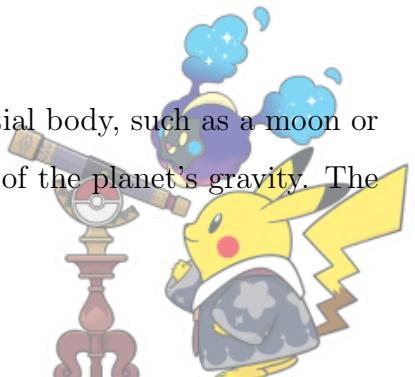
Therefore,

$$L_4 : \quad \left(\frac{1}{2} - \mu, \frac{\sqrt{3}}{2} \right)$$

$$L_5 : \quad \left(\frac{1}{2} - \mu, -\frac{\sqrt{3}}{2} \right)$$

6.6 Roche Limit

The Roche limit is the minimum distance from a planet within which a celestial body, such as a moon or asteroid, will experience tidal forces strong enough to break it apart because of the planet's gravity. The



Roche limit is given by the formula

$$d_{\text{Roche}} = 2.44 R \left(\frac{\rho_{\text{planet}}}{\rho_{\text{moon}}} \right)^{1/3}$$

Derivation of simplified model: Tidal force is the difference in gravitational force felt across an object due to another massive body.

Consider a small object of radius R located at a distance d from a much more massive body of mass M .

$$a_{\text{near}} = \frac{GM}{(d - R)^2}, \quad a_{\text{far}} = \frac{GM}{(d + R)^2}$$

The tidal acceleration is the difference between the accelerations on the near and far sides

$$a_{\text{tidal}} = a_{\text{near}} - a_{\text{far}} = \frac{GM}{(d - R)^2} - \frac{GM}{(d + R)^2}$$

Assume $R \ll d$. Then

$$\frac{1}{(d \pm R)^2} \approx \frac{1}{d^2} \mp \frac{2R}{d^3}$$

Therefore,

$$a_{\text{tidal}} \approx GM \left(\left(\frac{1}{d^2} + \frac{2R}{d^3} \right) - \left(\frac{1}{d^2} - \frac{2R}{d^3} \right) \right) = GM \cdot \frac{4R}{d^3}$$

$$a_{\text{tidal (per side)}} \approx \frac{2GMR}{d^3}$$

Note that

$$a_{\text{tidal}} \approx \frac{2GM_p R_s}{d^3}, \quad a_{\text{self}} = \frac{Gm_s}{R_s^2}$$

By $m_s = \frac{4}{3}\pi R_s^3 \rho_s$ and $M_p = \frac{4}{3}\pi R_p^3 \rho_p$,

$$\frac{2GM_p R_s}{d^3} = \frac{Gm_s}{R_s^2}$$

which can be simplified to

$$d = R_p \cdot \left(2 \cdot \frac{\rho_p}{\rho_s} \right)^{1/3}$$

For the derivation of non-simplified model,

one can refer to <https://dxwl.bnu.edu.cn/CN/abstract/abstract7639.shtml>.



7 Relativity

Newtonian mechanics, formulated in the 17th century, works exceptionally well for everyday speeds ($v \ll c$). However, as experiments in the late 19th century became more precise (e.g., Michelson-Morley experiment), inconsistencies emerged that required a new theoretical framework.

7.1 Postulates of Special Relativity

- **Principle of Relativity:** The laws of physics are the same in all inertial reference frames.
- **Invariance of Light Speed:** The speed of light in vacuum, c , is the same in all inertial frames, regardless of the motion of the source or observer.

7.2 Galileo Transformation

In classical mechanics, the Galilean transformation relates the coordinates of an event as observed in two inertial frames of reference moving at a constant relative velocity. Suppose we have two inertial frames:

- Frame S : stationary frame with coordinates (x, y, z, t)
- Frame S' : moving with constant velocity v along the x -axis relative to S , with coordinates (x', y', z', t')

The Galilean transformation equations are

$$x' = x - vt$$

$$y' = y$$

$$z' = z$$

$$t' = t$$

7.3 Lorentz Transformation

Definition. 7.1: Lorentz Factor

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \beta^2}}$$

where v is the speed of the object and $\beta := \frac{v}{c}$.



In special relativity, the Lorentz transformation relates the coordinates of an event as observed in two inertial frames moving at a constant velocity relative to each other. Unlike the Galilean transformation, it accounts for the constancy of the speed of light.

Theorem. 7.1: Lorentz Transformation

For two inertial frames S and S' with S' moving at velocity v along the x -axis relative to S , the Lorentz transformations are

$$\begin{aligned}t' &= \gamma \left(t - \frac{vx}{c^2} \right) \\x' &= \gamma(x - vt) \\y' &= y \\z' &= z\end{aligned}$$

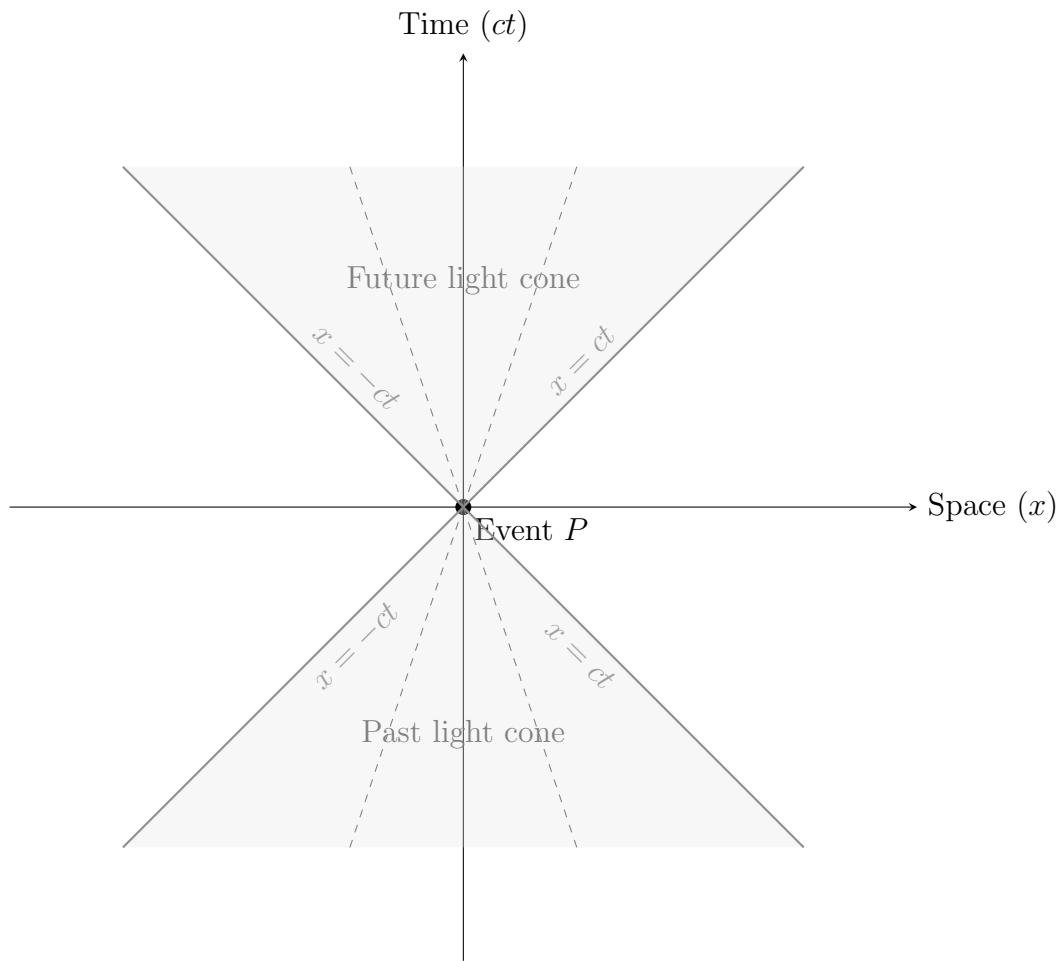
Theorem. 7.2: Inverse Lorentz Transformation

$$\begin{aligned}t &= \gamma \left(t' + \frac{vx'}{c^2} \right) \\x &= \gamma(x' + vt') \\y &= y' \\z &= z'\end{aligned}$$



7.4 Spacetime Diagram

A spacetime diagram is a graphical representation of events in spacetime.



7.5 Relativistic Kinematics and Mechanics

7.5.1 Time Dilation

Time dilation is one of the most fascinating consequences of special relativity. It states that time measured in a moving frame (proper time τ) appears to run slower compared to time measured in a stationary frame (coordinate time t). The relationship between proper time (τ) and coordinate time (t) is given by

$$\Delta t = \gamma \Delta \tau$$

Example. A spaceship travels at $v = 0.8c$ relative to Earth. If 5 years pass on the spaceship (proper time), how much time passes on Earth?



Solution.

$$\gamma = \frac{1}{\sqrt{1 - (0.8)^2}} \approx 1.667$$

$$\Delta t = \gamma \Delta \tau = 1.667 \times 5 \approx 8.33 \text{ years}$$

7.5.2 Length Contraction

Length is defined as the distance between simultaneous measurements of an object's endpoints. For a rod of proper length L_0 at rest in S', moving with velocity v in S. In S, measure endpoints simultaneously ($\Delta t = 0$):

$$\Delta x = L \quad (\text{measured length in S})$$

$$\Delta x' = L_0 \quad (\text{proper length in S'})$$

$$\Delta x' = \gamma(\Delta x - v\Delta t) = \gamma L$$

$$L = \frac{L_0}{\gamma}$$

Hence, moving objects appear shorter in their direction of motion.

7.5.3 Relativistic Momentum

Consider a particle with rest mass m_0 .

$$\mathbf{p} = \gamma m_0 \mathbf{v}$$

7.5.4 Relativistic Energy

The correct relativistic expression for kinetic energy is

$$K = (\gamma - 1)mc^2$$

Proof. Starting from the work-energy theorem and relativistic momentum:

$$\begin{aligned} K &= \int \mathbf{F} \cdot d\mathbf{r} = \int \frac{d}{dt}(\gamma m_0 \mathbf{v}) \cdot \mathbf{v} dt \\ &= \int \left(\gamma m_0 \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} + m_0 \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} \frac{d\gamma}{dv^2} \frac{dv^2}{dt} \right) dt \\ &= m_0 c^2 \int d\gamma = \gamma m_0 c^2 + C \end{aligned}$$



When $\gamma = 0$, $K = 0$. Therefore, $K = (\gamma - 1)m_0c^2$.

The total energy of a particle is

$$E = \gamma mc^2 = K + m_0c^2$$

This consists of two parts:

- **Rest energy:** $E_0 = m_0c^2$
- **Kinetic energy:** $K = (\gamma - 1)m_0c^2$

7.5.5 Relation between Relativistic Momentum and Energy

From our definitions:

$$E = \gamma m_0 c^2$$

$$\mathbf{p} = \gamma m_0 \mathbf{v}$$

Compute $E^2 - (pc)^2$:

$$\begin{aligned} E^2 - (pc)^2 &= (\gamma m_0 c^2)^2 - (\gamma m_0 v c)^2 \\ &= \gamma^2 m_0^2 c^4 \left(1 - \frac{v^2}{c^2}\right) \\ &= m_0^2 c^4 \frac{1 - \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}} \quad (\text{since } \gamma^2 = \frac{1}{1 - \frac{v^2}{c^2}}) \\ &= m_0^2 c^4 \end{aligned}$$

Hence we obtain the fundamental relation:

$$E^2 = (pc)^2 + (m_0 c^2)^2$$

7.6 Spacetime and Four Vector

7.6.1 Spacetime in Special Relativity

In special relativity, space and time are unified into a four-dimensional continuum called **spacetime**. Events are described by four coordinates:

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$$



where

- $x^0 = ct$ is the temporal coordinate (with c being the speed of light)
- x^1, x^2, x^3 are spatial coordinates

7.6.2 Four-Vector

A **four-vector** A^μ is a mathematical object that transforms under Lorentz transformations in the same way as spacetime coordinates:

$$A^\mu = (A^0, A^1, A^2, A^3)$$

The transformation rule for a Lorentz transformation Λ_ν^μ is

$$A'^\mu = \Lambda_\nu^\mu A^\nu$$

where

$$\Lambda_\nu^\mu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

7.6.3 Metric Tensor

The metric tensor $g_{\mu\nu}$ defines the geometry of spacetime. In special relativity with Cartesian coordinates, it takes a simple form known as the **Minkowski metric**:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

In component form:

$$g_{00} = 1, \quad g_{11} = g_{22} = g_{33} = -1, \quad g_{\mu\nu} = 0 \text{ for } \mu \neq \nu$$

The metric tensor allows conversion between contravariant (upper index) and covariant (lower index) vectors:



Lowering an index

$$A_\mu = g_{\mu\nu} A^\nu$$

Explicitly:

$$A_0 = g_{00} A^0 = A^0$$

$$A_1 = g_{11} A^1 = -A^1$$

$$A_2 = g_{22} A^2 = -A^2$$

$$A_3 = g_{33} A^3 = -A^3$$

Raising an index

$$A^\mu = g^{\mu\nu} A_\nu$$

where $g^{\mu\nu}$ is the inverse metric, which for Minkowski spacetime is identical to $g_{\mu\nu}$:

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The metric tensor and its inverse satisfy

$$g^{\mu\lambda} g_{\lambda\nu} = \delta_\nu^\mu = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu \end{cases}$$

where δ_ν^μ is the Kronecker delta.

7.6.4 Invariant Interval

The **invariant interval** between two infinitesimally close events in spacetime is defined as:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

Expanding in components for Minkowski spacetime:

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$



For finite separation between events, the interval is:

$$\Delta s^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

The most important property of ds^2 is that it is **invariant** under Lorentz transformations. If we transform to a new coordinate system x'^μ :

$$ds'^2 = g_{\mu\nu} dx'^\mu dx'^\nu = g_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma dx^\rho dx^\sigma = g_{\rho\sigma} dx^\rho dx^\sigma = ds^2$$

This invariance follows from the defining property of Lorentz transformations:

$$g_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma = g_{\rho\sigma}$$

7.6.5 Important Four-Vectors in Physics

Position Four-Vector

$$x^\mu = (ct, \mathbf{x}) = (ct, x, y, z)$$

Velocity Four-Vector The four-velocity is defined as the derivative of the position four-vector with respect to proper time τ :

$$u^\mu = \frac{dx^\mu}{d\tau} = (\gamma c, \gamma \mathbf{v})$$

where τ is proper time.

The components are

$$\begin{aligned} u^0 &= \frac{dx^0}{d\tau} = \frac{cdt}{d\tau} = \gamma c \\ u^i &= \frac{dx^i}{d\tau} = \frac{dx^i}{dt} \frac{dt}{d\tau} = \gamma v^i \end{aligned}$$

Magnitude of four-velocity is given by

$$u^\mu u_\mu = \gamma^2 (c^2 - v^2) = \frac{c^2 - v^2}{1 - v^2/c^2} = c^2$$

Momentum Four-Vector

$$p^\mu = mu^\mu = \left(\frac{E}{c}, \mathbf{p} \right)$$



where $E = \gamma mc^2$ is relativistic energy and $\mathbf{p} = \gamma m\mathbf{v}$ is relativistic momentum.

Explicitly,

$$p^\mu = \left(\frac{E}{c}, p_x, p_y, p_z \right)$$

Furthermore,

$$p^\mu p_\mu = \frac{E^2}{c^2} - p^2 = m^2 c^2$$

7.6.6 Introduction to General Relativity

Motivation Classical Newtonian gravity describes gravitation as a force acting instantaneously at a distance. However, this framework is incompatible with special relativity, which asserts that

- The speed of light is finite and invariant.
- Space and time form a unified structure: spacetime.

General Relativity, formulated by Albert Einstein in 1915, resolves this incompatibility by replacing the concept of gravitational force with the *geometry of spacetime*.

Operators in General Relativity There are different operators in General Relativity. One of the examples is Christoffel symbols, which describe how coordinate axes change from point to point. In curved spacetime (or even flat space with curved coordinates), basis vectors are not constant, taking an ordinary derivative of a vector does not give a tensor and we must correct for the “bending” of coordinates.

Definition. 7.2: Christoffel Symbols

The Christoffel symbols are defined as

$$\Gamma_{\mu\nu}^\lambda = \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$$

where $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ is called the 4-gradient.

Another example is d'Alembertian operator.

Related Questions

Romanian Master of Physics 2014
Problem 3 (**Black Holes Physics**),
Romanian Master of Physics 2012
Problem 3 (**Fundaments of General Relativity**)



Definition. 7.3: d'Alembertian Operator

$$\square = \partial_\mu \partial^\mu = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2$$

A scalar field $\phi(x^\mu)$ satisfies the relativistic wave equation

$$\square \phi = 0$$

This equation can describe

- Electromagnetic waves
- Gravitational waves
- Massless scalar fields

For a scalar field of mass m , the equation becomes

$$(\square + m^2 c^2 / \hbar^2) \phi = 0$$

This is the relativistic generalization of the Schrödinger equation.

In general relativity, spacetime is curved, and partial derivatives must be replaced by *covariant derivatives*.

The d'Alembertian becomes the **Laplace–Beltrami operator**:

$$\square \phi = \nabla^\mu \nabla_\mu \phi$$

In coordinates, this can be written as

$$\square \phi = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \phi)$$

where $g = \det(g_{\mu\nu})$.

Introduction to Spacetime and Differentiable Manifold In classical physics, space and time are treated as separate entities:

- Space: a three-dimensional Euclidean arena
- Time: a universal parameter flowing identically for all observers



However, experiments involving high velocities and electromagnetic waves reveal that:

- The speed of light is invariant
- Measurements of time and length depend on the observer

These facts force a radical conclusion:

Space and time are not independent; they form a unified structure called spacetime.

In the curved spacetime, we model spacetime as a *differentiable manifold* M :

- A manifold is a set of points that locally resembles \mathbb{R}^n , allowing us to define coordinates in a neighborhood of each point.
- For spacetime, $n = 4$ and each point represents an event. An event is something that occurs at a specific place and time.
- Smooth functions and tensors can be defined on the manifold, enabling calculus on curved spacetime.

Example. A sphere in \mathbb{R}^n is a manifold.

Proof. Let

$$N = (0, \dots, 0, 1), \quad S = (0, \dots, 0, -1)$$

be the north and south poles of S^n . Define the stereographic projection¹ from the north pole:

$$\pi_N : S^n \setminus \{N\} \rightarrow \mathbb{R}^n$$

by

$$\pi_N(x_1, \dots, x_{n+1}) = \left(\frac{x_1}{1-x_{n+1}}, \dots, \frac{x_n}{1-x_{n+1}} \right)$$

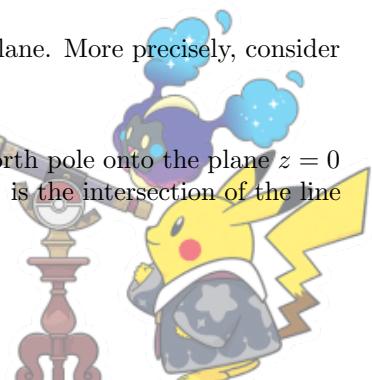
As the inverse map is given by

$$\pi_N^{-1}(y_1, \dots, y_n) = \left(\frac{2y_1}{\|y\|^2 + 1}, \dots, \frac{2y_n}{\|y\|^2 + 1}, \frac{\|y\|^2 - 1}{\|y\|^2 + 1} \right)$$

¹The **stereographic projection** is a mapping that projects points from a sphere onto a plane. More precisely, consider the unit sphere in \mathbb{R}^3 :

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

Let $N = (0, 0, 1)$ be the **north pole** of the sphere. The stereographic projection from the north pole onto the plane $z = 0$ is defined as follows: for any point $P = (x, y, z) \in S^2 \setminus \{N\}$, its projection $\pi(P) = (X, Y) \in \mathbb{R}^2$ is the intersection of the line through N and P with the plane $z = 0$.



and both π_N and π_N^{-1} are continuous, π_N is a homeomorphism². Similarly, define

$$\pi_S : S^n \setminus \{S\} \rightarrow \mathbb{R}^n$$

by

$$\pi_S(x_1, \dots, x_{n+1}) = \left(\frac{x_1}{1+x_{n+1}}, \dots, \frac{x_n}{1+x_{n+1}} \right)$$

The two charts

$$(S^n \setminus \{N\}, \pi_N), \quad (S^n \setminus \{S\}, \pi_S)$$

cover S^n . Each component is a rational function of y_1, \dots, y_n with denominator $\|y\|^2$, which is nonzero on the domain. Hence the transition map $\pi_S \circ \pi_N^{-1}$ is smooth. The stereographic charts provide a smooth atlas covering S^n , and the transition functions are smooth. Therefore, S^n satisfies all the requirements of a smooth manifold.

Metric Tensor On this 4-dimensional manifold M , the geometry is described by a metric tensor $g_{\mu\nu}(x)$, a smooth, symmetric, non-degenerate tensor field:

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$$

where the Einstein's notation uses Lorentzian signature $(-, +, +, +)$. Locally, at any point $p \in M$, there exists a coordinate system in which the metric reduces to the Minkowski form $\eta_{\mu\nu}$ and its first derivatives vanish, reflecting the *equivalence principle*. Free particles follow geodesics, which generalize straight lines:

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma_{\nu\rho}^\mu \frac{dx^\nu}{d\lambda} \frac{dx^\rho}{d\lambda} = 0$$

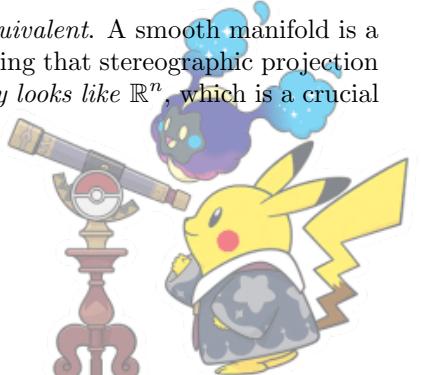
²Let X and Y be topological spaces. A function

$$f : X \rightarrow Y$$

is called a *homeomorphism* if

1. f is a *bijection* (one-to-one and onto),
2. f is *continuous*,
3. the inverse function $f^{-1} : Y \rightarrow X$ is also *continuous*.

If such a map exists, we say X and Y are *homeomorphic*, meaning they are *topologically equivalent*. A smooth manifold is a space that is locally *homeomorphic* to \mathbb{R}^n and has smooth transitions between charts. Showing that stereographic projection is a homeomorphism guarantees that each point of S^n has a neighborhood that *topologically looks like* \mathbb{R}^n , which is a crucial step in proving that S^n is a smooth manifold.



where $\Gamma_{\mu\nu}^\lambda$ are the Christoffel symbols computed from $g_{\mu\nu}$. In a flat space (Euclidean geometry), the shortest path between two points is a straight line. In a curved space (like the surface of the Earth), the shortest path is a geodesic. For timelike geodesics, the proper time τ is an example of an affine parameter λ . For null (lightlike) geodesics, there is no proper time, but an affine parameter still exists to describe the geodesic.

Riemann Curvature Tensor Curvature is measured by the Riemann tensor:

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma_{\nu\sigma}^\rho - \partial_\nu \Gamma_{\mu\sigma}^\rho + \Gamma_{\mu\lambda}^\rho \Gamma_{\nu\sigma}^\lambda - \Gamma_{\nu\lambda}^\rho \Gamma_{\mu\sigma}^\lambda$$

Contracting indices yields

$$R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$$

The Ricci scalar is

$$R = g^{\mu\nu} R_{\mu\nu}$$

Einstein's Field Equation The field equations can be written as

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

where

- $T_{\mu\nu}$ is the stress-energy tensor which describes the density and flux of energy and momentum.

In the weak-field, slow-motion limit:

$$g_{00} \approx - \left(1 + \frac{2\Phi}{c^2} \right)$$

Einstein's equations reduce to Poisson's equation:

$$\nabla^2 \Phi = 4\pi G \rho$$

Hence, Newtonian gravity emerges as an approximation of General Relativity.

Solution to Einstein's Field Equation **Schwarzschild solution** describes spacetime outside a spherical mass:

$$ds^2 = - \left(1 - \frac{2GM}{r} \right) c^2 dt^2 + \left(1 - \frac{2GM}{r} \right)^{-1} dr^2 + r^2 d\Omega^2$$



Friedmann–Lemaître–Robertson–Walker Metric models an expanding universe:

$$ds^2 = -c^2 dt^2 + a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right]$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$.

Define χ such that

$$d\chi = \frac{dr}{\sqrt{1 - kr^2}}$$

Integrating gives the relations

$$r = \begin{cases} \frac{1}{\sqrt{k}} \sin(\sqrt{k}\chi) & , k > 1 \\ \chi & , k = 0 \\ \frac{1}{\sqrt{-k}} \sinh(\sqrt{-k}\chi) & , k < -1 \end{cases} = \begin{cases} \sin(\chi) & , k = 1 \\ \chi & , k = 0 \\ \sinh(\chi) & , k = -1 \end{cases}$$

as $k = -1, 0, 1$ only. The metric then becomes

$$ds^2 = -c^2 dt^2 + a^2(t) [d\chi^2 + S_k^2(\chi) d\Omega^2]$$

By natural unit $c = 1$,

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{a^2}{1 - kr^2} & 0 & 0 \\ 0 & 0 & a^2 r^2 & 0 \\ 0 & 0 & 0 & a^2 r^2 \sin^2 \theta \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1 - kr^2}{a^2} & 0 & 0 \\ 0 & 0 & \frac{1}{a^2 r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{a^2 r^2 \sin^2 \theta} \end{pmatrix}$$



The non-zero Christoffel symbols are

$$\begin{aligned}\Gamma_{ij}^0 &= a\dot{a}\tilde{g}_{ij} = \frac{\dot{a}}{a}g_{ij} \\ \Gamma_{0j}^i &= \frac{\dot{a}}{a}\delta_j^i \\ \Gamma_{rr}^r &= \frac{kr}{1-kr^2}, \quad \Gamma_{\theta\theta}^r = -r(1-kr^2), \quad \Gamma_{\phi\phi}^r = -r(1-kr^2)\sin^2\theta \\ \Gamma_{r\theta}^\theta &= \Gamma_{r\phi}^\phi = \frac{1}{r} \\ \Gamma_{\phi\phi}^\theta &= -\sin\theta\cos\theta, \quad \Gamma_{\theta\phi}^\phi = \cot\theta\end{aligned}$$

$$\begin{aligned}R_{00} &= -3\frac{\ddot{a}}{a} \\ R_{ij} &= \left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{2k}{a^2} \right] g_{ij} \\ R &= g^{\mu\nu}R_{\mu\nu} = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} \right]\end{aligned}$$

For a perfect fluid in comoving coordinates,

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu}$$

with $U^\mu = (1, 0, 0, 0)$, so:

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & pg_{11} & 0 & 0 \\ 0 & 0 & pg_{22} & 0 \\ 0 & 0 & 0 & pg_{33} \end{pmatrix}$$



By $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi GT_{\mu\nu}$,

$$\begin{aligned} -3\frac{\ddot{a}}{a} - \frac{1}{2}(-1) \cdot 6 \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right] - \Lambda &= 8\pi G\rho \\ -3\frac{\ddot{a}}{a} + 3 \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right] - \Lambda &= 8\pi G\rho \\ 3 \left(\frac{\dot{a}}{a} \right)^2 + \frac{3k}{a^2} - \Lambda &= 8\pi G\rho \\ \left(\frac{\dot{a}}{a} \right)^2 &= \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \end{aligned}$$

This is the **First Friedmann Equation**.

$$\begin{aligned} \left[\frac{\ddot{a}}{a} + 2 \left(\frac{\dot{a}}{a} \right)^2 + \frac{2k}{a^2} \right] g_{ii} - \frac{1}{2}g_{ii} \cdot 6 \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right] + \Lambda g_{ii} &= 8\pi G p g_{ii} \\ \frac{\ddot{a}}{a} + 2 \left(\frac{\dot{a}}{a} \right)^2 + \frac{2k}{a^2} - 3 \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right] + \Lambda &= 8\pi G p \\ -2\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a} \right)^2 - \frac{k}{a^2} + \Lambda &= 8\pi G p \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \end{aligned}$$

This is the **Second Friedmann Equation**.

Covariant Derivative The **covariant derivative** ∇_μ generalizes the partial derivative ∂_μ to curved manifolds. The partial derivative fails to transform as a tensor under coordinate changes. The covariant derivative introduces correction terms, the **Christoffel symbols** $\Gamma^\lambda_{\mu\nu}$, to ensure tensorial behavior. For a contravariant vector field V^ν ,

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda.$$

Lie Derivative The Lie derivative \mathcal{L}_X measures how a tensor field changes along the flow of a vector field X . For a vector field Y along the flow of X ,

$$(\mathcal{L}_X Y)^\mu = X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu$$

In coordinate-free notation,

$$\mathcal{L}_X Y = [X, Y]$$



where $[X, Y]$ is the Lie bracket.

Killing Vector Field A vector field K^μ is a **Killing vector field** if its flow generates an **isometry**, which is a symmetry that preserves the spacetime metric. By vanishing the Lie derivative of the metric along K^μ , the condition is given by

$$\mathcal{L}_K g_{\mu\nu} = 0.$$

Using the Lie derivative formula for the metric, this becomes the **Killing equation**:

$$\boxed{\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0}$$

Theorem. 7.3: Conservation Law Along Geodesics

For a geodesic with tangent vector $u^\mu = \frac{dx^\mu}{d\lambda}$, if K^μ is a Killing vector, then the quantity $K_\mu u^\mu$ is constant along the geodesic.

Proof.

$$\frac{d}{d\lambda}(K_\mu u^\mu) = u^\nu \nabla_\nu(K_\mu u^\mu) = u^\nu u^\mu \nabla_\nu K_\mu + K_\mu u^\nu \nabla_\nu u^\mu$$

The second term vanishes by the geodesic equation $u^\nu \nabla_\nu u^\mu = 0$. The first term is symmetric in ν and μ , so we can write

$$u^\nu u^\mu \nabla_\nu K_\mu = \frac{1}{2} u^\nu u^\mu (\nabla_\nu K_\mu + \nabla_\mu K_\nu) = 0$$

by the Killing equation. Hence,

$$\boxed{\frac{d}{d\lambda}(K_\mu u^\mu) = 0}$$

Local Escape Speed Consider a static, spherically symmetric spacetime with metric

$$ds^2 = -g_{tt}(r) dt^2 + g_{rr}(r) dr^2 + r^2 d\Omega^2$$

where $g_{tt}(r) > 0$. As the metric is time-independent, it admits a timelike Killing vector

$$K^\mu = (\partial_t)^\mu = (1, 0, 0, 0)$$



Let a particle move along a geodesic with tangent vector

$$u^\mu = \frac{dx^\mu}{d\lambda}$$

If K^μ is a Killing vector, then the quantity

$$K_\mu u^\mu$$

is conserved along the geodesic. We define the conserved energy per unit mass

$$E \equiv -K_\mu u^\mu$$

Since $K^\mu = (1, 0, 0, 0)$, we have

$$K_\mu = g_{\mu\nu} K^\nu = (g_{tt}, 0, 0, 0)$$

and therefore

$$E = -g_{tt} u^t$$

For a massive particle, the four-velocity satisfies

$$u^\mu u_\mu = -1$$

For purely radial motion ($u^\theta = u^\phi = 0$), this gives

$$-g_{tt}(u^t)^2 + g_{rr}(u^r)^2 = -1$$

Substituting $u^t = E/g_{tt}$, we obtain

$$-g_{tt} \left(\frac{E}{g_{tt}} \right)^2 + g_{rr}(u^r)^2 = -1$$

which simplifies to

$$g_{rr}(u^r)^2 = \frac{E^2}{g_{tt}} - 1$$

A static observer at radius r has four-velocity

$$U^\mu = \frac{1}{\sqrt{g_{tt}}} (1, 0, 0, 0)$$



The locally measured speed v of the particle is defined by

$$v^2 = \frac{(\text{proper spatial distance})^2}{(\text{proper time})^2} = \frac{g_{rr}(u^r)^2}{g_{tt}(u^t)^2}$$

Using the previous results, this becomes

$$v^2 = \frac{\frac{E^2}{g_{tt}} - 1}{\frac{E^2}{g_{tt}}} = 1 - \frac{g_{tt}}{E^2}$$

For a particle that just escapes to infinity in an asymptotically flat spacetime,

$$g_{tt}(\infty) = 1 \quad u^r(\infty) = 0$$

The normalization condition then implies

$$E = 1$$

Substituting into the expression for v^2 , we obtain the local escape speed:

$$v_{\text{esc}}^2 = 1 - g_{tt}(r)$$

For the Schwarzschild metric,

$$g_{tt}(r) = 1 - \frac{2GM}{r}$$

Hence,

$$v_{\text{esc}}^2 = \frac{2GM}{r}$$

This is the locally measured escape speed by a static observer.



8 Thermodynamics

8.1 Pressure and Hydrostatic Equilibrium

Introduction Pressure is a scalar quantity defined as the force applied per unit area. Mathematically,

$$P = \frac{F}{A}$$

Hydrostatic Equilibrium: Plane-Parallel Case Consider a small cylindrical fluid element of cross-sectional area A and height dz .

- **Gravitational force (downward):** The mass of the element is $m = \rho A dz$. Hence,

$$F_g = mg = \rho g A dz$$

- **Pressure force (upward):** The pressure at the bottom is $P(z)$, while at the top it is $P(z + dz) = P + dP$. The net pressure force is therefore

$$F_{P,\text{net}} = P(z)A - P(z + dz)A = -AdP$$

For hydrostatic equilibrium, the total force must vanish:

$$\sum F = 0 \implies F_{P,\text{net}} - F_g = 0.$$

Hence,

$$-AdP - \rho g A dz = 0 \quad \boxed{\frac{dP}{dz} = -\rho g}$$

Hydrostatic Equilibrium: Spherical Symmetry (Stars) Now consider a thin spherical shell inside a star at radius r with thickness dr .

- **Gravitational force (inward):** The mass of the shell is

$$dm = \rho(r) dV = \rho(r) 4\pi r^2 dr$$



The gravitational acceleration at radius r is

$$g(r) = \frac{GM(r)}{r^2}$$

where $M(r)$ is the mass enclosed within radius r . Hence, the inward gravitational force is

$$F_g = dm g(r) = 4\pi G \rho(r) M(r) dr$$

- **Pressure force (outward):** The surface area of the shell is $A = 4\pi r^2$. The net pressure force is the difference between the outward pressure at r and that at $r + dr$:

$$F_{P,\text{net}} = P(r)A - P(r + dr)A = -\frac{dP}{dr} \cdot 4\pi r^2 dr$$

Hydrostatic equilibrium requires

$$\sum F = 0 \implies F_{P,\text{net}} - F_g = 0$$

Therefore,

$$-\frac{dP}{dr} 4\pi r^2 dr - 4\pi G \rho(r) M(r) dr = 0$$

which yields the stellar hydrostatic equilibrium equation:

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

Remark In addition to hydrostatic equilibrium, stars also satisfy *thermal equilibrium*, meaning that the energy generated in the core equals the energy radiated from the surface:

$$L_{\text{core}} = L_{\text{surface}}$$

where L denotes the luminosity. Stellar equilibrium refers to the combined balance of gravitational, pressure, and energy-transport processes that allow a star to maintain a stable structure over long timescales.



8.2 Ideal Gas Law

The Ideal Gas Law is a fundamental equation in thermodynamics that describes the behavior of an ideal gas. It establishes a relationship between the pressure P , volume V , temperature T , and the number of moles of gas n . The equation is expressed as

$$PV = nRT$$

where

- P is the pressure of the gas,
- V is the volume of the gas,
- n is the number of moles of gas,
- R is the universal gas constant ($R = 8.314 \text{ J/mol}\cdot\text{K}$),
- T is the temperature of the gas in Kelvin.

For a gas undergoing a change in volume at constant pressure, the work done by the gas is given by

$$W = P\Delta V$$

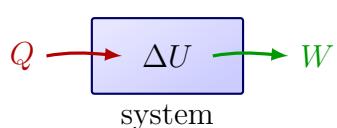
8.3 First Law of Thermodynamics

It states that energy cannot be created or destroyed, only transformed. It is expressed mathematically as

$$dU = \delta Q - \delta W$$

where

- (1) dU is the change in internal energy of the system,
- (2) δQ is the heat added to the system,
- (3) δW is the work done by the system.



Process	Condition	Description
Adiabatic	$\delta Q = 0$	No heat exchange with surroundings; temperature changes due to work done
Isothermal	$T = \text{constant}$	Temperature remains constant; heat absorbed equals work done
Isobaric	$P = \text{constant}$	Pressure remains constant; volume changes with temperature
Isochoric (Isovolumetric)	$V = \text{constant}$	No work done since volume is constant; heat changes internal energy
Polytropic	$PV^n = \text{constant}$	Generalized process covering isothermal, adiabatic, and isobaric cases
Cyclic	Final state = Initial state	Net change in internal energy over one cycle is zero
Reversible	Quasi-static with no losses	Idealized process that delivers maximum possible work
Irreversible	Finite gradients and friction	Real processes; entropy production is positive

Table 1: Thermodynamic processes, their conditions, and descriptions

8.4 Second Law of Thermodynamics

Definition. 8.1: Entropy

Entropy is a measure of the disorder or randomness of a system. For a reversible process, the change in entropy dS is given by the heat transfer dQ divided by the temperature T :

$$dS = \frac{dQ_{\text{rev}}}{T}$$

Theorem. 8.1: Sackur–Tetrode Equation

$$S = Nk_B \left(\ln \left(\frac{V}{N} \left(\frac{4\pi m E}{3h^2} \right)^{3/2} \right) + \frac{5}{2} \right)$$

where



- S is the entropy,
- N is the number of particles,
- V is the volume,
- m is the mass of a gas particle,
- E is the internal energy,

Theorem. 8.2: Second Law of Thermodynamics

The second law of thermodynamics states that the entropy of an isolated system tends to increase over time.

$$dS \geq 0$$

Theorem. 8.3: Gibbs Free Energy

The Gibbs free energy G is related to entropy through the following equation:

$$G = H - TS = U + PV - TS$$

where H is the enthalpy, T is the temperature (kept constant), P is the pressure (kept constant), and S is the entropy.

8.5 Heat Capacity

Definition. 8.2: Heat Capacity

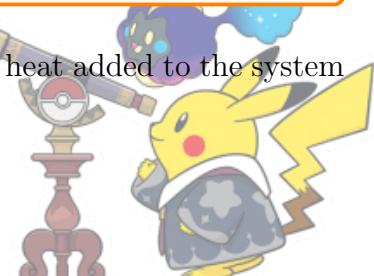
The heat capacity at constant volume, C_V , is defined as the amount of heat required to raise the temperature of a system by one degree Celsius (or one Kelvin) while maintaining constant volume.

Mathematically, it is expressed as

$$C_V = \left(\frac{\partial Q}{\partial T} \right)_V$$

where Q is the heat added to the system, and T is the temperature. The subscript V indicates that the volume is held constant.

Since the system is not allowed to do any work (because the volume is fixed), the heat added to the system



only changes the internal energy:

$$dQ = dU$$

where dU is the change in internal energy.

For an ideal gas, the heat capacity at constant volume is related to the molar heat capacity C_V by

$$C_V = \left(\frac{\partial U}{\partial T} \right)_V$$

For an ideal gas, the internal energy U is a function of temperature alone, and the specific heat capacity can be derived from the equation of state.

Definition. 8.3: Heat Capacity

The heat capacity at constant pressure, C_P , is defined as the amount of heat required to raise the temperature of a system by one degree Celsius (or one Kelvin) while maintaining constant pressure.

Mathematically, it is expressed as

$$C_P = \left(\frac{\partial Q}{\partial T} \right)_P$$

where Q is the heat added to the system, and T is the temperature. The subscript P indicates that the pressure is held constant.

In contrast to constant volume, when heat is added at constant pressure, the system may do work as it expands. Therefore, the total heat added is the sum of the change in internal energy and the work done by the system

$$dQ = dU + PdV$$

For an ideal gas, the heat capacity at constant pressure is related to the molar heat capacity C_P by

$$C_P = \left(\frac{\partial H}{\partial T} \right)_P$$

Theorem. 8.4: Mayer's Relation

$$c_P - c_V = nR$$

where n is the number of moles of the gas and R is the universal gas constant.



Proof. By the first law of thermodynamics,

$$dU = \delta Q - pdV = nC_VdT$$

where C_V is the heat capacity at constant volume, and n is the number of moles. For the heat added at constant pressure, we use:

$$\delta Q = nC_PdT$$

where C_P is the heat capacity at constant pressure.

By the ideal gas equation of state,

$$pV = nRT$$

Differentiating this equation with respect to temperature at constant pressure:

$$pdV + Vdp = nRdT$$

At constant pressure, $dp = 0$, so we have:

$$pdV = nRdT$$

Substitute $pdV = nRdT$ into the equation:

$$nC_VdT = nC_PdT - nRdT$$

$$C_V = C_P - R \implies \boxed{C_P - C_V = nR}$$

Definition. 8.4: Heat Capacity Ratio

$$\gamma = \frac{C_p}{C_V}$$

In an adiabatic process, there is no heat exchange with the surroundings ($dQ = 0$). The first law of thermodynamics gives:

$$dU = -PdV$$

For an ideal gas, the relationship between pressure and volume during an adiabatic process is governed by the equation:

$$PV^\gamma = \text{constant}$$



8.6 Blackhole Thermodynamics

Black hole entropy is a measure of the amount of information that is hidden inside a black hole. The famous Bekenstein-Hawking entropy formula links the entropy of a black hole to the area of its event horizon, rather than its volume. This result was first derived by Jacob Bekenstein and later confirmed by Stephen Hawking:

$$S = \frac{k_B c^3 A}{4G\hbar}$$

Proof. The first law of black hole thermodynamics is given by

$$dM = TdS + \Phi dQ + \Omega dJ$$

where

- M is the mass,
- T is the temperature,
- S is the entropy,
- Φ is the electrostatic potential,
- Q is the charge, and
- J is the angular momentum.

For a non-rotating, uncharged Schwarzschild black hole, the first law reduces to

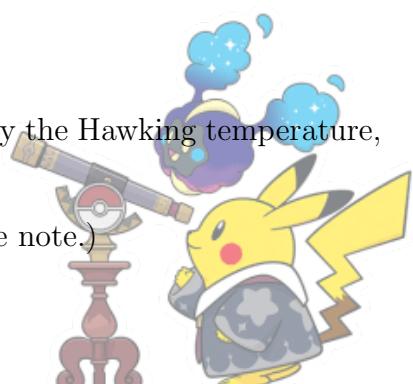
$$dM = TdS$$

which is similar to the first law of thermodynamics, and implies that the black hole mass is related to its entropy via its temperature. The area of the event horizon is

$$A = 16\pi \left(\frac{GM}{c^2} \right)^2$$

Using the fact that the temperature T of a Schwarzschild black hole is given by the Hawking temperature,

$$T = \frac{\hbar c^3}{8\pi GM} \quad (\text{The derivation is out of scope of this lecture note.})$$



The first law $dM = TdS$ implies

$$dS = \frac{dM}{T} = \frac{dM}{\frac{\hbar c^3}{8\pi GM}} = \frac{8\pi GM}{\hbar c^3} dM$$

Integrating with respect to M ,

$$S = \frac{8\pi GM^2}{\hbar c^3}$$

Using the expression for the area $A = 16\pi \left(\frac{GM}{c^2}\right)^2$,

$$M = \frac{c^2 \sqrt{A}}{4\pi G}$$

Substituting this into the entropy expression, we get

$$S = \frac{k_B c^3 A}{4G\hbar}$$

8.7 Kinetic Theory

8.7.1 Mean Free Path

Let the number density of the gas (the number of particles per unit volume) be n , and let the cross-sectional area for a collision between two particles be σ . The cross-sectional area σ depends on the type of collision and the physical properties of the particles involved. If we assume spherical particles with radius r , the collision cross-section is given by

$$\sigma = \pi(2r)^2 = 4\pi r^2$$

Next, the relative velocity between two particles in the gas is on the order of the mean speed of the particles v_{avg} . The total rate of collisions per unit time per particle is

$$\text{Collision rate} = n\sigma v_{\text{avg}}$$



The mean free path λ is defined as the average distance a particle travels before undergoing a collision. The relationship between the mean free path and the collision rate is given by

$$\lambda = \frac{1}{\text{Collision rate}} = \frac{1}{n\sigma v_{\text{avg}}} = \frac{1}{n \cdot 4\pi r^2 \cdot v_{\text{avg}}}$$

8.7.2 Boltzmann Distribution and Maxwell–Boltzmann Distribution

The Boltzmann distribution, also known as the Gibbs distribution, is a fundamental probability distribution in statistical mechanics that describes the statistical properties of a system in thermal equilibrium at a fixed temperature. It provides the probability that a system will be in a particular microstate with energy E_i when it is in contact with a heat bath at temperature T .

Formulation and Derivation Consider a system in thermal equilibrium with a large reservoir at constant temperature T . The probability P_i of finding the system in a particular microstate i with energy E_i is given by

$$P_i = \frac{1}{Z} e^{-\beta E_i}$$

where

- $\beta = \frac{1}{k_B T}$ is the thermodynamic beta
- $Z = \sum_i e^{-\beta E_i}$ is the partition function

The partition function Z serves as a normalization constant ensuring that $\sum_i P_i = 1$. It contains all thermodynamic information about the system.

Proof. Statistical mechanics often uses the principle of maximum entropy. The entropy of a discrete set of probabilities $\{P_i\}$ is

$$S = -k_B \sum_i P_i \ln P_i$$

as $S = k_B \ln \Omega$ (where Ω is the number of micro-states by Statistical Mechanics). We want to maximize S subject to the constraints:

$$\sum_i P_i = 1 \quad (\text{normalization})$$

$$\sum_i P_i E_i = \langle E \rangle \quad (\text{average energy fixed})$$



Using Lagrange multipliers α and β , define

$$\mathcal{L} = -k_B \sum_i P_i \ln P_i - \alpha \left(\sum_i P_i - 1 \right) - \beta \left(\sum_i P_i E_i - \langle E \rangle \right)$$

Setting the derivative with respect to P_i to zero:

$$\frac{\partial \mathcal{L}}{\partial P_i} = -k_B (\ln P_i + 1) - \alpha - \beta E_i = 0$$

which gives

$$P_i = e^{-1-\alpha/k_B} e^{-\beta E_i/k_B}$$

Defining $Z = e^{1+\alpha/k_B} = \sum_i e^{-\beta E_i/k_B}$, we get the familiar form:

$$P_i = \frac{e^{-\beta E_i}}{Z}$$

Maxwell-Boltzmann Distribution The Maxwell–Boltzmann distribution describes the statistical distribution of speeds (or velocities) of particles in an ideal gas at thermal equilibrium. It is a special case of the Boltzmann distribution applied to the translational kinetic energy of non-interacting particles.

For an ideal gas of N identical particles of mass m at temperature T , the probability density function for finding a particle with velocity $\mathbf{v} = (v_x, v_y, v_z)$ is

$$f(\mathbf{v}) = \left(\frac{m}{2\pi k_B T} \right)^{3/2} \exp \left(-\frac{m|\mathbf{v}|^2}{2k_B T} \right)$$

This three-dimensional distribution factorizes as $f(\mathbf{v}) = f(v_x)f(v_y)f(v_z)$, where each component distribution is Gaussian

$$f(v_\alpha) = \sqrt{\frac{m}{2\pi k_B T}} \exp \left(-\frac{mv_\alpha^2}{2k_B T} \right), \quad \alpha = x, y, z$$

More commonly used is the distribution of speeds $v = |\mathbf{v}|$, obtained by integrating over all directions in velocity space:

$$f(v) = 4\pi v^2 \left(\frac{m}{2\pi k_B T} \right)^{3/2} \exp \left(-\frac{mv^2}{2k_B T} \right)$$

The factor $4\pi v^2$ arises from the spherical shell volume element in velocity space.

Proof. Consider an ideal gas with N particles in thermal equilibrium at temperature T . Each particle has a mass m and a velocity vector $\mathbf{v} = (v_x, v_y, v_z)$. The total energy of a particle is purely kinetic:



$$E = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2).$$

According to the Boltzmann distribution, the probability of a particle having energy E is

$$P(E) \propto e^{-E/k_B T}$$

Note that

$$P(v_x, v_y, v_z) = f(v_x)f(v_y)f(v_z)$$

$$f(v_i) = A e^{-\frac{1}{2}mv_i^2/k_B T}, \quad i = x, y, z$$

As

$$\int_{-\infty}^{\infty} f(v_x) dv_x = 1, \quad \int_{-\infty}^{\infty} A e^{-\frac{mv_x^2}{2k_B T}} dv_x = 1$$

and by

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

we have

$$A \sqrt{\frac{2\pi k_B T}{m}} = 1 \implies A = \sqrt{\frac{m}{2\pi k_B T}}.$$

Hence,

$$f(v_i) = \sqrt{\frac{m}{2\pi k_B T}} e^{-\frac{mv_i^2}{2k_B T}}$$

The speed of a particle is

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2}.$$

The probability density function of speeds, $F(v)$, can be obtained by transforming to spherical coordinates in velocity space:

$$F(v)dv = 4\pi v^2 f(v_x)f(v_y)f(v_z)dv.$$

Hence,

$$F(v) = 4\pi \left(\frac{m}{2\pi k_B T} \right)^{3/2} v^2 e^{-\frac{mv^2}{2k_B T}}$$

Three important characteristic speeds are derived from this distribution:



1. **Most probable speed** (mode):

$$v_{\text{mp}} = \sqrt{\frac{2k_B T}{m}} = \sqrt{\frac{2RT}{M}}$$

2. **Mean speed:**

$$\langle v \rangle = \sqrt{\frac{8k_B T}{\pi m}} = \sqrt{\frac{8RT}{\pi M}}$$

3. **Root-mean-square speed:**

$$v_{\text{rms}} = \sqrt{\frac{3k_B T}{m}} = \sqrt{\frac{3RT}{M}}$$

where $R = N_A k_B$ is the gas constant and $M = N_A m$ is the molar mass.

8.8 Boundary Conditions

Solving the stellar structure equations requires proper boundary conditions:

- **At the Center ($r = 0$):**

$$M(0) = 0, \quad L(0) = 0$$

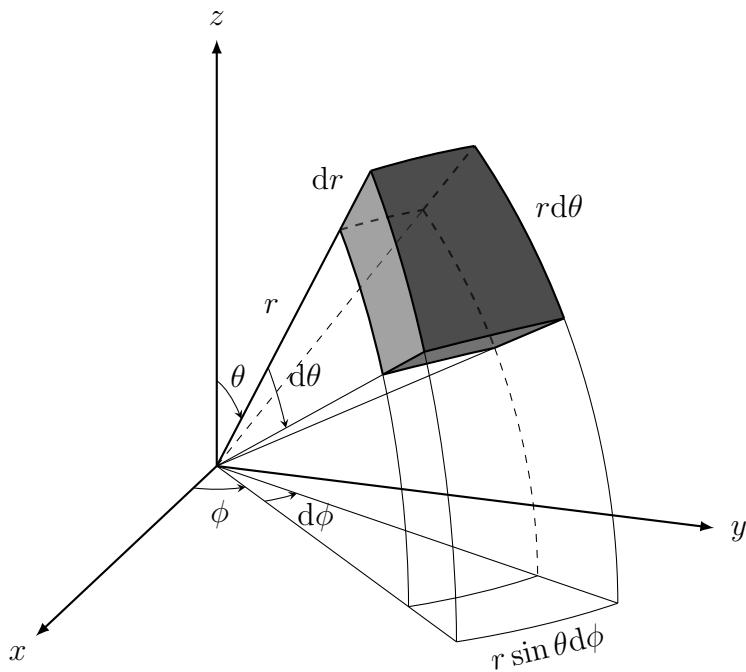
- **At the Surface ($r = R$):**

$$P(R) = P_{\text{surface}} \approx 0, \quad T(R) = T_{\text{eff}}$$

These conditions ensure the solution matches physical reality: zero mass at the center, and finite temperature and pressure at the surface.



8.9 Case Study



Suppose a static spherical star consists of N neutral particles with radius R with $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, satisfying the following equation of states

$$PV = Nk \frac{T_R - T_0}{\ln(T_R/T_0)} \quad (1)$$

where P and V are the pressure inside the star and the volume of the star respectively, k is the Boltzmann constant, T_R and T_0 are the temperatures at the surface $r = R$ and the temperature at the center $r = 0$ respectively. Assume that $T_R \leq T_0$.

- (a) Simplify the stellar equation of state (1) if $\Delta T = T_R - T_0 \rightarrow 0$ (this is called ideal star) (Hint: Use the approximation $\ln(1 + x) \approx x$ for small x).

Suppose the star undergoes a quasi-static process, in which it may slightly contract or expand, such that the above stellar equation of state (1) still holds.

The star satisfies the first law of thermodynamics:

$$Q = \Delta M c^2 + W \quad (2)$$

where Q , M , and W are heat, mass of the star, and work respectively, while c is the light speed in the vacuum and $\Delta M = M_{\text{final}} - M_{\text{initial}}$.



In the following, we assume T_0 to be constant, while $T_R \equiv T$ varies.

- (b) Find the heat capacity of the star at constant volume C_V and at constant pressure C_P , expressed in C_P and C_V (Hint: Use the approximation $1 + x^n \approx 1 + nx$ for small x).

Assuming that C_P is constant and the gas undergoes the isobaric process so the star produces the heat and radiates it outside to the space.

- (c) Find the heat produced by the isobaric process if the initial temperature and the final temperature are T_i and T_f , respectively.
- (d) For the next parts, assume the star is the Sun.
- (e) If the sunlight is monochromatic with frequency 5×10^{14} Hz, estimate the number of photons radiated by the Sun per second.
- (f) Calculate the heat capacity C_P of the Sun assuming its surface temperature varies from 5500 K to 6000 K in one second.

Solution.

- (a) Defining $\Delta T = T_f - T_0$ and $\Delta T \approx 0$, we have

$$\frac{P\Delta V}{Nk} = \frac{\Delta T}{\ln(1 + \Delta T/T_0)}$$

Using $\ln(1 + \Delta T/T_0) \approx \Delta T/T_0$, we then obtain

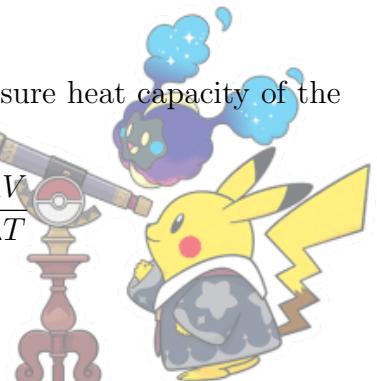
$$\frac{P\Delta V}{Nk} = T_0$$

- (b) The internal energy of the star is $U = Mc^2$ ($U(T) = M(T)c^2$ for ideal star). Hence, the constant volume heat capacity of the star has the form:

$$C_V = \left(\frac{\Delta Q}{\Delta T} \right)_V = \left(\frac{\Delta M}{\Delta T} \right)_V c^2$$

for small ΔT . Then, using first law of thermodynamics, the constant pressure heat capacity of the star is

$$C_P = \left(\frac{\Delta Q}{\Delta T} \right)_P = \left(\frac{\Delta M}{\Delta T} \right)_V c^2 + P \frac{\Delta V}{\Delta T} = C_V + P \frac{\Delta V}{\Delta T}$$



for small ΔT . Defining $\Delta T = T_2 - T_1$, then

$$\frac{P\Delta V}{Nk} = \frac{T_1 - T_0 + \Delta T}{\ln((T_1 + \Delta T)/T_0)} - \frac{T_1 - T_0}{\ln(T_1/T_0)}$$

Using the approximation

$$\ln((T_1 + \Delta T)/T_0) \approx \ln\left(\frac{T_1}{T_0}\right) + \frac{\Delta T}{T_1}$$

$$\frac{1}{\ln((T_1 + \Delta T)/T_0)} \approx \frac{1}{\ln(T_1/T_0)} - \frac{1}{T_1 \ln(T_1/T_0)^2} \frac{\Delta T}{T_1}$$

then we have

$$\frac{P\Delta V}{\Delta T} \approx Nk \left(1 - \frac{1}{\ln(T/T_0)}\right) \frac{(T - T_0)/T}{\ln(T/T_0)}$$

where $T_1 = T$. Finally, we obtain

$$C_P = \left(\frac{\Delta Q}{\Delta T}\right)_P = \left(\frac{\Delta M}{\Delta T}\right)_V c^2 + P \frac{\Delta V}{\Delta T} = C_V + \frac{Nk}{\ln(T/T_0)} \left(1 - \frac{(T - T_0)/T}{\ln(T/T_0)}\right)$$

(c) Since C_V is constant, the heat produced by the star is given by

$$Q_H = C_V(T_f - T_i) + P\Delta V$$

$$Q_H = C_V(T_f - T_i) + Nk \left[\frac{T_f - T_0}{\ln(T_f/T_0)} - \frac{T_i - T_0}{\ln(T_i/T_0)} \right]$$

(d) Energy per second radiated by the Sun $\dot{E} = L_\odot = N h\nu$ where N is the number of photons. Hence

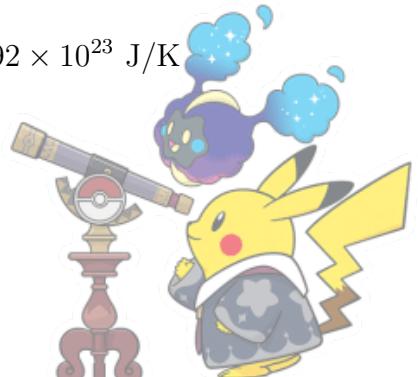
$$N = \frac{L_\odot}{h\nu} = \frac{3.90 \times 10^{26}}{6.626 \times 10^{-34} \times 5 \times 10^{14}} = 1.195 \times 10^{45} \text{ photons}$$

(e) Energy per second radiated by the Sun is proportional to mass defect of the Sun

$$L_\odot = \frac{\Delta M c^2}{\Delta t}$$

Hence,

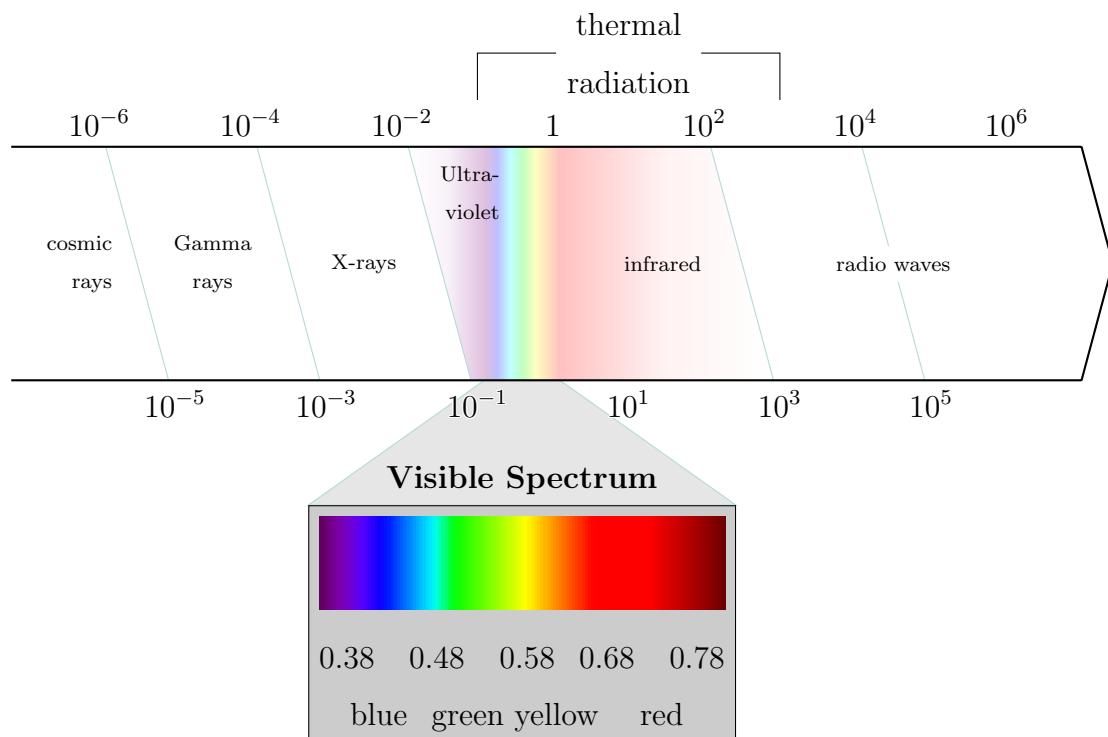
$$C_V = \frac{\Delta M c^2}{L_\odot} = \frac{3.96 \times 10^{26}}{\Delta T} \frac{1}{\frac{L_\odot}{\Delta t}} \approx \frac{3.96 \times 10^{26}}{6000 - 5500} \text{ J/K} = 7.92 \times 10^{23} \text{ J/K}$$



9 Spectroscopy

9.1 Basic Concepts

- **(Spectrum)** The diffraction of light produces a spectrum, which can be observed as a series of bright and dark fringes.



- **(Absorption)** Absorption occurs when atoms or molecules in a celestial object absorb photons of specific energies, raising electrons to higher energy levels.
- **(Emission)** Emission occurs when excited electrons drop to lower energy levels, releasing photons of specific energies, forming **emission lines**.

$$E_{\text{photon}} = E_{\text{upper}} - E_{\text{lower}}$$

This leads to **absorption lines** in the spectrum.

- (4) Spectral lines reveal chemical composition, temperature, pressure, and velocity fields (via Doppler shifts).
- (5) **(Scattering)** Scattering occurs when photons interact with particles, changing direction and sometimes energy:



- **Rayleigh scattering:** Elastic scattering by particles much smaller than wavelength.
- **Thomson scattering:** Elastic scattering by free electrons.
- **Compton scattering:** Inelastic scattering, photon loses energy. There is a relation derived by Compton:

$$\lambda' - \lambda = \frac{h}{mc}(1 - \cos \theta).$$

(6) (**Splitting**) Splitting occurs when a single spectral line divides into multiple components due to external or internal interactions. The Zeeman effect is the splitting of spectral lines in the presence of a magnetic field \mathbf{B} . For transitions with no spin, a single spectral line splits into three components:

$$\Delta E = m_l \mu_B B, \quad m_l = 0, \pm 1$$

where μ_B is the Bohr magneton, B is the magnetic field strength, and m_l is the magnetic quantum number.

The Stark effect is the splitting of spectral lines due to an external electric field \mathbf{E} :

$$\Delta E \propto E$$

(7) (**Broadening**) Broadening refers to the widening of spectral lines beyond their natural linewidth.

- Due to the finite lifetime τ of excited states, spectral lines have an intrinsic width governed by the uncertainty principle.
- Stellar rotation causes line broadening due to different Doppler shifts across the stellar disk.

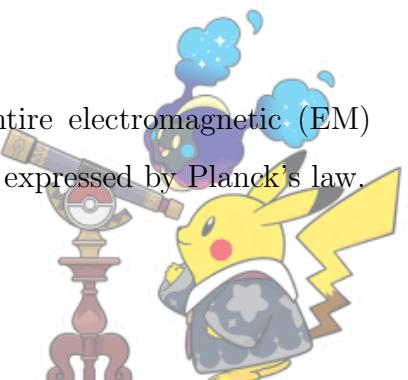
9.2 Spectral Radiance

The spectral radiance (power emitted per unit area per unit solid angle per unit wavelength) is given by Planck's law:

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/(\lambda k_B T)} - 1}$$

where λ is the wavelength.

Astronomy has evolved from visible-light observations to encompass the entire electromagnetic (EM) spectrum. Each wavelength band reveals unique astrophysical phenomena, as expressed by Planck's law.



Band	Wavelength Range	Energy Range	Temperature (K)	Primary Sources
Radio	> 10 cm	$< 1.24 \times 10^{-5}$ eV	< 0.1	Cold gas, pulsars
Microwave	1 mm – 10 cm	1.24×10^{-5} eV – 1.24×10^{-3} eV	0.1 – 10	CMB, molecular clouds
Infrared	700 nm – 1 mm	1.24×10^{-3} eV – 1.77 eV	$10 - 10^4$	Dust, planets, cool stars
Visible	400 nm – 700 nm	1.77 eV – 3.1 eV	10^4	Stars, galaxies, nebulae
Ultraviolet	10 nm – 400 nm	3.1 eV – 124 eV	$10^4 - 10^5$	Hot stars, quasars
X-ray	0.01 nm – 10 nm	124 eV – 124 keV	$10^5 - 10^8$	Black holes, supernovae
Gamma-ray	< 0.01 nm	> 124 keV	$> 10^8$	GRBs, nuclear processes

Figure 7: Electromagnetic spectrum bands in astronomy

Limiting cases:

- **Rayleigh-Jeans Law (long wavelengths, $hc \ll \lambda k_B T$):**

$$B_\lambda(T) \approx \frac{2ck_B T}{\lambda^4}$$

- **Wien's Law (short wavelengths, $hc \gg \lambda k_B T$):**

$$B_\lambda(T) \approx \frac{2hc^2}{\lambda^5} e^{-hc/(\lambda k_B T)}$$

The wavelength at which the emission is maximum:

$$\lambda_{\max} T \approx 2.898 \times 10^{-3} \text{ m}\cdot\text{K} \text{ (Wien's Displacement Law)}$$

9.3 Stefan–Boltzmann law

$$j^* = \sigma T^4$$

where



- j^* is the total radiative flux (power per unit area, (W/m^2)),
- T is the absolute temperature of the blackbody in Kelvin (K),
- σ is the Stefan–Boltzmann constant:

$$\sigma = 5.670374419 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$$

For a real object with emissivity (ϵ) ($0 \leq \epsilon \leq 1$), the law generalizes to:

$$j = \epsilon \sigma T^4$$

Proof. The total radiated power per unit area is obtained by integrating over all frequencies and solid angles from Planck's law:

$$\begin{aligned} j^* &= \int_0^\infty \int_{\Omega} B(\nu, T) \cos \theta d\Omega d\nu \\ &= \pi \int_0^\infty B(\nu, T) d\nu \end{aligned}$$

Let $x = \frac{h\nu}{k_B T}$.

$$\begin{aligned} j^* &= \frac{2\pi h}{c^2} \int_0^\infty \frac{\nu^3}{e^{h\nu/k_B T} - 1} d\nu \\ &= \frac{2\pi (k_B T)^4}{h^3 c^2} \int_0^\infty \frac{x^3}{e^x - 1} dx \end{aligned}$$

The integral evaluates to $\frac{\pi^4}{15}$, giving

$$j^* = \sigma T^4, \quad \sigma = \frac{2\pi^5 k_B^4}{15 h^3 c^2}$$

9.4 Doppler's Effect

The **Doppler effect** is a fundamental phenomenon in wave physics where the observed frequency of a wave changes due to relative motion between the source and observer.

For non-relativistic speeds ($v \ll c$), where c is the speed of light:

$$f_{\text{obs}} = \frac{f_{\text{src}}}{1 \pm \frac{v_s}{c}} \quad (\text{Source Moving, Observer Stationary})$$



where + for receding source, – for approaching source.

$$f_{\text{obs}} = f_{\text{src}} \left(1 \pm \frac{v_o}{c} \right) \text{ (Observer Moving, Source Stationary)}$$

In astrophysics, we typically measure the **radial velocity** v_r through the wavelength shift:

$$\frac{\Delta\lambda}{\lambda_0} = \frac{\lambda_{\text{obs}} - \lambda_0}{\lambda_0} = \frac{v_r}{c}$$

where

- λ_0 represents the rest wavelength,
- λ_{obs} represents the observed wavelength, and
- v_r represents the radial velocity (positive for recession)

For astronomical objects moving at significant fractions of light speed, we must use the relativistic formula:

$$\frac{\lambda_{\text{obs}}}{\lambda_0} = \sqrt{\frac{1+\beta}{1-\beta}} \quad \text{or} \quad \frac{f_{\text{obs}}}{f_0} = \sqrt{\frac{1-\beta}{1+\beta}}$$

The redshift z is defined as

$$z = \frac{\lambda_{\text{obs}} - \lambda_0}{\lambda_0} = \frac{\Delta\lambda}{\lambda_0}$$

For relativistic speeds,

$$1+z = \sqrt{\frac{1+\beta}{1-\beta}}$$



10 Electromagnetism

10.1 Lorentz's Force

The Lorentz force law describes the force exerted on a charged particle moving through an electromagnetic field. It is given by

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

- \mathbf{F} is the force on the charged particle,
- q is the charge of the particle,
- \mathbf{E} is the electric field,
- \mathbf{v} is the velocity of the particle, and
- \mathbf{B} is the magnetic field.

10.2 Maxwell's Equation

Theorem. 10.1: Maxwell's Equation

For free charge density ρ (S.I. unit: C/m³) and free current density \mathbf{J} (S.I. unit: A/m²),

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (\text{Gauss's Law for Electricity})$$

$$\nabla \cdot \mathbf{B} = 0 \quad (\text{Gauss's Law for Magnetism})$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (\text{Faraday's Law of Induction})$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad (\text{Ampère-Maxwell Law})$$

For continuously differentiable vector field $\mathbf{F} = F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k}$, define $\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \text{ and } \nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \mathbf{k}$$

which $\nabla \cdot \mathbf{F}$ measures the net flow of the field out of an infinitesimally small volume around a point.



10.3 Poynting Vector

The Poynting vector describes the directional energy flux (the energy transfer per unit area per unit time) or power flow of an electromagnetic field. It is given by

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$$

where

- \mathbf{S} is the Poynting vector,
- \mathbf{E} is the electric field,
- \mathbf{B} is the magnetic field,

10.4 Optics

10.4.1 Wavefunction

The displacement of a point on a string in simple harmonic motion can be modeled by a sinusoidal function. The solution to the wave equation for a string under tension is:

$$\phi(x, t) = A \sin(kx - \omega t + \delta)$$

where

- A is the amplitude of the wave,
- $k = \frac{2\pi}{\lambda}$ is the wave number (with λ being the wavelength),
- $\omega = 2\pi f$ is the angular frequency (with f being the frequency), and
- δ is a phase constant.



10.4.2 Lens

Theorem. 10.2: Refractive Index

The *refractive index* of a medium is defined as the ratio of the speed of light in vacuum to the phase velocity of light in the medium:

$$n \equiv \frac{c}{v}.$$

Without free charge, Ampère's law states that

$$\nabla \times \mathbf{B} = \mu\epsilon \frac{\partial \mathbf{E}}{\partial t}$$

Take the curl of Faraday's law:

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B})$$

Substituting the expression for $\nabla \times \mathbf{B}$ yields

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Using the vector identity

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$$

and Gauss's law, we obtain

$$-\nabla^2 \mathbf{E} = -\mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\nabla^2 \mathbf{E} = \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

The standard wave equation for a field \mathbf{E} propagating with speed v is

$$\nabla^2 \mathbf{E} = \frac{1}{v^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Comparing with the equation above, we identify

$$\frac{1}{v^2} = \mu\epsilon$$



Therefore, the speed of electromagnetic waves in the medium is

$$v = \frac{1}{\sqrt{\mu\varepsilon}}$$

For speed of light, $c = \frac{1}{\sqrt{\epsilon_0\mu_0}}$. Then $n = \sqrt{\epsilon_r\mu_r}$ where $\epsilon_r = \frac{\epsilon}{\epsilon_0}$ is the relative permittivity (dielectric constant) of the medium and $\mu_r = \frac{\mu}{\mu_0}$ is the relative permeability of the medium. For $\mu_r \approx 1$, $n \approx \sqrt{\epsilon_r}$.

Theorem. 10.3: Snell's Law

Consider an interface between two homogeneous, isotropic media with refractive indices n_1 and n_2 . Let θ_1 and θ_2 denote the angles that the incident and refracted rays make with the normal to the interface, respectively. Snell's law states that

$$n_1 \sin \theta_1 = n_2 \sin \theta_2.$$

Theorem. 10.4: Lens Formula

$$\frac{1}{f} = \frac{1}{v} + \frac{1}{u}$$

where

- f is the focal length of the lens, which is the distance from the optical element (lens or mirror) to the point where light converges to form an image and determines the magnification and field of view of the telescope,
- u is the object distance (distance from the object to the lens), and
- v is the image distance (distance from the lens to the image).

The most widely used convention is the **Cartesian sign convention**:



Parameter	Sign Convention
Object distance (u)	Negative (for real objects)
Image distance (v)	Positive (for real images) Negative (for virtual images)
Focal length (f)	Positive (for convex/ converging lenses) Negative (for concave/ diverging lenses)
Height of object (h_o)	Positive (upward from principal axis)
Height of image (h_i)	Positive (upward from principal axis) Negative (downward from principal axis)

Table 2: Sign conventions for lens formula

Definition. 10.1: Optical Power

For a lens with focal length f , the power is given by

$$P = \frac{1}{f}$$

Theorem. 10.5: Combination of Thin Lens

For two or more thin lenses close together, the effective power is given by $\sum P$.

Theorem. 10.6: Lens Maker's Equation

$$\frac{1}{f} = (n - 1) \left[\frac{1}{R_1} - \frac{1}{R_2} + \frac{(n - 1)d}{nR_1R_2} \right]$$

where

- f is the focal length of the lens,
- n is the refractive index of the lens material,
- R_1 is the radius of curvature of the first surface of the lens,
- R_2 is the radius of curvature of the second surface of the lens,
- d is the thickness of the lens.

Proof. We will derive the case of thin lens where $d \rightarrow 0$.



Consider a **spherical refracting surface** with radius of curvature R separating two media with refractive indices n_1 and n_2 . Let C be the **center of curvature** and V be the **vertex** of the spherical surface. We use the sign convention:

- Distances measured from V
- $R > 0$ if C is to the right of V (convex toward object)
- Object distance $u = -s_o$ (negative if object is left of V)
- Image distance $v = s_i$ (positive if image is right of V)

Using the **paraxial (small-angle) approximation**:

$$\sin \theta \approx \theta, \quad \tan \theta \approx \theta$$

for rays making small angles with the optical axis.

Consider a point object O on the optical axis, sending a ray to point A on the spherical surface at height h above the axis. From triangle OAC ,

$$\text{Angle of incidence: } \theta_1 = \alpha + \phi$$

From triangle IAC ,

$$\text{Angle of refraction: } \theta_2 = \phi - \beta$$

where

$$\alpha \approx \frac{h}{-u}, \quad \beta \approx \frac{h}{v}, \quad \phi \approx \frac{h}{R}$$

with $u = -s_o < 0$, $v > 0$, and R as given.

Snell's law in paraxial form:

$$n_1 \theta_1 = n_2 \theta_2$$

Substituting:

$$\begin{aligned} n_1(\alpha + \phi) &= n_2(\phi - \beta) \\ n_1 \left(\frac{h}{-u} + \frac{h}{R} \right) &= n_2 \left(\frac{h}{R} - \frac{h}{v} \right) \\ n_1 \left(\frac{1}{-u} + \frac{1}{R} \right) &= n_2 \left(\frac{1}{R} - \frac{1}{v} \right) \end{aligned}$$



Rewriting with Cartesian sign convention ($u = -s_o$, $v = s_i$, R signed):

$$\boxed{\frac{n_2}{v} - \frac{n_1}{u} = \frac{n_2 - n_1}{R}}$$

Consider a **thin lens** of refractive index n_l surrounded by medium n_m . The lens has:

- First spherical surface: radius R_1 (center C_1)
- Second spherical surface: radius R_2 (center C_2)
- Thickness negligible compared to object/image distances (thin lens approximation)

Sign convention:

- Light travels left to right
- $R > 0$ if center of curvature is to the right of surface
- Object distance $s_o > 0$ (real object left of lens)
- Image distance $s_i > 0$ (real image right of lens)

From medium 1 $n_m \Rightarrow$ to medium 2 n_l ,

$$\frac{n_l}{v_1} - \frac{n_m}{(-s_o)} = \frac{n_l - n_m}{R_1}$$

Since $u_1 = -s_o$,

$$\frac{n_l}{v_1} + \frac{n_m}{s_o} = \frac{n_l - n_m}{R_1} \quad (**)$$

From medium 1 n_l to medium 2 n_m , the object for second surface is the image from first surface. For thin lens, object distance is $-v_1$.

$$\frac{n_m}{s_i} - \frac{n_l}{(-v_1)} = \frac{n_m - n_l}{R_2}$$

$$\frac{n_m}{s_i} + \frac{n_l}{v_1} = \frac{n_m - n_l}{R_2} \quad (***)$$

From (**), $\frac{n_l}{v_1} = \frac{n_l - n_m}{R_1} - \frac{n_m}{s_o}$. From (***) $, \frac{n_l}{v_1} = \frac{n_m - n_l}{R_2} - \frac{n_m}{s_i}$. Equating both expressions for $\frac{n_l}{v_1}$ and simplifying gives

$$\frac{n_m}{s_o} - \frac{n_m}{s_i} = (n_l - n_m) \left(\frac{1}{R_2} - \frac{1}{R_1} \right)$$



For $s_i = f$ (image at focal point),

$$\frac{1}{f} = \frac{n_l - n_m}{n_m} \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

Let $n_m = 1$ and $n_l = n$.

$$\boxed{\frac{1}{f} = (n - 1) \left(\frac{1}{R_1} - \frac{1}{R_2} \right)}$$

10.5 Diffraction and Interference

10.5.1 Principle of Superposition

The **principle of superposition** states that when two or more waves overlap in space, the resultant displacement at any point is equal to the algebraic sum of the individual displacements at that point. Let two harmonic waves be given by

$$y_1(x, t) = A_1 \sin(kx - \omega t + \phi_1)$$

$$y_2(x, t) = A_2 \sin(kx - \omega t + \phi_2)$$

According to the principle of superposition, the resultant wave is

$$\begin{aligned} y(x, t) &= y_1(x, t) + y_2(x, t) \\ &= A_1 \sin(kx - \omega t + \phi_1) + A_2 \sin(kx - \omega t + \phi_2) \end{aligned}$$

Using the trigonometric identity

$$\sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2},$$

we get

$$y(x, t) = 2A \cos \left(\frac{\phi_2 - \phi_1}{2} \right) \sin \left(kx - \omega t + \frac{\phi_1 + \phi_2}{2} \right)$$

where A is the effective amplitude.

- Constructive interference occurs when $\phi_2 - \phi_1 = 2n\pi$.
- Destructive interference occurs when $\phi_2 - \phi_1 = (2n + 1)\pi$.



10.5.2 Complex Number

A complex number z is written as:

$$z = x + iy$$

where x is the real part, y is the imaginary part, and $i = \sqrt{-1}$.

Euler's formula links complex exponentials to trigonometric functions:

$$e^{i\theta} = \cos \theta + i \sin \theta$$

This is extremely useful in wave physics because oscillating quantities like $A \cos(\omega t + \phi)$ can be represented as the real part of a complex exponential:

$$A \cos(\omega t + \phi) = \operatorname{Re} \{ A e^{i(\omega t + \phi)} \}$$

10.5.3 Young's Double-Slit Experiment

Consider two slits separated by distance d illuminated by coherent light of wavelength λ . At a point on a screen at distance L , the path difference (which is defined as the difference in the distance traveled by two waves from their respective sources to a common point.) is

$$\delta = d \sin \theta$$

where θ is the angle from the central maximum.

The condition for constructive interference (bright fringes) is

$$d \sin \theta = m\lambda, \quad m = 0, \pm 1, \pm 2, \dots$$

and for destructive interference (dark fringes)

$$d \sin \theta = \left(m + \frac{1}{2} \right) \lambda$$

For small angles, $\sin \theta \approx \tan \theta = \frac{x}{L}$, where x is the fringe displacement. Then the fringe width is

$$\Delta x = \frac{\lambda L}{d}$$



10.5.4 Single-Slit Diffraction

Diffraction refers to the bending of waves around obstacles and apertures. Consider a slit of width a . The condition for minima in the diffraction pattern is

$$a \sin \theta = m\lambda, \quad m = \pm 1, \pm 2, \dots$$

The central maximum is twice as wide as the secondary maxima. The intensity at angle θ is given by

$$I(\theta) = I_0 \left(\frac{\sin(\beta)}{\beta} \right)^2, \quad \beta = \frac{\pi a \sin \theta}{\lambda}$$

10.5.5 Rayleigh's Criteria

Two point sources are said to be just resolved if the central maximum of one diffraction pattern coincides with the first minimum of the other. The Rayleigh criterion gives the limit at which two point sources can be resolved:

$$\theta_R = 1.22 \frac{\lambda}{D}$$

Derivation: Consider a circular aperture of diameter D . For monochromatic light of wavelength λ , the far-field (Fraunhofer) diffraction pattern of a point source is the **Airy pattern**, whose intensity is given by

$$I(\theta) = I_0 \left[\frac{2J_1(ka \sin \theta)}{ka \sin \theta} \right]^2$$

where $a = D/2$, $k = 2\pi/\lambda$, J_1 is the Bessel function of the first kind of order 1, and θ is the angular distance from the optical axis.

The first zero of $J_1(x)$ occurs at $x \approx 3.8317$. Let $x = ka \sin \theta$. Then

$$ka \sin \theta_{\min} = 3.8317 \implies \frac{2\pi a}{\lambda} \sin \theta_{\min} = 3.8317$$

Hence,

$$\sin \theta_{\min} \approx \frac{3.8317\lambda}{2\pi(D/2)} \approx \frac{1.22\lambda}{D}$$

For small angles (θ in radians),

$$\theta_{\min} \approx \frac{1.22\lambda}{D}$$

This θ_{\min} is the angular radius of the Airy disk (to the first dark ring).

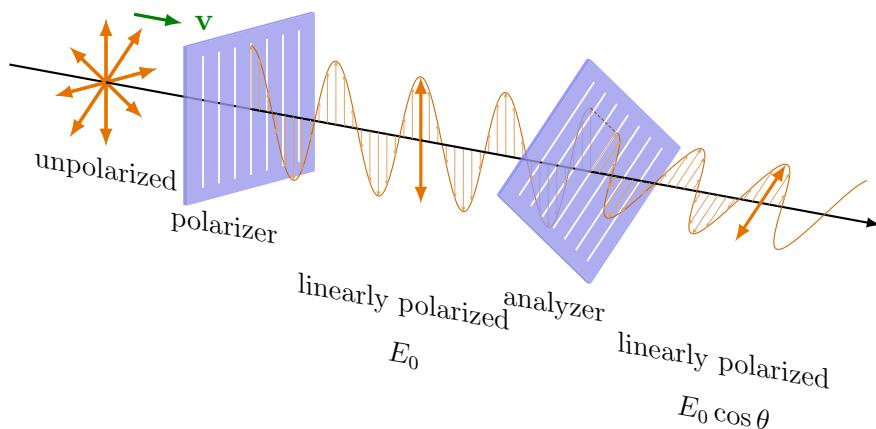


10.6 Polarization

10.6.1 Introduction

Polarization describes the direction in which the electric field vector of a light wave oscillates. Light can be

- **Unpolarized:** The electric field points in random directions (like sunlight).
- **Linearly polarized:** The electric field oscillates in a single direction.
- **Circularly polarized:** The electric field rotates in a circle as light travels.
- **Elliptically polarized:** A general case where the electric field traces an ellipse.



There are some methods for polarization:

- **Polarization by Reflection:** When light is reflected at a certain angle (Brewster's angle, θ_B), the reflected light becomes completely polarized:

$$\tan \theta_B = \frac{n_2}{n_1}$$

where n_1 and n_2 are refractive indices of the two media.

- **Polarization by Absorption:** Polarizing filters only allow electric field components in a particular direction to pass through. When linearly polarized light passes through a polarizing filter, the transmitted intensity I is given by

$$I = I_0 \cos^2 \theta \text{ (Malus's Law)}$$



where

- I_0 is the initial intensity of the light.
- θ is the angle between the light's polarization direction and the axis of the polarizer.

10.6.2 Faraday's Rotation

The Faraday effect, or Faraday rotation, is a magneto-optic phenomenon where the plane of polarization of linearly polarized light is rotated when the light propagates through a material subjected to a strong, static magnetic field aligned in the direction of propagation. This effect is one of the first historical pieces of evidence linking light with electromagnetism.

When linearly polarized light passes through a transparent material of length L that is immersed in a magnetic field \mathbf{B} (parallel to the direction of propagation), the angle of rotation β of the polarization plane is given by

$$\beta = VBL$$



11 Quantum Mechanics

11.1 Atomic Structure

11.1.1 Historical Development

- **Thomson's Model (1897):** "Plum pudding" model with electrons embedded in positive charge
- **Rutherford's Model (1911):** Nuclear model from alpha scattering experiments
- **Bohr's Model (1913):** Quantized electron orbits
- **Quantum Mechanical Model:** Electron clouds/orbitals

11.1.2 Introduction

The atom consists of

- **Nucleus:** Protons (p^+) and neutrons (n)
- **Electrons:** Negatively charged particles in orbitals

Particle	Symbol	Charge	Mass (u)
Proton	p or ${}_1^1H$	+1	1.007276
Neutron	n	0	1.008665
Electron	e^-	-1	0.000549

Table 3: Basic atomic particles

For an element X,



where

- A is the mass number (protons + neutrons),
- Z is the atomic number (number of protons), and
- we can define N to be neutron number ($N = A - Z$)



Example. 6_6C_6 has 6 protons and 6 neutrons.

The actual mass of a nucleus is **less** than the sum of its constituent particles:

$$\Delta m = (Zm_p + Nm_n) - m_{\text{nucleus}}$$

where

- Δm represents the mass defect,
- m_p represents the mass of proton,
- m_n represents the mass of neutron, and
- m_{nucleus} represents the measured nuclear mass.

According to Einstein's mass-energy equivalence

$$E = mc^2$$

The binding energy is

$$BE = \Delta m \cdot c^2$$

More conveniently, using atomic mass units (u) where $1 u = 931.5 \text{ MeV}/c^2$:

$$BE (\text{MeV}) = \Delta m (u) \times 931.5$$

11.1.3 Nuclear Decay

Type	Emitted Particle	Change in Nucleus	Penetration
Alpha (α)	4_2He nucleus	$Z \rightarrow Z - 2, A \rightarrow A - 4$	Low
Beta (β^-)	Electron (e^-)	$n \rightarrow p + e^- + \bar{\nu}$	Medium
Beta (β^+)	Positron (e^+)	$p \rightarrow n + e^+ + \nu$	Medium
Gamma (γ)	Photon (γ)	No change in Z or A	High

Table 4: Types of radioactive decay

Radioactive decay follows first-order kinetics:

$$N(t) = N_0 e^{-\lambda t}$$



where

- $N(t)$ represents the number of nuclei at time t ,
- N_0 represents the initial number of nuclei, and
- $\lambda = \frac{\ln 2}{t_{1/2}}$ represents the decay constant where $t_{1/2}$ is half-life (time for half the nuclei to decay).

11.1.4 Neutrinos

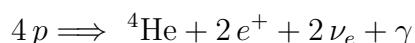
Neutrinos are elementary particles that are part of the lepton family. They are electrically neutral and have an extremely small mass, making them very difficult to detect. Neutrinos interact only through weak nuclear force and gravity, which is why they pass through matter almost unaffected. The three types of neutrinos correspond to their associated charged leptons:

- Electron neutrino (ν_e)
- Muon neutrino (ν_μ)
- Tau neutrino (ν_τ)

These particles are produced in various high-energy processes, such as nuclear reactions in the Sun and other stars, as well as in cosmic ray interactions and supernovae.

Solar Neutrinos The Sun is a primary source of neutrinos, particularly electron neutrinos (ν_e). These solar neutrinos are produced during nuclear fusion processes that take place in the Sun's core. The dominant fusion reaction in the Sun is the proton-proton chain, which is responsible for the majority of the energy production. In this process, four protons fuse to form a helium nucleus, releasing energy in the form of gamma rays, neutrinos, and positrons.

The overall process can be written as



11.2 Wave-Particle Duality

Louis de Broglie proposed that all matter has wave-like properties. The de Broglie wavelength is given by

$$\lambda = \frac{h}{p}$$



where

- λ is the de Broglie wavelength,
- $h = 6.626 \times 10^{-34} \text{ J}\cdot\text{s}$ is Planck's constant, and
- p is the particle's momentum.

11.3 Planck's Equation

For electromagnetic waves of frequency ν , the energy of photon is given by

$$E = h\nu$$

11.4 Bohr Model of the Hydrogen Atom

11.4.1 Bohr Postulates

1. The electron moves in circular orbits around the proton due to Coulomb attraction.
2. Only certain discrete orbits are allowed.
3. The angular momentum of the electron is quantized.
4. Radiation is emitted or absorbed only during transitions between allowed orbits.

11.4.2 Formulas

The electrostatic force between the proton and electron is

$$F = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{r^2}$$

For circular motion, the centripetal force is

$$F = \frac{mv^2}{r}$$

Equating the two,

$$\frac{mv^2}{r} = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{r^2}$$

Solving for velocity,

$$v^2 = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{mr}$$



Bohr postulated

$$L = mvr = n\hbar \quad n = 1, 2, 3, \dots$$

Solving for velocity,

$$v = \frac{n\hbar}{mr}$$

Substitute the expression for v :

$$\left(\frac{n\hbar}{mr}\right)^2 = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{mr}$$

Solving for r ,

$$r_n = \frac{4\pi\varepsilon_0\hbar^2}{me^2} n^2$$

Define the Bohr radius

$$a_0 = \frac{4\pi\varepsilon_0\hbar^2}{me^2}$$

Hence,

$$r_n = a_0 n^2$$

The total energy is

$$E = K + U = \frac{1}{2}mv^2 - \frac{1}{4\pi\varepsilon_0} \frac{e^2}{r}$$

Note that

$$K = \frac{1}{2} \frac{1}{4\pi\varepsilon_0} \frac{e^2}{r}$$

Hence, the total energy becomes

$$E = -\frac{1}{2} \frac{1}{4\pi\varepsilon_0} \frac{e^2}{r}$$

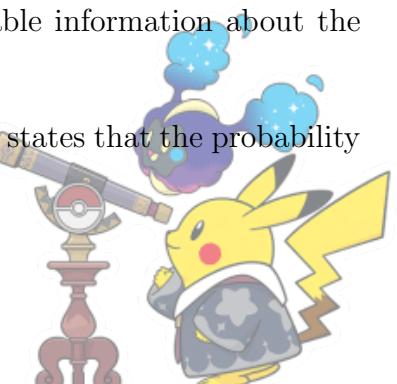
Substituting r_n ,

$$E_n = -\frac{me^4}{2(4\pi\varepsilon_0)^2\hbar^2} \frac{1}{n^2} = \frac{-13.6 \text{ eV}}{n^2}$$

11.4.3 Wavefunction

In quantum mechanics, the state of a particle is fully described by a complex-valued function called the *wavefunction*, denoted by $\psi(x, t)$. The wavefunction contains all the measurable information about the system.

The physical interpretation of the wavefunction is given by the Born rule, which states that the probability



density of finding a particle at position x at time t is

$$P(x, t) = |\psi(x, t)|^2.$$

Since the particle must be found somewhere in space, the total probability of finding the particle over all space must be equal to one. This requirement leads to the *normalisation condition*:

$$\int_{-\infty}^{\infty} |\psi(x, t)|^2 dx = 1.$$

If a wavefunction does not initially satisfy this condition, it can be normalised by introducing a constant A such that

$$\Psi(x, t) = A\psi(x, t),$$

where the constant A is chosen to ensure the normalisation condition is satisfied.

11.4.4 Time-Independent Schrödinger Equation

The Schrödinger equation is

$$-\frac{\hbar^2}{2m} \nabla^2 \psi + V(r)\psi = E\psi$$

with the Coulomb potential

$$V(r) = -\frac{1}{4\pi\epsilon_0} \frac{e^2}{r}$$

Using separation of variables,

$$\psi(r, \theta, \phi) = R(r)Y_{\ell m}(\theta, \phi)$$

the radial equation becomes

$$\frac{d^2R}{dr^2} + \frac{2}{r} \frac{dR}{dr} + \left[\frac{2m}{\hbar^2} \left(E + \frac{e^2}{4\pi\epsilon_0 r} \right) - \frac{\ell(\ell+1)}{r^2} \right] R = 0$$

For large r , the Coulomb term becomes negligible, yielding

$$\frac{d^2R}{dr^2} - \kappa^2 R = 0$$

where

$$\kappa = \sqrt{-\frac{2mE}{\hbar^2}}$$



The physically acceptable solution is

$$R(r) \sim e^{-\kappa r}$$

For small r , the equation has solution

$$R(r) \sim r^\ell$$

Motivated by the asymptotic behavior, write

$$R(r) = r^\ell e^{-\kappa r} F(r)$$

Define a dimensionless variable

$$\rho = 2\kappa r$$

and substitute into the radial equation to obtain

$$\rho \frac{d^2F}{d\rho^2} + (2\ell + 2 - \rho) \frac{dF}{d\rho} + \left(\frac{me^2}{2\pi\varepsilon_0\hbar^2\kappa} - \ell - 1 \right) F = 0$$

Assume a power series

$$F(\rho) = \sum_{k=0}^{\infty} a_k \rho^k$$

Substitution yields the recursion relation

$$a_{k+1} = \frac{k + \ell + 1 - \frac{me^2}{2\pi\varepsilon_0\hbar^2\kappa}}{(k+1)(k+2\ell+2)} a_k$$

For the wavefunction to remain normalizable, the series must terminate. Hence, there exists an integer n_r such that

$$\frac{me^2}{2\pi\varepsilon_0\hbar^2\kappa} = n_r + \ell + 1$$

Define the principal quantum number

$$n = n_r + \ell + 1$$

Solving for energy,

$$E_n = -\frac{me^4}{2(4\pi\varepsilon_0)^2\hbar^2} \frac{1}{n^2}$$



11.5 Uncertainty Principle

Werner Heisenberg showed that certain pairs of physical quantities cannot be simultaneously measured with arbitrary precision. The most famous uncertainty relation is between position and momentum:

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

where

- Δx is the uncertainty in position
- Δp is the uncertainty in momentum
- $\hbar = \frac{h}{2\pi}$ is the reduced Planck constant

Example. Consider a particle (e.g., an electron) passing through a narrow slit of width a along the x -direction. Due to the confinement in position, the particle's wavefunction is localized within the slit, giving

$$\Delta x \sim a.$$

After passing through the slit, the particle undergoes diffraction, resulting in a spread in its momentum in the x -direction. The diffraction pattern for a slit is given by the first minimum condition:

$$a \sin \theta \sim \lambda,$$

where θ is the diffraction angle and λ is the wavelength of the particle. For a particle with momentum p , its wavelength is related to the momentum by de Broglie's relation:

$$\lambda = \frac{h}{p},$$

where h is Planck's constant. The transverse momentum uncertainty is approximately

$$\Delta p_x \sim p \sin \theta \sim \frac{h}{a}.$$

Multiplying the uncertainties in position and momentum:

$$\Delta x \Delta p_x \sim a \cdot \frac{h}{a} \sim h \sim \hbar,$$



which reproduces the Heisenberg Uncertainty Principle:

$$\Delta x \Delta p_x \gtrsim \frac{\hbar}{2}.$$

Another important uncertainty relation is between energy and time:

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

The uncertainty principle isn't about measurement limitations, but about fundamental properties of quantum systems. Particles don't have precisely defined positions and momenta simultaneously.



12 Stellar Astrophysics

12.1 Stellar Classifications

Stars can be classified by their internal structure:

- **Main Sequence Stars:** Hydrogen-burning core, radiative or convective energy transport. Governed by mass-luminosity relation:

$$\frac{L}{L_\odot} \approx \begin{cases} \left(\frac{M}{M_\odot}\right)^{4.0} & \text{for } M > 10M_\odot \\ \left(\frac{M}{M_\odot}\right)^{3.5} & \text{for } 0.5 < M < 10M_\odot \\ \left(\frac{M}{M_\odot}\right)^{2.3} & \text{for } M < 0.5M_\odot \end{cases}$$

- **Giant Stars:** Expanded envelope, hydrogen shell burning around inert helium core.
- **Supergiants:** Massive stars with complex layered burning (H, He, C, O, Si shells).
- **White Dwarfs:** Electron-degenerate cores, supported by electron degeneracy pressure.
- **Neutron Stars:** Neutron-degenerate matter, extreme density ($\sim 10^{17}$ kg/m³).
- **Black Holes:** Gravitational collapse beyond neutron degeneracy pressure support.

Stars can also be classified by their composition:

Population	Metallicity [Fe/H]	Characteristics
Population I	≥ -0.5	Metal-rich, disk stars, young
Population II	-1.0 to -0.5	Metal-poor, halo stars, old
Population III	$\ll -1.0$	Zero metals, first generation

Table 5: Stellar Populations

Metallicity affects stellar evolution:

- Opacity changes with metal content
- Lower metallicity stars are hotter and bluer at given mass
- Metallicity influences mass loss rates



Stars can also be classified by their activity:

- **Flare Stars** (UV Ceti type): M-dwarfs with magnetic reconnection events: Magnetic reconnection is a fundamental plasma physics process where the topology of magnetic field lines is rearranged, converting magnetic energy into kinetic energy, thermal energy, and particle acceleration.

- **Rotational Classes:**

- Slow rotators: $v_{\text{eq}} < 10 \text{ km/s}$ (Sun: 2 km/s)
- Fast rotators: $v_{\text{eq}} > 50 \text{ km/s}$ (young stars)

- **Magnetic Activity:**

$$R'_{HK} = \frac{L_{HK}}{L_{\text{bol}}}$$

where L_{HK} is the luminosity in the Calcium II H & K lines and L_{bol} is the bolometric luminosity: Total power output across all wavelengths (from radio to X-rays). This ratio tells what fraction of the star's total energy output is being emitted specifically from its magnetically heated chromosphere via these Calcium lines as Calcium II is singly-ionized calcium.

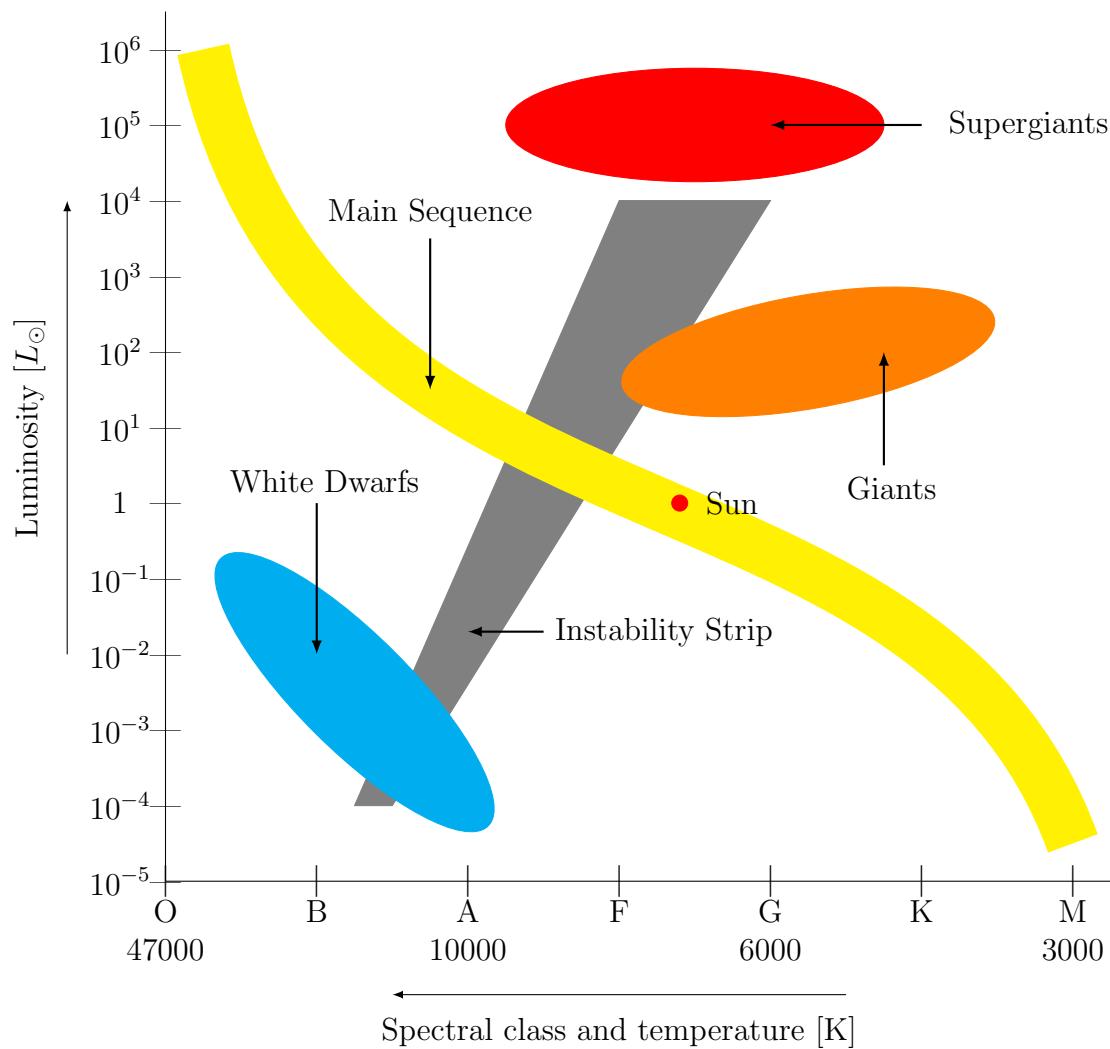
12.2 HR Diagram

The Hertzsprung-Russell diagram (HR diagram) is one of the most important tools in astrophysics, independently developed by Ejnar Hertzsprung (1905-1911) and Henry Norris Russell (1913). It revolutionized our understanding of stellar evolution by revealing patterns in stellar properties.

Spectral Type	Temp (K)	Color	Mass (M_{\odot})	Examples
O	30,000-50,000	Blue	15-90	ζ Puppis
B	10,000-30,000	Blue-white	2-15	Rigel, Spica
A	7,500-10,000	White	1.4-2	Sirius, Vega
F	6,000-7,500	Yellow-white	1.04-1.4	Procyon
G	5,200-6,000	Yellow	0.8-1.04	Sun, α Cen A
K	3,700-5,200	Orange	0.45-0.8	Arcturus, Aldebaran
M	2,400-3,700	Red	0.08-0.45	Betelgeuse, Proxima Cen

Table 6: MK Spectral Classification System





12.3 Stellar Evolution

12.3.1 Stellar Formation

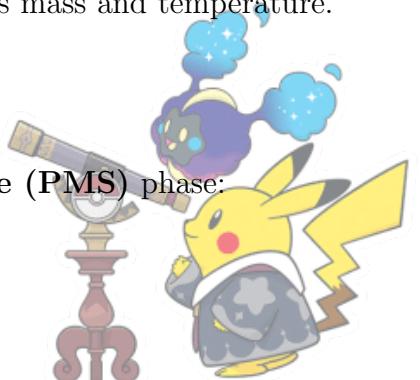
Stars form from giant clouds of gas and dust called **molecular clouds** or **nebulae**. The process includes:

- **Gravitational Collapse:** Regions with higher density collapse under gravity.
- **Protostar Formation:** As the cloud collapses, a dense core forms and heats up due to gravitational energy.
- **Accretion:** Surrounding material falls onto the protostar, increasing its mass and temperature.

12.3.2 Pre-Main Sequence Stars

Before reaching the main sequence, stars go through the **pre-main sequence (PMS)** phase:

- The protostar contracts and heats up.

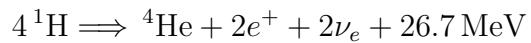


- Energy is produced mainly by gravitational contraction, not nuclear fusion.
- PMS stars follow **Hayashi tracks** (for lower-mass stars) or **Henyey tracks** (for higher-mass stars) on the Hertzsprung-Russell diagram.

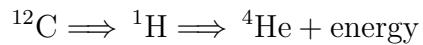
12.3.3 Main Sequence Stars

A star enters the **main sequence** when hydrogen fusion begins in its core.

- Core hydrogen fusion converts hydrogen into helium via the **proton-proton chain** (low-mass stars) or **CNO cycle** (high-mass stars).
 - **Proton-Proton Chain:**



- **CNO Cycle:**



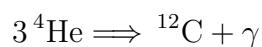
- The star achieves **hydrostatic equilibrium**: gravity balanced by radiation pressure from fusion.
- The main sequence lifetime depends on stellar mass; higher-mass stars burn faster and live shorter lives.

12.3.4 Post-Main Sequence Stars

After hydrogen in the core is exhausted, stars evolve differently depending on their mass.

Low to Intermediate-Mass Stars ($< 8M_\odot$)

- Expand into **Red Giants**.
- Helium fusion begins in the core (**triple-alpha process**):



- May go through **asymptotic giant branch (AGB)** phase before shedding outer layers.



High-Mass Stars ($> 8M_{\odot}$)

- Expand into **supergiants**.
- Fuse heavier elements in successive shells (C, O, Si, etc.).
- Form an **iron core**, which cannot undergo fusion to produce energy.

12.3.5 Supernovae

- Massive stars end their lives in a **core-collapse supernova** (Type II).
- The core collapses, and outer layers are expelled violently.
- Supernovae enrich the interstellar medium with heavy elements.

12.3.6 Planetary Nebulae

- Formed by low to intermediate-mass stars shedding their outer layers.
- The exposed core emits ultraviolet radiation, ionizing the ejected gas.
- Eventually fades to leave a **white dwarf**.

12.3.7 End States of Stars

- **White Dwarfs:** Remnants of low/intermediate-mass stars. Supported by electron degeneracy pressure.
- **Neutron Stars:** Remnants of core-collapse supernovae of stars with $8 - 20M_{\odot}$. Supported by neutron degeneracy pressure.
- **Black Holes:** Remnants of very massive stars ($> 20M_{\odot}$). Gravity overwhelms all forms of degeneracy pressure.

12.4 Magnitude System

Definition. 12.1: Luminosity

Luminosity is the *total amount of energy emitted by a source per unit time*.



Definition. 12.2: Luminance

Luminance is the amount of *visible light emitted or reflected in a given direction per unit area per unit solid angle*. It describes how bright a surface appears to the human eye.

Property	Luminosity	Luminance
Physical meaning	Total power emitted	Brightness per area per direction
Depends on distance	No	No (but depends on direction)
Depends on area	Indirectly	Directly
SI unit	Watt (W)	cd/m ²
Usually used in	Astrophysics	Optics

Definition. 12.3: Flux

The **flux** (F or Φ) is the total energy received from an astronomical object per unit time per unit area. It represents the power (energy per unit time) crossing a unit area oriented perpendicular to the direction of propagation.

For a source emitting total luminosity L isotropically, the flux measured at distance r is

$$F = \frac{L}{4\pi r^2}$$

Definition. 12.4: Solid Angle

The **solid angle** measures the three-dimensional angular size of an object as seen from a point. It is defined as

$$d\Omega = \frac{dA}{r^2}$$

where

- dA is an area element on a sphere of radius r ,
- $d\Omega$ is the corresponding solid angle.

Definition. 12.5: Intensity

The flux per unit solid angle is called **intensity**:

$$I = \frac{dF}{d\Omega}$$



Definition. 12.6: Spectral Flux Density

The **spectral flux density** (F_λ or F_ν) is the flux per unit wavelength or frequency interval. It describes how the flux is distributed across the electromagnetic spectrum.

The **solar constant** is the flux received from the Sun at Earth's distance:

$$F_\odot = 1361 \text{ W m}^{-2} = 1.361 \times 10^6 \text{ erg s}^{-1} \text{cm}^{-2}$$

This represents the total solar power (all wavelengths) incident on 1 m² at 1 AU.

Definition. 12.7: Apparent Magnitude

It refers to the brightness of an object **as observed from Earth**, given by

$$m = -2.5 \log_{10} \left(\frac{F}{F_0} \right)$$

where F_0 is the reference flux for zero magnitude. The zero-point F_0 is calibrated using standard stars. Originally, Vega (α Lyrae) was defined to have magnitude 0.0 in all filters. Modern systems use more precisely defined spectrophotometric standards.

Ancient astronomers (especially Hipparchus, 150 BCE) ranked stars by eye:

1. 1st magnitude means the brightest (e.g. Sirius and Vega)
2. 6th magnitude means faintest visible to the naked eye

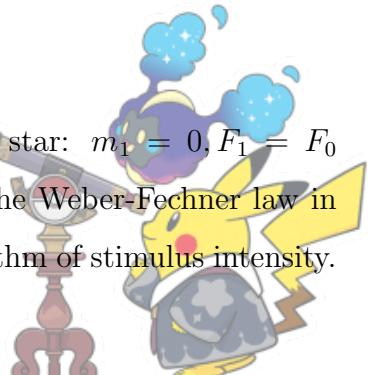
Much later (1856), Norman Pogson put this on a mathematical footing. He defined the scale so that a difference of 5 magnitudes corresponds to a brightness ratio of 100.

$$\frac{F_1}{F_2} = 100 \text{ when } m_2 - m_1 = 5$$

Assume magnitude difference is proportional to the logarithm of the flux ratio:

$$m_2 - m_1 = k \log_{10} \left(\frac{F_1}{F_2} \right)$$

By Pogson's condition, $5 = 2k \implies k = 2.5$. Take star 1 as the reference star: $m_1 = 0, F_1 = F_0$ gives $m = -2.5 \log_{10} \left(\frac{F}{F_0} \right)$. Later, the logarithmic scale was found to match the Weber-Fechner law in psychophysics, which states that perceived sensation is proportional to the logarithm of stimulus intensity.



Definition. 12.8: Absolute Magnitude

The brightness an object would have if placed at a standard distance of **10 parsecs** (32.6 light-years):

$$M = m - 5 \log_{10} \left(\frac{d}{10 \text{ pc}} \right)$$

where d is the distance to the object in parsecs.

Definition. 12.9: Distance Modulus

The difference between apparent and absolute magnitude relates directly to distance:

$$\mu = m - M = 5 \log_{10} d - 5$$

Definition. 12.10: Limiting Magnitude

The limiting magnitude is the faintest apparent magnitude of a celestial object that can be detected with a given instrument under specific observing conditions. It represents the threshold between what is observable and what is not. It is given by

$$m_{\text{lim}} = m_{\text{eye}} + 5 \log \left(\frac{D}{d_{\text{eye}}} \right) + 2.5 \log_{10} \left(\frac{\text{Transmission}}{0.95} \right)$$

where

- (1) m_{eye} is the naked-eye limiting magnitude (typically 6.0 under ideal conditions),
- (2) D is the telescope aperture,
- (3) d_{eye} is the dark-adapted pupil diameter (typically 7 mm), and
- (4) Transmission is the optical transmission coefficient.

Definition. 12.11: Bolometric Magnitude

The **bolometric magnitude** (m_{bol} or M_{bol}) is a measure of an astronomical object's total electromagnetic luminosity across all wavelengths, from gamma rays to radio waves. Unlike filter-specific magnitudes, it accounts for *all* emitted radiation.

The bolometric magnitude is defined through the **bolometric flux** F_{bol} , which is the integral of the



spectral flux density over all wavelengths:

$$F_{\text{bol}} = \int_0^{\infty} F_{\lambda} d\lambda = \int_0^{\infty} F_{\nu} d\nu$$

The apparent bolometric magnitude is then

$$m_{\text{bol}} = -2.5 \log_{10} \left(\frac{F_{\text{bol}}}{F_{\text{bol},0}} \right)$$

where $F_{\text{bol},0}$ is the reference bolometric flux for zero magnitude.

Definition. 12.12: Bolometric Correction

It refers to the difference between bolometric and visual magnitudes:

$$BC = M_{\text{bol}} - M_V$$

or for apparent magnitudes:

$$BC = m_{\text{bol}} - m_V$$

The Sun's magnitude is assumed to be bolometric by convention. Therefore, it is common to use

$$M_{\text{bol}} - M_{\odot} = -2.5 \log_{10} \left(\frac{L}{L_{\odot}} \right).$$

Example. (2013 IOAA) A star has visual apparent magnitude $m_V = 12.2$ mag, parallax $\pi = 0.001''$ and effective temperature $T_{\text{eff}} = 4000$ K. Its bolometric correction is B.C. = -0.6 mag.

(a) Find its luminosity as a function of the solar luminosity.

(b) What type of star is it?

(i) a red giant

(ii) a blue giant

(iii) a red dwarf

Please write (i), (ii) or (iii) in your answer sheet.



(a) First, the absolute visual magnitude is obtained from

$$M_V - m_V = 5 - 5 \log r$$

or equivalently,

$$M_V - m_V = 5 + 5 \log \pi$$

Thus,

$$M_V = 12.2 + 5 + 5 \log(0.001) = 12.2 + 5 - 15 = 2.2 \text{ mag}$$

The bolometric correction is defined as

$$\text{B.C.} = M_{\text{bol}} - M_V$$

so

$$M_{\text{bol}} = \text{B.C.} + M_V = -0.6 + 2.2 = 1.6 \text{ mag}$$

The luminosity is calculated using

$$M_{\odot} - M_{\text{bol}} = 2.5 \log \left(\frac{L}{L_{\odot}} \right)$$

where $M_{\odot} = 4.72$ mag. Hence,

$$4.72 - 1.6 = 2.5 \log \left(\frac{L}{L_{\odot}} \right)$$

$$\log \left(\frac{L}{L_{\odot}} \right) = 1.25$$

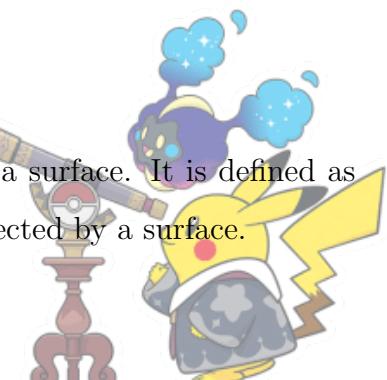
and therefore

$$L = 17.7 L_{\odot}$$

(b) A star with $M_{\text{bol}} = 1.6$ mag, luminosity $L = 17.7 L_{\odot}$, and effective temperature $T_{\text{eff}} = 4000$ K is much brighter and cooler than the Sun. Therefore, it is **(i) a red giant star.**

12.5 Albedo

Albedo is a dimensionless physical quantity that measures the reflectivity of a surface. It is defined as the fraction of incident electromagnetic radiation (usually sunlight) that is reflected by a surface.



If a surface receives an incident radiant power P_{in} and reflects a power P_{ref} , its albedo α is defined as

$$\alpha = \frac{P_{\text{ref}}}{P_{\text{in}}}, \quad 0 \leq \alpha \leq 1$$

Assuming the planet radiates as a black body and is in thermal equilibrium, the absorbed power equals emitted power:

$$(1 - \alpha)\pi R^2 S = 4\pi R^2 \sigma T^4$$

where σ is the Stefan–Boltzmann constant.

Solving for the equilibrium temperature T :

$$T = \left(\frac{(1 - \alpha)S}{4\sigma} \right)^{1/4}$$

12.6 Geometric Albedo

The **geometric albedo** is a dimensionless quantity that measures how bright an astronomical body appears when observed at *full phase* (i.e. zero phase angle), compared to an idealized reference surface. Let

- α be the **phase angle**, defined as the angle between the incident radiation from the source (e.g. the Sun) and the direction to the observer, as seen from the object,
- $\alpha = 0$ correspond to **full illumination** (observer directly behind the light source).

We introduce a reference surface:

A perfectly diffusing (Lambertian) flat disk with the same cross-sectional area as the object, illuminated and observed at normal incidence.

A Lambertian surface reflects radiation isotropically, obeying Lambert's cosine law:

Theorem. 12.1: Lambert's Cosine law

A surface obeys **Lambert's cosine law** if its radiance is independent of direction:

$$I(\theta, \phi) = I_0 = \text{constant}$$



For a Lambertian surface, the power emitted into solid angle $d\Omega$ is

$$dP = I_0 \cos \theta dA d\Omega$$

If the incident solar flux is F_{inc} , a perfectly reflecting disk intercepts power $P = F_{inc}A$. Integrating the Lambertian emission over a hemisphere:

$$P_{reflected} = \int_0^{2\pi} \int_0^{\pi/2} I_{ref} \cos \theta \sin \theta A d\theta d\phi = \pi A I_{ref}$$

Equating incident and reflected power ($P_{reflected} = F_{inc}A$), we find:

$$I_{ref} = \frac{F_{inc}}{\pi}$$

The geometric albedo p is defined as

$$p = \frac{I_{object}(\alpha = 0)}{I_{ref}} = \frac{\pi I_{object}(\alpha = 0)}{F_{inc}}$$

The total energy reflected in all directions is characterized by the **Bond Albedo** (A_B), related to p by the **phase integral** q :

$$A_B = p \cdot q, \quad q = 2 \int_0^\pi \Phi(\alpha) \sin \alpha d\alpha$$

where $\Phi(\alpha)$ is the phase function, which is the relative brightness of the object as a function of the phase angle, normalized such that $\Phi(0) = 1$.

12.7 Color Indices

Color indices quantify the color of astronomical objects by measuring the difference in magnitude between two different wavelength bands:

$$C = m_{\lambda_1} - m_{\lambda_2}$$

where m_{λ_1} and m_{λ_2} are apparent magnitudes measured through different filters.



Filter	Center (λ_c , nm)	FWHM (nm)	Typical Use
U	365	66	Ultraviolet continuum
B	445	94	Blue light, Balmer break
V	551	88	Visual (photopic)
R	658	138	Red continuum
I	806	149	Near-infrared

Table 7: Johnson-Cousins photometric system filters

The followings are the common color indices to use:

$$U - B = m_U - m_B$$

$$B - V = m_B - m_V$$

$$V - R = m_V - m_R$$

$$V - I = m_V - m_I$$

The observed color index is affected by interstellar extinction:

$$(B - V)_{\text{obs}} = (B - V)_0 + E(B - V)$$

where $E(B - V)$ is the color excess:

$$E(B - V) = A_B - A_V$$

where A_B is the extinction in B -band in mag and A_V is the extinction in V -band in mag.

12.8 Atmospheric Extinction

Atmospheric extinction reduces the observed flux:

$$m_{\lambda, \text{obs}} = m_{\lambda, \text{true}} + k'_{\lambda} \cdot X$$

where

- k'_{λ} is the extinction coefficient (mag/airmass) and
- X is the airmass (dimensionless) which quantifies the atmosphere's thickness along a light's path.



For a plane-parallel atmosphere,

$$X = \sec z$$

where z is the zenith distance ($z = 90^\circ - \text{altitude}$).

12.9 Optical Depth

Optical depth τ_ν describes attenuation by comparing the initial intensity $I_\nu(0)$ and the transmitted intensity $I_\nu(s)$:

$$\tau_\nu(s) \equiv -\ln \left(\frac{I_\nu(s)}{I_\nu(0)} \right).$$

Extinction in magnitudes A_λ is related to optical depth by

$$A_\lambda = 2.5 \log_{10}(e\tau_\lambda) = 1.086 \tau_\lambda.$$

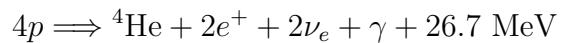
as

$$\frac{I}{I_0} = e^{-\tau_\lambda} = 10^{-0.4A_\lambda}.$$

12.10 Study of the Sun

12.10.1 Composition

The Sun is primarily composed of hydrogen and helium, with trace amounts of heavier elements. Hydrogen nuclei (protons) undergo nuclear fusion in the solar core:



powering the Sun through the proton-proton chain.

12.10.2 Internal Structure

1. Core:

- Radius: $\sim 0.2 R_\odot$
- Site of nuclear fusion: $4p \implies {}^4\text{He} + 2e^+ + 2\nu_e + 26.7 \text{ MeV}$
- Temperature: $\sim 1.5 \times 10^7 \text{ K}$

2. Radiative Zone:



- Energy transported by radiation
- Temperature gradient decreases outward

3. Convective Zone:

- Energy transported by convection
- Outer $\sim 30\%$ of Sun's radius
- Convection cells cause granulation on the photosphere

12.10.3 Atmosphere

- **Photosphere:** Visible surface; $T \sim 5800$ K
- **Chromosphere:** Above photosphere; hotter than photosphere; $T \sim 10^4$ K
- **Corona:** Outermost layer; $T \sim 10^6$ K; source of solar wind

12.10.4 Solar Surface Activities

Sunspots Sunspots are temporary dark regions on the solar photosphere (visible surface of the Sun) caused by strong magnetic fields.

12.10.5 Solar Wind

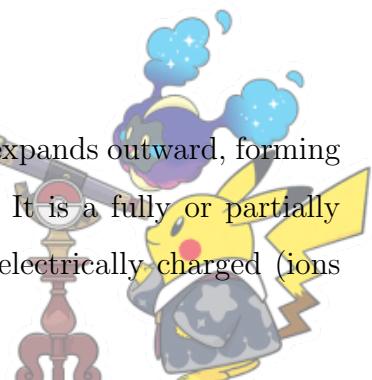
The solar wind is a continuous outflow of plasma from the solar corona. The solar wind consists mainly of

- Protons (p^+)
- Electrons (e^-)
- Alpha particles ($\alpha = {}^4\text{He}^{2+}$)

The solar corona has high temperature ($T \sim 1 \times 10^6$ K). The sound speed is

$$c_s \sim \sqrt{\frac{k_B T}{m_p}}$$

As the corona cannot remain static with such high thermal pressure, the plasma expands outward, forming the solar wind. (**Remark:** Plasma is often called the fourth state of matter. It is a fully or partially ionized gas, meaning that a significant fraction of the atoms or molecules are electrically charged (ions



and electrons).)

The **heliosphere** is the region of space dominated by the solar wind and the Sun's magnetic field, extending well beyond the orbit of Pluto.

The **magnetosphere** is the region around a planet where the planetary magnetic field dominates the motion of charged particles, protecting the planet from the solar wind.

12.11 Binary Star

Binary star systems consist of two stars orbiting around their common center of mass.

12.11.1 Different Types of Binary Stars

Visual Binaries They refer to the stars that can be resolved individually through telescopes. Their orbits can be directly observed over time.

Spectroscopic Binaries They refer to the stars that can be detected through periodic Doppler shifts in their spectral lines. They can be further identified as

- **Single-lined spectroscopic binaries (SB1):** Only one set of spectral lines is visible
- **Double-lined spectroscopic binaries (SB2):** Both sets of spectral lines are visible

Eclipsing Binaries They are the systems where the orbital plane is nearly edge-on to our line of sight, causing periodic eclipses. These provide the most complete information about stellar parameters.

Astrometric Binaries They can be detected through the wobble of one star's proper motion due to an unseen companion.

Interacting Binaries

- Mass transfer occurs when one star fills its Roche lobe.
- Examples: Cataclysmic variables, X-ray binaries.

Peculiar Binary Systems

- Systems with unusual properties such as extremely short orbital periods or highly eccentric orbits.
- Examples: Contact binaries, Algol-type binaries.



12.11.2 Modified Kepler's Third Law

For a binary system, Kepler's third law relates the orbital period P , semi-major axis a , and total mass M :

$$P^2 = \frac{4\pi^2 a^3}{G(M_1 + M_2)} \quad (5)$$

where

- P is the orbital period,
- a is the semi-major axis of the relative orbit, and
- M_1, M_2 are the masses of the two stars

12.11.3 Mass Function

For spectroscopic binaries, we measure the **mass function**. For single-lined spectroscopic binaries,

$$f(M) = \frac{M_2^3 \sin^3 i}{(M_1 + M_2)^2} = \frac{P}{2\pi G} v_{1,r}^3$$

For double-lined spectroscopic binaries, we can determine the mass ratio:

$$\frac{M_1}{M_2} = \frac{v_{2,r}}{v_{1,r}}$$

Proof. We begin with Kepler's third law for two bodies orbiting their common center of mass:

$$P^2 = \frac{4\pi^2 a^3}{G(M_1 + M_2)}$$

where $a = a_1 + a_2$ is the total separation between the stars, and a_1, a_2 are their distances from the center of mass.

The center of mass condition gives

$$a_1 M_1 = a_2 M_2$$

From this, we can write the mass ratio:

$$\frac{a_1}{a_2} = \frac{M_2}{M_1}$$



Also, the total separation can be expressed in terms of a_1 :

$$a = a_1 + a_2 \quad (7)$$

$$= a_1 + a_1 \frac{M_1}{M_2} \quad (\text{from Equation 6}) \quad (8)$$

$$= a_1 \left(\frac{M_1 + M_2}{M_2} \right) \quad (9)$$

For the visible star (star 1), we measure its **radial velocity amplitude** $v_{1,r}$. The true orbital speed v_1 is related to the observed radial velocity by the inclination:

$$v_{1,r} = v_1 \sin i \quad (10)$$

For a circular orbit ($e = 0$), the orbital speed is constant and given by

$$v_1 = \frac{2\pi a_1}{P} \quad (11)$$

Combining Equations 10 and 11:

$$v_{1,r} = \frac{2\pi a_1 \sin i}{P} \quad (12)$$

From Equation 12, we can solve for a_1 :

$$a_1 = \frac{P v_{1,r}}{2\pi \sin i}$$

Now substitute Equation 9 into Kepler's law (Equation 12.11.3):

$$P^2 = \frac{4\pi^2}{G(M_1 + M_2)} \left[a_1 \left(\frac{M_1 + M_2}{M_2} \right) \right]^3 \quad (13)$$

$$= \frac{4\pi^2 a_1^3}{G(M_1 + M_2)} \cdot \frac{(M_1 + M_2)^3}{M_2^3} \quad (14)$$

$$= \frac{4\pi^2 a_1^3}{G} \cdot \frac{(M_1 + M_2)^2}{M_2^3} \quad (15)$$



Now substitute a_1 from Equation 12.11.3 into Equation 15:

$$\begin{aligned} P^2 &= \frac{4\pi^2}{G} \cdot \frac{(M_1 + M_2)^2}{M_2^3} \cdot \left(\frac{Pv_{1,r}}{2\pi \sin i} \right)^3 \\ &= \frac{4\pi^2}{G} \cdot \frac{(M_1 + M_2)^2}{M_2^3} \cdot \frac{P^3 v_{1,r}^3}{8\pi^3 \sin^3 i} \\ &= \frac{P^3 v_{1,r}^3}{2\pi G \sin^3 i} \cdot \frac{(M_1 + M_2)^2}{M_2^3} \end{aligned}$$

Cancel P^2 from both sides (multiply both sides by $1/P^2$):

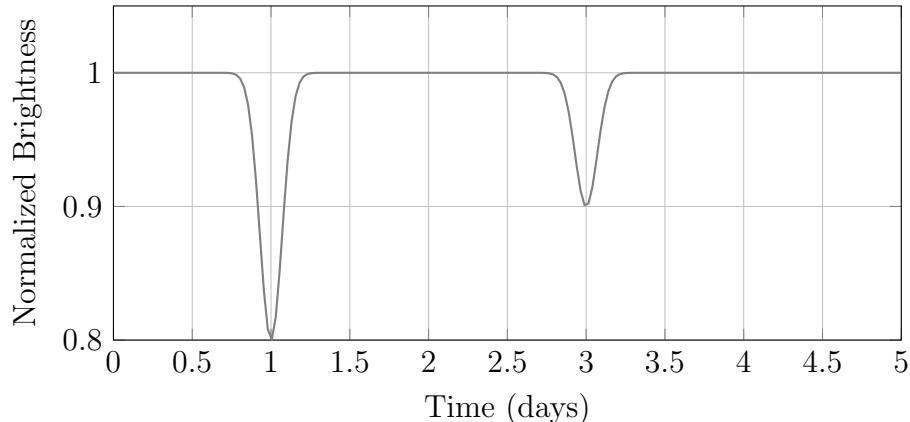
$$1 = \frac{Pv_{1,r}^3}{2\pi G \sin^3 i} \cdot \frac{(M_1 + M_2)^2}{M_2^3}$$

Now rearrange to isolate the mass terms on the left:

$$\frac{M_2^3 \sin^3 i}{(M_1 + M_2)^2} = \frac{P}{2\pi G} v_{1,r}^3$$

12.11.4 Light Curves

Eclipsing binaries exhibit characteristic light curves with periodic dips in brightness:

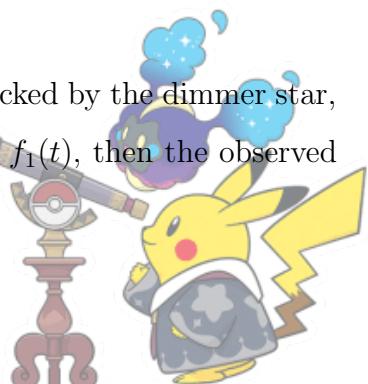


1. Let F_1 and F_2 be the flux of the two stars. The total observed flux when both stars are visible is

$$F_{\text{total}} = F_1 + F_2$$

2. The **primary eclipse** occurs when the brighter star is partially or fully blocked by the dimmer star, causing a significant dip. Let the obscured fraction of the primary star be $f_1(t)$, then the observed flux is

$$F_{\text{primary}}(t) = F_1 [1 - f_1(t)] + F_2$$



3. The **secondary eclipse** occurs when the dimmer star is blocked, causing a smaller dip. The fainter star is obscured by the brighter star. Let $f_2(t)$ be the obscured fraction of the secondary star, then:

$$F_{\text{secondary}}(t) = F_1 + F_2 [1 - f_2(t)]$$

4. If the eclipse is total, the flux reduces exactly by the luminosity of the obscured star.

In this plot,

- The first deep dip at $t = 1$ day corresponds to the **primary eclipse**.
- The smaller dip at $t = 3$ days corresponds to the **secondary eclipse**.

12.11.5 Radial Velocity Curves

A **binary star system** consists of two stars orbiting their common center of mass. If the orbital plane is inclined relative to the line of sight, the stars will alternately move toward and away from the observer. This motion causes a periodic **Doppler shift** in the spectral lines:

- Motion toward the observer will result in blueshift
- Motion away from the observer will result in redshift

The line-of-sight component of the orbital velocity is called the **radial velocity**. A **radial velocity curve** is a plot of radial velocity v_r versus time t (or orbital phase). It provides direct information about the orbital properties of the binary system.

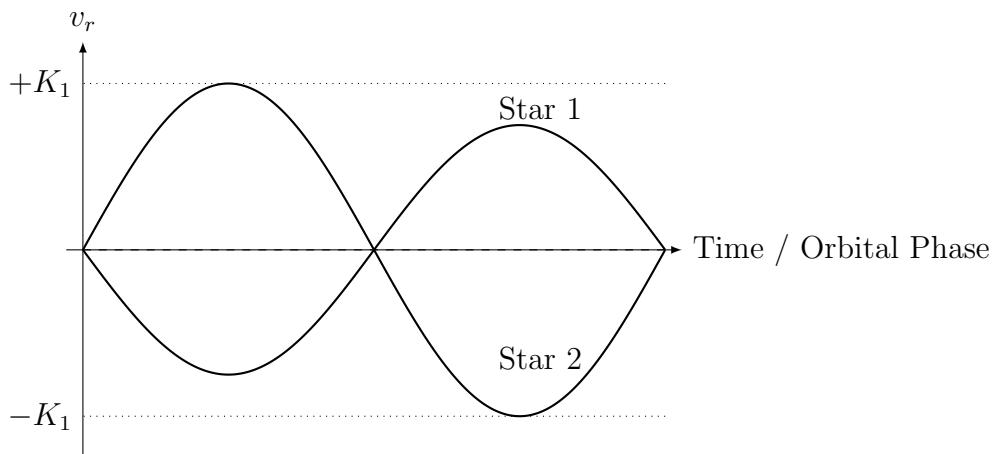
For a binary star in a circular orbit, the radial velocity varies sinusoidally:

$$v_r(t) = K \sin \left(\frac{2\pi t}{P} + \phi \right)$$

where

- K is the radial velocity amplitude
- P is the orbital period
- ϕ is the phase constant





- The curves are sinusoidal for circular orbits.
- The two stars have opposite phases due to conservation of momentum.
- The more massive star has a smaller velocity amplitude.
- The period of the curve equals the orbital period.

For a binary star in an **elliptical orbit**, the radial velocity of one component is given by:

$$v_r(t) = \gamma + K [\cos(\theta(t) + \omega) + e \cos \omega]$$

where

- γ is the systemic (center-of-mass) velocity
- K is the radial velocity semi-amplitude
- e is the orbital eccentricity
- ω is the argument of periastron
- $\theta(t)$ is the true anomaly

The semi-amplitude K is

$$K = \frac{2\pi a \sin i}{P \sqrt{1 - e^2}}$$

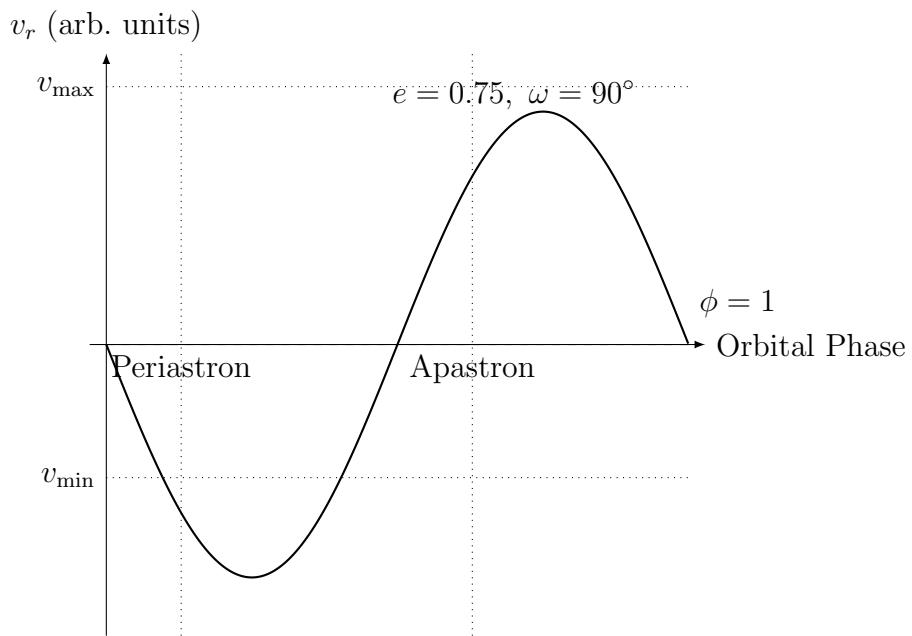
where

- a is the semi-major axis of the star's orbit about the barycenter



- i is the inclination angle
- P is the orbital period

The following shows an example of the radial velocity curve:



12.11.6 Roche Lobe

In a close binary star system, the gravitational field experienced by a test particle is determined by the combined gravity of both stars and the centrifugal effect in the rotating frame. The *Roche lobe* of a star is defined as the region around that star within which material is gravitationally bound to it. If a star fills or overflows its Roche lobe, mass transfer to its companion can occur, a key mechanism in interacting binaries such as X-ray binaries, cataclysmic variables, and some exoplanetary systems.



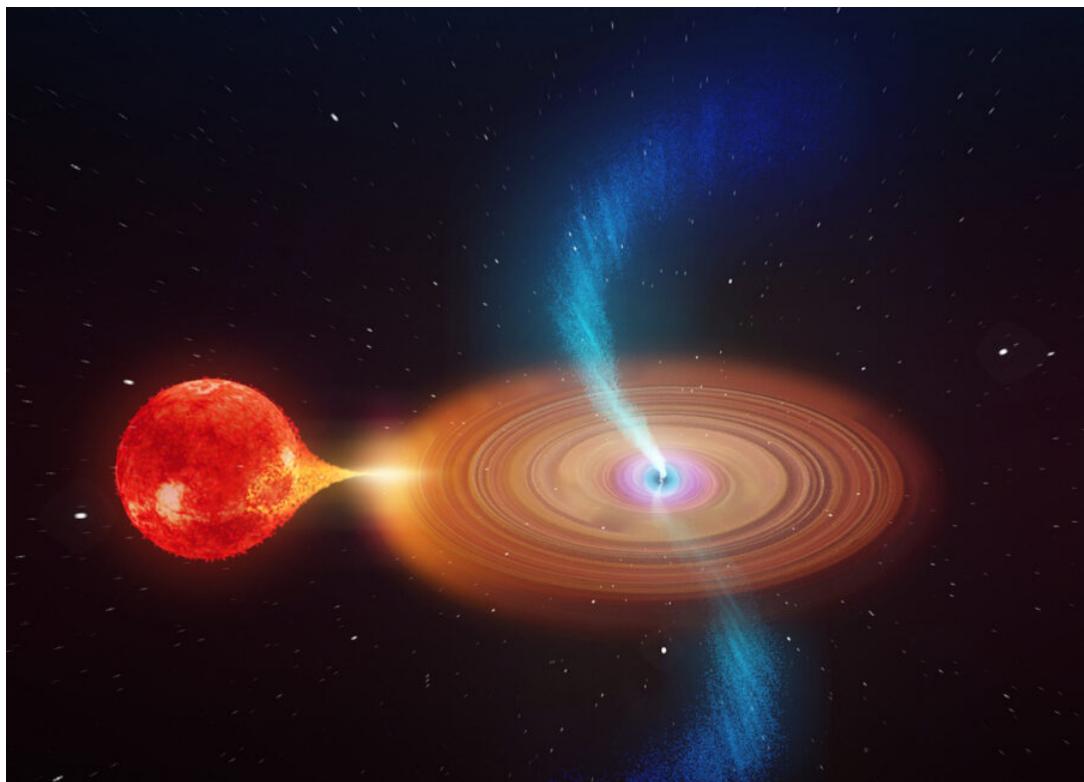


Figure 8: Source: <https://root-nation.com/ua/news-ua/it-news-ua/ua-blackhole-warped-accretion-disc/>

12.12 Exoplanet

12.12.1 Introduction

An **exoplanet** (or extrasolar planet) is a planet that orbits a star outside our Solar System. The term comes from the Greek "exo" (outside) and "planētēs" (wanderer).

12.12.2 Classes of Exoplanets

- **Hot Jupiters:** Gas giants very close to their stars.
- **Super-Earths:** Planets with masses larger than Earth but smaller than Neptune.
- **Terrestrial planets:** Rocky planets similar to Earth or Mars.
- **Ice giants:** Analogous to Uranus and Neptune.

12.12.3 Spectral Signatures of Possible Life

- Detection of biosignature gases like O₂, O₃, CH₄, and water vapor in exoplanet atmospheres.
- Observed via transmission spectroscopy during planetary transits.



12.12.4 Radial Velocity Method

It is also known as the Doppler method. This technique measures the star's wobble caused by an orbiting planet's gravitational pull.

$$\Delta v = K \left[\frac{P}{2\pi G} \right]^{1/3} \frac{m_p \sin i}{m_s^{2/3}} \frac{1}{\sqrt{1 - e^2}}$$

where

- Δv represents the velocity semi-amplitude,
- K is a constant,
- P represents the orbital period,
- m_p represents the planet mass,
- m_s represents the star mass,
- i represents the orbital inclination, and
- e represents the orbital eccentricity.

12.12.5 Transit Method

It measures the periodic dimming of a star as a planet passes in front of it.

$$\Delta F = \left(\frac{R_p}{R_s} \right)^2$$

where ΔF is the fractional flux decrease, R_p is planet radius, and R_s is star radius.

12.12.6 Habitable Zone

A habitable zone refers to the region around a star where liquid water could exist on a planet's surface.



13 Cosmology

13.1 Structure of the Universe

13.1.1 Star Clusters

Star clusters are groups of stars that are gravitationally bound and formed from the same molecular cloud. They provide important insights into stellar evolution and galactic structure. Star clusters are broadly classified into two types:

- **Open Clusters (Galactic Clusters):** These contain a few tens to a few thousand stars. They are loosely bound and typically found in the Galactic disk. Open clusters are relatively young (a few million to a few hundred million years) and often contain hot, massive stars.
- **Globular Clusters:** These are densely packed spherical collections of tens of thousands to millions of stars. They orbit the Galactic halo and are typically very old (10–13 billion years). Globular clusters are rich in low-mass stars and show little gas or dust.

Structurally, a star cluster has a *core* (densest region), a *halo* (more diffuse stars), and in some cases a *tidal radius* where stars may escape due to Galactic gravitational forces.

13.1.2 Galaxies

Galaxies are vast systems of stars, gas, dust, and dark matter, bound together by gravity. They are the fundamental building blocks of the Universe. Galaxies can be classified based on their structure, composition, and activity:

- **Elliptical galaxies:** Smooth, featureless light distribution, dominated by older stars, little gas or dust.
- **Spiral galaxies:** Flat, rotating disks with spiral arms, containing gas, dust, and young stars.
- **Barred spiral galaxies:** Spiral galaxies with a central bar-shaped structure of stars.
- **Irregular galaxies:** No definite shape, often rich in gas and dust.
- **Active galaxies:** Galaxies with energetic cores (AGN), often emitting strong radiation due to accretion onto supermassive black holes.



Our galaxy is a barred spiral galaxy with several distinct components:

- **Bulge:** Central region, high star density, mostly old stars.
- **Disk:** Contains spiral arms, gas, dust, and young stars.
- **Halo:** Spherical region with globular clusters and dark matter.

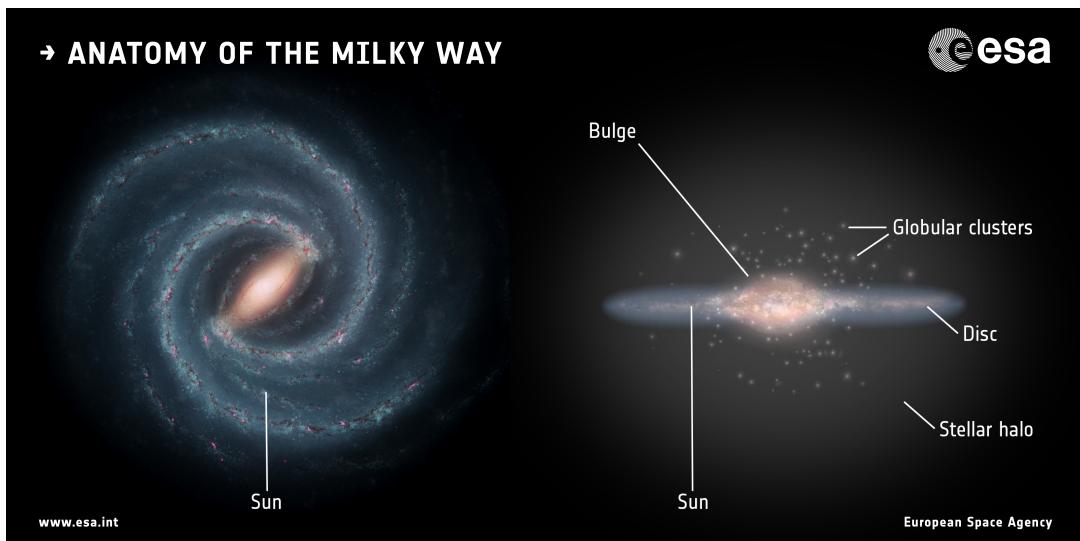


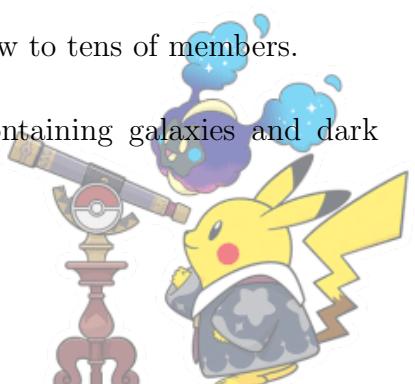
Figure 9: From <https://sci.esa.int/web/gaia/-/58206-anatomy-of-the-milky-way>

Satellites of the Milky Way include dwarf galaxies like the Large and Small Magellanic Clouds, and several ultra-faint dwarf galaxies. These satellites provide clues about the Milky Way's formation and dark matter distribution.

13.2 Large-scale Structure

The **large-scale structure** (LSS) of the Universe refers to the distribution of matter on scales larger than individual galaxies (typically $\gtrsim 1$ Mpc). On these scales, matter is not distributed uniformly, but forms a complex network known as the **cosmic web**:

- **Galaxy clusters:** Dense, gravitationally bound systems of hundreds to thousands of galaxies.
- **Galaxy groups:** Smaller associations of galaxies, often containing a few to tens of members.
- **Filaments:** Elongated structures connecting clusters and groups, containing galaxies and dark matter.
- **Voids:** Large, underdense regions with very few galaxies.



- **Walls / Sheets:** Flattened structures of galaxies separating voids.

Large-scale structure formed from tiny density fluctuations in the early Universe:

- Quantum fluctuations were stretched during cosmic inflation.
- Overdensities grew via **gravitational instability**.
- Dark matter collapsed first, forming gravitational potential wells.
- Baryonic matter later fell into these wells, forming galaxies.

On sufficiently large scales ($\gtrsim 100$ Mpc), the Universe is approximately homogeneous and isotropic, consistent with the **cosmological principle**.

13.3 Cosmological Principle

The **cosmological principle** states that on sufficiently large scales ($\gtrsim 100$ Mpc), the universe is

- **Homogeneous:** Matter and energy are uniformly distributed
- **Isotropic:** No preferred direction in space

13.4 Rotational Curve

In a galaxy, the stars or gas clouds orbit the galactic center due to the gravitational pull of the mass contained within the galaxy. According to Newtonian mechanics, the orbital velocity of an object at a given distance r from the center of a galaxy should behave as

$$v(r) = \sqrt{\frac{GM_{\text{enc}}(r)}{r}}$$

where

- $v(r)$ is the orbital velocity at a distance r from the center and
- $M_{\text{enc}}(r)$ is the enclosed mass within radius r .

The mass enclosed $M_{\text{enc}}(r)$ depends on the distribution of both visible matter (such as stars and gas) and dark matter within the galaxy.



For a galaxy dominated by visible matter (stars, gas, etc.), the enclosed mass increases with radius, but at larger distances from the center, the mass distribution becomes less dense. Then

$$v(r) \propto \frac{1}{\sqrt{r}}$$

This is called the **Keplerian decline** and is observed in the motion of planets in our solar system. However, observations of galaxies reveal that the rotation curves do not behave this way.

In the 1970s, astronomers like Vera Rubin and Kent Ford observed that the rotation curves of spiral galaxies remain nearly flat at large distances from the center, far beyond the region where visible matter is present. This observation was unexpected, as the rotation velocity should have decreased if the mass distribution followed the visible matter alone. The flatness of the rotation curve suggests that there is additional mass present that is not visible. This mass is what we refer to as **dark matter**.

The flat rotation curves observed at large radii imply that the mass within the galaxy continues to increase even in the outer regions. The orbital velocity $v(r)$ remains constant:

$$v(r) = v_{\text{flat}} \quad \text{for large } r$$

This indicates that the gravitational influence of dark matter is significant at large distances from the galactic center, where visible matter is sparse.

13.5 Hubble's Law

$$v = H_0 d$$

where v is the recession velocity, d is the proper distance to the galaxy, and H_0 is the present-day Hubble constant. One can notice that $t_H = \frac{1}{H_0}$ is the age of the universe.

Cosmic expansion is described by the **scale factor** $a(t)$, which measures how distances in the Universe change over time. The Hubble parameter is defined using the scale factor:

$$H(t) = \frac{\dot{a}(t)}{a(t)}$$

Light traveling through an expanding Universe also stretches with the expansion. If a photon is emitted at time t_{em} with wavelength λ_{em} and observed today at t_0 with wavelength λ_{obs} , the **cosmological redshift**



is defined as

$$1 + z = \frac{\lambda_{\text{obs}}}{\lambda_{\text{em}}}$$

The redshift is directly related to the scale factor:

$$1 + z = \frac{a(t_0)}{a(t_{\text{em}})}$$

13.6 Cosmological Distance Measures

13.6.1 Proper Distance

The proper distance is related to the scale factor $a(t)$ of the universe, which describes how distances between objects in the universe change with time due to the expansion. For an object at redshift z , the proper distance D_p at a given time is related to the comoving distance D_c by

$$D_p(t) = a(t)D_c$$

where

- D_p is the proper distance,
- D_c is the comoving distance,
- $a(t)$ is the scale factor at the time of observation.

13.6.2 Comoving Distance

The *comoving distance* is the distance between two objects as measured in a coordinate system that accounts for the expansion of the universe. Unlike the proper distance, the comoving distance remains constant over time for two objects that are at rest relative to each other in the expanding universe.

The comoving distance D_c at redshift z is related to the scale factor $a(t)$ by

$$D_c = \int_0^z \frac{c dz'}{H(z')}$$

where

- c is the speed of light,
- $H(z')$ is the Hubble parameter as a function of redshift z' , and



- z is the redshift at which the object is located.

13.6.3 Luminosity Distance

The *luminosity distance* is the distance to an object based on its observed brightness and its intrinsic luminosity. It is often used for objects like supernovae, which have a known intrinsic luminosity.

The luminosity distance D_L is related to the observed flux f_{obs} and the intrinsic luminosity L of an object by the inverse square law:

$$f_{\text{obs}} = \frac{L}{4\pi D_L^2}$$

From this relation, we can solve for the luminosity distance:

$$D_L = \sqrt{\frac{L}{4\pi f_{\text{obs}}}}$$

In a flat universe, the luminosity distance is related to the comoving distance by

$$D_L(z) = (1 + z)D_c(z)$$

13.6.4 Angular Diameter Distance

The angular diameter distance is related to the physical size r of an object and its angular size θ (in radians) by the relation

$$\theta = \frac{r}{D_A}$$

13.7 Friedmann Equation

Theorem. 13.1: First Friedmann Equation

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a(t)^2} + \frac{\Lambda c^2}{3}$$

where

- cosmological constant Λ represents homogeneous energy density inherent to empty space (dark energy): $\Lambda = 0$ for flat and static universe (Minkowski universe), $\Lambda > 0$ for expanding universe and $\Lambda < 0$ for contracting universe and
- curvature parameter $k = +1$ for positive curvature, $k = -1$ for negative curvature, $k = 0$ for



zero curvature.

Theorem. 13.2: Second Friedmann Equation

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3} \left(\rho - \frac{3p}{c^2} \right) + \frac{\Lambda c^2}{3}$$

- When $\Lambda = 0$ and $k = 0$, $\rho = \frac{3H^2}{8\pi G}$ which is called the critical density. We may denote ρ_c .
- Density parameter $\Omega_i := \frac{\rho_i}{\rho_c}$. Note that $\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_k = 1$. Also, by second Friedmann's equation and definition,

$$\Omega_m = \frac{8\pi G \rho_m}{3H^2}, \quad \Omega_k = -\frac{kc^2}{a^2 H^2}, \quad \Omega_\Lambda = \frac{\Lambda c^2}{8\pi G}$$

- From first and second Friedmann equation,

$$\dot{\rho} + 3H \left(\rho + \frac{p}{c^2} \right) = 0 \text{ (Fluid Equation)}$$

- The Λ CDM model, also known as the Lambda Cold Dark Matter model, is the current standard model of cosmology. It describes the evolution of the Universe from the early hot and dense state to its current large-scale structure. The model is based on the following components:

- **Cosmological Constant (Λ):** It is responsible for the accelerated expansion of the Universe. Dark energy has an opposing effect to gravity: instead of attracting matter (like gravity does), dark energy exerts a repulsive force. This repulsive force is responsible for causing the accelerated expansion of the Universe, pushing galaxies apart at an ever-increasing rate.
- **Cold Dark Matter (CDM):** It is also referred to as "cold" because it moves slowly compared to the speed of light. This is in contrast to "hot" dark matter, which would consist of fast-moving particles.

Cold dark matter is preferred in cosmological models because it allows for the formation of small structures (such as galaxies) early in the Universe's history, which is consistent with observational data.



Definition. 13.1: Hubble parameter

The Hubble parameter $H(t)$ is defined as the rate of change of the scale factor $a(t)$ and is given by

$$H(t) = \frac{\dot{a}(t)}{a(t)}$$

13.8 Equation of State

For different cosmic components,

Non-relativistic matter (dust): $p = 0 \implies \rho_m \propto a^{-3}$

Radiation: $p = \rho c^2/3 \implies \rho_r \propto a^{-4}$

Cosmological constant: $p = -\rho c^2 \implies \rho_\Lambda = \text{constant}$

Proof. Rewriting the fluid equation,

$$\frac{d\rho}{dt} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0$$

By chain rule $\dot{\rho} = \frac{d\rho}{da}\dot{a}$,

$$\dot{a}\frac{d\rho}{da} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0$$

Assuming $\dot{a} \neq 0$:

$$\frac{d\rho}{da} + \frac{3}{a}\left(\rho + \frac{p}{c^2}\right) = 0$$

By equation of states,

$$p = w\rho c^2$$

where w is a constant for each cosmological component.

$$\frac{d\rho}{da} + \frac{3}{a}(1+w)\rho = 0$$

$$\frac{d\rho}{\rho} = -3(1+w)\frac{da}{a}$$

$$\int_{\rho_i}^{\rho} \frac{d\rho'}{\rho'} = -3(1+w) \int_{a_i}^a \frac{da'}{a'}$$



$$\ln\left(\frac{\rho}{\rho_i}\right) = -3(1+w)\ln\left(\frac{a}{a_i}\right)$$

$$\frac{\rho}{\rho_i} = \left(\frac{a}{a_i}\right)^{-3(1+w)}$$

Choosing $a_i = 1$ for present epoch ($\rho_i = \rho_0$):

$$\boxed{\rho(a) = \rho_0 a^{-3(1+w)}}$$

13.9 Big Bang

13.9.1 Singularity

At the very beginning of the universe, all matter and energy were concentrated in a singularity, a point of infinite density and temperature. This singularity is thought to have contained all the space, time, and energy that would later expand to form the universe as we know it.

13.9.2 Cosmic Inflation

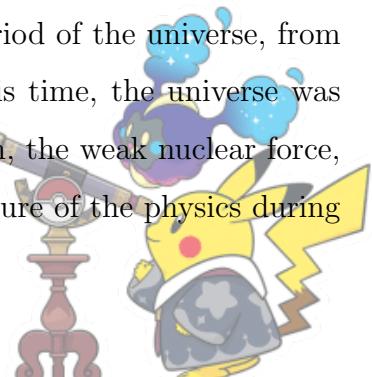
Cosmic inflation is a theory that explains the rapid expansion of the universe during the first fractions of a second after the Big Bang. During inflation, the universe expanded exponentially, increasing in size by a factor of at least 10^{26} in a fraction of a second. This theory helps explain several observed features of the universe, such as its large-scale homogeneity and isotropy, and the distribution of galaxies.

13.9.3 Expansion of Space

The Big Bang theory proposes that space itself is expanding. This expansion is not into pre-existing space; rather, it is the stretching of space itself. As space expands, the distances between distant galaxies increase, leading to the observed redshift of light from those galaxies. This expansion of the universe is described mathematically by the *Friedmann equations*.

13.9.4 Phases of the Big Bang

Planck Era (0 to 10^{-43} seconds) The Planck era represents the earliest period of the universe, from time $t = 0$ up to approximately 10^{-43} seconds after the Big Bang. During this time, the universe was incredibly hot and dense, and the fundamental forces (gravity, electromagnetism, the weak nuclear force, and the strong nuclear force) were likely unified in a single force. The exact nature of the physics during this period is unknown, as quantum gravity has yet to be fully understood.



Grand Unification Era (from 10^{-43} to 10^{-36} seconds) During the Grand Unification era, the fundamental forces separated. At the highest energies, the strong, weak, and electromagnetic forces were unified into a single force. As the universe cooled, these forces separated and the strong force, responsible for holding atomic nuclei together, emerged.

Inflationary Era (from 10^{-36} to 10^{-32} seconds) The universe underwent a brief period of exponential expansion during the inflationary era. During this time, the universe expanded by a factor of at least 10^{26} in a fraction of a second. This rapid inflation smoothed out the universe and led to the large-scale homogeneity and isotropy observed today. It also helped set the initial conditions for the formation of the first particles.

Quark Era (from 10^{-12} to 10^{-6} seconds) As the universe continued to cool, quarks, electrons, and other fundamental particles began to form. During the quark era, quarks combined to form protons and neutrons. The temperature and energy were still high enough for these particles to interact and decay frequently.

Hadron Era (from 10^{-6} seconds to 1 second) At around 10^{-6} seconds after the Big Bang, the temperature dropped enough for quarks to combine into hadrons, such as protons and neutrons. This era marked the formation of the first stable atomic nuclei.

Lepton Era (from 1 second to 10 seconds) During the lepton era, the universe was dominated by leptons (such as electrons and neutrinos). These particles were created and annihilated in large quantities. Neutrinos, which were created in abundance, decoupled from the rest of matter at around 10 seconds.

Photon Era (from 10 seconds to 380,000 years) As the universe continued to cool, photons dominated the universe. At this time, matter and radiation were tightly coupled. The universe was opaque because free electrons scattered photons, preventing light from traveling freely. However, the universe continued to expand and cool, and at about 380,000 years after the Big Bang, the universe had cooled enough for atoms to form and photons to travel freely, leading to the decoupling of matter and radiation. This event is known as the *recombination epoch* and is associated with the cosmic microwave background (CMB) radiation, which we observe today.

Recombination and the Formation of Atoms (380,000 years) During recombination, the universe cooled enough for protons and electrons to combine and form neutral hydrogen atoms. This allowed



photons to travel freely through space, marking the beginning of the era of *decoupling*. The release of these photons is known as the CMB radiation.

Dark Ages (380,000 years to 1 billion years) After the formation of atoms, the universe entered the "dark ages," a period in which there were no stars or galaxies. During this time, the universe continued to cool and matter began to clump together due to gravitational attraction. However, it was not until the formation of the first stars and galaxies that the universe became more active.

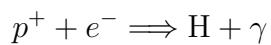
Reionization (1 billion years to 2 billion years) Reionization occurred when the first stars and galaxies formed and emitted ultraviolet light that reionized the hydrogen gas. This process ended the dark ages and allowed the universe to become transparent to ultraviolet light.

The Modern Universe (Present Day) Since the era of reionization, the universe has continued to expand and evolve. Galaxies, clusters of galaxies, and large-scale structures have formed over billions of years. The observable universe is currently about 93 billion light-years in diameter.

13.10 Cosmic Microwave Background

The Cosmic Microwave Background (CMB) is the oldest light in the universe, originating approximately 380,000 years after the Big Bang. It provides a snapshot of the infant universe when it transitioned from an opaque plasma to a transparent gas.

The CMB originates from the recombination epoch when the universe cooled sufficiently for electrons and protons to combine into neutral hydrogen atoms:



The cosmic microwave background is a faint radiation that fills the universe and is a remnant of the early hot, dense phase of the universe. It was first detected by Penzias and Wilson in 1965 and is often considered the strongest evidence for the Big Bang.



13.11 Gravitational Lensing

13.11.1 Introduction

Gravitational lensing is the bending of light by mass according to general relativity. A mass distribution between a distant source and an observer deflects light rays, producing phenomena such as multiple images, magnification, and distortion of the source.

For a point mass M (in Schwarzschild metric), a light ray with impact parameter b is deflected by an angle in weak-field. Then, the deflection angle is

$$\hat{\alpha}(b) = \frac{4GM}{bc^2}$$

where G is the gravitational constant and c the speed of light.

We use the thin-lens approximation: the deflection happens effectively at a single lens plane located at angular diameter distance D_L from the observer. Similarly, the source is at distance D_S and the distance between lens and source is D_{LS} . Let β be the true angular position of the source on the sky (unlensed), and θ the observed angular position of the image. Then

$$\beta = \theta - \alpha(\theta)$$

where the scaled deflection angle is

$$\alpha(\theta) = \frac{D_{LS}}{D_S} \hat{\alpha}(D_L \theta)$$

For a point mass M located on the optical axis, the scalar form of the lens equation (assuming alignment along a single axis) becomes

$$\beta = \theta - \frac{D_{LS}}{D_S} \frac{4GM}{c^2 D_L \theta}$$

When the source, lens and observer are perfectly aligned ($\beta = 0$) the image forms a ring (Einstein ring) with angular radius θ_E solving:

$$\theta_E = \sqrt{\frac{4GM}{c^2} \frac{D_{LS}}{D_L D_S}}$$

13.11.2 Derivation using Newtonian Mechanics

In Newtonian gravity, the force on a particle of mass m at distance r from M is

$$F = \frac{GMm}{r^2}.$$



Only the component perpendicular to the initial direction of motion contributes to the deflection. If the photon moves along the x -axis and passes the mass at distance b , then

$$r = \sqrt{x^2 + b^2}, \quad \sin \theta = \frac{b}{r}.$$

The transverse force is therefore

$$F_{\perp} = F \sin \theta = \frac{GMm}{r^2} \frac{b}{r} = \frac{GMmb}{(x^2 + b^2)^{3/2}}$$

The transverse acceleration is

$$a_{\perp} = \frac{F_{\perp}}{m} = \frac{GMb}{(x^2 + b^2)^{3/2}}$$

The photon moves approximately at constant speed c , so

$$x = ct, \quad dt = \frac{dx}{c}$$

The total change in transverse velocity is

$$\Delta v_{\perp} = \int_{-\infty}^{\infty} a_{\perp} dt = \frac{1}{c} \int_{-\infty}^{\infty} \frac{GMb}{(x^2 + b^2)^{3/2}} dx$$

Using the standard integral

$$\int_{-\infty}^{\infty} \frac{b dx}{(x^2 + b^2)^{3/2}} = \frac{2}{b}$$

we obtain

$$\Delta v_{\perp} = \frac{2GM}{bc}$$

For small deflections, the bending angle α is approximately the ratio of the transverse velocity change to the speed of light:

$$\alpha \approx \frac{\Delta v_{\perp}}{c}$$

Hence,

$$\alpha_{\text{Newton}} = \frac{2GM}{bc^2}$$

General Relativity predicts a deflection angle

$$\alpha_{\text{GR}} = \frac{4GM}{bc^2}$$



which is exactly twice the Newtonian result. The additional factor arises from the curvature of space, which is absent in Newtonian gravity.

13.11.3 Derivation using General Relativity

Definition. 13.2: Lagrangian

The **Lagrangian** \mathcal{L} is defined as

$$\mathcal{L}(q, \dot{q}, t) = T(\dot{q}) - V(q)$$

where T is the kinetic energy and V is the potential energy.

Theorem. 13.3: Euler-Lagrange Equation

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) = \frac{\partial \mathcal{L}}{\partial q}$$

Theorem. 13.4: Fermat's Principle

For light propagating in a medium with refractive index $n(\mathbf{r})$, the time taken to travel along a path \mathcal{C} from point A to point B is

$$T = \int_A^B dt = \int_A^B \frac{ds}{v} = \frac{1}{c} \int_A^B n(\mathbf{r}) ds$$

where

- $v = c/n$ is the speed of light in the medium
- $ds = \sqrt{dx^2 + dy^2 + dz^2}$ is the infinitesimal path length
- c is the speed of light in vacuum

Fermat's principle states that the actual path \mathcal{C} minimizes (or more generally, makes stationary) the **optical path length**:

$$\mathcal{S} = \int_A^B n(\mathbf{r}) ds$$



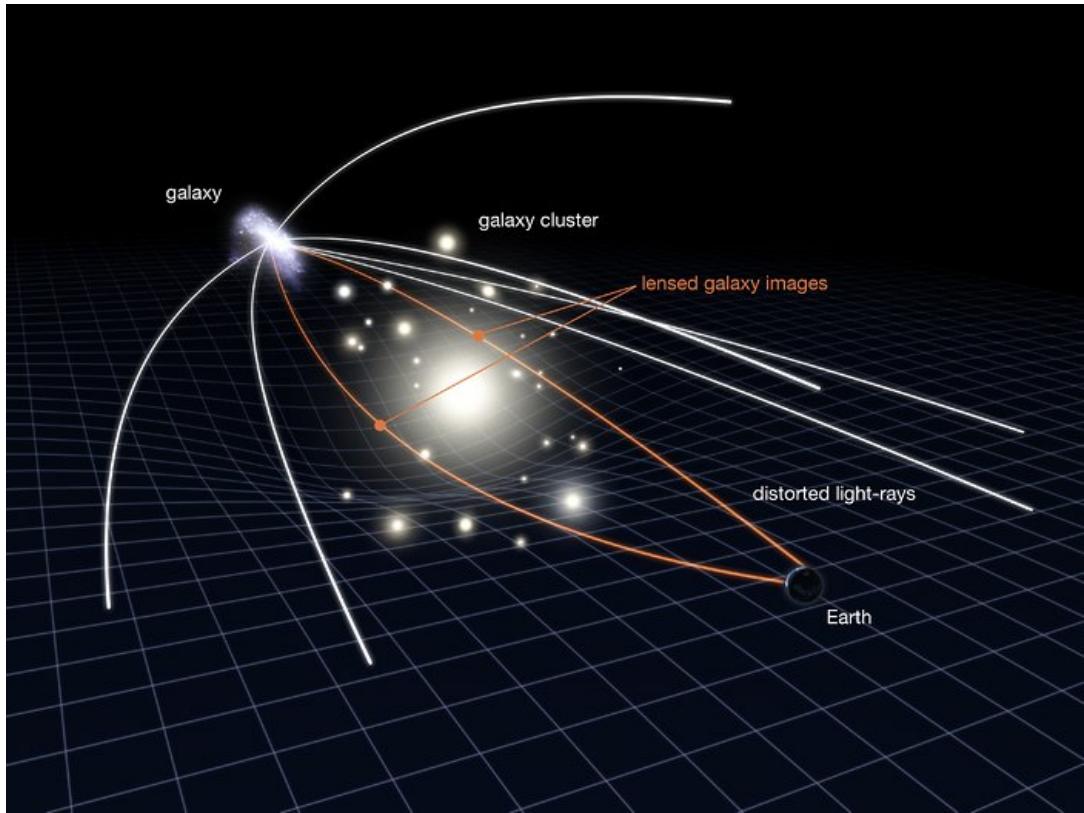


Figure 10: Source: <https://esahubble.org/images/heic1106c/>

In weak-field approximation of the spacetime metric (which is a situation in which the gravitational field is relatively weak and the spacetime curvature is small),

$$ds^2 = - \left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 + \left(1 - \frac{2\Phi}{c^2}\right) \delta_{ij} dx^i dx^j + \mathcal{O}\left(\frac{\Phi^2}{c^4}\right)$$

where $\Phi = -GM/r$ and $|\Phi| \ll c^2$. Set $c = 1$.

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)(dx^2 + dy^2 + dz^2) = 0 \implies v \approx 1 + 2\Phi$$

Consider a nearly straight path in the x - y plane with small deflection. Let $y = y(x)$, $z = 0$, with $|y'| \ll 1$. Then

$$dl = \sqrt{1 + (y')^2} dx \approx \left[1 + \frac{1}{2}(y')^2\right] dx$$

The optical path length is

$$\begin{aligned} S &= \int n dl \approx \int [1 - 2\Phi(x, y)] \left[1 + \frac{1}{2}(y')^2\right] dx \\ &\approx \int \left[1 - 2\Phi + \frac{1}{2}(y')^2\right] dx \end{aligned}$$



Consider

$$L = \frac{1}{2}(y')^2 - 2\Phi(x, y)$$

$$\frac{d}{dx} \left(\frac{\partial L}{\partial y'} \right) = \frac{\partial L}{\partial y} \implies \frac{d}{dx}(y') = -2 \frac{\partial \Phi}{\partial y} \implies \frac{d^2 y}{dx^2} = -2 \frac{\partial \Phi}{\partial y}$$

Let $y(x) = b + \varepsilon(x)$ with ε small:

$$\varepsilon''(x) = -2 \frac{\partial \Phi}{\partial y}(x, b)$$

The total deflection angle is:

$$\alpha = \Delta y' = \varepsilon'(+\infty) - \varepsilon'(-\infty) = \int_{-\infty}^{\infty} \varepsilon''(x) dx$$

For a point mass $\Phi = -GM/r$ with $r = \sqrt{x^2 + y^2}$:

$$\frac{\partial \Phi}{\partial y} = GM \frac{y}{(x^2 + y^2)^{3/2}}$$

At $y = b$:

$$\alpha = -2GMb \int_{-\infty}^{\infty} \frac{dx}{(x^2 + b^2)^{3/2}}$$

Note that

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 + b^2)^{3/2}} = \frac{2}{b^2}$$

Therefore,

$$\alpha = -2GMb \cdot \frac{2}{b^2} = -\frac{4GM}{b}$$

The magnitude of the deflection (toward the mass) is

$$|\alpha| = \frac{4GM}{b}$$

In the metric, $\Phi \rightarrow \Phi/c^2$. Then

$$\alpha = \frac{4GM}{bc^2}$$



13.12 Gravitational Wave

13.12.1 Introduction

Gravitational waves are disturbances in the curvature of spacetime caused by accelerated masses. They were first predicted by Albert Einstein in 1916 as a consequence of his General Relativity theory. Unlike electromagnetic waves, gravitational waves interact weakly with matter, making them challenging to detect but allowing them to carry information about cataclysmic cosmic events.

13.12.2 Chirp Mass

For a binary system with component masses m_1 and m_2 , the chirp mass is

$$\mathcal{M} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}} = \frac{c^3}{G} \left(\frac{5}{96} \pi^{-8/3} f^{-11/3} \dot{f} \right)^{3/5}$$

where f is the orbital frequency.

13.12.3 Binary System

For a binary system with masses m_1 and m_2 , and orbital frequency f_{orb} , the gravitational wave luminosity is

$$L_{GW} = \frac{32}{5Gc^5} (2\pi G \mathcal{M} f_{\text{orb}})^{10/3}$$

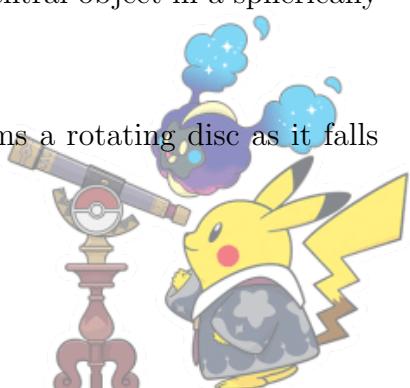
where r is the orbital separation.

13.13 Accretion Processes

13.13.1 Introduction

Accretion is the process by which matter falls onto a central object, such as a star, black hole, or neutron star, under the influence of gravity.

- Spherical accretion occurs when matter falls radially inward toward a central object in a spherically symmetric manner.
- Disc accretion occurs when matter, due to its angular momentum, forms a rotating disc as it falls toward a central object.



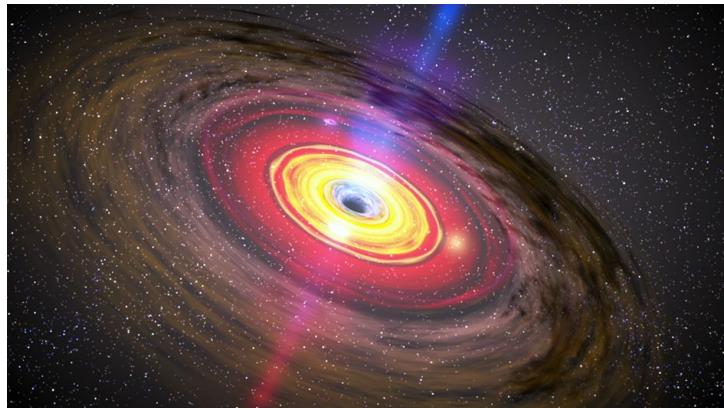


Figure 11: Source: <https://www.skyatnightmagazine.com/space-science/accretion-disk>

13.13.2 Eddington Luminosity

Consider a spherical surface with radius r centered around a light source, where the total luminosity of the source is L . The energy flux (the energy per unit time passing through a unit area) at a distance r from the source is given by the total luminosity divided by the surface area of a sphere with radius r . The surface area of a sphere is

$$A = 4\pi r^2$$

Hence, the energy flux at distance r is:

$$F = \frac{L}{4\pi r^2}$$

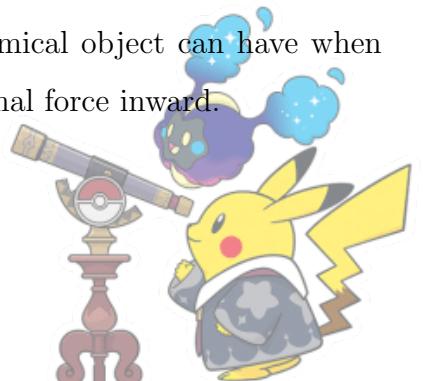
Now, consider the nature of radiation pressure. Photons carry momentum, and when they strike a surface, they transfer momentum to it. The radiation pressure is related to the energy flux by the relationship:

$$E = pc \implies P_{\text{rad}} = \frac{1}{A} \frac{dp}{dt} = \frac{F}{c}$$

where c is the speed of light, and this formula assumes that the radiation is isotropic and that the photon momentum transfer is fully efficient in transferring momentum to the surface. Then

$$P_{\text{rad}} = \frac{1}{c} \cdot \frac{L}{4\pi r^2} = \frac{L}{4\pi r^2 c}$$

The **Eddington luminosity**, L_{Edd} , is the maximum luminosity an astronomical object can have when there is a balance between the radiation pressure outward and the gravitational force inward.



The radiation pressure on an object is given by

$$P_{\text{rad}} = \frac{L}{4\pi r^2 c}$$

where L is the luminosity, r is the radius of the object, and c is the speed of light. The gravitational force is given by

$$F_{\text{grav}} = \frac{GMm}{r^2}$$

where M is the mass of the central object, m is the mass of the accreting material, and G is the gravitational constant. When balanced,

$$\frac{L}{4\pi r^2 c} = \frac{GMm}{r^2}$$

Simplifying this equation:

$$L = \frac{4\pi GMmc}{r^2}$$

For the Eddington luminosity, we consider the maximum luminosity for the material to remain bound to the central object without being blown away by radiation pressure. Using the fact that the material consists of hydrogen, for which the mass of an electron is m_e and the Thomson scattering cross-section is σ_T , we can calculate the Eddington luminosity as follows:

$$L_{\text{Edd}} = \frac{4\pi GMm_e c}{\sigma_T}$$

where $\sigma_T \approx 6.65 \times 10^{-25} \text{ m}^2$ is the effective area that quantifies the likelihood of an electron scattering a photon through Thomson scattering.

13.14 Cosmic Distance Ladder

13.14.1 Introduction

The cosmic distance ladder is a succession of methods by which astronomers determine the distances to celestial objects. Each rung of the ladder provides information that allows calibration of the next method, enabling measurements from the Solar System to the edge of the observable universe.



The Cosmic Distance Ladder

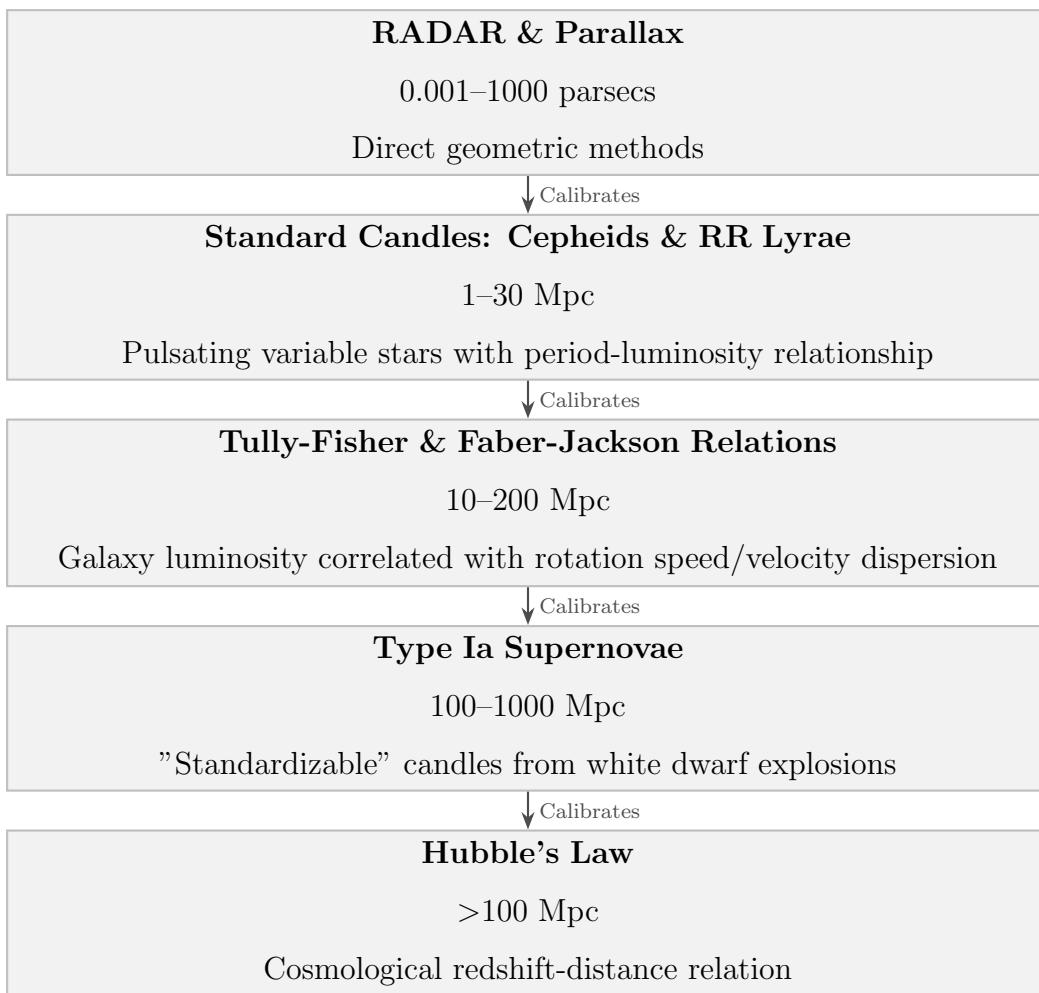


Figure 12: The hierarchy of distance measurement techniques in astronomy. Each rung calibrates the next.

13.14.2 Radar Ranging

For objects within our Solar System, we can use radar to measure distances directly:

- Transmit radio waves toward a planet or asteroid
- Measure time delay for echo to return: Δt
- Distance: $d = \frac{c \cdot \Delta t}{2}$ where c is the speed of light
- Limited to ~ 10 AU (within Solar System)



13.14.3 Stellar Parallax

Parallax uses Earth's orbit as a baseline to measure distances to nearby stars:

$$d(\text{parsecs}) = \frac{1}{\theta(\text{arcseconds})}$$

where

- d = distance in parsecs (1 pc \approx 3.26 light-years)
- θ = parallax angle in arcseconds
- Baseline = 1 AU (astronomical unit)

13.14.4 Standard Candles: Cepheid Variables

Cepheid variables are pulsating stars whose period correlates with luminosity:

$$M = a \cdot \log_{10}(P) + b$$

where

- M is the absolute magnitude (intrinsic brightness),
- P is the pulsation period in days, and
- a, b are the calibration constants.

Once the absolute magnitude M is known from the period, the distance modulus formula gives the distance:

$$m - M = 5 \log_{10} \left(\frac{d}{10 \text{ pc}} \right)$$

where

- m is the apparent magnitude (observed brightness),
- M is the absolute magnitude, and
- d is the distance in parsecs.



13.14.5 Faber-Jackson Relations

For elliptical galaxies, the velocity dispersion correlates with luminosity:

$$L \propto \sigma^\beta$$

where

- σ is the stellar velocity dispersion
- $\beta \approx 4$ empirically

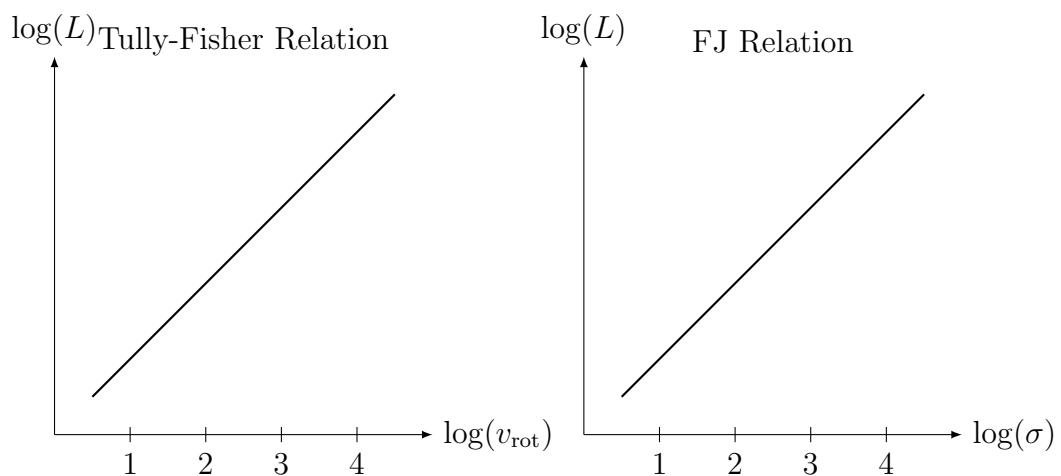


Figure 13: The Tully-Fisher and Faber-Jackson relations allow estimation of galaxy distances from measurable kinematic properties.

13.14.6 Type Ia Supernovae

Type Ia supernovae are extremely luminous and serve as excellent "standardizable candles":

- Result from thermonuclear explosion of white dwarf reaching Chandrasekhar limit ($\sim 1.4 M_\odot$)
- Peak luminosity: $\sim 10^{10} L_\odot$ (as bright as entire galaxy)
- Can be observed at cosmological distances
- Light curve shape correlates with peak luminosity (Phillips relationship)

The distance is calculated from

$$\mu = m_B - M_B + \alpha(s - 1) - \beta c$$

where



- μ is the distance modulus
- m_B is the apparent magnitude in B-band
- M_B is the absolute magnitude (calibrated)
- s is the light curve shape parameter
- c is the color correction
- α, β are the calibration coefficients



14 Interstellar Medium

14.1 Introduction

The **interstellar medium** refers to the matter that exists in the space between stars within a galaxy. It is composed of gas, dust, and cosmic rays.

It can be categorized into several distinct phases based on the temperature and density of the material:

- **Neutral Gas:** Consists of neutral hydrogen (H) and molecular hydrogen (H_2), often found in molecular clouds.
- **Ionized Gas:** Composed of ionized hydrogen (H^+) and other ionized elements, typically found in regions such as HII regions.
- **Dust:** Microscopic solid particles that can range in size from nanometers to microns, contributing to the absorption and scattering of light.
- **Cosmic Rays:** High-energy particles, primarily protons and atomic nuclei, that travel through the ISM.

Some important regions within the interstellar medium include:

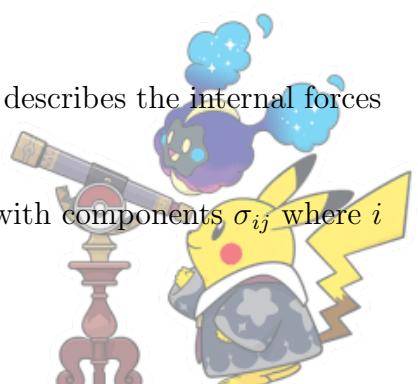
- **HII Regions:** These are regions of ionized hydrogen, created by the ultraviolet radiation from young, hot stars. They are often observed in emission lines such as $H\alpha$.
- **Molecular Clouds:** These are cold, dense regions of the interstellar medium where molecules such as H_2 are found. They are often the birthplaces of stars.
- **Warm Ionized Medium:** A diffuse component of the interstellar medium, where the gas is partially ionized and has a temperature around 10^4 K.

14.2 Fluid Dynamics

14.2.1 Stress Tensor

In continuum mechanics, the **stress tensor σ** is a fundamental concept that describes the internal forces acting within a material or fluid.

For a general three-dimensional space, the stress tensor σ is a 3×3 matrix, with components σ_{ij} where i and j refer to the directions (or axes) in space.



14.2.2 Tensor Product

Definition. 14.1: Tensor Product

Tensor product of two vectors \mathbf{A} and \mathbf{B} is denoted by $\mathbf{A} \otimes \mathbf{B}$. The resulting object is a second-order tensor, with components given by

$$(\mathbf{A} \otimes \mathbf{B})_{ij} = A_i B_j,$$

14.2.3 Divergence Theorem

Theorem. 14.1: Divergence Theorem

$$\int_{\partial V} \mathbf{F} \cdot \mathbf{n} dS = \int_V \nabla \cdot \mathbf{F} dV$$

where

- \mathbf{F} is a vector field,
- V is the volume enclosed by the surface ∂V ,
- \mathbf{n} is the outward-pointing unit normal vector on the surface ∂V ,
- dS is the surface area element on ∂V ,

14.2.4 Continuity Equation and Momentum Equation

Let $\rho(\mathbf{x}, t)$ denote the mass density and $\mathbf{v}(\mathbf{x}, t)$ the velocity field of a fluid. The total mass within a fixed control volume V with boundary ∂V and outward unit normal \mathbf{n} is

$$M_V(t) = \int_V \rho dV.$$

The principle of mass conservation states that the rate of change of mass within V equals the negative of the net mass flux through the boundary:

$$\frac{d}{dt} \int_V \rho dV = - \int_{\partial V} \rho \mathbf{v} \cdot \mathbf{n} dS$$

As V is fixed in space,

$$\int_V \frac{\partial \rho}{\partial t} dV = - \int_{\partial V} \rho \mathbf{v} \cdot \mathbf{n} dS$$



By the divergence theorem,

$$\int_V \frac{\partial \rho}{\partial t} dV = - \int_V \nabla \cdot (\rho \mathbf{v}) dV$$

Hence,

$$\int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right) dV = 0$$

As V is arbitrary, the integrand must vanish everywhere and

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0}$$

which is the local form of continuity equation.

By the material derivative $D/Dt = \partial/\partial t + \mathbf{v} \cdot \nabla$,

$$\boxed{\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0}$$

Consider the momentum of fluid within a fixed control volume V :

$$\int_V \rho \mathbf{v} dV$$

$$\frac{d}{dt} \int_V \rho \mathbf{v} dV = - \int_{\partial V} \rho \mathbf{v} (\mathbf{v} \cdot \mathbf{n}) dS + \int_{\partial V} \boldsymbol{\sigma} \cdot \mathbf{n} dS + \int_V \rho \mathbf{f} dV$$

where

- $\boldsymbol{\sigma}$ is the stress tensor,
- \mathbf{f} is the body force per unit mass (e.g. gravity).

Using the divergence theorem,

$$\int_{\partial V} \rho \mathbf{v} (\mathbf{v} \cdot \mathbf{n}) dS = \int_V \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) dV, \quad \int_{\partial V} \boldsymbol{\sigma} \cdot \mathbf{n} dS = \int_V \nabla \cdot \boldsymbol{\sigma} dV$$

Therefore,

$$\int_V \left(\frac{\partial}{\partial t} (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) - \nabla \cdot \boldsymbol{\sigma} - \rho \mathbf{f} \right) dV = 0$$

As V is arbitrary,

$$\boxed{\frac{\partial}{\partial t} (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) = \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{f}}$$



which is the **Cauchy momentum equation in conservation form**. Using the continuity equation, it can be rewritten in material-derivative form:

$$\rho \frac{D\mathbf{v}}{Dt} = \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{f}$$

14.2.5 Euler's Equation (Special Case of Navier-Stokes Equation)

From

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla \cdot (\rho\mathbf{v} \otimes \mathbf{v}) = \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{f}$$

where ρ is the density of the fluid, \mathbf{v} is the velocity of the fluid, $\boldsymbol{\sigma}$ is the stress tensor, and \mathbf{f} represents external body forces per unit mass.

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) = \rho \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \frac{\partial \rho}{\partial t}$$

$$\nabla \cdot (\rho\mathbf{v} \otimes \mathbf{v}) = \mathbf{v} \cdot \nabla(\rho\mathbf{v}) = \mathbf{v} \cdot \nabla(\rho)\mathbf{v} + \rho\mathbf{v} \cdot \nabla\mathbf{v}$$

For an inviscid fluid (no viscosity), the stress tensor only contains the pressure term:

$$\boldsymbol{\sigma} = -p\mathbf{I}$$

where p is the pressure and \mathbf{I} is the identity matrix. The divergence of the stress tensor is

$$\nabla \cdot \boldsymbol{\sigma} = -\nabla p$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho \mathbf{v} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \rho \mathbf{f}$$

For an incompressible flow (constant ρ):

$$\frac{\partial \rho}{\partial t} = 0 \quad \text{and} \quad \nabla \rho = 0.$$

This implies that the terms $\mathbf{v} \frac{\partial \rho}{\partial t}$ and $\mathbf{v} \cdot \nabla \rho \mathbf{v}$ vanish. Hence,

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \rho \mathbf{f}$$

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p + \mathbf{f}$$



14.3 Case Study

14.3.1 Background

The interstellar medium consists of gas and dust at various densities and temperatures, organized into molecular clouds, atomic gas, and ionized phases. Gravitational collapse occurs when self-gravity overcomes pressure support, leading to star formation. The interstellar medium exhibits a wide range of densities:

Phase	Density, ρ (g cm $^{-3}$)	Collapse Timescale, t_{ff}
Molecular Cloud (diffuse)	4.2×10^{-22}	$\sim 10^7$ yr
Molecular Cloud (dense core)	4.2×10^{-20}	$\sim 10^6$ yr
Atomic HI Cloud	4.2×10^{-24}	$\sim 10^8$ yr
HII Region	4.2×10^{-25}	$\sim 10^9$ yr

14.3.2 Derivation

- Continuity equation (mass conservation):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

- Euler equation (momentum conservation):

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla p - \nabla \Phi$$

- Poisson equation for the gravitational potential where $\nabla^2 f = \nabla \cdot (\nabla f)$ and $\nabla^2 \Phi = f$:

$$\nabla^2 \Phi = 4\pi G \rho$$

Consider an uniform background:

$$\rho = \rho_0 + \delta\rho, \quad \mathbf{v} = \delta\mathbf{v}, \quad p = p_0 + \delta p, \quad \Phi = \Phi_0 + \delta\Phi$$

To isolate pure gravitational collapse, we consider the pressure-less limit:



- Linearized continuity equation:

$$\frac{\partial \delta \rho}{\partial t} + \rho_0 \nabla \cdot \delta \mathbf{v} = 0$$

- Linearized Euler equation (no pressure):

$$\frac{\partial \delta \mathbf{v}}{\partial t} = -\nabla \delta \Phi.$$

- Linearized Poisson equation:

$$\nabla^2 \delta \Phi = 4\pi G \delta \rho$$

Therefore,

$$\frac{\partial}{\partial t} (\nabla \cdot \delta \mathbf{v}) = -\nabla^2 \delta \Phi$$

and then

$$\frac{\partial}{\partial t} (\nabla \cdot \delta \mathbf{v}) = -4\pi G \delta \rho$$

Also,

$$\frac{\partial^2 \delta \rho}{\partial t^2} + \rho_0 \frac{\partial}{\partial t} (\nabla \cdot \delta \mathbf{v}) = 0$$

Finally,

$$\frac{\partial^2 \delta \rho}{\partial t^2} + \rho_0 [-4\pi G \delta \rho] = 0$$

Hence,

$$\frac{\partial^2 \delta \rho}{\partial t^2} = 4\pi G \rho_0 \delta \rho$$

which is in the form of $\ddot{\delta \rho} = \omega^2 \delta \rho$ with

$$\omega^2 \equiv 4\pi G \rho_0 > 0$$

The $e^{\omega t}$ mode represents exponential growth of density perturbations which represent gravitational collapse.

The $e^{-\omega t}$ mode decays and is typically not physically relevant for collapse initial conditions. The **e-folding time** (characteristic growth/collapse timescale) is

$$\tau \equiv \frac{1}{\omega} = \frac{1}{\sqrt{4\pi G \rho_0}} \sim \frac{1}{\sqrt{G \rho_0}}$$



15 Study of the Earth

15.1 Tides

Tides are the periodic rise and fall of sea levels caused by the gravitational forces exerted by the Moon and the Sun on Earth.

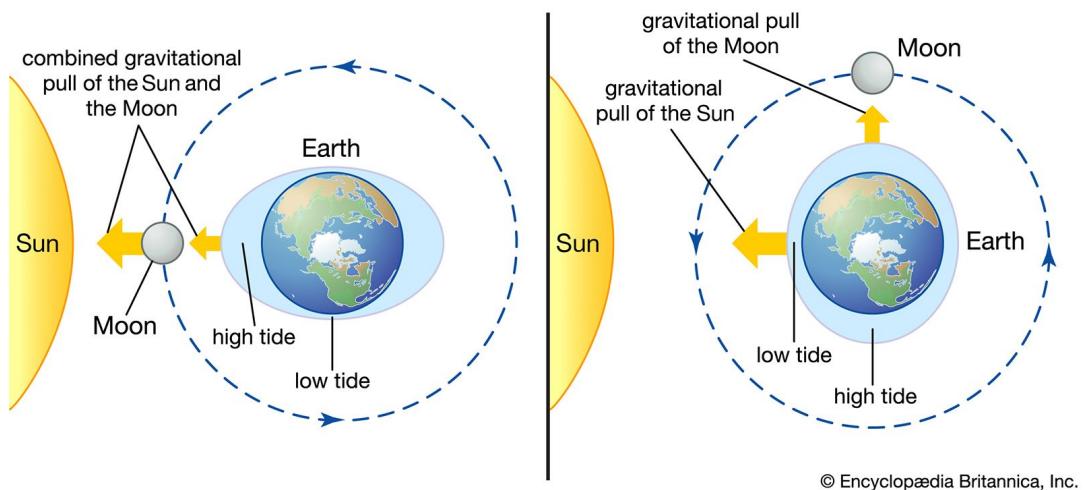
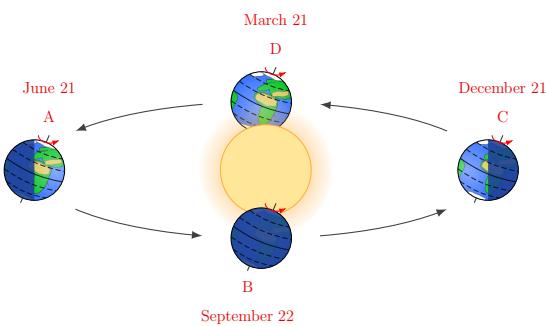


Figure 14: Source: <https://www.britannica.com/science/tide>

15.2 Seasons

Seasons are caused by the tilt of Earth's axis (23.5°) relative to its orbit around the Sun.

- Summer occurs in the hemisphere tilted toward the Sun.
- Winter occurs in the hemisphere tilted away from the Sun.
- Spring and Autumn occur when neither hemisphere is tilted toward the Sun.



15.3 Factors Influencing Climate

Climate depends on several natural factors:

- Latitude: Distance from the equator affects temperature.
- Altitude: Higher altitudes are cooler.
- Ocean Currents: Warm or cold currents can affect coastal climates.
- Topography: Mountains can block winds and affect rainfall.
- Human Activities: Urbanization and greenhouse gases influence climate.

15.4 Eclipses

15.4.1 Solar Eclipse

It occurs when the Moon comes between the Earth and Sun, blocking sunlight.

15.4.2 Lunar Eclipse

It occurs when the Earth comes between the Sun and Moon, casting a shadow on the Moon.

15.5 Space Weather

15.5.1 Introduction

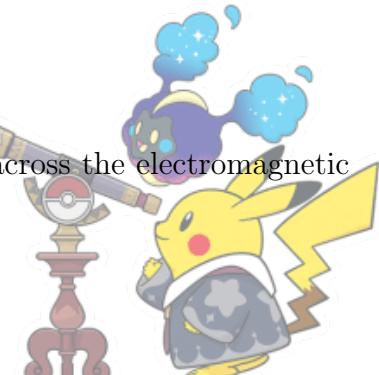
Space weather refers to the dynamic conditions in Earth's space environment, primarily influenced by the Sun.

15.5.2 Solar Wind

The solar wind is a stream of charged particles released from the upper atmosphere of the Sun, called the corona. It affects the Earth's magnetosphere and can disrupt satellite communications.

15.5.3 Solar Flares

Solar flares are sudden bursts of radiation from the Sun. They release energy across the electromagnetic spectrum, which can impact radio communications and GPS systems on Earth.



15.5.4 Coronal Mass Ejections (CMEs)

CMEs are massive bursts of solar wind and magnetic fields rising above the solar corona. They can trigger geomagnetic storms that affect satellites, power grids, and auroras.

15.5.5 Aurorae

Aurorae (Northern and Southern Lights) are caused by charged particles from the Sun interacting with Earth's magnetic field and atmosphere.

15.6 Meteor Showers

Meteor showers occur when Earth passes through the debris left by a comet.

16 Study of the Universe

16.1 Human Exploration of the Solar System

Human exploration of the Solar System involves sending astronauts beyond Earth to explore, study, and potentially inhabit other celestial bodies.

- **Purpose:** advancing scientific knowledge, developing space technology, inspiring societies, and ensuring long-term survival of humanity.
- **Major challenges:** long-duration exposure to microgravity, cosmic radiation, limited medical support, psychological isolation, and safe re-entry.
- **Key destinations:** the Moon for testing technologies, Mars for long-term exploration, and near-Earth asteroids for scientific and resource studies.
- **Approach:** step-by-step expansion using space stations, lunar missions, and sustainable life-support systems.

16.2 Planetary Missions

Planetary missions are robotic or crewed missions designed to explore planets, moons, asteroids, and comets.



- **Flyby missions:** provide brief but valuable observations with minimal fuel and mission complexity.
- **Orbiter missions:** allow long-term monitoring, global mapping, and atmospheric studies.
- **Landers and rovers:** enable direct surface analysis, geology, and chemical investigations, but require complex entry, descent, and landing systems.
- **Scientific value:** reveal planetary formation history, climate evolution, and potential habitability.

17 Study of the Moon

17.1 Precession

Precession is the slow, conical motion of the Earth's rotation axis caused primarily by the gravitational torque of the Sun and Moon on Earth's equatorial bulge.

$$\text{Rate of precession} \approx 50.3'' \text{ per year}$$

Derivation. Consider the Earth as an oblate spheroid with equatorial radius R_e and polar radius R_p . The Earth's equatorial bulge experiences a gravitational torque due to the Sun (or Moon):

$$\tau = \mathbf{r} \times \mathbf{F}$$

The magnitude of the torque is proportional to the Earth's moment of inertia difference and the gravitational force:

$$\tau \approx \frac{3GM_s}{2r^3}(C - A) \sin 2\theta$$

where

- G is the gravitational constant,
- M_s is the mass of the Sun,
- r is the Earth-Sun distance,
- C and A are the Earth's principal moments of inertia about polar and equatorial axes respectively, and
- θ is the obliquity of the ecliptic ($\approx 23.5^\circ$).



The precessional angular velocity Ω_p is given by the ratio of the torque to the Earth's spin angular momentum $L = C\omega$:

$$\Omega_p = \frac{\tau}{C\omega} = \frac{3GM_s}{2r^3} \frac{C - A}{C\omega} \cos \theta$$

Here, ω is the Earth's spin angular velocity. The $\cos \theta$ factor appears due to the component of torque perpendicular to the spin axis. For an oblate Earth,

$$C - A = \frac{2}{5}M_e R_e^2 J_2$$

where

- M_e is the Earth's mass,
- $J_2 \approx 1.0826 \times 10^{-3}$ is the dynamical flattening coefficient.

The precessional angular velocity becomes

$$\Omega_p = \frac{3GM_s}{2r^3} \cdot \frac{\frac{2}{5}M_e R_e^2 J_2}{C\omega} \cos \theta$$

Using $C \approx \frac{2}{5}M_e R_e^2$, we simplify:

$$\Omega_p \approx \frac{3GM_s}{2r^3\omega} J_2 \cos \theta$$

Substitute the values

$$G = 6.674 \times 10^{-11} \text{ m}^3 \text{kg}^{-1} \text{s}^{-2}$$

$$M_s = 1.989 \times 10^{30} \text{ kg}$$

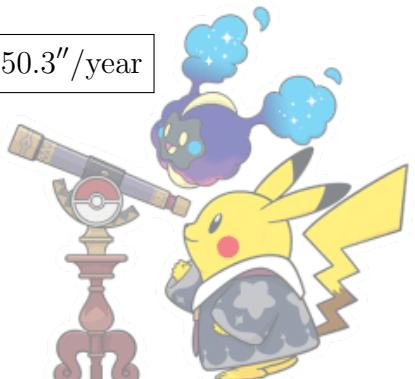
$$r = 1.496 \times 10^{11} \text{ m}$$

$$\omega = 7.292 \times 10^{-5} \text{ rad/s}$$

$$J_2 = 1.0826 \times 10^{-3}$$

$$\theta = 23.5^\circ$$

$$\text{Rate of precession} = \Omega_p \times 206264.8'' \times 3.156 \times 10^7 \text{ s/year} \approx 50.3''/\text{year}$$



17.2 Nutation

Nutation refers to small periodic oscillations superimposed on the precessional motion. These are caused by the varying positions of the Moon and Sun relative to Earth, leading to deviations in the tilt of Earth's axis.

17.3 Libration

Libration is the apparent oscillation of the Moon that allows observers on Earth to see slightly more than half of its surface over time. There are three main types:

- **Longitudinal libration:** Due to the eccentricity of the Moon's orbit.
- **Latitudinal libration:** Due to the tilt of the Moon's axis relative to its orbital plane.
- **Diurnal libration:** Due to the rotation of the Earth and the observer's changing viewpoint.



Part II: Data Analysis

18 Probability

18.1 Introduction

The probability of an event A is a number between 0 and 1 that represents the likelihood of the event occurring. It is defined as

$$P(A) = \frac{\text{Number of favorable outcomes}}{\text{Total number of possible outcomes}}$$

For example, the probability of getting heads in a fair coin toss is:

$$P(\text{Heads}) = \frac{1}{2}$$

The probability of the complement of an event A , denoted as A^c , is:

$$P(A^c) = 1 - P(A)$$

For example, if $P(\text{Heads}) = \frac{1}{2}$, then the probability of getting tails, $P(\text{Tails}) = \frac{1}{2}$.

The probability of event A occurring given that event B has occurred is called conditional probability and is denoted as $P(A|B)$. It is given by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

This is the probability of A given B , assuming $P(B) > 0$.

18.2 Random Variables

A random variable is a numerical outcome of a random phenomenon. It can be classified as either discrete or continuous:

- **Discrete Random Variable:** Takes distinct values (e.g., number of heads in 10 coin tosses).
- **Continuous Random Variable:** Takes any value within a given range (e.g., height of individuals).



The probability mass function (PMF) gives the probability of each possible outcome for discrete random variables. For example, the PMF of a fair die roll (with outcomes 1, 2, 3, 4, 5, 6) is

$$P(X = x) = \frac{1}{6}, \quad x \in \{1, 2, 3, 4, 5, 6\}$$

The probability density function (PDF) is used for continuous random variables. The probability that a continuous random variable X takes a value in the interval $[a, b]$ is given by the integral of the PDF over that interval:

$$P(a \leq X \leq b) = \int_a^b f_X(x) dx$$

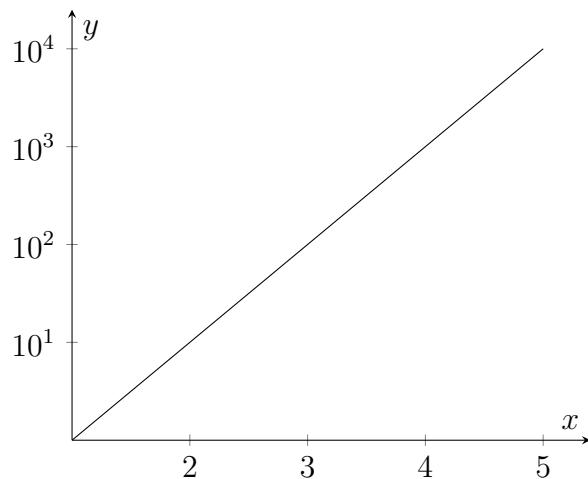
where $f_X(x)$ is the PDF of X .

19 Linear and Logarithmic Scale

In a linear scale, the distance between points is the same for each increment. The axis values increase by a constant amount.

In a logarithmic scale, the distance between values is proportional to the logarithm of the values:

Example



20 Measure of Central Tendency

Definition. 20.1: Mean

The mean \bar{x} is the sum of all values divided by the number of observations. For a sample of size n ,

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

Definition. 20.2: Median

The median is the middle value when the data is ordered from least to greatest.

- If n is odd: The median is the value at position $\frac{n+1}{2}$.
- If n is even: The median is the average of the two middle values.

Definition. 20.3: Mode

The mode is the value that appears most frequently in the dataset.

21 Measure of Dispersion

21.1 Basic Concepts

Definition. 21.1: Quartile

Quartiles are statistical measures that divide an ordered dataset into four equal parts. They help describe the spread and distribution of the data. The three quartiles are:

$$Q_1 \text{ (first quartile)}, \quad Q_2 \text{ (second quartile or median)}, \quad Q_3 \text{ (third quartile)}$$

These quartiles divide the data into four segments, each containing 25% of the observations.

Definition. 21.2: Standard Deviation

Standard deviation is a measure of the spread or dispersion of a set of numerical data. It tells us how much the values deviate, on average, from the mean (average). A small standard deviation indicates that the data points are clustered close to the mean, while a large standard deviation



indicates that the data are spread out. For the data points

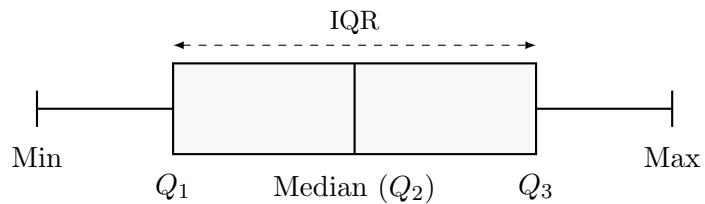
$$x_1, x_2, x_3, \dots, x_N,$$

the standard deviation is defined as

$$\sigma = \sqrt{\frac{1}{N} \sum_{i=1}^N (x_i - \bar{x})^2}$$

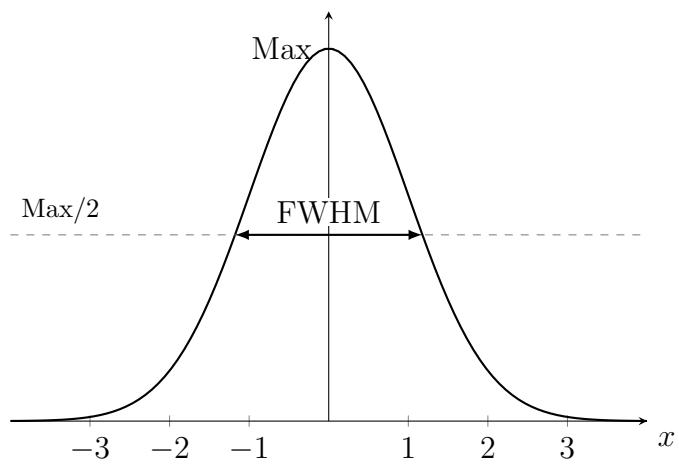
21.2 Box Plots (Box-and-Whisker)

A box plot is a graphical method for displaying the distribution of numerical data using five key summary values:



22 Full Width at Half Maximum

FWHM is the width of a curve measured at half of its maximum amplitude.



Example. For Gaussian function

$$f(x) = Ae^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where



- A is the amplitude (maximum value) of the Gaussian,
- μ is the mean (center) of the distribution,
- σ is the standard deviation (width) of the distribution.

From

$$Ae^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{A}{2},$$

$$e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{1}{2}.$$

$$-\frac{(x-\mu)^2}{2\sigma^2} = \ln \frac{1}{2} = -\ln 2.$$

$$(x-\mu)^2 = 2\sigma^2 \ln 2.$$

The points x_1 and x_2 are

$$x = \mu \pm \sigma \sqrt{2 \ln 2}.$$

Therefore, the FWHM is

$$\text{FWHM} = x_2 - x_1 = 2\sigma \sqrt{2 \ln 2}$$

23 Error Analysis

In experimental measurements, quantities have uncertainties or errors. If a function depends on multiple variables:

$$z = f(x, y, \dots),$$

and x, y, \dots have uncertainties $\Delta x, \Delta y, \dots$, then the approximate uncertainty in z is

$$\boxed{\Delta z \approx \sqrt{\left(\frac{\partial z}{\partial x} \Delta x\right)^2 + \left(\frac{\partial z}{\partial y} \Delta y\right)^2 + \dots}}$$

24 Regression Analysis

24.1 Linear regression

Linear regression is a fundamental supervised learning algorithm used to model the relationship between a **scalar response** (dependent variable, Y) and **one or more explanatory variables** (independent



variables, X). For **simple linear regression** (one independent variable), the relationship is modeled by a straight line:

$$\hat{y} = \beta_0 + \beta_1 x$$

Where \hat{y} is the predicted response, x is the independent variable, β_0 is the intercept, and β_1 is the slope.

24.1.1 Least Squares Method

The goal is to find the optimal coefficients (β_0, β_1) that make the line best fit the observed data points (x_i, y_i) . The best fit is defined by the least squares method, which minimizes the sum of the squares of the residuals (errors).

Definition. 24.1: Residual

A residual, e_i , for the i -th data point is the difference between the actual value y_i and the predicted value \hat{y}_i :

$$e_i = y_i - \hat{y}_i = y_i - (\beta_0 + \beta_1 x_i)$$

Definition. 24.2: Cost Function

The cost function, $J(\beta_0, \beta_1)$, is the quantity we want to minimize:

$$J(\beta_0, \beta_1) = \sum_{i=1}^n e_i^2 = \sum_{i=1}^n (y_i - (\beta_0 + \beta_1 x_i))^2$$

Theorem. 24.1: Linear Regression

To find the minimum of $J(\beta_0, \beta_1)$, we set the partial derivatives with respect to β_0 and β_1 to zero. This leads to the following solutions for the optimal coefficients, $\hat{\beta}_0$ and $\hat{\beta}_1$:

- The optimal slope coefficient is given by

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2}$$

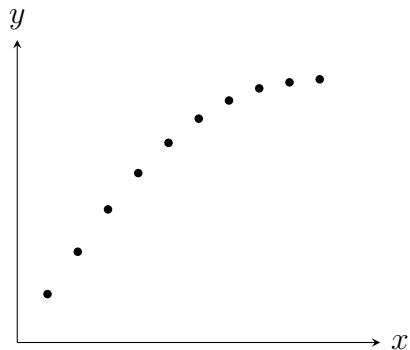
- The optimal intercept coefficient is then found using the fact that the regression line must pass through the point of means (\bar{x}, \bar{y}) :

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$



24.2 Nonlinear Regression

Nonlinear regression aims to fit a curve to data where the relationship between variables is **not linear**. While computers can do this accurately, it is possible to get an approximate fit manually using **eyes and pen**.



Draw a smooth curve that visually passes as close as possible to all points. Adjust the shape until it captures the overall trend.



Part III: Observational Astronomy

25 Instrumentation and Space Technologies

25.1 Plate Scale

The plate scale S is given by the formula

$$S = \frac{206265}{f} \quad (\text{arcseconds per mm})$$

where

- f is the focal length of the telescope in millimeters,
- 206265 is the number of arcseconds in a radian.

If the detector has a pixel size of p mm, the plate scale in arcseconds per pixel is:

$$S_{\text{pixel}} = \frac{206265 \cdot p}{f} \quad (\text{arcseconds per pixel})$$

25.2 Telescope

25.2.1 Type of Telescope

25.2.2 Linear Magnification

The magnification of a telescope is the factor by which the telescope increases the apparent size of an object. It is given by

$$M = \frac{f_{\text{obj}}}{f_{\text{eye}}}$$

where f_{obj} is the focal length of the objective lens or mirror and f_{eye} is the focal length of the eyepiece.

25.2.3 Angular Magnification

Consider an object of height h placed at a distance D from the eye. The angle θ subtended at the eye (called the *visual angle*) is

$$\theta \approx \tan \theta = \frac{h}{D}, \quad (\theta \ll 1)$$



The approximation $\tan \theta \approx \theta$ is valid for small angles (measured in radians), which is typical in optical systems.

Angular magnification M is defined as

$$M = \frac{\theta'}{\theta},$$

where

- θ' is the angle subtended at the eye when viewing the object through the lens,
- θ is the angle subtended at the eye when viewing the object directly (unaided eye).

25.2.4 Chromatic Aberration

Chromatic aberration is an optical defect of lenses that arises due to the *dispersion* of light. It occurs because the refractive index of lens material depends on the wavelength of light.

The refractive index n of a transparent medium is a function of wavelength λ :

$$n = n(\lambda)$$

In general, shorter wavelengths (violet light) experience a higher refractive index than longer wavelengths (red light):

$$n_{\text{violet}} > n_{\text{red}}$$

The focal length f of a thin lens is given by the lens maker's formula (for thin lens)

$$\frac{1}{f} = (n - 1) \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

Since n depends on λ , the focal length also depends on wavelength:

$$f = f(\lambda)$$

Hence, different colors of light are brought to focus at different positions along the optical axis.

25.2.5 f -number

The focal ratio (or f -number) is the ratio of the telescope's focal length f to the diameter D of the aperture:

$$f/\# = \frac{f}{D}$$



A smaller focal ratio corresponds to a wider field of view and faster exposure times, which is important for observing faint objects.

25.3 Detector

25.3.1 Photometers

A photometer measures the intensity of light from a source in a specific wavelength band. It typically uses a single detector element and filters to isolate desired wavelengths.

25.3.2 Charge-Coupled Devices (CCDs)

CCDs are widely used digital detectors that convert photons into electrons and then into a measurable voltage. They offer high quantum efficiency and the ability to create 2D images. The number of electrons generated in a pixel can be calculated as

$$N_e = F_\gamma A t \text{QE}$$

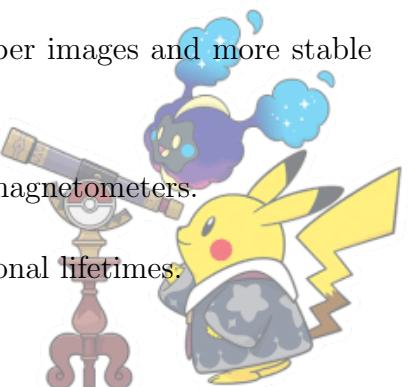
where

- F_γ is the photon flux ($\text{photons s}^{-1} \text{ m}^{-2}$),
- A is the telescope collecting area (m^2),
- t is the exposure time (s),
- QE is the quantum efficiency of the detector.

25.4 Space-Based Instruments

Space-based instruments operate above Earth's atmosphere to observe the universe with high precision.

- **Atmospheric limitations:** Earth's atmosphere absorbs or distorts many wavelengths such as ultraviolet, X-ray, and infrared radiation.
- **Improved resolution:** absence of atmospheric turbulence allows sharper images and more stable measurements.
- **Instrument types:** telescopes, spectrometers, particle detectors, and magnetometers.
- **Constraints:** high cost, limited repair opportunities, and finite operational lifetimes.



25.5 Signal-to-Noise Ratio

25.5.1 Introduction

Signal-to-Noise Ratio (SNR) is a measure of the quality of a signal in the presence of noise. It quantifies how much the signal stands out from the background noise. High SNR means a clear signal, while low SNR indicates that noise dominates.

SNR is defined as

$$\text{SNR} = \frac{S}{N}$$

where

- S is the signal strength (e.g., number of detected photons from the source),
- N is the noise (e.g. standard deviation of the background).

25.5.2 Pure Poisson Noise

Definition. 25.1: Poisson Distribution

The Poisson distribution is a discrete probability distribution that expresses the probability of a number of events occurring in a fixed interval of time or space, given the average number of occurrences. It is commonly used to model random events like the number of phone calls received at a call center, the number of accidents at an intersection, or the number of particles decaying per unit time. The probability mass function of a Poisson distribution is given by

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2, \dots$$

where

- X is the random variable representing the number of events,
- λ is the average number of occurrences in the given interval (also called the rate or intensity),
- $k!$ is the factorial of k , i.e., $k! = k(k - 1)(k - 2) \dots 1$.

The Poisson distribution is characterized by the mean and variance both being equal to λ , i.e.,

$$\mu = \sigma^2 = \lambda.$$



For photon-counting detectors, the dominant noise is often Poisson. If the expected signal is S photons, then Poisson statistics give

$$\sigma = \sqrt{S}$$

Hence,

$$\text{SNR} = \frac{S}{\sqrt{S}} = \sqrt{S}$$

25.5.3 Background Noise

Suppose the measurement includes both the object signal S and the background B . The total number of detected photons is

$$S_{\text{tot}} = S + B$$

Poisson statistics give total variance

$$\sigma^2 = S + B$$

Hence,

$$\text{SNR} = \frac{S}{\sqrt{S + B}}$$

25.5.4 Readout Noise

A CCD or CMOS detector adds readout noise σ_{read} per pixel. If n_{pix} pixels are used, the read noise contribution is

$$\sigma_{\text{read,tot}}^2 = n_{\text{pix}}\sigma_{\text{read}}^2$$

The total variance becomes

$$\sigma^2 = S + B + n_{\text{pix}}\sigma_{\text{read}}^2$$

Hence,

$$\text{SNR} = \frac{S}{\sqrt{S + B + n_{\text{pix}}\sigma_{\text{read}}^2}}$$

25.5.5 Complete Equation

$$\text{SNR} = \frac{S}{\sqrt{S + B + n_{\text{pix}} \left(1 + \frac{n_{\text{pix}}}{n_B} \right) (N_s + N_D + \sigma_{\text{read}}^2 + G^2 \sigma_f^2)}}$$



where

- n_B is the background pixels in the image, which correspond to regions with no object or light source.
- N_s represents the signal from the source, representing the number of photons from the object being observed.
- N_D represents the dark noise, which arises from thermally generated electrons in the CCD detector, even in the absence of light.
- G is the gain of the CCD detector, which relates the number of electrons collected by the pixel to the digital output recorded by the system.
- σ_f is the fluctuations or Fano noise, which arises from the statistical nature of electron counting processes.

