All orientable smooth surfaces support a complex structure

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I want to show that any oriented smooth surface M is in fact a complex manifold of dimension one (which will in fact be Kahler).
The first step is to observe that any smooth manifold M supports a Riemannian metric g, which we can create by defining it locally and then patching using a partition of unity [1] So without loss of generality, we may assume that M is in fact a Riemannian manifold (M, g). We recall the definition, as well as some elementary properties of the holonomy group:

1 Holonomy

Suppose we are given a Riemannian manifold (M, g) and a curve $\gamma : [0, 1] \to M$. The Levi-Civita connection ∇ gives us the notion of *parallel transport*

Definition 1.1 (Parallel Transport)**.** *For any curve $\gamma : [0, 1] \to M$ we have the linear ordinary differential equation:*

$$\nabla_{\dot{\gamma}(t)} V(t) = 0$$

Because this equation is linear, we know that for any initial value $X \in T_{\gamma(0)}M$ the solution to the initial value problem:

$$\begin{aligned} &\nabla_{\dot{\gamma}(t)} V(t) = 0 \\ &V(0) = X \end{aligned} \tag{1}$$

is defined for all $t \in [0, 1]$ (This is Theorem 4.12 on page 60 of [Lee97]). Now for any vector $X \in T_\gamma(0)M$ we define the parallel transport *of X along γ as:*

$$P_\gamma X = V(1)$$

This map is a linear isomorphism from $T_{\gamma(0)}M$ to $T_{\gamma(1)}M$.

[1] Interesting things happen if we relax the assumption that M is paracompact. See the remark at the end of this paper.

In addition, if we denote by $V_1(t)$ and $V_2(t)$ the solutions to (1) with initial data X_1 and X_2, by a defining property of the Levi-Civita connection we have that:

$$\dot{\gamma}g(V_1(t), V_2(t)) = g(\nabla_{\dot{\gamma}}V_1(t), V_2(t)) + g(V_1(t), \nabla_{\dot{\gamma}}V_2(t)) = 0$$

Hence:

$$g(X_1, X_2) = g(V_1(0), V_2(0)) = g(V_1(1), V_2(1)) = g(P_\gamma X_1, P_\gamma X_2)$$

and so P_γ is an isometry. If we choose γ to be a closed curve centred at x ,that is, $\gamma(0) = \gamma(1) = x$, P_γ becomes a linear isometry of the vector space T_xM:

$$P_\gamma \in O(T_xM) \cong O(m, \mathbb{R})$$

Thus we may define:

Definition 1.2 (The Holonomy group)**.** *The holonomy group of (M, g) at x is the group of all such P_γ, where γ is a curve starting and ending at x. We denote this group by $Hol_x(M, g)$. It is a subgroup of $O(T_xM)$ and is in fact a* Lie group *(see [Zil10] page 133). If M is connected, the holonomy groups at x and y are conjugate as subgroups of $O(m, \mathbb{R})$ and hence we shall frequently drop the index x and just talk about the holonomy group of (M, g), $Hol(M, g)$.*

Note that we can define parallel transport of elements of T^*M using the transpose of P_γ:

$$P_\gamma^{-t} : T_p^*M \to T_p^*M \tag{2}$$

and so we may parallel transport a tensor $A \in (T_p^*M)^{\otimes k} \otimes (T_pM)^{\otimes l}$ by defining:

$$(P_\gamma A)(X_1, \ldots, X_k, \omega^1, \ldots, \omega^l) = A(P_\gamma X_1, \ldots, P_\gamma X_k, P_\gamma^{-t}\omega^1, \ldots, P_\gamma^{-t}\omega^l) \tag{3}$$

Thus we have a *representation* of $Hol(M, g)$ on $T_p^*M)^{\otimes k} \otimes (T_pM)^{\otimes l}$ for any k and l [2] We also recall the holonomy principle:

Theorem 1.3 (The Holonomy Principle)**.** *Let $A \in \Gamma\left((TM)^{\otimes k} \otimes (T^*M)^{\otimes l}\right)$ be a tensor field on a connected Riemannian manifold (M, g). A is parallel (that is, $\nabla A = 0$) if and only if for any point $x \in M$, $A_x \in (T_xM)^k \otimes (T_x^*M)^l$ is invariant under $Hol_x(M, g)$.*

Proof. Suppose that A_p is invariant under $Hol_p(M)$. Then for any $q \in M$, choose a path γ_1 such that $\gamma_1(0) = p$ and $\gamma_1(1) = q$. Now define

$$A_q = P_\gamma A_p \tag{4}$$

This is well defined since if we choose another path γ_2 satisfying $\gamma_2(0) = p$ and $\gamma_2(1) = q$ then we have that $\gamma_2^{-1}\gamma_1$ is a closed curve centred at p. Then:

$$P_{\gamma_2^{-1}\gamma_1} A_p = A_p \tag{5}$$

$$\Rightarrow P_{\gamma_2}^{-1} P_{\gamma_1}(A_p) = A_p \tag{6}$$

$$P_{\gamma_1}(A_p) = P_{\gamma_2}(A_p) \tag{7}$$

$$\tag{8}$$

[2]It is an interesting question to ask when this is an *irreducible* representation. This leads to the De Rham decomposition theorem.

To show that A is a parallel tensor field it suffices to show that, for all $x \in M$, $\nabla_{\dot\gamma} A = 0$ for arbitrary path γ through x. But this is true since by definition $A_{\gamma(t)}$ satisfies the differential equation:

$$\nabla_{\dot\gamma} A_{\gamma(t)} = 0 \tag{9}$$

For the converse, observe that for any path γ with $\gamma(0) = p$ and $\gamma(1) = q$, $A_{\gamma(t)}$ is a solution to (1) with initial data A_p since:

$$\nabla_{\dot\gamma} A_{\gamma(t)} = 0 \tag{10}$$
$$A_{\gamma(0)} = A_p \tag{11}$$

By the uniqueness of solutions to ODE's it is *the* solution to (1), and hence:

$$P_\gamma A_p = A_{\gamma(1)} = A_q \tag{12}$$

So if we take γ to be a closed curve with $\gamma(0) = \gamma(1) = p$ then:

$$P_\gamma A_p = A_p \tag{13}$$

That is, A is invariant under the holonomy group. See also Theorem 2.3 pg.8 of [GHJ03]. $\square$

2 Main result

Let's formulate the claim made in the introduction precisely and prove it.

Theorem 2.1. *Suppose that M is a smooth, orientable 2 dimensional (paracompact) manifold, i.e. a smooth surface. Then we may define a complex structure J on M making (M, J, g) into a Kähler manifold.*

Proof. Lets place a Riemannian metric g on M. Because M is orientable we know that $Hol(M, g) = SO(2) \subset O(2)$. But, and here's the trick, $SO(2) = U(1)$, and $U(1)$, acting on the tangent space at any point $p \in M$, will preserve a complex structure $J_p \in T_pM \otimes T_p^*M$. We can even check this directly; lets choose a g_p-orthonormal basis for T_pM and define J_p with respect to this basis as $J_p = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Observe that since $\det(J_p) = 1$ $J_p \in SO(2) \cong Hol(M, g)_p$. Recall that the representation of $SO(2)$ on T_pM is the standard, or defining representation [3]. Thus $SO(2)$ acts on $T_p^*M \otimes T_pM \cong End(T_pM)$ as:

$$(g \cdot A)(X) = gA(g^{-1}X) \quad A \in End(T_pM) \cong T_pM \otimes T_p^*M$$

Because $J_p \in SO(2)$ and this group is commutative, we have that:

$$g \cdot J_p(g^{-1}X) = gJ_pg^{-1}(X) = J_p(X)$$

[3] Why? The defining representation is the unique irreducible representation of $SO(2)$ of dimension 2

So J_p is holonomy-invariant and so by the holonomy principle (cf. theorem 1.3) it extends to a parallel, almost complex structure on our manifold M. Observe also that J is compatible with g in the sense that $g(JX, JY) = g(X, Y)$ for all vector fields X and Y [4]

Now the Newlander-Nirenburg theorem gives a necessary and sufficient condition for this almost complex structure to be integrable, thus making M into an honest complex manifold:

Theorem 2.2 (Newlander-Nirenburg). *J is an integrable complex structure if and only if the Nijenhuis tensor:*

$$N_J(X, Y) = [X, Y] + J([JX, Y] + [X, JY]) - [JX, JY] \tag{14}$$

vanishes.

There is a somewhat inelegant, but effective way to show that $\nabla J = 0 \Rightarrow N_J = 0$ by using the fact that, by definition of the Levi-Civita connection, ∇ is torsion free, hence $[X, Y] = \nabla_X Y - \nabla_Y X$ for all $X, Y \in \Gamma(TM)$. Using this identity we have:

$$\begin{aligned} N_J(X, Y) =& - J^2(\nabla_X Y - \nabla_Y X) + J(\nabla_{JX} Y - \nabla_Y JX) \\ &+ J(\nabla_X JY - \nabla_{JY} X) - \nabla_{JX} JY + \nabla_{JY} JX \\ \Rightarrow N_J(X, Y) =& J(-J\nabla_X Y + \nabla_X JY) - J(-J\nabla_Y X + J\nabla_Y JX) \\ &+ (J\nabla_{JX} Y - \nabla_{JX} JY) + (J\nabla_{JY} X - \nabla_{JY} JX) \end{aligned}$$

Now observe that each bracketed term can be rewritten in terms of the covariant derivative of J, since, for example: $(\nabla_X J)(Y) = \nabla_X(JY) - J(\nabla_X Y)$ and the same goes for all the other bracketed terms. But by construction $(\nabla_X J) = 0$ for all X, hence $N_J(X, Y) = 0$. Thus our complex structure is integrable, and M is in fact a complex manifold. The fact that $\nabla J = 0$ is also sufficient to show that (M, J, g) is Kähler (see Prop. 6.4 on pg. 168 of [Zil10]), but we can prove that (M, J, g) is Kähler in a simpler fashion by appealing to the fact that the Kähler form $\omega = g(J\cdot, \cdot)$ is a 2-form on a manifold of real dimension two, hence $d\omega$, being a 3-form, must vanish. □

3 A remark on paracompactness

At the beginning of the note we assumed that our smooth, oriented manifold M supported a Riemannian metric g. To prove this, we needed a partition of unity on M. This exists if and only if M is paracompact. The most common definition of a smooth manifold requires M to be second countable (cf. [Lee03]), in which case M is indeed paracompact (cf. Prop. 2.17 pg. 49 of [Lee03]). However there are examples of topological spaces which are locally Euclidean and support a smooth atlas but are not second countable (for example the Prüfer surface). For such a surface the above argument *will not work*.

[4] Why is this exactly? By construction this is true at p: $g_p(J_pX, J_pY) = g_p(X, Y)$ for all $X, Y \in T_pM$. But how do we extend this to all of M?

References

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