and that is,

$$2t(t-1)-1 < t(t-1)+t\sqrt{t^2-2t+\frac{2}{t}} < 2t(t-1)+1.$$

In addition, since n is an integer $n \geq 2t(t-1)$. But we assumed that $(t-1) \nmid n$ which implies that $n \neq 2t(t-1)$.
Thus, for $t>1$, $\mathsf{diff}>0$ if and only if $n > 2t(t-1)$. $\square$

Theorem 30. *Let n be an integer. It holds that*

$$\mathsf{T} = \arg\min_{t\in\mathbb{N}} \left\{ \left| t - \frac{1}{2}\sqrt{1+2n} \right| \right\},$$

and the maximum ℓ-balls of radius one are the balls centered at the t-balanced sequences of length n, for $t \in \mathsf{T}$. In addition, the size of the maximum ℓ-balls of radius one is given by

$$\max_{\boldsymbol{x}\in\Sigma_2^n}\{|L_1(\boldsymbol{x})|\} = n^2-n(t+1)+t+2 \quad - \frac{t-k}{2}\left(\left\lceil\frac{n}{t}\right\rceil-2\right)\left(\left\lceil\frac{n}{t}\right\rceil-3\right)-\frac{k}{2}\left(\left\lceil\frac{n}{t}\right\rceil-1\right)\left(\left\lceil\frac{n}{t}\right\rceil-2\right)$$

where $k = n(\mathrm{mod}\ t)$ and the residues are taken from the set $\{1,\dots,t\}$.

Proof. Let n be a positive integer. By Lemma 27,

$$\max_{\boldsymbol{x}\in\Sigma_2^n}|L_1(\boldsymbol{x})| = \max_{t\in\{1,\dots,n\}}\left\{\max_{\substack{\boldsymbol{x}\in\Sigma_2^n\\ \alpha(\boldsymbol{x})=t}}|L_1(\boldsymbol{x})|\right\} = \max_{t\in\{1,\dots,n\}}\left\{\max_{\substack{\boldsymbol{x}\in\Sigma_2^n\\ \boldsymbol{x}\text{ is } t\text{-balanced}}}|L_1(\boldsymbol{x})|\right\}.$$

(d) Efficient algorithms to computationally calculate the sizes of the ℓ-balls and Levenshtein

all t-subsequences of $\boldsymbol{x}$.

Definition 2. *The radius-t insertion ball centered at $\boldsymbol{x} \in \Sigma_q^n$, $I_t(\boldsymbol{x}) \subseteq \Sigma_m^{n+t}$, is the set of all t-supersequences of $\boldsymbol{x}$.*

Let $\boldsymbol{x} \in \Sigma_q^n$ be a sequence. The size of the insertion ball $|I_t(\boldsymbol{x})|$ does not depend on $\boldsymbol{x}$ for any $0 \leq t \leq n$. To be exact, it was shown by Levenshtein [31] that

$$|I_t(\boldsymbol{x})| = \sum_{i=0}^{t} \binom{n+t}{i} (q-1)^i. \tag{1.1}$$

On the other hand, calculating the exact size of the deletion ball is one of the more intriguing problems when studying codes for deletions. Deletion balls, unlike substitutions and insertions balls, are not regular. That is, the size of the deletion ball, $|D_t(\boldsymbol{x})|$, depends on the choice of the sequence $\boldsymbol{x}$. It was shown in [23] that the alternating sequences have the largest deletion ball, denoted by $D_q(n,t)$, which is given by

$$D_q(n,t) = \sum_{i=0}^{t} \binom{n-t}{i} D_{q-1}(t, t-i)$$

for any $0 \le t \le n$. To be exact, it was shown by Levenshtein [31] that

$$|I_t(\boldsymbol{x})| = \sum_{i=0}^{t} \binom{n+t}{i} (q-1)^i. \tag{1.1}$$

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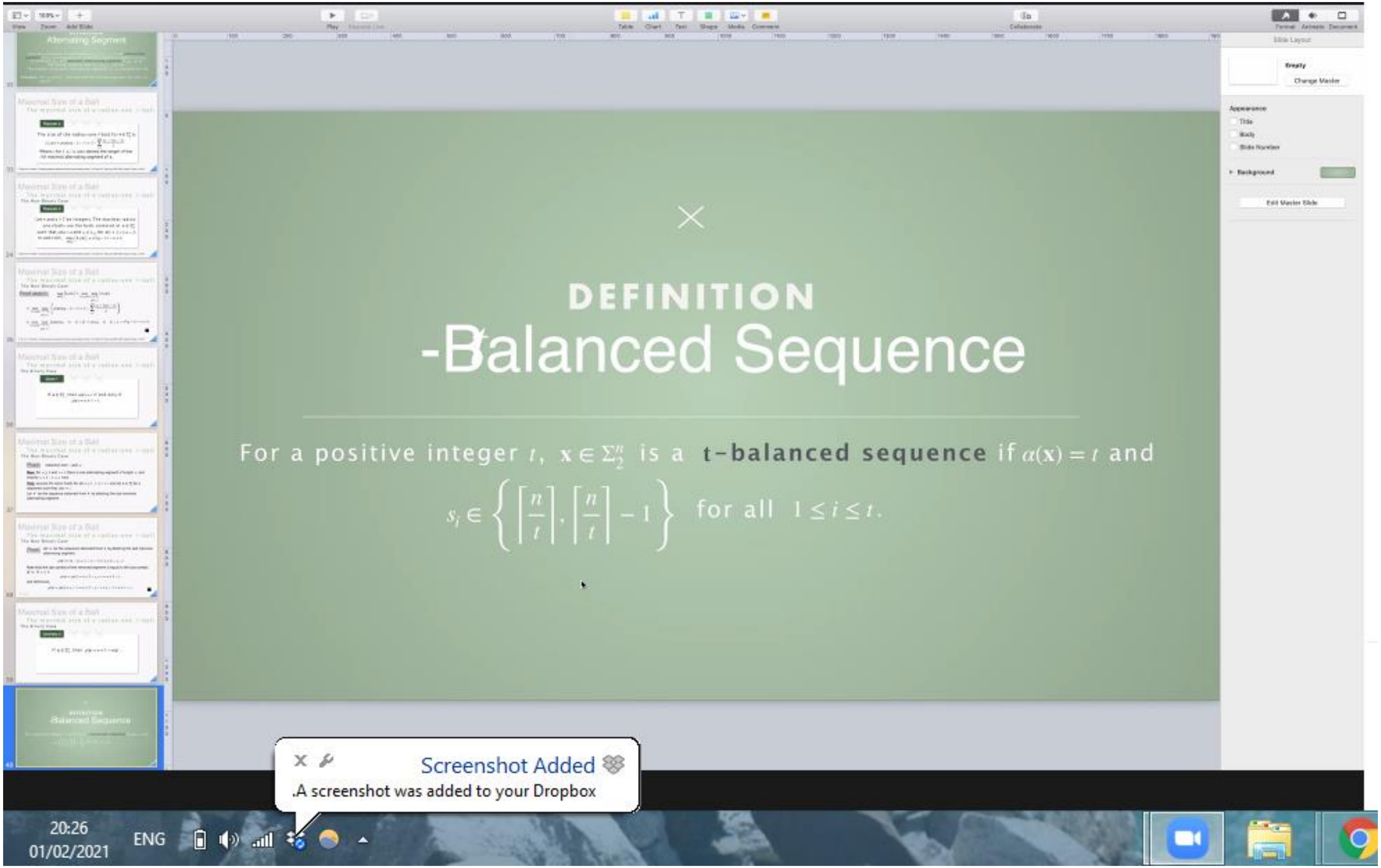
$$D_q(n,t) = \sum_{i=0}^{t} \binom{n-t}{i} D_{q-1}(t, t-i)$$

where n is the sequence length, q is the alphabet size and t is the number of deletions. In particular, $D_2(n,t) = \sum_{i=0}^{t} \binom{n-t}{i}$ and $D_3(n,t) = \sum_{i=0}^{t} \binom{n-t}{i} \sum_{j=0}^{t-i} \binom{i}{j}$. The value $D_2(n,t)$ satisfies also the following recursion

$$D_2(n,t) = D_2(n-1,t) + D_2(n-2,t-1).$$


DEFINITION
t-Balanced Sequence
For a positive integer t, $\mathbf{x} \in \Sigma_2^n$ is a t-balanced sequence if $\alpha(\mathbf{x}) = t$ and
$s_i \in \left\{ \left\lceil \frac{n}{t} \right\rceil, \left\lceil \frac{n}{t} \right\rceil - 1 \right\}$ for all $1 \le i \le t$.
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ENG

Zoom Meeting

Maximal Size of a Ball

The maximal size of a radius-one ℓ-ball

The Non-Binary Case

Proof sketch:

$$\max_{\mathbf{x}\in\Sigma_q^n} \left|L_1(\mathbf{x})\right| = \max_{1\le r\le\rho(\mathbf{x})} \max_{\substack{\mathbf{x}\in\Sigma_q^n \\ \rho(\mathbf{x})=r}} |L_1(\mathbf{x})|$$

$$= \max_{1\le r\le\rho(\mathbf{x})} \max_{\substack{\mathbf{x}\in\Sigma_q^n \\ \rho(\mathbf{x})=r}} \left\{\rho(\mathbf{x})(n(q-1)-1)+2-\sum_{i=1}^{\alpha(\mathbf{x})}\frac{(s_i-1)(s_i-2)}{2}\right\}$$

$$\le \max_{1\le r\le\rho(\mathbf{x})} \max_{\substack{\mathbf{x}\in\Sigma_q^n \\ \rho(\mathbf{x})=r}} \left\{\rho(\mathbf{x})(n(q-1)-1)+2\right\} = n(n(q-1)-1)+2 = n^2(q-1)-n+2. \quad \blacksquare$$

F. Sala and L. Dolecek, “Counting sequences obtained from the synchronization channel,” Int. Symp. Inf. Theory, pp. 2925–2929, Istanbul, Turkey, Jul. 2013.

Chat

From daniel to Everyone:

שומעים ?

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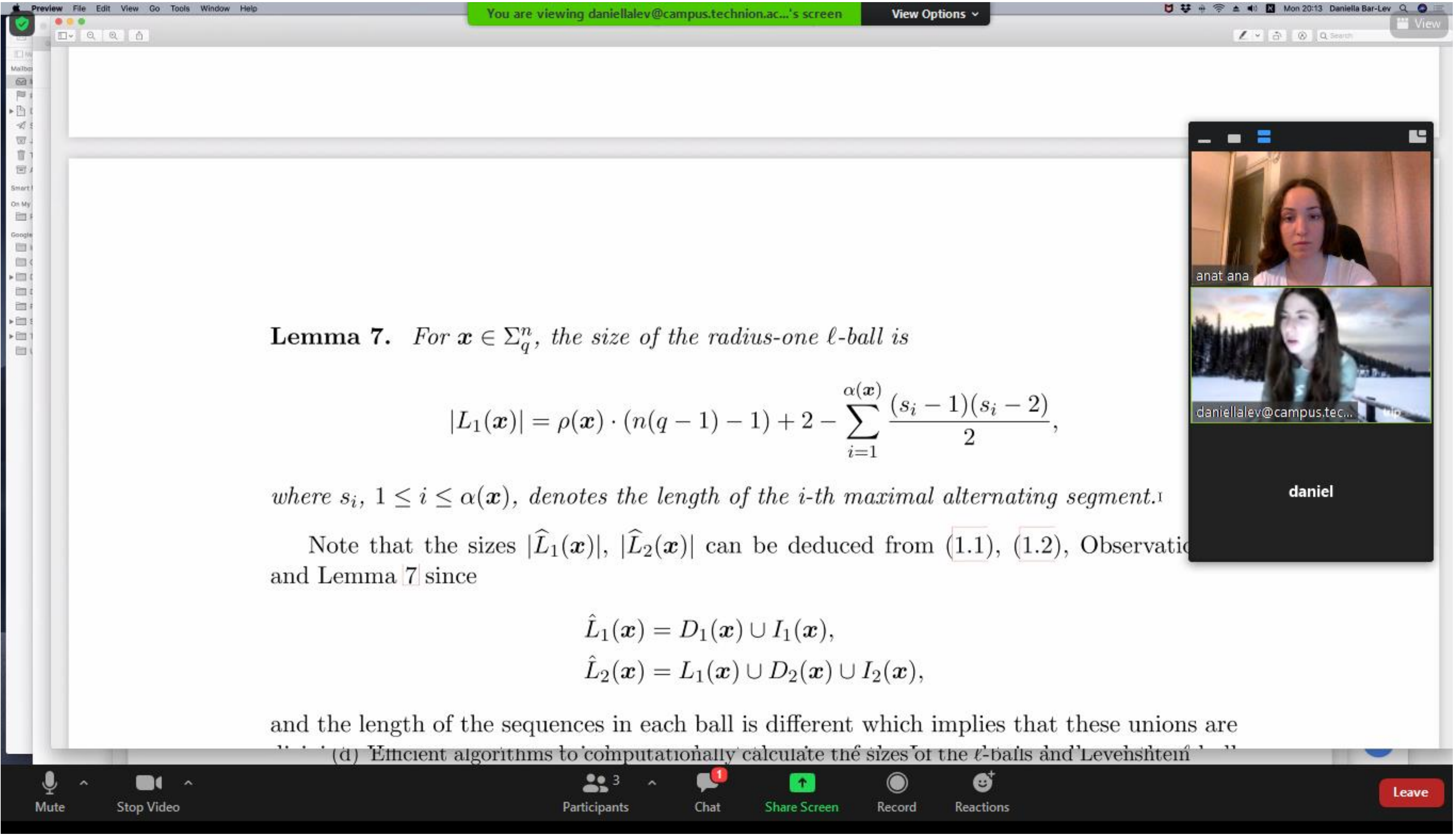

Lemma 7. *For $\boldsymbol{x} \in \Sigma_q^n$, the size of the radius-one ℓ-ball is*

$$|L_1(\boldsymbol{x})| = \rho(\boldsymbol{x}) \cdot (n(q-1)-1) + 2 - \sum_{i=1}^{\alpha(\boldsymbol{x})} \frac{(s_i-1)(s_i-2)}{2},$$

where s_i, $1 \leq i \leq \alpha(\boldsymbol{x})$, denotes the length of the i-th maximal alternating segment.

Note that the sizes $|\widehat{L}_1(\boldsymbol{x})|$, $|\widehat{L}_2(\boldsymbol{x})|$ can be deduced from (1.1), (1.2), Observatio
and Lemma 7 since

$$\hat{L}_1(\boldsymbol{x}) = D_1(\boldsymbol{x}) \cup I_1(\boldsymbol{x}),$$
$$\hat{L}_2(\boldsymbol{x}) = L_1(\boldsymbol{x}) \cup D_2(\boldsymbol{x}) \cup I_2(\boldsymbol{x}),$$

and the length of the sequences in each ball is different which implies that these unions are

(d) Efficient algorithms to computationally calculate the sizes of the ℓ-balls and Levenshtein

Corollary 25. *If $\boldsymbol{x} \in \Sigma_2^n$ then $\rho(\boldsymbol{x}) = n + 1 - \alpha(\boldsymbol{x})$.*

Notice that Lemma 24 and Corollary 25 do not hold for alphabet size $q > 2$. To clarify, consider the sequences $\boldsymbol{x}_1 = 0120$, $\boldsymbol{x}_2 = 0101$ and $\boldsymbol{x}_3 = 0102$, each of the sequences has four runs even though they differ in the number of maximal alternating segments.

Definition 26. *For a positive integer t, $\boldsymbol{x} \in \Sigma_2^n$ is a t-balanced sequence if $\alpha(x) = t$ and $s_i \in \{\lceil \frac{n}{t} \rceil, \lceil \frac{n}{t} \rceil - 1\}$ for all $i \in \{1, \ldots, \alpha\}$.*

Lemma 27. *If n is a positive integer and $t \in \{1, \ldots, n\}$ then*

$$\underset{\substack{\boldsymbol{x} \in \Sigma_2^n \\ \alpha(\boldsymbol{x}) = t}}{\arg\max} |L_1(\boldsymbol{x})| = \{\boldsymbol{x} \in \Sigma_2^n : \boldsymbol{x} \text{ is a } t\text{-balanced sequence}\}.$$

Proof. Let $\boldsymbol{x} \in \Sigma_2^n$ be a sequence such that $\alpha(\boldsymbol{x}) = t$. By Lemma 24, $\rho(\boldsymbol{x}) = n + 1 - t$ and

(d) Efficient algorithms to computationally calculate the sizes of the t-balls and Levenshtein

$$2t(t-1)-1 < t(t-1)+t\sqrt{t^2-2t+\frac{2}{t}} < 2t(t-1)+1.$$

In addition, since n is an integer $n \geq 2t(t-1)$. But we assumed that $(t-1) \nmid n$ which implies that $n \neq 2t(t-1)$.
Thus, for $t > 1$, $\mathsf{diff} > 0$ if and only if $n > 2t(t-1)$. $\square$

Theorem 30. *Let n be an integer. It holds that*

$$\mathsf{T} = \arg\min_{t\in\mathbb{N}} \left\{ \left| t - \frac{1}{2}\sqrt{1+2n} \right| \right\},$$

and the maximum ℓ-balls of radius one are the balls centered at the t-balanced sequences of length n, for $t \in \mathsf{T}$. In addition, the size of the maximum ℓ-balls of radius one is given by

$$\max_{\boldsymbol{x}\in\Sigma_2^n} \{|L_1(\boldsymbol{x})|\} = n^2 - n(t+1) + t + 2 \quad - \frac{t-k}{2}\left(\left\lceil\frac{n}{t}\right\rceil - 2\right)\left(\left\lceil\frac{n}{t}\right\rceil - 3\right) - \frac{k}{2}\left(\left\lceil\frac{n}{t}\right\rceil - 1\right)\left(\left\lceil\frac{n}{t}\right\rceil - 2\right)$$

where $k = n(\text{mod } t)$ and the residues are taken from the set $\{1,\dots,t\}$.

Proof. Let n be a positive integer. By Lemma 27,

$$\max_{\boldsymbol{x}\in\Sigma_2^n} |L_1(\boldsymbol{x})| = \max_{t\in\{1,\dots,n\}} \left\{ \max_{\substack{\boldsymbol{x}\in\Sigma_2^n \\ \alpha(\boldsymbol{x})=t}} |L_1(\boldsymbol{x})| \right\} = \max_{t\in\{1,\dots,n\}} \left\{ \max_{\substack{\boldsymbol{x}\in\Sigma_2^n \\ \boldsymbol{x} \text{ is } t\text{-balanced}}} |L_1(\boldsymbol{x})| \right\}.$$

If there exists an integer $t \in \{1,\dots,n\}$ such that $n = 2t(t-1)$, then, by Lemma 28,

(d) Efficient algorithms to computationally calculate the sizes of the ℓ-balls and Levenshtein

Maximal Size of a Ball

The maximal size of a radius-one ℓ-ball

The Non-Binary Case

Proof: Let $\mathbf{x}'$ be the sequence obtained from $\mathbf{x}$ by deleting the last maximal alternating segment.

$$\rho(\mathbf{x}') = (n - s_t) + 1 - (t-1) = n + 2 - s_t - t$$

Note that the last symbol of the removed segment is equal to the last symbol of $\mathbf{x}'$. If $s_t = 1$,

$$\rho(\mathbf{x}) = \rho(\mathbf{x}') = n + 2 - s_t - t = n + 1 - t,$$

and otherwise,

$$\rho(\mathbf{x}) = \rho(\mathbf{x}') + s_t - 1 = n + 2 - s_t - t + s_t - 1 = n + 1 - t.$$

■

Lemma 7. *For $\boldsymbol{x} \in \Sigma_q^n$, the size of the radius-one ℓ-ball is*

$$|L_1(\boldsymbol{x})| = \rho(\boldsymbol{x}) \cdot (n(q-1)-1) + 2 - \sum_{i=1}^{\alpha(\boldsymbol{x})} \frac{(s_i-1)(s_i-2)}{2},$$

where s_i, $1 \leq i \leq \alpha(\boldsymbol{x})$, denotes the length of the i-th maximal alternating segment.

Note that the sizes $|\widehat{L}_1(\boldsymbol{x})|$, $|\widehat{L}_2(\boldsymbol{x})|$ can be deduced from (1.1), (1.2), Observation 5, and Lemma 7 since

$$\hat{L}_1(\boldsymbol{x}) = D_1(\boldsymbol{x}) \cup I_1(\boldsymbol{x}),$$
$$\hat{L}_2(\boldsymbol{x}) = L_1(\boldsymbol{x}) \cup D_2(\boldsymbol{x}) \cup I_2(\boldsymbol{x}),$$

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