Swing Contracts in Energy Markets

Presented as Part of MATH8630: Stochastic Analysis

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3. In the case where the asset is energy (either a fuel or electricity) lack of storeability and susceptibility to markets, weather etc. make the spot price extremely volatile.
4. Thus buyer would like more flexibility in amount to purchase, while seller would still like an amount of future income to be guaranteed.

It's all about the swing

Definition (Basei et al[1])

A swing contract is an agreement whereby for any time $s \in [0, T]$ the buyer may buy energy at a rate of $u(s) \in [0, \bar{u}]$ at fixed price K. Total energy purchased, $Z(T) = \int_0^T u(s)ds$ is constrained by either:

1. *Penalized constraint: If $Z(T) \notin [m, M]$ buyer pays fine of $\Phi(P(T), Z(T))$.*
2. *Strict constraint: $Z(T) \in [m, M]$.*

[1] Basei, Cesaroni, and Vargiolu 2014.

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We follow Basei et al. (*in. loc. cit*) in examining penalized case first, and then discussing how strict case can be relaxed to the penalized case.

[1] Basei, Cesaroni, and Vargiolu 2014.

Penalized Swing Contracts - The diffusion

Fix a filtered probability space $(\Omega, \mathcal{F}_T, \{\mathcal{F}_s\}_{s\in[t,T]}, \mathbb{P})$ with $\{\mathcal{F}_s\}$-adapted Brownian motion W_s Spot price given by[2]

$$dP^{t,p}(s) = f(s, P^{t,p}(s))ds + \sigma(s, P^{t,p}(s))dW(s) \quad s \in [t, T]\; P^{t,p}(t) = p$$

[2] Assume that f, and σ are 'nice enough' to guarantee a solution satisfying $\mathbb{E}[\int_t^T Pt, p(s)ds] < \infty$ exists

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Let $Z^{t,z;u}$ denote total energy purchased up to s. Model this using control variable $u(s)$:

$$dZ^{t,z;u}(s) = u(s) \quad s \in [t, T] \quad Z^{t,z;u}(t) = z$$

Then:

$$dX^{t,z,p;u}(s) = d\begin{bmatrix} P^{t,p} \\ Z^{t,z;u} \end{bmatrix} = \begin{bmatrix} f(s, P^{t,p}(s)) \\ u(s) \end{bmatrix} ds + \begin{bmatrix} \sigma(s, P^{t,p}(s)) \\ 0 \end{bmatrix}$$

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Penalized Swing Contracts - The Net Profit

Recall that buyer purchases at price K. This give instantaneous profit of $(P(s) - K)u(s)$. Let $\tilde{\Phi}(P(T), Z(T))$ denote the penalty, where [3]

$$\tilde{\Phi}(p, z) = -Ap(z - M)^+ - Bp(m - z)^+ \tag{1}$$

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Net profit resulting from buying regime u:

$$\begin{aligned} J(t,p,z;u) =& \mathbb{E}\left[\int_t^T e^{-r(s-t)}(P^{t,p}(s)-K)u(s)ds \right. \\ & \left. + e^{-r(T-t)}\tilde{\Phi}(P^{t,p}(T), Z^{t,z;u}(T))\right] \\ & = e^{rt}\mathbb{E}\left[\int_t^T e^{rs}(P^{t,p}(s)-K)u(s)ds \right. \\ & \left. + e^{-rT}\tilde{\Phi}(P^{t,p}(T), Z^{t,z;u}(T))\right] = e^{rt}\tilde{J}(t,p,z;u) \end{aligned}$$

[3] Here $(f)^{+} = \chi_{\{f \geq 0\}} f$

Penalized Swing Contracts - The Value Function

Let $\mathcal{U}_t = \{u : [t, T] \times \Omega \to [0, \bar{u}] : \; u(s) \; \{\mathcal{F}_s\} - \text{adapted}\}$ denote the set of all admissable controls.

Penalized Swing Contracts - The Value Function

Let $\mathcal{U}_t = \{u : [t, T] \times \Omega \to [0, \bar{u}] : \ u(s)\ \{\mathcal{F}_s\} - \text{adapted}\}$ denote the set of all admissable controls. Let $V(t, p, z)$ denote maximum expected profit:

$$V(t, p, z) = \sup_{u \in \mathcal{U}_t} J(t, p, z; u) = \sup_{u \in \mathcal{U}_t} e^{rt} \tilde{J}(t, p, z; u) \qquad (2)$$

$$= e^{rt} \sup_{u \in \mathcal{U}_t} \tilde{J}(t, p, z; u) =: e^{rt} \tilde{V}(t, p, z; u) \qquad (3)$$

Clearly our goal is to find the control $u(s) = u(X(s), s)$ maximising V. Will suffice to maximise $\tilde{V}$.

Setting up the HJB equation

Theorem 1 (See Chpt. 4 and 5 of Fleming and Soner 2006)

If $\tilde{J} = \mathbb{E}[\int_t^T \tilde{L}(u, s, X)ds + \tilde{\Phi}(X(T))]$ *and* $V = \sup_{u \in \mathcal{U}_t} J(t, p, z; u)$
Then V *is the unique viscosity solution to*

$$\frac{\partial}{\partial t} V + \sup_{w \in \mathcal{U}} [\mathcal{A}^w V + L(w, t, x)] = 0 \quad V(T, x) = \Phi(x) \tag{4}$$

Provided that this sup is achieved.

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$$X(s) = (P(s), Z(s))^T \text{ and } \tilde{\Phi}(X) = e^{-rT}\Phi(P,Z) \tag{5}$$

$$\tilde{L}(t,p,z;w) = e^{-rt}(p-K)w \tag{6}$$

$$\mathcal{A}^w \tilde{V} = f\frac{\partial}{\partial p}\tilde{V} + w\frac{\partial}{\partial z}\tilde{V} + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial p^2}\tilde{V} \tag{7}$$

Penalized Swing Contract - HJB cont.

The HJB equation:

$$\tilde{V}_t + \sup_{w \in \mathcal{U}_t} \left[f \tilde{V}_p + w \tilde{V}_z + \frac{\sigma^2}{2} \tilde{V}_{pp} + e^{-rt}(p - K)w \right] = 0 \tag{8}$$

$$\tilde{V}(T, p, z) = e^{-rT} \Phi(p, z) \tag{9}$$

Can simplify a little bit by switching from $\tilde{V}$ to $V = e^{-rt} V$:

$$(\tilde{V})_t = -re^{-rt} V + e^{-rt} V_t \quad \mathcal{A}^w \tilde{V} = e^{-rt} \mathcal{A}^w V \tag{10}$$

and removing terms not depending on w from the sup:

$$-rV + V_t + fV_p + \frac{\sigma^2}{2} V_{pp} + \sup_{w \in \mathcal{U}_t} [wV_z + (p - K)w] = 0 \tag{11}$$

$$V(T, p, z) = \Phi(p, z) \tag{12}$$

Penalized Swing Contracts - Solutions and Regularity

'Standard Theory'[4] tells us that there is a unique viscosity solution to (11) satisfying $|V(t,p,z)| \leq C(1+|p|^2+|z|^2)$. Moreover Basei et al provide additional regularity:

1. $V(t,\cdot,z)$ is Lipschitz continuous and $V_p(t,p,z)$ exists a.e.[5].
2. $V(t,p,\cdot)$ is Lipschitz continuous, concave and a.e. twice differentiable (wrt z)[6].

[4] For example, that of Chpt. 4 and 5 of Fleming and Soner 2006.

[5] Prop. 2.2 of Basei, Cesaroni, and Vargiolu 2014.

[6] Prop. 2.3 of Basei, Cesaroni, and Vargiolu 2014.

Penalized Swing Contracts - an optimal buying policy

Recall that if

$$\mathcal{A}^{u^*} V + L(u^*, t, x) = \sup_{w \in \mathcal{U}_t} [\mathcal{A}^w V + L(w, t, x)]$$

Then u^* is an optimal control.

Penalized Swing Contracts - an optimal buying policy

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Then u^* is an optimal control.
In this case only need:

$$u^* V_z + (p - K)u^* = \sup_w [wV_z + (p - K)w]$$

and so an optimal policy is:

$$u^*(t, p, z) = \begin{cases} \bar{u} & \text{if } V_z(t, p, z) \geq -(p - K) \\ 0 & \text{if } V_z(t, p, z) < -(p - K) \end{cases} \tag{13}$$

Strict Swing Contracts - The Problem

Will now focus on strict swing contracts, i.e. we add the integral constraint $Z(T) = z + \int_t^T u(s)ds \in [m, M]$. Note that the diffusion is still the same!

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The problem: our admissable control set is now more complicated:

$$\mathcal{U}_{tz}^{adm} = \{u \in \mathcal{U}_t : \ \mathbb{P}_{tz}\left[z + \int_t^T u(s)ds \in [m, M]\right] = 1\} \qquad (14)$$

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Thus we cannot näively apply standard results from Optimal Control to:

$$J(t, p, z; u) = \mathbb{E}_{tpz}\left[\int_t^T e^{-r(s-t)}(P^{t,p} - K)u(s)ds\right] \tag{15}$$

$$V(t, p, z) = \sup_{u \in \mathcal{U}_{tz}^{adm}} J(t, p, z; u) \tag{16}$$

Strict Swing Contracts - A solution

Roughly: Approximate with Penalized contracts.

$$I^c = \left[m + \frac{1}{\sqrt{c}}, M - \frac{1}{\sqrt{c}}\right] \tag{17}$$

$$\Phi^c(z) = -c\left[\left((m + \frac{1}{\sqrt{c}}) - z\right)^+ + \left(z - (M - \frac{1}{\sqrt{c}})\right)^+\right] \tag{18}$$

$$J^c = J + \mathbb{E}_{tpz}\left[e^{-r(T-t)}\Phi^c(Z(T))\right] \tag{19}$$

$$V^c(t,p,z) = \sup_{u \in \mathcal{U}_t} J^c(t,p,z;u) \tag{20}$$

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The diffusion is still:

$$dX^{t,z,p;u}(s) = d\begin{bmatrix} P^{t,p} \\ Z^{t,z;u} \end{bmatrix} = \begin{bmatrix} f(s, P^{t,p}(s)) \\ u(s) \end{bmatrix} ds + \begin{bmatrix} \sigma(s, P^{t,p}(s)) \\ 0 \end{bmatrix}$$

Strict Swing Contracts - Characterization and Regularity of V^c

Because $\mathcal{U}_t$ is 'nice', V^c is unique viscosity solution to the HJB equation:

$$V_t^c + rV^c + \inf_{w \in \mathcal{U}_t} \left[(p - K)w + wV_z^c + fV_{pp}^c \right] = 0 \qquad (21)$$

$$V^c(T, p, z) = \Phi^c(z) \qquad (22)$$

And have regularity properties much as in the penalized case (V^c is continuous, Lipschitz in z and p etc.)

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Note that all V^c are solutions to same equation (21), but with different boundary conditions, and:

$$\lim_{c\to\infty} \Phi^c(z) = \mathcal{I}_{[m,M]}(z) = \begin{cases} 0 & \text{if } z \in [m,M] \\ -\infty & \text{if } z \notin [m,M] \end{cases} \qquad (23)$$

Strict Swing Contracts - The Approximation Result

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$$\mathbb{E}_{tpz}\left[\left|\int_t^T e^{-r(s-t)}(P^{t,p}-K)u(s)ds\right|\right]$$
$$\leq e^{rt}\mathbb{E}_{tpz}\left[\bar{u}\int_t^T |P^{t,p}(s)|\right] + e^{rt}K\bar{u}(T-t)$$
$$\leq C$$

Where the bound on $\mathbb{E}_{tpz}\left[\int_t^T |P^{t,p}(s)|\right]$ comes from standard existence and uniqueness results for diffusions (assume f, σ have sublinear growth)

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Where the bound on $\mathbb{E}_{tpz}\left[\int_t^T |P^{t,p}(s)|\right]$ comes from standard existence and uniqueness results for diffusions (assume f, σ have sublinear growth)

Hence $V(t,p,z) = \sup_{u\in\mathcal{U}_{tz}^{adm}} J(t,p,z;u) \leq C$ is well defined, assuming $\mathcal{U}_{tz}^{adm} \neq \emptyset$

Strict Swing Contracts - The Approximation Result cont.

Say $[m, M]$ is *reachable* from (t, z) if exists a Borel measurable function $u : [t, T] \to [0, \bar{u}]$ such that $z + \int_t^T u(s)ds \in [m, M]$.

[7] Basei, Cesaroni, and Vargiolu 2014.

Strict Swing Contracts - The Approximation Result cont.

Say $[m, M]$ is *reachable* from (t, z) if exists a Borel measurable function $u : [t, T] \to [0, \bar{u}]$ such that $z + \int_t^T u(s)ds \in [m, M]$.

$\mathcal{D} = \{(t, p, z) \in [0, T] \times \mathbb{R} \times \mathbb{R} : \ [m, M]$ reachable from $(t, z)\}$

$\mathcal{D}^\rho = \{(t, p, z) \in [0, T] \times \mathbb{R} \times \mathbb{R} : \ [m + \rho, M - \rho]$ reachable from $(t, z)\}$

$\tilde{\mathcal{D}} = \cup_\rho \mathcal{D}^\rho$

Theorem 2 (Lemma 3.3 and §4.1 in[7])

1. $\mathcal{D}^\rho = \mathcal{D}^\rho_{tz} \times \mathbb{R}_p$ *is non-empty for all* $0 < \rho < (M - m)/2$.
2. *Hence* $\mathcal{D}^\rho \subset \mathcal{D} = \mathcal{D}_{tz} \times \mathbb{R}_p$ *is nonempty.*
3. *Thus* $\mathcal{U}_{tz}$ *is non-empty for all* $(t, z) \in \mathcal{D}_{tz}$.
4. *So* $V(t, p, z)$ *is well defined on* $\mathcal{D}$.

[7] Basei, Cesaroni, and Vargiolu 2014.

Strict Swing Contracts - Picture of feasible region

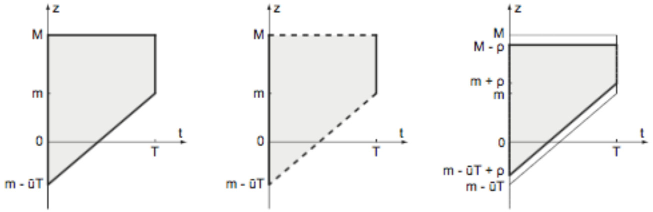

Figure: $\mathcal{D}_{tz}$, $\tilde{\mathcal{D}}_{tz}$ and $\mathcal{D}^{\rho}_{tz}$ for typical ρ.[8]

[8] Basei, Cesaroni, and Vargiolu 2014, Figure 2 in.

Strict Swing Contracts - The Approximation Theorem cont.

Theorem 3 (Theorem 3.5 in[9])

With notation as above, as $c \to \infty$ the V^c converge uniformly on compact sets of $\mathcal{D}^\rho$ to V, for $0 < \rho < (M-m)/2$.

Proof.

$\square$

[9] Basei, Cesaroni, and Vargiolu 2014.

Strict Swing Contracts - Consequences of Approx. Theorem

1. V is continuous on $\tilde{\mathcal{D}}$ (Corollary 3.6).
2. V is a viscosity solution to HJB equation (21), with boundary conditions TBD. (follows from stability of viscosity solutions, Corollary 3.7)
3. Explicit boundary conditions for $V(t, p, z)$ can be determined (See Theorem 4.4).
4. $V(t, \cdot, z)$ is Lipschitz, and a.e. twice differentiable (wrt p) (Prop. 4.5)
5. $V(t, p, \cdot)$ is Lipschitz, concave and a.e. twice differentiable (wrt z) (Prop 4.6)

Strict Swing Contracts - Optimal Buying Policy

Again, $u^*(s)$ is optimal if it achieves the inf:

$$\mathcal{A}^{u^*} V + L(u^*, t, x) = \inf_{w \in \mathcal{U}_t^{adm}} \left[\mathcal{A}^w V + L(w, t, x) \right]$$

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Observe that optimal control for penalized case:

$$u^*(t,p,z) = \begin{cases} \bar{u} & \text{if } V_z(t,p,z) \geq -(p-K) \\ 0 & \text{if } V_z(t,p,z) < -(p-K) \end{cases} \tag{24}$$

is well defined as $V_z(t,p,z)$ exists a.e.

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Moreover if $z \in [m, M-(T-t)\bar{u}]$ (and $M-(T-t)\bar{u} \geq m$) then:

$$m \leq Z(T) = z + \int_t^T u(s)ds \leq z + \bar{u}(T-t) \leq M \quad \text{for all } u \in \mathcal{U}_t$$

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Moreover if $z \in [m, M - (T - t)\bar{u}]$ (and $M - (T - t)\bar{u} \geq m$) then:

$$m \leq Z(T) = z + \int_t^T u(s) ds \leq z + \bar{u}(T - t) \leq M \quad \text{for all } u \in \mathcal{U}_t$$

Hence $\mathcal{U}_{tz}^{adm} = \mathcal{U}_t$ and so $u^*(s)$ above is also an optimal buying policy for Strict Swing Contract.

Thanks for listening!

References I

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