Deep Learning (IST, 2022-23)

Practical 2: Perceptron

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Pen-and-Paper Exercises

The following questions should be solved by hand. You can use, of course, tools for auxiliary numerical computations.

Question 1

Consider the following linearly separable training set:

$$\boldsymbol{x}^{(1)} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \boldsymbol{x}^{(2)} = \begin{bmatrix} 0 \\ 0.25 \end{bmatrix}, \boldsymbol{x}^{(3)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \boldsymbol{x}^{(4)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$y^{(1)} = -1, y^{(2)} = +1, y^{(3)} = +1, y^{(4)} = -1.$$

1. Initialize all weights to zero (including the bias). Assume $\text{sign}(z) = +1$ iff $z \geq 0$, and -1 if $z < 0$. Use a learning rate of one. Compute two epochs of the perceptron learning algorithm.

Solution: We start by adding a constant feature of 1 to the inputs, leading to:

$$\boldsymbol{x}^{(1)} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \boldsymbol{x}^{(2)} = \begin{bmatrix} 1 \\ 0 \\ 0.25 \end{bmatrix}, \boldsymbol{x}^{(3)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \boldsymbol{x}^{(4)} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

The weight vector is initialized as $\boldsymbol{w} = [0, 0, 0]$, where the first dimension corresponds to the bias (intercept) parameter.

First epoch (3 mistakes in total):

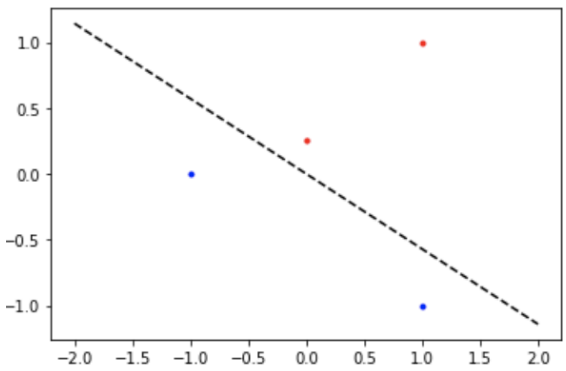
- $\boldsymbol{w} = [0, 0, 0]^\top$
- $\hat{y}^{(1)} = \text{sign}(\boldsymbol{w}^\top \boldsymbol{x}^{(1)}) = \text{sign}(0) = +1 \neq y^{(1)}$ – **mistake!**
- Perceptron update: $\boldsymbol{w} \leftarrow \boldsymbol{w} + y^{(1)}\boldsymbol{x}^{(1)} = [0, 0, 0]^\top - [1, -1, 0]^\top = [-1, 1, 0]^\top$
- $\hat{y}^{(2)} = \text{sign}(\boldsymbol{w}^\top \boldsymbol{x}^{(2)}) = \text{sign}(-1) = -1 \neq y^{(2)}$ – **mistake!**
- Perceptron update: $\boldsymbol{w} \leftarrow \boldsymbol{w} + y^{(2)}\boldsymbol{x}^{(2)} = [-1, 1, 0]^\top + [1, 0, 0.25]^\top = [0, 1, 0.25]^\top$
- $\hat{y}^{(3)} = \text{sign}(\boldsymbol{w}^\top \boldsymbol{x}^{(3)}) = \text{sign}(1.25) = +1 = y^{(3)}$ – **correct, no update.**
- $\hat{y}^{(4)} = \text{sign}(\boldsymbol{w}^\top \boldsymbol{x}^{(4)}) = \text{sign}(0.75) = +1 \neq y^{(4)}$ – **mistake!**
- Perceptron update: $\boldsymbol{w} \leftarrow \boldsymbol{w} + y^{(4)}\boldsymbol{x}^{(4)} = [0, 1, 0.25]^\top - [1, 1, -1]^\top = [-1, 0, 1.25]^\top$

Second epoch (1 mistake in total):

- $\boldsymbol{w} = [-1, 0, 1.25]^\top$
- $\hat{y}^{(1)} = \text{sign}(\boldsymbol{w}^\top \boldsymbol{x}^{(1)}) = \text{sign}(-1) = -1 = y^{(1)}$ – **correct, no update.**
- $\hat{y}^{(2)} = \text{sign}(\boldsymbol{w}^\top \boldsymbol{x}^{(2)}) = \text{sign}(-0.6875) = -1 \neq y^{(2)}$ – **mistake!**
- Perceptron update: $\boldsymbol{w} \leftarrow \boldsymbol{w} + y^{(2)}\boldsymbol{x}^{(2)} = [-1, 0, 1.25]^\top + [1, 0, 0.25]^\top = [0, 0, 1.5]^\top$
- $\hat{y}^{(3)} = \text{sign}(\boldsymbol{w}^\top \boldsymbol{x}^{(3)}) = \text{sign}(1.5) = +1 = y^{(3)}$ – **correct, no update.**
- $\hat{y}^{(4)} = \text{sign}(\boldsymbol{w}^\top \boldsymbol{x}^{(4)}) = \text{sign}(-1.5) = -1 = y^{(4)}$ – **correct, no update.**

2. After applying the algorithm until convergence the final trained weights were:$\boldsymbol{w} = [0, 1, 1.75]^\top$. Draw the separation hyperplane.

   Solution: The trained weight vector is $\boldsymbol{w} = [0, 1, 1.75]^\top$, resulting in the following separating boundary, which corresponds to the line with equation $w_0 + w_1 x_1 + w_2 x_2 = 0$:



3. What is the perceptron output for the query point $\begin{bmatrix} 0 & 1 \end{bmatrix}^\top$?

   Solution: We have $\hat{y} = \text{sign}(\boldsymbol{w}^\top \boldsymbol{x}_{\text{new}}) = \text{sign}([0, 1, 1.75]^\top [1, 0, 1]) = \text{sign}(1.75) = +1$.

Question 2

The perceptron can learn a relatively large number of functions. In this exercise, we focus on simple logical functions.

1. Show graphically that a perceptron can learn the logical NOT function. Give an example with specific weights.

   Solution: The NOT function receives as input a logical value $x \in \{-1, +1\}$ and outputs its logical negation $y \in \{-1, +1\}$. We can enumerate all possible inputs and their inputs:

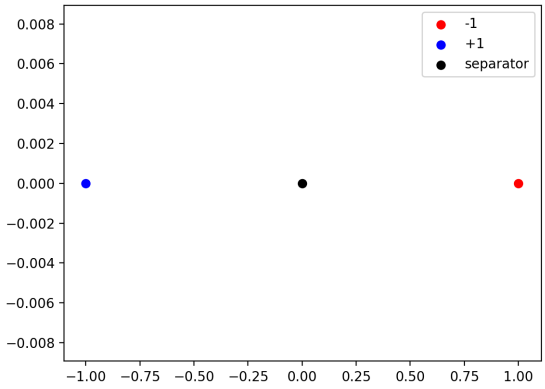
   - For $x = -1$ the output is $y = +1$
   - For $x = +1$ the output is $y = -1$

To show that a perceptron can learn a given function we just need to show that all points that require a positive output (+1) can be separated from all points that require a negative output by an hyperplane.

Since we are working with 1-dimensional inputs, an hyperplane is, in this case, a point.

So, is there a point that accurately separates the points? Yes, any point between −1 and +1 will achieve this.

An example hyperplane is given by the weight −1 and bias 0, leading to $\hat{y} = -x$.



2. Show graphically that a perceptron can learn the logical AND function for two inputs. Give an example with specific weights.

Solution: The AND function receives as input a pair of logical values $\boldsymbol{x} \in \mathbb{R}^2$ and outputs its logical conjuntion $y \in \{-1, +1\}$. We can enumerate all possible inputs and their inputs:

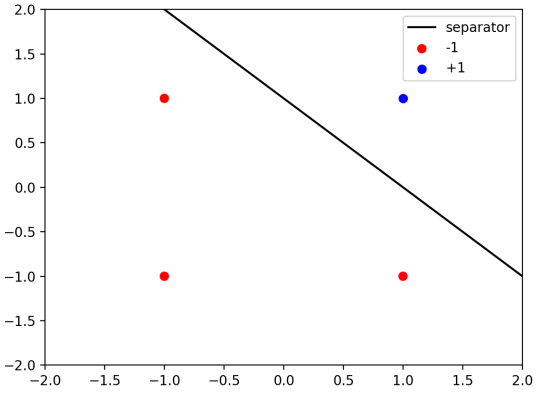
- For $\boldsymbol{x} = \begin{bmatrix} -1 & -1 \end{bmatrix}^\top$ the output is $y = -1$
- For $\boldsymbol{x} = \begin{bmatrix} -1 & +1 \end{bmatrix}^\top$ the output is $y = -1$
- For $\boldsymbol{x} = \begin{bmatrix} +1 & -1 \end{bmatrix}^\top$ the output is $y = -1$
- For $\boldsymbol{x} = \begin{bmatrix} +1 & +1 \end{bmatrix}^\top$ the output is $y = +1$

To show that a perceptron can learn a given function we just need to show that all points that require a positive output (+1) can be separated from all points that require a negative output by an hyperplane.

Since we are working with 2-dimensional inputs, an hyperplane is, in this case, a line.

So, is there a weight vector that defines a boundary line that accurately separates the points?

Yes, for instance $\boldsymbol{w} = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix}^\top$ achieves:



3. Show graphically that a perceptron can learn the logical OR function for two inputs. Give an example with specific weights.

   Solution: The OR function receives as input a pair of logical values $\boldsymbol{x} \in \mathbb{R}^2$ and outputs its logical conjuntion $y \in \{-1, +1\}$. We can enumerate all possible inputs and their inputs:

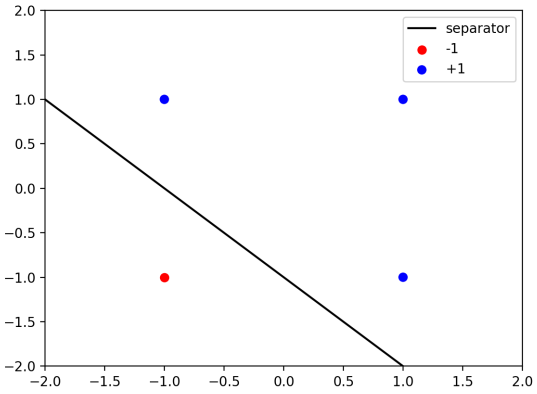
   - For $\boldsymbol{x} = \begin{bmatrix} -1 & -1 \end{bmatrix}^\top$ the output is $y = -1$
   - For $\boldsymbol{x} = \begin{bmatrix} -1 & +1 \end{bmatrix}^\top$ the output is $y = +1$
   - For $\boldsymbol{x} = \begin{bmatrix} +1 & -1 \end{bmatrix}^\top$ the output is $y = +1$
   - For $\boldsymbol{x} = \begin{bmatrix} +1 & +1 \end{bmatrix}^\top$ the output is $y = +1$

   To show that a perceptron can learn a given function we just need to show that all points that require a positive output (+1) can be separated from all points that require a negative output by an hyperplane.

   Since we are working with 2-dimensional inputs, an hyperplane is, in this case, a line.

   So, is there a weight vector that defines a boundary line that accurately separates the points?

   Yes, for instance $\boldsymbol{w} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top$ achieves:



4. Show graphically that a perceptron can not learn the logical XOR function for two inputs.

   Solution: The XOR function receives as input a pair of logical values $\boldsymbol{x} \in \mathbb{R}^2$ and outputs its logical conjuntion $y \in \{-1, +1\}$. We can enumerate all possible inputs and their inputs:

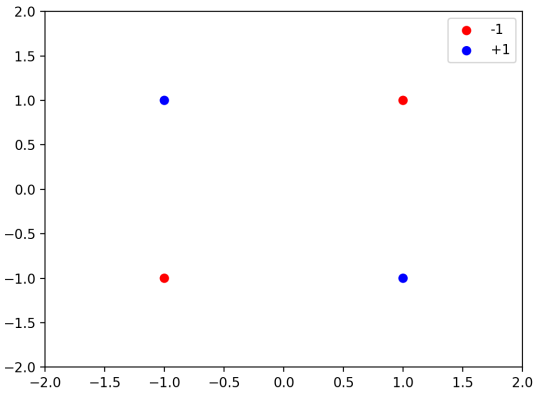
   - For $\boldsymbol{x} = \begin{bmatrix} -1 & -1 \end{bmatrix}^\top$ the output is $y = -1$
   - For $\boldsymbol{x} = \begin{bmatrix} -1 & +1 \end{bmatrix}^\top$ the output is $y = +1$
   - For $\boldsymbol{x} = \begin{bmatrix} +1 & -1 \end{bmatrix}^\top$ the output is $y = +1$
   - For $\boldsymbol{x} = \begin{bmatrix} +1 & +1 \end{bmatrix}^\top$ the output is $y = -1$

   To show that a perceptron can learn a given function we just need to show that all points that require a positive output (+1) can be separated from all points that require a negative output by an hyperplane.

   Since we are working with 2-dimensional inputs, an hyperplane is, in this case, a line.

   So, is there a weight vector that defines a boundary line that accurately separates the points?

   No... If we plot the points we see right away that no line can separate the positive instances from the negative ones:



Programming Exercises

The following exercises should be solved using Python, you can use the corresponding practical's notebook for guidance.

1. Let us consider **Question 1**.
   (a) Initialize all weights to zero (including the bias). Assume $\text{sign}(z) = +1$ iff $z \geq 0$, and -1 if $z < 0$. Use a learning rate of one. Apply the perceptron learning algorithm until convergence. How many epochs does it take to converge?
   (b) Change the initialization of weights and biases to be random with a standard normal distribution $\mathcal{N}(0, 1)$. Try multiple times. Does it always converge?
2. Generate a balanced dataset with 30 examples in $\mathbb{R}^2$ and 3 classes. Assume each of the 10 inputs associated to class $k \in \{0, 1, 2\}$ is generated as $x \sim \mathcal{N}(\mu_k, \sigma_k^2 I)$, with $\sigma_0 = \sigma_1 = \sigma_2 = 1$, $\mu_0 = [0, 0]^\top$, $\mu_1 = [0, 3]^\top$, and $\mu_2 = [2, 2]^\top$. Plot the data.

(a) Implement the multi-class perceptron algorithm and run 100 iterations. Initialize all the weights to zero and use a learning rate of one. What is the training accuracy (fraction of points that are correctly classified)?

3. Now it's time to try the perceptron on real data and see what happens. Load the UCI handwritten digits dataset using `scikit-learn`

   (a) Randomly split this data into training (80%) and test (20%) partitions.

   (b) Create and run your implementation of the multi-class perceptron algorithm on this dataset. Measure the training and test accuracy.

   (c) Use `scikit-learn`'s implementation of the perceptron algorithm. Compare the resulting accuracies.