PHYS1150 Final Mock Paper

Answer ALL 9 problems.

Notes: You should always give precise and adequate explanations to support your conclusions. Clarity of your argument counts so think carefully before you write.

Problem 1

A *vector field* on a set $U \subseteq \mathbb{R}^n$ is a function

$$\mathbf{F} : U \to \mathbb{R}^n$$

that assigns to each point $\mathbf{x} \in U$ a vector $\mathbf{F}(\mathbf{x})$ in $\mathbb{R}^n$.

In coordinates, if $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, then a vector field can be written as

$$\mathbf{F}(\mathbf{x}) = \big(F_1(x_1, \ldots, x_n),\ F_2(x_1, \ldots, x_n),\ \ldots,\ F_n(x_1, \ldots, x_n)\big).$$

The partial derivative of the vector field $\mathbf{A}$ with respect to a scalar variable s is found by differentiating each components with respect to s, while treating the constant basis vectors as constants:

$$\frac{\partial \mathbf{A}}{\partial s} = \frac{\partial}{\partial s}\left(A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}\right) = \left(\frac{\partial A_x}{\partial s}\right)\mathbf{i} + \left(\frac{\partial A_y}{\partial s}\right)\mathbf{j} + \left(\frac{\partial A_z}{\partial s}\right)\mathbf{k}$$

Given a regular surface in $\mathbb{R}^3$ parameterised by

$$\mathbf{r}(u, v),$$

where $u, v \in \mathbb{R}$, its tangent vectors are defined to be

$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}, \qquad \mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}.$$

Define the coefficients of the first fundamental form

$$E = \mathbf{r}_u \cdot \mathbf{r}_u, \qquad F = \mathbf{r}_u \cdot \mathbf{r}_v, \qquad G = \mathbf{r}_v \cdot \mathbf{r}_v.$$

Thus, the first fundamental form is

$$I = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

The second fundamental form describes how the surface bends in $\mathbb{R}^3$. Let

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}$$

be the unit normal vector field on the surface. Then the coefficients of the second fundamental form are

$$e = \mathbf{r}_{uu} \cdot \mathbf{n}, \qquad f = \mathbf{r}_{uv} \cdot \mathbf{n}, \qquad g = \mathbf{r}_{vv} \cdot \mathbf{n}.$$

The second fundamental form is

$$II = \begin{pmatrix} e & f \\ f & g \end{pmatrix}.$$

The Gaussian curvature at a point is defined by

$$K = \frac{\det II}{\det I}.$$

Consider the parametrisation of the sphere:

$$\mathbf{r}(u, v) = \big(R \sin u \cos v,\ R \sin u \sin v,\ R \cos u\big), \qquad u \in (0, \pi),\ v \in (0, 2\pi).$$

where $u, v \in \mathbb{R}$ R is a constant.

Find the Gaussian curvature of the sphere. **(12 marks)**

Problem 2

Consider a particle of mass m confined in a one-dimensional infinite potential well of width L:

$$V(x) = \begin{cases} 0, & 0 < x < L, \\ \infty, & \text{otherwise.} \end{cases}$$

Given that the time-independent Schrödinger equation is satisfied:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V\psi(x) = E\psi(x),$$

for some functions ψ.

(a) Using the transformation

$$p(x) = \frac{\psi'(x)}{\psi(x)}$$

where it is assumed that $\psi(x) \neq 0$, solve the Schrödinger equation for **the general solution of** $\psi(x)$ for $0 < x < L$.

(b) Verify your answer in (a) using a characteristics equation.

(c) Given that $\psi(0) = \psi(L) = 0$ (as it is impossible to find the particle at $x = 0$ and $x = L$) is the initial condition of the equation. Then E can be quantized. Derive the quantized E in terms of arbitrary integers n.

(16 marks)

Problem 3

In Quantum Mechanics, one may be interested in studying a spin-$\frac{1}{2}$ particle. Define $\mathbf{n} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$ to be a unit vector that specifies a direction in space. The spin operator in the direction $\mathbf{n}$ is

$$\hat{S}_{\mathbf{n}} = \frac{\hbar}{2}\mathbf{n}\cdot\boldsymbol{\sigma} = \frac{\hbar}{2}\big(\sin\theta\cos\phi\,\sigma_x + \sin\theta\sin\phi\,\sigma_y + \cos\theta\,\sigma_z\big),$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

are the Pauli matrices. Let

$$|\psi_+\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

be the eigenvector of $\hat{S}_{\mathbf{n}}$ for eigenvalue $\hbar/2$. Express $|\psi_+\rangle$ in terms of $|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, θ and ϕ. Normalize your answer such that the magnitude of $|\psi_+\rangle$ is 1. **(8 marks)**

Problem 4

A *Möbius transformation* (also called a *fractional linear transformation*) is a map

$$T : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}, \qquad T(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{C}$ satisfy $ad - bc \neq 0$. Here

$$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

is the extended complex plane (the Riemann sphere). By convention,

$$T\left(-\frac{d}{c}\right) = \infty, \qquad T(\infty) = \frac{a}{c} \quad (c \neq 0), \qquad T(\infty) = \infty \quad (c = 0).$$

For distinct complex numbers z_1, z_2, z_3, z_4, the cross ratio is defined to be

$$[z_1, z_2; z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}.$$

Prove that if T is a Möbius transformation, then the cross ratio is invariant i.e.

$$[T(z_1), T(z_2); T(z_3), T(z_4)] = [z_1, z_2; z_3, z_4].$$

(5 marks)

Problem 5

(a) Use complex number to find the exact form of $\tan \dfrac{\pi}{12}$. No marks will be given to those who ignore the application of complex numbers.

(b) Find the general solution for $(2+\sqrt{3})\sin x + \cos x = \sqrt{2+\sqrt{3}}$.

(15 marks)

Problem 6

A ball is projected horizontally with $v = 1$ at $(20, 10)$. To intercept the ball, another ball is projected at the origin with $v' = 5$ to intercept. What is the launching angle of the ball for interception? Solve using two methods:

Method 1: Solve using the concept of relative motion.

Method 2: Solve without using the concept of relative motion.

(8 marks)

Problem 7

Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = (625 + x^4)^{-\frac{3}{4}}$$

Let $f^{(n)}(x)$ denote the n-th derivative of $f(x)$, with $f^{(0)}(x) = f(x)$.

(a) Prove that $f(x)$ satisfies the first-order differential equation:

$$(625 + x^4)f'(x) + 3x^3 f(x) = 0.$$

(b) Using Leibniz rule, prove that for all positive integers $n \geq 3$,

$$(625 + x^4)f^{(n+1)}(x) + A_n x^3 f^{(n)}(x) + B_n x^2 f^{(n-1)}(x) + C_n x f^{(n-2)}(x) + D_n f^{(n-3)}(x) = 0$$

where A_n, B_n, C_n, and D_n are polynomials in n to be determined.

(c) Derive a recurrence relation that connects $f^{(n+1)}(0)$ with $f^{(n-3)}(0)$.

(d) Find $f^{(8)}(0)$ and $f^{(9)}(0)$.

(e) Let the Maclaurin series for $f(x)$ be given by $f(x) = \displaystyle\sum_{k=0}^{\infty} c_k x^k$. Show that $c_k = 0$ unless k is a multiple of 4 (i.e., $k = 4m$ for some integer $m \geq 0$). Find a recurrence relation for the coefficient c_{4m} in terms of $c_{4(m-1)}$. Derive a general formula for c_{4m} in terms of m and c_0, using product ($\prod$) notation. **(14 marks)**

Problem 8

The Biot-Savart Law describes the magnetic field $\vec{B}$ produced by an electric current I. The law states that the differential magnetic field $d\vec{B}$ generated by a differential current element $Id\vec{l}$ is given by:

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{l} \times \hat{r}}{r^2}$$

where

- I is the current.
- $d\vec{l}$ is the differential length vector in the direction of the current.
- $\hat{r}$ is the unit vector pointing from the current element $d\vec{l}$ to the point P where the field is measured.
- r is the distance from $d\vec{l}$ to P.
- μ_0 is the permeability of free space ($\mu_0 = 4\pi \times 10^{-7}\,\mathrm{T \cdot m/A}$).

The total magnetic field $\vec{B}$ is found by integrating this expression over the entire length of the current path:

$$\vec{B} = \int d\vec{B} = \frac{\mu_0 I}{4\pi} \oint \frac{d\vec{l} \times \hat{r}}{r^2}$$

A rigid wire, carrying a steady current I, is bent into a 'U' shape and lies entirely in the xy-plane. The wire consists of three segments:

1. Two straight, semi-infinite segments parallel to the y-axis, located at $x = -R$ and $x = +R$. These segments run from $y = 0$ to $y = +\infty$.
2. A semicircular arc of radius R that connects the two straight segments at $y = 0$. The arc is centered at the origin $(0, 0, 0)$.

The current I flows in the direction shown in the diagram: down the left wire ($x = -R$), around the semicircular arc, and up the right wire ($x = +R$).

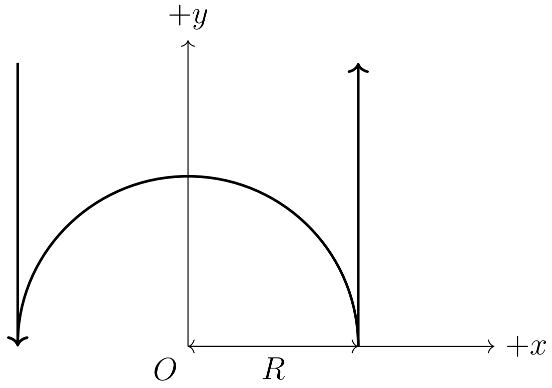

Figure 1

Using the Biot-Savart Law, calculate the magnitude and direction of the total magnetic field vector $\vec{B}$ at the origin, $O(0,0,0)$. **(12 marks)**

Problem 9

In Astrophysics, the L1 point is one of the Lagrange point that lies on the line between the Sun and the Earth. It's a position where the Sun's gravitational pull is partially canceled by the Earth's gravity, allowing a spacecraft to orbit the Sun with the same period as Earth, even though it's closer to the Sun. It is described by the equation

$$\underbrace{\frac{GM_S m}{(R-r)^2}}_{\text{Sun's Gravity}} - \underbrace{\frac{GM_E m}{r^2}}_{\text{Earth's Gravity}} = \underbrace{m\omega^2(R-r)}_{\text{Centripetal Force}}$$

where

- $G \approx 6.674 \times 10^{-11}\,\text{N m}^2\text{kg}^{-2}$ is the gravitational constant.
- $M_S \approx 1.989 \times 10^{30}\,\text{kg}$ is the mass of the Sun.
- $M_E \approx 5.972 \times 10^{24}\,\text{kg}$ is the mass of the Earth.
- $R \approx 1.496 \times 10^{11}\,\text{m}$ is the (average) distance from the Sun to the Earth.
- r is the distance from the Earth to the L1 point.
- m is the mass of the spacecraft.
- $\omega \approx 1.991 \times 10^{-7}\,\text{rad/s}$ is the angular velocity of the Earth-Sun system.

From Kepler's Third Law (approximating $M_S + M_E \approx M_S$), we know: $\omega^2 \approx \dfrac{GM_S}{R^3}$. By substituting ω^2 into the force equation and canceling G and m, we get the final equation to solve for r:

$$\frac{M_S}{(R-r)^2} - \frac{M_E}{r^2} = \frac{M_S(R-r)}{R^3}$$

Using Newton's method with 8 iterations to approximate r and correct your answer to 7 significant figures. Use the initial guess $r_0 = 122700000\,\text{m}$. Tabulate r_n (correct to 8 significant figures) for each iteration. **(10 marks)**