Simulation of random variables

1. The time T, measured in days, during which a manufacturing system remains non-operational each time it breaks down, is described by a cumulative distribution function (cdf) given by:

$$F_T(t) = \begin{cases} 1 - (2/t)^3 & \text{if} \quad t > 2 \\ 0 & \quad\quad\quad \text{otherwise} \end{cases}$$

   Apply the inverse transform method and write a function that simulates samples from the above cdf.

2. A random variable T follows a Pareto distribution with scale parameter equal to 2 and shape parameter equal to 3. We can use its density function as an instrumental density function to simulate samples from another Pareto model whose tail is not as heavy.

   Consider now a random variable S (Pareto with scale parameter 2 and shape parameter 4) with cdf

$$F_S(t) = \begin{cases} 1 - (2/t)^4 & \text{if} \quad t > 2 \\ 0 & \quad\quad\quad \text{otherwise} \end{cases}$$

   (a) Determine M such that $f_S(t) \leq M f_T(t)$, where f_S and f_T are the density functions of S and T.

   (b) Write an acceptance-rejection algorithm to simulate from F_S.

   (c) Use your simulation algorithm to check that $E[S] = 8/3 = 2.6667$.

   Show a 99% confidence interval algorithm based on $MC = 10000$ observations.

3. A random variable X follows a truncated normal distribution within the interval $[a, b]$ characterized by the parameters μ (mean) and σ (standard deviation). Thus, X can be denoted as $TN(\mu, \sigma, a, b)$ if its probability density function $f_X(x)$ is defined as:

$$f_X(x) = \frac{\frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)},$$

   where $a \leq x \leq b$ holds and $\phi(\cdot)$ represents the normal density function, while $\Phi(\cdot)$ represents the cumulative distribution function (cdf) of a standard normal distribution.

   To simulate observations from a truncated normal random variable $TN(0, 1; -1, 1)$, write three functions utilizing:

(a) Rejection sampling algorithm with the density function of a standard normal distribution as a candidate function.

(b) A rejection sampling algorithm with the density function of a uniform distribution $U(-1, 1)$ as the candidate function.

(c) An inverse transform technique.

Compare their behavior in terms of efficiency.

Observation: In these exercises write the solutions, at least one of them, in `Rcpp`.