

# Astronomy and Astrophysics Olympiad (Theory and Data Analysis)

A Comprehensive Note Created by Former Hong Kong Team Members of IOAA

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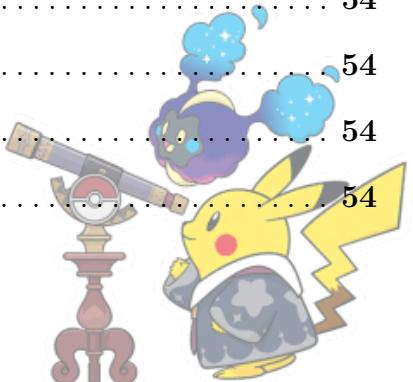
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## Theory

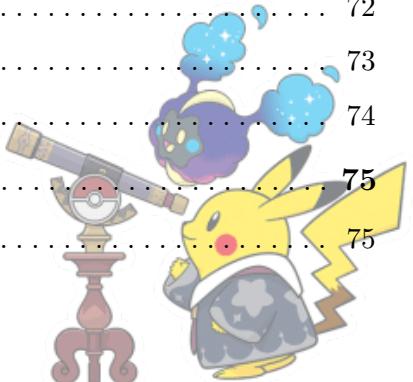
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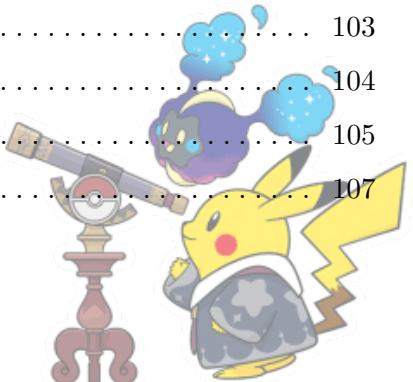
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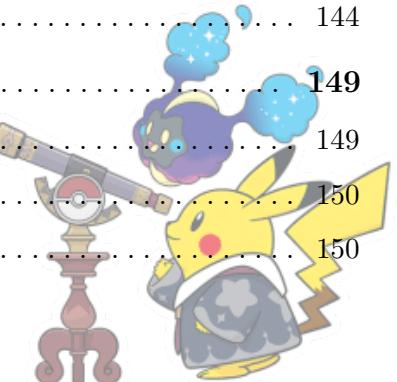
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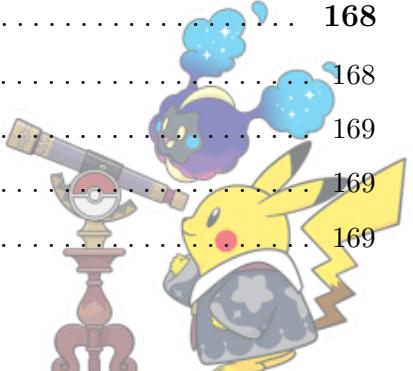
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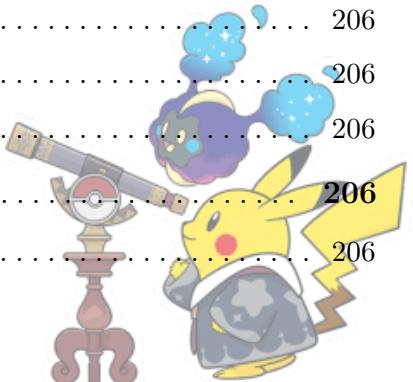
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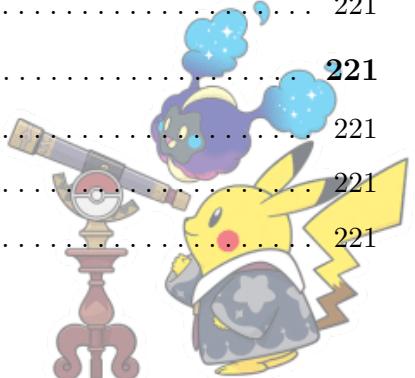
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## Part I: Theory

# 1 Mathematical Physics

## 1.1 Basic Set Theory

A **set** is any collection of distinct objects, considered as an object in its own right. The objects in a set can be anything like real numbers  $\mathbb{R}$ . We denote a set by curly braces:

$$A = \{a_1, a_2, a_3, \dots\}$$

where  $a_1, a_2, a_3, \dots$  are elements of the set  $A$ . We define several common operations on sets:

- **Union:** The union of two sets  $A$  and  $B$  is the set of all elements that are in either  $A$  or  $B$  (or both).

Denoted as:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

- **Intersection:** The intersection of two sets  $A$  and  $B$  is the set of all elements that are in both  $A$  and  $B$ . Denoted as:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

- **Difference:** The difference of two sets  $A$  and  $B$ , denoted  $A \setminus B$ , is the set of all elements that are in  $A$  but not in  $B$ :

$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$$

- **Complement:** The complement of a set  $A$ , denoted  $A^c$ , is the set of all elements not in  $A$ .

$$A^c = \{x \mid x \notin A\}$$

We say that a set  $A$  is a **subset** of a set  $B$ , denoted  $A \subseteq B$ , if every element of  $A$  is also an element of  $B$ :

$$A \subseteq B \quad \text{if} \quad \forall x (x \in A \implies x \in B)$$

where  $\forall$  means for all. Conversely,  $B$  is a **superset** of  $A$ , denoted  $B \supseteq A$ , if every element of  $A$  is in  $B$ .



## 1.2 Function

A **function** is a relationship between two sets, where each element of the first set (called the **domain**) is associated with exactly one element of the second set (called the **codomain**). A function assigns a unique output to every input from the domain. Formally, a function  $f$  from a set  $A$  to a set  $B$  is denoted as

$$f : A \rightarrow B$$

where  $A$  is the **domain** of the function,  $B$  is the **codomain** of the function. For each  $a \in A$ , there exists a unique  $b \in B$  such that  $f(a) = b$ . There are several important types of functions that are commonly used in mathematics. Below are some of the most important ones:

- A function  $f : A \rightarrow B$  is called **one-to-one** or **injective** if distinct elements in the domain  $A$  map to distinct elements in the codomain  $B$ . In other words:

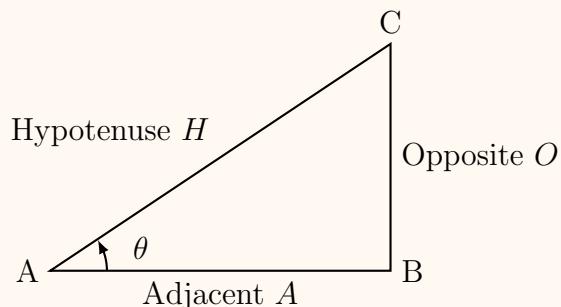
$$f(x_1) = f(x_2) \implies x_1 = x_2 \quad \text{for all } x_1, x_2 \in A$$

- A function  $f : A \rightarrow B$  is called **onto** or **surjective** if for every element  $b \in B$ , there exists at least one element  $a \in A$  such that  $f(a) = b$ . In other words, the range of  $f$  is the entire codomain  $B$ .
- A function  $f : A \rightarrow B$  is called **bijective** if it is both injective (one-to-one) and surjective (onto). This means that every element of the domain is mapped to a unique element in the codomain, and every element of the codomain has a corresponding element in the domain.

## 1.3 Trigonometry

### Definition. 1.1: Trigonometric Function

Consider a right triangle with an angle  $\theta$ , opposite side  $O$ , adjacent side  $A$ , and hypotenuse  $H$ . Recall that



For the angle  $\theta$  in a right triangle:

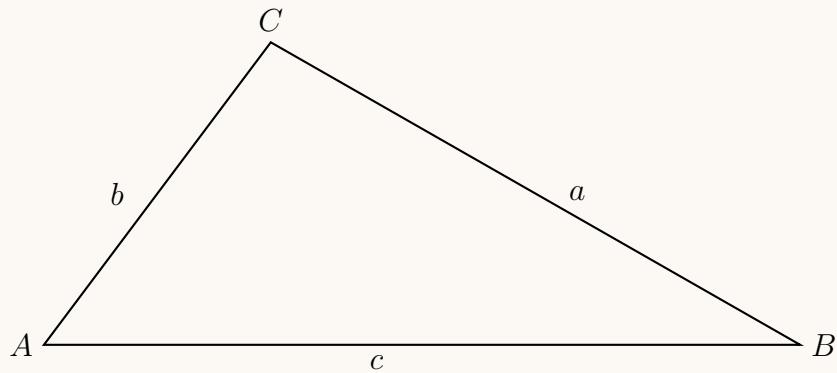
$$\sin \theta = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{O}{H}, \quad \csc \theta = \frac{H}{O}$$

$$\cos \theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{A}{H}, \quad \sec \theta = \frac{H}{A}$$

$$\tan \theta = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{O}{A}, \quad \cot \theta = \frac{A}{O}$$

### Theorem. 1.1: Law of Sine and Law of Cosine

Consider any triangle  $ABC$  with sides  $a$ ,  $b$ , and  $c$  opposite angles  $A$ ,  $B$ , and  $C$ , respectively:



The law of sine states

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The law of cosine states

$$c^2 = a^2 + b^2 - 2ab \cos C$$

### Definition. 1.2: Degree

Degrees are often divided into smaller units for more precise measurements. The most common smaller units are minutes and seconds:

- 1 degree ( $^\circ$ ) = 60 minutes ( $'$ )
- 1 minute ( $'$ ) = 60 seconds ( $''$ )



**Definition. 1.3: Radian**

A **radian** is defined as the angle subtended at the center of a circle by an arc whose length is equal to the radius of the circle:

$$\theta = \frac{s}{r}$$

where

- $\theta$  is the angle in radians,
- $s$  is the length of the arc, and
- $r$  is the radius of the circle.

Radians are closely related to degrees, but they are more natural for mathematical calculations. The relationship between radians and degrees is given by the following conversion formulas:

$$\theta_{\text{degrees}} = \theta_{\text{radians}} \times \frac{180}{\pi}$$

$$\theta_{\text{radians}} = \theta_{\text{degrees}} \times \frac{\pi}{180}$$

**1.4 Logarithm**

Recall that

$$\log_b x = y \iff b^y = x$$

**Definition. 1.4: Natural Number  $e$** 

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$$

which is  $\left(1 + \frac{1}{n}\right)^n$  when  $n \rightarrow \infty$ .

Define natural logarithm  $\log_e x$  by  $\ln$ . Note that

$$1. \ln(xy) = \ln x + \ln y$$

$$2. \ln\left(\frac{x}{y}\right) = \ln x - \ln y$$

$$3. \ln(x^n) = n \ln x$$



## 1.5 Summation

The summation symbol  $\sum$  is used to denote the sum of a sequence of numbers:

$$\sum_{i=1}^n a_i = a_1 + a_2 + \cdots + a_n$$

Einstein summation notation is a compact and powerful way to write sums over indexed quantities in tensor calculus. Instead of writing explicit summation signs, repeated indices in a term imply a sum over all values of that index. If an index appears twice in a single term (once as an upper index and once as a lower index, or in the same position in Euclidean space), it is implicitly summed over all its possible values:

$$A_i B_i \equiv \sum_{i=1}^n A_i B_i$$

**Example.** For two vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ , the dot product can be written using Einstein notation as:

$$\mathbf{u} \cdot \mathbf{v} = u_i v_i = \sum_{i=1}^3 u_i v_i$$

## 1.6 Conic Section

A **conic section** is the curve obtained by intersecting a plane with a double-napped cone. The general form of the equation of a conic section in Cartesian coordinates is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where  $A, B, C, D, E$ , and  $F$  are constants that define the shape and orientation of the conic.



Conic Section Equation and Properties	
<b>Circle</b>	Equation: $(x - h)^2 + (y - k)^2 = r^2$ Center: $(h, k)$ Radius: $r$
<b>Ellipse</b>	Equation: $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$ Center: $(h, k)$ Semi-major axis: $a$ , Semi-minor axis: $b$ Orientation: If $a > b$ , major axis along $x$ -axis; if $b > a$ , major axis along $y$ -axis
<b>Parabola</b>	Equation: $y - k = a(x - h)^2$ Vertex: $(h, k)$ Axis: Vertical
<b>Hyperbola</b>	Equation: $\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$ Center: $(h, k)$ Transverse axis along $x$ -axis, Conjugate axis along $y$ -axis

Table 1: Different Conic Sections and Their Properties

## 1.7 Dimensional Analysis

### 1.7.1 Introduction

Dimensional analysis is used to

- check the consistency of equations, and
- predict how different physical quantities relate to one another.

In essence, it is based on the idea that any physical quantity can be expressed in terms of a few fundamental dimensions, such as:

$[M]$  (mass),  $[L]$  (length),  $[T]$  (time),  $[I]$  (electric current),  $[\Theta]$  (temperature).



Every valid physical equation must be **dimensionally homogeneous**, meaning all additive terms must have the same dimensions.

Consider Newton's second law:

$$F = ma$$

The dimensions are:

$$[F] = [MLT^{-2}], \quad [m] = [M], \quad [a] = [LT^{-2}]$$

Since  $[F] = [m][a]$ , the equation is dimensionally consistent.

### 1.7.2 Buckingham Pi Theorem (Optional)

Given a physical relation described by:

$$f(Q_1, Q_2, \dots, Q_n) = 0$$

where  $Q_i$  are the  $n$  physical variables, and these  $n$  variables involve  $k$  independent fundamental dimensions, the equation can be rewritten as a function of  $(n - k)$  independent dimensionless  $\Pi$  groups:

$$F(\Pi_1, \Pi_2, \dots, \Pi_{n-k}) = 0$$

Each  $\Pi$  group is a dimensionless product of the original variables raised to some powers:

$$\Pi = Q_1^a Q_2^b Q_3^c \dots$$

**Example.** Find the dimensionless relation for the drag force  $F_D$  on a sphere in a fluid, considering diameter  $D$ , fluid velocity  $V$ , fluid density  $\rho$ ,  $g$ , and fluid dynamic viscosity  $\mu$  with dimension  $[ML^{-1}T^{-1}]$ .

**Solution.** Note that  $n = 6$  and  $k = 3$  so number of pi terms  $p = 3$  so we expect 3 dimensionless groups. Choose Repeating Variables ( $k = 3$ ):  $D$ ,  $V$  and  $\rho$  (which are dimensionally independent and cover  $M$ ,  $L$  and  $T$ ). Then

$$\Pi_1 = F_D \cdot D^a V^b \rho^c$$

Solving,

$$\Pi_1 = \frac{F_D}{\rho V^2 D^2}$$



This is related to the drag coefficient  $C_D$ .

$$\Pi_2 = \mu \cdot D^a V^b \rho^c$$

Solving,

$$\Pi_2 = \frac{\mu}{\rho V D} = \text{Re}$$

where Re is the Reynolds number.

$$\Pi_3 = g \cdot D^a V^b \rho^c$$

Solving,

$$\Pi_3 = \frac{g D}{V^2}$$

This is the inverse of the Froude number (Fr) squared.

$$F(C_D, \text{Re}, \text{Fr}) = 0 \implies C_D = f(\text{Re}, \text{Fr})$$

### 1.7.3 Natural Unit

In physics, natural units are systems of units in which certain fundamental physical constants are set equal to 1. The most commonly used ones set the following fundamental constants to 1:

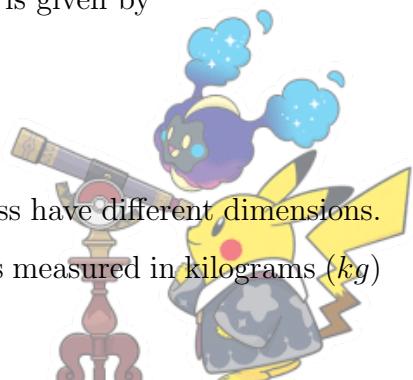
- $\hbar$  (The reduced Planck's constant)
- $c$  (The speed of light in vacuum)
- $G$  (The gravitational constant)
- $k_B$  (Boltzmann constant)
- $\epsilon_0$  (The permittivity of free space)

Dimensional analysis is used to restore the correct physical dimensions of quantities when transitioning between different unit systems.

**Example.** In natural units, the famous equation of mass-energy equivalence is given by

$$E = m$$

where  $E$  is energy and  $m$  is mass. However, in standard units, energy and mass have different dimensions. Energy is measured in joules ( $J$ ) with the dimensions  $[ML^2T^{-2}]$ , while mass is measured in kilograms ( $kg$ )



with the dimension  $[M]$ .

Using dimensional analysis, we restore the correct relationship by introducing the speed of light  $c$ . The energy and mass are related by the equation:

$$E = mc^2$$

Here,  $c$  (speed of light) has the dimension of  $[LT^{-1}]$ . Now the dimensions of both sides match because

$$[E] = [ML^2T^{-2}], \quad [mc^2] = [M][L^2T^{-2}]$$

Hence, the restored equation in standard units is

$$E = mc^2$$

#### 1.7.4 Exercise

The Planck unit system includes the following fundamental quantities: Planck Length ( $l_P$ ), Planck Time ( $t_P$ ), Planck Mass ( $m_P$ ), Planck Charge ( $q_P$ ), Planck Temperature ( $T_P$ ), Planck Energy ( $E_P$ ), Planck Force ( $F_P$ ) and Planck Power ( $P_P$ ). Each of these units is derived from combinations of the fundamental constants:

- The speed of light ( $c$ )
- The gravitational constant ( $G$ , with  $[G] = [M^{-1}L^3T^{-2}]$ )
- Planck's constant ( $\hbar = h/2\pi$ , with  $[h] = [ML^2T^{-1}]$ )
- Coulomb constant ( $\epsilon_0$ , with  $[\epsilon] = [ML^3T^{-4}I^{-2}]$ )
- Boltzmann constant ( $k_B$ , with  $[ML^2T^{-2}\Theta^{-1}]$ )

Derive each of the quantities above.



## 1.8 Tensor

### 1.8.1 Vector and Vector Space

#### Definition. 1.5: Vector

A vector (denoted by an arrow above the letter ( $\mathbf{A}$ ) or by a boldface letter ( $\mathbf{A}$ )) is a mathematical object that has both a magnitude (denoted by  $|\mathbf{A}|$  or  $A$ ) (size or length) and a direction. This is in contrast to a scalar, which only has magnitude (e.g., temperature, mass).

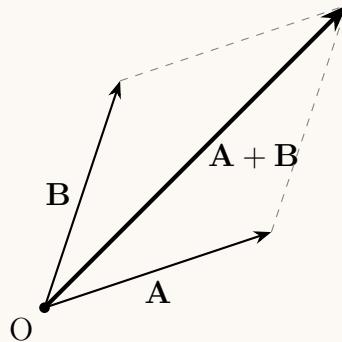
#### Definition. 1.6: Unit Vector

A unit vector ( $\hat{\mathbf{u}}$ ) has a magnitude of 1 and defines direction. The standard Cartesian unit vectors are  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  along the  $x, y$ , and  $z$  axes, respectively. A vector  $\mathbf{A}$  in 3D space can be expressed in terms of its Cartesian components ( $A_x, A_y, A_z$ ):

$$\mathbf{A} = A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k}.$$

#### Theorem. 1.2: Tail-to-tail Method

The sum of two vectors can be visualized as the diagonal of a parallelogram formed by the two vectors placed tail-to-tail:



#### Definition. 1.7: Dot Product

Given that angle between two vectors is  $\theta$ . If  $\mathbf{A} = (A_x, A_y, A_z)$  and  $\mathbf{B} = (B_x, B_y, B_z)$ , then

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}| |\mathbf{B}| \cos(\theta) = A_x B_x + A_y B_y + A_z B_z$$



**Theorem. 1.3: Projection**

The projection of  $\mathbf{a}$  onto  $\mathbf{b}$  is given by the following formula:

$$\text{proj}_{\mathbf{b}} \mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2} \mathbf{b}$$

**Example.** The effective area  $A_{\text{eff}}$  of a surface can be defined as the projection of the surface area in the direction of a particular vector. If  $\mathbf{A}$  is a vector representing the surface area and  $\hat{v}$  is a unit vector in the direction of interest, the effective area is given by the dot product:

$$A_{\text{eff}} = \mathbf{A} \cdot \hat{v}$$

where

- $\mathbf{A}$  is the vector representing the surface area (with magnitude representing the area and direction perpendicular to the surface),
- $\hat{v}$  is the unit vector in the direction of interest, and
- $A_{\text{eff}}$  is the projection of the area in the direction of  $\hat{v}$ .

**Definition. 1.8: Cross Product**

Given that angle between two vectors is  $\theta$ . If  $\mathbf{A} = (A_x, A_y, A_z)$  and  $\mathbf{B} = (B_x, B_y, B_z)$ , then the component form of cross product is given by

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2) \mathbf{i} - (a_1 b_3 - a_3 b_1) \mathbf{j} + (a_1 b_2 - a_2 b_1) \mathbf{k}$$

and the vector form of cross product is given by

$$\mathbf{A} \times \mathbf{B} = |\mathbf{A}| |\mathbf{B}| \sin(\theta) \hat{n}$$

where  $\hat{n}$  is a unit vector perpendicular to the plane formed by  $\mathbf{a}$  and  $\mathbf{b}$ , and its direction is given by the right-hand rule.



**Example.** Given two vectors  $\mathbf{a}$  and  $\mathbf{b}$ , which form two sides of a triangle, the area  $A$  of the triangle is half the magnitude of the cross product of  $\mathbf{a}$  and  $\mathbf{b}$ . Mathematically, the area is given by

$$A = \frac{1}{2}|\mathbf{a} \times \mathbf{b}|$$

### Definition. 1.9: Vector Space

A **vector space**  $V$  over a field  $\mathbb{F}$  (e.g.,  $\mathbb{R}$  or  $\mathbb{C}$ ) is a set of vectors equipped with two operations:

- **Vector addition:**  $+ : V \times V \rightarrow V$
- **Scalar multiplication:**  $\cdot : \mathbb{F} \times V \rightarrow V$

satisfying the following axioms for all  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$  and scalars  $\alpha, \beta \in \mathbb{F}$ :

- **Associativity of addition:**  $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- **Commutativity of addition:**  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- **Existence of zero vector:** There exists  $\mathbf{0} \in V$  such that  $\mathbf{v} + \mathbf{0} = \mathbf{v}$
- **Existence of additive inverse:** For each  $\mathbf{v} \in V$ , there exists  $-\mathbf{v} \in V$  such that  $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$
- **Compatibility of scalar multiplication:**  $\alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v}$
- **Identity element of scalar multiplication:**  $1 \cdot \mathbf{v} = \mathbf{v}$
- **Distributivity over vector addition:**  $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$
- **Distributivity over scalar addition:**  $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$

### 1.8.2 Matrix

**Definition** A **matrix** is an ordered rectangular array of numbers arranged in  $m$  rows and  $n$  columns. A matrix is typically denoted as  $A$ , and its elements are written as  $a_{ij}$ , where  $i$  is the row index and  $j$  is



the column index.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Here,  $A$  is an  $m \times n$  matrix (with  $m$  rows and  $n$  columns).

**Matrix Multiplication** Matrix multiplication is defined as the dot product of rows of the first matrix with columns of the second matrix. If  $A$  is an  $m \times n$  matrix and  $B$  is an  $n \times p$  matrix, then their product  $C = AB$  is an  $m \times p$  matrix, and its elements are given by

$$C_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$$

**Scalar Multiplication** A matrix  $A$  can be multiplied by a scalar  $\lambda$  (a number), which scales all the elements of the matrix:

$$\lambda A = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{m1} & \lambda a_{m2} & \cdots & \lambda a_{mn} \end{pmatrix}$$

**Determinant of a Matrix** For a  $2 \times 2$  matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the determinant is computed as

$$\det(A) = ad - bc$$



For larger matrices, the determinant can be computed using cofactor expansion. The determinant of an  $n \times n$  matrix  $A$  is given by

$$\det(A) = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det(A_{ij})$$

where  $A_{ij}$  is the matrix obtained by removing the  $i$ -th row and  $j$ -th column of  $A$ , and  $a_{ij}$  is the element in the  $i$ -th row and  $j$ -th column.

## Inverse of a Matrix

### Definition. 1.10: Minor, Cofactor and Adjoint

Let  $A = (a_{ij})$  be an  $n \times n$  matrix. The **minor**  $M_{ij}$  of the entry  $a_{ij}$  is the determinant of the  $(n-1) \times (n-1)$  matrix obtained by deleting the  $i$ -th row and  $j$ -th column of  $A$ .

The **cofactor**  $C_{ij}$  of  $a_{ij}$  is defined by

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Let  $A$  be an  $n \times n$  matrix.

The **adjoint** (or **adjugate**) of  $A$ , denoted by  $\text{adj}(A)$ , is given by

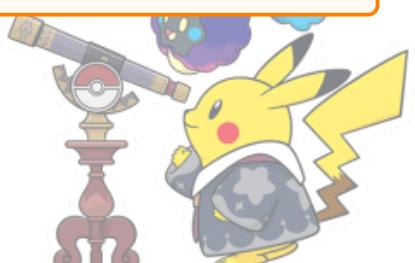
$$\text{adj}(A) = (C_{ji})_{1 \leq i,j \leq n}$$

### Definition. 1.11: Inverse Matrix

Let  $A$  be an  $n \times n$  matrix. A matrix  $B$  is called the **inverse** of  $A$  if

$$AB = BA = I_n$$

where  $I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$  is a  $n \times n$  matrix. If such a matrix exists,  $A$  is called **invertible** and the inverse is denoted by  $A^{-1}$ .



**Theorem. 1.4: Existence of the Inverse**

An  $n \times n$  matrix  $A$  is invertible if and only if  $\det A \neq 0$ .

**Theorem. 1.5: Inverse via the Adjoint**

If  $A$  is an  $n \times n$  matrix with  $\det A \neq 0$ , then

$$A^{-1} = \frac{1}{\det A} \text{adj}(A)$$

**1.8.3 Tensor**

**Basis** For a vector space  $V$ , a set of vectors  $\{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$  is a basis of  $V$  if

- **Linear Independence:** No vector in the set can be written as a linear combination of the others.
- **Spanning:** Every vector in  $V$  can be expressed as a linear combination of the basis vectors.

**Example.**

$$\mathcal{B} = \left\{ \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for  $\mathbb{R}^2$  because the vectors are linearly independent, and any vector  $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$  can be written as

$$\mathbf{v} = x\mathbf{e}_1 + y\mathbf{e}_2$$

**Change of Basis** Let  $V$  be a vector space, and let  $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$  be a basis for  $V$ . A vector  $\mathbf{v} \in V$  can be written as a linear combination of the basis vectors:

$$\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$$

where  $c_1, c_2, \dots, c_n$  are the coordinates of  $\mathbf{v}$  with respect to the basis  $\mathcal{B}$ . These coordinates  $\{c_1, c_2, \dots, c_n\}$  form the coordinate vector  $[\mathbf{v}]_{\mathcal{B}}$  of  $\mathbf{v}$  in the basis  $\mathcal{B}$ .

Consider two bases:

$$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$$



and

$$\mathcal{B}' = \{\mathbf{b}'_1, \mathbf{b}'_2, \dots, \mathbf{b}'_n\}$$

Let  $\mathbf{v}$  be a vector in  $V$ . The goal is to express the coordinate vector  $[\mathbf{v}]_{\mathcal{B}}$  of  $\mathbf{v}$  with respect to the new basis  $\mathcal{B}'$ . To do this, we need to find the relationship between the two coordinate systems. Suppose the new basis vectors  $\mathbf{b}'_i$  are expressed as linear combinations of the old basis vectors  $\mathbf{b}_i$ :

$$\mathbf{b}'_i = \sum_{j=1}^n P_{ij} \mathbf{b}_j$$

where  $P = [P_{ij}]$  is the change of basis matrix that transforms the old basis  $\mathcal{B}$  to the new basis  $\mathcal{B}'$ .

Then, if  $\mathbf{v}$  has coordinates  $[\mathbf{v}]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$  in the basis  $\mathcal{B}$ , its coordinates  $[\mathbf{v}]_{\mathcal{B}'}$  in the new basis  $\mathcal{B}'$  can be computed as:

$$[\mathbf{v}]_{\mathcal{B}'} = P^{-1} [\mathbf{v}]_{\mathcal{B}}$$

where  $P^{-1}$  is the inverse of the change of basis matrix.

**Tensor** A tensor is a mathematical object that generalizes scalars, vectors, and matrices. It can be thought of as a multi-dimensional array of numbers that transforms according to certain rules under a change of coordinates.

The order (or rank) of a tensor refers to the number of indices needed to describe it:

- A scalar is a tensor of order 0.
- A vector is a tensor of order 1.
- A matrix is a tensor of order 2.
- A general tensor may have order 3, 4, or higher, depending on the number of indices.

The components of a tensor are typically denoted with indices. For example, a second-order tensor  $T$  with components  $T_{ij}$  can be written as:

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$



**Contravariant Tensor** A contravariant tensor has components that transform in a certain way when we change the coordinates. Specifically, the components of a contravariant tensor transform according to the inverse of the transformation matrix. Let  $T$  be a contravariant vector. When we change coordinates, the components  $T^i$  transform as:

$$T'^i = \sum_j A_j^i T^j$$

where  $A_j^i$  is the transformation matrix for the new coordinates. In index notation, the upper index (superscript) indicates that the tensor is contravariant.

For example, consider a contravariant vector  $\mathbf{v} = (v^1, v^2)$ . Under a change of coordinates, the components transform as

$$v'^1 = A_1^1 v^1 + A_2^1 v^2, \quad v'^2 = A_1^2 v^1 + A_2^2 v^2$$

**Covariant Tensor** In contrast, a covariant tensor has components that transform differently. The components of a covariant tensor transform according to the transformation matrix itself. Let  $S$  be a covariant vector. Under a change of coordinates, the components  $S_i$  transform as

$$S'_i = \sum_j A_i^j S_j$$

where  $A_i^j$  is the transformation matrix for the new coordinates. In index notation, the lower index (subscript) indicates that the tensor is covariant. For example, consider a covariant vector  $\mathbf{w} = (w_1, w_2)$ . Under a change of coordinates, the components transform as

$$w'_1 = A_1^1 w_1 + A_2^1 w_2, \quad w'_2 = A_1^2 w_1 + A_2^2 w_2$$

#### 1.8.4 Scalar Field and Vector Field

##### Definition. 1.12: Scalar Field

A **scalar field** is a mathematical function that assigns a scalar value to every point in a space. It is represented as

$$\phi : \mathbb{R}^n \rightarrow \mathbb{R}, \quad \mathbf{x} \mapsto \phi(\mathbf{x})$$

where  $\mathbb{R}^n$  is the space in which the field is defined, and  $\phi(\mathbf{x})$  is a scalar value at the point  $\mathbf{x}$ .

**Example.** The temperature at every point in a room is a scalar field. If  $T(\mathbf{x})$  gives the temperature at point  $\mathbf{x}$ , then  $T$  is a scalar field.

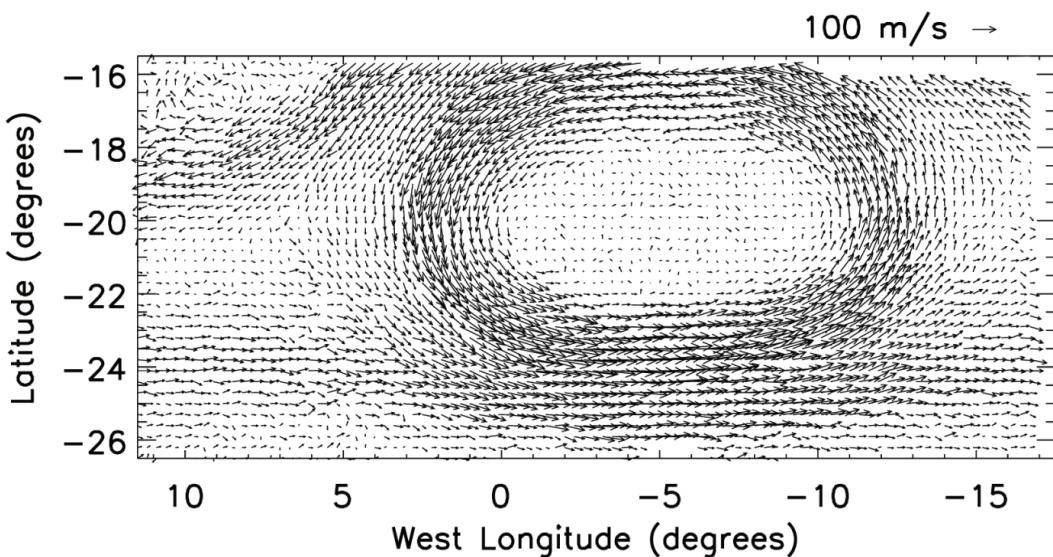


**Definition. 1.13: Vector Field**

A **vector field** is a function that assigns a vector to every point in a space. It is represented as

$$\mathbf{V} : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad \mathbf{x} \mapsto \mathbf{V}(\mathbf{x})$$

where  $\mathbf{V}(\mathbf{x})$  is the vector field at point  $\mathbf{x}$ , and  $\mathbb{R}^n$  is the space in which the vector field is defined.

**Exercise. (2020 GeCAA)**

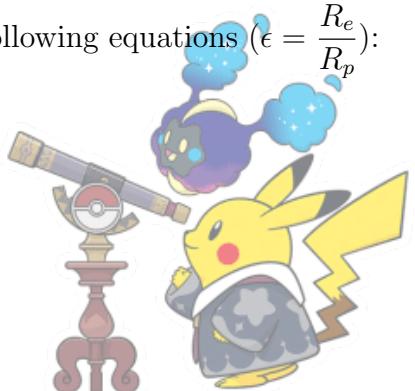
In the following problem, the fluid mechanics of Jupiter's Great Red Spot (GRS) is studied based on the velocity field data. The diagram above shows a map of relative velocity for GRS and the surrounding region. The arrows are oriented and scaled as per the directions and magnitudes of winds at different points. Due to the combined effects of gravity and rotation, Jupiter is slightly flattened at its poles. The equation of a spheroid approximating for the shape of Jupiter can be stated as:

$$\frac{x^2}{R_e^2} + \frac{y^2}{R_p^2} = 1$$

where  $R_e = 7.15 \times 10^7 \text{ m}$  is the equatorial radius of Jupiter, and  $R_p = 6.69 \times 10^7 \text{ m}$  the polar radius. The radii of curvature of this spheroid in any direction can be calculated by the following equations ( $\epsilon = \frac{R_e}{R_p}$ ):

$$r(\phi) = R_e(1 + \epsilon^2 \tan^2 \phi)^{1/2}$$

$$R(\phi) = R_e \epsilon^2 \left( \frac{r(\phi)}{R_e \cos \phi} \right)^3$$



where  $r(\phi)$  and  $R(\phi)$  are the zonal (aka the zone of a particular latitude) and meridional (aka longitudinal) radii of curvature, respectively, as a function of planetographic latitude  $\phi$ . The sidereal rotation period of Jupiter is  $P = 3.57 \times 10^4$  s.

- (a) Calculate the zonal and meridional radii values ( $\bar{r}$  and  $\bar{R}$ ) respectively at the location of the centre of the GRS.
- (b) Estimate the eccentricity of the GRS.
- (c) 'Vorticity' at any point is a measure of local spinning of the fluid as measured by an observer situated in the reference frame of the fluid. Mathematically, it is calculated as 'curl' (vector derivative product) of the velocity field. In this case, the average relative vorticity may be estimated by the equation:

$$\zeta = \frac{V_v}{L_{GRS} A_{GRS}}$$

where  $V_v$  is the maximum value of winds as per the velocity field,  $L_{GRS}$  is the length of the circumference of the GRS and  $A_{GRS}$  is the area of the GRS. Estimate average relative vorticity of the GRS.

**Hint:** The circumference of an ellipse is well approximated by an average of circumferences of the corresponding auxiliary and minor circles.

- (d) Find the absolute vorticity  $\zeta_a = (\zeta + f)$  by adding the Coriolis parameter:

$$f = 2\Omega \sin \phi$$

where  $\Omega$  is the angular velocity of Jupiter (due to axial rotation) and  $\phi$  is the appropriate latitude.

- (e) If the absolute vorticity has the same sign as the latitude, we call the storm a 'cyclonic storm'. If they have opposite signs, the system is 'anticyclonic'. Is the GRS cyclonic or anticyclonic?
- (f) Imagine that the GRS moves to another latitude  $\phi_1$ , where the absolute vorticity changes the sign (changes from anti-cyclonic to cyclonic or vice versa). Assuming minimum possible displacement of the GRS, at what value of  $\phi_1$  do we expect this change?

In your analysis, assume that the GRS at the new location would occupy the same angular span in latitude, as well as have the same wind velocities and eccentricity as the original.



## 1.9 Differentiation and Partial Differentiation

### 1.9.1 Introduction

Differentiation is used for functions of a single independent variable,  $y = f(x)$ . It measures the instantaneous rate of change of  $y$  with respect to  $x$ . The derivative of a function  $f(x)$  with respect to  $x$  is defined using a limit and is denoted by  $\frac{df}{dx}$  or  $f'(x)$ .

$$\frac{df}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \text{ (First principle)}$$

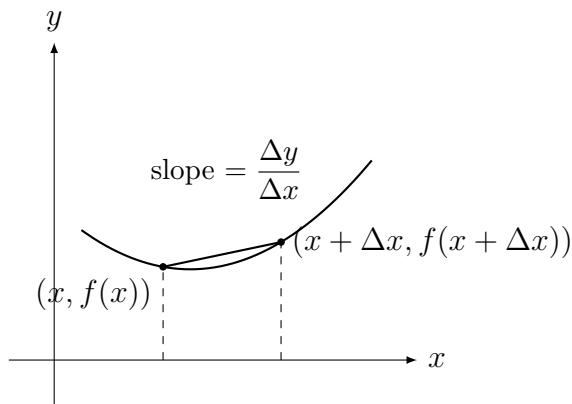


Figure 1: Differentiation is the slope when  $\Delta x \rightarrow 0$

### 1.9.2 Properties and Formula

#### Definition. 1.14: Basic Properties

If  $f(x)$  and  $g(x)$  are differentiable functions (the derivative exists),  $c$  and  $n$  are any real numbers,

$$(cf)' = cf'(x)$$

$$\frac{d}{dx}(c) = 0$$

$$(f \pm g)' = f'(x) \pm g'(x)$$

$$\frac{d}{dx}(x^n) = nx^{n-1} \text{ (Power Rule)}$$

$$(fg)' = f'g + fg' \text{ (Product Rule)}$$

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x) \text{ (Chain Rule)}$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \text{ (Quotient Rule)}$$



**Definition. 1.15: Formulas**

$$\frac{d}{dx}(x) = 1$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(a^x) = a^x \ln(a), \quad a > 0$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}, \quad x > 0$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}, \quad x \neq 0$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\log_a(x)) = \frac{1}{x \ln a}, \quad x > 0$$

**1.9.3 Partial Differentiation**

Partial differentiation is used for functions of **two or more independent variables**, such as  $z = f(x, y)$ . A partial derivative measures the rate of change of the function with respect to **one** variable, while **holding all other variables constant**. For example,  $\frac{\partial z}{\partial x}$  is the derivative with respect to  $x$  while keeping  $y$  constant.

**Example.** Consider the function

$$f(x, y) = x^2y + e^{xy}$$

$$\frac{\partial}{\partial x} (x^2y + e^{xy}) = 2xy + ye^{xy}$$

**1.9.4 Numerical Analysis****Definition. 1.16: Bisection Method**

Let the function  $f(x)$  be continuous on the interval  $[a, b]$ , and suppose that  $f(a)$  and  $f(b)$  have opposite signs, i.e.,

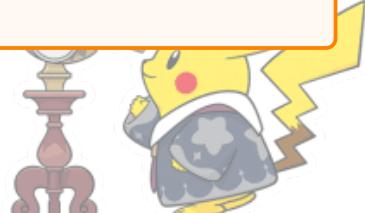
$$f(a) \cdot f(b) < 0$$

The root  $r$  lies between  $a$  and  $b$ , and we can approximate the root by iteratively halving the interval.

The midpoint  $m$  of the interval is given by

$$m = \frac{a+b}{2}$$

We then check the sign of  $f(m)$ :



- If  $f(m) = 0$ , then  $m$  is the root.
- If  $f(a) \cdot f(m) < 0$ , the root lies between  $a$  and  $m$ , so we set  $b = m$ .
- If  $f(m) \cdot f(b) < 0$ , the root lies between  $m$  and  $b$ , so we set  $a = m$ .

This process is repeated until the interval becomes sufficiently small.

**Example.** To find the root of  $f(x) = x^2 - 2$  on the interval  $[1, 2]$ :

- $f(1) = 1^2 - 2 = -1$
- $f(2) = 2^2 - 2 = 2$
- Since  $f(1) \cdot f(2) < 0$ , the root lies in  $[1, 2]$ .
- First approximation:  $m_1 = \frac{1+2}{2} = 1.5$ .
- $f(1.5) = 1.5^2 - 2 = 0.25$ . Since  $f(1) \cdot f(1.5) < 0$ , the new interval is  $[1, 1.5]$ .

### Definition. 1.17: Newton–Raphson method

Let  $f(x)$  be a differentiable function, and let  $x_0$  be an initial guess for the root. The next approximation  $x_1$  is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

The method continues iteratively:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

**Example.** To solve  $f(x) = x^2 - 2 = 0$ , we find the derivative  $f'(x) = 2x$ . Using an initial guess  $x_0 = 1.5$ :

$$\begin{aligned} x_1 &= x_0 - \frac{x_0^2 - 2}{2x_0} \\ x_1 &= 1.5 - \frac{1.5^2 - 2}{2(1.5)} \approx 1.4167 \end{aligned}$$



### 1.9.5 Linear Approximation

To find the linear approximation, we begin by recalling the definition of the tangent line to a curve at a point. The equation of the tangent line at  $x = a$  is

$$y = f'(a)(x - a) + f(a)$$

which provides the best linear approximation to the function near  $x = a$ :

$$f(x) \approx f(a) + f'(a)(x - a)$$

**Example.**  $\sin x \approx x$  for small  $x$  in radian.

### 1.9.6 Taylor's Series

The Taylor Series of a function  $f(x)$  about a point  $a$  is given by

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f^{(3)}(a)}{3!}(x - a)^3 + \dots$$

where  $f^{(n)}(x)$  is the  $n$ -th derivative of  $f(x)$ .

**Example.** Consider the function  $f(x) = e^x$ . The derivatives of  $e^x$  are

$$f'(x) = e^x, \quad f''(x) = e^x, \quad f^{(3)}(x) = e^x, \quad \dots$$

Evaluating these at  $x = 0$ , we get

$$f(0) = 1, \quad f'(0) = 1, \quad f''(0) = 1, \quad f^{(n)}(0) = 1 \text{ for all } n.$$

Hence, the Taylor series of  $e^x$  around  $x = 0$  (also known as the Maclaurin series) is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$



### 1.9.7 First Derivative Test

#### Definition. 1.18: Critical Point

Let  $f(x)$  be a function that is differentiable on an interval containing  $c$ , except possibly at  $c$  itself.

**Critical points** of  $f$  are points  $c$  where either  $f'(c) = 0$  or  $f'(c)$  does not exist.

#### Theorem. 1.6: First Derivative Test

Suppose  $c$  is a critical point of a function  $f$ .

- If  $f'(x) > 0$  for  $x < c$  and  $f'(x) < 0$  for  $x > c$ , then  $f$  has a **local maximum** at  $c$ .
- If  $f'(x) < 0$  for  $x < c$  and  $f'(x) > 0$  for  $x > c$ , then  $f$  has a **local minimum** at  $c$ .
- If  $f'(x)$  has the same sign on both sides of  $c$ , then  $f$  has **no local extremum** at  $c$ .

**Example. (2025 IOAA)** The Event Horizon Telescope (EHT) has released an image of the supermassive black hole at the centre of the M87 galaxy, as shown in the left panel of Fig. 2. To understand some simple features of this image, we consider a simplified model of a non-rotating, static, spherically symmetric black hole of mass

$$M = 6.5 \times 10^9 M_\odot,$$

surrounded by a massless, thin, planar accretion disk of inner and outer radii

$$a_{\text{inner}} = 6R_{\text{SC}}, \quad a_{\text{outer}} = 10R_{\text{SC}},$$

respectively, where  $R_{\text{SC}}$  is the Schwarzschild radius. A face-on view sketch of this system is shown in the right panel of Fig. 2. (The figure is not to scale.)

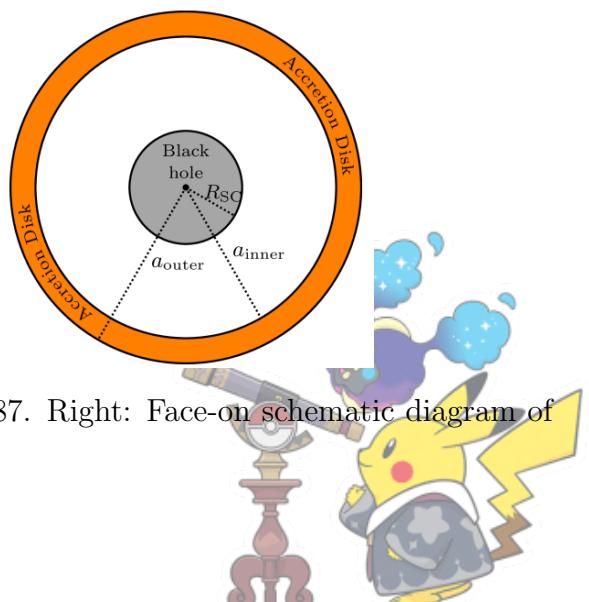
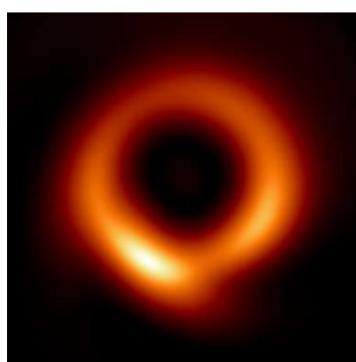


Figure 2: Left: EHT image of the black hole at the centre of M87. Right: Face-on schematic diagram of a black hole surrounded by a thin accretion disk.

We assume that the accretion disk is the only source of light to be considered. Every point on the disk emits light in all directions. This light travels under the influence of the gravitational field of the black hole. The path of the light rays is governed by the following two equations (which are similar to those describing the motion of an object around the Sun):

$$\frac{1}{2}v_r^2 + \frac{L^2}{2r^2} \left(1 - \frac{2GM}{c^2r}\right) = E \quad (1)$$

$$v_\phi = r\omega = \frac{L}{r} \quad (2)$$

where  $r \in (R_{\text{SC}}, \infty)$  is the radial coordinate,  $\phi \in [0, 2\pi]$  is the azimuthal angle, and  $E$  and  $L$  are constants related to the conserved energy and conserved angular momentum, respectively. Here,  $v_r = \frac{dr}{dt}$  is the magnitude of the radial velocity,  $v_\phi$  is the magnitude of the tangential velocity, and  $\omega = \frac{d\phi}{dt}$  is the angular velocity. We define the impact parameter  $b$  for a trajectory as

$$b = \frac{L}{\sqrt{2E}}.$$

Time dilation effects are neglected in this problem. Another useful equation is obtained by differentiating the first equation with respect to time:

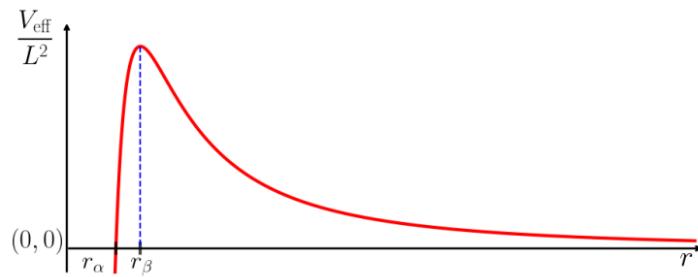
$$\frac{dv_r}{dt} - \frac{L^2}{r^3} + \frac{3GML^2}{c^2r^4} = 0. \quad (3)$$

- (a) Circular light trajectories can exist around the black hole. Find the radius  $r_{\text{ph}}$  and the impact parameter  $b_{\text{ph}}$  for such photon trajectories in terms of  $M$  and relevant constants.
- (b) Calculate the time  $T_{\text{ph}}$  taken to complete one full orbit of the circular light trajectory, in seconds.
- (c) The radial velocity equation for light trajectories can be compared with an equation of the form

$$\frac{v_r^2}{2} + V_{\text{eff}}(r) = E. \quad (4)$$

A schematic plot of  $V_{\text{eff}}/L^2$  as a function of  $r$  is shown in the following figure. The plot indicates two special radii,  $r_\alpha$  and  $r_\beta$ . Obtain expressions for  $r_\alpha$  and  $r_\beta$  in terms of  $M$  and relevant constants.





## 1.10 Integration

### 1.10.1 Primitive Function and Indefinite Integration

The **primitive function** (or antiderivative) of a given function  $f(x)$  is a function  $F(x)$  such that

$$F'(x) = f(x)$$

**Example.** Let  $f(x) = 2x$ . The corresponding primitive function  $F(x)$  is

$$F(x) = x^2 + C$$

where  $C$  is a constant, called the constant of integration. This constant appears because the derivative of any constant is zero, and thus there are infinitely many functions that differ only by a constant.

The **indefinite integral** of a function  $f(x)$  is the operation of finding its primitive function. It is written as

$$\int f(x) dx = F(x) + C$$

Some important properties of indefinite integrals include

#### Theorem. 1.7: Properties

- **Linearity:**

$$\int (f(x) + g(x)) dx = \int f(x) dx + \int g(x) dx$$

- **Constant factor:**

$$\int cf(x) dx = c \int f(x) dx$$

where  $c$  is a constant.



**Theorem. 1.8: Basic Primitive Function**

- **Power Rule:**

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad \text{for } n \neq -1$$

- **Exponential Functions:**

$$\int e^x dx = e^x + C$$

- **Trigonometric Functions:**

$$\int \sin(x) dx = -\cos(x) + C$$

$$\int \cos(x) dx = \sin(x) + C$$

**1.10.2 Definite Integral****Definition. 1.19: Riemann Sum**

Let  $f(x)$  be a function defined on the interval  $[a, b]$ , and let the interval be divided into  $n$  subintervals:

$$[a, b] = [x_0, x_1] \cup [x_1, x_2] \cup \dots \cup [x_{n-1}, x_n]$$

where  $x_0 = a$  and  $x_n = b$ , and the partition points

$$x_0, x_1, \dots, x_n$$

are chosen such that  $a = x_0 < x_1 < \dots < x_n = b$ . The Riemann sum is defined as

$$S_n = \sum_{i=1}^n f(x_i^*) \Delta x_i$$

where:

- $x_i^*$  is a sample point chosen from the  $i$ -th subinterval  $[x_{i-1}, x_i]$ ,
- $\Delta x_i = x_i - x_{i-1}$  is the width of the  $i$ -th subinterval.

There are several ways to choose the sample points  $x_i^*$  in each subinterval, which leads to different types of Riemann sums:



- **Left Riemann Sum:**  $x_i^* = x_{i-1}$ , the left endpoint of each subinterval.
- **Right Riemann Sum:**  $x_i^* = x_i$ , the right endpoint of each subinterval.
- **Midpoint Riemann Sum:**  $x_i^* = \frac{x_{i-1} + x_i}{2}$ , the midpoint of each subinterval.

### Definition. 1.20: Definite Integral

The definite integral is defined as the limit of the Riemann sum as the number of subintervals  $n \rightarrow \infty$  and the maximum subinterval width  $\Delta x_i \rightarrow 0$ :

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x_i$$

The definite integral can be interpreted as the signed area under the curve  $y = f(x)$  from  $x = a$  to  $x = b$ . If  $f(x)$  is above the x-axis, the area is positive, and if  $f(x)$  is below the x-axis, the area is negative.

### Theorem. 1.9: Fundamental Theorem of Calculus

Alternatively, the definite integral can be computed using the Fundamental Theorem of Calculus:

$$\int_a^b f(x) dx = F(b) - F(a)$$

where  $F(x)$  is any antiderivative of  $f(x)$ .

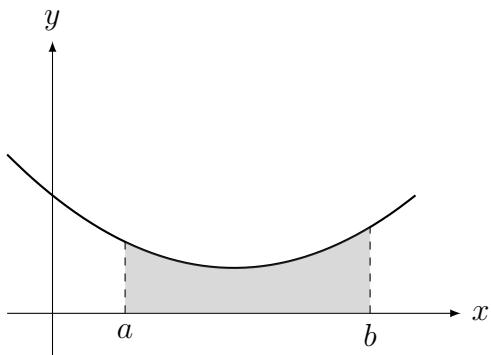


Figure 3: Signed area under the curve  $y = f(x)$  from  $x = a$  to  $x = b$  is given by  $\int_a^b f(x) dx$ .

Some important properties of definite integrals include



**Definition. 1.21: Properties**

- **Linearity:**

$$\int_a^b (c_1 f(x) + c_2 g(x)) dx = c_1 \int_a^b f(x) dx + c_2 \int_a^b g(x) dx$$

where  $c_1$  and  $c_2$  are constants.

- **Additivity:**

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

for any point  $c$  between  $a$  and  $b$ .

- **Reversal of Limits:**

$$\int_a^b f(x) dx = - \int_b^a f(x) dx$$

Reversing the limits of integration changes the sign of the integral.

- **Zero Width Integral:**

$$\int_a^a f(x) dx = 0$$

The integral over a single point is always zero.

**1.10.3 Integration by Substitution****Theorem. 1.10: Integration by Substitution in Indefinite Integral**

Suppose we have the following indefinite integral:

$$\int f(g(x)) \cdot g'(x) dx$$

We can use substitution by letting  $u = g(x)$ , so that  $du = g'(x) dx$ . This changes the integral into

$$\int f(u) du$$

Now, we can integrate with respect to  $u$  and then substitute back in terms of  $x$ .

**Example.** Consider the integral

$$\int x e^{x^2} dx$$



Let

$$u = x^2, \quad du = 2x \, dx$$

Hence, the integral becomes

$$\frac{1}{2} \int e^u \, du = \frac{1}{2} e^u + C = \frac{1}{2} e^{x^2} + C$$

### Theorem. 1.11: Integration by Substitution in Definite Integral

For definite integrals, the limits of integration change according to the substitution. Suppose we have

$$\int_a^b f(g(x)) \cdot g'(x) \, dx$$

Let  $u = g(x)$ , then the limits of integration transform as follows:

$$u(a) = g(a) \quad \text{and} \quad u(b) = g(b)$$

The integral becomes

$$\int_{g(a)}^{g(b)} f(u) \, du$$

We can now evaluate the definite integral in terms of  $u$ .

#### 1.10.4 Integration by Parts

### Theorem. 1.12: Integration by Part in Definite Integral

$$\int u(x)v'(x) \, dx = u(x)v(x) - \int v(x)u'(x) \, dx$$

where  $u(x)$  is chosen to simplify  $u'(x)$  and  $v'(x)$  is easily integrable.

**Proof.** Integration by parts is based on the product rule for differentiation:

$$\frac{d}{dx}[u(x)v(x)] = u'(x)v(x) + u(x)v'(x)$$

Rewriting and integrating both sides gives the formula for integration by parts:

$$\int u(x)v'(x) \, dx = u(x)v(x) - \int v(x)u'(x) \, dx$$



**Example.** Evaluate the integral

$$\int xe^x dx$$

Let  $u = x$ , so that  $du = dx$ , and let  $dv = e^x dx$ , so that  $v = e^x$ . Using the integration by parts formula:

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$$

### Theorem. 1.13: Integration by Part in Indefinite Integral

To solve an integral using integration by parts, we choose  $u$  and  $dv$  from the integrand. Differentiating  $u$  gives  $du$ , and integrating  $dv$  gives  $v$ . Substituting into the formula, we obtain:

$$\int u dv = uv - \int v du$$

## 1.10.5 Multiple Integration, Line Integration and Surface Integration

### Definition. 1.22: $n$ -dimensional Integration

In  $n$  dimensions, the integral of a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  over a region  $D \subset \mathbb{R}^n$  is given by

$$\int_D f(\mathbf{x}) d\mathbf{x} = \int_{x_1=a_1}^{b_1} \int_{x_2=a_2}^{b_2} \cdots \int_{x_n=a_n}^{b_n} f(x_1, x_2, \dots, x_n) dx_n \dots dx_2 dx_1$$

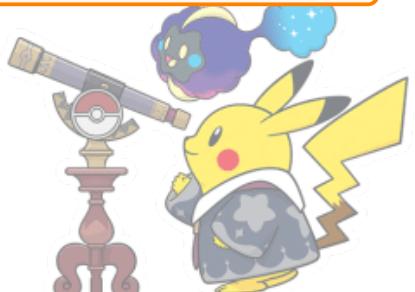
**Example.** the 2-dimensional integral of a function  $f(x, y)$  over a rectangle  $[a_1, b_1] \times [a_2, b_2]$  is

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} f(x, y) dy dx$$

### Definition. 1.23: Line Integration

A line integral is an integral of a function along a curve or path. Let  $\mathbf{r}(t) = (x(t), y(t), z(t))$  be a parametrization of the curve  $C$ , with  $t$  ranging from  $a$  to  $b$ . Then, the line integral of a scalar function  $f$  along  $C$  is given by

$$\int_C f(x, y, z) ds = \int_a^b f(x(t), y(t), z(t)) \left| \frac{d\mathbf{r}}{dt} \right| dt$$



If the integrand involves a vector field  $\mathbf{F} = (F_1, F_2, F_3)$ , the line integral becomes

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \left( F_1(x(t), y(t), z(t)) \frac{dx}{dt} + F_2(x(t), y(t), z(t)) \frac{dy}{dt} + F_3(x(t), y(t), z(t)) \frac{dz}{dt} \right) dt$$

### Definition. 1.24: Surface Integration

A surface integral involves integrating over a surface  $S$ . The surface integral of a scalar function  $f(x, y, z)$  over a surface  $S$  is

$$\int_S f(x, y, z) dS = \iint_D f(x(u, v), y(u, v), z(u, v)) \left| \frac{\partial(x, y, z)}{\partial(u, v)} \right| du dv$$

where  $(x(u, v), y(u, v), z(u, v))$  is the parametrization of the surface, and the determinant term is the Jacobian of the transformation from the  $(u, v)$  parameter space to the  $(x, y, z)$  space.

For a vector field  $\mathbf{F} = (F_1, F_2, F_3)$ , the surface integral becomes

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(x(u, v), y(u, v), z(u, v)) \cdot \mathbf{n}(u, v) \left| \frac{\partial(x, y, z)}{\partial(u, v)} \right| du dv$$

where  $\mathbf{n}(u, v)$  is the unit normal vector to the surface at each point.

### 1.10.6 Trapezoidal Rule and Simpson's Rule

For more complex functions where integration is difficult, numerical methods such as the Trapezoidal Rule can be used.

### Theorem. 1.14: Trapezoidal Rule

Let  $f$  be a continuous function on  $[a, b]$ . Partition the interval into  $n$  equal subintervals of width

$$h = \frac{b - a}{n},$$

and define the partition points by  $x_i = a + ih$  for  $i = 0, 1, \dots, n$ .

The Trapezoidal Rule approximates the definite integral

$$\int_a^b f(x) dx$$



by

$$\int_a^b f(x) dx \approx \frac{h}{2} \left[ f(a) + 2 \sum_{i=1}^{n-1} f(x_i) + f(b) \right]$$

This method corresponds to approximating the graph of  $f$  on each subinterval by a straight line and summing the areas of the resulting trapezoids.

Another more accurate numerical approximation is Simpson's Rule, which approximates the area under the curve by fitting parabolas to segments of the curve.

### Theorem. 1.15: Simpson's Rule

Let  $f$  be a continuous function on  $[a, b]$ , and suppose that  $n$  is an **even** positive integer. With  $h = \frac{b-a}{n}$  and partition points  $x_i = a + ih$ , Simpson's Rule approximates

$$\int_a^b f(x) dx$$

by

$$\int_a^b f(x) dx \approx \frac{h}{3} \left[ f(a) + 4 \sum_{\substack{i=1 \\ i \text{ odd}}}^{n-1} f(x_i) + 2 \sum_{i=2}^{n-2} f(x_i) + f(b) \right]$$

This method is derived by approximating  $f$  on each pair of subintervals using a parabola, resulting in significantly higher accuracy for sufficiently smooth functions.

## 1.10.7 Differential Equation

**Introduction** An **ordinary differential equation** (ODE) is an equation that involves a function and its derivatives. A general first-order ODE has the form

$$\frac{dy}{dx} = f(x, y)$$

**Separable ODE** A first-order ODE is **separable** if it can be written as

$$\frac{dy}{dx} = g(x)h(y)$$

Then we solve by separating variables:

$$\frac{1}{h(y)} dy = g(x) dx$$



and integrate both sides.

**Integrating Factor** A first-order linear ODE has the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

can be solved by introducing

$$\mu(x) = e^{\int P(x)dx}$$

$$y\mu(x) = \int Q(x)\mu(x)dx$$

**Initial Condition** An **initial condition** is a specification of the value of the solution (and possibly its derivatives) at a particular point, usually denoted  $t_0$  or  $x_0$ . For an  $n$ -th order ODE, we typically need  $n$  initial conditions to determine a unique solution.



## 1.11 Coordinates System

Coordinate System	Definition and Conversions
Polar Coordinates (2D)	<p>Represents a point in the plane as <math>(r, \theta)</math>, where:</p> <p><math>r</math>: Distance from the origin</p> <p><math>\theta</math>: Angle with the positive <math>x</math>-axis</p> <p>Conversion to Cartesian: <math>x = r \cos \theta</math>, <math>y = r \sin \theta</math></p>
Cylindrical Coordinates (3D)	<p>Extension of polar coordinates: <math>(r, \theta, z)</math></p> <p><math>r</math>: Radial distance from <math>z</math>-axis</p> <p><math>\theta</math>: Azimuthal angle from positive <math>x</math>-axis in <math>xy</math>-plane</p> <p><math>z</math>: Vertical coordinate (same as Cartesian)</p> <p>Conversion to Cartesian: <math>x = r \cos \theta</math>, <math>y = r \sin \theta</math>, <math>z = z</math></p>
Spherical Coordinates (3D)	<p>Represents a point in 3D space as <math>(\rho, \theta, \phi)</math>, where:</p> <p><math>\rho</math>: Distance from the origin</p> <p><math>\theta</math>: Polar angle measured from positive <math>z</math>-axis (<math>0 \leq \theta \leq \pi</math>)</p> <p><math>\phi</math>: Azimuthal angle measured from positive <math>x</math>-axis in <math>xy</math>-plane (<math>0 \leq \phi &lt; 2\pi</math>)</p> <p style="text-align: right;"><math>x = \rho \sin \theta \cos \phi</math>,</p> <p>Conversion to Cartesian: <math>y = \rho \sin \theta \sin \phi</math>,</p> <p style="text-align: right;"><math>z = \rho \cos \theta</math></p>

Table 2: Coordinate Systems and Their Properties



## 2 Spherical Trigonometry & Astronomical Coordinate Systems

### 2.1 Spherical Trigonometry

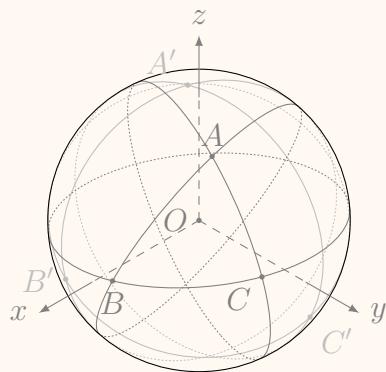
#### Definition. 2.1: Terminologies

On a sphere of radius  $R$ , we define

- a **great circle** is a circle on the sphere whose center is the sphere's center,
- a **spherical triangle** is formed by three arcs of great circles,
- the **sides** of a spherical triangle are arcs of great circles, and
- the **angles** of a spherical triangle are the angles between the tangents to the great circles at the vertices.

Consider a spherical triangle with vertices  $A$ ,  $B$ , and  $C$  on a unit sphere (radius = 1). We denote

- Sides:  $a = BC$ ,  $b = AC$ ,  $c = AB$  (arc lengths)
- Angles:  $A = \angle BAC$ ,  $B = \angle ABC$ ,  $C = \angle ACB$



#### Definition. 2.2: Excess Angle

The sum of angles in a spherical triangle exceeds  $\pi$  radians ( $180^\circ$ ). The difference is called the spherical excess  $E$

$$E = A + B + C - \pi$$



**Theorem. 2.1: Area of the Spherical Triangle**

The area of the spherical triangle on a sphere of radius  $R$  is

$$\text{Area} = R^2 \cdot E$$

**Theorem. 2.2: Law of Cosine**

For a spherical triangle with sides  $a, b, c$  and opposite angles  $A, B, C$ ,

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

$$\cos b = \cos c \cos a + \sin c \sin a \cos B$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

For the same triangle,

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$

$$\cos B = -\cos C \cos A + \sin C \sin A \cos b$$

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c$$

## 2.2 Astronomical Coordinate Systems

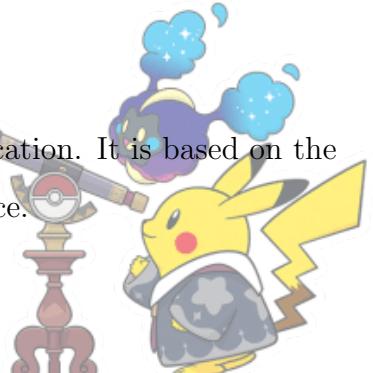
### 2.2.1 Horizontal (Alt-Az) Coordinate System

The horizontal coordinate system is based on the observer's location on Earth. It is oriented with respect to the local horizon and is dynamic, changing as the observer's position changes.

- **Altitude (Alt):** The angle above the horizon, ranging from  $0^\circ$  at the horizon to  $90^\circ$  at the zenith (directly above the observer).
- **Azimuth (Az):** The angle measured clockwise from the North, where  $0^\circ$  is North,  $90^\circ$  is East,  $180^\circ$  is South, and  $270^\circ$  is West.

### 2.2.2 Equatorial Coordinate System

This system is fixed relative to the stars and is independent of the observer's location. It is based on the celestial sphere, which extends the Earth's equator and prime meridian into space.



- **Right Ascension (RA):** Analogous to longitude, measured eastward along the celestial equator from the vernal equinox. It is expressed in hours, minutes, and seconds, with 24 hours corresponding to  $360^\circ$ .
- **Declination (Dec):** Analogous to latitude, measured north or south of the celestial equator. Declination ranges from  $-90^\circ$  (celestial South Pole) to  $+90^\circ$  (celestial North Pole).

### 2.2.3 Ecliptic Coordinate System

The ecliptic coordinate system is based on the plane of Earth's orbit around the Sun. It is particularly useful for describing planetary positions and the motion of the Sun, Moon, and planets.

- **Ecliptic Longitude ( $\lambda$ ):** Measured along the ecliptic from the vernal equinox. It ranges from  $0^\circ$  to  $360^\circ$ , with  $0^\circ$  corresponding to the vernal equinox.
- **Ecliptic Latitude ( $\beta$ ):** Measured perpendicular to the ecliptic plane, ranging from  $-90^\circ$  to  $+90^\circ$ .

### 2.2.4 Galactic Coordinate System

The galactic coordinate system is used to map objects relative to the plane of the Milky Way galaxy.

- **Galactic Longitude ( $l$ ):** Measured in the plane of the Milky Way from the Galactic Center, with values ranging from  $0^\circ$  to  $360^\circ$ .
- **Galactic Latitude ( $b$ ):** Measured perpendicular to the Galactic plane, ranging from  $-90^\circ$  to  $+90^\circ$ .

### 2.2.5 Circumpolar Stars

Circumpolar stars are stars that never set below the horizon from a given location. Their position is close enough to one of the celestial poles that they appear to trace circular paths around the pole throughout the night. These stars are visible year-round in regions near the pole, such as in the northern hemisphere where the North Star ( $\alpha$  Ursae Minoris) is circumpolar. Mathematically, the declination  $\delta$  of a star must satisfy:

$$\delta \geq 90^\circ - \phi$$

where  $\phi$  is the latitude of the observer.



## 3 Astronomical Time

### 3.1 Solar Time

Solar time is based on the position of the Sun in the sky. It differs from the standard time used in daily life (which is based on the Earth's rotation relative to distant stars) and is important in astronomical observations.

#### 3.1.1 Apparent Solar Time

Apparent solar time is based on the actual observed position of the Sun. It is determined by the local meridian and the position of the Sun in the sky, which varies due to the elliptical orbit of the Earth. The apparent solar time is given by

$$\text{Apparent Solar Time} = \text{Local Mean Time} + \text{Equation of Time}$$

The **Equation of Time** corrects for the Earth's orbital eccentricity and axial tilt.

#### 3.1.2 Mean Solar Time

Mean solar time is an average time based on the Earth's rotation. Unlike apparent solar time, it ignores irregularities in the Earth's motion. The length of the mean solar day is constant at 86,400 seconds. The equation for mean solar time is

$$\text{Mean Solar Time} = \text{Universal Time (UT)} + \text{Longitude Correction}$$

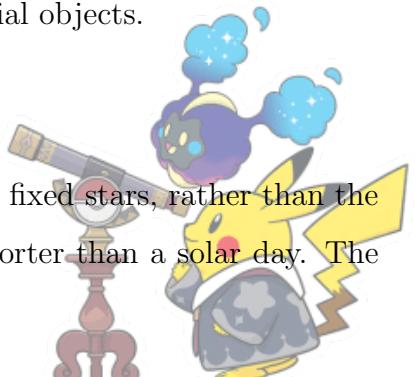
## 3.2 Sidereal Time

### 3.2.1 Introduction

Sidereal time is based on the Earth's rotation relative to distant stars rather than the Sun. This is particularly important in astronomy, as it allows astronomers to locate celestial objects.

#### 3.2.2 Sidereal Day

A sidereal day is the time it takes for the Earth to rotate once relative to the fixed stars, rather than the Sun. This period is about 23 hours, 56 minutes, and 4.1 seconds, slightly shorter than a solar day. The



relationship between solar time and sidereal time is given by

$$\text{Sidereal Time} = \text{Solar Time} - \text{Longitude Correction}$$

To calculate sidereal time at a given location, the following formula is used:

$$\text{Sidereal Time (ST)} = \text{GMST} + \text{Longitude Correction}$$

where

- **GMST** is the Greenwich Mean Sidereal Time, which is the sidereal time at the Prime Meridian.
- **Longitude Correction** is based on the observer's longitude.

The GMST is computed as follows:

$$\text{GMST} = 280.46061837^\circ + 360.98564736629^\circ \times (JD - 2451545.0)$$

where

- **JD** is the Julian Date.
- The terms account for the Earth's rotation and the passage of time.

### 3.3 Heliocentric Julian Date (HJD)

The **Heliocentric Julian Date** is the Julian Date (JD) corrected for the light travel time between the Earth and the Sun. This correction accounts for the position of the Earth in its orbit and is essential in high-precision astronomical observations, such as timing of variable stars or exoplanet transits.

$$\text{HJD} = \text{JD} + \frac{\mathbf{r} \cdot \hat{\mathbf{s}}}{c}$$

where

- **r** is the vector from the Sun to the Earth,
- **$\hat{s}$**  is the unit vector pointing toward the observed object,
- **c** is the speed of light.



### 3.4 Time Zones

Time zones are regions of the Earth that share the same standard time. Local civil time is defined relative to the **Greenwich Mean Time (GMT)** or **Coordinated Universal Time (UTC)**:

$$\text{Local Time} = \text{UTC} + \text{Time Zone Offset}$$

### 3.5 Universal Time (UT)

**Universal Time** is a time standard based on the rotation of the Earth relative to distant celestial objects. The most commonly used form is **UT1**, which is directly related to the Earth's rotation angle.

### 3.6 Local Mean Time (LMT)

**Local Mean Time** is the mean solar time at a particular longitude. It is the local time that would be measured by a clock set to the average solar day rather than the apparent solar day:

$$\text{LMT} = \text{UT} + \frac{\lambda}{15^\circ} \text{ hours}$$

where  $\lambda$  is the longitude east of the Greenwich meridian (positive east, negative west).

### 3.7 Different Definitions of a Year

There are several astronomical definitions of a year:

- **Sidereal Year:** Time taken for the Earth to complete one orbit relative to the fixed stars.  $\approx 365.25636$  days.
- **Tropical Year:** Time between successive vernal equinoxes.  $\approx 365.24219$  days. Used in calendars to keep seasons aligned.
- **Anomalistic Year:** Time between successive perihelion passages.  $\approx 365.25964$  days.
- **Julian Year:** Exactly 365.25 days. Used in astronomical calculations for simplicity.



### 3.8 Equation of Time

The **Equation of Time** quantifies the difference between apparent solar time (measured by the Sun's actual position in the sky) and mean solar time (based on a uniform 24-hour day):

$$\text{Equation of Time} = \text{Apparent Solar Time} - \text{Mean Solar Time}$$

This difference arises due to

- The **ellipticity of Earth's orbit** (Kepler's second law),
- The **tilt of Earth's axis** relative to the orbital plane (obliquity).



## 4 Measurement in Astrophysics

Quantity	Value / Definition
<b>Distance Units</b>	
Astronomical Unit (AU)	$1 \text{ AU} = 1.496 \times 10^{11} \text{ m}$ (Mean distance between Earth and the Sun)
Parsec (pc)	$1 \text{ pc} = 3.086 \times 10^{16} \text{ m} \approx 3.26 \text{ ly}$ (Distance at which 1 AU subtends 1 arcsecond)
Light-year (ly)	$1 \text{ ly} = 9.461 \times 10^{15} \text{ m}$
Kiloparsec (kpc)	$1 \text{ kpc} = 10^3 \text{ pc}$
Megaparsec (Mpc)	$1 \text{ Mpc} = 10^6 \text{ pc}$
<b>Energy Units</b>	
Electron volt	$1 \text{ eV} = 1.602 \times 10^{-19} \text{ J}$
Erg	$1 \text{ erg} = 10^{-7} \text{ J}$
<b>Luminosity and Flux</b>	
Solar luminosity	$1 L_\odot = 3.828 \times 10^{26} \text{ W}$
Flux	$\text{W m}^{-2}$ or $\text{erg s}^{-1} \text{ cm}^{-2}$
<b>Spectral Units</b>	
Wavelength	$1 \text{ \AA} = 10^{-10} \text{ m}$
<b>Temperature</b>	
Kelvin	$K = {}^\circ C + 273.15$ (Standard unit for blackbody radiation and star temperature)

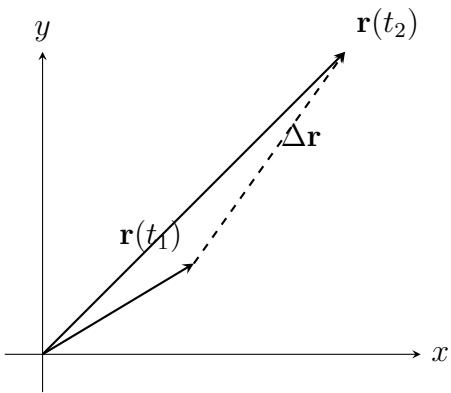
## 5 Fundamental Mechanics

### 5.1 Kinematics

#### 5.1.1 Linear Motion

We can use the **position vector**  $\mathbf{r}(t)$  to specify the position of a particle in a plane at time  $t$ .





The figure above shows the position of a particle at two different times,  $t_1$  and  $t_2$ . The **displacement vector** is given by

$$\Delta \mathbf{r} = \mathbf{r}(t_2) - \mathbf{r}(t_1)$$

The **average velocity** during the time interval  $\Delta t = t_2 - t_1$  is:

$$\mathbf{v}_{\text{av}} = \frac{\Delta \mathbf{r}}{\Delta t}$$

The **instantaneous velocity** at time  $t_0$  is obtained by taking the limit as  $\Delta t \rightarrow 0$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

The **instantaneous speed** is the magnitude of the velocity vector

$$v = |\mathbf{v}|$$

The **average acceleration** is the rate of change of velocity

$$\mathbf{a}_{\text{av}} = \frac{\Delta \mathbf{v}}{\Delta t}$$

and the **instantaneous acceleration** is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$$

Suppose a particle moves with a **constant acceleration**  $\mathbf{a}$ . At time  $t_0$ , let its velocity and position be  $\mathbf{v}_0$  and  $\mathbf{x}_0$ , respectively. Then, the motion can be described by:

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{v}_0(t - t_0) + \frac{1}{2}\mathbf{a}(t - t_0)^2$$



$$\mathbf{v} = \mathbf{v}_0 + \mathbf{a}(t - t_0)$$

$$|\mathbf{v}|^2 - |\mathbf{v}_0|^2 = 2 \mathbf{a} \cdot (\mathbf{x} - \mathbf{x}_0)$$

The time derivative of a vector provides information about the rate of change of the vector's magnitude and direction.

### 5.1.2 Relative Motion

**Introduction** Let us consider two objects,  $A$  and  $B$ , moving in a straight line or in a plane. The velocity of object  $A$  relative to object  $B$  is given by

$$\mathbf{v}_{A/B} = \mathbf{v}_A - \mathbf{v}_B$$

where

- $\mathbf{v}_A$  is the velocity of object  $A$  relative to a fixed reference frame.
- $\mathbf{v}_B$  is the velocity of object  $B$  relative to the same reference frame.
- $\mathbf{v}_{A/B}$  is the velocity of object  $A$  relative to object  $B$ .

When two bodies are rotating relative to each other, their angular velocities need to be combined to determine the relative angular velocity. If object  $A$  has angular velocity  $\boldsymbol{\omega}_A$  and object  $B$  has angular velocity  $\boldsymbol{\omega}_B$ , the relative angular velocity of  $A$  with respect to  $B$ , denoted as  $\boldsymbol{\omega}_{A/B}$ , is given by:

$$\boldsymbol{\omega}_{A/B} = \boldsymbol{\omega}_A - \boldsymbol{\omega}_B$$

It is assumed that both objects are rotating about the same axis.

**Retrograde Motion** Retrograde motion is the apparent backward motion of a planet relative to the background stars, caused by the relative orbital motion of Earth and the planet. Earth moves faster in its inner orbit. Earth moves faster in its inner orbit.



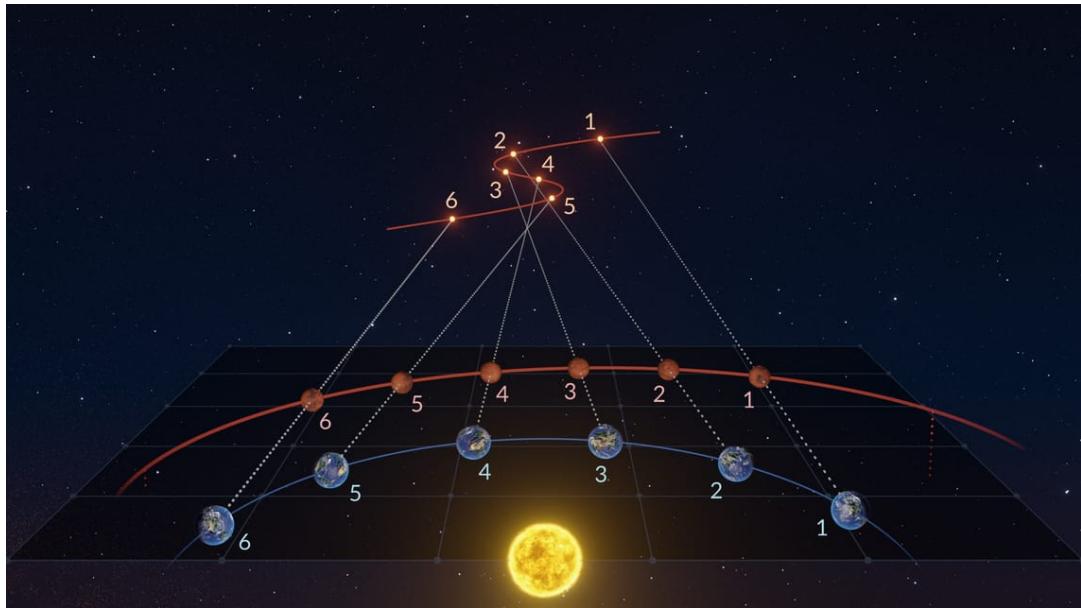


Figure 4: Source: <https://starwalk.space/en/news/what-is-retrograde-motion>

**Example.** Given:

$$T_E = 365.25 \text{ days}$$

$$T_{Mars} = 686.98 \text{ days}$$

Angular speeds:

$$\omega_E = \frac{360^\circ}{365.25} = 0.9856^\circ/\text{day}$$

$$\omega_{Mars} = \frac{360^\circ}{686.98} = 0.524^\circ/\text{day}$$

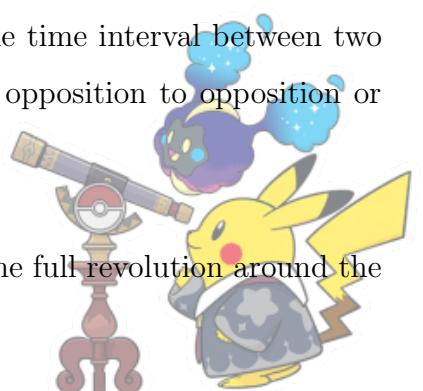
Relative angular speed:

$$\omega_{rel} = \omega_E - \omega_{Mars} = 0.4616^\circ/\text{day}$$

This relative angular motion causes Mars to appear to move backward against the stars during opposition. The duration between successive retrograde motions of a planet is equal to its **synodic period**.

**Synodic Period and Sidereal Period** The **synodic period** ( $T_{syn}$ ) is the time interval between two successive identical configurations of a planet as observed from Earth (e.g., opposition to opposition or conjunction to conjunction).

The **sidereal period** ( $T_s$ ) of a celestial body is the time taken to complete one full revolution around the



Sun with respect to the fixed background stars.

**Example.** Let

- $T_E$  = sidereal period of Earth
- $T_P$  = sidereal period of the planet

For **inferior planets** (Mercury, Venus):

$$\frac{1}{T_{syn}} = \frac{1}{T_P} - \frac{1}{T_E}$$

For **superior planets** (Mars, Jupiter, Saturn):

$$\frac{1}{T_{syn}} = \frac{1}{T_E} - \frac{1}{T_P}$$

**Slingshot Effect of Gravity** The **slingshot effect**, also known as **gravitational assist**, occurs when a spacecraft uses the gravity of a planet or moon to change its trajectory and increase its velocity. This is achieved by flying close to a planet or moon, which causes the spacecraft to accelerate as it is pulled by the planet's gravitational field. The effect is used to save fuel and to direct spacecraft to their destinations. Mathematically, this can be described by the following equation for velocity change during the flyby

$$\Delta v = v_{\text{spacecraft}} + v_{\text{planet}}$$

where

1.  $v_{\text{spacecraft}}$  is the velocity of the spacecraft, and
2.  $v_{\text{planet}}$  is the velocity of the planet as seen from the spacecraft's reference frame.

## 5.2 Newton's Law

**First Law (Law of Inertia)** An object remains at rest, or moves in a straight line at constant velocity, unless acted upon by an external force.

$$\sum \mathbf{F} = 0 \implies \mathbf{a} = 0$$



**Second Law** The net force on a particle is equal to the time rate of change of its momentum:

$$\sum \mathbf{F} = \frac{d\mathbf{p}}{dt}$$

For constant mass,

$$\sum \mathbf{F} = m\mathbf{a}$$

This is the most commonly used form of Newton's Second Law.

**Third Law** For every action, there is an equal and opposite reaction:

$$\mathbf{F}_{AB} = -\mathbf{F}_{BA}$$

## 5.3 Linear Momentum

### 5.3.1 Introduction

Linear momentum is a fundamental concept in mechanics that describes the motion of an object in terms of its mass and velocity. It is defined as the product of an object's mass and velocity:

$$\mathbf{p} = m\mathbf{v}$$

where

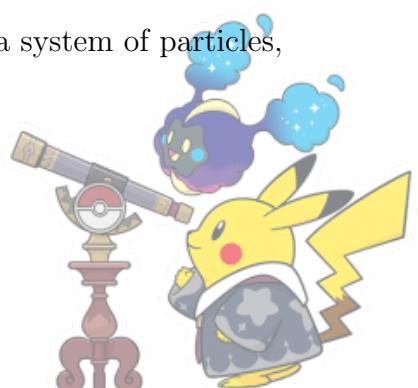
- $\mathbf{p}$  is the linear momentum of the object,
- $m$  is the mass of the object, and
- $\mathbf{v}$  is the velocity of the object.

### 5.3.2 Conservation of Linear Momentum

In an isolated system (no external forces), the total linear momentum of the system remains constant. This is known as the conservation of linear momentum. Mathematically, for a system of particles,

$$\sum_{i=1}^n \mathbf{p}_i = \text{constant}$$

where  $\mathbf{p}_i$  is the linear momentum of the  $i$ -th particle.



### 5.3.3 Impulse

The impulse of a force  $\mathbf{F}$  acting on an object is defined as the change in the object's momentum. Mathematically,

$$\mathbf{J} = \Delta\mathbf{p} = \mathbf{F}\Delta t$$

where  $\mathbf{J}$  is the impulse,  $\Delta\mathbf{p}$  is the change in momentum,  $\mathbf{F}$  is the average force, and  $\Delta t$  is the time interval during which the force acts.

### 5.3.4 Rocket Equation

Let the rocket expel a small mass of fuel  $dm$  in a short time interval  $dt$ . By conservation of momentum,

$$\text{Initial momentum} = \text{Final momentum}$$

Initially, the momentum of the rocket is

$$p_{\text{initial}} = m(t)v(t)$$

After expelling  $dm$  of fuel, the rocket mass becomes  $m(t) - dm$  and its velocity becomes  $v(t) + dv$ . The expelled fuel has momentum  $dm \cdot (v - u)$ , because it moves at velocity  $v - u$  in the inertial frame. Hence the final momentum is

$$p_{\text{final}} = (m(t) - dm)(v + dv) + dm(v - u)$$

Using differential approximation (ignoring higher-order terms  $dm dv$ ) and equating momentum:

$$m dv = u dm$$

Integrating from the initial mass  $m_0$  to the final mass  $m_f$  and initial velocity  $v_0$  to final velocity  $v_f$ :

$$\int_{v_0}^{v_f} dv = u \int_{m_0}^{m_f} \frac{dm}{m}$$

$$v_f - v_0 = u \ln \frac{m_0}{m_f}$$



If external forces (like gravity  $g$  or drag  $F_{\text{drag}}$ ) act on the rocket, the generalized form is

$$m \frac{dv}{dt} = u \frac{dm}{dt} + F_{\text{ext}}(t)$$

where  $F_{\text{ext}}(t)$  represents any additional force.

## 5.4 Force, Torque and Equilibrium of Body

The force is any interaction that causes a change in an object's motion. It is a vector quantity, having both magnitude and direction. In vector notation:

$$\mathbf{F} = (F_x, F_y, F_z)$$

**SI Unit:** Newton (N) which is defined as

$$1 \text{ N} = 1 \text{ kg m s}^{-1}$$

### Example of Forces

- Gravitational force:  $\mathbf{F}_g = mg$
- Normal force: the perpendicular reaction from a surface
- Friction:  $\mathbf{f} = \mu N$  (opposes motion)
- Tension: force transmitted through a rope

The torque  $\tau$  produced by a force  $\mathbf{F}$  acting at a distance  $\mathbf{r}$  from the pivot is the cross product:

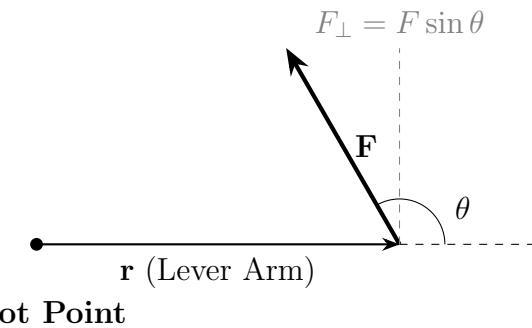
$$\tau = \mathbf{r} \times \mathbf{F}$$

The magnitude is given by

$$\tau = rF \sin(\theta)$$

**SI Unit:** Newton-meter ( $\text{N} \cdot \text{m}$ )





A body is said to be in equilibrium when the net force acting on it is zero, and the net torque acting on it is also zero. There are two types of equilibrium:

- **Translational equilibrium** occurs when the sum of all forces acting on a body is zero:

$$\sum \mathbf{F} = 0$$

- **Rotational equilibrium** occurs when the sum of all torques acting on a body about any point is zero:

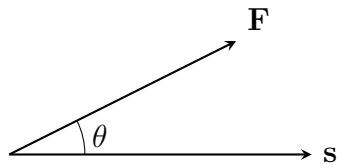
$$\sum \tau = 0$$

## 5.5 Energy

When a constant force  $\mathbf{F}$  acts on an object and causes a displacement  $\mathbf{s}$ , the work done by the force is defined as the dot product

$$W = \mathbf{F} \cdot \mathbf{s} = F s \cos \theta$$

where  $\theta$  is the angle between the force and the displacement.



If the force is not constant, then

$$W = \int_C \mathbf{F} \cdot d\mathbf{s}$$

where  $C$  is the path of the object. Note that

- Work is a **scalar quantity**.
- $W > 0$  if force and displacement are in the same direction.



- $W < 0$  if force opposes motion (e.g. friction).
- $W = 0$  if  $\mathbf{F} \perp \mathbf{s}$  (no work done).

For an object of mass  $m$  moving vertically through a height  $h$ :

$$W_g = -mgh$$

if the object is lifted upward, and

$$W_g = +mgh$$

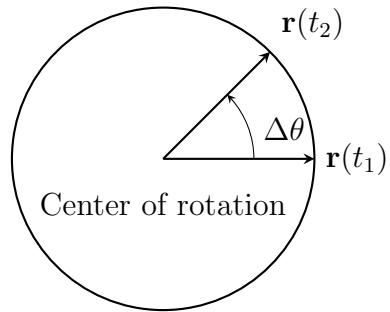
if it moves downward. The **kinetic energy** of a body of mass  $m$  moving with speed  $v$  is

$$K = \frac{1}{2}mv^2$$

## 5.6 Circular Motion

### 5.6.1 Kinematics

Consider an object (for example, a dog on a rope) rotating about a fixed point with a radius  $r$ .



- **Angular displacement:**  $\Delta\theta$  (in radians)

- **Angular velocity:**  $\omega = \frac{d\theta}{dt}$

- **Angular acceleration:**  $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$

The rotational kinematic equations (analogous to linear motion) are

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\omega = \omega_0 + \alpha t$$



$$\omega^2 - \omega_0^2 = 2\alpha(\theta - \theta_0)$$

For an object moving in a circle of radius  $r$ ,

$$v = r\omega$$

is the **tangential (linear) speed**.

The **tangential acceleration** is

$$a_{\tan} = r\alpha$$

and the **radial (centripetal) acceleration** is

$$a_{\text{rad}} = \frac{v^2}{r} = r\omega^2$$

### 5.6.2 Centripetal Force

Centripetal force is the force that acts on an object moving in a circular path. It is directed towards the center of the circle or axis of rotation. The magnitude of the centripetal force is given by

$$F_c = \frac{mv^2}{r}$$

where

- $m$  is the mass of the object,
- $v$  is the velocity of the object,
- $r$  is the radius of the circular path.

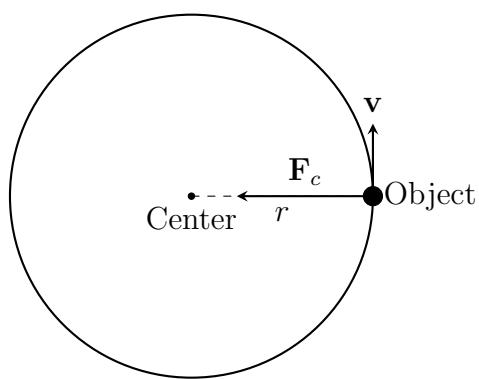


Figure 5: Centripetal force always points toward the center of the circle.



### 5.6.3 Centrifugal Force

Centrifugal force is a fictitious or pseudo force that appears to act on an object when viewed from a rotating reference frame. It appears to push the object away from the center of rotation. It is equal in magnitude and opposite in direction to the centripetal force in the rotating frame of reference. The magnitude of the centrifugal force is

$$F_{\text{cf}} = \frac{mv^2}{r}$$

### 5.6.4 Coriolis Force

Coriolis force is an apparent force that acts on a mass moving in a rotating system, such as the Earth. It arises due to the rotation of the reference frame. The Coriolis force is given by

$$F_{\text{Coriolis}} = 2m\mathbf{v} \times \boldsymbol{\omega}$$

where

- $m$  is the mass of the object,
- $\mathbf{v}$  is the velocity of the object relative to the rotating frame,
- $\boldsymbol{\omega}$  is the angular velocity vector of the rotating reference frame.

### 5.6.5 Angular Momentum

For a particle at position  $\mathbf{r}$  (relative to the origin) with linear momentum  $\mathbf{p}$ :

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

For a rigid body rotating around a fixed axis with moment of inertia  $I$  and angular velocity  $\omega$ :

$$\mathbf{L} = I\boldsymbol{\omega}$$

Just as force changes linear momentum, torque ( $\boldsymbol{\tau}$ ) changes angular momentum:

$$\boldsymbol{\tau}_{\text{net}} = \frac{d\mathbf{L}}{dt}$$



**Theorem. 5.1: Conservation of Angular Momentum**

If the net external torque on a system is zero ( $\tau_{ext} = 0$ ), the total angular momentum is conserved:

$$L_i = L_f$$

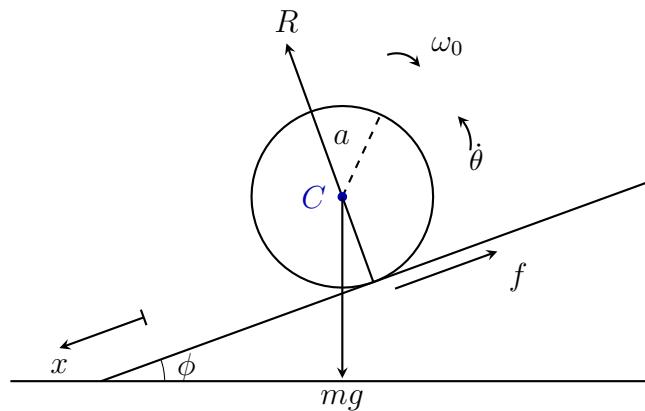


## 5.7 Case Study

A circular disk rotating about its axis with angular speed  $\omega_0$  is placed gently with its axis horizontal on a rough and inclined plane such that the friction acts up the plane. Given that the coefficient of friction is  $\mu$  and the inclination angle of the plane to the horizontal is  $\phi$ .

- (a) Show that the disk will move upwards if  $\mu > \tan \phi$ .
- (b) Find the time that elapses before rolling takes place.

**Solution.** (a)



**Equations of Motion:** Take downward as positive and take counterclockwise as positive. Using Newton's Second Law for translation along the plane and rotation about the center of mass:

$$m\ddot{x} = mg \sin \phi - f \quad (5)$$

$$R - mg \cos \phi = 0 \implies R = mg \cos \phi \quad (6)$$

$$I\ddot{\theta} = \tau \quad (7)$$

Since the disk is slipping, kinetic friction applies:

$$f = \mu R = \mu mg \cos \phi \quad (8)$$

Substituting (8) into (5):

$$m\ddot{x} = mg \sin \phi - \mu mg \cos \phi \implies \ddot{x} = g(\sin \phi - \mu \cos \phi) < 0$$



$$g(\sin \phi - \mu \cos \phi) < 0$$

$$\sin \phi < \mu \cos \phi$$

$$\tan \phi < \mu$$

Hence, the disk will move upwards if  $\boxed{\mu > \tan \phi}$ .

(b) For pure rolling,

$$v_{\text{contact}} = v + a\omega = 0$$

where  $v = \dot{x}$  is the linear velocity and  $\omega = \dot{\theta}$  is the angular velocity. Note that

$$v(t) = v_0 + \ddot{x}t = 0 + g(\sin \phi - \mu \cos \phi)t.$$

Also,

$$\tau = -f \cdot a = -(\mu mg \cos \phi)a$$

Using  $I = \frac{1}{2}ma^2$  for a disk,

$$\frac{1}{2}ma^2\ddot{\theta} = -\mu m g a \cos \phi \implies \ddot{\theta} = -\frac{2\mu g \cos \phi}{a}$$

Note that

$$\omega(t) = \omega_0 + \ddot{\theta}t = \omega_0 - \frac{2\mu g \cos \phi}{a}t$$

Substitute (5.7) and (5.7) into the rolling condition (5.7):

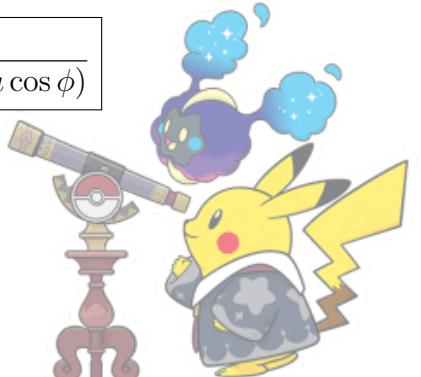
$$\begin{aligned} v(t) + a\omega(t) &= 0 \\ [g(\sin \phi - \mu \cos \phi)t] + a \left[ \omega_0 - \frac{2\mu g \cos \phi}{a}t \right] &= 0 \end{aligned}$$

$$gt(\sin \phi - \mu \cos \phi) + a\omega_0 - 2\mu gt \cos \phi = 0$$

$$gt(\sin \phi - \mu \cos \phi - 2\mu \cos \phi) = -a\omega_0$$

$$gt(\sin \phi - 3\mu \cos \phi) = -a\omega_0$$

$$t = \boxed{\frac{-a\omega_0}{g(\sin \phi - 3\mu \cos \phi)}}$$



## 6 Orbital Mechanics

### 6.1 Newton's Law of Universal Gravitation

#### 6.1.1 Introduction

##### Theorem. 6.1: Newton's Law of Universal Gravitation

The Universal Law of Gravity states that every point mass attracts every other point mass with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between their centers. This is mathematically expressed as

$$F = G \frac{m_1 m_2}{r^2}$$

where

- $F$  is the magnitude of the gravitational force between the two masses,
- $G$  is the gravitational constant,
- $m_1$  and  $m_2$  are the two masses,
- $r$  is the distance between the centers of the two masses.

##### Theorem. 6.2: Newton's Law of Universal Gravitation (Vector Form)

The gravitational force between two point masses  $m_1$  and  $m_2$  can be written in vector form as

$$\mathbf{F} = -G \frac{m_1 m_2}{r^2} \hat{r}$$

where

- $\mathbf{F}$  is the gravitational force vector on  $m_1$  due to  $m_2$ ,
- $G$  is the gravitational constant,
- $m_1$  and  $m_2$  are the masses of the two objects,
- $r$  is the distance between the two masses,
- $\hat{r}$  is the unit vector pointing from  $m_1$  to  $m_2$ , i.e., the direction of the force.



The negative sign indicates that the force is attractive, meaning the force is directed towards the other mass.

### Theorem. 6.3: Gauss's Law of Gravity

$$\nabla \cdot \mathbf{g} = -4\pi G\rho$$

where

- $\nabla \cdot \mathbf{g}$  is the divergence of the gravitational field  $\mathbf{g}$ , and
- $\rho$  is the mass density of the object.

This law can be integrated over a closed surface to give the integral form of Gauss's Law for gravity:

$$\oint_S \mathbf{g} \cdot d\mathbf{A} = -4\pi GM_{\text{enc}}$$

where  $M_{\text{enc}}$  is the mass enclosed by the surface.

#### 6.1.2 Barycentre

The **barycentre** or **centre of mass** is the weighted average position of all the mass in a system. It is the point where, if all the mass were concentrated, the system would balance. For a system of particles with masses  $m_1, m_2, \dots, m_n$  located at positions  $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n$ , the position of the barycentre  $\mathbf{R}$  is given by

$$\mathbf{R} = \frac{1}{M} \sum_{i=1}^n m_i \mathbf{r}_i$$

where  $M = \sum_{i=1}^n m_i$  is the total mass of the system.

For a continuous mass distribution, the barycentre is calculated by integrating over the mass distribution. If the mass density is  $\rho(\mathbf{r})$ , the barycentre position is given by

$$\mathbf{R} = \frac{1}{M} \int_V \mathbf{r} \rho(\mathbf{r}) dV$$

where  $M = \int_V \rho(\mathbf{r}) dV$  is the total mass of the system, and the integral is taken over the volume  $V$  of the mass distribution.



### 6.1.3 2-body Problem

Consider two point masses,  $m_1$  at position vector  $\mathbf{r}_1$  and  $m_2$  at position vector  $\mathbf{r}_2$ , interacting via a central force:

$$m_1 \ddot{\mathbf{r}}_1 = \mathbf{F}_{21}, \quad m_2 \ddot{\mathbf{r}}_2 = \mathbf{F}_{12} = -\mathbf{F}_{21}$$

The center of mass (barycenter) is located at

$$\mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}$$

and since the internal forces cancel by Newton's third law, the acceleration of the center of mass is

$$\ddot{\mathbf{R}} = 0$$

Hence, the center of mass moves in a straight line with constant velocity. Define the relative position vector

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$

Differentiating twice with respect to time gives:

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = \frac{\mathbf{F}_{21}}{m_1} - \frac{\mathbf{F}_{12}}{m_2} = \frac{\mathbf{F}(\mathbf{r})}{m_1} + \frac{\mathbf{F}(\mathbf{r})}{m_2} = \left( \frac{1}{m_1} + \frac{1}{m_2} \right) \mathbf{F}(\mathbf{r})$$

Define the reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

Then

$$\mu \ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r})$$

which is an effective one-body problem where a particle with mass  $\mu$  moves under the force  $\mathbf{F}(\mathbf{r})$ . The angular momentum about the center of mass for the effective one-body problem with reduced mass  $\mu$  is

$$\mathbf{L} = \mu \mathbf{r} \times \mathbf{v}$$

For a **central force**

$$\mathbf{F}(\mathbf{r}) = f(r) \hat{\mathbf{r}}$$



the torque is given by

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}(\mathbf{r}) = \mathbf{r} \times f(r)\hat{\mathbf{r}} = \mathbf{0}$$

Hence,

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau} = \mathbf{0} \quad \Rightarrow \quad \mathbf{L} = \text{constant}$$

The total energy of the system is

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2 + U(r)$$

where  $U(r)$  is the potential energy of the central force. Note that

$$L = |\mathbf{L}| = \mu r^2\dot{\theta}$$

Using

$$\dot{\theta} = \frac{L}{\mu r^2}$$

we have

$$\begin{aligned} \frac{1}{2}\mu r^2\dot{\theta}^2 &= \frac{L^2}{2\mu r^2} \\ E &= \frac{1}{2}\mu\dot{r}^2 + \underbrace{\left[ U(r) + \frac{L^2}{2\mu r^2} \right]}_{V_{\text{eff}}(r)} \end{aligned}$$

#### 6.1.4 *n*-body Problem

Consider  $N$  bodies with masses  $m_i$  and position vectors  $\mathbf{r}_i(t)$  in three-dimensional space. According to Newton's law of gravitation, the acceleration of the  $i$ -th body is given by

$$\ddot{\mathbf{r}}_i = \sum_{\substack{j=1 \\ j \neq i}}^N G \frac{m_j(\mathbf{r}_j - \mathbf{r}_i)}{\|\mathbf{r}_j - \mathbf{r}_i\|^3}, \quad i = 1, 2, \dots, N, \quad (9)$$

where  $\|\mathbf{r}_j - \mathbf{r}_i\|$  is the Euclidean distance between bodies  $i$  and  $j$ .



## 6.2 Kepler's Law

### 6.2.1 Three Kepler's Laws

#### Theorem. 6.4: Kepler's First Law

The orbit of every planet is an ellipse with the Sun at one of the two foci. The general equation for an ellipse in polar coordinates  $(r, \theta)$  with one focus at the origin is

$$r = \frac{a(1 - e^2)}{1 + e \cos \theta}$$

where  $a$  is the semi-major axis, and  $e$  is the eccentricity ( $0 \leq e < 1$ ). The eccentricity of an orbit describes how elliptical it is. For a perfectly circular orbit, the eccentricity is 0, while for an elliptical orbit, it is between 0 and 1. The closer the eccentricity is to 1, the more elongated the orbit.

1. Inner Solar System: The four terrestrial planets are located here, and their orbits are closer to the Sun with relatively small eccentricities.
2. Outer Solar System: The gas giants and icy bodies like comets have more elongated orbits and are located farther from the Sun.

**Proof.** Let the system be a planet of mass  $m$  orbiting a star of mass  $M$  (assumed fixed at the origin).

The gravitational force is

$$\mathbf{F} = -G \frac{Mm}{r^2} \hat{r} = m\mathbf{a}$$

where  $\mu = GM$  is the gravitational parameter and  $\mathbf{r} = r\hat{r}$ . The acceleration is  $\mathbf{a} = -\frac{\mu}{r^2} \hat{r}$ .

- **Angular Momentum L (Kepler's Second Law):** The torque is  $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = 0$  (since  $\mathbf{r}$  is parallel to  $\mathbf{F}$ ). Therefore, the angular momentum is constant:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m(\mathbf{r} \times \mathbf{v}) = \text{constant}$$

The orbit is confined to the plane perpendicular to  $\mathbf{L}$ .

- **Laplace-Runge-Lenz (LRL) Vector A:** The LRL vector is a constant of motion only for an inverse-square force. It is defined as

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - m\mu\hat{r}$$



To prove it's conserved, we show  $\frac{d\mathbf{A}}{dt} = 0$ . Using the product rule,

$$\frac{d\mathbf{A}}{dt} = \frac{d\mathbf{p}}{dt} \times \mathbf{L} + \mathbf{p} \times \frac{d\mathbf{L}}{dt} - m\mu \frac{d\hat{r}}{dt}$$

As  $\frac{d\mathbf{p}}{dt} = \mathbf{F}$  and  $\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau} = 0$ ,

$$\frac{d\mathbf{A}}{dt} = \mathbf{F} \times \mathbf{L} - m\mu \frac{d\hat{r}}{dt}$$

By  $\mathbf{F} = -\frac{m\mu}{r^2}\hat{r}$  and  $\mathbf{L} = m(\mathbf{r} \times \mathbf{v})$ :

$$\mathbf{F} \times \mathbf{L} = -\frac{m\mu}{r^2}\hat{r} \times m(\mathbf{r} \times \mathbf{v}) = -\frac{m^2\mu}{r^2} [(\hat{r} \cdot \mathbf{v})\mathbf{r} - (\hat{r} \cdot \mathbf{r})\mathbf{v}]$$

By  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ ,  $\hat{r} \cdot \mathbf{r} = r$  and  $\hat{r} \cdot \mathbf{v} = \dot{r}$ ,

$$\mathbf{F} \times \mathbf{L} = -\frac{m^2\mu}{r^2} [(\dot{r})\mathbf{r} - (r)\mathbf{v}] = m\mu \left[ \frac{r\mathbf{v}}{r^2} - \frac{\dot{r}\mathbf{r}}{r^2} \right]$$

Note that

$$\frac{d\hat{r}}{dt} = \frac{d}{dt} \left( \frac{\mathbf{r}}{r} \right) = \frac{\dot{\mathbf{r}}r - \mathbf{r}\dot{r}}{r^2} = \frac{\mathbf{v}r - \mathbf{r}\dot{r}}{r^2}$$

Hence,  $\mathbf{F} \times \mathbf{L} = m\mu \frac{d\hat{r}}{dt}$ . Therefore,

$$\frac{d\mathbf{A}}{dt} = m\mu \frac{d\hat{r}}{dt} - m\mu \frac{d\hat{r}}{dt} = 0$$

The LRL vector  $\mathbf{A}$  lies in the orbital plane. We find the orbit equation by taking the dot product of  $\mathbf{A}$  with the position vector  $\mathbf{r}$ :

$$\mathbf{r} \cdot \mathbf{A} = \mathbf{r} \cdot (\mathbf{p} \times \mathbf{L} - m\mu \hat{r}) = \mathbf{r} \cdot (\mathbf{p} \times \mathbf{L}) - m\mu(\mathbf{r} \cdot \hat{r})$$

Using the scalar triple product identity  $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$ ,

$$\mathbf{r} \cdot (\mathbf{p} \times \mathbf{L}) = \mathbf{L} \cdot (\mathbf{r} \times \mathbf{p})$$

Since  $\mathbf{r} \times \mathbf{p} = \mathbf{L}$ , the identity simplifies to

$$\mathbf{r} \cdot (\mathbf{p} \times \mathbf{L}) = \mathbf{L} \cdot \mathbf{L} = L^2$$



Therefore,

$$\mathbf{r} \cdot \mathbf{A} = L^2 - m\mu r$$

The LRL vector  $\mathbf{A}$  is constant. Let  $A = |\mathbf{A}|$ . We define the angle  $\theta$  between  $\mathbf{r}$  and  $\mathbf{A}$ . Then  $\mathbf{r} \cdot \mathbf{A} = rA \cos \theta$ :

$$rA \cos \theta = L^2 - m\mu r$$

$$r = \frac{L^2}{m\mu} \frac{1}{1 + \frac{A}{m\mu} \cos \theta}$$

Normalizing by  $m = 1$  gives

$$r = \frac{\ell}{1 + e \cos \theta}$$

which is the equation of conic section in polar coordinates, where

- The **semi-latus rectum** is  $\ell = \frac{L^2}{m\mu}$ .
- The **eccentricity** is  $e = \frac{A}{m\mu}$ .

As  $A$ ,  $m$ , and  $\mu$  are all positive, the eccentricity  $e$  is a non-negative constant.

- If the total energy  $E < 0$ , then  $0 \leq e < 1$ , which defines an ellipse.
- If  $E = 0$ , then  $e = 1$ , defining a parabola.
- If  $E > 0$ , then  $e > 1$ , defining a hyperbola.

Planetary orbits are bound orbit, meaning the total mechanical energy  $E$  is negative ( $E < 0$ ). This guarantees  $0 \leq e < 1$ , proving that the orbit is an ellipse with the Sun (the central mass) at one focus.



**Theorem. 6.5: Kepler's Second Law**

A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.

**Proof.**

- The gravitational force  $\mathbf{F}$  is always directed along the line connecting the planet and the Sun (it's a **central force**).

$$\mathbf{F} = -G \frac{Mm}{r^2} \hat{r}$$

- The **torque** ( $\boldsymbol{\tau}$ ) on the planet with respect to the Sun (the origin) is given by

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \left( -G \frac{Mm}{r^2} \hat{r} \right)$$

Since  $\mathbf{r}$  and  $\hat{r}$  are in the same direction,  $\mathbf{r} \times \hat{r} = 0$ . Hence, the torque is zero:  $\boldsymbol{\tau} = 0$ .

- By Newton's second law for rotation, the torque is equal to the rate of change of the angular momentum ( $\mathbf{L}$ ):

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}$$

Since  $\boldsymbol{\tau} = 0$ , the angular momentum  $\mathbf{L}$  is conserved:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times (m\mathbf{v}) = \text{constant}$$

where  $\mathbf{p}$  is the linear momentum.

- The angular momentum can be expressed in terms of the area swept out per unit time. Consider an infinitesimal time  $dt$ . The area  $dA$  swept by the radius vector  $\mathbf{r}$  is approximately the area of a triangle:

$$dA = \frac{1}{2} |\mathbf{r} \times d\mathbf{r}|$$

Since  $d\mathbf{r} = \mathbf{v}dt$ , we have:

$$dA = \frac{1}{2} |\mathbf{r} \times \mathbf{v}dt| = \frac{1}{2} |\mathbf{r} \times \mathbf{v}| dt$$

- The rate at which area is swept out is  $\frac{dA}{dt}$ :

$$\frac{dA}{dt} = \frac{1}{2} |\mathbf{r} \times \mathbf{v}| = \frac{1}{2m} |\mathbf{r} \times m\mathbf{v}| = \frac{|\mathbf{L}|}{2m} = \text{constant}$$



### Theorem. 6.6: Kepler's Third Law

Kepler's Third Law states that the square of the orbital period  $T$  of a planet is directly proportional to the cube of the semi-major axis  $a$  of its orbit. Mathematically,

$$T^2 \propto a^3$$

In the case of elliptical orbits, where  $a$  is the semi-major axis, the law still holds and it can be expressed as

$$T^2 = \frac{4\pi^2}{GM}a^3$$

where

- $T$  is the orbital period of the planet.
- $a$  is the semi-major axis of the ellipse.
- $G$  is the gravitational constant.
- $M$  is the mass of the central body (e.g., the Sun).

**Proof.** By Kepler's 2nd law,

$$\frac{\pi ab}{T} = \frac{L}{2m}$$

as the area of ellipse is given by  $\pi ab$ . By Kepler's 1st law,  $r = a(1 - e)$  and

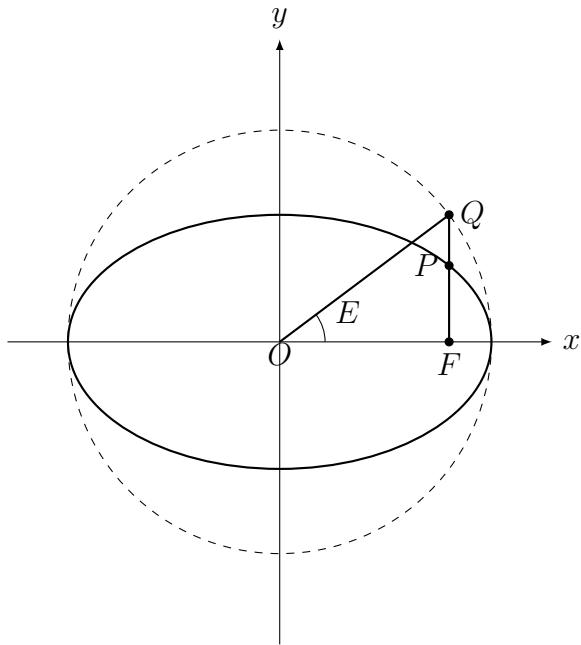
$$r(\theta) = \frac{L^2}{GMm^2(1 + e \cos \theta)}.$$

Then  $\frac{L^2}{m^2} = a(1 - e^2)GM$ . Substitute the formula involving the area,  $T^2 = \frac{4\pi^2}{GM}a^3$ .

In some regions of the Solar System, objects have synchronized orbital periods due to gravitational interactions. This phenomenon, known as orbital resonance, is particularly important in the asteroid belt and among some moons of the giant planets.



### 6.2.2 Kepler's Equation



The eccentric anomaly  $E$  is defined using an auxiliary circle:

- Draw a circle of radius  $a$  centered at the origin (the auxiliary circle).
- For a point  $P$  on the ellipse, draw a vertical line to intersect the auxiliary circle at  $Q$ .
- The eccentric anomaly  $E$  is the angle  $\angle X O Q$ .

where  $O$  is the center of the ellipse and  $X$  is the positive  $x$ -axis direction. Although  $E$  is not a physical angle measured from the focus, it provides a convenient parametrization of the ellipse. Using the eccentric anomaly  $E$ , the position of the orbiting body is

$$\boxed{\begin{aligned}x &= a(\cos E - e) \\y &= a\sqrt{1 - e^2} \sin E\end{aligned}}$$

The **true anomaly**  $\nu$  is the physical polar angle of the orbiting body measured from periapsis. The relation between  $\nu$  and  $E$  is

$$\tan \frac{\nu}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2}$$

The area swept from periapsis to point  $P$  on the ellipse is

$$A = \frac{a^2}{2}(E - e \sin E)$$



The total area of the ellipse is  $\pi a^2 \sqrt{1 - e^2}$ , but by Kepler's second law, the fraction of area swept equals the fraction of time elapsed:

$$\frac{A}{\pi a^2} = \frac{t - \tau}{T}$$

where  $T$  is the orbital period. Using mean anomaly

$$M \equiv n(t - \tau)$$

(where **mean motion**  $n = 2\pi/T$  and  $\tau$  is the time of periapsis passage), we obtain

$$M = E - e \sin E$$

which is the Kepler's equation.

### 6.2.3 Poisson Bracket

#### Definition. 6.1: Poisson Bracket

The Poisson bracket of two functions  $f(q, p)$  and  $g(q, p)$ , where  $q$  and  $p$  represent generalized coordinates and momenta, is defined as

$$\{f, g\} = \sum_i \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$$

Here,  $q_i$  and  $p_i$  represent the generalized coordinates and momenta for the  $i$ -th degree of freedom.

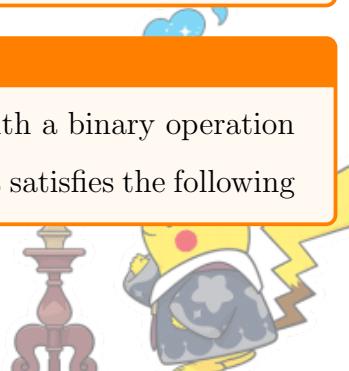
#### Definition. 6.2: Levi-Civita Symbol

The Levi-Civita symbol is defined as

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } (i, j, k) \text{ is an even permutation of } (1, 2, 3), \\ -1 & \text{if } (i, j, k) \text{ is an odd permutation of } (1, 2, 3), \\ 0 & \text{if any two indices are equal.} \end{cases}$$

#### Definition. 6.3: Lie Algebra

A **Lie algebra** is a vector space  $\mathfrak{g}$  over a field (typically  $\mathbb{R}$  or  $\mathbb{C}$ ) equipped with a binary operation called the **Lie bracket**, denoted by  $[\cdot, \cdot]$ . The Lie bracket is a bilinear map that satisfies the following



two properties:

- **Bilinearity:**

$$[aX+bY, Z] = a[X, Z]+b[Y, Z], \quad [Z, aX+bY] = a[Z, X]+b[Z, Y] \quad \text{for all } X, Y, Z \in \mathfrak{g}, \quad a, b \in \mathbb{F}$$

- **Antisymmetry:**

$$[X, Y] = -[Y, X] \quad \text{for all } X, Y \in \mathfrak{g}$$

- **Jacobi Identity:**

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \quad \text{for all } X, Y, Z \in \mathfrak{g}$$

For the angular momentum  $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ , the components of the angular momentum are

$$L_i = \epsilon_{ijk} r_j p_k$$

where  $\epsilon_{ijk}$  is the Levi-Civita symbol, and we are summing over repeated indices.

$$\{p_i, L_j\} = \{p_i, \epsilon_{jkl} r_k p_l\} = \epsilon_{jkl} (\{p_i, r_k p_l\})$$

As the position  $r_k$  and momentum  $p_l$  are conjugate variables, we have

$$\{p_i, r_k\} = -\delta_{ik}$$

and

$$\{p_i, p_l\} = 0$$

Note that

$$\{p_i, r_k p_l\} = r_k \{p_i, p_l\} + p_l \{p_i, r_k\} = -p_l \delta_{ik}$$

Finally,

$$\{p_i, L_j\} = \epsilon_{jkl} (-p_l \delta_{ik}) = -\epsilon_{jkl} p_l \delta_{ik} = \epsilon_{ijk} p_k$$



Define  $\mathbf{W} = \left\{ \frac{\mathbf{p}}{m}, \mathbf{L} \right\} - \frac{GMm}{r}\mathbf{r}$  which is generalized LRL vector. One can prove that

$$\{W_i, W_j\} = \epsilon_{ijk} \left( \frac{-2E}{m} L_k \right)$$

and show that it forms lie algebra.

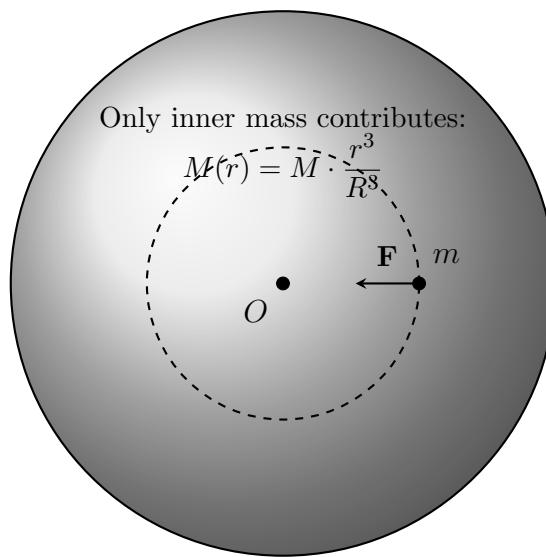
### 6.3 Shell Theorem

#### Theorem. 6.7: Shell Theorem

- A spherically symmetric shell of mass exerts **no net gravitational force** on a particle located **inside** the shell.
- A spherically symmetric shell of mass exerts a **gravitational force on an external point mass** as if all the mass were concentrated at the shell's center.

For a point mass located at a distance  $r < R$  inside a uniform solid sphere (e.g., a planet), the Shell Theorem implies:

- Only the mass enclosed within radius  $r$  contributes to the gravitational force.
- Mass at radii greater than  $r$  exerts no net force by First Shell Theorem by considering it as infinitesimal shells.



Assuming uniform density, the mass enclosed within radius  $r$  is

$$M(r) = M \cdot \frac{r^3}{R^3}$$



Then, the gravitational force on a test mass  $m$  located at distance  $r$  from the center is

$$F = \frac{GM(r)m}{r^2} = \frac{GMm}{R^3} \cdot r$$

by second shell theorem.

**Example. (Simple Gravity Train)** Mass outside the spherical surface of radius  $r$  produces **zero net force**. The mass enclosed is

$$M(r) = M \cdot \frac{r^3}{R^3}$$

The gravitational force magnitude is therefore

$$F = -G \frac{M(r)m}{r^2} = -\frac{GMm}{R^3} r = m\ddot{r}$$

which is in Simple Harmonic Form (SHM) form  $a = -\omega^2 x$ . Therefore,  $\omega = \sqrt{\frac{GM}{R^3}}$  and time of travel  $t = \frac{\pi}{\sqrt{\frac{GM}{R^3}}} = \pi \sqrt{\frac{R}{g}}$ .

**Remark.** The gravity train (also known as the Gauss gun, gravity elevator, or vacuum train) is a fascinating thought experiment in classical mechanics. First proposed by Robert Hooke in the 17th century and later analyzed by Isaac Newton, it explores what would happen if a tunnel were drilled through the Earth and an object were dropped through it.

**Example. (Generalised Gravity Train)** Consider a **straight tunnel** connecting two arbitrary points  $A$  and  $B$  on Earth's surface, not necessarily antipodal. The tunnel forms a **chord** of Earth's circular cross-section.

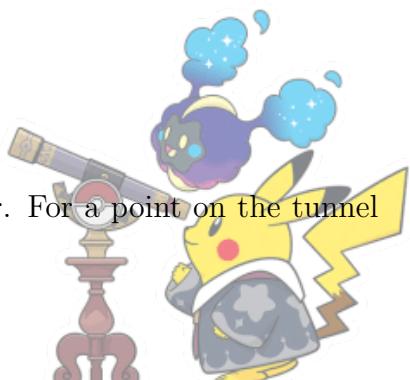
Let  $s$  be the coordinate along the tunnel, with  $s = 0$  at the midpoint  $M$ . The endpoints are at  $s = \pm L/2$ . For a point **inside a homogeneous Earth** at position  $\mathbf{r}$  from the center, the gravitational force is radial:

$$\mathbf{F} = -\frac{GM(r)}{r^2} \hat{\mathbf{r}}, \quad \text{with } M(r) = \frac{r^3}{R^3} M$$

Hence,

$$\mathbf{F} = -\frac{GM}{R^3} r \hat{\mathbf{r}}$$

Let the tunnel be at a constant perpendicular distance  $y$  from Earth's center. For a point on the tunnel



at coordinate  $s$ , its radial distance from Earth's center is:

$$r(s) = \sqrt{s^2 + y^2}$$

The **component of gravitational force along the tunnel** (in the  $\hat{\mathbf{s}}$  direction) is

$$F_s = \mathbf{F} \cdot \hat{\mathbf{s}} = \left( -\frac{GM}{R^3} r \hat{\mathbf{r}} \right) \cdot \hat{\mathbf{s}}$$

Since  $\hat{\mathbf{r}} = \frac{s\hat{\mathbf{s}} + y\hat{\mathbf{y}}}{r}$ , we have:

$$F_s = -\frac{GM}{R^3} r \left( \frac{s}{r} \right) = -\frac{GM}{R^3} s = m\ddot{s}$$

which is also in SHM form.

## 6.4 Gravitational Potential Energy

### Theorem. 6.8: Gravitational Potential Energy

The gravitational potential energy ( $U$ ) of an object in a gravitational field is defined as the work done in moving the object from infinity to a point at distance  $r$  from the source of the gravitational field. For two point masses, the gravitational potential energy is given by

$$U = -G \frac{m_1 m_2}{r}$$

where

- $U$  is the gravitational potential energy,
- $G$  is the gravitational constant,
- $m_1$  and  $m_2$  are the two masses,
- $r$  is the distance between the centers of the two masses.

**Proof.** The gravitational force is:

$$F = G \frac{Mm}{r^2}$$



Then,

$$\begin{aligned}
 U &= \int_{\infty}^r F dr \\
 &= \int_{\infty}^r G \frac{Mm}{r^2} dr \\
 &= GMm \int_{\infty}^r \frac{1}{r^2} dr \\
 &= GMm \left[ -\frac{1}{r} \right]_{\infty}^r \\
 &= -G \frac{Mm}{r}
 \end{aligned}$$

#### 6.4.1 Vis-viva Equation

##### Theorem. 6.9: Vis-viva Equation

The Vis-viva equation, also known as the orbital energy conservation equation, relates the velocity of an orbiting body to its position and the geometry of its orbit. It is expressed as

$$v^2 = \mu \left( \frac{2}{r} - \frac{1}{a} \right)$$

where

- $v$  is the orbital speed of the orbiting body.
- $r$  is the distance between the two bodies.
- $a$  is the semi-major axis of the orbit.
- $\mu$  is the standard gravitational parameter ( $GM$ ).

**Proof.** Let  $v_1$  be the furthest distance from the object to the planet and  $v_2$  be the closest distance from the object to the planet.

By conservation of angular momentum,  $m_1 r_1 v_1 = m_2 r_2 v_2 \implies v_2 = \frac{r_1}{r_2} v_1$ . By conservation of mechanical energy  $\frac{1}{2}mv^2 - \frac{GMm}{r} = \text{constant}$ .

$$v_1^2 = \frac{2GMr_2}{r_1(r_1 + r_2)}$$



By  $m_1 r_1 v_1 = m r_2 v_2 \implies v_2 = \frac{r_1}{r_2} v_1$  and  $r_1 + r_2 = 2a$ ,

$$v^2 = \mu \left( \frac{2}{r} - \frac{1}{a} \right)$$

where  $G$  is the gravitational constant.

#### 6.4.2 First Cosmic Speed, Second Cosmic Speed and Third Cosmic Speed

##### Definition. 6.4: First Cosmic Speed

The first cosmic speed is the minimum velocity required for an object to orbit the Earth in a circular trajectory at the lowest possible altitude (negligible height above the surface).

For a satellite to remain in a circular orbit, the gravitational force must provide the required centripetal force.

$$\begin{aligned} F_{grav} &= F_{centripetal} \\ \frac{GMm}{R^2} &= m \frac{v_1^2}{R} \end{aligned}$$

where

- $M$ : Mass of Earth ( $\approx 5.97 \times 10^{24}$  kg)
- $m$ : Mass of the satellite
- $R$ : Radius of Earth ( $\approx 6.37 \times 10^6$  m)
- $G$ : Gravitational Constant

$$v_1 = \sqrt{\frac{GM}{R}}$$

##### Definition. 6.5: Second Cosmic Speed

The second cosmic speed is the minimum velocity required for an object to break free from Earth's gravitational attraction, effectively escaping to infinity. It is commonly known as **escape velocity**.

The total energy at Earth's surface must equal the total energy at infinity (where both potential and kinetic energy are zero for a barely escape scenario).



$$K_{surface} + U_{surface} = K_\infty + U_\infty$$

$$\frac{1}{2}mv_2^2 - \frac{GMm}{R} = 0$$

$$v_2 = \sqrt{\frac{2GM}{R}}$$

Note that  $v_2 = \sqrt{2}v_1$ .

#### Definition. 6.6: Third Cosmic Speed

The third Cosmic Speed is the launch velocity required from Earth to escape the Solar System.

This requires overcoming Earth's gravity and then having enough residual velocity to escape the Sun's gravity.

- **Hyperbolic Excess Velocity ( $v_\infty$ ):** Utilizing Earth's orbital speed ( $v_E \approx 29.8 \text{ km/s}$ ), the spacecraft needs an excess velocity relative to Earth of

$$v_\infty = v_{esc,\odot} - v_E$$

- **Launch Calculation:** Applying energy conservation from Earth's surface:

$$\frac{1}{2}mv_3^2 - \frac{GMm}{R} = \frac{1}{2}mv_\infty^2$$

$$v_3 = \sqrt{v_\infty^2 + \frac{2GM}{R}}$$

Since  $\frac{2GM}{R} = v_2^2$ :

$$v_3 = \sqrt{v_\infty^2 + v_2^2}$$

#### 6.4.3 Hohmann Transfer Orbit

The **Hohmann transfer orbit** is the most fuel-efficient two-impulse maneuver for transferring a spacecraft between two coplanar circular orbits around the same central body.



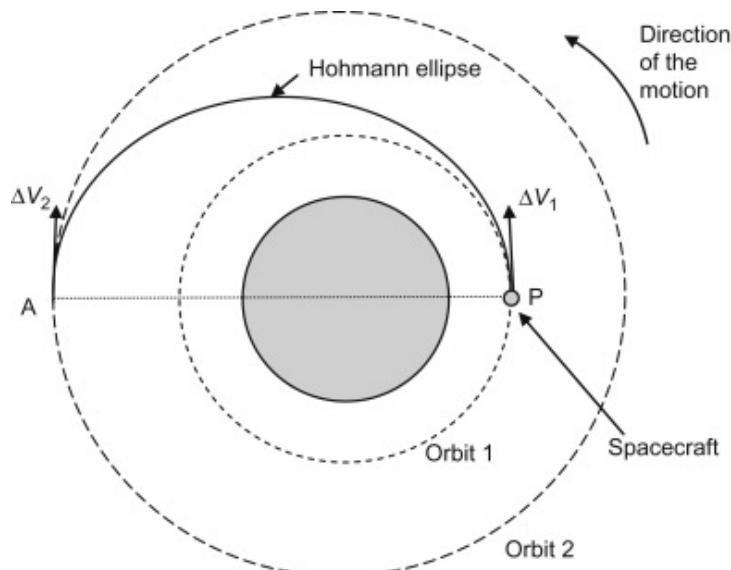


Figure 6: Source: <https://www.sciencedirect.com/topics/engineering/hohmann-transfer>

For a spacecraft in a circular orbit of radius  $r$  around a central body of mass  $M$ , the orbital speed is

$$v_c = \sqrt{\frac{GM}{r}}$$

For any Keplerian orbit, the speed at distance  $r$  is given by the vis-viva equation:

$$v^2 = GM \left( \frac{2}{r} - \frac{1}{a} \right)$$

where  $a$  is the semi-major axis of the orbit. Consider two circular orbits of radii  $r_1$  (initial) and  $r_2$  (final), with  $r_2 > r_1$ . The Hohmann transfer orbit is an ellipse with:

1. Periapsis at  $r_1$ ,
2. Apoapsis at  $r_2$ .

The semi-major axis of the transfer ellipse is

$$a = \frac{r_1 + r_2}{2}$$

The spacecraft initially moves with circular speed

$$v_1 = \sqrt{\frac{GM}{r_1}}$$



At periapsis of the transfer ellipse, the required speed is obtained from the vis-viva equation:

$$v_p = \sqrt{GM \left( \frac{2}{r_1} - \frac{1}{a} \right)}$$

The first velocity increment is therefore

$$\Delta v_1 = v_p - v_1$$

This burn places the spacecraft onto the elliptical transfer orbit.

At apoapsis of the transfer ellipse ( $r = r_2$ ), the spacecraft speed is

$$v_a = \sqrt{GM \left( \frac{2}{r_2} - \frac{1}{a} \right)}$$

The circular speed in the final orbit is

$$v_2 = \sqrt{\frac{GM}{r_2}}$$

The second velocity increment is

$$\Delta v_2 = v_2 - v_a$$

After this burn, the spacecraft is inserted into the final circular orbit.

The total fuel cost of the maneuver is measured by the total velocity increment:

$$\Delta v_{\text{total}} = \Delta v_1 + \Delta v_2$$

The Hohmann transfer minimizes  $\Delta v_{\text{total}}$  among all two-impulse transfers between coplanar circular orbits.

The spacecraft travels half of the elliptical orbit during the transfer. The orbital period of an ellipse is

$$T = 2\pi \sqrt{\frac{a^3}{GM}}$$

Hence, the Hohmann transfer time is

$$t_{\text{transfer}} = \pi \sqrt{\frac{a^3}{GM}}$$

This relatively long transfer time is the main trade-off for fuel efficiency.



#### 6.4.4 Blackhole

The event horizon is a boundary in spacetime beyond which events cannot affect an observer. It is commonly associated with black holes, and marks the point at which the escape velocity exceeds the speed of light. This means that not even light can escape from inside the event horizon, which is why it appears black. For a non-rotating, spherically symmetric black hole, the event horizon is given by the Schwarzschild radius  $r_s$ , which is the radius at which the escape velocity equals the speed of light. The Schwarzschild radius is defined as

$$r_s = \frac{2GM}{c^2}$$

where

- $G$  is the gravitational constant,
- $M$  is the mass of the black hole,
- $c$  is the speed of light.

Beyond the event horizon, all paths of particles and light curves back into the black hole, and no signal can escape to the outside universe.

#### 6.4.5 Virial Theorem

For a gravitationally bound system in dynamical equilibrium, such as an isolated galaxy, the virial theorem states that the time-averaged total kinetic energy  $\langle T \rangle$  and the time-averaged total gravitational potential energy  $\langle U \rangle$  satisfy

$$2\langle T \rangle + \langle U \rangle = 0$$

For a system of particles of total mass  $M$  in virial equilibrium, the virial theorem gives

$$2\langle T \rangle + \langle U \rangle = 0$$

The total kinetic energy may be written in terms of the velocity dispersion as

$$\langle T \rangle \sim \frac{1}{2}M\langle v^2 \rangle \sim \frac{3}{2}M\sigma^2$$



where  $\sigma$  is the one-dimensional velocity dispersion.

For a self-gravitating system of characteristic size  $R$ , the gravitational potential energy is of order

$$\langle U \rangle \sim -\frac{GM^2}{R}$$

up to a dimensionless factor of order unity that depends on the mass distribution.

Substituting into the virial theorem gives

$$3M\sigma^2 \sim \frac{GM^2}{R}$$

Solving for  $M$  yields

$$M \sim \frac{R\sigma^2}{G}$$

which is the virial mass estimate for a galaxy.

#### 6.4.6 The Tully-Fisher Relation

In 1977, R. Brent Tully and J. Richard Fisher discovered an empirical relationship between the luminosity  $L$  of spiral galaxies and their maximum rotational velocity  $v_{\max}$ :

$$L \propto v_{\max}^\alpha$$

where observations give  $\alpha \approx 3.5 - 4$  in the infrared and  $\alpha \approx 2.5 - 3$  in the blue band.



## 6.5 Lagrange Point

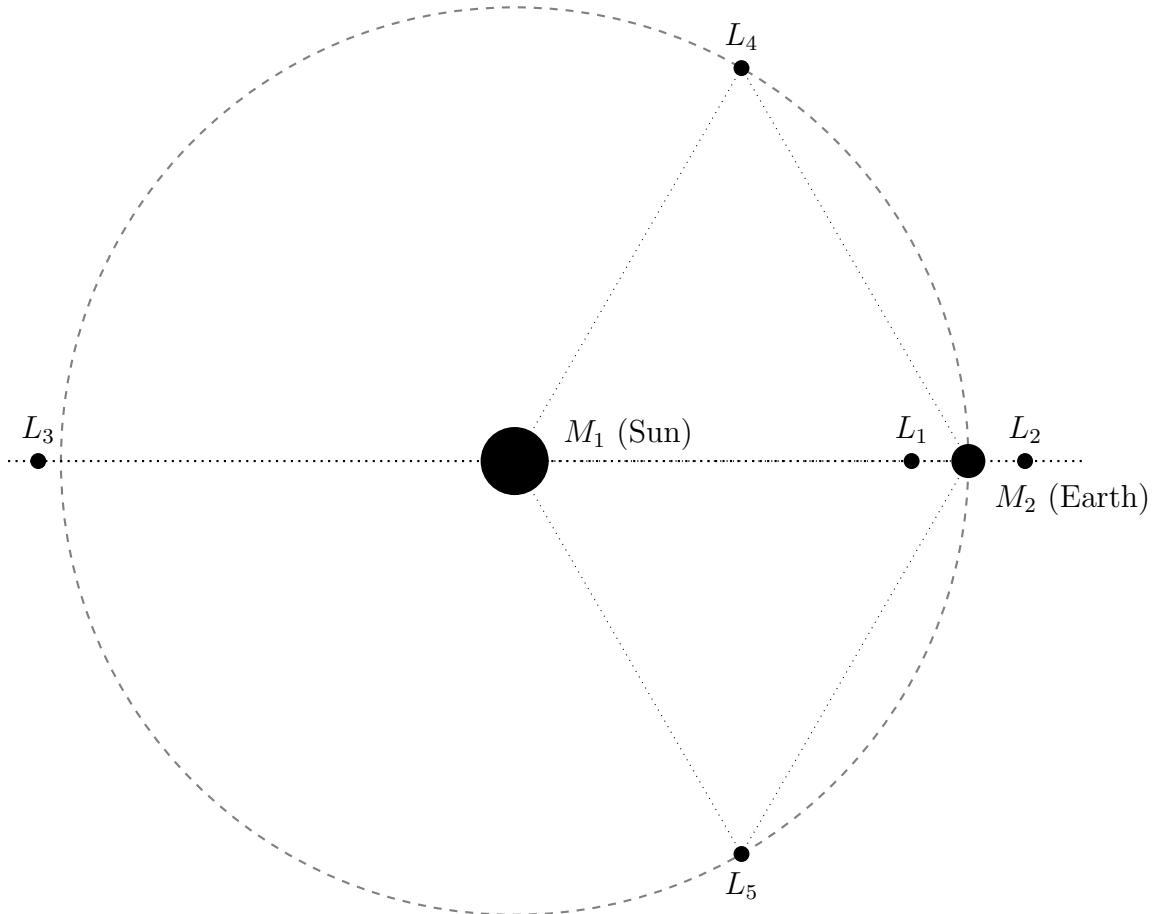


Figure 7: The Five Lagrangian Points ( $L_1$  to  $L_5$ ) in the Sun-Earth System's Rotating Frame

Two primary masses  $M_1$  and  $M_2$  orbit their common center of mass in circular motion. Let  $M_1 > M_2$ . Define the reduced mass

$$\mu = \frac{M_2}{M_1 + M_2}$$

Normalize  $M_1 + M_2 = 1$ , and the gravitational constant  $G = 1$ . In the rotating frame centerline at the center of mass,  $M_1$  is at  $(-\mu, 0)$  and  $M_2$  is at  $(1 - \mu, 0)$ . The angular velocity  $\omega$  of the rotating frame satisfies (from Kepler's 3rd law for  $M_1$  and  $M_2$ )

$$\omega^2 = \frac{G(M_1 + M_2)}{a^3} = 1$$

since  $a = 1$ ,  $G = 1$ ,  $M_1 + M_2 = 1$ . So  $\omega = 1$ . In the rotating frame, the equations of motion for a test particle at  $\mathbf{r} = (x, y)$  are

$$\ddot{x} - 2\omega\dot{y} = \frac{\partial U}{\partial x}, \quad \ddot{y} + 2\omega\dot{x} = \frac{\partial U}{\partial y}$$



where

$$U(x, y) = \underbrace{\frac{1}{2}\omega^2(x^2 + y^2)}_{\text{Centrifugal Potential}} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$$

with

$$r_1 = \sqrt{(x + \mu)^2 + y^2}, \quad r_2 = \sqrt{(x - 1 + \mu)^2 + y^2}$$

Lagrange points are stationary points of  $U$  in the rotating frame (equilibrium points where  $\dot{x} = \dot{y} = 0$ ):

$$\frac{\partial U}{\partial x} = 0, \quad \frac{\partial U}{\partial y} = 0$$

$$\frac{\partial U}{\partial x} = x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3} = 0$$

$$\frac{\partial U}{\partial y} = y \left[ 1 - \frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3} \right] = 0$$

For  $y = 0$ , we have  $r_1 = |x + \mu|$ ,  $r_2 = |x - 1 + \mu|$ . The  $x$ -equation becomes

$$x - \frac{1-\mu}{(x+\mu)^2} \operatorname{sgn}(x+\mu) - \frac{\mu}{(x-1+\mu)^2} \operatorname{sgn}(x-1+\mu) = 0$$

- $x > 1 - \mu$  (beyond  $M_2$ ):

$$x - \frac{1-\mu}{(x+\mu)^2} - \frac{\mu}{(x-1+\mu)^2} = 0$$

- $\mu < x < 1 - \mu$  (between  $M_1$  and  $M_2$ ):

$$x - \frac{1-\mu}{(x+\mu)^2} + \frac{\mu}{(1-\mu-x)^2} = 0$$

- $x < -\mu$  (opposite side of  $M_1$  from  $M_2$ ):

$$x + \frac{1-\mu}{(x+\mu)^2} + \frac{\mu}{(x-1+\mu)^2} = 0$$

They have series solution:

$$L_1 : \quad x \approx 1 - \mu - \left(\frac{\mu}{3}\right)^{1/3} + \frac{1}{3} \left(\frac{\mu}{3}\right)^{2/3} + \dots$$

$$L_2 : \quad x \approx 1 - \mu + \left(\frac{\mu}{3}\right)^{1/3} + \frac{1}{3} \left(\frac{\mu}{3}\right)^{2/3} + \dots$$



$$L_3 : \quad x \approx -1 - \frac{5}{12}\mu + \dots$$

From  $\frac{\partial U}{\partial y} = 0$ ,

$$y \left[ 1 - \frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3} \right] = 0$$

If  $y \neq 0$ , then

$$\frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3} = 1 \quad (*)$$

From  $\frac{\partial U}{\partial x} = 0$ ,

$$x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3} = 0$$

Multiply (\*) by  $x$  and subtract from above:

$$-\frac{(1-\mu)\mu}{r_1^3} + \frac{\mu(1-\mu)}{r_2^3} = 0 \quad \Rightarrow \quad \frac{1-\mu}{r_1^3} = \frac{1-\mu}{r_2^3}$$

Hence, for  $\mu \neq 1$ , we get  $r_1 = r_2$ . From  $r_1 = r_2$ , (\*) gives

$$\frac{1-\mu}{r_1^3} + \frac{\mu}{r_1^3} = \frac{1}{r_1^3} = 1 \quad \Rightarrow \quad r_1 = 1$$

So  $r_1 = r_2 = 1$ . Now  $(x+\mu)^2 + y^2 = 1$  and  $(x-1+\mu)^2 + y^2 = 1$ . Solving,

$$x = \frac{1}{2} - \mu, y = \pm \frac{\sqrt{3}}{2}$$

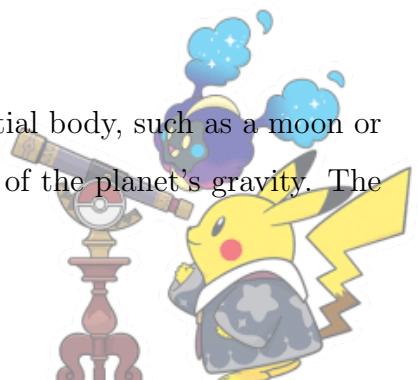
Therefore,

$$L_4 : \quad \left( \frac{1}{2} - \mu, \frac{\sqrt{3}}{2} \right)$$

$$L_5 : \quad \left( \frac{1}{2} - \mu, -\frac{\sqrt{3}}{2} \right)$$

## 6.6 Roche Limit

The Roche limit is the minimum distance from a planet within which a celestial body, such as a moon or asteroid, will experience tidal forces strong enough to break it apart because of the planet's gravity. The



Roche limit is given by the formula

$$d_{\text{Roche}} = 2.44 R \left( \frac{\rho_{\text{planet}}}{\rho_{\text{moon}}} \right)^{1/3}$$

**Derivation of simplified model:** Tidal force is the difference in gravitational force felt across an object due to another massive body.

Consider a small object of radius  $R$  located at a distance  $d$  from a much more massive body of mass  $M$ .

$$a_{\text{near}} = \frac{GM}{(d - R)^2}, \quad a_{\text{far}} = \frac{GM}{(d + R)^2}$$

The tidal acceleration is the difference between the accelerations on the near and far sides

$$a_{\text{tidal}} = a_{\text{near}} - a_{\text{far}} = \frac{GM}{(d - R)^2} - \frac{GM}{(d + R)^2}$$

Assume  $R \ll d$ . Then

$$\frac{1}{(d \pm R)^2} \approx \frac{1}{d^2} \mp \frac{2R}{d^3}$$

Therefore,

$$a_{\text{tidal}} \approx GM \left( \left( \frac{1}{d^2} + \frac{2R}{d^3} \right) - \left( \frac{1}{d^2} - \frac{2R}{d^3} \right) \right) = GM \cdot \frac{4R}{d^3}$$

$$a_{\text{tidal (per side)}} \approx \frac{2GMR}{d^3}$$

Note that

$$a_{\text{tidal}} \approx \frac{2GM_p R_s}{d^3}, \quad a_{\text{self}} = \frac{Gm_s}{R_s^2}$$

By  $m_s = \frac{4}{3}\pi R_s^3 \rho_s$  and  $M_p = \frac{4}{3}\pi R_p^3 \rho_p$ ,

$$\frac{2GM_p R_s}{d^3} = \frac{Gm_s}{R_s^2}$$

which can be simplified to

$$d = R_p \cdot \left( 2 \cdot \frac{\rho_p}{\rho_s} \right)^{1/3}$$

For the derivation of non-simplified model,

one can refer to <https://dxwl.bnu.edu.cn/CN/abstract/abstract7639.shtml>.



## 7 Relativity

Newtonian mechanics, formulated in the 17th century, works exceptionally well for everyday speeds ( $v \ll c$ ). However, as experiments in the late 19th century became more precise (e.g., Michelson-Morley experiment), inconsistencies emerged that required a new theoretical framework.

### 7.1 Postulates of Special Relativity

- **Principle of Relativity:** The laws of physics are the same in all inertial reference frames.
- **Invariance of Light Speed:** The speed of light in vacuum,  $c$ , is the same in all inertial frames, regardless of the motion of the source or observer.

### 7.2 Galileo Transformation

In classical mechanics, the Galilean transformation relates the coordinates of an event as observed in two inertial frames of reference moving at a constant relative velocity. Suppose we have two inertial frames:

- Frame  $S$ : stationary frame with coordinates  $(x, y, z, t)$
- Frame  $S'$ : moving with constant velocity  $v$  along the  $x$ -axis relative to  $S$ , with coordinates  $(x', y', z', t')$

The Galilean transformation equations are

$$x' = x - vt$$

$$y' = y$$

$$z' = z$$

$$t' = t$$

### 7.3 Lorentz Transformation

#### Definition. 7.1: Lorentz Factor

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \beta^2}}$$

where  $v$  is the speed of the object and  $\beta := \frac{v}{c}$ .



In special relativity, the Lorentz transformation relates the coordinates of an event as observed in two inertial frames moving at a constant velocity relative to each other. Unlike the Galilean transformation, it accounts for the constancy of the speed of light.

### Theorem. 7.1: Lorentz Transformation

For two inertial frames  $S$  and  $S'$  with  $S'$  moving at velocity  $v$  along the  $x$ -axis relative to  $S$ , the Lorentz transformations are

$$\begin{aligned} t' &= \gamma \left( t - \frac{vx}{c^2} \right) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned}$$

### Theorem. 7.2: Inverse Lorentz Transformation

For two inertial frames  $S$  and  $S'$  with  $S'$  moving at velocity  $v$  along the  $x$ -axis relative to  $S$ , the inverse Lorentz transformations are

$$\begin{aligned} t &= \gamma \left( t' + \frac{vx'}{c^2} \right) \\ x &= \gamma(x' + vt') \\ y &= y' \\ z &= z' \end{aligned}$$

### Theorem. 7.3: Lorentz Transformation for Velocity

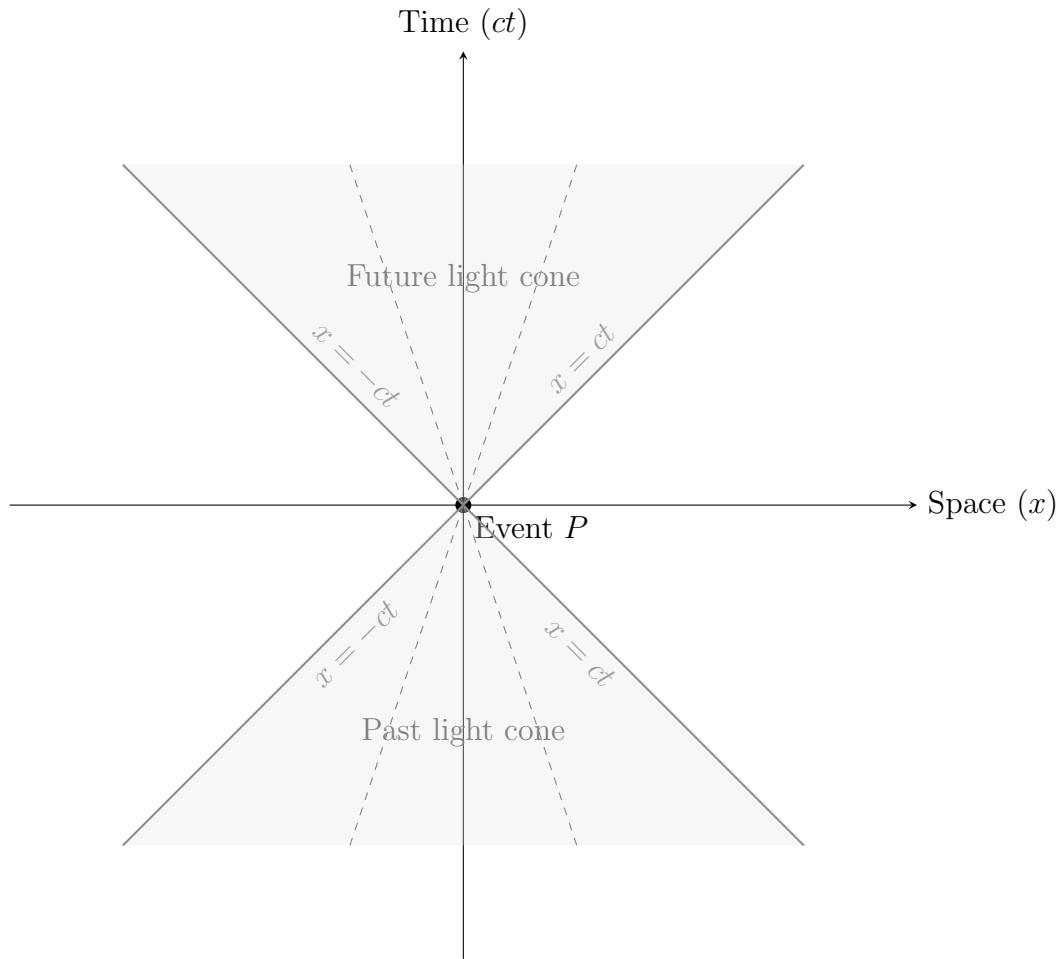
For two inertial frames  $S$  and  $S'$  with  $S'$  moving at velocity  $v$  along the  $x$ -axis relative to  $S$ , the velocity transformations are

$$\begin{aligned} v'_x &= \frac{v_x - v}{1 - \frac{vv_x}{c^2}} \\ v'_y &= \frac{v_y}{\gamma \left( 1 - \frac{vv_x}{c^2} \right)} \\ v'_z &= \frac{v_z}{\gamma \left( 1 - \frac{vv_x}{c^2} \right)} \end{aligned}$$



## 7.4 Spacetime Diagram

A spacetime diagram is a graphical representation of events in spacetime.



## 7.5 Relativistic Kinematics and Mechanics

### 7.5.1 Time Dilation

Time dilation is one of the most fascinating consequences of special relativity. It states that time measured in a moving frame (proper time  $\tau$ ) appears to run slower compared to time measured in a stationary frame (coordinate time  $t$ ). The relationship between proper time ( $\tau$ ) and coordinate time ( $t$ ) is given by

$$\Delta t = \gamma \Delta \tau$$

**Example.** A spaceship travels at  $v = 0.8c$  relative to Earth. If 5 years pass on the spaceship (proper time), how much time passes on Earth?



**Solution.**

$$\gamma = \frac{1}{\sqrt{1 - (0.8)^2}} \approx 1.667$$

$$\Delta t = \gamma \Delta \tau = 1.667 \times 5 \approx 8.33 \text{ years}$$

### 7.5.2 Length Contraction

Length is defined as the distance between simultaneous measurements of an object's endpoints. For a rod of proper length  $L_0$  at rest in S', moving with velocity  $v$  in S. In S, measure endpoints simultaneously ( $\Delta t = 0$ ):

$$\Delta x = L \quad (\text{measurable length in S})$$

$$\Delta x' = L_0 \quad (\text{proper length in S'})$$

$$\Delta x' = \gamma(\Delta x - v\Delta t) = \gamma L$$

$$L = \frac{L_0}{\gamma}$$

Hence, moving objects appear shorter in their direction of motion.

### 7.5.3 Relativistic Momentum

Consider a particle with rest mass  $m_0$ .

$$\mathbf{p} = \gamma m_0 \mathbf{v}$$

### 7.5.4 Relativistic Energy

The correct relativistic expression for kinetic energy is

$$K = (\gamma - 1)mc^2$$

**Proof.** Starting from the work-energy theorem and relativistic momentum:

$$\begin{aligned} K &= \int \mathbf{F} \cdot d\mathbf{r} = \int \frac{d}{dt}(\gamma m_0 \mathbf{v}) \cdot \mathbf{v} dt \\ &= \int \left( \gamma m_0 \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} + m_0 \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} \frac{d\gamma}{dv^2} \frac{dv^2}{dt} \right) dt \\ &= m_0 c^2 \int d\gamma = \gamma m_0 c^2 + C \end{aligned}$$



When  $\gamma = 0$ ,  $K = 0$ . Therefore,  $K = (\gamma - 1)m_0c^2$ .

The total energy of a particle is

$$E = \gamma mc^2 = K + m_0c^2$$

This consists of two parts:

- **Rest energy:**  $E_0 = m_0c^2$
- **Kinetic energy:**  $K = (\gamma - 1)m_0c^2$

### 7.5.5 Relation between Relativistic Momentum and Energy

From our definitions:

$$E = \gamma m_0 c^2$$

$$\mathbf{p} = \gamma m_0 \mathbf{v}$$

Compute  $E^2 - (pc)^2$ :

$$\begin{aligned} E^2 - (pc)^2 &= (\gamma m_0 c^2)^2 - (\gamma m_0 v c)^2 \\ &= \gamma^2 m_0^2 c^4 \left(1 - \frac{v^2}{c^2}\right) \\ &= m_0^2 c^4 \frac{1 - \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}} \quad (\text{since } \gamma^2 = \frac{1}{1 - \frac{v^2}{c^2}}) \\ &= m_0^2 c^4 \end{aligned}$$

Hence we obtain the fundamental relation:

$$E^2 = (pc)^2 + (m_0 c^2)^2$$

## 7.6 Spacetime and Four Vector

### 7.6.1 Spacetime in Special Relativity

In special relativity, space and time are unified into a four-dimensional continuum called **spacetime**. Events are described by four coordinates:

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$$



where

- $x^0 = ct$  is the temporal coordinate (with  $c$  being the speed of light)
- $x^1, x^2, x^3$  are spatial coordinates

**Example.** A frame  $S'$ , is moving with velocity  $v$  along the  $y$ -axis of another frame  $S$ . The Lorentz transformation in terms of  $x^0 \equiv ct, x^1 \equiv x, x^2 \equiv y, x^3 \equiv z, x^0 \equiv ct'$  is

$$x'^0 = \gamma(x^0 - \beta x^1)$$

$$x'^1 = \gamma(x^1 - \beta x^0)$$

$$x'^2 = x^2$$

$$x'^3 = x^3$$

where  $\beta = \frac{v}{c}$  and  $\gamma = \frac{1}{\sqrt{1 - \beta^2}}$ .

### 7.6.2 Four-Vector

A **four-vector**  $A^\mu$  is a mathematical object that transforms under Lorentz transformations in the same way as spacetime coordinates:

$$A^\mu = (A^0, A^1, A^2, A^3)$$

The transformation rule for a Lorentz transformation  $\Lambda_\nu^\mu$  is

$$A'^\mu = \Lambda_\nu^\mu A^\nu$$

where

$$\Lambda_\nu^\mu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



**Example. (Continued)** A second-rank tensor has only one non-zero component in the  $S$ -frame,  $T^{00} \equiv \rho$ . The components in the  $S'$ -frame are obtained using the Lorentz transformation:

$$T'^{00} = \gamma^2 T^{00}, \quad T'^{01} = \gamma\beta T^{00}, \quad T'^{10} = \gamma\beta T^{00}, \quad T'^{11} = T^{00}$$

$$T'^{22} = T^{22}, \quad T'^{33} = T^{33}.$$

Hence, the components of  $T^{\mu\nu}$  in  $S'$ -frame are

$$T'^{00} = \gamma^2 \rho, \quad T'^{01} = T'^{10} = \gamma\beta\rho, \quad T'^{11} = \rho.$$

### 7.6.3 Metric Tensor

The metric tensor  $g_{\mu\nu}$  defines the geometry of spacetime. In special relativity with Cartesian coordinates, it takes a simple form known as the **Minkowski metric**:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

In component form:

$$g_{00} = 1, \quad g_{11} = g_{22} = g_{33} = -1, \quad g_{\mu\nu} = 0 \text{ for } \mu \neq \nu$$

The metric tensor allows conversion between contravariant (upper index) and covariant (lower index) vectors.

#### Lowering an index

$$A_\mu = g_{\mu\nu} A^\nu$$



Explicitly:

$$A_0 = g_{00} A^0 = A^0$$

$$A_1 = g_{11} A^1 = -A^1$$

$$A_2 = g_{22} A^2 = -A^2$$

$$A_3 = g_{33} A^3 = -A^3$$

## Raising an index

$$A^\mu = g^{\mu\nu} A_\nu$$

where  $g^{\mu\nu}$  is the inverse metric, which for Minkowski spacetime is identical to  $g_{\mu\nu}$ :

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The metric tensor and its inverse satisfy

$$g^{\mu\lambda} g_{\lambda\nu} = \delta_\nu^\mu = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu \end{cases}$$

where  $\delta_\nu^\mu$  is the Kronecker delta.

### 7.6.4 Invariant Interval

The **invariant interval** between two infinitesimally close events in spacetime is defined as:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

Expanding in components for Minkowski spacetime:

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$



For finite separation between events, the interval is:

$$\Delta s^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

The most important property of  $ds^2$  is that it is **invariant** under Lorentz transformations. If we transform to a new coordinate system  $x'^\mu$ :

$$ds'^2 = g_{\mu\nu} dx'^\mu dx'^\nu = g_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma dx^\rho dx^\sigma = g_{\rho\sigma} dx^\rho dx^\sigma = ds^2$$

This invariance follows from the defining property of Lorentz transformations:

$$g_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma = g_{\rho\sigma}$$

### 7.6.5 Important Four-Vectors in Physics

#### Position Four-Vector

$$x^\mu = (ct, \mathbf{x}) = (ct, x, y, z)$$

**Velocity Four-Vector** The four-velocity is defined as the derivative of the position four-vector with respect to proper time  $\tau$ :

$$u^\mu = \frac{dx^\mu}{d\tau} = (\gamma c, \gamma \mathbf{v})$$

where  $\tau$  is proper time.

The components are

$$\begin{aligned} u^0 &= \frac{dx^0}{d\tau} = \frac{cdt}{d\tau} = \gamma c \\ u^i &= \frac{dx^i}{d\tau} = \frac{dx^i}{dt} \frac{dt}{d\tau} = \gamma v^i \end{aligned}$$

Magnitude of four-velocity is given by

$$u^\mu u_\mu = \gamma^2 (c^2 - v^2) = \frac{c^2 - v^2}{1 - v^2/c^2} = c^2$$

#### Momentum Four-Vector

$$p^\mu = mu^\mu = \left( \frac{E}{c}, \mathbf{p} \right)$$



where  $E = \gamma mc^2$  is relativistic energy and  $\mathbf{p} = \gamma m\mathbf{v}$  is relativistic momentum.

Explicitly,

$$p^\mu = \left( \frac{E}{c}, p_x, p_y, p_z \right)$$

Furthermore,

$$p^\mu p_\mu = \frac{E^2}{c^2} - p^2 = m^2 c^2$$

**Example.** Consider two spatial and one temporal dimensions. A particle emits a photon, making an angle  $\theta'$  with the  $x'$ -axis in a frame ( $S'$ -frame). Hence, the trajectory of the photon can be written as  $x' = ct' \cos \theta'$  and  $y' = ct' \sin \theta'$ .

- (a) If the  $S'$ -frame is moving with velocity  $v$  with respect to another frame ( $S$ -frame), what is the trajectory of the photon in the  $S$ -frame?
- (b) What is the energy-momentum four-vector of the photon in the  $S'$ -frame, if the frequency measured in this frame is  $f$ ? (Of course, the  $z$ -component,  $p'_z$ , is zero.)
- (c) Express the energy of the photon in the  $S$ -frame in terms of  $\theta'$ .
- (d) At which direction does the photon have maximum energy in the  $S$ -frame?

To find the trajectory in the  $S$ -frame, we apply the inverse Lorentz transformations. Assuming the  $S'$ -frame moves with velocity  $v$  along the  $x$ -axis of the  $S$ -frame, the coordinates transform as:

$$x = \gamma(x' + vt')$$

$$y = y'$$

$$t = \gamma \left( t' + \frac{vx'}{c^2} \right)$$

where  $\gamma = \frac{1}{\sqrt{1 - \beta^2}}$  and  $\beta = \frac{v}{c}$ . Substituting the  $S'$ -frame trajectory  $x' = ct' \cos \theta'$  and  $y' = ct' \sin \theta'$ :

$$(1) \quad x = \gamma(ct' \cos \theta' + vt') = \gamma t'(c \cos \theta' + v)$$

$$(2) \quad y = ct' \sin \theta'$$

$$(3) \quad t = \gamma \left( t' + \frac{v(ct' \cos \theta')}{c^2} \right) = \gamma t'(1 + \beta \cos \theta')$$



To find  $x(t)$  and  $y(t)$ , we solve for  $t'$  in terms of  $t$  from (3):  $t' = \frac{t}{\gamma(1 + \beta \cos \theta')}$ . Substituting this into (1) and (2):

$$x(t) = \frac{c \cos \theta' + v}{1 + \beta \cos \theta'} t, \quad y(t) = \frac{c \sin \theta'}{\gamma(1 + \beta \cos \theta')} t$$

For a photon,  $E' = hf$  and  $p' = \frac{E'}{c} = \frac{hf}{c}$ . The components of the momentum vector in the  $S'$ -frame are  $p'_x = p' \cos \theta'$  and  $p'_y = p' \sin \theta'$ . The four-vector  $P'^\mu$  is

$$P'^\mu = \left( \frac{E'}{c}, p'_x, p'_y, p'_z \right) = \left( \frac{hf}{c}, \frac{hf}{c} \cos \theta', \frac{hf}{c} \sin \theta', 0 \right)$$

The energy  $E$  in the  $S$ -frame is obtained from the transformation  $E = \gamma(E' + vp'_x)$ :

$$E = \gamma \left( hf + v \frac{hf}{c} \cos \theta' \right) = \gamma hf(1 + \beta \cos \theta')$$

The energy  $E(\theta') = \gamma hf(1 + \beta \cos \theta')$  is maximized when  $\cos \theta'$  is at its maximum value. This occurs when  $\cos \theta' = 1$ , which corresponds to  $\theta' = 0$ . Therefore, the photon has maximum energy when emitted in the forward direction (the same direction as the velocity  $v$ ).

### 7.6.6 Introduction to General Relativity

**Motivation** Classical Newtonian gravity describes gravitation as a force acting instantaneously at a distance. However, this framework is incompatible with special relativity, which asserts that

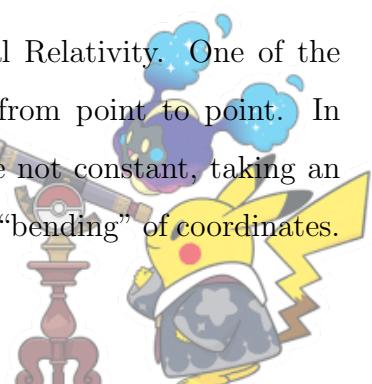
- The speed of light is finite and invariant.
- Space and time form a unified structure: spacetime.

General Relativity, formulated by Albert Einstein in 1915, resolves this incompatibility by replacing the concept of gravitational force with the **geometry of spacetime**.

**Operators in General Relativity** There are different operators in General Relativity. One of the examples is Christoffel symbols, which describe how coordinate axes change from point to point. In curved spacetime (or even flat space with curved coordinates), basis vectors are not constant, taking an ordinary derivative of a vector does not give a tensor and we must correct for the “bending” of coordinates.

#### Relevant Questions

- Romanian Master of Physics 2014  
Problem 3 (**Black Holes Physics**),  
Romanian Master of Physics 2012  
Problem 3 (**Fundaments of General Relativity**)



**Definition. 7.2: Christoffel Symbols**

The Christoffel symbols are defined as

$$\Gamma_{\mu\nu}^{\lambda} = \frac{1}{2}g^{\lambda\rho}(\partial_{\mu}g_{\nu\rho} + \partial_{\nu}g_{\mu\rho} - \partial_{\rho}g_{\mu\nu})$$

where  $\partial_{\mu} \equiv \frac{\partial}{\partial x^{\mu}}$  is called the 4-gradient.

Another example is d'Alembertian operator.

**Definition. 7.3: d'Alembertian Operator**

$$\square = \partial_{\mu}\partial^{\mu} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

A scalar field  $\phi(x^{\mu})$  satisfies the relativistic wave equation

$$\square\phi = 0$$

This equation can describe

- Electromagnetic waves
- Gravitational waves
- Massless scalar fields

For a scalar field of mass  $m$ , the equation becomes

$$\left(\square + \frac{m^2c^2}{\hbar^2}\right)\phi = 0$$

This is the relativistic generalization of the Schrödinger equation.

In general relativity, spacetime is curved, and partial derivatives must be replaced by **covariant derivatives**. The d'Alembertian becomes the **Laplace–Beltrami operator**:

$$\square\phi = \nabla^{\mu}\nabla_{\mu}\phi$$



In coordinates, this can be written as

$$\square\phi = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \phi)$$

where  $g = \det(g_{\mu\nu})$ .

**Introduction to Spacetime and Differentiable Manifold** In classical physics, space and time are treated as separate entities:

- Space: a three-dimensional Euclidean arena
- Time: a universal parameter flowing identically for all observers

However, experiments involving high velocities and electromagnetic waves reveal that:

- The speed of light is invariant
- Measurements of time and length depend on the observer

These facts force a radical conclusion:

Space and time are not independent; they form a unified structure called spacetime.

In the curved spacetime, we model spacetime as a **differentiable manifold**  $M$ :

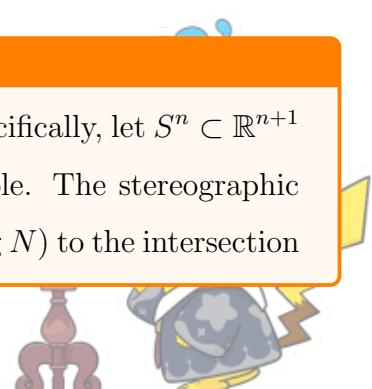
- A manifold is a set of points that locally resembles  $\mathbb{R}^n$ , allowing us to define coordinates in a neighborhood of each point.
- For spacetime,  $n = 4$  and each point represents an event. An event is something that occurs at a specific place and time.
- Smooth functions and tensors can be defined on the manifold, enabling calculus on curved spacetime.

**Example.** A sphere in  $\mathbb{R}^n$  is a manifold.

**Proof.** We introduce some mathematical concepts first:

#### Definition. 7.4: Stereographic Projection

Stereographic projection is a mapping that projects a sphere onto a plane. Specifically, let  $S^n \subset \mathbb{R}^{n+1}$  be the  $n$ -dimensional unit sphere, and let  $N = (0, \dots, 0, 1)$  be the north pole. The stereographic projection  $\pi : S^n \setminus \{N\} \rightarrow \mathbb{R}^n$  is defined by mapping a point  $P$  on  $S^n$  (excluding  $N$ ) to the intersection



of the line through  $N$  and  $P$  with the hyperplane  $x_{n+1} = 0$ :

$$\pi(P) = \frac{1}{1 - x_{n+1}}(x_1, x_2, \dots, x_n),$$

where  $P = (x_1, x_2, \dots, x_{n+1}) \in S^n$ .

### Definition. 7.5: Stereographic Projection

A **topological space** is a pair  $(X, \mathcal{T})$ , where  $X$  is a set and  $\mathcal{T}$  is a collection of subsets of  $X$  (called open sets) satisfying:

1.  $\emptyset, X \in \mathcal{T}$ ,
2. Any union of elements of  $\mathcal{T}$  is in  $\mathcal{T}$ ,
3. Any finite intersection of elements of  $\mathcal{T}$  is in  $\mathcal{T}$ .

The collection  $\mathcal{T}$  is called a topology on  $X$ .

### Definition. 7.6: Homomorphism

Let  $X$  and  $Y$  be topological spaces. A function

$$f : X \rightarrow Y$$

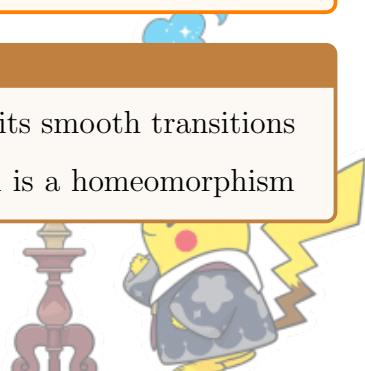
is called a **homeomorphism** if

1.  $f$  is a **bijection** (one-to-one and onto),
2.  $f$  is **continuous**,
3. the inverse function  $f^{-1} : Y \rightarrow X$  is also **continuous**.

If such a map exists, we say  $X$  and  $Y$  are **homeomorphic**, meaning they are **topologically equivalent**.

### Theorem. 7.4: Criteria of a Smooth Manifold

A smooth manifold is a space that is locally **homeomorphic** to  $\mathbb{R}^n$  and admits smooth transitions between overlapping charts. Demonstrating that the stereographic projection is a homeomorphism



ensures that each point on  $S^n$  possesses a neighborhood that topologically resembles  $\mathbb{R}^n$ , which is important in establishing that  $S^n$  is a smooth manifold.

Let

$$N = (0, \dots, 0, 1), \quad S = (0, \dots, 0, -1)$$

be the north and south poles of  $S^n$ . Define the stereographic projection from the north pole:

$$\pi_N : S^n \setminus \{N\} \rightarrow \mathbb{R}^n$$

by

$$\pi_N(x_1, \dots, x_{n+1}) = \left( \frac{x_1}{1-x_{n+1}}, \dots, \frac{x_n}{1-x_{n+1}} \right)$$

As the inverse map is given by

$$\pi_N^{-1}(y_1, \dots, y_n) = \left( \frac{2y_1}{\|y\|^2 + 1}, \dots, \frac{2y_n}{\|y\|^2 + 1}, \frac{\|y\|^2 - 1}{\|y\|^2 + 1} \right)$$

and both  $\pi_N$  and  $\pi_N^{-1}$  are continuous,  $\pi_N$  is a homeomorphism. Similarly, define

$$\pi_S : S^n \setminus \{S\} \rightarrow \mathbb{R}^n$$

by

$$\pi_S(x_1, \dots, x_{n+1}) = \left( \frac{x_1}{1+x_{n+1}}, \dots, \frac{x_n}{1+x_{n+1}} \right)$$

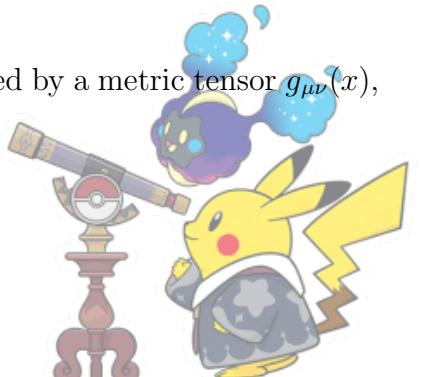
The two charts

$$(S^n \setminus \{N\}, \pi_N), \quad (S^n \setminus \{S\}, \pi_S)$$

cover  $S^n$ . Each component is a rational function of  $y_1, \dots, y_n$  with denominator  $\|y\|^2$ , which is nonzero on the domain. Hence the transition map  $\pi_S \circ \pi_N^{-1}$  is smooth. The stereographic charts provide a smooth atlas covering  $S^n$ , and the transition functions are smooth. Therefore,  $S^n$  satisfies all the requirements of a smooth manifold.

**Metric Tensor** On this 4-dimensional manifold  $M$ , the geometry is described by a metric tensor  $g_{\mu\nu}(x)$ , a smooth, symmetric, non-degenerate tensor field:

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$$



where the Einstein's notation uses Lorentzian signature  $(-, +, +, +)$ . Locally, at any point  $p \in M$ , there exists a coordinate system in which the metric reduces to the Minkowski form  $\eta_{\mu\nu}$  and its first derivatives vanish, reflecting the **equivalence principle**

**Equivalence Principle** It states that

Locally, the effects of gravity are indistinguishable from the effects of acceleration.

**Geodesics** In General Relativity, free-falling particles follow the **straightest possible paths** in curved spacetime, called **geodesics**:

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma_{\nu\rho}^\mu \frac{dx^\nu}{d\lambda} \frac{dx^\rho}{d\lambda} = 0$$

In a flat space (Euclidean geometry), the shortest path between two points is a straight line. In a curved space (like the surface of the Earth), the shortest path is a geodesic.

For timelike geodesics, the proper time  $\tau$  is the affine parameter  $\lambda$ . For null (lightlike) geodesics, there is no proper time, but an affine parameter still exists to describe the geodesic.

**Riemann Curvature Tensor** Curvature is measured by the Riemann tensor:

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma_{\nu\sigma}^\rho - \partial_\nu \Gamma_{\mu\sigma}^\rho + \Gamma_{\mu\lambda}^\rho \Gamma_{\nu\sigma}^\lambda - \Gamma_{\nu\lambda}^\rho \Gamma_{\mu\sigma}^\lambda$$

Contracting indices yields

$$R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$$

The Ricci scalar is

$$R = g^{\mu\nu} R_{\mu\nu}$$

**Einstein's Field Equation** The field equations can be written as

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

where

- $T_{\mu\nu}$  is the stress-energy tensor which describes the density and flux of energy and momentum.

In the weak-field, slow-motion limit:

$$g_{00} \approx - \left( 1 + \frac{2\Phi}{c^2} \right)$$



Einstein's equations reduce to Poisson's equation:

$$\nabla^2 \Phi = 4\pi G\rho$$

Hence, Newtonian gravity emerges as an approximation of General Relativity.

**Solution to Einstein's Field Equation** **Schwarzschild solution** describes spacetime outside a spherical mass:

$$ds^2 = - \left(1 - \frac{2GM}{r}\right) c^2 dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

**Friedmann–Lemaître–Robertson–Walker Metric** models an expanding universe:

$$ds^2 = -c^2 dt^2 + a(t)^2 \left[ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right]$$

where  $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$ .

Define  $\chi$  such that

$$d\chi = \frac{dr}{\sqrt{1 - kr^2}}$$

Integrating gives the relations

$$r = \begin{cases} \frac{1}{\sqrt{k}} \sin(\sqrt{k}\chi) & , k > 1 \\ \chi & , k = 0 \\ \frac{1}{\sqrt{-k}} \sinh(\sqrt{-k}\chi) & , k < -1 \end{cases} = \begin{cases} \sin(\chi) & , k = 1 \\ \chi & , k = 0 \\ \sinh(\chi) & , k = -1 \end{cases}$$

as  $k = -1, 0, 1$  only. The metric then becomes

$$ds^2 = -c^2 dt^2 + a^2(t) [d\chi^2 + S_k^2(\chi) d\Omega^2]$$



By natural unit  $c = 1$ ,

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{a^2}{1-kr^2} & 0 & 0 \\ 0 & 0 & a^2r^2 & 0 \\ 0 & 0 & 0 & a^2r^2 \sin^2 \theta \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1-kr^2}{a^2} & 0 & 0 \\ 0 & 0 & \frac{1}{a^2r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{a^2r^2 \sin^2 \theta} \end{pmatrix}$$

The non-zero Christoffel symbols are

$$\Gamma_{ij}^0 = a\dot{a}\tilde{g}_{ij} = \frac{\dot{a}}{a}g_{ij}$$

$$\Gamma_{0j}^i = \frac{\dot{a}}{a}\delta_j^i$$

$$\Gamma_{rr}^r = \frac{kr}{1-kr^2}, \quad \Gamma_{\theta\theta}^r = -r(1-kr^2), \quad \Gamma_{\phi\phi}^r = -r(1-kr^2)\sin^2 \theta$$

$$\Gamma_{r\theta}^\theta = \Gamma_{r\phi}^\phi = \frac{1}{r}$$

$$\Gamma_{\phi\phi}^\theta = -\sin \theta \cos \theta, \quad \Gamma_{\theta\phi}^\phi = \cot \theta$$

$$R_{00} = -3\frac{\ddot{a}}{a}$$

$$R_{ij} = \left[ \frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{2k}{a^2} \right] g_{ij}$$

$$R = g^{\mu\nu}R_{\mu\nu} = 6\left[ \frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} \right]$$

For a perfect fluid in comoving coordinates,

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu}$$



with  $U^\mu = (1, 0, 0, 0)$ , so:

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & pg_{11} & 0 & 0 \\ 0 & 0 & pg_{22} & 0 \\ 0 & 0 & 0 & pg_{33} \end{pmatrix}$$

By  $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi GT_{\mu\nu}$ ,

$$\begin{aligned} -3\frac{\ddot{a}}{a} - \frac{1}{2}(-1) \cdot 6 \left[ \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right] - \Lambda &= 8\pi G\rho \\ -3\frac{\ddot{a}}{a} + 3 \left[ \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right] - \Lambda &= 8\pi G\rho \\ 3 \left( \frac{\dot{a}}{a} \right)^2 + \frac{3k}{a^2} - \Lambda &= 8\pi G\rho \\ \left( \frac{\dot{a}}{a} \right)^2 &= \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \end{aligned}$$

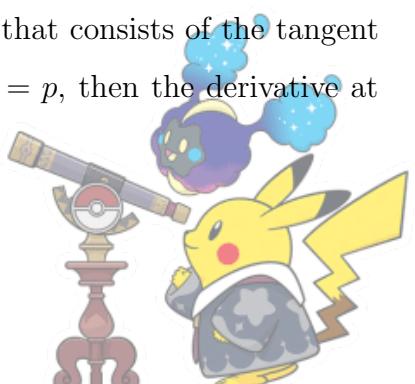
This is the **First Friedmann Equation**.

$$\begin{aligned} \left[ \frac{\ddot{a}}{a} + 2 \left( \frac{\dot{a}}{a} \right)^2 + \frac{2k}{a^2} \right] g_{ii} - \frac{1}{2}g_{ii} \cdot 6 \left[ \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right] + \Lambda g_{ii} &= 8\pi G p g_{ii} \\ \frac{\ddot{a}}{a} + 2 \left( \frac{\dot{a}}{a} \right)^2 + \frac{2k}{a^2} - 3 \left[ \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right] + \Lambda &= 8\pi G p \\ -2\frac{\ddot{a}}{a} - \left( \frac{\dot{a}}{a} \right)^2 - \frac{k}{a^2} + \Lambda &= 8\pi G p \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \end{aligned}$$

This is the **Second Friedmann Equation**.

**Tangent Space** Let  $M$  be a smooth manifold and  $p$  a point on  $M$ . Let  $M$  be a smooth manifold and  $p$  a point on  $M$ . The **tangent space** at  $p$ , denoted by  $T_p M$ , is a vector space that consists of the tangent vectors at the point  $p$ . If  $\gamma : (-\epsilon, \epsilon) \rightarrow M$  is a smooth curve such that  $\gamma(0) = p$ , then the derivative at  $t = 0$ , denoted  $\gamma'(0)$ , is a tangent vector. The tangent space is given by

$$T_p M = \{ \gamma'(t) \mid \gamma(0) = p, \gamma \text{ smooth} \}$$



**Tensor Space** Tensor space refers to the set of all tensors that can be constructed from the tangent space of a manifold.

**Covariant Derivative** The **covariant derivative**  $\nabla_\mu$  generalizes the partial derivative  $\partial_\mu$  to curved manifolds. The partial derivative fails to transform as a tensor under coordinate changes. The covariant derivative introduces correction terms, the **Christoffel symbols**  $\Gamma_{\mu\nu}^\lambda$ , to ensure tensorial behavior. For a contravariant vector field  $V^\nu$ ,

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma_{\mu\lambda}^\nu V^\lambda.$$

**Lie Derivative** The Lie derivative  $\mathcal{L}_X$  measures how a tensor field changes along the flow of a vector field  $X$ . For a vector field  $Y$  along the flow of  $X$ ,

$$(\mathcal{L}_X Y)^\mu = X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu$$

In coordinate-free notation,

$$\mathcal{L}_X Y = [X, Y]$$

where  $[X, Y]$  is the Lie bracket.

**Killing Vector Field** A vector field  $K^\mu$  is a **Killing vector field** if its flow generates an **isometry**, which is a symmetry that preserves the spacetime metric. By vanishing the Lie derivative of the metric along  $K^\mu$ , the condition is given by

$$\mathcal{L}_K g_{\mu\nu} = 0.$$

Using the Lie derivative formula for the metric, this becomes the **Killing equation**:

$$\boxed{\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0}$$

### Theorem. 7.5: Conservation Law Along Geodesics

For a geodesic with tangent vector  $u^\mu = \frac{dx^\mu}{d\lambda}$ , if  $K^\mu$  is a Killing vector, then the quantity  $K_\mu u^\mu$  is constant along the geodesic.

### Proof.

$$\frac{d}{d\lambda}(K_\mu u^\mu) = u^\nu \nabla_\nu(K_\mu u^\mu) = u^\nu u^\mu \nabla_\nu K_\mu + K_\mu u^\nu \nabla_\nu u^\mu$$



The second term vanishes by the geodesic equation  $u^\nu \nabla_\nu u^\mu = 0$ . The first term is symmetric in  $\nu$  and  $\mu$ , so we can write

$$u^\nu u^\mu \nabla_\nu K_\mu = \frac{1}{2} u^\nu u^\mu (\nabla_\nu K_\mu + \nabla_\mu K_\nu) = 0$$

by the Killing equation. Hence,

$$\boxed{\frac{d}{d\lambda}(K_\mu u^\mu) = 0}$$

**Local Escape Speed** Consider a static, spherically symmetric spacetime with metric

$$ds^2 = -g_{tt}(r) dt^2 + g_{rr}(r) dr^2 + r^2 d\Omega^2$$

where  $g_{tt}(r) > 0$ . As the metric is time-independent, it admits a timelike Killing vector

$$K^\mu = (\partial_t)^\mu = (1, 0, 0, 0)$$

Let a particle move along a geodesic with tangent vector

$$u^\mu = \frac{dx^\mu}{d\lambda}$$

If  $K^\mu$  is a Killing vector, then the quantity

$$K_\mu u^\mu$$

is conserved along the geodesic. We define the conserved energy per unit mass

$$E \equiv -K_\mu u^\mu$$

Since  $K^\mu = (1, 0, 0, 0)$ , we have

$$K_\mu = g_{\mu\nu} K^\nu = (g_{tt}, 0, 0, 0)$$

and therefore

$$E = -g_{tt} u^t$$

For a massive particle, the four-velocity satisfies

$$u^\mu u_\mu = -1$$



For purely radial motion ( $u^\theta = u^\phi = 0$ ), this gives

$$-g_{tt}(u^t)^2 + g_{rr}(u^r)^2 = -1$$

Substituting  $u^t = E/g_{tt}$ , we obtain

$$-g_{tt} \left( \frac{E}{g_{tt}} \right)^2 + g_{rr}(u^r)^2 = -1$$

which simplifies to

$$g_{rr}(u^r)^2 = \frac{E^2}{g_{tt}} - 1$$

A static observer at radius  $r$  has four-velocity

$$U^\mu = \frac{1}{\sqrt{g_{tt}}} (1, 0, 0, 0)$$

The locally measured speed  $v$  of the particle is defined by

$$v^2 = \frac{(\text{proper spatial distance})^2}{(\text{proper time})^2} = \frac{g_{rr}(u^r)^2}{g_{tt}(u^t)^2}$$

Using the previous results, this becomes

$$v^2 = \frac{\frac{E^2}{g_{tt}} - 1}{\frac{E^2}{g_{tt}}} = 1 - \frac{g_{tt}}{E^2}$$

For a particle that just escapes to infinity in an asymptotically flat spacetime,

$$g_{tt}(\infty) = 1 \quad u^r(\infty) = 0$$

The normalization condition then implies

$$E = 1$$

Substituting into the expression for  $v^2$ , we obtain the local escape speed:

$$v_{\text{esc}}^2 = 1 - g_{tt}(r)$$



For the Schwarzschild metric,

$$g_{tt}(r) = 1 - \frac{2GM}{r}$$

Hence,

$$v_{\text{esc}}^2 = \frac{2GM}{r}$$

This is the locally measured escape speed by a static observer.



## 8 Thermodynamics

### 8.1 Pressure and Hydrostatic Equilibrium

**Introduction** Pressure is a scalar quantity defined as the force applied per unit area. Mathematically,

$$P = \frac{F}{A}$$

**Hydrostatic Equilibrium: Plane-Parallel Case** Consider a small cylindrical fluid element of cross-sectional area  $A$  and height  $dz$ .

- **Gravitational force (downward):** The mass of the element is  $m = \rho A dz$ . Hence,

$$F_g = mg = \rho g A dz$$

- **Pressure force (upward):** The pressure at the bottom is  $P(z)$ , while at the top it is  $P(z + dz) = P + dP$ . The net pressure force is therefore

$$F_{P,\text{net}} = P(z)A - P(z + dz)A = -AdP$$

For hydrostatic equilibrium, the total force must vanish:

$$\sum F = 0 \implies F_{P,\text{net}} - F_g = 0.$$

Hence,

$$-AdP - \rho g A dz = 0 \implies \boxed{\frac{dP}{dz} = -\rho g}$$

**Hydrostatic Equilibrium: Spherical Symmetry (Stars)** Now consider a thin spherical shell inside a star at radius  $r$  with thickness  $dr$ .

- **Gravitational force (inward):** The mass of the shell is

$$dm = \rho(r) dV = \rho(r) 4\pi r^2 dr$$



The gravitational acceleration at radius  $r$  is

$$g(r) = \frac{GM(r)}{r^2}$$

where  $M(r)$  is the mass enclosed within radius  $r$ . Hence, the inward gravitational force is

$$F_g = dm g(r) = 4\pi G \rho(r) M(r) dr$$

- **Pressure force (outward):** The surface area of the shell is  $A = 4\pi r^2$ . The net pressure force is the difference between the outward pressure at  $r$  and that at  $r + dr$ :

$$F_{P,\text{net}} = P(r)A - P(r + dr)A = -\frac{dP}{dr} \cdot 4\pi r^2 dr$$

Hydrostatic equilibrium requires

$$\sum F = 0 \implies F_{P,\text{net}} - F_g = 0$$

Therefore,

$$-\frac{dP}{dr} 4\pi r^2 dr - 4\pi G \rho(r) M(r) dr = 0$$

which yields the stellar hydrostatic equilibrium equation:

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

**Remark** In addition to hydrostatic equilibrium, stars also satisfy **thermal equilibrium**, meaning that the energy generated in the core equals the energy radiated from the surface:

$$L_{\text{core}} = L_{\text{surface}}$$

where  $L$  denotes the luminosity. Stellar equilibrium refers to the combined balance of gravitational, pressure, and energy-transport processes that allow a star to maintain a stable structure over long timescales.



## 8.2 Ideal Gas Law

The Ideal Gas Law is a fundamental equation in thermodynamics that describes the behavior of an ideal gas. It establishes a relationship between the pressure  $P$ , volume  $V$ , temperature  $T$ , and the number of moles of gas  $n$ . The equation is expressed as

$$PV = nRT$$

where

- $P$  is the pressure of the gas,
- $V$  is the volume of the gas,
- $n$  is the number of moles of gas,
- $R$  is the universal gas constant ( $R = 8.314 \text{ J/mol}\cdot\text{K}$ ),
- $T$  is the temperature of the gas in Kelvin.

For a gas undergoing a change in volume at constant pressure, the work done by the gas is given by

$$W = P\Delta V$$

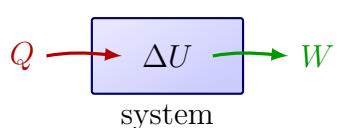
## 8.3 First Law of Thermodynamics

It states that energy cannot be created or destroyed, only transformed. It is expressed mathematically as

$$dU = \delta Q - \delta W$$

where

- $dU$  is the change in internal energy of the system,
- $\delta Q$  is the heat added to the system,
- $\delta W$  is the work done by the system.



Process	Condition	Description
Adiabatic	$\delta Q = 0$	No heat exchange with surroundings; temperature changes due to work done
Isothermal	$T = \text{constant}$	Temperature remains constant; heat absorbed equals work done
Isobaric	$P = \text{constant}$	Pressure remains constant; volume changes with temperature
Isochoric (Isovolumetric)	$V = \text{constant}$	No work done since volume is constant; heat changes internal energy
Polytropic	$PV^n = \text{constant}$	Generalized process covering isothermal, adiabatic, and isobaric cases
Cyclic	Final state = Initial state	Net change in internal energy over one cycle is zero
Reversible	Quasi-static with no losses	Idealized process that delivers maximum possible work
Irreversible	Finite gradients and friction	Real processes; entropy production is positive

Table 3: Thermodynamic processes, their conditions, and descriptions

## 8.4 Second Law of Thermodynamics

### Definition. 8.1: Entropy

Entropy is a measure of the disorder or randomness of a system. For a reversible process, the change in entropy  $dS$  is given by the heat transferred  $dQ$  divided by the temperature  $T$ :

$$dS = \frac{dQ_{\text{rev}}}{T}$$



**Theorem. 8.1: Sackur–Tetrode Equation**

$$S = Nk_B \left( \ln \left( \frac{V}{N} \left( \frac{4\pi m E}{3h^2} \right)^{3/2} \right) + \frac{5}{2} \right)$$

where

- $S$  is the entropy,
- $N$  is the number of particles,
- $V$  is the volume,
- $m$  is the mass of a gas particle,
- $E$  is the internal energy,

**Theorem. 8.2: Second Law of Thermodynamics**

The second law of thermodynamics states that the entropy of an isolated system tends to increase over time.

$$dS \geq 0$$

**Theorem. 8.3: Gibbs Free Energy**

The Gibbs free energy  $G$  is related to entropy through the following equation:

$$G = H - TS = U + PV - TS$$

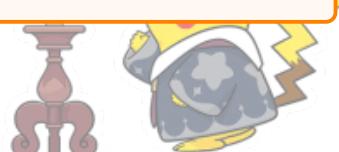
where  $H$  is the enthalpy,  $T$  is the temperature (kept constant),  $P$  is the pressure (kept constant), and  $S$  is the entropy.

## 8.5 Heat Capacity

**Definition. 8.2: Heat Capacity**

The heat capacity at constant volume,  $C_V$ , is defined as the amount of heat required to raise the temperature of a system by one degree Celsius (or one Kelvin) while maintaining constant volume. Mathematically, it is expressed as

$$C_V = \left( \frac{\partial Q}{\partial T} \right)_V$$



where  $Q$  is the heat added to the system, and  $T$  is the temperature. The subscript  $V$  indicates that the volume is held constant.

Since the system is not allowed to do any work (because the volume is fixed), the heat added to the system only changes the internal energy:

$$dQ = dU$$

where  $dU$  is the change in internal energy.

For an ideal gas, the heat capacity at constant volume is related to the molar heat capacity  $C_V$  by

$$C_V = \left( \frac{\partial U}{\partial T} \right)_V$$

For an ideal gas, the internal energy  $U$  is a function of temperature alone, and the specific heat capacity can be derived from the equation of state.

### Definition. 8.3: Heat Capacity

The heat capacity at constant pressure,  $C_P$ , is defined as the amount of heat required to raise the temperature of a system by one degree Celsius (or one Kelvin) while maintaining constant pressure.

Mathematically, it is expressed as

$$C_P = \left( \frac{\partial Q}{\partial T} \right)_P$$

where  $Q$  is the heat added to the system, and  $T$  is the temperature. The subscript  $P$  indicates that the pressure is held constant.

In contrast to constant volume, when heat is added at constant pressure, the system may do work as it expands. Therefore, the total heat added is the sum of the change in internal energy and the work done by the system

$$dQ = dU + PdV$$

For an ideal gas, the heat capacity at constant pressure is related to the molar heat capacity  $C_P$  by

$$C_P = \left( \frac{\partial H}{\partial T} \right)_P$$



**Theorem. 8.4: Mayer's Relation**

$$c_P - c_V = nR$$

where  $n$  is the number of moles of the gas and  $R$  is the universal gas constant.

**Proof.** By the first law of thermodynamics,

$$dU = \delta Q - pdV = nC_VdT$$

where  $C_V$  is the heat capacity at constant volume, and  $n$  is the number of moles. For the heat added at constant pressure, we use:

$$\delta Q = nC_PdT$$

where  $C_P$  is the heat capacity at constant pressure.

By the ideal gas equation of state,

$$pV = nRT$$

Differentiating this equation with respect to temperature at constant pressure:

$$pdV + Vdp = nRdT$$

At constant pressure,  $dp = 0$ , so we have:

$$pdV = nRdT$$

Substitute  $pdV = nRdT$  into the equation:

$$nC_VdT = nC_PdT - nRdT$$

$$C_V = C_P - R \implies \boxed{C_P - C_V = nR}$$

**Definition. 8.4: Heat Capacity Ratio**

$$\gamma = \frac{C_p}{C_V}$$

In an adiabatic process, there is no heat exchange with the surroundings ( $dQ = 0$ ). The first law of



thermodynamics gives:

$$dU = -PdV$$

For an ideal gas, the relationship between pressure and volume during an adiabatic process is governed by the equation:

$$PV^\gamma = \text{constant}$$

## 8.6 Blackhole Thermodynamics

Black hole entropy is a measure of the amount of information that is hidden inside a black hole. The famous Bekenstein-Hawking entropy formula links the entropy of a black hole to the area of its event horizon, rather than its volume. This result was first derived by Jacob Bekenstein and later confirmed by Stephen Hawking:

$$S = \frac{k_B c^3 A}{4G\hbar}$$

**Proof.** The first law of black hole thermodynamics is given by

$$dM = TdS + \Phi dQ + \Omega dJ$$

where

- $M$  is the mass,
- $T$  is the temperature,
- $S$  is the entropy,
- $\Phi$  is the electrostatic potential,
- $Q$  is the charge, and
- $J$  is the angular momentum.

For a non-rotating, uncharged Schwarzschild black hole, the first law reduces to

$$dM = TdS$$



which is similar to the first law of thermodynamics, and implies that the black hole mass is related to its entropy via its temperature. The area of the event horizon is

$$A = 16\pi \left( \frac{GM}{c^2} \right)^2$$

Using the fact that the temperature  $T$  of a Schwarzschild black hole is given by the Hawking temperature,

$$T = \frac{\hbar c^3}{8\pi GM} \quad (\text{The derivation is out of scope of this lecture note.})$$

The first law  $dM = TdS$  implies

$$dS = \frac{dM}{T} = \frac{dM}{\frac{\hbar c^3}{8\pi GM}} = \frac{8\pi GM}{\hbar c^3} dM$$

Integrating with respect to  $M$ ,

$$S = \frac{8\pi GM^2}{\hbar c^3}$$

Using the expression for the area  $A = 16\pi \left( \frac{GM}{c^2} \right)^2$ ,

$$M = \frac{c^2 \sqrt{A}}{4\pi G}$$

Substituting this into the entropy expression, we get

$$S = \frac{k_B c^3 A}{4G\hbar}$$

## 8.7 Kinetic Theory

### 8.7.1 Mean Free Path

Let the number density of the gas (the number of particles per unit volume) be  $n$ , and let the cross-sectional area for a collision between two particles be  $\sigma$ . The cross-sectional area  $\sigma$  depends on the type of collision and the physical properties of the particles involved. If we assume spherical particles with radius  $r$ , the collision cross-section is given by

$$\sigma = \pi(2r)^2 = 4\pi r^2$$



Next, the relative velocity between two particles in the gas is on the order of the mean speed of the particles  $v_{\text{avg}}$ . The total rate of collisions per unit time per particle is

$$\text{Collision rate} = n\sigma v_{\text{avg}}$$

The mean free path  $\lambda$  is defined as the average distance a particle travels before undergoing a collision. The relationship between the mean free path and the collision rate is given by

$$\lambda = \frac{1}{\text{Collision rate}} = \frac{1}{n\sigma v_{\text{avg}}} = \frac{1}{n \cdot 4\pi r^2 \cdot v_{\text{avg}}}$$

### 8.7.2 Boltzmann Distribution and Maxwell–Boltzmann Distribution

The Boltzmann distribution, also known as the Gibbs distribution, is a fundamental probability distribution in statistical mechanics that describes the statistical properties of a system in thermal equilibrium at a fixed temperature. It provides the probability that a system will be in a particular microstate with energy  $E_i$  when it is in contact with a heat bath at temperature  $T$ .

**Formulation and Derivation** Consider a system in thermal equilibrium with a large reservoir at constant temperature  $T$ . The probability  $P_i$  of finding the system in a particular microstate  $i$  with energy  $E_i$  is given by

$$P_i = \frac{1}{Z} e^{-\beta E_i}$$

where

- $\beta = \frac{1}{k_B T}$  is the thermodynamic beta
- $Z = \sum_i e^{-\beta E_i}$  is the partition function

The partition function  $Z$  serves as a normalization constant ensuring that  $\sum_i P_i = 1$ . It contains all thermodynamic information about the system.

**Proof.** Statistical mechanics often uses the principle of maximum entropy. The entropy of a discrete set of probabilities  $\{P_i\}$  is

$$S = -k_B \sum_i P_i \ln P_i$$



as  $S = k_B \ln \Omega$  (where  $\Omega$  is the number of micro-states by Statistical Mechanics). We want to maximize  $S$  subject to the constraints:

$$\sum_i P_i = 1 \quad (\text{normalization})$$

$$\sum_i P_i E_i = \langle E \rangle \quad (\text{average energy fixed})$$

Using Lagrange multipliers  $\alpha$  and  $\beta$ , define

$$\mathcal{L} = -k_B \sum_i P_i \ln P_i - \alpha \left( \sum_i P_i - 1 \right) - \beta \left( \sum_i P_i E_i - \langle E \rangle \right)$$

Setting the derivative with respect to  $P_i$  to zero:

$$\frac{\partial \mathcal{L}}{\partial P_i} = -k_B(\ln P_i + 1) - \alpha - \beta E_i = 0$$

which gives

$$P_i = e^{-1-\alpha/k_B} e^{-\beta E_i/k_B}$$

Defining  $Z = e^{1+\alpha/k_B} = \sum_i e^{-\beta E_i/k_B}$ , we get the familiar form:

$$P_i = \frac{e^{-\beta E_i}}{Z}$$

**Maxwell-Boltzmann Distribution** The Maxwell–Boltzmann distribution describes the statistical distribution of speeds (or velocities) of particles in an ideal gas at thermal equilibrium. It is a special case of the Boltzmann distribution applied to the translational kinetic energy of non-interacting particles.

For an ideal gas of  $N$  identical particles of mass  $m$  at temperature  $T$ , the probability density function for finding a particle with velocity  $\mathbf{v} = (v_x, v_y, v_z)$  is

$$f(\mathbf{v}) = \left( \frac{m}{2\pi k_B T} \right)^{3/2} \exp \left( -\frac{m|\mathbf{v}|^2}{2k_B T} \right)$$

This three-dimensional distribution factorizes as  $f(\mathbf{v}) = f(v_x)f(v_y)f(v_z)$ , where each component distribution is Gaussian

$$f(v_\alpha) = \sqrt{\frac{m}{2\pi k_B T}} \exp \left( -\frac{mv_\alpha^2}{2k_B T} \right), \quad \alpha = x, y, z$$



More commonly used is the distribution of speeds  $v = |\mathbf{v}|$ , obtained by integrating over all directions in velocity space:

$$f(v) = 4\pi v^2 \left( \frac{m}{2\pi k_B T} \right)^{3/2} \exp \left( -\frac{mv^2}{2k_B T} \right)$$

The factor  $4\pi v^2$  arises from the spherical shell volume element in velocity space.

**Proof.** Consider an ideal gas with  $N$  particles in thermal equilibrium at temperature  $T$ . Each particle has a mass  $m$  and a velocity vector  $\mathbf{v} = (v_x, v_y, v_z)$ . The total energy of a particle is purely kinetic:

$$E = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2).$$

According to the Boltzmann distribution, the probability of a particle having energy  $E$  is

$$P(E) \propto e^{-E/k_B T}$$

Note that

$$P(v_x, v_y, v_z) = f(v_x)f(v_y)f(v_z)$$

$$f(v_i) = Ae^{-\frac{1}{2}mv_i^2/k_B T}, \quad i = x, y, z$$

As

$$\int_{-\infty}^{\infty} f(v_x) dv_x = 1, \quad \int_{-\infty}^{\infty} Ae^{-\frac{mv_x^2}{2k_B T}} dv_x = 1$$

and by

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

we have

$$A\sqrt{\frac{2\pi k_B T}{m}} = 1 \implies A = \sqrt{\frac{m}{2\pi k_B T}}.$$

Hence,

$$f(v_i) = \sqrt{\frac{m}{2\pi k_B T}} e^{-\frac{mv_i^2}{2k_B T}}$$

The speed of a particle is

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2}.$$

The probability density function of speeds,  $F(v)$ , can be obtained by transforming to spherical coordinates in velocity space:

$$F(v)dv = 4\pi v^2 f(v_x)f(v_y)f(v_z)dv.$$



Hence,

$$F(v) = 4\pi \left( \frac{m}{2\pi k_B T} \right)^{3/2} v^2 e^{-\frac{mv^2}{2k_B T}}$$

Three important characteristic speeds are derived from this distribution:

1. **Most probable speed (mode):**

$$v_{\text{mp}} = \sqrt{\frac{2k_B T}{m}} = \sqrt{\frac{2RT}{M}}$$

2. **Mean speed:**

$$\langle v \rangle = \sqrt{\frac{8k_B T}{\pi m}} = \sqrt{\frac{8RT}{\pi M}}$$

3. **Root-mean-square speed:**

$$v_{\text{rms}} = \sqrt{\frac{3k_B T}{m}} = \sqrt{\frac{3RT}{M}}$$

where  $R = N_A k_B$  is the gas constant and  $M = N_A m$  is the molar mass.

## 8.8 Boundary Conditions

Solving the stellar structure equations requires proper boundary conditions:

- **At the Center ( $r = 0$ ):**

$$M(0) = 0, \quad L(0) = 0$$

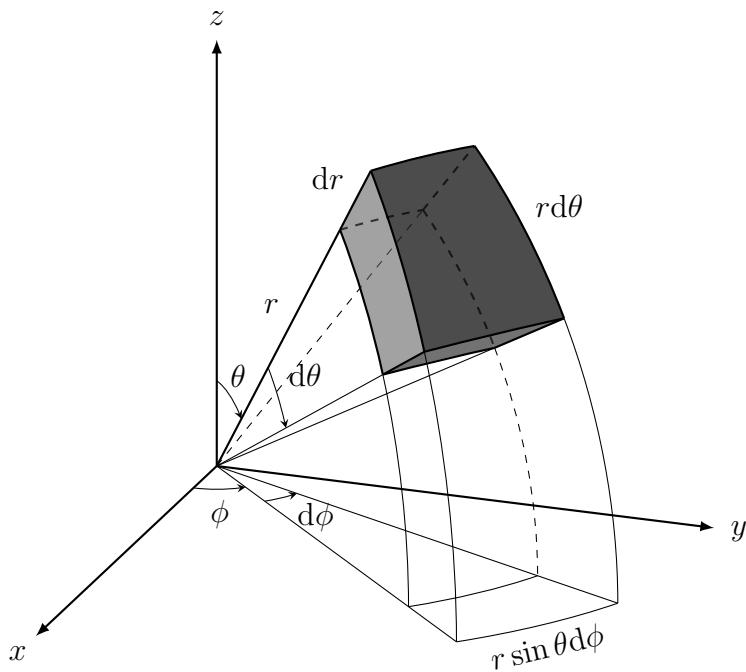
- **At the Surface ( $r = R$ ):**

$$P(R) = P_{\text{surface}} \approx 0, \quad T(R) = T_{\text{eff}}$$

These conditions ensure the solution matches physical reality: zero mass at the center, and finite temperature and pressure at the surface.



## 8.9 Case Study



Suppose a static spherical star consists of  $N$  neutral particles with radius  $R$  with  $0 \leq \theta \leq \pi$ ,  $0 \leq \phi \leq 2\pi$ , satisfying the following equation of states

$$PV = Nk \frac{T_R - T_0}{\ln(T_R/T_0)} \quad (1)$$

where  $P$  and  $V$  are the pressure inside the star and the volume of the star respectively,  $k$  is the Boltzmann constant,  $T_R$  and  $T_0$  are the temperatures at the surface  $r = R$  and the temperature at the center  $r = 0$  respectively. Assume that  $T_R \leq T_0$ .

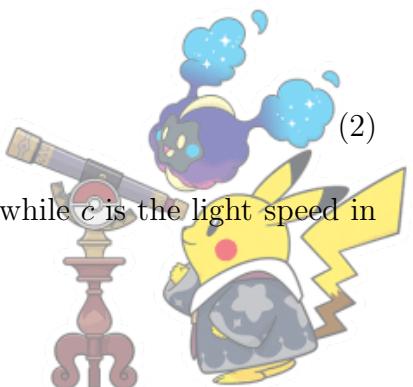
- (a) Simplify the stellar equation of state (1) if  $\Delta T = T_R - T_0 \rightarrow 0$  (this is called ideal star) (Hint: Use the approximation  $\ln(1 + x) \approx x$  for small  $x$ ).

Suppose the star undergoes a quasi-static process, in which it may slightly contract or expand, such that the above stellar equation of state (1) still holds.

The star satisfies the first law of thermodynamics:

$$Q = \Delta M c^2 + W \quad (2)$$

where  $Q$ ,  $M$ , and  $W$  are heat, mass of the star, and work respectively, while  $c$  is the light speed in the vacuum and  $\Delta M = M_{\text{final}} - M_{\text{initial}}$ .



In the following, we assume  $T_0$  to be constant, while  $T_R \equiv T$  varies.

- (b) Find the heat capacity of the star at constant volume  $C_V$  and at constant pressure  $C_P$ , expressed in  $C_P$  and  $C_V$  (Hint: Use the approximation  $(1 + x)^n \approx 1 + nx$  for small  $x$ ).

Assuming that  $C_P$  is constant and the gas undergoes the isobaric process so the star produces the heat and radiates it outside to the space.

- (c) Find the heat produced by the isobaric process if the initial temperature and the final temperature are  $T_i$  and  $T_f$ , respectively.
- (d) For the next parts, assume the star is the Sun.
- (e) If the sunlight is monochromatic with frequency  $5 \times 10^{14}$  Hz, estimate the number of photons radiated by the Sun per second.
- (f) Calculate the heat capacity  $C_P$  of the Sun assuming its surface temperature varies from 5500 K to 6000 K in one second.

### Solution.

- (a) Defining  $\Delta T = T_f - T_0$  and  $\Delta T \approx 0$ , we have

$$\frac{P\Delta V}{Nk} = \frac{\Delta T}{\ln(1 + \Delta T/T_0)}$$

Using  $\ln(1 + \Delta T/T_0) \approx \Delta T/T_0$ , we then obtain

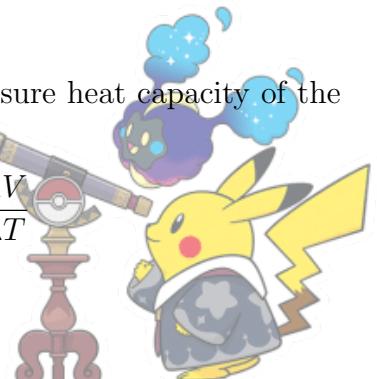
$$\frac{P\Delta V}{Nk} = T_0$$

- (b) The internal energy of the star is  $U = Mc^2$  ( $U(T) = M(T)c^2$  for ideal star). Hence, the constant volume heat capacity of the star has the form:

$$C_V = \left( \frac{\Delta Q}{\Delta T} \right)_V = \left( \frac{\Delta M}{\Delta T} \right)_V c^2$$

for small  $\Delta T$ . Then, using first law of thermodynamics, the constant pressure heat capacity of the star is

$$C_P = \left( \frac{\Delta Q}{\Delta T} \right)_P = \left( \frac{\Delta M}{\Delta T} \right)_V c^2 + P \frac{\Delta V}{\Delta T} = C_V + P \frac{\Delta V}{\Delta T}$$



for small  $\Delta T$ . Defining  $\Delta T = T_2 - T_1$ , then

$$\frac{P\Delta V}{Nk} = \frac{T_1 - T_0 + \Delta T}{\ln((T_1 + \Delta T)/T_0)} - \frac{T_1 - T_0}{\ln(T_1/T_0)}$$

Using the approximation

$$\ln((T_1 + \Delta T)/T_0) \approx \ln\left(\frac{T_1}{T_0}\right) + \frac{\Delta T}{T_1}$$

$$\frac{1}{\ln((T_1 + \Delta T)/T_0)} \approx \frac{1}{\ln(T_1/T_0)} - \frac{1}{T_1 \ln(T_1/T_0)^2} \frac{\Delta T}{T_1}$$

then we have

$$\frac{P\Delta V}{\Delta T} \approx Nk \left(1 - \frac{1}{\ln(T/T_0)}\right) \frac{(T - T_0)/T}{\ln(T/T_0)}$$

where  $T_1 = T$ . Finally, we obtain

$$C_P = \left(\frac{\Delta Q}{\Delta T}\right)_P = \left(\frac{\Delta M}{\Delta T}\right)_V c^2 + P \frac{\Delta V}{\Delta T} = C_V + \frac{Nk}{\ln(T/T_0)} \left(1 - \frac{(T - T_0)/T}{\ln(T/T_0)}\right)$$

(c) Since  $C_V$  is constant, the heat produced by the star is given by

$$Q_H = C_V(T_f - T_i) + P\Delta V$$

$$Q_H = C_V(T_f - T_i) + Nk \left[ \frac{T_f - T_0}{\ln(T_f/T_0)} - \frac{T_i - T_0}{\ln(T_i/T_0)} \right]$$

(d) Energy per second radiated by the Sun  $\dot{E} = L_\odot = N h\nu$  where  $N$  is the number of photons. Hence

$$N = \frac{L_\odot}{h\nu} = \frac{3.90 \times 10^{26}}{6.626 \times 10^{-34} \times 5 \times 10^{14}} = 1.195 \times 10^{45} \text{ photons}$$

(e) Energy per second radiated by the Sun is proportional to mass defect of the Sun

$$L_\odot = \frac{\Delta M c^2}{\Delta t}$$

Hence,

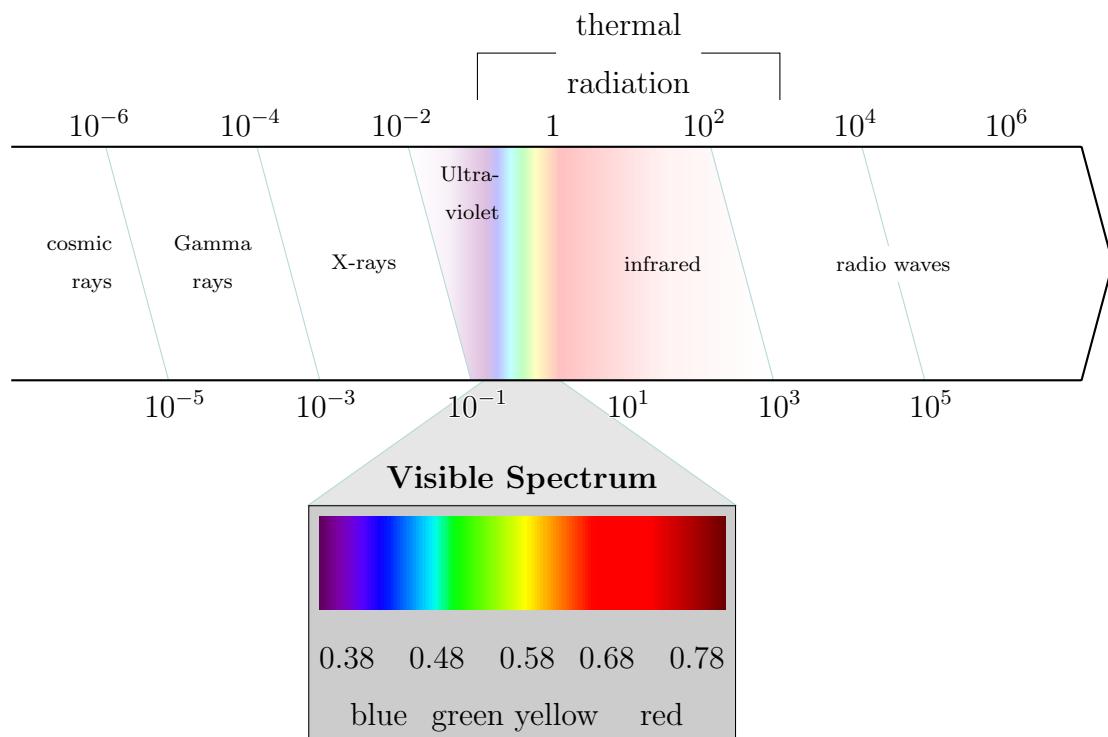
$$C_V = \frac{\Delta M c^2}{L_\odot} = \frac{3.96 \times 10^{26}}{\Delta T} \frac{1}{\frac{L_\odot}{\Delta t}} \approx \frac{3.96 \times 10^{26}}{6000 - 5500} \text{ J/K} = 7.92 \times 10^{23} \text{ J/K}$$



## 9 Spectroscopy

### 9.1 Basic Concepts

- **(Spectrum)** The diffraction of light produces a spectrum, which can be observed as a series of bright and dark fringes.



- **(Absorption)** Absorption occurs when atoms or molecules in a celestial object absorb photons of specific energies, raising electrons to higher energy levels.
- **(Emission)** Emission occurs when excited electrons drop to lower energy levels, releasing photons of specific energies, forming **emission lines**.

$$E_{\text{photon}} = E_{\text{upper}} - E_{\text{lower}}$$

This leads to **absorption lines** in the spectrum.

- (4) Spectral lines reveal chemical composition, temperature, pressure, and velocity fields (via Doppler shifts).
- (5) **(Scattering)** Scattering occurs when photons interact with particles, changing direction and sometimes energy:



- **Rayleigh scattering:** Elastic scattering by particles much smaller than wavelength.
- **Thomson scattering:** Elastic scattering by free electrons.
- **Compton scattering:** Inelastic scattering, photon loses energy. There is a relation derived by Compton:

$$\lambda' - \lambda = \frac{h}{m_e c} (1 - \cos \theta)$$

where

- (i)  $\lambda$  is the initial wavelength of the photon,
- (ii)  $\lambda'$  is the wavelength of the scattered photon,
- (iii)  $h$  is Planck's constant,
- (iv)  $m_e$  is the mass of the electron,
- (v)  $c$  is the speed of light,
- (vi)  $\theta$  is the scattering angle, which is the angle between the direction of the incident and scattered photon.

- (6) (**Splitting**) Splitting occurs when a single spectral line divides into multiple components due to external or internal interactions. The Zeeman effect is the splitting of spectral lines in the presence of a magnetic field  $\mathbf{B}$ . For transitions with no spin, a single spectral line splits into three components:

$$\Delta E = m_l \mu_B B, \quad m_l = 0, \pm 1$$

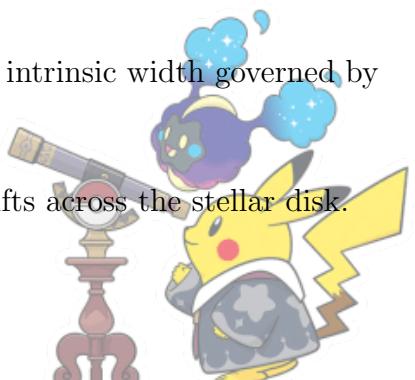
where  $\mu_B$  is the Bohr magneton,  $B$  is the magnetic field strength, and  $m_l$  is the magnetic quantum number.

The Stark effect is the splitting of spectral lines due to an external electric field  $\mathbf{E}$ :

$$\Delta E \propto E$$

- (7) (**Broadening**) Broadening refers to the widening of spectral lines beyond their natural linewidth.

- Due to the finite lifetime  $\tau$  of excited states, spectral lines have an intrinsic width governed by the uncertainty principle.
- Stellar rotation causes line broadening due to different Doppler shifts across the stellar disk.



## 9.2 Spectral Radiance

The spectral radiance (power emitted per unit area per unit solid angle per unit wavelength) is given by Planck's law:

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/(\lambda k_B T)} - 1}$$

where  $\lambda$  is the wavelength.

Astronomy has evolved from visible-light observations to encompass the entire electromagnetic (EM) spectrum. Each wavelength band reveals unique astrophysical phenomena, as expressed by Planck's law.

Band	Wavelength Range	Energy Range	Temperature (K)	Primary Sources
Radio	> 10 cm	$< 1.24 \times 10^{-5}$ eV	< 0.1	Cold gas, pulsars
Microwave	1 mm – 10 cm	$1.24 \times 10^{-5}$ eV – $1.24 \times 10^{-3}$ eV	0.1 – 10	CMB, molecular clouds
Infrared	700 nm – 1 mm	$1.24 \times 10^{-3}$ eV – 1.77 eV	$10 - 10^4$	Dust, planets, cool stars
Visible	400 nm – 700 nm	1.77 eV – 3.1 eV	$10^4$	Stars, galaxies, nebulae
Ultraviolet	10 nm – 400 nm	3.1 eV – 124 eV	$10^4 - 10^5$	Hot stars, quasars
X-ray	0.01 nm – 10 nm	124 eV – 124 keV	$10^5 - 10^8$	Black holes, supernovae
Gamma-ray	< 0.01 nm	> 124 keV	$> 10^8$	GRBs, nuclear processes

Figure 8: Electromagnetic spectrum bands in astronomy

**Limiting cases:**

- Rayleigh-Jeans Law (long wavelengths,  $hc \ll \lambda k_B T$ ):

$$B_\lambda(T) \approx \frac{2ck_B T}{\lambda^4}$$



- Wien's Law (short wavelengths,  $hc \gg \lambda k_B T$ ):

$$B_\lambda(T) \approx \frac{2hc^2}{\lambda^5} e^{-hc/(\lambda k_B T)}$$

The wavelength at which the emission is maximum:

$$\lambda_{\max} T \approx 2.898 \times 10^{-3} \text{ m}\cdot\text{K} \text{ (Wien's Displacement Law)}$$

### 9.3 Stefan–Boltzmann law

$$j^* = \sigma T^4$$

where

- $j^*$  is the total radiative flux (power per unit area,  $(\text{W}/\text{m}^2)$ ),
- $T$  is the absolute temperature of the blackbody in Kelvin (K),
- $\sigma$  is the Stefan–Boltzmann constant:

$$\sigma = 5.670374419 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$$

For a real object with emissivity ( $\epsilon$ ) ( $0 \leq \epsilon \leq 1$ ), the law generalizes to:

$$j = \epsilon \sigma T^4$$

**Proof.** The total radiated power per unit area is obtained by integrating over all frequencies and solid angles from Planck's law:

$$\begin{aligned} j^* &= \int_0^\infty \int_{\Omega} B(\nu, T) \cos \theta d\Omega d\nu \\ &= \pi \int_0^\infty B(\nu, T) d\nu \end{aligned}$$



Let  $x = \frac{h\nu}{k_B T}$ .

$$\begin{aligned} j^* &= \frac{2\pi h}{c^2} \int_0^\infty \frac{\nu^3}{e^{h\nu/k_B T} - 1} d\nu \\ &= \frac{2\pi (k_B T)^4}{h^3 c^2} \int_0^\infty \frac{x^3}{e^x - 1} dx \end{aligned}$$

The integral evaluates to  $\frac{\pi^4}{15}$ , giving

$$j^* = \sigma T^4, \quad \sigma = \frac{2\pi^5 k_B^4}{15 h^3 c^2}$$

## 9.4 Doppler's Effect

The **Doppler effect** is a fundamental phenomenon in wave physics where the observed frequency of a wave changes due to relative motion between the source and observer.

For non-relativistic speeds ( $v \ll c$ ), where  $c$  is the speed of light:

$$f_{\text{obs}} = \frac{f_{\text{src}}}{1 \pm \frac{v_s}{c}} \quad (\text{Source Moving, Observer Stationary})$$

where + for receding source, – for approaching source.

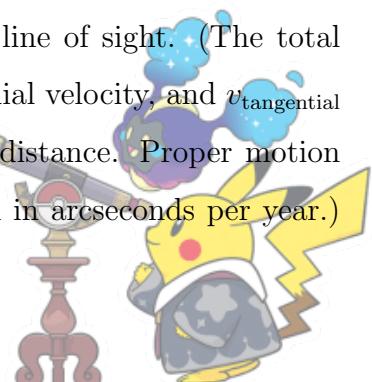
$$f_{\text{obs}} = f_{\text{src}} \left( 1 \pm \frac{v_o}{c} \right) \quad (\text{Observer Moving, Source Stationary})$$

In astrophysics, we typically measure the **radial velocity**  $v_r$  through the wavelength shift:

$$\frac{\Delta\lambda}{\lambda_0} = \frac{\lambda_{\text{obs}} - \lambda_0}{\lambda_0} = \frac{v_r}{c}$$

where

- $\lambda_0$  represents the rest wavelength,
- $\lambda_{\text{obs}}$  represents the observed wavelength, and
- $v_r$  represents the radial velocity which is the motion of a star along the line of sight. (The total velocity of a star is given by  $v = \sqrt{v_{\text{radial}}^2 + v_{\text{tangential}}^2}$  where  $v_{\text{radial}}$  is the radial velocity, and  $v_{\text{tangential}}$  is the tangential velocity, which can be derived from proper motion and distance. Proper motion refers to the angular movement of a star across the sky, usually measured in arcseconds per year.) (positive for recession)



For astronomical objects moving at significant fractions of light speed, we must use the relativistic formula:

$$\frac{\lambda_{\text{obs}}}{\lambda_0} = \sqrt{\frac{1 + \beta}{1 - \beta}} \quad \text{or} \quad \frac{f_{\text{obs}}}{f_0} = \sqrt{\frac{1 - \beta}{1 + \beta}}$$

The redshift  $z$  is defined as

$$z = \frac{\lambda_{\text{obs}} - \lambda_0}{\lambda_0} = \frac{\Delta\lambda}{\lambda_0}$$

For relativistic speeds,

$$1 + z = \sqrt{\frac{1 + \beta}{1 - \beta}}$$



## 10 Electromagnetism

### 10.1 Lorentz's Force

The Lorentz force law describes the force exerted on a charged particle moving through an electromagnetic field. It is given by

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

- $\mathbf{F}$  is the force on the charged particle,
- $q$  is the charge of the particle,
- $\mathbf{E}$  is the electric field,
- $\mathbf{v}$  is the velocity of the particle, and
- $\mathbf{B}$  is the magnetic field.

### 10.2 Maxwell's Equation

#### Definition. 10.1: Del Operator

For continuously differentiable vector field  $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$ , define  $\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$ . Then

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \text{ and } \nabla \times \mathbf{F} = \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \mathbf{i} + \left( \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \mathbf{j} + \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \mathbf{k}$$

which  $\nabla \cdot \mathbf{F}$  measures the net flow of the field out of an infinitesimally small volume around a point.

#### Theorem. 10.1: Differential Form of Maxwell's Equation

For free charge density  $\rho$  (S.I. unit: C/m<sup>3</sup>) and free current density  $\mathbf{J}$  (S.I. unit: A/m<sup>2</sup>),

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (\text{Gauss's Law for Electricity})$$

$$\nabla \cdot \mathbf{B} = 0 \quad (\text{Gauss's Law for Magnetism})$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (\text{Faraday's Law of Induction})$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad (\text{Ampère-Maxwell Law})$$



### Theorem. 10.2: Integral Form of Maxwell's Equation

For free charge density  $\rho$  and free current density  $\mathbf{J}$ , the Maxwell's equations in integral form are:

$$\int_S \mathbf{E} \cdot d\mathbf{A} = \frac{1}{\epsilon_0} \int_V \rho dV \quad (\text{Gauss's Law for Electricity})$$

$$\int_S \mathbf{B} \cdot d\mathbf{A} = 0 \quad (\text{Gauss's Law for Magnetism})$$

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{A} \quad (\text{Faraday's Law of Induction})$$

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{A} + \mu_0 \epsilon_0 \frac{d}{dt} \int_S \mathbf{E} \cdot d\mathbf{A} \quad (\text{Ampère-Maxwell Law})$$

## 10.3 Poynting Vector

The Poynting vector describes the directional energy flux (the energy transfer per unit area per unit time) or power flow of an electromagnetic field. It is given by

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$$

where

- $\mathbf{S}$  is the Poynting vector,
- $\mathbf{E}$  is the electric field, and
- $\mathbf{B}$  is the magnetic field.

## 10.4 Optics

### 10.4.1 Wavefunction

The displacement of a point on a string in simple harmonic motion can be modeled by a sinusoidal function. The solution to the wave equation for a string under tension is:

$$\phi(x, t) = A \sin(kx - \omega t + \delta)$$

where

- $A$  is the amplitude of the wave,



- $k = \frac{2\pi}{\lambda}$  is the wave number (with  $\lambda$  being the wavelength),
- $\omega = 2\pi f$  is the angular frequency (with  $f$  being the frequency), and
- $\delta$  is a phase constant.

### 10.4.2 Lens

#### Theorem. 10.3: Refractive Index

The **refractive index** of a medium is defined as the ratio of the speed of light in vacuum to the phase velocity of light in the medium:

$$n \equiv \frac{c}{v}.$$

Without free charge, Ampère's law states that

$$\nabla \times \mathbf{B} = \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

Take the curl of Faraday's law:

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B})$$

Substituting the expression for  $\nabla \times \mathbf{B}$  yields

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Using the vector identity

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$$

and Gauss's law, we obtain

$$-\nabla^2 \mathbf{E} = -\mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\nabla^2 \mathbf{E} = \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

The standard wave equation for a field  $\mathbf{E}$  propagating with speed  $v$  is

$$\nabla^2 \mathbf{E} = \frac{1}{v^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}$$



Comparing with the equation above, we identify

$$\frac{1}{v^2} = \mu\epsilon$$

Therefore, the speed of electromagnetic waves in the medium is

$$v = \frac{1}{\sqrt{\mu\epsilon}}$$

For speed of light,  $c = \frac{1}{\sqrt{\epsilon_0\mu_0}}$ . Then  $n = \sqrt{\epsilon_r\mu_r}$  where  $\epsilon_r = \frac{\epsilon}{\epsilon_0}$  is the relative permittivity (dielectric constant) of the medium and  $\mu_r = \frac{\mu}{\mu_0}$  is the relative permeability of the medium. For  $\mu_r \approx 1$ ,  $n \approx \sqrt{\epsilon_r}$ .

#### Theorem. 10.4: Snell's Law

Consider an interface between two homogeneous, isotropic media with refractive indices  $n_1$  and  $n_2$ . Let  $\theta_1$  and  $\theta_2$  denote the angles that the incident and refracted rays make with the normal to the interface, respectively. Snell's law states that

$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$

#### Theorem. 10.5: Lens Formula

$$\frac{1}{f} = \frac{1}{v} + \frac{1}{u}$$

where

- $f$  is the focal length of the lens, which is the distance from the optical element (lens or mirror) to the point where light converges to form an image and determines the magnification and field of view of the telescope,
- $u$  is the object distance (distance from the object to the lens), and
- $v$  is the image distance (distance from the lens to the image).

The most widely used convention is the **Cartesian sign convention**:



Parameter	Sign Convention
Object distance ( $u$ )	Negative (for real objects)
Image distance ( $v$ )	Positive (for real images) Negative (for virtual images)
Focal length ( $f$ )	Positive (for convex/ converging lenses) Negative (for concave/ diverging lenses)
Height of object ( $h_o$ )	Positive (upward from principal axis)
Height of image ( $h_i$ )	Positive (upward from principal axis) Negative (downward from principal axis)

Table 4: Sign conventions for lens formula

**Definition. 10.2: Optical Power**

For a lens with focal length  $f$ , the power is given by

$$P = \frac{1}{f}$$

**Theorem. 10.6: Combination of Thin Lens**

For two or more thin lenses close together, the effective power is given by  $\sum P$ .

**Theorem. 10.7: Lens Maker's Equation**

$$\frac{1}{f} = (n - 1) \left[ \frac{1}{R_1} - \frac{1}{R_2} + \frac{(n - 1)d}{nR_1R_2} \right]$$

where

- $f$  is the focal length of the lens,
- $n$  is the refractive index of the lens material,
- $R_1$  is the radius of curvature of the first surface of the lens,
- $R_2$  is the radius of curvature of the second surface of the lens,



- $d$  is the thickness of the lens.

**Proof.** We will derive the case of thin lens where  $d \rightarrow 0$ .

Consider a **spherical refracting surface** with radius of curvature  $R$  separating two media with refractive indices  $n_1$  and  $n_2$ . Let  $C$  be the **center of curvature** and  $V$  be the **vertex** of the spherical surface. We use the sign convention:

- Distances measured from  $V$
- $R > 0$  if  $C$  is to the right of  $V$  (convex toward object)
- Object distance  $u = -s_o$  (negative if object is left of  $V$ )
- Image distance  $v = s_i$  (positive if image is right of  $V$ )

Using the **paraxial (small-angle) approximation**:

$$\sin \theta \approx \theta, \quad \tan \theta \approx \theta$$

for rays making small angles with the optical axis.

Consider a point object  $O$  on the optical axis, sending a ray to point  $A$  on the spherical surface at height  $h$  above the axis. From triangle  $OAC$ ,

$$\text{Angle of incidence: } \theta_1 = \alpha + \phi$$

From triangle  $IAC$ ,

$$\text{Angle of refraction: } \theta_2 = \phi - \beta$$

where

$$\alpha \approx \frac{h}{-u}, \quad \beta \approx \frac{h}{v}, \quad \phi \approx \frac{h}{R}$$

with  $u = -s_o < 0$ ,  $v > 0$ , and  $R$  as given.

Snell's law in paraxial form:

$$n_1 \theta_1 = n_2 \theta_2$$

Substituting:

$$n_1(\alpha + \phi) = n_2(\phi - \beta)$$



$$n_1 \left( \frac{h}{-u} + \frac{h}{R} \right) = n_2 \left( \frac{h}{R} - \frac{h}{v} \right)$$

$$n_1 \left( \frac{1}{-u} + \frac{1}{R} \right) = n_2 \left( \frac{1}{R} - \frac{1}{v} \right)$$

Rewriting with Cartesian sign convention ( $u = -s_o$ ,  $v = s_i$ ,  $R$  signed):

$$\boxed{\frac{n_2}{v} - \frac{n_1}{u} = \frac{n_2 - n_1}{R}}$$

Consider a **thin lens** of refractive index  $n_l$  surrounded by medium  $n_m$ . The lens has:

- First spherical surface: radius  $R_1$  (center  $C_1$ )
- Second spherical surface: radius  $R_2$  (center  $C_2$ )
- Thickness negligible compared to object/image distances (thin lens approximation)

### Sign convention:

- Light travels left to right
- $R > 0$  if center of curvature is to the right of surface
- Object distance  $s_o > 0$  (real object left of lens)
- Image distance  $s_i > 0$  (real image right of lens)

From medium 1  $n_m \implies$  to medium 2  $n_l$ ,

$$\frac{n_l}{v_1} - \frac{n_m}{(-s_o)} = \frac{n_l - n_m}{R_1}$$

Since  $u_1 = -s_o$ ,

$$\frac{n_l}{v_1} + \frac{n_m}{s_o} = \frac{n_l - n_m}{R_1} \quad (**)$$

From medium 1  $n_l$  to medium 2  $n_m$ , the object for second surface is the image from first surface. For thin lens, object distance is  $-v_1$ .

$$\begin{aligned} \frac{n_m}{s_i} - \frac{n_l}{(-v_1)} &= \frac{n_m - n_l}{R_2} \\ \frac{n_m}{s_i} + \frac{n_l}{v_1} &= \frac{n_m - n_l}{R_2} \quad (***) \end{aligned}$$



From (\*\*),  $\frac{n_l}{v_1} = \frac{n_l - n_m}{R_1} - \frac{n_m}{s_o}$ . From (\*\*\*) $, \frac{n_l}{v_1} = \frac{n_m - n_l}{R_2} - \frac{n_m}{s_i}$ . Equating both expressions for  $\frac{n_l}{v_1}$  and simplifying gives

$$\frac{n_m}{s_o} - \frac{n_m}{s_i} = (n_l - n_m) \left( \frac{1}{R_2} - \frac{1}{R_1} \right)$$

For  $s_i = f$  (image at focal point),

$$\frac{1}{f} = \frac{n_l - n_m}{n_m} \left( \frac{1}{R_1} - \frac{1}{R_2} \right)$$

Let  $n_m = 1$  and  $n_l = n$ .

$$\boxed{\frac{1}{f} = (n - 1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right)}$$

## 10.5 Diffraction and Interference

### 10.5.1 Principle of Superposition

The **principle of superposition** states that when two or more waves overlap in space, the resultant displacement at any point is equal to the algebraic sum of the individual displacements at that point. Let two harmonic waves be given by

$$y_1(x, t) = A_1 \sin(kx - \omega t + \phi_1)$$

$$y_2(x, t) = A_2 \sin(kx - \omega t + \phi_2)$$

According to the principle of superposition, the resultant wave is

$$\begin{aligned} y(x, t) &= y_1(x, t) + y_2(x, t) \\ &= A_1 \sin(kx - \omega t + \phi_1) + A_2 \sin(kx - \omega t + \phi_2) \end{aligned}$$

Using the trigonometric identity

$$\sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2},$$

we get

$$y(x, t) = 2A \cos \left( \frac{\phi_2 - \phi_1}{2} \right) \sin \left( kx - \omega t + \frac{\phi_1 + \phi_2}{2} \right)$$

where  $A$  is the effective amplitude.



- Constructive interference occurs when  $\phi_2 - \phi_1 = 2n\pi$ .
- Destructive interference occurs when  $\phi_2 - \phi_1 = (2n + 1)\pi$ .

### 10.5.2 Complex Number

Recall that a complex number  $z$  is written as

$$z = x + iy$$

where  $x$  is the real part,  $y$  is the imaginary part, and  $i = \sqrt{-1}$ .

#### Theorem. 10.8: Euler's Formula

$$e^{i\theta} = \cos \theta + i \sin \theta$$

This is extremely useful in wave physics because oscillating quantities like  $A \cos(\omega t + \phi)$  can be represented as the real part of a complex exponential:

$$A \cos(\omega t + \phi) = \Re(A e^{i(\omega t + \phi)})$$

### 10.5.3 Young's Double-Slit Experiment

Consider two slits separated by distance  $d$  illuminated by coherent light of wavelength  $\lambda$ . At a point on a screen at distance  $L$ , the path difference (which is defined as the difference in the distance traveled by two waves from their respective sources to a common point.) is

$$\delta = d \sin \theta$$

where  $\theta$  is the angle from the central maximum.

The condition for constructive interference (bright fringes) is

$$d \sin \theta = m\lambda, \quad m = 0, \pm 1, \pm 2, \dots$$

and for destructive interference (dark fringes)

$$d \sin \theta = \left(m + \frac{1}{2}\right) \lambda$$



For small angles,  $\sin \theta \approx \tan \theta = \frac{x}{L}$ , where  $x$  is the fringe displacement. Then the fringe width is

$$\Delta x = \frac{\lambda L}{d}$$

#### 10.5.4 Single-Slit Diffraction

Diffraction refers to the bending of waves around obstacles and apertures. Consider a slit of width  $a$ . The condition for minima in the diffraction pattern is

$$a \sin \theta = m\lambda, \quad m = \pm 1, \pm 2, \dots$$

The central maximum is twice as wide as the secondary maxima. The intensity at angle  $\theta$  is given by

$$I(\theta) = I_0 \left( \frac{\sin(\beta)}{\beta} \right)^2, \quad \beta = \frac{\pi a \sin \theta}{\lambda}$$

#### 10.5.5 Rayleigh's Criteria

Two point sources are said to be just resolved if the central maximum of one diffraction pattern coincides with the first minimum of the other. The Rayleigh criterion gives the limit at which two point sources can be resolved:

$$\theta_R = 1.22 \frac{\lambda}{D}$$

**Derivation:** Consider a circular aperture of diameter  $D$ . For monochromatic light of wavelength  $\lambda$ , the far-field (Fraunhofer) diffraction pattern of a point source is the **Airy pattern**, whose intensity is given by

$$I(\theta) = I_0 \left[ \frac{2J_1(ka \sin \theta)}{ka \sin \theta} \right]^2$$



where  $a = D/2$ ,  $k = 2\pi/\lambda$ ,  $J_1$  is the Bessel function of the first kind of order 1<sup>1</sup>, and  $\theta$  is the angular distance from the optical axis.

The first zero of  $J_1(x)$  occurs at  $x \approx 3.8317$ . Let  $x = ka \sin \theta$ . Then

$$ka \sin \theta_{\min} = 3.8317 \implies \frac{2\pi a}{\lambda} \sin \theta_{\min} = 3.8317$$

Hence,

$$\sin \theta_{\min} \approx \frac{3.8317\lambda}{2\pi(D/2)} \approx \frac{1.22\lambda}{D}$$

For small angles ( $\theta$  in radians),

$$\theta_{\min} \approx \frac{1.22\lambda}{D}$$

This  $\theta_{\min}$  is the angular radius of the Airy disk (to the first dark ring).

## 10.6 Polarization

### 10.6.1 Introduction

Polarization describes the direction in which the electric field vector of a light wave oscillates. Light can be

- **Unpolarized:** The electric field points in random directions (like sunlight).
- **Linearly polarized:** The electric field oscillates in a single direction.
- **Circularly polarized:** The electric field rotates in a circle as light travels.
- **Elliptically polarized:** A general case where the electric field traces an ellipse.

---

<sup>1</sup>Bessel functions are solutions to Bessel's differential equation:

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - \nu^2)y = 0,$$

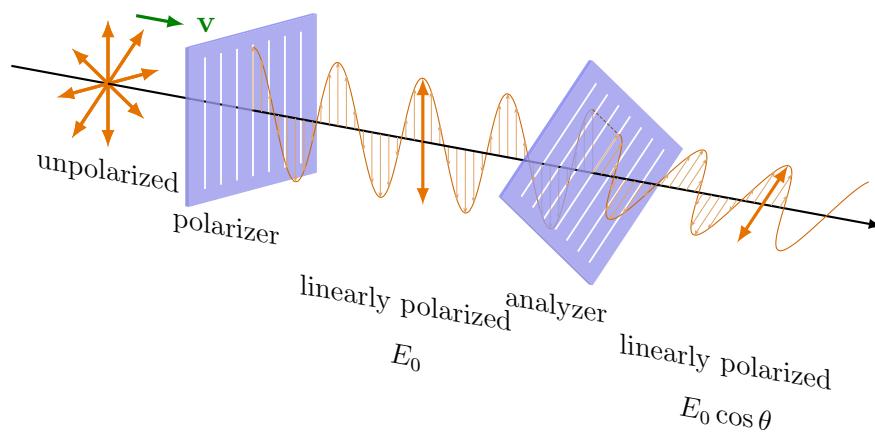
where  $\nu$  is a constant, often referred to as the order of the Bessel function.

The Bessel function of the first kind  $J_\nu(x)$  is defined by the following series expansion:

$$J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \nu + 1)} \left(\frac{x}{2}\right)^{2m+\nu}$$

where  $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$  for  $\Re(z) > 0$





There are some methods for polarization:

- **Polarization by Reflection:** When light is reflected at a certain angle (Brewster's angle,  $\theta_B$ ), the reflected light becomes completely polarized:

$$\tan \theta_B = \frac{n_2}{n_1}$$

where  $n_1$  and  $n_2$  are refractive indices of the two media.

- **Polarization by Absorption:** Polarizing filters only allow electric field components in a particular direction to pass through. When linearly polarized light passes through a polarizing filter, the transmitted intensity  $I$  is given by

$$I = I_0 \cos^2 \theta \text{ (Malus's Law)}$$

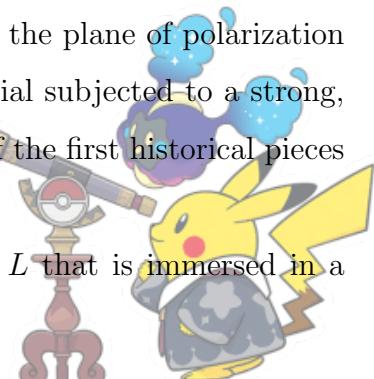
where

- $I_0$  is the initial intensity of the light.
- $\theta$  is the angle between the light's polarization direction and the axis of the polarizer.

### 10.6.2 Faraday's Rotation

The Faraday effect, or Faraday rotation, is a magneto-optic phenomenon where the plane of polarization of linearly polarized light is rotated when the light propagates through a material subjected to a strong, static magnetic field aligned in the direction of propagation. This effect is one of the first historical pieces of evidence linking light with electromagnetism.

When linearly polarized light passes through a transparent material of length  $L$  that is immersed in a



magnetic field  $\mathbf{B}$  (parallel to the direction of propagation), the angle of rotation  $\beta$  of the polarization plane is given by

$$\beta = VBL$$



# 11 Quantum Mechanics

## 11.1 Atomic Structure

### 11.1.1 Historical Development

- **Thomson's Model (1897):** "Plum pudding" model with electrons embedded in positive charge
- **Rutherford's Model (1911):** Nuclear model from alpha scattering experiments
- **Bohr's Model (1913):** Quantized electron orbits
- **Quantum Mechanical Model:** Electron clouds/orbitals

### 11.1.2 Introduction

The atom consists of

- **Nucleus:** Protons ( $p^+$ ) and neutrons ( $n$ )
- **Electrons:** Negatively charged particles in orbitals

Particle	Symbol	Charge	Mass (u)
Proton	$p$ or ${}_1^1H$	+1	1.007276
Neutron	$n$	0	1.008665
Electron	$e^-$	-1	0.000549

Table 5: Basic atomic particles

For an element X,



where

- $A$  is the mass number (protons + neutrons),
- $Z$  is the atomic number (number of protons), and
- we can define  $N$  to be neutron number ( $N = A - Z$ )



**Example.**  ${}^6_6C_6$  has 6 protons and 6 neutrons.

The actual mass of a nucleus is **less** than the sum of its constituent particles:

$$\Delta m = (Zm_p + Nm_n) - m_{\text{nucleus}}$$

where

- $\Delta m$  represents the mass defect,
- $m_p$  represents the mass of proton,
- $m_n$  represents the mass of neutron, and
- $m_{\text{nucleus}}$  represents the measured nuclear mass.

According to Einstein's mass-energy equivalence

$$E = mc^2$$

The binding energy is

$$BE = \Delta m \cdot c^2$$

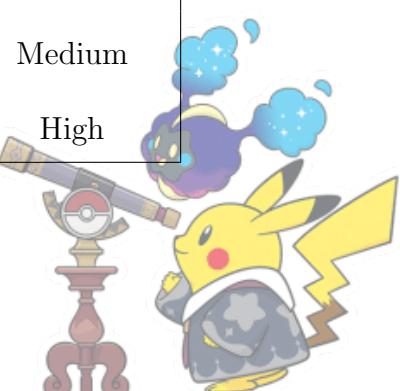
More conveniently, using atomic mass units (u) where  $1 u = 931.5 \text{ MeV}/c^2$ :

$$BE (\text{MeV}) = \Delta m (u) \times 931.5$$

### 11.1.3 Nuclear Decay

Type	Emitted Particle	Change in Nucleus	Penetration
Alpha ( $\alpha$ )	${}^4_2He$ nucleus	$Z \rightarrow Z - 2, A \rightarrow A - 4$	Low
Beta ( $\beta^-$ )	Electron ( $e^-$ )	$n \rightarrow p + e^- + \bar{\nu}$	Medium
Beta ( $\beta^+$ )	Positron ( $e^+$ )	$p \rightarrow n + e^+ + \nu$	Medium
Gamma ( $\gamma$ )	Photon ( $\gamma$ )	No change in $Z$ or $A$	High

Table 6: Types of radioactive decay



Radioactive decay follows first-order kinetics:

$$N(t) = N_0 e^{-\lambda t}$$

where

- $N(t)$  represents the number of nuclei at time  $t$ ,
- $N_0$  represents the initial number of nuclei, and
- $\lambda = \frac{\ln 2}{t_{1/2}}$  represents the decay constant where  $t_{1/2}$  is half-life (time for half the nuclei to decay).

#### 11.1.4 Neutrinos

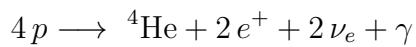
Neutrinos are elementary particles that are part of the lepton family. They are electrically neutral and have an extremely small mass, making them very difficult to detect. Neutrinos interact only through weak nuclear force and gravity, which is why they pass through matter almost unaffected. The three types of neutrinos correspond to their associated charged leptons:

- Electron neutrino ( $\nu_e$ )
- Muon neutrino ( $\nu_\mu$ )
- Tau neutrino ( $\nu_\tau$ )

These particles are produced in various high-energy processes, such as nuclear reactions in the Sun and other stars, as well as in cosmic ray interactions and supernovae.

**Solar Neutrinos** The Sun is a primary source of neutrinos, particularly electron neutrinos ( $\nu_e$ ). These solar neutrinos are produced during nuclear fusion processes that take place in the Sun's core. The dominant fusion reaction in the Sun is the proton-proton chain, which is responsible for the majority of the energy production. In this process, four protons fuse to form a helium nucleus, releasing energy in the form of gamma rays, neutrinos, and positrons.

The overall process can be written as



## 11.2 Wave-Particle Duality

Louis de Broglie proposed that all matter has wave-like properties. The de Broglie wavelength is given by

$$\lambda = \frac{h}{p}$$

where

- $\lambda$  is the de Broglie wavelength,
- $h = 6.626 \times 10^{-34} \text{ J}\cdot\text{s}$  is Planck's constant, and
- $p$  is the particle's momentum.

## 11.3 Planck's Equation

For electromagnetic waves of frequency  $\nu$ , the energy of photon is given by

$$E = h\nu$$

## 11.4 Bohr Model of the Hydrogen Atom

### 11.4.1 Bohr Postulates

1. The electron moves in circular orbits around the proton due to Coulomb attraction.
2. Only certain discrete orbits are allowed.
3. The angular momentum of the electron is quantized.
4. Radiation is emitted or absorbed only during transitions between allowed orbits.

### 11.4.2 Formulas

The electrostatic force between the proton and electron is

$$F = \frac{1}{4\pi\epsilon_0} \frac{e^2}{r^2}$$

For circular motion, the centripetal force is

$$F = \frac{mv^2}{r}$$



Equating the two,

$$\frac{mv^2}{r} = \frac{1}{4\pi\epsilon_0} \frac{e^2}{r^2}$$

Solving for velocity,

$$v^2 = \frac{1}{4\pi\epsilon_0} \frac{e^2}{mr}$$

Bohr postulated

$$L = mvr = n\hbar \quad n = 1, 2, 3, \dots$$

Solving for velocity,

$$v = \frac{n\hbar}{mr}$$

Substitute the expression for  $v$ :

$$\left(\frac{n\hbar}{mr}\right)^2 = \frac{1}{4\pi\epsilon_0} \frac{e^2}{mr}$$

Solving for  $r$ ,

$$r_n = \frac{4\pi\epsilon_0\hbar^2}{me^2} n^2$$

Define the Bohr radius

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{me^2}$$

Hence,

$$r_n = a_0 n^2$$

The total energy is

$$E = K + U = \frac{1}{2}mv^2 - \frac{1}{4\pi\epsilon_0} \frac{e^2}{r}$$

Note that

$$K = \frac{1}{2} \frac{1}{4\pi\epsilon_0} \frac{e^2}{r}$$

Hence, the total energy becomes

$$E = -\frac{1}{2} \frac{1}{4\pi\epsilon_0} \frac{e^2}{r}$$

Substituting  $r_n$ ,

$$E_n = -\frac{me^4}{2(4\pi\epsilon_0)^2\hbar^2} \frac{1}{n^2} = \frac{-13.6 \text{ eV}}{n^2}$$



### 11.4.3 Wavefunction

In quantum mechanics, the state of a particle is fully described by a complex-valued function called the **wavefunction**, denoted by  $\psi(x, t)$ . The wavefunction contains all the measurable information about the system.

The physical interpretation of the wavefunction is given by the Born rule, which states that the probability density of finding a particle at position  $x$  at time  $t$  is

$$P(x, t) = |\psi(x, t)|^2.$$

Since the particle must be found somewhere in space, the total probability of finding the particle over all space must be equal to one. This requirement leads to the **normalisation condition**:

$$\int_{-\infty}^{\infty} |\psi(x, t)|^2 dx = 1.$$

If a wavefunction does not initially satisfy this condition, it can be normalised by introducing a constant  $A$  such that

$$\Psi(x, t) = A\psi(x, t),$$

where the constant  $A$  is chosen to ensure the normalisation condition is satisfied.

### 11.4.4 Time-Independent Schrödinger Equation

The Schrödinger equation is

$$-\frac{\hbar^2}{2m} \nabla^2 \psi + V(r)\psi = E\psi$$

with the Coulomb potential

$$V(r) = -\frac{1}{4\pi\epsilon_0} \frac{e^2}{r}$$

Using separation of variables,

$$\psi(r, \theta, \phi) = R(r)Y_{\ell m}(\theta, \phi)$$

the radial equation becomes

$$\frac{d^2R}{dr^2} + \frac{2}{r} \frac{dR}{dr} + \left[ \frac{2m}{\hbar^2} \left( E + \frac{e^2}{4\pi\epsilon_0 r} \right) - \frac{\ell(\ell+1)}{r^2} \right] R = 0$$



For large  $r$ , the Coulomb term becomes negligible, yielding

$$\frac{d^2R}{dr^2} - \kappa^2 R = 0$$

where

$$\kappa = \sqrt{-\frac{2mE}{\hbar^2}}$$

The physically acceptable solution is

$$R(r) \sim e^{-\kappa r}$$

For small  $r$ , the equation has solution

$$R(r) \sim r^\ell$$

Motivated by the asymptotic behavior, write

$$R(r) = r^\ell e^{-\kappa r} F(r)$$

Define a dimensionless variable

$$\rho = 2\kappa r$$

and substitute into the radial equation to obtain

$$\rho \frac{d^2F}{d\rho^2} + (2\ell + 2 - \rho) \frac{dF}{d\rho} + \left( \frac{me^2}{2\pi\varepsilon_0\hbar^2\kappa} - \ell - 1 \right) F = 0$$

Assume a power series

$$F(\rho) = \sum_{k=0}^{\infty} a_k \rho^k$$

Substitution yields the recursion relation

$$a_{k+1} = \frac{k + \ell + 1 - \frac{me^2}{2\pi\varepsilon_0\hbar^2\kappa}}{(k+1)(k+2\ell+2)} a_k$$

For the wavefunction to remain normalizable, the series must terminate. Hence, there exists an integer  $n_r$  such that

$$\frac{me^2}{2\pi\varepsilon_0\hbar^2\kappa} = n_r + \ell + 1$$

Define the principal quantum number

$$n = n_r + \ell + 1$$



Solving for energy,

$$E_n = -\frac{me^4}{2(4\pi\epsilon_0)^2\hbar^2} \frac{1}{n^2}$$

## 11.5 Uncertainty Principle

Werner Heisenberg showed that certain pairs of physical quantities cannot be simultaneously measured with arbitrary precision. The most famous uncertainty relation is between position and momentum:

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

where

- $\Delta x$  is the uncertainty in position
- $\Delta p$  is the uncertainty in momentum
- $\hbar = \frac{h}{2\pi}$  is the reduced Planck constant

**Example.** Consider a particle (e.g., an electron) passing through a narrow slit of width  $a$  along the  $x$ -direction. Due to the confinement in position, the particle's wavefunction is localized within the slit, giving

$$\Delta x \sim a.$$

After passing through the slit, the particle undergoes diffraction, resulting in a spread in its momentum in the  $x$ -direction. The diffraction pattern for a slit is given by the first minimum condition:

$$a \sin \theta \sim \lambda,$$

where  $\theta$  is the diffraction angle and  $\lambda$  is the wavelength of the particle. For a particle with momentum  $p$ , its wavelength is related to the momentum by de Broglie's relation:

$$\lambda = \frac{h}{p},$$

where  $h$  is Planck's constant. The transverse momentum uncertainty is approximately

$$\Delta p_x \sim p \sin \theta \sim \frac{h}{a}.$$



Multiplying the uncertainties in position and momentum:

$$\Delta x \Delta p_x \sim a \cdot \frac{h}{a} \sim h \sim \hbar,$$

which reproduces the Heisenberg Uncertainty Principle:

$$\Delta x \Delta p_x \gtrsim \frac{\hbar}{2}.$$

Another important uncertainty relation is between energy and time:

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

The uncertainty principle isn't about measurement limitations, but about fundamental properties of quantum systems. Particles don't have precisely defined positions and momenta simultaneously.



## 12 Stellar Astrophysics

### 12.1 Stellar Classifications

Stars can be classified by their internal structure:

- **Main Sequence Stars:** Hydrogen-burning core, radiative or convective energy transport. Governed by mass-luminosity relation:

$$\frac{L}{L_\odot} \approx \begin{cases} \left(\frac{M}{M_\odot}\right)^{4.0} & \text{for } M > 10M_\odot \\ \left(\frac{M}{M_\odot}\right)^{3.5} & \text{for } 0.5 < M < 10M_\odot \\ \left(\frac{M}{M_\odot}\right)^{2.3} & \text{for } M < 0.5M_\odot \end{cases}$$

- **Giant Stars:** Expanded envelope, hydrogen shell burning around inert helium core.
- **Supergiants:** Massive stars with complex layered burning (H, He, C, O, Si shells).
- **White Dwarfs:** Electron-degenerate cores, supported by electron degeneracy pressure.
- **Neutron Stars:** Neutron-degenerate matter, extreme density ( $\sim 10^{17}$  kg/m<sup>3</sup>).
- **Black Holes:** Gravitational collapse beyond neutron degeneracy pressure support.

Stars can also be classified by their composition:

Population	Metallicity [Fe/H]	Characteristics
Population I	$\geq -0.5$	Metal-rich, disk stars, young
Population II	-1.0 to -0.5	Metal-poor, halo stars, old
Population III	$\ll -1.0$	Zero metals, first generation

Table 7: Stellar Populations

Metallicity affects stellar evolution:

- Opacity changes with metal content
- Lower metallicity stars are hotter and bluer at given mass



- Metallicity influences mass loss rates

Stars can also be classified by their activity:

- **Flare Stars** (UV Ceti type): M-dwarfs with magnetic reconnection events: Magnetic reconnection is a fundamental plasma physics process where the topology of magnetic field lines is rearranged, converting magnetic energy into kinetic energy, thermal energy, and particle acceleration.

- **Rotational Classes:**

- Slow rotators:  $v_{\text{eq}} < 10 \text{ km/s}$  (Sun: 2 km/s)
- Fast rotators:  $v_{\text{eq}} > 50 \text{ km/s}$  (young stars)

- **Magnetic Activity:**

$$R'_{HK} = \frac{L_{HK}}{L_{\text{bol}}}$$

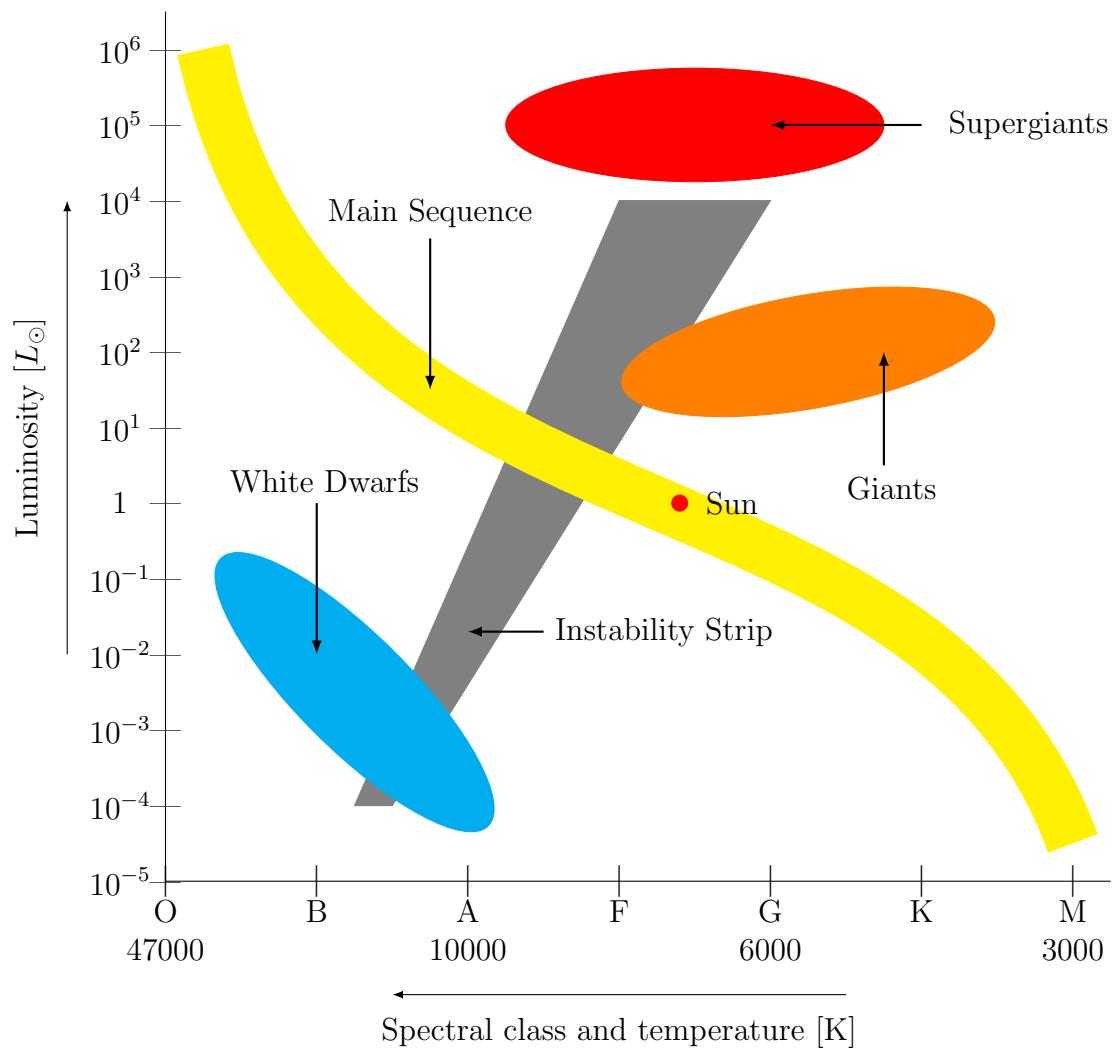
where  $L_{HK}$  is the luminosity in the Calcium II H & K lines and  $L_{\text{bol}}$  is the bolometric luminosity: Total power output across all wavelengths (from radio to X-rays). This ratio tells what fraction of the star's total energy output is being emitted specifically from its magnetically heated chromosphere via these Calcium lines as Calcium II is singly-ionized calcium.

## 12.2 HR Diagram

### 12.2.1 Introduction

The Hertzsprung-Russell diagram (HR diagram) is one of the most important tools in astrophysics, independently developed by Ejnar Hertzsprung (1905-1911) and Henry Norris Russell (1913). It revolutionized our understanding of stellar evolution by revealing patterns in stellar properties.





### 12.2.2 Spectral Types and Temperature

Each spectral type corresponds to a particular range of temperatures and characteristics. The order from hottest to coolest stars is as follows:



Spectral Type	Temp (K)	Color	Mass ( $M_{\odot}$ )	Examples
O	30,000-50,000	Blue	15-90	$\zeta$ Puppis
B	10,000-30,000	Blue-white	2-15	Rigel, Spica
A	7,500-10,000	White	1.4-2	Sirius, Vega
F	6,000-7,500	Yellow-white	1.04-1.4	Procyon
G	5,200-6,000	Yellow	0.8-1.04	Sun, $\alpha$ Cen A
K	3,700-5,200	Orange	0.45-0.8	Arcturus, Aldebaran
M	2,400-3,700	Red	0.08-0.45	Betelgeuse, Proxima Cen

Table 8: MK Spectral Classification System

The sequence "O, B, A, F, G, K, M" can be difficult to remember due to the variety of letters. To aid in memorization, astronomers and students often use mnemonics. A common mnemonic for remembering the spectral types in order is

### “Oh Be A Fine Girl/Guy, Kiss Me!”

This mnemonic associates each letter with a word:

- **O** - Oh
- **B** - Be
- **A** - A
- **F** - Fine
- **G** - Girl/Guy
- **K** - Kiss
- **M** - Me

In addition to spectral types, stars are also classified based on their luminosity, which is related to their size and brightness. This is done using Roman numerals from I to V:

- **I**: Supergiants (e.g., Betelgeuse)



- **II:** Bright giants
- **III:** Giants (e.g., Alpha Centauri)
- **IV:** Subgiants
- **V:** Main sequence stars (e.g., the Sun)

The full classification of a star would be a combination of both spectral type and luminosity class. For example, our Sun is classified as a **G2V**<sup>2</sup> star, meaning it is a G-type main sequence star.

### 12.2.3 Turn-Off Point

The **turn-off point** is an important feature in the HR diagram that helps astronomers determine the age of a star cluster. It represents the point where stars, after exhausting the hydrogen in their cores, begin to leave the main sequence and evolve into red giants. The position of the turn-off point on the diagram depends on the mass of the stars in the cluster.

Note that

$$\text{Age of Cluster} = \frac{M}{L} \propto \frac{M}{M^{3.5}} = \frac{1}{M^{2.5}}$$

where we use the mass of the star at the turn-off point  $M$ .

## 12.3 Stellar Evolution

### 12.3.1 Stellar Formation

Stars form from giant clouds of gas and dust called **molecular clouds** or **nebulae**. The process includes:

- **Gravitational Collapse:** Regions with higher density collapse under gravity.
- **Protostar Formation:** As the cloud collapses, a dense core forms and heats up due to gravitational energy.
- **Accretion:** Surrounding material falls onto the protostar, increasing its mass and temperature.

<sup>2</sup>2 is a subclass number that further refines the classification. The scale goes from 0 to 9, with 0 being the hottest and 9 being the coolest in each spectral type. The number 2 indicates that the star is towards the middle of the G-type range. In this case, a G2 star has a temperature closer to 5,800 K.



### 12.3.2 Pre-Main Sequence Stars

Before reaching the main sequence, stars go through the **pre-main sequence (PMS)** phase:

- The protostar contracts and heats up.
- Energy is produced mainly by gravitational contraction, not nuclear fusion.
- PMS stars follow **Hayashi tracks** (for lower-mass stars) or **Henyey tracks** (for higher-mass stars) on the Hertzsprung-Russell diagram.

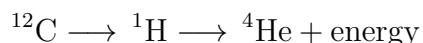
### 12.3.3 Main Sequence Stars

A star enters the **main sequence** when hydrogen fusion begins in its core.

- Core hydrogen fusion converts hydrogen into helium via the **proton-proton chain** (low-mass stars) or **CNO cycle** (high-mass stars).
  - **Proton-Proton Chain:**



- **CNO Cycle:**



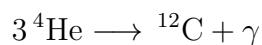
- The star achieves **hydrostatic equilibrium**: gravity balanced by radiation pressure from fusion.
- The main sequence lifetime depends on stellar mass; higher-mass stars burn faster and live shorter lives.

### 12.3.4 Post-Main Sequence Stars

After hydrogen in the core is exhausted, stars evolve differently depending on their mass.

#### Low to Intermediate-Mass Stars ( $< 8M_\odot$ )

- Expand into **Red Giants**.
- Helium fusion begins in the core (**triple-alpha process**):



- May go through **asymptotic giant branch (AGB)** phase before shedding outer layers.

### High-Mass Stars ( $> 8M_{\odot}$ )

- Expand into **supergiants**.
- Fuse heavier elements in successive shells (C, O, Si, etc.).
- Form an **iron core**, which cannot undergo fusion to produce energy.

#### 12.3.5 Supernovae

- Massive stars end their lives in a **core-collapse supernova** (Type II).
- The core collapses, and outer layers are expelled violently.
- Supernovae enrich the interstellar medium with heavy elements.

#### 12.3.6 Planetary Nebulae

- Formed by low to intermediate-mass stars shedding their outer layers.
- The exposed core emits ultraviolet radiation, ionizing the ejected gas.
- Eventually fades to leave a **white dwarf**.

#### 12.3.7 End States of Stars

- **White Dwarfs:** Remnants of low/intermediate-mass stars. Supported by electron degeneracy pressure.
- **Neutron Stars:** Remnants of core-collapse supernovae of stars with  $8 - 20M_{\odot}$ . Supported by neutron degeneracy pressure.
- **Black Holes:** Remnants of very massive stars ( $> 20M_{\odot}$ ). Gravity overwhelms all forms of degeneracy pressure.



## 12.4 Magnitude System

### Definition. 12.1: Luminosity

**Luminosity** is the **total amount of energy emitted by a source per unit time**.

### Definition. 12.2: Luminance

**Luminance** is the amount of **visible light emitted or reflected in a given direction per unit area per unit solid angle**. It describes how bright a surface appears to the human eye.

Property	Luminosity	Luminance
Physical meaning	Total power emitted	Brightness per area per direction
Depends on distance	No	No (but depends on direction)
Depends on area	Indirectly	Directly
SI unit	Watt (W)	cd/m <sup>2</sup>
Usually used in	Astrophysics	Optics

### Definition. 12.3: Flux

The **flux** ( $F$  or  $\Phi$ ) is the total energy received from an astronomical object per unit time per unit area. It represents the power (energy per unit time) crossing a unit area oriented perpendicular to the direction of propagation.

For a source emitting total luminosity  $L$  isotropically, the flux measured at distance  $r$  is

$$F = \frac{L}{4\pi r^2}$$

### Definition. 12.4: Solid Angle

The **solid angle** measures the three-dimensional angular size of an object as seen from a point. It is defined as

$$d\Omega = \frac{dA}{r^2}$$

where

- $dA$  is an area element on a sphere of radius  $r$ ,
- $d\Omega$  is the corresponding solid angle.



**Definition. 12.5: Intensity**

The flux per unit solid angle is called **intensity**:

$$I = \frac{dF}{d\Omega}$$

**Definition. 12.6: Spectral Flux Density**

The **spectral flux density** ( $F_\lambda$  or  $F_\nu$ ) is the flux per unit wavelength or frequency interval. It describes how the flux is distributed across the electromagnetic spectrum.

The **solar constant** is the flux received from the Sun at Earth's distance:

$$F_\odot = 1361 \text{ W m}^{-2} = 1.361 \times 10^6 \text{ erg s}^{-1} \text{cm}^{-2}$$

This represents the total solar power (all wavelengths) incident on 1 m<sup>2</sup> at 1 AU.

**Definition. 12.7: Apparent Magnitude**

It refers to the brightness of an object **as observed from Earth**, given by

$$m = -2.5 \log_{10} \left( \frac{F}{F_0} \right)$$

where  $F_0$  is the reference flux for zero magnitude. The zero-point  $F_0$  is calibrated using standard stars. Originally, Vega ( $\alpha$  Lyrae) was defined to have magnitude 0.0 in all filters. Modern systems use more precisely defined spectrophotometric standards.

Ancient astronomers (especially Hipparchus, 150 BCE) ranked stars by eye:

1. 1st magnitude means the brightest (e.g. Sirius and Vega)
2. 6th magnitude means faintest visible to the naked eye

Much later (1856), Norman Pogson put this on a mathematical footing. He defined the scale so that a difference of 5 magnitudes corresponds to a brightness ratio of 100.

$$\frac{F_1}{F_2} = 100 \text{ when } m_2 - m_1 = 5$$



Assume magnitude difference is proportional to the logarithm of the flux ratio:

$$m_2 - m_1 = k \log_{10} \left( \frac{F_1}{F_2} \right)$$

By Pogson's condition,  $5 = 2k \implies k = 2.5$ . Take star 1 as the reference star:  $m_1 = 0, F_1 = F_0$  gives  $m = -2.5 \log_{10} \left( \frac{F}{F_0} \right)$ . Later, the logarithmic scale was found to match the Weber-Fechner law in psychophysics, which states that perceived sensation is proportional to the logarithm of stimulus intensity.

#### Definition. 12.8: Absolute Magnitude

The brightness an object would have if placed at a standard distance of **10 parsecs** (32.6 light-years):

$$M = m - 5 \log_{10} \left( \frac{d}{10 \text{ pc}} \right)$$

where  $d$  is the distance to the object in parsecs.

#### Definition. 12.9: Distance Modulus

The difference between apparent and absolute magnitude relates directly to distance:

$$\mu = m - M = 5 \log_{10} d - 5$$

#### Definition. 12.10: Limiting Magnitude

The limiting magnitude is the faintest apparent magnitude of a celestial object that can be detected with a given instrument under specific observing conditions. It represents the threshold between what is observable and what is not. It is given by

$$m_{\text{lim}} = m_{\text{eye}} + 5 \log \left( \frac{D}{d_{\text{eye}}} \right) + 2.5 \log_{10} \left( \frac{\text{Transmission}}{0.95} \right)$$

where

1.  $m_{\text{eye}}$  is the naked-eye limiting magnitude (typically 6.0 under ideal conditions),
2.  $D$  is the telescope aperture,
3.  $d_{\text{eye}}$  is the dark-adapted pupil diameter (typically 7 mm), and
4. Transmission is the optical transmission coefficient.



**Definition. 12.11: Bolometric Magnitude**

The **bolometric magnitude** ( $m_{\text{bol}}$  or  $M_{\text{bol}}$ ) is a measure of an astronomical object's total electromagnetic luminosity across all wavelengths, from gamma rays to radio waves. Unlike filter-specific magnitudes, it accounts for **all** emitted radiation.

The bolometric magnitude is defined through the **bolometric flux**  $F_{\text{bol}}$ , which is the integral of the spectral flux density over all wavelengths:

$$F_{\text{bol}} = \int_0^{\infty} F_{\lambda} d\lambda = \int_0^{\infty} F_{\nu} d\nu$$

The apparent bolometric magnitude is then

$$m_{\text{bol}} = -2.5 \log_{10} \left( \frac{F_{\text{bol}}}{F_{\text{bol},0}} \right)$$

where  $F_{\text{bol},0}$  is the reference bolometric flux for zero magnitude.

**Definition. 12.12: Bolometric Correction**

It refers to the difference between bolometric and visual magnitudes:

$$BC = M_{\text{bol}} - M_V$$

or for apparent magnitudes:

$$BC = m_{\text{bol}} - m_V$$

The Sun's magnitude is assumed to be bolometric by convention. Therefore, it is common to use

$$M_{\text{bol}} - M_{\odot} = -2.5 \log_{10} \left( \frac{L}{L_{\odot}} \right).$$

**Example. (2013 IOAA)** A star has visual apparent magnitude  $m_V = 12.2$  mag, parallax  $\pi = 0.001''$  and effective temperature  $T_{\text{eff}} = 4000$  K. Its bolometric correction is B.C. =  $-0.6$  mag.

(a) Find its luminosity as a function of the solar luminosity.

(b) What type of star is it?

(i) a red giant

(ii) a blue giant



(iii) a red dwarf

Please write (i), (ii) or (iii) in your answer sheet.

(a) First, the absolute visual magnitude is obtained from

$$M_V - m_V = 5 - 5 \log r$$

or equivalently,

$$M_V - m_V = 5 + 5 \log \pi$$

Thus,

$$M_V = 12.2 + 5 + 5 \log(0.001) = 12.2 + 5 - 15 = 2.2 \text{ mag}$$

The bolometric correction is defined as

$$\text{B.C.} = M_{\text{bol}} - M_V$$

so

$$M_{\text{bol}} = \text{B.C.} + M_V = -0.6 + 2.2 = 1.6 \text{ mag}$$

The luminosity is calculated using

$$M_{\odot} - M_{\text{bol}} = 2.5 \log \left( \frac{L}{L_{\odot}} \right)$$

where  $M_{\odot} = 4.72$  mag. Hence,

$$4.72 - 1.6 = 2.5 \log \left( \frac{L}{L_{\odot}} \right)$$

$$\log \left( \frac{L}{L_{\odot}} \right) = 1.25$$

and therefore

$$L = 17.7 L_{\odot}$$

(b) A star with  $M_{\text{bol}} = 1.6$  mag, luminosity  $L = 17.7 L_{\odot}$ , and effective temperature  $T_{\text{eff}} = 4000 \text{ K}$  is much brighter and cooler than the Sun. Therefore, it is **(i) a red giant star**.

## 12.5 Albedo

**Albedo** is a dimensionless physical quantity that measures the reflectivity of a surface. It is defined as the fraction of incident electromagnetic radiation (usually sunlight) that is reflected by a surface.

If a surface receives an incident radiant power  $P_{\text{in}}$  and reflects a power  $P_{\text{ref}}$ , its albedo  $\alpha$  is defined as

$$\alpha = \frac{P_{\text{ref}}}{P_{\text{in}}}, \quad 0 \leq \alpha \leq 1$$

Assuming the planet radiates as a black body and is in thermal equilibrium, the absorbed power equals emitted power:

$$(1 - \alpha)\pi R^2 S = 4\pi R^2 \sigma T^4$$

where  $\sigma$  is the Stefan–Boltzmann constant.

Solving for the equilibrium temperature  $T$ :

$$T = \left( \frac{(1 - \alpha)S}{4\sigma} \right)^{1/4}$$

## 12.6 Geometric Albedo

The **geometric albedo** is a dimensionless quantity that measures how bright an astronomical body appears when observed at **full phase** (i.e. zero phase angle), compared to an idealized reference surface.

Let

- $\alpha$  be the **phase angle**, defined as the angle between the incident radiation from the source (e.g. the Sun) and the direction to the observer, as seen from the object,
- $\alpha = 0$  correspond to **full illumination** (observer directly behind the light source).

We introduce a reference surface:

A perfectly diffusing (Lambertian) flat disk with the same cross-sectional area as the object, illuminated and observed at normal incidence.

A Lambertian surface reflects radiation isotropically, obeying Lambert's cosine law:



**Theorem. 12.1: Lambert's Cosine law**

A surface obeys **Lambert's cosine law** if its radiance is independent of direction:

$$I(\theta, \phi) = I_0 = \text{constant}$$

For a Lambertian surface, the power emitted into solid angle  $d\Omega$  is

$$dP = I_0 \cos \theta dA d\Omega$$

If the incident solar flux is  $F_{inc}$ , a perfectly reflecting disk intercepts power  $P = F_{inc}A$ . Integrating the Lambertian emission over a hemisphere:

$$P_{reflected} = \int_0^{2\pi} \int_0^{\pi/2} I_{ref} \cos \theta \sin \theta A d\theta d\phi = \pi A I_{ref}$$

Equating incident and reflected power ( $P_{reflected} = F_{inc}A$ ), we find:

$$I_{ref} = \frac{F_{inc}}{\pi}$$

The geometric albedo  $p$  is defined as

$$p = \frac{I_{object}(\alpha = 0)}{I_{ref}} = \frac{\pi I_{object}(\alpha = 0)}{F_{inc}}$$

The total energy reflected in all directions is characterized by the **Bond Albedo** ( $A_B$ ), related to  $p$  by the **phase integral**  $q$ :

$$A_B = p \cdot q, \quad q = 2 \int_0^\pi \Phi(\alpha) \sin \alpha d\alpha$$

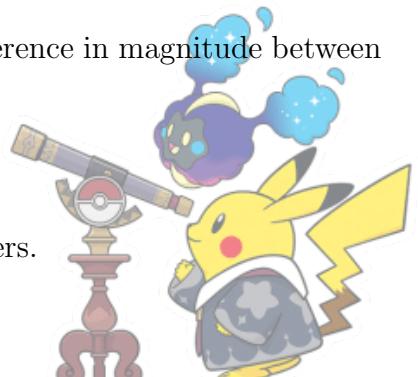
where  $\Phi(\alpha)$  is the phase function, which is the relative brightness of the object as a function of the phase angle, normalized such that  $\Phi(0) = 1$ .

## 12.7 Color Indices

Color indices quantify the color of astronomical objects by measuring the difference in magnitude between two different wavelength bands:

$$C = m_{\lambda_1} - m_{\lambda_2}$$

where  $m_{\lambda_1}$  and  $m_{\lambda_2}$  are apparent magnitudes measured through different filters.



Filter	Center ( $\lambda_c$ , nm)	FWHM (nm)	Typical Use
U	365	66	Ultraviolet continuum
B	445	94	Blue light, Balmer break
V	551	88	Visual (photopic)
R	658	138	Red continuum
I	806	149	Near-infrared

Table 9: Johnson-Cousins photometric system filters

The followings are the common color indices to use:

$$U - B = m_U - m_B$$

$$B - V = m_B - m_V$$

$$V - R = m_V - m_R$$

$$V - I = m_V - m_I$$

The observed color index is affected by interstellar extinction:

$$(B - V)_{\text{obs}} = (B - V)_0 + E(B - V)$$

where  $E(B - V)$  is the color excess:

$$E(B - V) = A_B - A_V$$

where  $A_B$  is the extinction in  $B$ -band in mag and  $A_V$  is the extinction in  $V$ -band in mag.

## 12.8 Atmospheric Extinction

Atmospheric extinction reduces the observed flux:

$$m_{\lambda, \text{obs}} = m_{\lambda, \text{true}} + k'_{\lambda} \cdot X$$

where

- $k'_{\lambda}$  is the extinction coefficient (mag/airmass) and



- $X$  is the airmass (dimensionless) which quantifies the atmosphere's thickness along a light's path.

For a plane-parallel atmosphere,

$$X = \sec z$$

where  $z$  is the zenith distance ( $z = 90^\circ - \text{altitude}$ ).

## 12.9 Optical Depth

Optical depth  $\tau_\nu$  describes attenuation by comparing the initial intensity  $I_\nu(0)$  and the transmitted intensity  $I_\nu(s)$ :

$$\tau_\nu(s) \equiv -\ln \left( \frac{I_\nu(s)}{I_\nu(0)} \right).$$

Extinction in magnitudes  $A_\lambda$  is related to optical depth by

$$A_\lambda = 2.5 \log_{10}(e\tau_\lambda) = 1.086 \tau_\lambda.$$

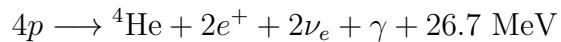
as

$$\frac{I}{I_0} = e^{-\tau_\lambda} = 10^{-0.4A_\lambda}.$$

## 12.10 Study of the Sun

### 12.10.1 Composition

The Sun is primarily composed of hydrogen and helium, with trace amounts of heavier elements. Hydrogen nuclei (protons) undergo nuclear fusion in the solar core:



powering the Sun through the proton-proton chain.

### 12.10.2 Internal Structure

#### 1. Core:

- Radius:  $\sim 0.2 R_\odot$
- Site of nuclear fusion:  $4p \Longrightarrow {}^4\text{He} + 2e^+ + 2\nu_e + 26.7 \text{ MeV}$
- Temperature:  $\sim 1.5 \times 10^7 \text{ K}$



## 2. Radiative Zone:

- Energy transported by radiation
- Temperature gradient decreases outward

## 3. Convective Zone:

- Energy transported by convection
- Outer  $\sim 30\%$  of Sun's radius
- Convection cells cause granulation on the photosphere

### 12.10.3 Atmosphere

- **Photosphere:** Visible surface;  $T \sim 5800$  K
- **Chromosphere:** Above photosphere; hotter than photosphere;  $T \sim 10^4$  K
- **Corona:** Outermost layer;  $T \sim 10^6$  K; source of solar wind

### 12.10.4 Solar Surface Activities

**Sunspots** Sunspots are temporary dark regions on the solar photosphere (visible surface of the Sun) caused by strong magnetic fields.

### 12.10.5 Solar Wind

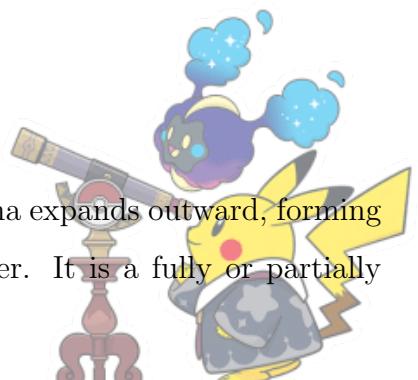
The solar wind is a continuous outflow of plasma from the solar corona. The solar wind consists mainly of

- Protons ( $p^+$ )
- Electrons ( $e^-$ )
- Alpha particles ( $\alpha = {}^4\text{He}^{2+}$ )

The solar corona has high temperature ( $T \sim 1 \times 10^6$  K). The sound speed is

$$c_s \sim \sqrt{\frac{k_B T}{m_p}}$$

As the corona cannot remain static with such high thermal pressure, the plasma expands outward, forming the solar wind. (Remark: Plasma is often called the fourth state of matter. It is a fully or partially



ionized gas, meaning that a significant fraction of the atoms or molecules are electrically charged (ions and electrons).)

The **heliosphere** is the region of space dominated by the solar wind and the Sun's magnetic field, extending well beyond the orbit of Pluto.

The **magnetosphere** is the region around a planet where the planetary magnetic field dominates the motion of charged particles, protecting the planet from the solar wind.

## 12.11 Binary Star

Binary star systems consist of two stars orbiting around their common center of mass.

### 12.11.1 Different Types of Binary Stars

**Visual Binaries** They refer to the stars that can be resolved individually through telescopes. Their orbits can be directly observed over time.

**Spectroscopic Binaries** They refer to the stars that can be detected through periodic Doppler shifts in their spectral lines. They can be further identified as

- **Single-lined spectroscopic binaries (SB1):** Only one set of spectral lines is visible
- **Double-lined spectroscopic binaries (SB2):** Both sets of spectral lines are visible

**Eclipsing Binaries** They are the systems where the orbital plane is nearly edge-on to our line of sight, causing periodic eclipses. These provide the most complete information about stellar parameters.

**Astrometric Binaries** They can be detected through the wobble of one star's proper motion due to an unseen companion.

### Interacting Binaries

- Mass transfer occurs when one star fills its Roche lobe.
- Examples: Cataclysmic variables, X-ray binaries.

### Peculiar Binary Systems

- Systems with unusual properties such as extremely short orbital periods or **highly eccentric orbits**.
- Examples: Contact binaries, Algol-type binaries.



### 12.11.2 Modified Kepler's Third Law

For a binary system, Kepler's third law relates the orbital period  $P$ , semi-major axis  $a$ , and total mass  $M$ :

$$P^2 = \frac{4\pi^2 a^3}{G(M_1 + M_2)} \quad (10)$$

where

- $P$  is the orbital period,
- $a$  is the semi-major axis of the relative orbit, and
- $M_1, M_2$  are the masses of the two stars

### 12.11.3 Mass Function

For spectroscopic binaries, we measure the **mass function**. For single-lined spectroscopic binaries,

$$f(M) = \frac{M_2^3 \sin^3 i}{(M_1 + M_2)^2} = \frac{P}{2\pi G} v_{1,r}^3$$

For double-lined spectroscopic binaries, we can determine the mass ratio:

$$\frac{M_1}{M_2} = \frac{v_{2,r}}{v_{1,r}}$$

**Proof.** We begin with Kepler's third law for two bodies orbiting their common center of mass:

$$P^2 = \frac{4\pi^2 a^3}{G(M_1 + M_2)}$$

where  $a = a_1 + a_2$  is the total separation between the stars, and  $a_1, a_2$  are their distances from the center of mass.

The center of mass condition gives

$$a_1 M_1 = a_2 M_2$$

From this, we can write the mass ratio:

$$\frac{a_1}{a_2} = \frac{M_2}{M_1}$$



Also, the total separation can be expressed in terms of  $a_1$ :

$$a = a_1 + a_2 \quad (12)$$

$$= a_1 + a_1 \frac{M_1}{M_2} \quad (\text{from Equation 11}) \quad (13)$$

$$= a_1 \left( \frac{M_1 + M_2}{M_2} \right) \quad (14)$$

For the visible star (star 1), we measure its **radial velocity amplitude**  $v_{1,r}$ . The true orbital speed  $v_1$  is related to the observed radial velocity by the inclination:

$$v_{1,r} = v_1 \sin i \quad (15)$$

For a circular orbit ( $e = 0$ ), the orbital speed is constant and given by

$$v_1 = \frac{2\pi a_1}{P} \quad (16)$$

Combining Equations 15 and 16:

$$v_{1,r} = \frac{2\pi a_1 \sin i}{P} \quad (17)$$

From Equation 17, we can solve for  $a_1$ :

$$a_1 = \frac{P v_{1,r}}{2\pi \sin i}$$

Now substitute Equation 14 into Kepler's law (Equation 12.11.3):

$$P^2 = \frac{4\pi^2}{G(M_1 + M_2)} \left[ a_1 \left( \frac{M_1 + M_2}{M_2} \right) \right]^3 \quad (18)$$

$$= \frac{4\pi^2 a_1^3}{G(M_1 + M_2)} \cdot \frac{(M_1 + M_2)^3}{M_2^3} \quad (19)$$

$$= \frac{4\pi^2 a_1^3}{G} \cdot \frac{(M_1 + M_2)^2}{M_2^3} \quad (20)$$



Now substitute  $a_1$  from Equation 12.11.3 into Equation 20:

$$\begin{aligned} P^2 &= \frac{4\pi^2}{G} \cdot \frac{(M_1 + M_2)^2}{M_2^3} \cdot \left( \frac{Pv_{1,r}}{2\pi \sin i} \right)^3 \\ &= \frac{4\pi^2}{G} \cdot \frac{(M_1 + M_2)^2}{M_2^3} \cdot \frac{P^3 v_{1,r}^3}{8\pi^3 \sin^3 i} \\ &= \frac{P^3 v_{1,r}^3}{2\pi G \sin^3 i} \cdot \frac{(M_1 + M_2)^2}{M_2^3} \end{aligned}$$

Cancel  $P^2$  from both sides (multiply both sides by  $1/P^2$ ):

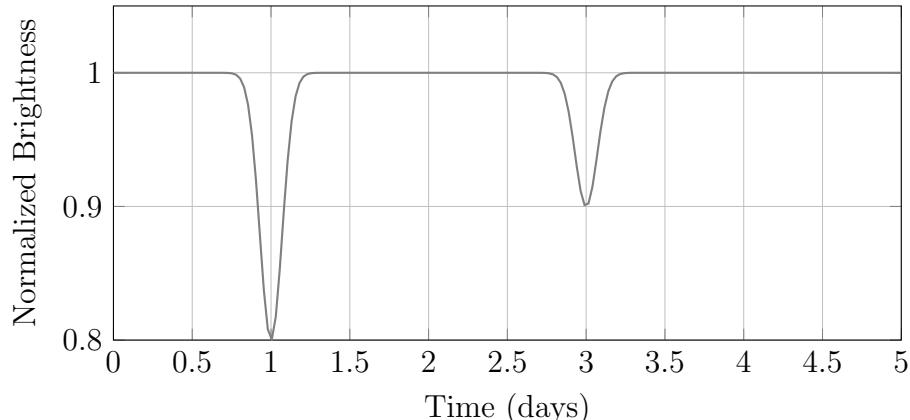
$$1 = \frac{Pv_{1,r}^3}{2\pi G \sin^3 i} \cdot \frac{(M_1 + M_2)^2}{M_2^3}$$

Now rearrange to isolate the mass terms on the left:

$$\frac{M_2^3 \sin^3 i}{(M_1 + M_2)^2} = \frac{P}{2\pi G} v_{1,r}^3$$

#### 12.11.4 Light Curves

Eclipsing binaries exhibit characteristic light curves with periodic dips in brightness:

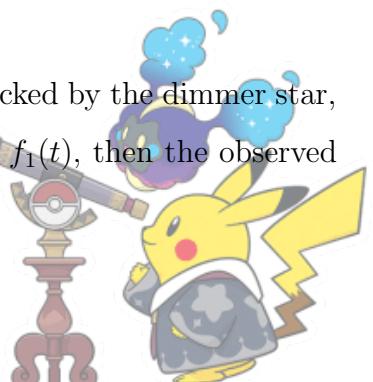


1. Let  $F_1$  and  $F_2$  be the flux of the two stars. The total observed flux when both stars are visible is

$$F_{\text{total}} = F_1 + F_2$$

2. The **primary eclipse** occurs when the brighter star is partially or fully blocked by the dimmer star, causing a significant dip. Let the obscured fraction of the primary star be  $f_1(t)$ , then the observed flux is

$$F_{\text{primary}}(t) = F_1 [1 - f_1(t)] + F_2$$



3. The **secondary eclipse** occurs when the dimmer star is blocked, causing a smaller dip. The fainter star is obscured by the brighter star. Let  $f_2(t)$  be the obscured fraction of the secondary star, then:

$$F_{\text{secondary}}(t) = F_1 + F_2 [1 - f_2(t)]$$

4. If the eclipse is total, the flux reduces exactly by the luminosity of the obscured star.

In this plot,

- The first deep dip at  $t = 1$  day corresponds to the **primary eclipse**.
- The smaller dip at  $t = 3$  days corresponds to the **secondary eclipse**.

#### 12.11.5 Radial Velocity Curves

A **binary star system** consists of two stars orbiting their common center of mass. If the orbital plane is inclined relative to the line of sight, the stars will alternately move toward and away from the observer. This motion causes a periodic **Doppler shift** in the spectral lines:

- Motion toward the observer will result in blueshift
- Motion away from the observer will result in redshift

The line-of-sight component of the orbital velocity is called the **radial velocity**. A **radial velocity curve** is a plot of radial velocity  $v_r$  versus time  $t$  (or orbital phase). It provides direct information about the orbital properties of the binary system.

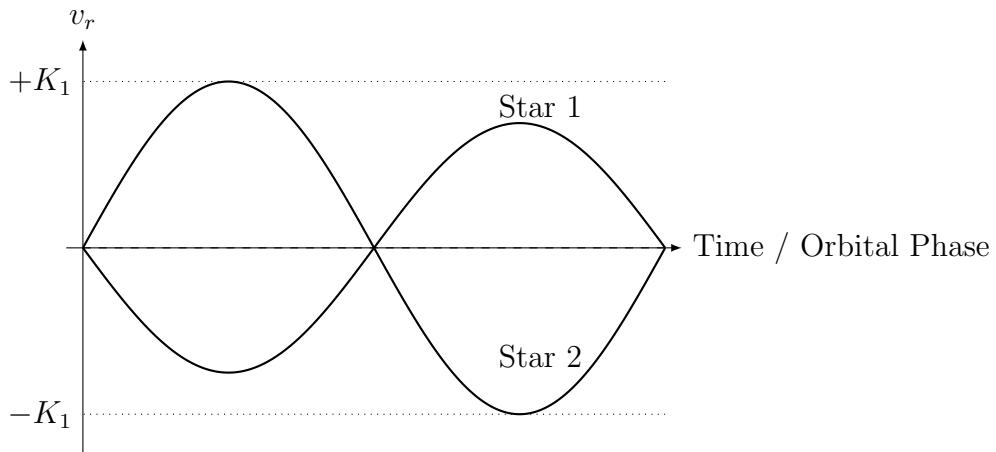
For a binary star in a circular orbit, the radial velocity varies sinusoidally:

$$v_r(t) = K \sin \left( \frac{2\pi t}{P} + \phi \right)$$

where

- $K$  is the radial velocity amplitude
- $P$  is the orbital period
- $\phi$  is the phase constant





- The curves are sinusoidal for circular orbits.
- The two stars have opposite phases due to conservation of momentum.
- The more massive star has a smaller velocity amplitude.
- The period of the curve equals the orbital period.

For a binary star in an **elliptical orbit**, the radial velocity of one component is given by:

$$v_r(t) = \gamma + K [\cos(\theta(t) + \omega) + e \cos \omega]$$

where

- $\gamma$  is the systemic (center-of-mass) velocity
- $K$  is the radial velocity semi-amplitude
- $e$  is the orbital eccentricity
- $\omega$  is the argument of periastron
- $\theta(t)$  is the true anomaly

The semi-amplitude  $K$  is

$$K = \frac{2\pi a \sin i}{P \sqrt{1 - e^2}}$$

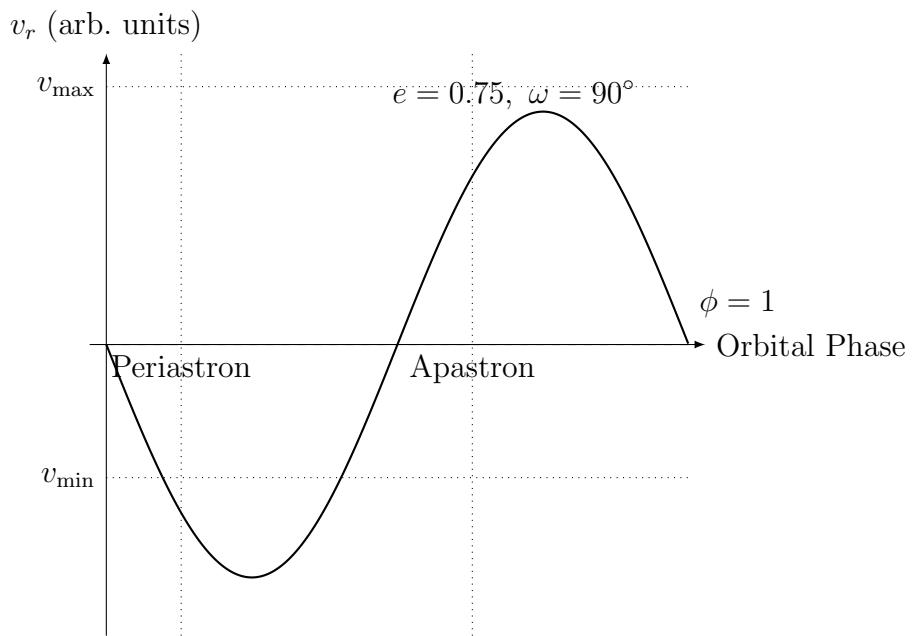
where

- $a$  is the semi-major axis of the star's orbit about the barycenter



- $i$  is the inclination angle
- $P$  is the orbital period

The following shows an example of the radial velocity curve:



### 12.11.6 Roche Lobe

In a close binary star system, the gravitational field experienced by a test particle is determined by the combined gravity of both stars and the centrifugal effect in the rotating frame. The **Roche lobe** of a star is defined as the region around that star within which material is gravitationally bound to it. If a star fills or overflows its Roche lobe, mass transfer to its companion can occur, a key mechanism in interacting binaries such as X-ray binaries, cataclysmic variables, and some exoplanetary systems.



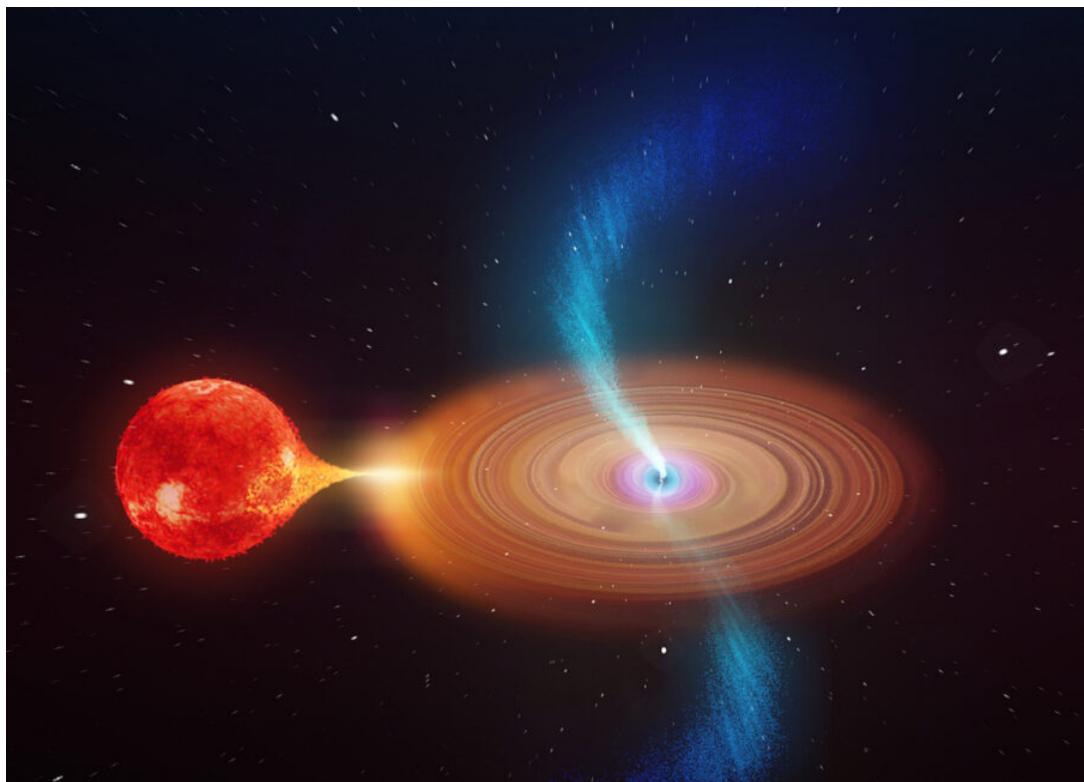


Figure 9: Source: <https://root-nation.com/ua/news-ua/it-news-ua/ua-blackhole-warped-accretion-disc/>

## 12.12 Exoplanet

### 12.12.1 Introduction

An **exoplanet** (or extrasolar planet) is a planet that orbits a star outside our Solar System. The term comes from the Greek "exo" (outside) and "planētēs" (wanderer).

### 12.12.2 Classes of Exoplanets

- **Hot Jupiters:** Gas giants very close to their stars.
- **Super-Earths:** Planets with masses larger than Earth but smaller than Neptune.
- **Terrestrial planets:** Rocky planets similar to Earth or Mars.
- **Ice giants:** Analogous to Uranus and Neptune.

### 12.12.3 Spectral Signatures of Possible Life

- Detection of biosignature gases like O<sub>2</sub>, O<sub>3</sub>, CH<sub>4</sub>, and water vapor in exoplanet atmospheres.
- Observed via transmission spectroscopy during planetary transits.



#### 12.12.4 Radial Velocity Method

It is also known as the Doppler method. This technique measures the star's wobble caused by an orbiting planet's gravitational pull.

$$\Delta v = K \left[ \frac{P}{2\pi G} \right]^{1/3} \frac{m_p \sin i}{m_s^{2/3}} \frac{1}{\sqrt{1 - e^2}}$$

where

- $\Delta v$  represents the velocity semi-amplitude,
- $K$  is a constant,
- $P$  represents the orbital period,
- $m_p$  represents the planet mass,
- $m_s$  represents the star mass,
- $i$  represents the orbital inclination, and
- $e$  represents the orbital eccentricity.

#### 12.12.5 Transit Method

It measures the periodic dimming of a star as a planet passes in front of it.

$$\Delta F = \left( \frac{R_p}{R_s} \right)^2$$

where  $\Delta F$  is the fractional flux decrease,  $R_p$  is planet radius, and  $R_s$  is star radius.

#### 12.12.6 Habitable Zone

A habitable zone refers to the region around a star where liquid water could exist on a planet's surface.



# 13 Cosmology

## 13.1 Structure of the Universe

### 13.1.1 Star Clusters

**Introduction** Star clusters are groups of stars that are gravitationally bound and formed from the same molecular cloud. They provide important insights into stellar evolution and galactic structure. Star clusters are broadly classified into two types:

- **Open Clusters (Galactic Clusters):** These contain a few tens to a few thousand stars. They are loosely bound and typically found in the Galactic disk. Open clusters are relatively young (a few million to a few hundred million years) and often contain hot, massive stars.
- **Globular Clusters:** These are densely packed spherical collections of tens of thousands to millions of stars. They orbit the Galactic halo and are typically very old (10–13 billion years). Globular clusters are rich in low-mass stars and show little gas or dust.

Structurally, a star cluster has a **core** (densest region), a **halo** (more diffuse stars), and in some cases a **tidal radius** where stars may escape due to Galactic gravitational forces.

**Luminosity** The luminosity  $L$  of a star cluster can be calculated by summing the luminosities of all the stars in the cluster:

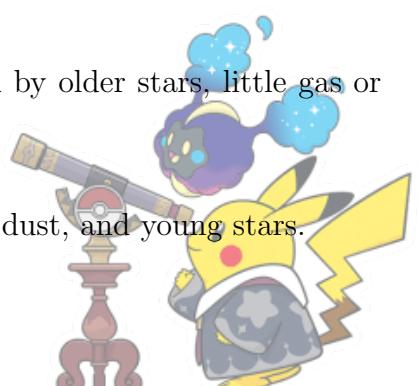
$$L_{\text{cluster}} = \sum L_{\text{stars}}$$

Where  $L_{\text{stars}}$  is the luminosity of each star in the cluster.

### 13.1.2 Galaxies

Galaxies are vast systems of stars, gas, dust, and dark matter, bound together by gravity. They are the fundamental building blocks of the Universe. Galaxies can be classified based on their structure, composition, and activity:

- **Elliptical galaxies:** Smooth, featureless light distribution, dominated by older stars, little gas or dust.
- **Spiral galaxies:** Flat, rotating disks with spiral arms, containing gas, dust, and young stars.



- **Barred spiral galaxies:** Spiral galaxies with a central bar-shaped structure of stars.
- **Irregular galaxies:** No definite shape, often rich in gas and dust.
- **Active galaxies:** Galaxies with energetic cores (AGN), often emitting strong radiation due to accretion onto supermassive black holes.

Our galaxy is a barred spiral galaxy with several distinct components:

- **Bulge:** Central region, high star density, mostly old stars.
- **Disk:** Contains spiral arms, gas, dust, and young stars.
- **Halo:** Spherical region with globular clusters and dark matter.

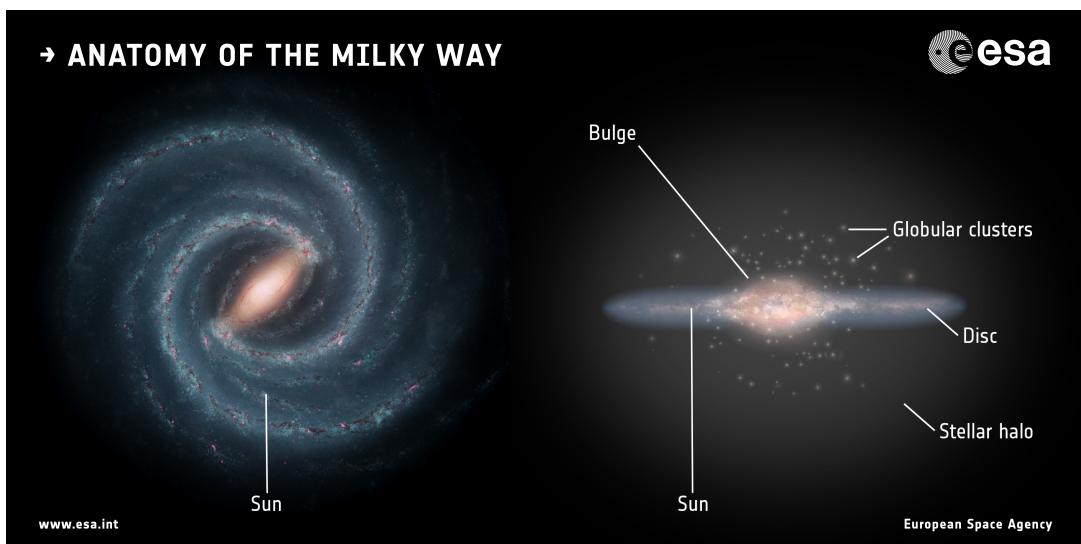


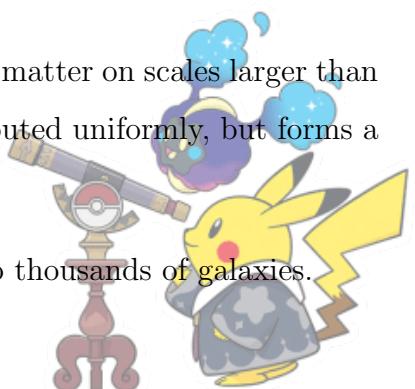
Figure 10: From <https://sci.esa.int/web/gaia/-/58206-anatomy-of-the-milky-way>

Satellites of the Milky Way include dwarf galaxies like the Large and Small Magellanic Clouds, and several ultra-faint dwarf galaxies. These satellites provide clues about the Milky Way's formation and dark matter distribution.

## 13.2 Large-scale Structure

The **large-scale structure** (LSS) of the Universe refers to the distribution of matter on scales larger than individual galaxies (typically  $\gtrsim 1$  Mpc). On these scales, matter is not distributed uniformly, but forms a complex network known as the **cosmic web**:

- **Galaxy clusters:** Dense, gravitationally bound systems of hundreds to thousands of galaxies.



- **Galaxy groups:** Smaller associations of galaxies, often containing a few to tens of members.
- **Filaments:** Elongated structures connecting clusters and groups, containing galaxies and dark matter.
- **Voids:** Large, underdense regions with very few galaxies.
- **Walls / Sheets:** Flattened structures of galaxies separating voids.

Large-scale structure formed from tiny density fluctuations in the early Universe:

- Quantum fluctuations were stretched during cosmic inflation.
- Overdensities grew via **gravitational instability**.
- Dark matter collapsed first, forming gravitational potential wells.
- Baryonic matter later fell into these wells, forming galaxies.

On sufficiently large scales ( $\gtrsim 100$  Mpc), the Universe is approximately homogeneous and isotropic, consistent with the **cosmological principle**.

### 13.3 Cosmological Principle

The **cosmological principle** states that on sufficiently large scales ( $\gtrsim 100$  Mpc), the universe is

- **Homogeneous:** Matter and energy are uniformly distributed
- **Isotropic:** No preferred direction in space

### 13.4 Rotational Curve

In a galaxy, the stars or gas clouds orbit the galactic center due to the gravitational pull of the mass contained within the galaxy. According to Newtonian mechanics, the orbital velocity of an object at a given distance  $r$  from the center of a galaxy should behave as

$$v(r) = \sqrt{\frac{GM_{\text{enc}}(r)}{r}}$$

where

- $v(r)$  is the orbital velocity at a distance  $r$  from the center and



- $M_{\text{enc}}(r)$  is the enclosed mass within radius  $r$ .

The mass enclosed  $M_{\text{enc}}(r)$  depends on the distribution of both visible matter (such as stars and gas) and dark matter within the galaxy.

For a galaxy dominated by visible matter (stars, gas, etc.), the enclosed mass increases with radius, but at larger distances from the center, the mass distribution becomes less dense. Then

$$v(r) \propto \frac{1}{\sqrt{r}}$$

This is called the **Keplerian decline** and is observed in the motion of planets in our solar system. However, observations of galaxies reveal that the rotation curves do not behave this way.

In the 1970s, astronomers like Vera Rubin and Kent Ford observed that the rotation curves of spiral galaxies remain nearly flat at large distances from the center, far beyond the region where visible matter is present. This observation was unexpected, as the rotation velocity should have decreased if the mass distribution followed the visible matter alone. The flatness of the rotation curve suggests that there is additional mass present that is not visible. This mass is what we refer to as **dark matter**.

The flat rotation curves observed at large radii imply that the mass within the galaxy continues to increase even in the outer regions. The orbital velocity  $v(r)$  remains constant:

$$v(r) = v_{\text{flat}} \quad \text{for large } r$$

This indicates that the gravitational influence of dark matter is significant at large distances from the galactic center, where visible matter is sparse.

### 13.5 Hubble's Law

$$v = H_0 d$$

where  $v$  is the recession velocity,  $d$  is the proper distance to the galaxy, and  $H_0$  is the present-day Hubble constant. One can notice that  $t_H = \frac{1}{H_0}$  is the age of the universe.

Cosmic expansion is described by the **scale factor**  $a(t)$ , which measures how distances in the Universe change over time. The Hubble parameter is defined using the scale factor:

$$H(t) = \frac{\dot{a}(t)}{a(t)}$$



Light traveling through an expanding Universe also stretches with the expansion. If a photon is emitted at time  $t_{\text{em}}$  with wavelength  $\lambda_{\text{em}}$  and observed today at  $t_0$  with wavelength  $\lambda_{\text{obs}}$ , the **cosmological redshift** is defined as

$$1 + z = \frac{\lambda_{\text{obs}}}{\lambda_{\text{em}}}$$

The redshift is directly related to the scale factor:

$$1 + z = \frac{a(t_0)}{a(t_{\text{em}})}$$

## 13.6 Cosmological Distance Measures

### 13.6.1 Proper Distance

The proper distance is related to the scale factor  $a(t)$  of the universe, which describes how distances between objects in the universe change with time due to the expansion. For an object at redshift  $z$ , the proper distance  $D_p$  at a given time is related to the comoving distance  $D_c$  by

$$D_p(t) = a(t)D_c$$

where

- $D_p$  is the proper distance,
- $D_c$  is the comoving distance,
- $a(t)$  is the scale factor at the time of observation.

### 13.6.2 Comoving Distance

The **comoving distance** is the distance between two objects as measured in a coordinate system that accounts for the expansion of the universe. Unlike the proper distance, the comoving distance remains constant over time for two objects that are at rest relative to each other in the expanding universe.

The comoving distance  $D_c$  at redshift  $z$  is related to the scale factor  $a(t)$  by

$$D_c = \int_0^z \frac{c dz'}{H(z')}$$

where



- $c$  is the speed of light,
- $H(z')$  is the Hubble parameter as a function of redshift  $z'$ , and
- $z$  is the redshift at which the object is located.

### 13.6.3 Luminosity Distance

The **luminosity distance** is the distance to an object based on its observed brightness and its intrinsic luminosity. It is often used for objects like supernovae, which have a known intrinsic luminosity.

The luminosity distance  $D_L$  is related to the observed flux  $f_{\text{obs}}$  and the intrinsic luminosity  $L$  of an object by the inverse square law:

$$f_{\text{obs}} = \frac{L}{4\pi D_L^2}$$

From this relation, we can solve for the luminosity distance:

$$D_L = \sqrt{\frac{L}{4\pi f_{\text{obs}}}}$$

In a flat universe, the luminosity distance is related to the comoving distance by

$$D_L(z) = (1 + z)D_c(z)$$

### 13.6.4 Angular Diameter Distance

The angular diameter distance is related to the physical size  $r$  of an object and its angular size  $\theta$  (in radians) by the relation

$$\theta = \frac{r}{D_A}$$

## 13.7 Friedmann Equation

### Theorem. 13.1: First Friedmann Equation

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a(t)^2} + \frac{\Lambda c^2}{3}$$

where

- cosmological constant  $\Lambda$  represents homogeneous energy density inherent to empty space (dark energy):  $\Lambda = 0$  for flat and static universe (Minkowski universe),  $\Lambda > 0$  for expanding universe



and  $\Lambda < 0$  for contracting universe and

- curvature parameter  $k = +1$  for positive curvature,  $k = -1$  for negative curvature,  $k = 0$  for zero curvature.

### Theorem. 13.2: Second Friedmann Equation

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3} \left( \rho - \frac{3p}{c^2} \right) + \frac{\Lambda c^2}{3}$$

- When  $\Lambda = 0$  and  $k = 0$ ,  $\rho = \frac{3H^2}{8\pi G}$  which is called the important density. We may denote  $\rho_c$ .
- Density parameter  $\Omega_i := \frac{\rho_i}{\rho_c}$ . Note that  $\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_k = 1$ . Also, by second Friedmann's equation and definition,

$$\Omega_m = \frac{8\pi G \rho_m}{3H^2}, \quad \Omega_k = -\frac{kc^2}{a^2 H^2}, \quad \Omega_\Lambda = \frac{\Lambda c^2}{8\pi G}$$

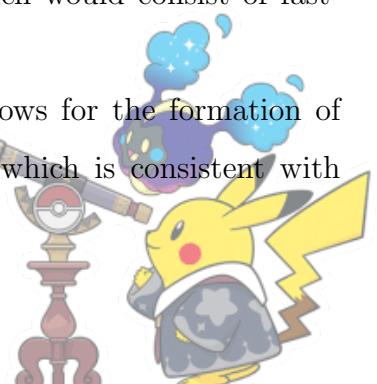
- From first and second Friedmann equation,

$$\dot{\rho} + 3H \left( \rho + \frac{p}{c^2} \right) = 0 \text{ (Fluid Equation)}$$

- The  $\Lambda$ CDM model, also known as the Lambda Cold Dark Matter model, is the current standard model of cosmology. It describes the evolution of the Universe from the early hot and dense state to its current large-scale structure. The model is based on the following components:

- **Cosmological Constant ( $\Lambda$ ):** It is responsible for the accelerated expansion of the Universe. Dark energy has an opposing effect to gravity: instead of attracting matter (like gravity does), dark energy exerts a repulsive force. This repulsive force is responsible for causing the accelerated expansion of the Universe, pushing galaxies apart at an ever-increasing rate.
- **Cold Dark Matter (CDM):** It is also referred to as "cold" because it moves slowly compared to the speed of light. This is in contrast to "hot" dark matter, which would consist of fast-moving particles.

Cold dark matter is preferred in cosmological models because it allows for the formation of small structures (such as galaxies) early in the Universe's history, which is consistent with observational data.



**Definition. 13.1: Hubble parameter**

The Hubble parameter  $H(t)$  is defined as the rate of change of the scale factor  $a(t)$  and is given by

$$H(t) = \frac{\dot{a}(t)}{a(t)}$$

## 13.8 Equation of State

For different cosmic components,

Non-relativistic matter (dust):  $p = 0 \implies \rho_m \propto a^{-3}$

Radiation:  $p = \rho c^2/3 \implies \rho_r \propto a^{-4}$

Cosmological constant:  $p = -\rho c^2 \implies \rho_\Lambda = \text{constant}$

**Proof.** Rewriting the fluid equation,

$$\frac{d\rho}{dt} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0$$

By chain rule  $\dot{\rho} = \frac{d\rho}{da}\dot{a}$ ,

$$\dot{a}\frac{d\rho}{da} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0$$

Assuming  $\dot{a} \neq 0$ :

$$\frac{d\rho}{da} + \frac{3}{a}\left(\rho + \frac{p}{c^2}\right) = 0$$

By equation of states,

$$p = w\rho c^2$$

where  $w$  is a constant for each cosmological component.

$$\frac{d\rho}{da} + \frac{3}{a}(1+w)\rho = 0$$

$$\frac{d\rho}{\rho} = -3(1+w)\frac{da}{a}$$

$$\int_{\rho_i}^{\rho} \frac{d\rho'}{\rho'} = -3(1+w) \int_{a_i}^a \frac{da'}{a'}$$



$$\ln\left(\frac{\rho}{\rho_i}\right) = -3(1+w)\ln\left(\frac{a}{a_i}\right)$$

$$\frac{\rho}{\rho_i} = \left(\frac{a}{a_i}\right)^{-3(1+w)}$$

Choosing  $a_i = 1$  for present epoch ( $\rho_i = \rho_0$ ):

$$\boxed{\rho(a) = \rho_0 a^{-3(1+w)}}$$

## 13.9 Big Bang

### 13.9.1 Singularity

At the very beginning of the universe, all matter and energy were concentrated in a singularity, a point of infinite density and temperature. This singularity is thought to have contained all the space, time, and energy that would later expand to form the universe as we know it.

### 13.9.2 Cosmic Inflation

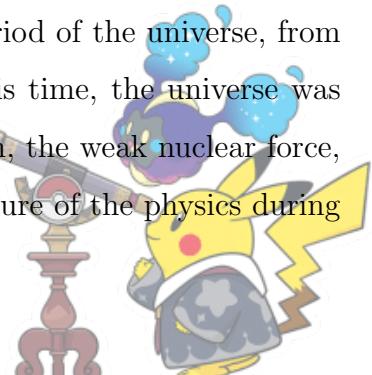
Cosmic inflation is a theory that explains the rapid expansion of the universe during the first fractions of a second after the Big Bang. During inflation, the universe expanded exponentially, increasing in size by a factor of at least  $10^{26}$  in a fraction of a second. This theory helps explain several observed features of the universe, such as its large-scale homogeneity and isotropy, and the distribution of galaxies.

### 13.9.3 Expansion of Space

The Big Bang theory proposes that space itself is expanding. This expansion is not into pre-existing space; rather, it is the stretching of space itself. As space expands, the distances between distant galaxies increase, leading to the observed redshift of light from those galaxies. This expansion of the universe is described mathematically by the **Friedmann equations**.

### 13.9.4 Phases of the Big Bang

**Planck Era (0 to  $10^{-43}$  seconds)** The Planck era represents the earliest period of the universe, from time  $t = 0$  up to approximately  $10^{-43}$  seconds after the Big Bang. During this time, the universe was incredibly hot and dense, and the fundamental forces (gravity, electromagnetism, the weak nuclear force, and the strong nuclear force) were likely unified in a single force. The exact nature of the physics during this period is unknown, as quantum gravity has yet to be fully understood.



**Grand Unification Era (from  $10^{-43}$  to  $10^{-36}$  seconds)** During the Grand Unification era, the fundamental forces separated. At the highest energies, the strong, weak, and electromagnetic forces were unified into a single force. As the universe cooled, these forces separated and the strong force, responsible for holding atomic nuclei together, emerged.

**Inflationary Era (from  $10^{-36}$  to  $10^{-32}$  seconds)** The universe underwent a brief period of exponential expansion during the inflationary era. During this time, the universe expanded by a factor of at least  $10^{26}$  in a fraction of a second. This rapid inflation smoothed out the universe and led to the large-scale homogeneity and isotropy observed today. It also helped set the initial conditions for the formation of the first particles.

**Quark Era (from  $10^{-12}$  to  $10^{-6}$  seconds)** As the universe continued to cool, quarks, electrons, and other fundamental particles began to form. During the quark era, quarks combined to form protons and neutrons. The temperature and energy were still high enough for these particles to interact and decay frequently.

**Hadron Era (from  $10^{-6}$  seconds to 1 second)** At around  $10^{-6}$  seconds after the Big Bang, the temperature dropped enough for quarks to combine into hadrons, such as protons and neutrons. This era marked the formation of the first stable atomic nuclei.

**Lepton Era (from 1 second to 10 seconds)** During the lepton era, the universe was dominated by leptons (such as electrons and neutrinos). These particles were created and annihilated in large quantities. Neutrinos, which were created in abundance, decoupled from the rest of matter at around 10 seconds.

**Photon Era (from 10 seconds to 380,000 years)** As the universe continued to cool, photons dominated the universe. At this time, matter and radiation were tightly coupled. The universe was opaque because free electrons scattered photons, preventing light from traveling freely. However, the universe continued to expand and cool, and at about 380,000 years after the Big Bang, the universe had cooled enough for atoms to form and photons to travel freely, leading to the decoupling of matter and radiation. This event is known as the **recombination epoch** and is associated with the cosmic microwave background (CMB) radiation, which we observe today.

**Recombination and the Formation of Atoms (380,000 years)** During recombination, the universe cooled enough for protons and electrons to combine and form neutral hydrogen atoms. This allowed



photons to travel freely through space, marking the beginning of the era of **decoupling**. The release of these photons is known as the CMB radiation.

**Dark Ages (380,000 years to 1 billion years)** After the formation of atoms, the universe entered the "dark ages," a period in which there were no stars or galaxies. During this time, the universe continued to cool and matter began to clump together due to gravitational attraction. However, it was not until the formation of the first stars and galaxies that the universe became more active.

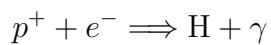
**Reionization (1 billion years to 2 billion years)** Reionization occurred when the first stars and galaxies formed and emitted ultraviolet light that reionized the hydrogen gas. This process ended the dark ages and allowed the universe to become transparent to ultraviolet light.

**The Modern Universe (Present Day)** Since the era of reionization, the universe has continued to expand and evolve. Galaxies, clusters of galaxies, and large-scale structures have formed over billions of years. The observable universe is currently about 93 billion light-years in diameter.

### 13.10 Cosmic Microwave Background

The Cosmic Microwave Background (CMB) is the oldest light in the universe, originating approximately 380,000 years after the Big Bang. It provides a snapshot of the infant universe when it transitioned from an opaque plasma to a transparent gas.

The CMB originates from the recombination epoch when the universe cooled sufficiently for electrons and protons to combine into neutral hydrogen atoms:



The cosmic microwave background is a faint radiation that fills the universe and is a remnant of the early hot, dense phase of the universe. It was first detected by Penzias and Wilson in 1965 and is often considered the strongest evidence for the Big Bang.



## 13.11 Gravitational Lensing

### 13.11.1 Introduction

Gravitational lensing is the bending of light by mass according to general relativity. A mass distribution between a distant source and an observer deflects light rays, producing phenomena such as multiple images, magnification, and distortion of the source.

For a point mass  $M$  (in Schwarzschild metric), a light ray with impact parameter  $b$  is deflected by an angle in weak-field. Then, the deflection angle is

$$\hat{\alpha}(b) = \frac{4GM}{bc^2}$$

where  $G$  is the gravitational constant and  $c$  the speed of light.

We use the thin-lens approximation: the deflection happens effectively at a single lens plane located at angular diameter distance  $D_L$  from the observer. Similarly, the source is at distance  $D_S$  and the distance between lens and source is  $D_{LS}$ . Let  $\beta$  be the true angular position of the source on the sky (unlensed), and  $\theta$  the observed angular position of the image. Then

$$\beta = \theta - \alpha(\theta)$$

where the scaled deflection angle is

$$\alpha(\theta) = \frac{D_{LS}}{D_S} \hat{\alpha}(D_L \theta)$$

For a point mass  $M$  located on the optical axis, the scalar form of the lens equation (assuming alignment along a single axis) becomes

$$\beta = \theta - \frac{D_{LS}}{D_S} \frac{4GM}{c^2 D_L \theta}$$

When the source, lens and observer are perfectly aligned ( $\beta = 0$ ) the image forms a ring (Einstein ring) with angular radius  $\theta_E$  solving:

$$\theta_E = \sqrt{\frac{4GM}{c^2} \frac{D_{LS}}{D_L D_S}}$$

### 13.11.2 Derivation using Newtonian Mechanics

In Newtonian gravity, the force on a particle of mass  $m$  at distance  $r$  from  $M$  is

$$F = \frac{GMm}{r^2}.$$



Only the component perpendicular to the initial direction of motion contributes to the deflection. If the photon moves along the  $x$ -axis and passes the mass at distance  $b$ , then

$$r = \sqrt{x^2 + b^2}, \quad \sin \theta = \frac{b}{r}.$$

The transverse force is therefore

$$F_{\perp} = F \sin \theta = \frac{GMm}{r^2} \frac{b}{r} = \frac{GMmb}{(x^2 + b^2)^{3/2}}$$

The transverse acceleration is

$$a_{\perp} = \frac{F_{\perp}}{m} = \frac{GMb}{(x^2 + b^2)^{3/2}}$$

The photon moves approximately at constant speed  $c$ , so

$$x = ct, \quad dt = \frac{dx}{c}$$

The total change in transverse velocity is

$$\Delta v_{\perp} = \int_{-\infty}^{\infty} a_{\perp} dt = \frac{1}{c} \int_{-\infty}^{\infty} \frac{GMb}{(x^2 + b^2)^{3/2}} dx$$

Using the standard integral

$$\int_{-\infty}^{\infty} \frac{b dx}{(x^2 + b^2)^{3/2}} = \frac{2}{b}$$

we obtain

$$\Delta v_{\perp} = \frac{2GM}{bc}$$

For small deflections, the bending angle  $\alpha$  is approximately the ratio of the transverse velocity change to the speed of light:

$$\alpha \approx \frac{\Delta v_{\perp}}{c}$$

Hence,

$$\alpha_{\text{Newton}} = \frac{2GM}{bc^2}$$

General Relativity predicts a deflection angle

$$\alpha_{\text{GR}} = \frac{4GM}{bc^2}$$



which is exactly twice the Newtonian result. The additional factor arises from the curvature of space, which is absent in Newtonian gravity.

### 13.11.3 Derivation using General Relativity

#### Definition. 13.2: Lagrangian

The **Lagrangian**  $\mathcal{L}$  is defined as

$$\mathcal{L}(q, \dot{q}, t) = T(\dot{q}) - V(q)$$

where  $T$  is the kinetic energy and  $V$  is the potential energy.

#### Theorem. 13.3: Euler-Lagrange Equation

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) = \frac{\partial \mathcal{L}}{\partial q}$$

#### Theorem. 13.4: Fermat's Principle

For light propagating in a medium with refractive index  $n(\mathbf{r})$ , the time taken to travel along a path  $\mathcal{C}$  from point  $A$  to point  $B$  is

$$T = \int_A^B dt = \int_A^B \frac{ds}{v} = \frac{1}{c} \int_A^B n(\mathbf{r}) ds$$

where

- $v = c/n$  is the speed of light in the medium
- $ds = \sqrt{dx^2 + dy^2 + dz^2}$  is the infinitesimal path length
- $c$  is the speed of light in vacuum

Fermat's principle states that the actual path  $\mathcal{C}$  minimizes (or more generally, makes stationary) the **optical path length**:

$$\mathcal{S} = \int_A^B n(\mathbf{r}) ds$$



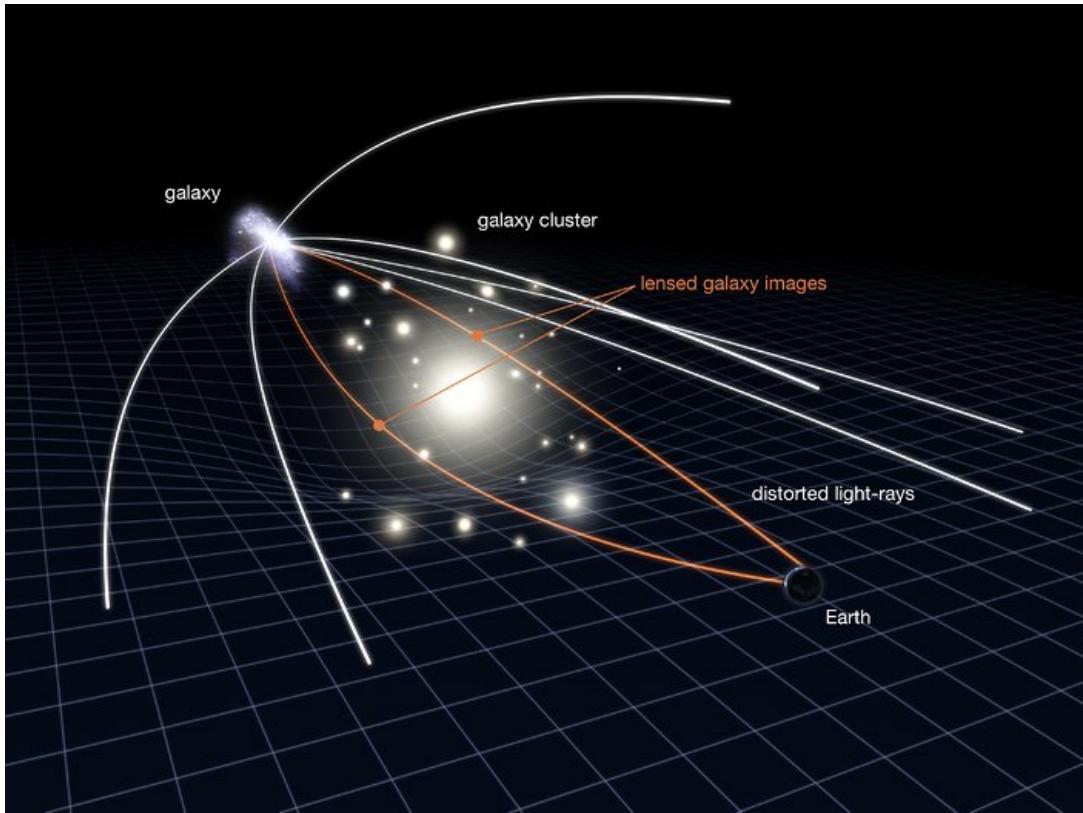


Figure 11: Source: <https://esahubble.org/images/heic1106c/>

In weak-field approximation of the spacetime metric (which is a situation in which the gravitational field is relatively weak and the spacetime curvature is small),

$$ds^2 = - \left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 + \left(1 - \frac{2\Phi}{c^2}\right) \delta_{ij} dx^i dx^j + \mathcal{O}\left(\frac{\Phi^2}{c^4}\right)$$

where  $\Phi = -GM/r$  and  $|\Phi| \ll c^2$ . Set  $c = 1$ .

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)(dx^2 + dy^2 + dz^2) = 0 \implies v \approx 1 + 2\Phi$$

Consider a nearly straight path in the  $x$ - $y$  plane with small deflection. Let  $y = y(x)$ ,  $z = 0$ , with  $|y'| \ll 1$ . Then

$$dl = \sqrt{1 + (y')^2} dx \approx \left[1 + \frac{1}{2}(y')^2\right] dx$$

The optical path length is

$$\begin{aligned} S &= \int n dl \approx \int [1 - 2\Phi(x, y)] \left[1 + \frac{1}{2}(y')^2\right] dx \\ &\approx \int \left[1 - 2\Phi + \frac{1}{2}(y')^2\right] dx \end{aligned}$$



Consider

$$L = \frac{1}{2}(y')^2 - 2\Phi(x, y)$$

$$\frac{d}{dx} \left( \frac{\partial L}{\partial y'} \right) = \frac{\partial L}{\partial y} \implies \frac{d}{dx}(y') = -2 \frac{\partial \Phi}{\partial y} \implies \frac{d^2 y}{dx^2} = -2 \frac{\partial \Phi}{\partial y}$$

Let  $y(x) = b + \varepsilon(x)$  with  $\varepsilon$  small:

$$\varepsilon''(x) = -2 \frac{\partial \Phi}{\partial y}(x, b)$$

The total deflection angle is:

$$\alpha = \Delta y' = \varepsilon'(+\infty) - \varepsilon'(-\infty) = \int_{-\infty}^{\infty} \varepsilon''(x) dx$$

For a point mass  $\Phi = -GM/r$  with  $r = \sqrt{x^2 + y^2}$ :

$$\frac{\partial \Phi}{\partial y} = GM \frac{y}{(x^2 + y^2)^{3/2}}$$

At  $y = b$ :

$$\alpha = -2GMb \int_{-\infty}^{\infty} \frac{dx}{(x^2 + b^2)^{3/2}}$$

Note that

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 + b^2)^{3/2}} = \frac{2}{b^2}$$

Therefore,

$$\alpha = -2GMb \cdot \frac{2}{b^2} = -\frac{4GM}{b}$$

The magnitude of the deflection (toward the mass) is

$$|\alpha| = \frac{4GM}{b}$$

In the metric,  $\Phi \rightarrow \Phi/c^2$ . Then

$$\boxed{\alpha = \frac{4GM}{bc^2}}$$



## 13.12 Gravitational Wave

### 13.12.1 Introduction

Gravitational waves are disturbances in the curvature of spacetime caused by accelerated masses. They were first predicted by Albert Einstein in 1916 as a consequence of his General Relativity theory. Unlike electromagnetic waves, gravitational waves interact weakly with matter, making them challenging to detect but allowing them to carry information about cataclysmic cosmic events.

### 13.12.2 Chirp Mass

For a binary system with component masses  $m_1$  and  $m_2$ , the chirp mass is

$$\mathcal{M} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}} = \frac{c^3}{G} \left( \frac{5}{96} \pi^{-8/3} f^{-11/3} \dot{f} \right)^{3/5}$$

where  $f$  is the orbital frequency.

### 13.12.3 Binary System

For a binary system with masses  $m_1$  and  $m_2$ , and orbital frequency  $f_{\text{orb}}$ , the gravitational wave luminosity is

$$L_{GW} = \frac{32}{5Gc^5} (2\pi G \mathcal{M} f_{\text{orb}})^{10/3}$$

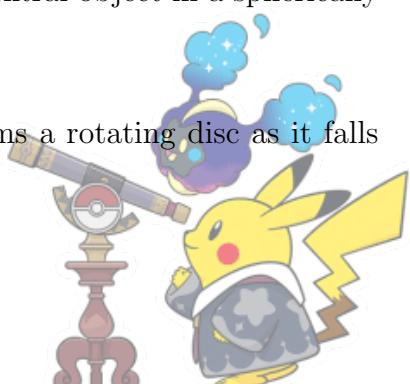
where  $r$  is the orbital separation.

## 13.13 Accretion Processes

### 13.13.1 Introduction

Accretion is the process by which matter falls onto a central object, such as a star, black hole, or neutron star, under the influence of gravity.

- Spherical accretion occurs when matter falls radially inward toward a central object in a spherically symmetric manner.
- Disc accretion occurs when matter, due to its angular momentum, forms a rotating disc as it falls toward a central object.



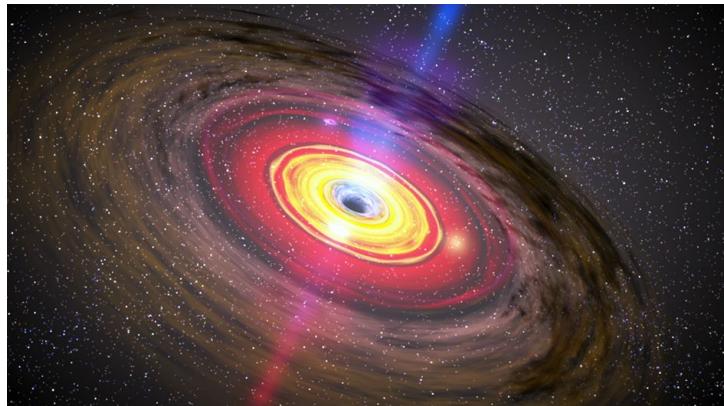


Figure 12: Source: <https://www.skyatnightmagazine.com/space-science/accretion-disk>

### 13.13.2 Eddington Luminosity

Consider a spherical surface with radius  $r$  centered around a light source, where the total luminosity of the source is  $L$ . The energy flux (the energy per unit time passing through a unit area) at a distance  $r$  from the source is given by the total luminosity divided by the surface area of a sphere with radius  $r$ . The surface area of a sphere is

$$A = 4\pi r^2$$

Hence, the energy flux at distance  $r$  is:

$$F = \frac{L}{4\pi r^2}$$

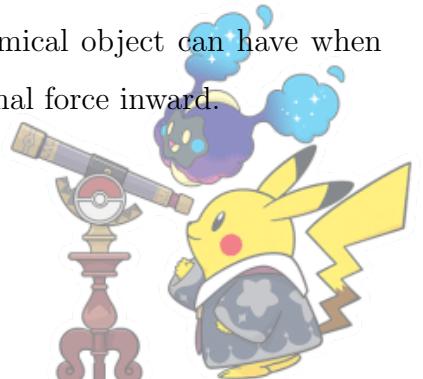
Now, consider the nature of radiation pressure. Photons carry momentum, and when they strike a surface, they transfer momentum to it. The radiation pressure is related to the energy flux by the relationship:

$$E = pc \implies P_{\text{rad}} = \frac{1}{A} \frac{dp}{dt} = \frac{F}{c}$$

where  $c$  is the speed of light, and this formula assumes that the radiation is isotropic and that the photon momentum transfer is fully efficient in transferring momentum to the surface. Then

$$P_{\text{rad}} = \frac{1}{c} \cdot \frac{L}{4\pi r^2} = \frac{L}{4\pi r^2 c}$$

The **Eddington luminosity**,  $L_{\text{Edd}}$ , is the maximum luminosity an astronomical object can have when there is a balance between the radiation pressure outward and the gravitational force inward.



The radiation pressure on an object is given by

$$P_{\text{rad}} = \frac{L}{4\pi r^2 c}$$

where  $L$  is the luminosity,  $r$  is the radius of the object, and  $c$  is the speed of light. The gravitational force is given by

$$F_{\text{grav}} = \frac{GMm}{r^2}$$

where  $M$  is the mass of the central object,  $m$  is the mass of the accreting material, and  $G$  is the gravitational constant. When balanced,

$$\frac{L}{4\pi r^2 c} = \frac{GMm}{r^2}$$

Simplifying this equation:

$$L = \frac{4\pi GMmc}{r^2}$$

For the Eddington luminosity, we consider the maximum luminosity for the material to remain bound to the central object without being blown away by radiation pressure. Using the fact that the material consists of hydrogen, for which the mass of an electron is  $m_e$  and the Thomson scattering cross-section is  $\sigma_T$ , we can calculate the Eddington luminosity as follows:

$$L_{\text{Edd}} = \frac{4\pi GMm_e c}{\sigma_T}$$

where  $\sigma_T \approx 6.65 \times 10^{-25} \text{ m}^2$  is the effective area that quantifies the likelihood of an electron scattering a photon through Thomson scattering.

## 13.14 Cosmic Distance Ladder

### 13.14.1 Introduction

The cosmic distance ladder is a succession of methods by which astronomers determine the distances to celestial objects. Each rung of the ladder provides information that allows calibration of the next method, enabling measurements from the Solar System to the edge of the observable universe.



## The Cosmic Distance Ladder

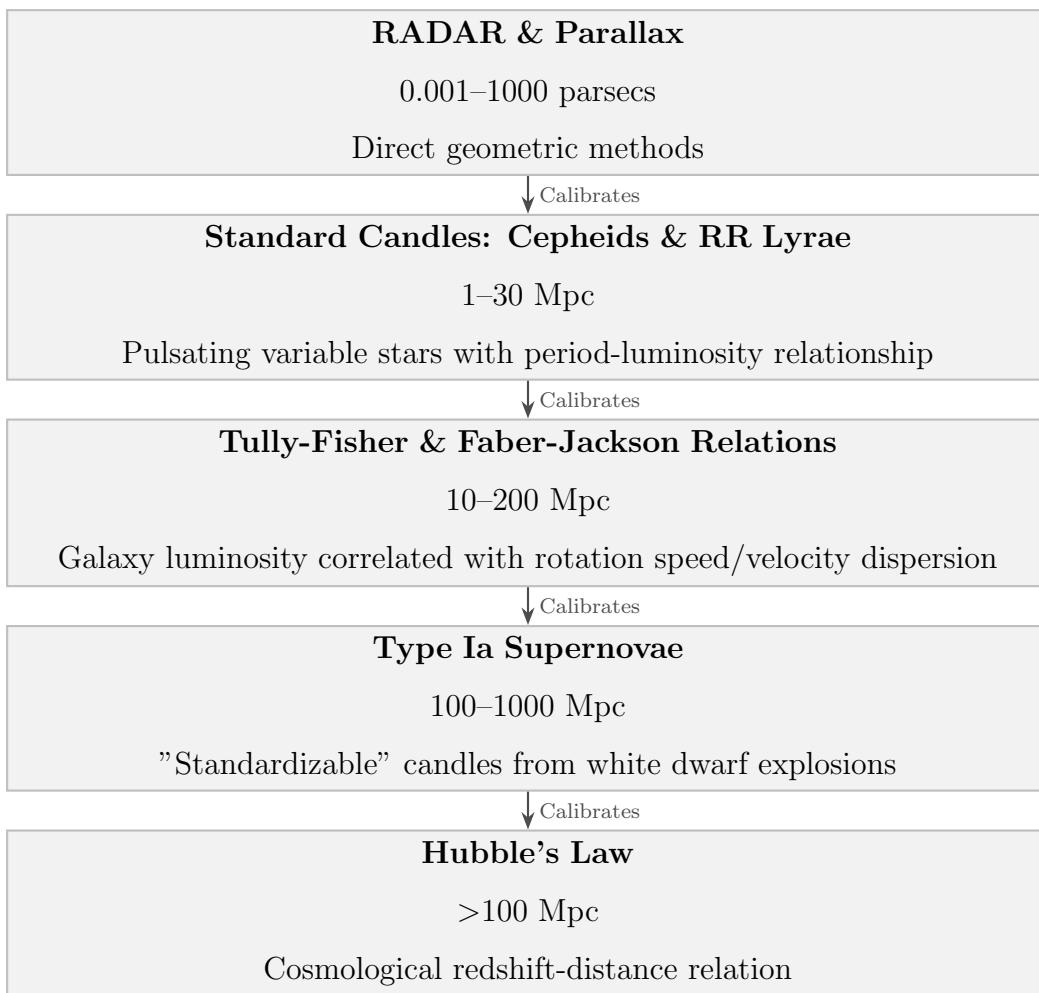


Figure 13: The hierarchy of distance measurement techniques in astronomy. Each rung calibrates the next.

### 13.14.2 Radar Ranging

For objects within our Solar System, we can use radar to measure distances directly:

- Transmit radio waves toward a planet or asteroid
- Measure time delay for echo to return:  $\Delta t$
- Distance:  $d = \frac{c \cdot \Delta t}{2}$  where  $c$  is the speed of light
- Limited to  $\sim 10$  AU (within Solar System)



### 13.14.3 Stellar Parallax

Parallax uses Earth's orbit as a baseline to measure distances to nearby stars:

$$d(\text{parsecs}) = \frac{1}{\theta(\text{arcseconds})}$$

where

- $d$  = distance in parsecs (1 pc  $\approx$  3.26 light-years)
- $\theta$  = parallax angle in arcseconds
- Baseline = 1 AU (astronomical unit)

### 13.14.4 Standard Candles: Cepheid Variables

Cepheid variables are pulsating stars<sup>3</sup> whose period correlates with luminosity:

$$M = a \cdot \log_{10}(P) + b$$

where

- $M$  is the absolute magnitude (intrinsic brightness),
- $P$  is the pulsation period in days, and
- $a, b$  are the calibration constants.

Once the absolute magnitude  $M$  is known from the period, the distance modulus formula gives the distance:

$$m - M = 5 \log_{10} \left( \frac{d}{10 \text{ pc}} \right)$$

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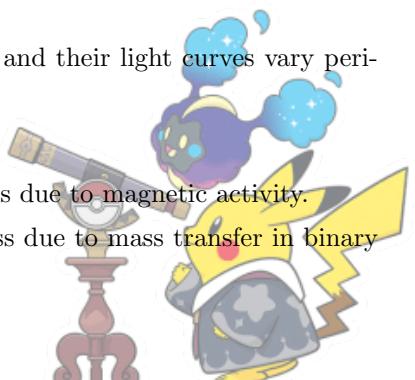
<sup>3</sup>Stellar variability is classified into two broad categories:

#### 1. Regular Variability:

- **Pulsating Stars:** These stars, such as Cepheid variables, exhibit periodic changes in brightness due to expansions and contractions of their outer layers.
- **Eclipsing Binaries:** These systems consist of two stars orbiting each other, and their light curves vary periodically due to one star eclipsing the other.

#### 2. Irregular Variability:

- **Flare Stars:** These stars exhibit sudden, unpredictable increases in brightness due to magnetic activity.
- **Cataclysmic Variables:** These stars experience large variations in brightness due to mass transfer in binary systems.



where

- $m$  is the apparent magnitude (observed brightness),
- $M$  is the absolute magnitude, and
- $d$  is the distance in parsecs.

### 13.14.5 Faber-Jackson Relations

For elliptical galaxies, the velocity dispersion correlates with luminosity:

$$L \propto \sigma^\beta$$

where

- $\sigma$  is the stellar velocity dispersion
- $\beta \approx 4$  empirically

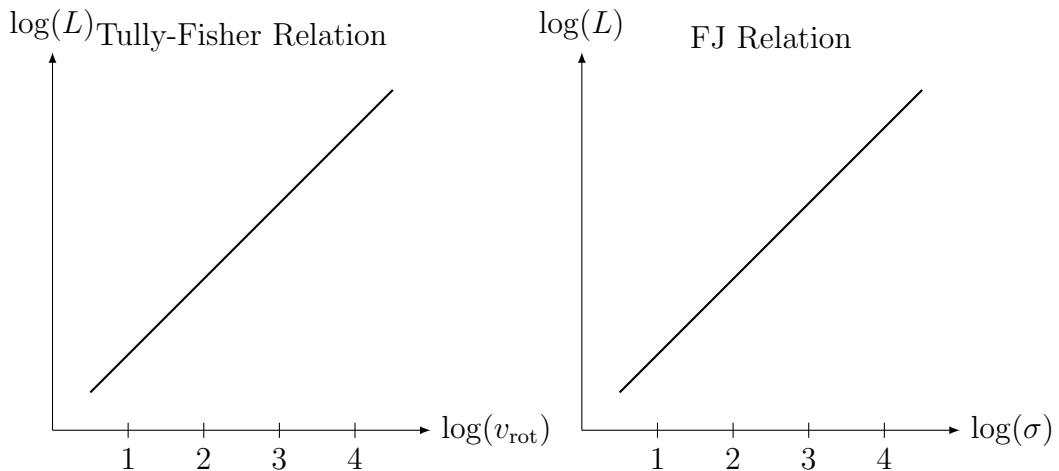


Figure 14: The Tully-Fisher and Faber-Jackson relations allow estimation of galaxy distances from measurable kinematic properties.

### 13.14.6 Type Ia Supernovae

Type Ia supernovae are extremely luminous and serve as excellent "standardizable candles":

- Result from thermonuclear explosion of white dwarf reaching Chandrasekhar limit ( $\sim 1.4 M_\odot$ )
- Peak luminosity:  $\sim 10^{10} L_\odot$  (as bright as entire galaxy)



- Can be observed at cosmological distances
- Light curve shape correlates with peak luminosity (Phillips relationship)

The distance is calculated from

$$\mu = m_B - M_B + \alpha(s - 1) - \beta c$$

where

- $\mu$  is the distance modulus
- $m_B$  is the apparent magnitude in B-band
- $M_B$  is the absolute magnitude (calibrated)
- $s$  is the light curve shape parameter
- $c$  is the color correction
- $\alpha, \beta$  are the calibration coefficients



# 14 Interstellar Medium

## 14.1 Introduction

The **interstellar medium** refers to the matter that exists in the space between stars within a galaxy. It is composed of gas, dust, and cosmic rays.

It can be categorized into several distinct phases based on the temperature and density of the material:

- **Neutral Gas:** Consists of neutral hydrogen ( $H$ ) and molecular hydrogen ( $H_2$ ), often found in molecular clouds.
- **Ionized Gas:** Composed of ionized hydrogen ( $H^+$ ) and other ionized elements, typically found in regions such as HII regions.
- **Dust:** Microscopic solid particles that can range in size from nanometers to microns, contributing to the absorption and scattering of light.
- **Cosmic Rays:** High-energy particles, primarily protons and atomic nuclei, that travel through the ISM.

Some important regions within the interstellar medium include:

- **HII Regions:** These are regions of ionized hydrogen, created by the ultraviolet radiation from young, hot stars. They are often observed in emission lines such as  $H\alpha$ .
- **Molecular Clouds:** These are cold, dense regions of the interstellar medium where molecules such as  $H_2$  are found. They are often the birthplaces of stars.
- **Warm Ionized Medium:** A diffuse component of the interstellar medium, where the gas is partially ionized and has a temperature around  $10^4$  K.

## 14.2 Fluid Dynamics

### 14.2.1 Stress Tensor

In continuum mechanics, the **stress tensor  $\sigma$**  is a fundamental concept that describes the internal forces acting within a material or fluid.



For a general three-dimensional space, the stress tensor  $\sigma$  is a  $3 \times 3$  matrix, with components  $\sigma_{ij}$  where  $i$  and  $j$  refer to the directions (or axes) in space:

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

where

- $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$  are the normal stress components in the  $x, y$ , and  $z$  directions.
- $\sigma_{xy}, \sigma_{xz}, \sigma_{yz}$  are the shear stress components.
- The matrix is symmetric:  $\sigma_{ij} = \sigma_{ji}$ .

### 14.2.2 Tensor Product

#### Definition. 14.1: Tensor Product

Let  $V$  and  $W$  be vector spaces over a field  $\mathbb{F}$ . The **tensor product**  $V \otimes W$  is a vector space equipped with a bilinear map

$$\otimes : V \times W \longrightarrow V \otimes W$$

satisfying the following universal property: For every bilinear map  $\phi : V \times W \longrightarrow U$  to any vector space  $U$ , there exists a unique linear map  $\tilde{\phi} : V \otimes W \longrightarrow U$  such that the following diagram commutes:

$$\begin{array}{ccc} V \times W & \xrightarrow{\otimes} & V \otimes W \\ & \searrow \phi & \downarrow \tilde{\phi} \\ & & U \end{array}$$

Tensor product of two vectors  $\mathbf{A}$  and  $\mathbf{B}$  is given by

$$(\mathbf{A} \otimes \mathbf{B})_{ij} = A_i B_j$$



### 14.2.3 Divergence Theorem

**Theorem. 14.1: Divergence Theorem**

$$\int_{\partial V} \mathbf{F} \cdot \mathbf{n} dS = \int_V \nabla \cdot \mathbf{F} dV$$

where

- $\mathbf{F}$  is a vector field,
- $V$  is the volume enclosed by the surface  $\partial V$ ,
- $\mathbf{n}$  is the outward-pointing unit normal vector on the surface  $\partial V$ ,
- $dS$  is the surface area element on  $\partial V$ ,

### 14.2.4 Continuity Equation and Momentum Equation

Let  $\rho(\mathbf{x}, t)$  denote the mass density and  $\mathbf{v}(\mathbf{x}, t)$  the velocity field of a fluid. The total mass within a fixed control volume  $V$  with boundary  $\partial V$  and outward unit normal  $\mathbf{n}$  is

$$M_V(t) = \int_V \rho dV.$$

The principle of mass conservation states that the rate of change of mass within  $V$  equals the negative of the net mass flux through the boundary:

$$\frac{d}{dt} \int_V \rho dV = - \int_{\partial V} \rho \mathbf{v} \cdot \mathbf{n} dS$$

As  $V$  is fixed in space,

$$\int_V \frac{\partial \rho}{\partial t} dV = - \int_{\partial V} \rho \mathbf{v} \cdot \mathbf{n} dS$$

By the divergence theorem,

$$\int_V \frac{\partial \rho}{\partial t} dV = - \int_V \nabla \cdot (\rho \mathbf{v}) dV$$

Hence,

$$\int_V \left( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right) dV = 0$$



As  $V$  is arbitrary, the integrand must vanish everywhere and

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0}$$

which is the local form of continuity equation.

By the material derivative  $D/Dt = \partial/\partial t + \mathbf{v} \cdot \nabla$ ,

$$\boxed{\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0}$$

Consider the momentum of fluid within a fixed control volume  $V$ :

$$\int_V \rho \mathbf{v} dV$$

$$\frac{d}{dt} \int_V \rho \mathbf{v} dV = - \int_{\partial V} \rho \mathbf{v} (\mathbf{v} \cdot \mathbf{n}) dS + \int_{\partial V} \boldsymbol{\sigma} \cdot \mathbf{n} dS + \int_V \rho \mathbf{f} dV$$

where

- $\boldsymbol{\sigma}$  is the stress tensor,
- $\mathbf{f}$  is the body force per unit mass (e.g. gravity).

Using the divergence theorem,

$$\int_{\partial V} \rho \mathbf{v} (\mathbf{v} \cdot \mathbf{n}) dS = \int_V \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) dV, \quad \int_{\partial V} \boldsymbol{\sigma} \cdot \mathbf{n} dS = \int_V \nabla \cdot \boldsymbol{\sigma} dV$$

Therefore,

$$\int_V \left( \frac{\partial}{\partial t} (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) - \nabla \cdot \boldsymbol{\sigma} - \rho \mathbf{f} \right) dV = 0$$

As  $V$  is arbitrary,

$$\boxed{\frac{\partial}{\partial t} (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) = \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{f}}$$

which is the **Cauchy momentum equation in conservation form**. Using the continuity equation, it can be rewritten in material-derivative form:

$$\boxed{\rho \frac{D \mathbf{v}}{Dt} = \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{f}}$$



### 14.2.5 Euler's Equation (Special Case of Navier-Stokes Equation)

From

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) = \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{f}$$

where  $\rho$  is the density of the fluid,  $\mathbf{v}$  is the velocity of the fluid,  $\boldsymbol{\sigma}$  is the stress tensor, and  $\mathbf{f}$  represents external body forces per unit mass.

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) = \rho \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \frac{\partial \rho}{\partial t}$$

$$\nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) = \mathbf{v} \cdot \nabla(\rho \mathbf{v}) = \mathbf{v} \cdot \nabla(\rho) \mathbf{v} + \rho \mathbf{v} \cdot \nabla \mathbf{v}$$

For an inviscid fluid (no viscosity), the stress tensor only contains the pressure term:

$$\boldsymbol{\sigma} = -p \mathbf{I}$$

where  $p$  is the pressure and  $\mathbf{I}$  is the identity matrix. The divergence of the stress tensor is

$$\nabla \cdot \boldsymbol{\sigma} = -\nabla p$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho \mathbf{v} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \rho \mathbf{f}$$

For an incompressible flow (constant  $\rho$ ):

$$\frac{\partial \rho}{\partial t} = 0 \quad \text{and} \quad \nabla \rho = 0.$$

This implies that the terms  $\mathbf{v} \frac{\partial \rho}{\partial t}$  and  $\mathbf{v} \cdot \nabla \rho \mathbf{v}$  vanish. Hence,

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \rho \mathbf{f}$$

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p + \mathbf{f}$$



## 14.3 Case Study

### 14.3.1 Background

The interstellar medium consists of gas and dust at various densities and temperatures, organized into molecular clouds, atomic gas, and ionized phases. Gravitational collapse occurs when self-gravity overcomes pressure support, leading to star formation. The interstellar medium exhibits a wide range of densities:

Phase	Density, $\rho$ (g cm $^{-3}$ )	Collapse Timescale, $t_{\text{ff}}$
Molecular Cloud (diffuse)	$4.2 \times 10^{-22}$	$\sim 10^7$ yr
Molecular Cloud (dense core)	$4.2 \times 10^{-20}$	$\sim 10^6$ yr
Atomic HI Cloud	$4.2 \times 10^{-24}$	$\sim 10^8$ yr
HII Region	$4.2 \times 10^{-25}$	$\sim 10^9$ yr

### 14.3.2 Derivation

- Continuity equation (mass conservation):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

- Euler equation (momentum conservation):

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla p - \nabla \Phi$$

- Poisson equation for the gravitational potential where  $\nabla^2 f = \nabla \cdot (\nabla f)$  and  $\nabla^2 \Phi = f$ :

$$\nabla^2 \Phi = 4\pi G \rho$$

Consider an uniform background:

$$\rho = \rho_0 + \delta\rho, \quad \mathbf{v} = \delta\mathbf{v}, \quad p = p_0 + \delta p, \quad \Phi = \Phi_0 + \delta\Phi$$

To isolate pure gravitational collapse, we consider the pressure-less limit:



- Linearized continuity equation:

$$\frac{\partial \delta \rho}{\partial t} + \rho_0 \nabla \cdot \delta \mathbf{v} = 0$$

- Linearized Euler equation (no pressure):

$$\frac{\partial \delta \mathbf{v}}{\partial t} = -\nabla \delta \Phi.$$

- Linearized Poisson equation:

$$\nabla^2 \delta \Phi = 4\pi G \delta \rho$$

Therefore,

$$\frac{\partial}{\partial t} (\nabla \cdot \delta \mathbf{v}) = -\nabla^2 \delta \Phi$$

and then

$$\frac{\partial}{\partial t} (\nabla \cdot \delta \mathbf{v}) = -4\pi G \delta \rho$$

Also,

$$\frac{\partial^2 \delta \rho}{\partial t^2} + \rho_0 \frac{\partial}{\partial t} (\nabla \cdot \delta \mathbf{v}) = 0$$

Finally,

$$\frac{\partial^2 \delta \rho}{\partial t^2} + \rho_0 [-4\pi G \delta \rho] = 0$$

Hence,

$$\frac{\partial^2 \delta \rho}{\partial t^2} = 4\pi G \rho_0 \delta \rho$$

which is in the form of  $\ddot{\delta \rho} = \omega^2 \delta \rho$  with

$$\omega^2 \equiv 4\pi G \rho_0 > 0$$

The  $e^{\omega t}$  mode represents exponential growth of density perturbations which represent gravitational collapse.

The  $e^{-\omega t}$  mode decays and is typically not physically relevant for collapse initial conditions. The **e-folding time** (characteristic growth/collapse timescale) is

$$\tau \equiv \frac{1}{\omega} = \frac{1}{\sqrt{4\pi G \rho_0}} \sim \frac{1}{\sqrt{G \rho_0}}$$



## 15 Study of the Earth

### 15.1 Tides

Tides are the periodic rise and fall of sea levels caused by the gravitational forces exerted by the Moon and the Sun on Earth.

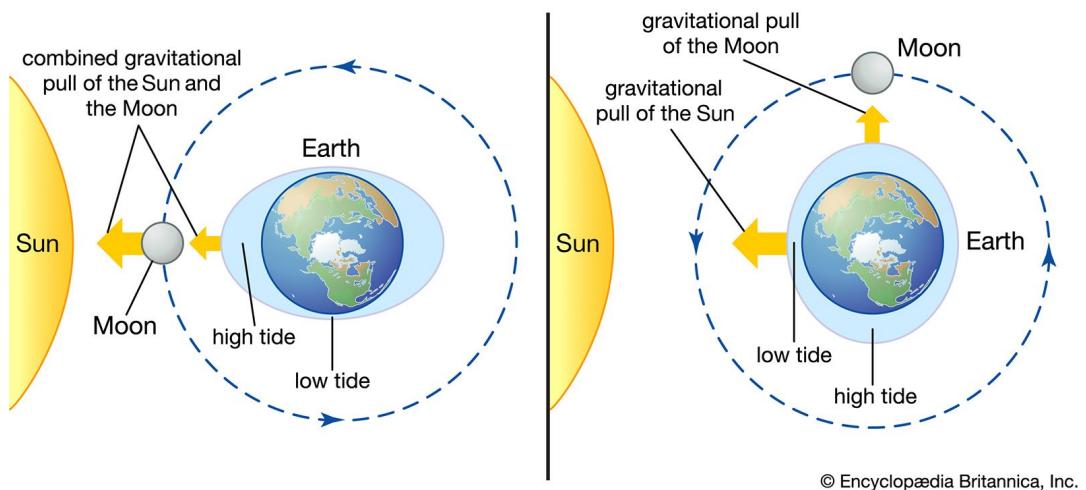
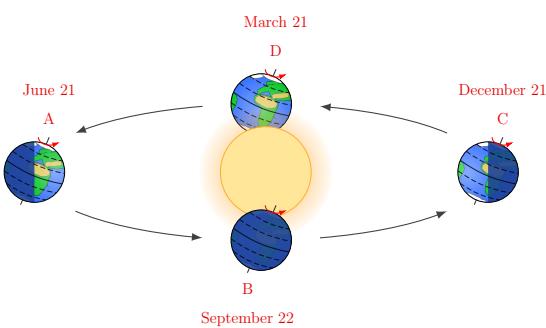


Figure 15: Source: <https://www.britannica.com/science/tide>

### 15.2 Seasons

Seasons are caused by the tilt of Earth's axis ( $23.5^\circ$ ) relative to its orbit around the Sun.

- Summer occurs in the hemisphere tilted toward the Sun.
- Winter occurs in the hemisphere tilted away from the Sun.
- Spring and Autumn occur when neither hemisphere is tilted toward the Sun.



## 15.3 Factors Influencing Climate

Climate depends on several natural factors:

- Latitude: Distance from the equator affects temperature.
- Altitude: Higher altitudes are cooler.
- Ocean Currents: Warm or cold currents can affect coastal climates.
- Topography: Mountains can block winds and affect rainfall.
- Human Activities: Urbanization and greenhouse gases influence climate.

## 15.4 Eclipses

### 15.4.1 Solar Eclipse

It occurs when the Moon comes between the Earth and Sun, blocking sunlight.

### 15.4.2 Lunar Eclipse

It occurs when the Earth comes between the Sun and Moon, casting a shadow on the Moon.

## 15.5 Space Weather

### 15.5.1 Introduction

Space weather refers to the dynamic conditions in Earth's space environment, primarily influenced by the Sun.

### 15.5.2 Solar Wind

The solar wind is a stream of charged particles released from the upper atmosphere of the Sun, called the corona. It affects the Earth's magnetosphere and can disrupt satellite communications.

### 15.5.3 Solar Flares

Solar flares are sudden bursts of radiation from the Sun. They release energy across the electromagnetic spectrum, which can impact radio communications and GPS systems on Earth.



### 15.5.4 Coronal Mass Ejections (CMEs)

CMEs are massive bursts of solar wind and magnetic fields rising above the solar corona. They can trigger geomagnetic storms that affect satellites, power grids, and auroras.

### 15.5.5 Aurorae

Aurorae (Northern and Southern Lights) are caused by charged particles from the Sun interacting with Earth's magnetic field and atmosphere.

## 15.6 Meteor Showers

Meteor showers occur when Earth passes through the debris left by a comet.

## 15.7 Equinoxes

An **equinox** occurs twice a year, when the Sun crosses the celestial equator. On these dates, day and night are approximately equal in length at all latitudes. The two equinoxes are:

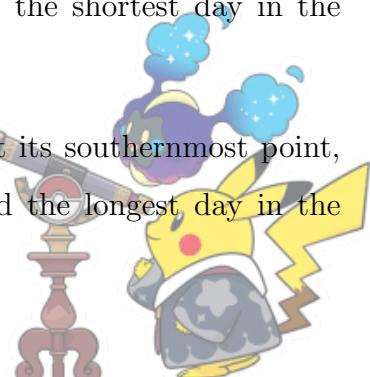
- **Vernal Equinox:** Occurs around March 20th or 21st, marking the start of spring in the northern hemisphere.
- **Autumnal Equinox:** Occurs around September 22nd or 23rd, marking the start of autumn in the northern hemisphere.

At the equinoxes, the Sun rises directly in the east and sets directly in the west.

## 15.8 Solstices

A **solstice** occurs twice a year when the Sun reaches its highest or lowest point in the sky at noon, relative to the celestial equator. This results in the longest and shortest days of the year.

- **Summer Solstice:** Occurs around June 21st or 22nd. The Sun is at its northernmost point, resulting in the longest day of the year in the northern hemisphere and the shortest day in the southern hemisphere.
- **Winter Solstice:** Occurs around December 21st or 22nd. The Sun is at its southernmost point, resulting in the shortest day of the year in the northern hemisphere and the longest day in the southern hemisphere.



## 15.9 Solar Declination

**Solar declination** is the angle between the rays of the Sun and the plane of the Earth's equator. It varies throughout the year, reaching  $+23.5^\circ$  during the summer solstice and  $-23.5^\circ$  during the winter solstice. At the equinoxes, the solar declination is  $0^\circ$ , meaning the Sun is directly above the equator.

# 16 Study of the Universe

## 16.1 Human Exploration of the Solar System

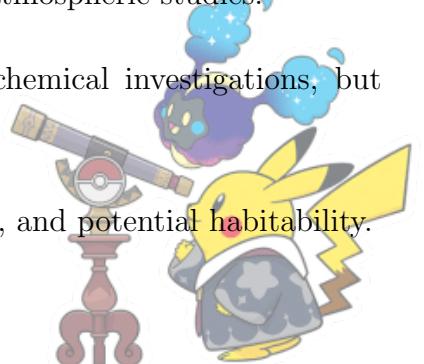
Human exploration of the Solar System involves sending astronauts beyond Earth to explore, study, and potentially inhabit other celestial bodies.

- **Purpose:** advancing scientific knowledge, developing space technology, inspiring societies, and ensuring long-term survival of humanity.
- **Major challenges:** long-duration exposure to microgravity, cosmic radiation, limited medical support, psychological isolation, and safe re-entry.
- **Key destinations:** the Moon for testing technologies, Mars for long-term exploration, and near-Earth asteroids for scientific and resource studies.
- **Approach:** step-by-step expansion using space stations, lunar missions, and sustainable life-support systems.

## 16.2 Planetary Missions

Planetary missions are robotic or crewed missions designed to explore planets, moons, asteroids, and comets.

- **Flyby missions:** provide brief but valuable observations with minimal fuel and mission complexity.
- **Orbiter missions:** allow long-term monitoring, global mapping, and atmospheric studies.
- **Landers and rovers:** enable direct surface analysis, geology, and chemical investigations, but require complex entry, descent, and landing systems.
- **Scientific value:** reveal planetary formation history, climate evolution, and potential habitability.



## 17 Study of the Moon

### 17.1 Precession

Precession is the slow, conical motion of the Earth's rotation axis caused primarily by the gravitational torque of the Sun and Moon on Earth's equatorial bulge.

$$\text{Rate of precession} \approx 50.3'' \text{ per year}$$

**Derivation.** Consider the Earth as an oblate spheroid with equatorial radius  $R_e$  and polar radius  $R_p$ . The Earth's equatorial bulge experiences a gravitational torque due to the Sun (or Moon):

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

The magnitude of the torque is proportional to the Earth's moment of inertia difference and the gravitational force:

$$\tau \approx \frac{3GM_s}{2r^3}(C - A) \sin 2\theta$$

where

- $G$  is the gravitational constant,
- $M_s$  is the mass of the Sun,
- $r$  is the Earth-Sun distance,
- $C$  and  $A$  are the Earth's principal moments of inertia about polar and equatorial axes respectively, and
- $\theta$  is the obliquity of the ecliptic ( $\approx 23.5^\circ$ ).

The precessional angular velocity  $\Omega_p$  is given by the ratio of the torque to the Earth's spin angular momentum  $L = C\omega$ :

$$\Omega_p = \frac{\tau}{C\omega} = \frac{3GM_s}{2r^3} \frac{C - A}{C\omega} \cos \theta$$

Here,  $\omega$  is the Earth's spin angular velocity. The  $\cos \theta$  factor appears due to the component of torque perpendicular to the spin axis. For an oblate Earth,

$$C - A = \frac{2}{5}M_e R_e^2 J_2$$



where

- $M_e$  is the Earth's mass,
- $J_2 \approx 1.0826 \times 10^{-3}$  is the dynamical flattening coefficient.

The precessional angular velocity becomes

$$\Omega_p = \frac{3GM_s}{2r^3} \cdot \frac{\frac{2}{5}M_eR_e^2J_2}{C\omega} \cos\theta$$

Using  $C \approx \frac{2}{5}M_eR_e^2$ , we simplify:

$$\Omega_p \approx \frac{3GM_s}{2r^3\omega} J_2 \cos\theta$$

Substitute the values

$$G = 6.674 \times 10^{-11} \text{ m}^3\text{kg}^{-1}\text{s}^{-2}$$

$$M_s = 1.989 \times 10^{30} \text{ kg}$$

$$r = 1.496 \times 10^{11} \text{ m}$$

$$\omega = 7.292 \times 10^{-5} \text{ rad/s}$$

$$J_2 = 1.0826 \times 10^{-3}$$

$$\theta = 23.5^\circ$$

$\text{Rate of precession} = \Omega_p \times 206264.8'' \times 3.156 \times 10^7 \text{ s/year} \approx 50.3''/\text{year}$

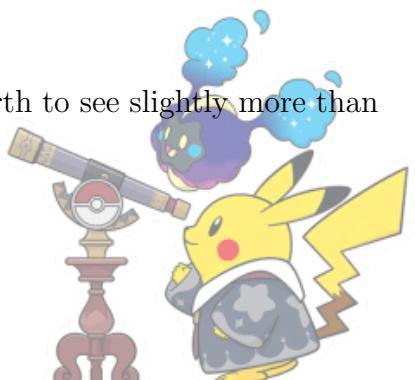
## 17.2 Nutation

Nutation refers to small periodic oscillations superimposed on the precessional motion. These are caused by the varying positions of the Moon and Sun relative to Earth, leading to deviations in the tilt of Earth's axis.

## 17.3 Libration

Libration is the apparent oscillation of the Moon that allows observers on Earth to see slightly more than half of its surface over time. There are three main types:

- **Longitudinal libration:** Due to the eccentricity of the Moon's orbit.



- **Latitudinal libration:** Due to the tilt of the Moon's axis relative to its orbital plane.
- **Diurnal libration:** Due to the rotation of the Earth and the observer's changing viewpoint.

## 18 Study of the Solar System

### 18.1 Formation

The Solar System formed about 4.6 billion years ago from a giant molecular cloud composed of gas and dust. The process can be divided into several stages.

#### 18.1.1 Nebular Hypothesis

The nebular hypothesis explains that the Solar System formed from a rotating disk of gas and dust. This cloud, known as the solar nebula, collapsed under its own gravity, leading to the formation of the Sun at its center and the planets from the remaining material. The key stages in the formation of the Solar System are as follows:P

1. **Collapse of the Solar Nebula:** The gas and dust cloud began to contract due to gravity. As it contracted, it started to rotate faster, forming a flat, rotating disk.
2. **Formation of the Sun:** At the center of the disk, the temperature and pressure increased, leading to nuclear fusion, which ignited the Sun.
3. **Accretion of Planets:** In the outer regions of the disk, dust and gas began to clump together to form planetesimals, which further collided and merged to form planets, moons, and other small bodies.
4. **Clearing the Nebula:** The young Sun's solar wind cleared away the remaining gas and dust, leaving behind the current structure of the Solar System.

#### 18.1.2 Differentiation and Evolution

After the initial formation, the Solar System underwent several evolutionary processes:

- **Differentiation:** The early planets were molten, and heavier materials sank toward their cores while lighter materials rose to the surface.



- **Late Heavy Bombardment:** During the early stages, the planets were frequently bombarded by leftover planetesimals, causing cratering on their surfaces.
- **Orbital Evolution:** Gravitational interactions between planets and other bodies in the Solar System led to changes in their orbits over time.

## 18.2 Structure and Components of the Solar System

The Solar System is composed of the Sun and all the objects that are bound by its gravitational field, including planets, moons, asteroids, comets, and other small bodies.

### 18.2.1 The Sun

The Sun is the central star of the Solar System, providing the gravitational force that holds the system together. It is composed primarily of hydrogen and helium and accounts for approximately 99.86% of the total mass of the Solar System. The Sun's core is where nuclear fusion occurs, generating the energy that powers the Sun and supports life on Earth.

### 18.2.2 The Planets

There are eight planets in the Solar System, divided into two main categories:

**Terrestrial Planets** These are the inner planets, composed mainly of rock and metal. They are

- Mercury
- Venus
- Earth
- Mars

**Gas Giants and Ice Giants** These are the outer planets, which are primarily composed of hydrogen, helium, and other gases. The gas giants include

- Jupiter
- Saturn

The ice giants, which have a higher proportion of ice (water, ammonia, methane), are:



- **Uranus**
- **Neptune**

### 18.2.3 Dwarf Planets and Small Bodies

The Solar System also contains numerous smaller bodies, including

- **Dwarf Planets:** These are similar to planets but do not clear their orbits of other objects. Examples include **Pluto**, **Ceres**, and **Eris**.
- **Asteroids:** Rocky bodies that primarily orbit between Mars and Jupiter in the asteroid belt.
- **Comets:** Icy bodies that often have highly elliptical orbits, and develop tails when they approach the Sun.
- **Meteoroids:** Small fragments of asteroids or comets that can enter Earth's atmosphere and cause meteor showers.

### 18.2.4 The Kuiper Belt and Oort Cloud

The outer regions of the Solar System are populated by icy bodies and dwarf planets:

- **Kuiper Belt:** A region beyond Neptune that contains icy bodies, including dwarf planets like Pluto.
- **Oort Cloud:** A hypothetical cloud of icy bodies that is believed to surround the Solar System at great distances, thought to be the source of long-period comets.



## Part II: Data Analysis

# 19 Probability

## 19.1 Introduction

The probability of an event  $A$  is a number between 0 and 1 that represents the likelihood of the event occurring. It is defined as

$$P(A) = \frac{\text{Number of favorable outcomes}}{\text{Total number of possible outcomes}}$$

For example, the probability of getting heads in a fair coin toss is:

$$P(\text{Heads}) = \frac{1}{2}$$

The probability of the complement of an event  $A$ , denoted as  $A^c$ , is:

$$P(A^c) = 1 - P(A)$$

For example, if  $P(\text{Heads}) = \frac{1}{2}$ , then the probability of getting tails,  $P(\text{Tails}) = \frac{1}{2}$ .

The probability of event  $A$  occurring given that event  $B$  has occurred is called conditional probability and is denoted as  $P(A|B)$ . It is given by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

This is the probability of  $A$  given  $B$ , assuming  $P(B) > 0$ .

## 19.2 Random Variables

A random variable is a numerical outcome of a random phenomenon. It can be classified as either discrete or continuous:

- **Discrete Random Variable:** Takes distinct values (e.g., number of heads in 10 coin tosses).
- **Continuous Random Variable:** Takes any value within a given range (e.g., height of individuals).



The probability mass function (PMF) gives the probability of each possible outcome for discrete random variables. For example, the PMF of a fair die roll (with outcomes 1, 2, 3, 4, 5, 6) is

$$P(X = x) = \frac{1}{6}, \quad x \in \{1, 2, 3, 4, 5, 6\}$$

The probability density function (PDF) is used for continuous random variables. The probability that a continuous random variable  $X$  takes a value in the interval  $[a, b]$  is given by the integral of the PDF over that interval:

$$P(a \leq X \leq b) = \int_a^b f_X(x) dx$$

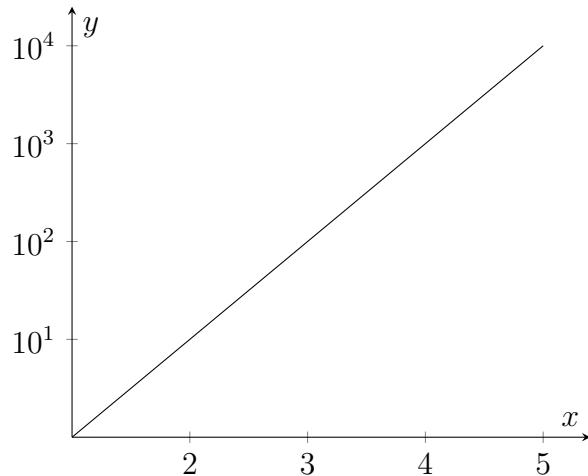
where  $f_X(x)$  is the PDF of  $X$ .

## 20 Linear and Logarithmic Scale

In a linear scale, the distance between points is the same for each increment. The axis values increase by a constant amount.

In a logarithmic scale, the distance between values is proportional to the logarithm of the values:

Example



## 21 Measure of Central Tendency

### Definition. 21.1: Mean

The mean  $\bar{x}$  is the sum of all values divided by the number of observations. For a sample of size  $n$ ,

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

### Definition. 21.2: Median

The median is the middle value when the data is ordered from least to greatest.

- If  $n$  is odd: The median is the value at position  $\frac{n+1}{2}$ .
- If  $n$  is even: The median is the average of the two middle values.

### Definition. 21.3: Mode

The mode is the value that appears most frequently in the dataset.

## 22 Measure of Dispersion

### 22.1 Basic Concepts

#### Definition. 22.1: Quartile

Quartiles are statistical measures that divide an ordered dataset into four equal parts. They help describe the spread and distribution of the data. The three quartiles are:

$$Q_1 \text{ (first quartile)}, \quad Q_2 \text{ (second quartile or median)}, \quad Q_3 \text{ (third quartile)}$$

These quartiles divide the data into four segments, each containing 25% of the observations.

#### Definition. 22.2: Standard Deviation

Standard deviation is a measure of the spread or dispersion of a set of numerical data. It tells us how much the values deviate, on average, from the mean (average). A small standard deviation indicates that the data points are clustered close to the mean, while a large standard deviation indicates that



the data are spread out. For the data points

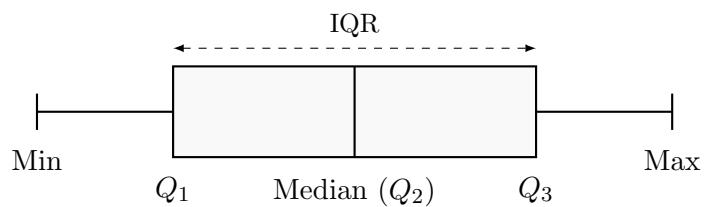
$$x_1, x_2, x_3, \dots, x_N,$$

the standard deviation is defined as

$$\sigma = \sqrt{\frac{1}{N} \sum_{i=1}^N (x_i - \bar{x})^2}$$

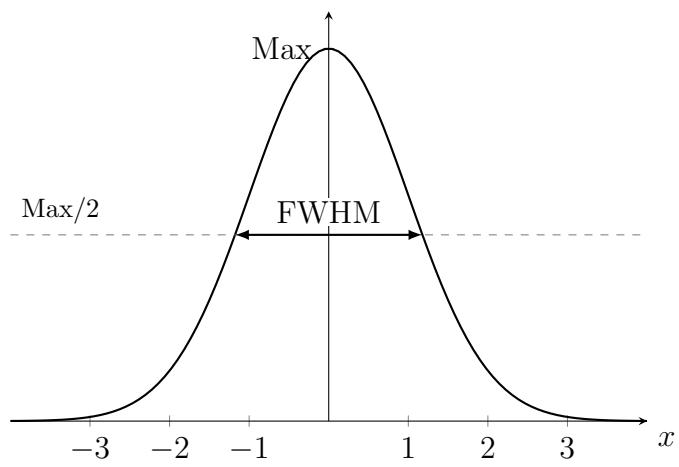
## 22.2 Box Plots (Box-and-Whisker)

A box plot is a graphical method for displaying the distribution of numerical data using five key summary values:



## 23 Full Width at Half Maximum

FWHM is the width of a curve measured at half of its maximum amplitude.



**Example.** For Gaussian function

$$f(x) = Ae^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where



- $A$  is the amplitude (maximum value) of the Gaussian,
- $\mu$  is the mean (center) of the distribution,
- $\sigma$  is the standard deviation (width) of the distribution.

From

$$Ae^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{A}{2},$$

$$e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{1}{2}.$$

$$-\frac{(x-\mu)^2}{2\sigma^2} = \ln \frac{1}{2} = -\ln 2.$$

$$(x-\mu)^2 = 2\sigma^2 \ln 2.$$

The points  $x_1$  and  $x_2$  are

$$x = \mu \pm \sigma \sqrt{2 \ln 2}.$$

Therefore, the FWHM is

$$\text{FWHM} = x_2 - x_1 = 2\sigma \sqrt{2 \ln 2}$$

## 24 Error Analysis

In experimental measurements, quantities have uncertainties or errors. If a function depends on multiple variables:

$$z = f(x, y, \dots),$$

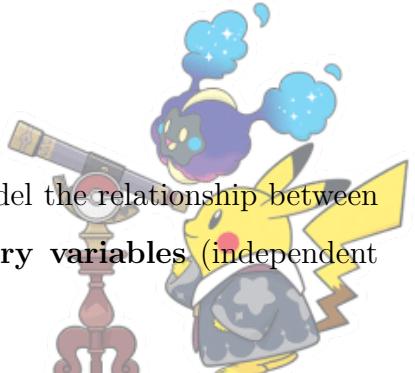
and  $x, y, \dots$  have uncertainties  $\Delta x, \Delta y, \dots$ , then the approximate uncertainty in  $z$  is

$$\boxed{\Delta z \approx \sqrt{\left(\frac{\partial z}{\partial x} \Delta x\right)^2 + \left(\frac{\partial z}{\partial y} \Delta y\right)^2 + \dots}}$$

## 25 Regression Analysis

### 25.1 Linear regression

Linear regression is a fundamental supervised learning algorithm used to model the relationship between a **scalar response** (dependent variable,  $Y$ ) and **one or more explanatory variables** (independent



variables,  $X$ ). For **simple linear regression** (one independent variable), the relationship is modeled by a straight line:

$$\hat{y} = \beta_0 + \beta_1 x$$

Where  $\hat{y}$  is the predicted response,  $x$  is the independent variable,  $\beta_0$  is the intercept, and  $\beta_1$  is the slope.

### 25.1.1 Least Squares Method

The goal is to find the optimal coefficients  $(\beta_0, \beta_1)$  that make the line best fit the observed data points  $(x_i, y_i)$ . The best fit is defined by the least squares method, which minimizes the sum of the squares of the residuals (errors).

#### Definition. 25.1: Residual

A residual,  $e_i$ , for the  $i$ -th data point is the difference between the actual value  $y_i$  and the predicted value  $\hat{y}_i$ :

$$e_i = y_i - \hat{y}_i = y_i - (\beta_0 + \beta_1 x_i)$$

#### Definition. 25.2: Cost Function

The cost function,  $J(\beta_0, \beta_1)$ , is the quantity we want to minimize:

$$J(\beta_0, \beta_1) = \sum_{i=1}^n e_i^2 = \sum_{i=1}^n (y_i - (\beta_0 + \beta_1 x_i))^2$$

#### Theorem. 25.1: Linear Regression

To find the minimum of  $J(\beta_0, \beta_1)$ , we set the partial derivatives with respect to  $\beta_0$  and  $\beta_1$  to zero. This leads to the following solutions for the optimal coefficients,  $\hat{\beta}_0$  and  $\hat{\beta}_1$ :

- The optimal slope coefficient is given by

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2}$$

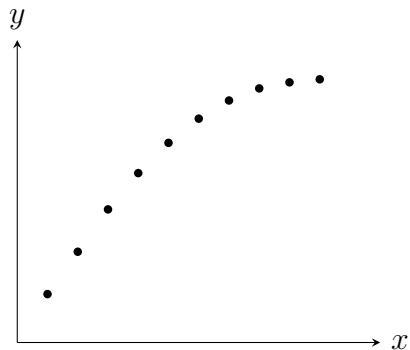
- The optimal intercept coefficient is then found using the fact that the regression line must pass through the point of means  $(\bar{x}, \bar{y})$ :

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$



## 25.2 Nonlinear Regression

Nonlinear regression aims to fit a curve to data where the relationship between variables is **not linear**. While computers can do this accurately, it is possible to get an approximate fit manually using **eyes and pen**.



Draw a smooth curve that visually passes as close as possible to all points. Adjust the shape until it captures the overall trend.



## Part III: Observational Astronomy

# 26 Instrumentation and Space Technologies

## 26.1 Telescope

### 26.1.1 Type of Telescope

**Optical Telescope** Refracting telescope uses lenses to bend (refract) light to a focal point.

- **Advantages:** Sealed tube, no central obstruction, good for planetary observation
- **Disadvantages:** Chromatic aberration, size limitations, expensive
- **Structure:** Objective lens → Tube → Eyepiece

Reflecting telescope uses mirrors to reflect light to a focal point.

- **Types by Optical Design:**
  1. **Newtonian:** Primary parabolic mirror and flat secondary diagonal
  2. **Cassegrain:** Primary hyperbolic and secondary convex mirror
  3. **Ritchey-Chrétien:** Both mirrors hyperbolic (no spherical aberration)
  4. **Nasmyth:** Light directed to side of telescope
- **Advantages:** No chromatic aberration, easier to make large apertures
- **Disadvantages:** Central obstruction, coma in some designs

**Radio Telescopes** Radio telescopes collect radio waves using large parabolic dishes or arrays.

- **Examples:** Arecibo (305 m), FAST (500 m), ALMA (array)
- **Components:** Dish, feed antenna, receiver, amplifier, recorder



Mount Type	Advantages	Disadvantages
Altazimuth	Simple, compact	Field rotation
Equatorial	Natural tracking	Complex, heavy
Fork	Stable for Cassegrains	Limited sky access
German Equatorial	Good balance	Meridian flip required

Table 10: Comparison of telescope mount types

### 26.1.2 Mount Types

### 26.1.3 Key Components

1. **Primary Mirror/Objective:** Main light-gathering element
2. **Secondary Mirror:** Redirects light path (in reflectors)
3. **Focuser:** Adjusts position of eyepiece or instrument
4. **Eyepiece/Instrument:** Final light analysis
5. **Mount:** Supports and points the telescope
6. **Drive System:** Tracks celestial objects

### 26.1.4 Linear Magnification

The magnification of a telescope is the factor by which the telescope increases the apparent size of an object. It is given by

$$M = \frac{f_{\text{obj}}}{f_{\text{eye}}}$$

where  $f_{\text{obj}}$  is the focal length of the objective lens or mirror and  $f_{\text{eye}}$  is the focal length of the eyepiece.

### 26.1.5 Angular Magnification

Consider an object of height  $h$  placed at a distance  $D$  from the eye. The angle  $\theta$  subtended at the eye (called the **visual angle**) is

$$\theta \approx \tan \theta = \frac{h}{D} \quad (\theta \ll 1)$$

The approximation  $\tan \theta \approx \theta$  is valid for small angles (measured in radians), which is typical in optical systems.



Angular magnification  $M$  is defined as

$$M = \frac{\theta'}{\theta},$$

where

- $\theta'$  is the angle subtended at the eye when viewing the object through the lens,
- $\theta$  is the angle subtended at the eye when viewing the object directly (unaided eye).

### 26.1.6 Chromatic Aberration

Chromatic aberration is an optical defect of lenses that arises due to the **dispersion** of light. It occurs because the refractive index of lens material depends on the wavelength of light.

The refractive index  $n$  of a transparent medium is a function of wavelength  $\lambda$ :

$$n = n(\lambda)$$

In general, shorter wavelengths (violet light) experience a higher refractive index than longer wavelengths (red light):

$$n_{\text{violet}} > n_{\text{red}}$$

The focal length  $f$  of a thin lens is given by the lens maker's formula (for thin lens)

$$\frac{1}{f} = (n - 1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right)$$

Since  $n$  depends on  $\lambda$ , the focal length also depends on wavelength:

$$f = f(\lambda)$$

Hence, different colors of light are brought to focus at different positions along the optical axis.

### 26.1.7 $f$ -number

The focal ratio (or  $f$ -number) is the ratio of the telescope's focal length  $f$  to the diameter  $D$  of the aperture:

$$f/\# = \frac{f}{D}$$



A smaller focal ratio corresponds to a wider field of view and faster exposure times, which is important for observing faint objects.

### 26.1.8 Light-gathering Power

The light-gathering power of a telescope is its ability to collect light from an astronomical object. This is important because the more light a telescope can gather, the fainter objects it can detect. The light-gathering power is proportional to the area of the telescope's aperture,  $A$ , which is typically circular. The formula for the area of a circular aperture is

$$A = \pi \left( \frac{D}{2} \right)^2$$

where  $D$  is the diameter of the aperture. Therefore, the light-gathering power is proportional to the square of the aperture diameter:

$$L \propto D^2$$

This relationship means that a telescope with a larger aperture collects more light, allowing for observations of fainter objects.

### 26.1.9 Adaptive Optics

Adaptive optics is a technology used to improve the performance of optical systems by compensating for distortions caused by the Earth's atmosphere. These distortions, known as atmospheric turbulence, can blur images taken with ground-based telescopes. Adaptive optics systems use deformable mirrors to correct for these distortions in real-time. The process involves

1. A guide star or laser is used to create a reference point in the sky.
2. A wavefront sensor measures the distortion of the light coming from the guide star.
3. A computer calculates the necessary corrections to the wavefront.
4. A deformable mirror adjusts the light path, compensating for the distortion.

The result is sharper images with significantly reduced atmospheric distortion.



### 26.1.10 Artificial Light

Artificial light pollution affects astronomical observations, particularly in urban areas. The brightness of the night sky due to artificial lighting can drown out faint celestial objects, making it harder to observe stars, planets, and galaxies. Efforts such as **light pollution mitigation** are crucial for preserving the quality of astronomical observations.

## 26.2 Interferometer

**Geometric Model of a Two-Element Interferometer** A two-element interferometer uses two telescopes to observe the same astronomical object simultaneously. The signals from the two telescopes are combined, and the interference pattern is analyzed to extract higher resolution data than a single telescope could provide. For two telescopes separated by a distance  $D$ , the resolution of the interferometer is related to the wavelength  $\lambda$  of the observed light and the baseline  $D$  by the following formula for the angular resolution  $\theta$ :

$$\theta \sim \frac{\lambda}{D}$$

**Aperture Synthesis** Aperture synthesis is a technique used in radio astronomy and interferometry to create an image with the resolution of an aperture much larger than the physical size of the telescope. The method involves observing the same object with multiple telescopes at different positions and combining the data to simulate the effect of a much larger telescope.

The resolution of a synthetic aperture is determined by the maximum separation between the telescopes, referred to as the baseline. By changing the positions of the telescopes, astronomers can collect data at multiple baselines, which allows them to improve the resolution over time.

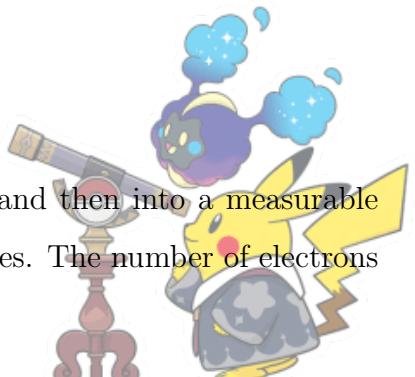
## 26.3 Detector

### 26.3.1 Photometers

A photometer measures the intensity of light from a source in a specific wavelength band. It typically uses a single detector element and filters to isolate desired wavelengths.

### 26.3.2 Charge-Coupled Devices (CCDs)

CCDs are widely used digital detectors that convert photons into electrons and then into a measurable voltage. They offer high quantum efficiency and the ability to create 2D images. The number of electrons



generated in a pixel can be calculated as

$$N_e = F_\gamma A t \text{QE}$$

where

- $F_\gamma$  is the photon flux (photons  $s^{-1}m^{-2}$ ),
- $A$  is the telescope collecting area ( $m^2$ ),
- $t$  is the exposure time (s),
- QE is the quantum efficiency of the detector.

## 26.4 Plate Scale

### 26.4.1 Introduction

For small angles (where  $\tan \theta \approx \theta$ ), the plate scale  $P$  for telescope is given by

$$P = \frac{s}{\theta} = \frac{1}{f}$$

where

- $\theta$  is the angular size on sky (radians)
- $s$  is the linear size in focal plane
- $f$  is the telescope focal length

Since astronomers work with arcseconds and millimeters (or microns for pixels):

$$P_{\text{arcsec/mm}} = \frac{206265}{f_{\text{mm}}}$$

The constant 206,265 comes from:

$$1 \text{ radian} = \frac{180}{\pi} \times 3600 = 206264.8 \text{ arcseconds}$$

For digital detectors like CCDs,

$$P_{\text{arcsec/px}} = \frac{206265 \times p_{\mu m}}{f_{\text{mm}} \times 1000}$$



where  $p_{\mu m}$  is the pixel size in micrometers.

### 26.4.2 Field of View

The total angular field covered by a detector is

$$\text{FOV}_{\text{width}} = N_x \cdot P_{\text{px}} \text{ [arcseconds]}$$

$$\text{FOV}_{\text{height}} = N_y \cdot P_{\text{px}} \text{ [arcseconds]}$$

where  $N_x$  and  $N_y$  are the number of pixels in each dimension.

## 26.5 Space-Based Instruments

Space-based instruments operate above Earth's atmosphere to observe the universe with high precision.

- **Atmospheric limitations:** Earth's atmosphere absorbs or distorts many wavelengths such as ultraviolet, X-ray, and infrared radiation.
- **Improved resolution:** absence of atmospheric turbulence allows sharper images and more stable measurements.
- **Instrument types:** telescopes, spectrometers, particle detectors, and magnetometers.
- **Constraints:** high cost, limited repair opportunities, and finite operational lifetimes.

## 26.6 Signal-to-Noise Ratio

### 26.6.1 Introduction

Signal-to-Noise Ratio (SNR) is a measure of the quality of a signal in the presence of noise. It quantifies how much the signal stands out from the background noise. High SNR means a clear signal, while low SNR indicates that noise dominates.

SNR is defined as

$$\text{SNR} = \frac{S}{N}$$

where

- $S$  is the signal strength (e.g., number of detected photons from the source),
- $N$  is the noise (e.g. standard deviation of the background).



### 26.6.2 Pure Poisson Noise

#### Definition. 26.1: Poisson Distribution

The Poisson distribution is a discrete probability distribution that expresses the probability of a number of events occurring in a fixed interval of time or space, given the average number of occurrences. It is commonly used to model random events like the number of phone calls received at a call center, the number of accidents at an intersection, or the number of particles decaying per unit time. The probability mass function of a Poisson distribution is given by

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2, \dots$$

where

- $X$  is the random variable representing the number of events,
- $\lambda$  is the average number of occurrences in the given interval (also called the rate or intensity),
- $k!$  is the factorial of  $k$ , i.e.,  $k! = k(k - 1)(k - 2) \dots 1$ .

The Poisson distribution is characterized by the mean and variance both being equal to  $\lambda$ , i.e.,

$$\mu = \sigma^2 = \lambda.$$

For photon-counting detectors, the dominant noise is often Poisson. If the expected signal is  $S$  photons, then Poisson statistics give

$$\sigma = \sqrt{S}$$

Hence,

$$\text{SNR} = \frac{S}{\sqrt{S}} = \sqrt{S}$$

### 26.6.3 Background Noise

Suppose the measurement includes both the object signal  $S$  and the background  $B$ . The total number of detected photons is

$$S_{\text{tot}} = S + B$$



Poisson statistics give total variance

$$\sigma^2 = S + B$$

Hence,

$$\text{SNR} = \frac{S}{\sqrt{S + B}}$$

#### 26.6.4 Readout Noise

A CCD or CMOS detector adds readout noise  $\sigma_{\text{read}}$  per pixel. If  $n_{\text{pix}}$  pixels are used, the read noise contribution is

$$\sigma_{\text{read,tot}}^2 = n_{\text{pix}}\sigma_{\text{read}}^2$$

The total variance becomes

$$\sigma^2 = S + B + n_{\text{pix}}\sigma_{\text{read}}^2$$

Hence,

$$\text{SNR} = \frac{S}{\sqrt{S + B + n_{\text{pix}}\sigma_{\text{read}}^2}}$$

#### 26.6.5 Complete Equation

$$\text{SNR} = \frac{S}{\sqrt{S + B + n_{\text{pix}} \left( 1 + \frac{n_{\text{pix}}}{n_B} \right) (N_s + N_D + \sigma_{\text{read}}^2 + G^2\sigma_f^2)}}$$

where

- $n_B$  is the background pixels in the image, which correspond to regions with no object or light source.
- $N_s$  represents the signal from the source, representing the number of photons from the object being observed.
- $N_D$  represents the dark noise, which arises from thermally generated electrons in the CCD detector, even in the absence of light.
- $G$  is the gain of the CCD detector, which relates the number of electrons collected by the pixel to the digital output recorded by the system.
- $\sigma_f$  is the fluctuations or Fano noise, which arises from the statistical nature of electron counting processes.

