Random vectors, stochastic processes and discrete event simulation

1. **Random permutations** Alice, Bob, Charly, and Dave share an apartment that has only one bathroom. They have decided that every morning from 7 to 8 each one of them can use the bathroom for 15 minutes.

   The bath turns (1st turn 7 to 7:15, 2nd turn 7:15 to 7:30, 3rd turn 7:30 to 7:45, 4th turn 7:45 to 8) must be completely random, with the single restriction that Alice must have one of the first turns since she must leave home at 7:50. They have decided to proceed as follows: they select (at random) who takes the 1st turn, then they select at random who takes the 2nd turn if Alice is not taking any of them, she takes the third and whoever has not yet used the bathroom at 7:45 takes the fourth.

   If Alice has taken one of the two first turns, they select that random who takes the 3rd turn, and the other flatmate takes the 4th turn.

   a) Is that method fair in the sense that all allowed permutations have the same probability?, why?

   b) Build a fair algorithm to distribute the turns (with the given restriction).

2. **Compound Poisson process** The number of purchases at a webpage follows a nonhomogeneous Poisson process with intensity function $\lambda(x) = x$ during the first 10 days and $\lambda(x) = 10$ purchases per day ever after.

   a) Simulate 10000 processes until 100 purchases are executed. What is the average time until those 100 purchases?

   b) Three different products are sold, one for 200 euro, another for 300 euro, and the third one for 500 euro. Half of the purchases correspond to the first product, 30% to the second, and the remaining to the third.

   What is the expected time until they sell products for a total value of 25000 euros?

3. **Pricing European options** European options may only be exercised at expiration. That is, if we buy today a European call option with strike price $k = 100$ and maturity $t_m = 1$ year, in one year (at maturity) we have the right to buy the asset for the fixed strike price $k = 100$, which we will do if the asset price at that precise moment is greater than k.

   If the price of the asset at that moment is below k, we will not exercise the option. Suppose we want to price a European call option with the initial asset price $S(0) = 100$, strike price $k = 100$, risk-free rate $r = 0.02$, volatility $\sigma = 0.25$, and maturity $t_m = 1$ year.

   The risk-neutral pricing process is

$$S(t) = S(0) \exp\left(\left(r - \sigma^2/2\right) t + \sigma Z \sqrt{t}\right)$$

   where $Z \sim N(0, 1)$.

   Use $MC = 10000$ simulations to price the option and give a 95% CI on the price.

   The option will only be exercised if the asset's price at maturity is greater than 100, so its payoff will be $\max S\{(t_m) - k, 0\}$, while the price is the expected payoff.

   Observe that the price is to be paid today, so it must be given in today's price of money, and we can use the risk-free rate to determine today's price of 100 monetary units in one year, which is $100 \exp(-r) = 98.01987$.