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Masters Thesis

On Uniformization of Compact Kähler Manifolds with Negative First Chern Class by Bounded Symmetric Domains

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UNIVERSITY OF CAPE TOWN

Abstract

Faculty of Science
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Master of Science

On Uniformization of Compact Kähler Manifolds with Negative First Chern Class by Bounded Symmetric Domains

by Daniel Mckenzie

We consider two complementary problems: given a compact Kähler manifold with negative first Chern Class, when is its universal cover a Bounded Symmetric Domain? And if it is, which Bounded Symmetric Domain is it? Existing literature is discussed, with particular attention given to two recent papers of Catanese and Di Scala ([CDS12] and [CDS]) which answer both questions first for Bounded Symmetric Domains of Tube Type, and then for all Bounded Symmetric Domains without Ball Factors. Using work of Yau and others on ball quotients we extend the main result of [CDS] to all bounded Symmetric Domains, including those with ball factors, thus answering the two questions posed in full generality.

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Contents

Chapter 1

Introduction

This thesis is concerned with the following problem:

> Given a compact Kähler manifold M, of dimension at least 2, can we determine its universal cover $\tilde{M}$?[1]

This question is motivated by the fact that for complex manifolds of dimension one[2] (Riemann surfaces) this question was is elegantly answered using the *genus* of M, written $g(M)$:

Theorem 1.1. *Suppose that M is a compact connected Riemann surface. Then $\tilde{M}$ is:*

1. *$\mathbb{CP}^1$ if $g(M) = 0$. Actually if $g(M) = 0$ M is simply connected.*
2. *$\mathbb{C}$ if $g(M = 1$.*
3. *The unit disk $\mathcal{D}$ equipped with the Poincaré metric (cf. §4.1.1) if $g(M) > 1$.*

Proof. See Chapter 10 of [Don11]. □

$g(M)$ may be defined topologically or geometrically as the number of 'holes' in M considered as a surface (i.e. a real two dimensional manifold) or more formally as:

$$g(M) = \frac{2 - 2\chi(M)}{2}$$

[1]In this thesis, we shall always denote the universal cover of M N etc. by $\tilde{M}$, $\tilde{N}$ etc.

[2]One dimnesional complex manifolds are automatically Kähler

where $\chi(M)$ is the Euler characteristic of M. But $g(M)$ may also be defined algebraically as the dimension of the vector space of holomorphic one-forms of M:

$$g(M) = \dim(H^0(\Omega_M))$$

and so already we see some of the interplay between Algebraic Geometry and Differential Geometry that we shall exploit in this thesis. In the first case the Fubini-Study metric on $\mathbb{CP}^1$ descends to a metric of constant *positive* Gaussian curvature on M. In the second and third cases the Euclidean and Poincaré metrics on $\mathbb{C}$ and $\mathcal{D}$ descend to metric of constant zero or negative Gaussian curvature on M respectively. Of these three cases the third is of most interest to us, as 'most' Riemann Surfaces will have genus greater than 1. Note that $\mathcal{D}$ is a bounded domain in a complex vector space, and so is a particularly easy to describe and work with. $\mathcal{D}$ is also *symmetric* in a sense to be made precise in Chapter 4. For this, and other reasons elaborated on in the introduction to [CDS12], we restrict our focus to the following question:

Question 1. Given a compact Kähler manifold M, when is its universal cover, $\tilde{M}$ a bounded symmetric domain $\Omega \subset \mathbb{C}^n$?

In Chapter 4 we shall show that there exists a classification of bounded symmetric domains, and that there are finitely many in each dimension. This motivates our second question:

Question 2. We know that the irreducible bounded symmetric domains are completely classified. So, if $\tilde{M}$ is a bounded symmetric domain, can we determine its irreducible components with multiplicities?

A necessary condition for $\tilde{M}$ to be a bounded symmetric domain, generalising the observation that Riemann surfaces having $\mathcal{D}$ as their universal cover have a metric of constant negative curvature is the following:

Theorem 1.2. *If M has a bounded symmetric domain as its universal cover, then $c_1(M) < 0$, where $c_1(M)$ denotes the first Chern class (defined in §3.8)*

Proof. See Chapter 3 or the discussion in the introduction of [CDS12] □

We note that we have the following easy but unsatisfactory answer to 1:

Theorem 1.3. *If M is a complex manifold with $c_1(M) < 0$ and a Kähler metric g such that the curvature, F_∇ of the Levi Civita connection associated to g is parallel: $\nabla F_\nabla = 0$, then $\tilde{M}$ is a bounded symmetric domain.*

Proof. $\nabla F_\nabla = 0$ implies that $\tilde{M}$ is a Hermitian Symmetric space (cf. Theorem 1.1 on pg. 198 of [Hel78]) or Theorem 4.1 in Chapter 4. Negative first Chern class implies that $\tilde{M}$ is of non-compact type, and hence by the Harish-Chandra embedding theorem (cf. Theorem 7.1 on pg. 383 of [Hel78]) □

However this theorem is not very useful for two reason. Firstly, to compute F_∇ and ∇F_∇ is in most cases computationally infeasible. Secondly,and more importantly, there may exist infinitely many Kähler metrics on M, thus we don't have much chance of finding the 'right' one, for which the curvature tensor is indeed parallel. What we are looking for is a characterization that avoids appealing directly to metric properties of M.
Our primary technical tools, discussed in chapter 3, are:

Theorem 1.4 (De Rham Decomposition theorem, Theorem 3.16)**.** *If $\tilde{g}$ is a Kähler metric on $\tilde{M}$ and $Hol_{\tilde{x}}(\tilde{M}, \tilde{g})$ denotes the holonomy group of $\tilde{M}$ at $\tilde{x} \in \tilde{M}$, and*

$$T_{\tilde{x}}M = T_1 \oplus \ldots \oplus T_r$$

is the decomposition of $T_{\tilde{x}}M$ into $Hol_{\tilde{x}}(\tilde{M}, g)$ irreducible subspaces, then there is a corresponding decomposition of $\tilde{M}$ as a Riemannian product:

$$(\tilde{M}, \tilde{g}) = (M_1, g_1) \times \ldots \times (M_r, g_r) \tag{1.1}$$

as a product of simply connected Kähler manifolds. Moreover:

$$Hol(\tilde{M}, g) \cong Hol(M_1, g_1) \times \ldots \times Hol(M_r, g_r)$$

with $Hol(M_i, g_i)$ acting irreducibly on T_i for all i.

and the:

Theorem 1.5 ((modified) Berger Holonomy Theorem, Theorem 3.13)**.** *Suppose that $\tilde{M}$ is a simply connected Kähler manifold of dimension m[3] with Kähler metric g. Unless $\tilde{M}$ is a bounded symmetric domain, $Hol(\tilde{M}, g)$ is equal to $U(m)$, $SU(m)$ or $Sp(m/2)$, with the third case possible only if m is even.*

Recalling that $c_1(M) < 0$ is a necessary condition for $\tilde{M}$ to be a bounded symmetric domain, we sharpen the above theorem to:

Theorem 1.6 (Theorem 3.31)**.** *If $\tilde{M}$ is a simply conected Kähler manifold with $c_1(\tilde{M}) < 0$, either $Hol(\tilde{M}, g) = U(m)$ for every Kähler metric on $\tilde{M}$ or M is a bounded symmetric domain.*

[3]In this thesis we shall always denote the dimension of M, M', M_i etc as m, m', m_i etc. This will usually be the dimension over $\mathbb{C}$

So in order to show that one of the factors M_i in equation (1.1) is a bounded symmetric domain we need to show that its holonomy group cannot be $U(m_i)$. We do this by appealing to the holonomy principle:

Theorem 1.7 (Theorem 3.14)**.** *Suppose that (M, g) is a Riemannian manifold and that $A \in \Gamma((TM)^{\otimes k} \otimes (T^*M)^{\otimes l})$*[4] *is a parallel tensor field (that is, $\nabla A = 0$ where ∇ is the Levi-Civita connection associated to g). For any $x \in M$, $A_x \in (TM)^{\otimes k} \otimes (T^*M)^{\otimes l})$ is $Hol_x(M, g)$-invariant.*

and the following result from representation theory:

Theorem 1.8 (Theorem 2.8)**.** *If $U(m)$ acts on T_xM irreducibly, then $S^kT^*_xM \otimes (K_M)^l_x$ is an irreducible representation for any $k \in \mathbb{Z}^+$ and $l \in \mathbb{Z}$. Here K_M is the top exterior power of TM*

Thus, assuming $c_1(M) < 0$ and that $Hol(M, g)$ acts irreducible on T_xM, if we can produce a parallel tensor field $A \in \Gamma(S^lT^*M \otimes (K_M)^m)$ we can conclude that M is a bounded symmetric domain. We strengthen this approach further by using a result of Kobayashi's (see [Kob80]) to show in chapter 5 that if $l = -q$ and $k = mq$ then *any* $A \in \Gamma(S^lT^*M \otimes (K_M)^m)$ is parallel. This approach, outlined in Kobayashi ([Kob80]), and Yau ([Yau88]) and given as a theorem in [Yau93]) (see also [VZ05], Theorem 1.4 for a clear exposition of the results in [Yau93]), is, as discussed in Chapter 6, taken to its logical conclusion in [CDS12]. There the authors point out that only some bounded symmetric domains (those of *tube type*) have such tensor fields.
We may also take a dual viewpoint here. Since $S^kT^*_xM$ is the vector space of all homogeneous polynomials of degree k on T_xM, the fact that it is an irreducible $U(m)$ representation for all k means that there are no proper $U(m)$ invariant varieties[5] $V \subset \mathbb{P}(T_pM)$. Thus if there exists a holonomy invariant variety $V \subset \mathbb{P}(T_pM)$ we can conclude that M is a bounded symmetric domain. This turns out to be a more fruitful approach. As shown in [CDS] and discussed in Chapter 7 a larger class of bounded symmetric domains, those not of ball-type possess such varieties. Moreover, if we know the dimension of the holonomy invariant variety and the dimension of M, we can classify M as a bounded symmetric domain. We then extend this result to include bounded symmetric domains of ball type (see Theorem 7.11) providing a (somewhat) satisfactory answer to questions 1 and 2.

[4]For any vector bundle $E \to M$, $\Gamma(E)$ denotes its global sections. Whether we are considering smooth or holomorphic sections is important, but usually clear from the context or explicitly stated. In this introductory section we shall be a bit vague and not specify which we are considering.

[5]When speaking of varieties, we shall usually be thinking of analytic, not algebraic varieties, but because of the GAGA correspondence (cf. Theorem A on pg. 75 of [GA74]) it doesn't really matter

Chapter 2

Lie Groups, Lie Algebras and Representation Theory

In this section we record the properties of Lie groups that will be necessary in later chapters. For a more thorough account of Lie groups, the reader is referred to [Zil10], [FH91] or [Kna96]. A Lie group G is a smooth manifold equipped with a group structure:

$$\begin{aligned} m : & G \times G \to G \\ & (g, h) \mapsto gh \\ (\cdot)^{-1} : & G \to G \\ & g \mapsto g^{-1} \end{aligned}$$

and an identity element $e \in G$, such that m and g are smooth maps. We shall try be as consistent as possible and use G, H or K to denote a Lie group, with the symbol K being used to denote a compact lie group. Stated more abstractly, a Lie group is a group object in the category of smooth manifolds. A *Lie group homomorphism* is a map $f : G \to H$ that respects both the smooth and the group structure of G and H. That is, $f \in \mathcal{C}^\infty(G, H)$ and f is a group homomorphism. Associated to every Lie group is a Lie algebra:

Definition 2.1. A Lie algebra is a vector space V (we shall consider only the cases where V is a $\mathbb{R}$- or a $\mathbb{C}$- vector space in this thesis) equipped with a binary operation called the Lie bracket, satisfying

$$[X, Y] = -[Y, X] \text{ Skew-commutativity} \tag{2.1}$$

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0 \text{ The Jacobi Identity} \tag{2.2}$$

for all $X, Y, Z \in V$.

We shall always denote the Lie algebra by the lower case gothic script version of the letter denoting the Lie group. Thus the Lie algebra of G would be $\mathfrak{g}$, the Lie algebra of K would be $\mathfrak{k}$, and so on. There is a one-to-one correspondence between Lie algebras and connected, simply connected Lie groups, and if G, H are Lie groups and $\pi : G \to H$ is a covering map, then $\mathfrak{g} \cong \mathfrak{h}$. A nice invariant way to construct $\mathfrak{g}$ given G is to consider the tangent space at the identity, T_eG, and equip it with a Lie bracket as follows:

1. First, for any fixed $g \in G$, consider the inner automorphism:

   $$\Psi_g : G \to G$$
   $$\Psi_g(h) = ghg^{-1}$$

   We shall be as consistent as possible and try to always use Ψ_g to denote conjugation by g.

2. Now, consider the derivative of Ψ_g:

   $$d\Psi_g : T_eG \to T_eG$$
   $$d\Psi_g|_e(X) = \frac{d}{dt}|_{t=0}\Psi_g(\gamma(t))$$

   Where $\gamma : (-\epsilon, \epsilon) \to G$ is any path in G satisfying $\gamma(0) = e$ and $\gamma^{'}(0) = X$. Henceforth we shall denote the map $d\Psi_g|_e$ as Ad_g. Note that we can think of Ad as a map $G \to Gl(T_eG)$, sending g to $Ad_g = d\Psi_g|_e$.

3. Finally we take the derivative of Ad, to get a map:

   $$ad : T_eG \to \mathrm{End}(T_eG)$$
   $$ad_X = \frac{d}{dt}|_{t=0}Ad_{\gamma(t)}$$

   where again $\gamma : (-\epsilon, \epsilon) \to G$ is any path in G satisfying $\gamma(0) = e$ and $\gamma^{'}(0) = X$. Now define $\mathfrak{g}$ to be T_eG with $[X, Y] := ad_X(Y)$. One should check that the bracket defined in this way is skew-commutative and satisfies the Jacobi identity (see 2.1), but this is a standard and fairly easy exercise (see [FH91] exercise 8.10 pg. 109).

The maps Ψ, Ad and ad are of independent interest, and we shall return to them shortly. But first, let us mention that one can connect a Lie algebra to its Lie group using the *exponential map*.

2.1 The exponential map

Given a Lie group G, we call any Lie group homomorphism $\varphi : \mathbb{R} \to G$ a *one-parameter subgroup*. It is a fact that to every $X \in \mathfrak{g}$ there corresponds a unique one-parameter subgroup $\varphi_X(t)$ satisfying $\varphi_X^{'}(0) = X$ (see [FH91] pg. 115)

Definition 2.2. The exponential is the map:

$$\begin{aligned} \mathfrak{g} &:\to G \\ X &\mapsto \varphi_X(1) \end{aligned}$$

exp is characterised by being the unique map from $\mathfrak{g}$ to G taking 0 to e whose differential at the origin is the identity $Id : \mathfrak{g} \to \mathfrak{g}$ and which carries lines through the origin in $\mathfrak{g}$ to one-parameter subgroups of G (see [FH91] Prop. 8.33 pg. 116).

Remark 2.3. If G is a subgroup of $Gl(n, \mathbb{R})$ then exp is given by the matrix exponential:

$$\exp(X) = I + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{X^n}{n!}$$

We shall find this concrete description of exp useful later.

Given a Lie algebra $\mathfrak{g}$, we denote by $\mathrm{Aut}(\mathfrak{g})$ the group of vector space automorphisms of $\mathfrak{g}$ satisfying $\varphi([X, Y]) = [\varphi X, \varphi Y]$. We say a linear map $A : \mathfrak{g} \to \mathfrak{g}$ is a *derivation* if it satisfies:

$$A([X, Y]) = [AX, Y] + [X, AY]$$

We denote the set of all derivations as $\mathrm{Der}(\mathfrak{g})$.

Proposition 2.4. 1. *Aut*$(\mathfrak{g})$ *is a Lie group with Lie algebra Der*$(\mathfrak{g})$*, where the Lie bracket is given by the commutator:*

$$[A, B] = AB - BA$$

2. $Ad_g \in Aut(\mathfrak{g})\ \forall g \in G$

3. $ad_X \in Der(\mathfrak{g})\ \forall X \in \mathfrak{g}$

4. $Ad_{\exp(X)} = e^{ad_X} = \sum_{n=0}^{\infty} \frac{X^n}{n!} \quad \forall X \in \mathfrak{g}$

Proof. See the discussion on pg. 126-128 of [Hel78] □

The above proposition tells us that $ad(\mathfrak{g})$ is a subalgebra of $\mathrm{Der}(\mathfrak{g})$. We call the subgroup of $\mathrm{Aut}(\mathfrak{g})$ corresponding to $ad(\mathfrak{g})$ the group of *inner automorphisms* and denote it as $\mathrm{Int}(\mathfrak{g})$. If G is any Lie group corresponding to $\mathfrak{g}$, note that $Ad(G) = G/Z(G) = \mathrm{Int}(\mathfrak{g})$ (see the discussion after Cor. 5.2 on pg. 129 pf [Hel78]) but that $\mathrm{Int}(\mathfrak{g})$ can be defined using only the Lie algebra $\mathfrak{g}$ and no knowledge of any Lie group corresponding to it. Before moving on, we generalize item 4 of Prop. 2.4 to the following:

Theorem 2.5. *Suppose that G and H are Lie groups. Let $f : G \to H$ be a Lie group homomorphism. Then for all $X \in \mathfrak{g}$,*

$$f(\exp_G(X)) = \exp_H(df(X))$$

where $exp_G : \mathfrak{g} \to G$ and $\exp_H : \mathfrak{h} \to H$.

Proof. Observe that $\psi(t) = f(\exp_G(tX))$ gives a one parameter subgroup of H satisfying $\psi'(0) = df(X)$. Since $\varphi_{df(X)}(t) = \exp_H(tdf(X))$ is the unique one parameter subgroup of H satisfying $\varphi'_{df(X)}(0) = df(X)$ we must have that:

$$\psi(t) = \varphi_{df(X)}(t) \quad \forall t \in \mathbb{R}$$

and so in particular $f(\exp_G(X)) = \psi(1) = \varphi_{df(X)}(1) = \exp_H(df(X))$. $\square$

2.2 Semisimple and Reductive Lie algebras

A *subalgebra* of a Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{h} \subset \mathfrak{g}$ which is closed under Lie bracket. We shall usually write this as $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$. Subalgebras of $\mathfrak{g}$ are in one-to-one correspondence with connected, closed subgroups of G (see Ch. II §2 pg. 112-120 in[Hel78]). Given a closed Lie subgroup $H < G$ a useful characterisation of its Lie algebra $\mathfrak{H}$ is the following (Prop. 2.7 pg. 118 in[Hel78]):

$$\mathfrak{h} = \{X \in \mathfrak{g} : \exp(tX) \in H \ \forall t \in \mathbb{R}\}$$

We define the centre of a Lie algebra $\mathfrak{g}$ as:

$$\mathfrak{z}(\mathfrak{g}) = \{X \in \mathfrak{g} : [X, Y] = 0 \ \forall Y \in \mathfrak{g}\}$$

We remark that it is an easy exercise to show that if $\mathfrak{g}$ is the Lie algebra of G, then $\mathfrak{z}(\mathfrak{g})$ is the Lie algebra of the centre of G, $Z(G)$. An ideal of $\mathfrak{g}$ is a subalgebra $\mathfrak{h}$ satisfying:

$$[X, Y] \in \mathfrak{h} \ \forall X \in \mathfrak{h}, \ Y \in \mathfrak{g}$$

We shall frequently write this as $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$. We say $\mathfrak{g}$ is simple if it contains no non-trivial ideals, and semisimple if it contains no non-trivial abelian ideals. As we shall see later, the semisimplicity assumption is very powerful. For example, since any semisimple Lie algebra is the direct sum of simple Lie algebras, each of which forms an ideal in $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_r$$

we shall frequently reduce arguments involving semisimple Lie algebras to arguments about their constituent simple parts. This is useful as the simple Lie algebras (over $\mathbb{R}$ or $\mathbb{C}$) are completely classified. But we shall also have occasion to consider slightly more general Lie algebras, the reductive Lie algebras:

Definition 2.6. $\mathfrak{g}$ is *reductive* if $\mathfrak{g} = \mathfrak{g}_{ss} \oplus \mathfrak{z}(\mathfrak{g})$ where $\mathfrak{g}_{ss}$ is semisimple. (See [FH91] pg. 131 for alternative definitions and discussion)

If G is a Lie group corresponding to $\mathfrak{g}$, we shall frequently refer to G as semisimple (respectively reductive) if $\mathfrak{g}$ is semisimple (respectively reductive).

2.3 Representations of Lie groups

Given a Lie group G, a *representation* of G is a map:

$$\rho : G \to Gl(V, \mathbb{K})$$

where V is a vector space over a field $\mathbb{K}$ (from now on we shall implicitly be assuming that $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and that V is finite dimensional). We shall denote this representation by (ρ, V), or ρ, when it is clear what V is, or V, when it is clear what G and ρ are. Given a representation ρ there is also a representation of $\mathfrak{g}$ given by the differential of ρ:

$$\begin{aligned} d\rho &: \mathfrak{g} \to \mathfrak{gl}(V) \\ &: X \mapsto \frac{d}{dt}|_{t=0} \exp(tX) \end{aligned}$$

We shall denote this representation as $(d\rho, V)$, or $d\rho$ for short. A *sub-representation* is a subspace $W \subset V$ such that:

$$\rho(g)(W) \subset W \; \forall g \in G$$

We say that ρ (respectively $d\rho$) is irreducible if V has no proper sub-representations. We say that ρ (respectively $d\rho$) is faithful if it is injective.

Theorem 2.7. *If V is a representation of a Lie group G, and G is either compact or semisimple, then V splits into the direct sum of irreducible representations:*

$$V = V_1 \oplus \ldots \oplus V_r$$

Proof. This is a consequence of Weyl's unitary trick and is discussed on pg. 130 of [FH91]. □

2.3.1 Irreducible representations of $U(n)$

In general, classifying the irreducible finite dimensional representations of an arbitrary Lie Group G is a non-trivial problem. However, the case of the unitary group $U(n)$ is fairly straightforward. Before we discuss the classification of irreducible representations of $U(n)$, we first note that if we have a representation $\rho : G \to Gl(V)$, we also have representations[1]:

1. V^*. This representation is given by $\rho^*(g) = \rho(g)^{-t}$ so as to preserve the pairing between V and V^*. That is, if $v \in V$ and $v^* \in V^*$ and $\langle v, v^* \rangle$ denotes the pairing then

$$\langle \rho^*(g)(v^*), \rho(g)(v) \rangle = \langle \rho(g)^{-t} v^*, \rho(g)(v) \rangle = \langle v^*, \rho(g)^{-1} \rho(g)(v) \rangle = \langle v^*, v \rangle$$

2. $V^{\otimes k}$ for any $k \in \mathbb{Z}^+$. Defined simply as

$$\rho^k(g)(v_1 \otimes \ldots \otimes v_k) = (\rho(g)(v_1)) \otimes \ldots \otimes (\rho(g)(v_k))$$

   This representation commutes with the representation of S_k given by:

$$\sigma \cdot v_1 \otimes \ldots \otimes v_k = v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(k)}$$

   and hence it induces representations on the symmetric and anti-symmetric powers of V.

[1] This is discussed in more detail in chapter 1 of [FH91]

For $U(n)$ the *determinental representation*:

$$\rho_{det} : U(n) \to Gl(\bigwedge^{n} \mathbb{C}^n) \cong \mathbb{C}^*$$
$$\rho_{det}(g)(v_1 \wedge \ldots \wedge v_n) = det(g)v_1 \wedge \ldots \wedge v_n$$

is a non-trivial one dimensional representation. We denote this representation by D_1. We denote by D_{-1} its dual representation, and naturally enough by D_k (respectively D_{-k}) the k-th tensor power of D_1 (respectively D_{-1}).

Theorem 2.8. *Let $U(n)$ act on $\mathbb{C}^n$ in the usual manner, that is, by left matrix multiplication. For all $k \in \mathbb{Z}^+$ and $l \in \mathbb{Z}$, $S^k(\mathbb{C}^n) \otimes D_l$ is an irreducible $U(n)$ representation.*

Proof. This is standard. See for example page 231 - 233 of [FH91]. □

2.3.2 Orthogonal, unitary and complex representations

Suppose we have a representation:

$$\rho : G \to Gl(V, \mathbb{R})$$

and moreover that V is equipped with an extra structure such as an inner product g, or perhaps V is an even-dimensional $\mathbb{R}$-vector space equipped with a complex structure J (that is, $J \in Gl(V)$ and $J^2 = -id$). We can now ask how ρ interacts with this extra structure.

Definition 2.9. If V is an even dimensional $\mathbb{R}$-vector space equipped with a complex structure J, then $\rho : G \to Gl(V, \mathbb{R})$ is said to be a *complex representation* if $\rho(\varphi)J = J\rho(\varphi)$ for all $\varphi \in G$. Equivalently, if we make V into a $\mathbb{C}$-vector space by defining:

$$(a + ib)v = av + bJv$$

then ρ is complex if $\rho(G) \subset Gl(V, \mathbb{C})$.

If V is equipped with an inner product g, then we have:

Definition 2.10. ρ is said to be an *orthogonal representation* if

$$g(\rho(\varphi)(X), \rho(\varphi)(Y)) = g(X, Y) \ \forall X, Y \in V, \ \forall \varphi \in G$$

Equivalently, $\rho(G) \subset O(V, g) \cong O(n)$ where $dim_{\mathbb{R}}(V) = n$.

Now suppose that V is a $\mathbb{C}$-vector space equipped with a Hermitian inner product h. As we would expect;

Definition 2.11. ρ is said to be an *unitary* representation if:

$$h(\rho(\varphi)(X), \rho(\varphi)(Y)) = h(X, Y)\ \forall X, Y \in V,\ \forall \varphi \in G$$

Equivalently, $\rho(G) \subset U(V, h) \cong U(n)$ where $dim_{\mathbb{C}}(V) = n$

If ρ preserves a symmetric (or hermitian symmetric) bilinear form g then $d\rho$ is skew-symmetric with respect to this form:

$$\begin{aligned}
&g(\rho(\exp(tX))(\xi_1), \rho(\exp(tX))(\xi_2)) = g(\xi_1, \xi_2)\ \forall \xi_1, \xi_2 \in V \\
\Rightarrow &\frac{d}{dt}|_{t=0} g(\rho(\exp(tX))(\xi_1), \rho(\exp(tX))(\xi_2)) = 0 \\
\Rightarrow &g(\frac{d}{dt}|_{t=0}\rho(\exp(tX))(\xi_1), \xi_2) + g(\xi_1, \frac{d}{dt}|_{t=0}\rho(\exp(tX))(\xi_2)) = 0 \\
\Rightarrow &g(d\rho(X)(\xi_1), \xi_2) = -g(\xi_1, d\rho(X)(\xi_2))
\end{aligned}$$

2.3.3 Schur's lemma

Suppose that (V_1, ρ_1) and (V_2, ρ_2) are two representations of a Lie group G. If $\tau : V_1 \to V_2$ is a linear map satisfying:

$$\tau(\rho_1(g)(X)) = \rho_2(\tau(X)) \quad \forall X \in V_1\ \forall g \in G \tag{2.3}$$

We say that τ is *G-equivariant*. Observe that $ker(\tau) \subset V_1$ is a ρ_1-invariant subspace, since for any $v \in \ker(\tau)$ and $\varphi \in G$

$$\tau(\rho_1(\varphi)(v)) = \rho_2(\varphi)(\tau(v)) = 0$$

Similarly, $\text{coker}(\tau) \subset V_2$ is ρ_2-invariant. So, if V_1 and V_2 are both irreducible, τ is either the zero map or an isomorphism. This simple conclusion is known as *Schur's lemma*. Suppose that V is defined over the field $\mathbb{K}$ (which is either $\mathbb{R}$ or $\mathbb{C}$). If $\tau : V \to V$ is any G-equivariant endomorphism with an eigenvalue λ. Then:

$$\tau - \lambda I : V \to V$$

is a G-equivariant map with a non-trivial kernel. But then by the above $\ker(\tau - \lambda I) = V$ and so $\tau = \lambda I$. If $\mathbb{K} = \mathbb{C}$ then any endomorphism τ has an eigenvalue, and so:

Corollary 2.12. *All G-equivariant endomorphisms of a complex representation (V, ρ) are of the form λI for some $\lambda \in \mathbb{C}$*

We note that there are real representations having G-equivariant endomorphisms not of the form aI for $a \in \mathbb{R}$. Further discussion of this result may be found as Lemma 5.1 on page 93 of [Zil10]).

If V is a complex vector space, recall that a map $A : V \times V \to V$ is said to be a *sesquilinear form* if it satisfies:

$$\begin{aligned} A(\alpha X + \beta Y, Z) &= \alpha A(X, Z) + \beta A(Y, Z) \\ \text{and } A(X, Y) &= \overline{A(Y, X)} \end{aligned}$$

Corollary 2.13. *Suppose that $\rho : G \to Gl(V)$ is an orthogonal (resp. unitary) irreducible representation of G with inner (resp. Hermitian inner) product $< \cdot, \cdot >$. Let A be a non-trivial G-invariant symmetric bilinear (resp. sesquilinear) form on V. Then A is non-degenerate and $A = a < \cdot, \cdot >$ for some $a \in \mathbb{R}$.*

Proof. $\ker(A) = \{X \in V : \ A(X, Y) = 0 \ \forall Y \in V\}$ is a ρ-invariant subspace of V. So, either $\ker(A) = V$ (in which case A is trivial) or $\ker(A) = \{0\}$ (in which case A is non-degenerate).

For any $X \in V$, define $A_X \in V^*$(resp. $A_X \in \bar{V}^*$) by $A_X(Y) = A(X, Y)$. The map $X \mapsto A_X$ is linear and injective since A is non-degenerate, so it is an isomorphism. Because $< \cdot, \cdot >$ also gives an isomorphism $V \to V^*$ (respectively $V \to \overline{V}^*$) by $X \mapsto < X, \cdot >$, for all $X \in V$ there exists an element in V (call it $\tilde{A}X$) such that

$$< \tilde{A}X, Y >= A_X(Y) = A(X, Y)$$

The map $X \mapsto \tilde{A}X$ is linear so $\tilde{A}$ is given by a matrix. Moreover $\tilde{A}$ is ρ-invariant since for any $\varphi \in G$, $X, Y \in V$, we have:

$$\begin{aligned} < \tilde{A}(\varphi X), Y > &= A(\varphi(X), Y) \\ &= A(X, \varphi^{-1}Y) \text{ since } Z \text{ is } \rho\text{-invariant} \\ &=< \tilde{A}X, \varphi^{-1}Y > \\ &=< \varphi(AX), Y > \ \text{ since } < \cdot, \cdot > \text{ is } \rho\text{-invariant} \end{aligned}$$

For any $X, Y \in V$, in the orthogonal case we have:

$$< \tilde{A}^T X, Y >=< X, \tilde{A}Y >=< \tilde{A}Y, X >= A(Y, X) = A(X, Y) =< \tilde{A}X, Y >$$

and so $\tilde{A}$ is symmetric. In the unitary case we have:

$$< \tilde{A}^H X, Y >=< X, \tilde{A}Y >= \overline{< \tilde{A}Y, X >} = \overline{A(Y, X)} = A(X, Y) =< \tilde{A}X, Y >$$

And so $\tilde{A}$ is Hermitian symmetric. In both cases $\tilde{A}$ has a real eigenvalue a, and so by Corollary 2.12 $\tilde{A} = aI$. But then:

$$A(X,Y) =< \tilde{A}X, Y >=< aX, Y >= a < X, Y >$$

as required. □

If A is positive definite then $a \in \mathbb{R}^+$

2.4 The Killing form and Compact Lie algebras

We now return to the structural theory of Lie algebras.

Definition 2.14. the *Killing form* of a lie algebra $\mathfrak{g}$ is defined as

$$B(X,Y) = \mathrm{tr}(ad_X \circ ad_Y)$$

One can easily check that B is a symmetric bilinear form. Moreover, B is $\mathrm{Aut}(\mathfrak{g})$-invariant, since if $\varphi([X,Y]) = [\varphi(X), \varphi(Y)]$ we have that

$$ad_{\varphi(X)}(Y) = [\varphi(X), Y] = \varphi([X, \varphi^{-1}(Y)] = (\varphi \circ ad_X \circ \varphi^{-1})(Y)$$

and so:

$$\begin{aligned} B(\varphi(X), \varphi(Y)) &= \mathrm{tr}\left((\varphi \circ ad_X \circ \varphi^{-1}) \circ (\varphi \circ ad_Y \circ \varphi^{-1})\right) \\ &= \mathrm{tr}(ad_X \circ ad_Y) = B(X,Y) \end{aligned}$$

And thus by the discussion in section 2.3.2, every derivation is skew-symmetric with respect to B:

$$B(AX, Y) + B(X, AY) = 0 \ \forall X, Y \in \mathfrak{g}, \ \forall A \in \mathrm{Der}(\mathfrak{g})$$

One immediate use of the Killing form is the following:

Theorem 2.15 (Cartan's second criterion)**.** *$\mathfrak{g}$ is semisimple if and only if B is non-degenerate*

Proof. see Theorem 3.19 on pg. 42 of [Zil10] □

One might then ask under what conditions B is positive, or negative, definite. In fact there is a precise condition on a Lie algebra which makes its Killing form negative definite.

Definition 2.16. A Lie algebra $\mathfrak{k}$ is *compact* if it is the Lie algebra of a compact Lie group.

Theorem 2.17 (Proposition 3.24 and 3.25 on pg. 44-45 of [Zil10])**.** *If $\mathfrak{k}$ is a compact Lie algebra then its Killing form is negative semi-definite. We write this as $B \leq 0$. If $\mathfrak{k}$ is in addition semisimple then its Killing form is negative definite. We write this as $B < 0$*

Proof. Let K be a compact Lie group with Lie algebra $\mathfrak{k}$. Then there exists an $Ad(K)$-invariant inner product on $\mathfrak{k}$ which we may create as follows.
Let $< \cdot, \cdot >_0$ be any inner product on $\mathfrak{k}$. Define a new inner product as follows:

$$< X, Y >= \int_K < Ad_k(X), Ad_k(Y) >_0 \, d\mu(k)$$

Where μ is the Haar measure on K. Since K is compact this integral converges, and $< \cdot, \cdot >$ is well-defined. Moreover $< \cdot, \cdot >$ is $Ad(K)$-invariant since μ is $Ad(K)$-invariant. So Ad is an orthogonal representation and hence ad_X is skew-symmetric for all $X \in \mathfrak{k}$ (see the discussion at the end of 2.3.2). This implies that the eigenvalues of ad_X; $\lambda_1, \ldots, \lambda_n$ are all either imaginary or 0. Hence:

$$B(X, X) = \mathrm{tr}(ad_X \circ ad_X) = \lambda_1^2 + \ldots + \lambda_n^2 \leq 0$$

If $\mathfrak{k}$ is semisimple then B is non-degenerate, and so $B(ad_X, ad_X) < 0$ for all $X \in \mathfrak{k}$. ☐

We note that in the semisimple case a stronger statement is true:

Theorem 2.18. *If $\mathfrak{k}$ is semisimple and compact any Lie group having $\mathfrak{k}$ as its Lie algebra is compact.*

Proof. See [Zil10] pg. 46 ☐

We say that a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is *compactly imbedded* if the subgroup of $\mathrm{Int}(\mathfrak{g})$ corresponding to $ad_{\mathfrak{g}}(\mathfrak{k})$ (cf. §) is compact.

Proposition 2.19 (Prop. 6.8 on pg 133 of [Hel78])**.** *Suppose that $\mathfrak{g}$ is a Lie algebra over $\mathbb{R}$ and $\mathfrak{k} \subset \mathfrak{g}$ is a compactly imbedded subalgebra. If $\mathfrak{k} \cap \mathfrak{z}(\mathfrak{g}) = \{0\}$ then $B_{\mathfrak{g}}|_{\mathfrak{k}} < 0$.*

Proof. Let $K < \mathrm{Int}(\mathfrak{g})$ be the subgroup corresponding to $ad_{\mathfrak{g}}(\mathfrak{k})$. As in the proof of 2.17 we construct an $Ad_{\mathrm{Int}(\mathfrak{g})}(K)$ invariant inner product on $\mathfrak{g}$. Again this implies that for any $X \in \mathfrak{k}$, the eigenvalues $\lambda_1, \ldots, \lambda_n$ of ad_X are all either imaginary or 0. Hence :

$$B_{\mathfrak{g}}(X, X) = \mathrm{tr}(ad_X \circ ad_X) = \lambda_1^2 + \ldots + \lambda_n^2 \leq 0$$

If $B_{\mathfrak{g}}(X, X) = 0$ then $\lambda_i = 0$ for all i and hence $ad_X = 0$. But then $X \in \mathfrak{k} \cap \mathfrak{z}(\mathfrak{g})$ and so by assumption $X = 0$. $\square$

A final remark about compact Lie algebras. If $\mathfrak{k}$ is compact then the exponential map $\exp : \mathfrak{k} \to K$ is surjective for any K corresponding to $\mathfrak{k}$ (see Corollary 3.29 pg. 47 of [Zil10]).

2.5 Cartan Decompositions

A *Cartan decomposition* of $\mathfrak{g}$ (cf. page 359-360 of [Kna96]) is a direct sum decomposition:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

Such that $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$ (that is, $\mathfrak{k}$ is a Lie subalgebra), $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ and $B|_{\mathfrak{k}}$ is negative definite while $B|_{\mathfrak{p}}$ is positive definite.
If $\mathfrak{g}$ is semisimple, then it has a Cartan decomposition (See the discussion on pg. 182-183 of [Hel78]). Moreover this decomposition is unique up to inner automorphism. That is, if

$$\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{p}_1 \tag{2.4}$$

$$\mathfrak{g} = \mathfrak{k}_2 \oplus \mathfrak{p}_2 \tag{2.5}$$

are two Cartan decompositions of $\mathfrak{g}$, there exists $\varphi \in \mathrm{Int}(\mathfrak{g})$ such that

$$\varphi(\mathfrak{k}_1) = \mathfrak{k}_2 \text{ and } \varphi(\mathfrak{p}_1) = \mathfrak{p}_2$$

This is Theorem 7.2 on pg. 183 of [Hel78]. Moreover $\mathfrak{k}$ is a maximal, compactly imbedded subalgebra of $\mathfrak{g}$ (This is part of Prop. 7.4 on pg. 184 of [Hel78]).Observe that since $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, the adjoint representation $ad : \mathfrak{k} \to \mathfrak{gl}(\mathfrak{g})$ restricts to a representation $ad : \mathfrak{k} \to \mathfrak{gl}(\mathfrak{p})$. If $\mathfrak{g}$ is non-compact, then $\mathfrak{p} \neq \{0\}$. We can now characterise simple non-compact Lie algebras in terms of this representation.

Theorem 2.20. *Let $\mathfrak{g}$ be a semi-simple non-compact Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then $\mathfrak{g}$ is simple if and only if $ad : \mathfrak{k} \to \mathfrak{gl}(\mathfrak{p})$ is faithful and irreducible.*

Proof. Suppose that $\mathfrak{g}$ is simple. If $ad : \mathfrak{k} \to \mathfrak{gl}(\mathfrak{p})$ is not faithful then $\mathfrak{k}_1 = \{X \in \mathfrak{k} : ad_X|_{\mathfrak{p}} = 0\}$ is non-empty. Obviously if $X \in \mathfrak{k}_1$ and $Y \in \mathfrak{p}$ then $[X, Y] = 0 \in \mathfrak{k}_1$. If $X \in \mathfrak{k}_1$ and $Y \in \mathfrak{k}$ then for any $Z \in \mathfrak{p}$ we have:

$$\begin{aligned} ad_{[X,Y]}(Z) &= [[X, Y], Z] \\ &= -[[Y, Z], X] - [[Z, X], Y] \text{ by } (2.2) \end{aligned}$$

But $[Z, X] = 0$ and since $[Y, Z] \in \mathfrak{p}$ it follows that $[[Y, Z], X] = 0$. Since Z was arbitrary we conclude that $ad_{[X,Y]} = 0$ and so $[X, Y] \in \mathfrak{k}_1$. Hence $\mathfrak{k}_1$ is an ideal, contradicting the assumption that $\mathfrak{g}$ is simple.
Now suppose that $\mathfrak{p}_1 \subset \mathfrak{p}$ is a proper, k-invariant subspace. Since B gives inner product on $\mathfrak{p}$ making all ad_X skew-symmetric, the orthogonal complement of $\mathfrak{p}_1$ (call it $\mathfrak{p}_2$) is also $\mathfrak{k}$-invariant. Define $\mathfrak{k}_1 \subset \mathfrak{k}$ as:

$$\mathfrak{k}_1 = \{X \in \mathfrak{k} : \ ad_X|_{\mathfrak{p}_2} = 0\}$$

Then we claim that $\mathfrak{g}_1 = \mathfrak{k}_1 \oplus \mathfrak{p}_1$ is an ideal of $\mathfrak{g}$. By linearity it suffices to show that:

$$\begin{aligned} [\mathfrak{p}_1, \mathfrak{k}] &\subset \mathfrak{g}_1 \\ [\mathfrak{k}_1, \mathfrak{p}] &\subset \mathfrak{g}_1 \\ [\mathfrak{k}_1, \mathfrak{k}] &\subset \mathfrak{g}_1 \\ \text{and } [\mathfrak{p}_1, \mathfrak{p}] &\subset \mathfrak{g}_1 \end{aligned}$$

The first containment follows from the fact that $\mathfrak{p}_1$ is by assumption $\mathfrak{k}$-invariant, while the second follows since $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$ and $[\mathfrak{k}_1, \mathfrak{p}_1] \subset \mathfrak{p}_1$ and by definition $[\mathfrak{k}_1, \mathfrak{p}_2] = 0$. If $X \in \mathfrak{k}_1$, $Y \in \mathfrak{k}$ and Z is any element of $\mathfrak{p}_2$ observe that:

$$\begin{aligned} ad_{[X,Y]}(Z) &= [[X, Y], Z] \\ &= -[[Y, Z], X] - [[Z, X], Y] \text{ by } (2.2) \\ &= 0 - [0, Y] \text{ since } [Y, Z] \in \mathfrak{p}_2 \\ &= 0 \end{aligned}$$

Thus $[X, Y] \in \mathfrak{k}_1$. This shows the third containment. To show the fourth containment we shall show that $[\mathfrak{p}_1, \mathfrak{p}_1] = \mathfrak{k}_1$ and $[\mathfrak{p}_1, \mathfrak{p}_2] = 0$.
Given $X, Y \in \mathfrak{p}_1$ and any $Z \in \mathfrak{p}_2$ observe that $[[X, Y], Z] \in \mathfrak{p}_2$ since $[X, Y] \in \mathfrak{k}$. But

$$ad_{[X,Y]}(Z) = [[X, Y], Z] = -[[Y, Z], X] - [[Z, X], Y] \text{ by } (2.2)$$

and $[Y,Z],[Z,X] \in \mathfrak{k}$ so both terms on the right hand side are in $\mathfrak{p}_1$. Hence:

$$ad_{[X,Y]}(Z) \in \mathfrak{p}_1 \cap \mathfrak{p}_2 = \{0\}$$

Thus $[X,Y] \in \mathfrak{k}_1$. Finally if $X \in \mathfrak{p}_1$ and $Y \in \mathfrak{p}_2$, we know that $[X,Y] \in \mathfrak{k}$ and so $[X,[X,Y]] \in \mathfrak{p}_1$. Now observe that:

$$B([X,Y],[X,Y]) = -B(Y,[X,[X,Y]]) = 0$$

since $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are orthogonal. Because B is non-degenerate (as $\mathfrak{g}$ was assumed to be semi-simple) we conclude that $[X,Y]=0$. Thus $\mathfrak{g}_1$ is indeed an ideal of $\mathfrak{g}$, contradicting the assumption that $\mathfrak{g}$ is simple.

Conversely, suppose that $\mathfrak{g}$ is semi-simple but not simple. Then there exists an ideal $\mathfrak{a} \subset \mathfrak{g}$ and we may write $\mathfrak{a} = \mathfrak{a}\cap\mathfrak{k} \oplus \mathfrak{a}\cap\mathfrak{p}$. Since

$$[\mathfrak{a}\cap\mathfrak{p}, \mathfrak{a}\cap\mathfrak{p}] \subset \mathfrak{a}\cap\mathfrak{k}$$

Observe that since

$$[\mathfrak{k}, \mathfrak{a}\cap\mathfrak{p}] \subset \mathfrak{a}\cap\mathfrak{p}$$

either $\mathfrak{a}\cap\mathfrak{p} = \{0\}$, $\mathfrak{a}\cap\mathfrak{p} = \mathfrak{p}$ or the representation $ad : \mathfrak{k} \to \mathfrak{gl}(\mathfrak{p})$ is reducible.

Observe that $\mathfrak{a}\cap\mathfrak{k} \neq \{0\}$ since if this were true

$$[\mathfrak{a},\mathfrak{a}] = [\mathfrak{a}\cap\mathfrak{p}, \mathfrak{a}\cap\mathfrak{p}] \subset \mathfrak{a}\cap\mathfrak{k} = \{0\}$$

implying that $\mathfrak{a}$ is an abelian ideal. But this would contradict the assumption that $\mathfrak{g}$ is semi-simple.

For all $X \in \mathfrak{a}\cap\mathfrak{k}$ and $Y \in \mathfrak{p}$ we have $[X,Y] \in \mathfrak{a}\cap\mathfrak{p}$. If $\mathfrak{a}\cap\mathfrak{p} = \{0\}$ then $ad_X|_{\mathfrak{p}} = 0$ and so $ad : \mathfrak{k} \to \mathfrak{gl}(\mathfrak{p})$ is not faithful.

If $\mathfrak{a}\cap\mathfrak{p} = \mathfrak{p}$ then

$$\mathfrak{k}_2 = \{X \in \mathfrak{g} : B(X,Y) = 0 \;\forall\; Y \in \mathfrak{a}\} \subset \mathfrak{k}$$

is an ideal. It must be non-trivial since $\mathfrak{a}$ was assumed to be proper. But for all $X \in \mathfrak{k}_2$ and $Y \in \mathfrak{p}$:

$$ad_X(Y) = [X,Y] \in \mathfrak{k}_2 \cap \mathfrak{p} = \{0\}$$

so again $ad : \mathfrak{k} \to \mathfrak{gl}(\mathfrak{p})$ is not faithful. □

As a corollary we have:

Corollary 2.21. *In addition to the hypotheses of theorem 2.20, suppose that G is a Lie group with Lie algebra $\mathfrak{g}$, and $K < G$ is a compact connected subgroup with Lie algebra*

$\mathfrak{k}$. *Then* $Ad : K \to Gl(\mathfrak{p})$ *is a representation. Moreover,* Ad *is faithful and irreducible if and only if* G *is simple.*

Chapter 3

Complex Geometry and Kähler manifolds

3.1 Complex manifolds

Let us introduce the main geometric object of study in this thesis, the complex manifold. We shall assume that the reader already has a working knowledge of smooth manifolds. The most common definition of a complex manifold [1] is:

Definition 3.1. A *Complex manifold* is a smooth manifold M of dimension $2n$ with an atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ such that, identifying $\varphi_\alpha(U_\alpha) \subset \mathbb{R}^{2n}$ with a domain in $\mathbb{C}^n$:

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is a biholomorphic map.

However, in this thesis we wish to emphasise the importance of a differential geometric property of M, the holonomy group. Hence we shall follow Joyce (see [GHJ03]) and Huybrechts (see [Huy05]) in *defining* complex manifolds (and later Kähler manifolds) as smooth manifolds with an additional structure satisfying an integrability condition.

Definition 3.2. Let M be a smooth real manifold with dimension $2n$. An *almost complex structure* on M is an endomorphism of the tangent bundle: $J : TM \to TM$ satisfying $J_x^2 = -Id_x$ for all $x \in M$. For any $a + ib \in \mathbb{C}$ and any $X \in T_xM$ we may now define:

$$(a + ib) \cdot X = aX + bJ(X)$$

thus turning T_xM into a complex vector space.

[1] As given in [GH78], page 14, for example

We call (M, J) an *almost complex manifold.* A complex manifold is an almost complex manifold satisfying an integrability condition:

Definition 3.3. Let (M, J) be an almost complex manifold. The Nijenhuis tensor of J is defined as:

$$N_J(X,Y) = [X,Y] + J([JX,Y]) + J([X,JY]) - [JX,JY]$$

The almost complex structure J is a *complex structure* and (M, J) is a *complex manifold* if and only if $N_J \equiv 0$.

Remark 3.4. The fact that this definition of complex manifold coincides with the more common definition in terms of a holomorphic atlas is the content of the Newlander-Nirenburg theorem. See for example page 355-356 of [Hel78] for a discussion.

3.2 The holomorphic tangent bundle

Let us examine the tangent space to a complex manifold M at some point x more closely. First we complexify it:

$$T_x^{\mathbb{C}}M = T_xM \otimes_{\mathbb{R}} \mathbb{C}$$

Recalling the definition of T_xM as the vector space of all derivations on germs of real valued smooth functions at x we see that $T_x^{\mathbb{C}}M$ corresponds to the space of all derivations on germs of complex valued smooth functions at x. The complex structure endomorphism J extends by complex linearity to an endomorphism of $T_x^{\mathbb{C}}M$, which we shall also denote as J. Now this J has two eigenvalues $\pm i$, thus we get a splitting of $T_x^{\mathbb{C}}M$ into eigenspaces:

$$T_x^{\mathbb{C}}M = T_x^{1,0}M \oplus T_x^{0,1}M$$

where $T_x^{1,0}M$ is the $+i$ eigenspace, and is called the *holomorphic tangent space.* We remark that specifying a splitting of E_x for every $x \in M$ does not in general give a splitting of E into sub-bundles, but because $T^{1,0}M = \cup_{x\in M} T_x^{1,0}M$ (respectively $T^{0,1}M = \cup_{x\in M} T_x^{0,1}M$) can be viewed as the kernel of the *constant rank* bundle endomorphism $J - iId$ (respectively $J + iId$) these are in fact holomorphic sub-bundles of $T_x^{\mathbb{C}}M$ and moreover we have the (global) decomposition:

$$T_x^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M \tag{3.1}$$

Now J defines an endomorphism of $(T_x^*)^{\mathbb{C}}M$ (that is, the complexification of the cotangent space) given by, for $\alpha \in (T_x^*)^{\mathbb{C}}M$:

$$(J\alpha)(X) = \alpha(JX)$$

and so we have a decomposition of $(T_x^*)^{\mathbb{C}}M$:

$$(T_x^*)^{\mathbb{C}}M = (T_x^*)^{1,0}M \oplus (T_x^*)^{0,1}M$$

where as before, $(T_x^*)^{1,0}M$ is the $+i$ eigenspace and this extends to a vector bundle decomposition into holomorphic vector bundles:

$$(T^*)^{\mathbb{C}}M = (T^*)^{1,0}M \oplus (T^*)^{0,1}M$$

Note that we could have equally defined $(T_x^*)^{1,0}M$ as the subspace of all covectors α vanishing on $T_x^{0,1}M$ since for $X \in T_x^{0,1}M$:

$$i\alpha(X) = J\alpha(X) = \alpha(JX) = \alpha(-iX) = -i\alpha(X)$$

hence $\alpha(X) = 0$, and similarly $(T_x^*)^{0,1}M$ as the subspace of all co-vectors vanishing on $T_x^{1,0}M$. Using this characterisation we have a decomposition of the k-th exterior power of $(T_x^*)^{\mathbb{C}}M$ as:

$$\bigwedge^k (T_x^*)^{\mathbb{C}}M = \bigoplus_{p+q=k} (T_x^*)^{p,q}$$

where $\beta \in (T*_x)^{p,q}$ if and only if $\beta(X_1, \ldots, X_k) = 0$ unless p of the X_i are in $T_x^{1,0}M$ and q are in $T_x^{0,1}M$. By the same argument as previously, this extends to a decomposition of the k-th exterior power of the co-tangent bundle:

$$\bigwedge^k (T^*)^{\mathbb{C}}M = \bigoplus_{p+q=k} (T^*)^{p,q}$$

and a section of $(T^*)^{p,q}$ is called a (p,q)-form. Now consider the exterior derivative on $\bigwedge^k (T^*)^{\mathbb{C}}M$. If we restrict it to a single summand $(T^*)^{p,q}$ we see that:

$$d : (T^*)^{p,q} \to (T^*)^{p+1,q} \oplus (T^*)^{p,q+1}$$

and so d decomposes as the direct sum of two operators: $d = \partial + \bar{\partial}$ where:

$$\begin{aligned} \partial &: (T^*)^{p,q} \to (T^*)^{p+1,q} \\ \bar{\partial} &: (T^*)^{p,q} \to (T^*)^{p,q+1} \end{aligned} \tag{3.2}$$

One of the main tools we shall use to connect the real differential geometry of M with the complex geometry of (M, J) is the following canonical ($\mathbb{R}$-linear) isomorphism:

$$\begin{aligned} \xi &: TM \to T^{1,0}M \\ X &\mapsto (X - iJ(X)) \end{aligned} \tag{3.3}$$

we can check this is an isomorphism by noting that it is $\mathbb{R}$-linear and has an inverse given by taking the real part of $v \in T^{1,0}M$:

$$\xi^{-1}(v) = \mathbf{Re}(v)$$

One final piece of notation. We know that to any smooth vector bundle E we can associate a locally free sheaf $\mathcal{E}$ by simply letting $\mathcal{E}(U)$ be the smooth sections of E over U. So, we can associate a sheaf to $(T^*)^{\mathbb{C}}M$, which we shall denote by $\mathcal{A}^1_M$. Similarly we can associate a sheaf of smooth sections to any exterior power, $\bigwedge^k (T^*)^{\mathbb{C}}M$ and to any of the sub-bundles $(T^*)^{p,q}$, which we shall denote by $\mathcal{A}^k_M$ and $\mathcal{A}^{(p,q)}_M$ respectively. For any smooth complex bundle E on M we can consider the new bundle $E \otimes (T^*)^{\mathbb{C}}M$ (respectively $E \otimes \bigwedge^k (T^*)^{\mathbb{C}}M$ or $E \otimes (T^*)^{p,q}$) and the sheaf of its smooth sections, denoted $\mathcal{A}^1_M(E)$ (respectively $\mathcal{A}^k_M(E)$ and $\mathcal{A}^{(p,q)}_M(E)$. In light of this, it is a useful convention to denote the sheaf of smooth sections of E as $\mathcal{A}^0_M(E)$. If E is a real vector bundle on the underlying smooth manifold we shall abuse notation slightly and use $\mathcal{A}^k_M(E)$ to denote the sheaf of sections of $E \otimes \bigwedge^k (T^*)M$. Note that even when E is a holomorphic vector bundle, we are considering smooth, not holomorphic sections!

3.3 Hermitian and Kähler manifolds

Given a complex manifold (M, J), since M is a smooth manifold we may equip it with a Riemannian metric g. We say that g is *Hermitian* and call the triple (M, J, g) a *Hermitian manifold* if g is compatible with the complex structure J in the sense that:

$$g(JX, JY) = g(X, Y)$$

for all real vector fields X and Y. Note this unusual nomenclature; g still gives a real inner product on each T_xM, not a Hermitian inner product, although as we shall see shortly one can extend g to a *bone fide* Hermitian inner product on each tangent space. We may also define an alternating 2-form:

$$\omega(X,Y) = g(JX,Y)$$

One easily checks that this is indeed alternating:

$$\omega(Y,X) = g(JY,X) = g(X,JY) = g(JX,J^2Y) = -g(JX,Y)$$

We denote by h the extension to $T^{\mathbb{C}}M$ by $\mathbb{C}$-*sesquilinearity* of g. That is, for all $X,Y,Z,W \in TM$:

$$\begin{aligned} h(X+iY, Z+iW) &= g(X,Y) + ig(Y,Z) - ig(X,W) + i(-i)g(Y,W) \\ &= g(X,Y) + g(Y,W) + i(g(Y,Z) - g(X,W)) \end{aligned}$$

So that, for all $X^{'}, Y^{'}, Z^{'} \in T^{\mathbb{C}}M$ and $\alpha, \beta \in \mathbb{C}$ we have that:

$$\begin{aligned} h(\alpha X^{'} + \beta Y^{'}, Z^{'}) &= \alpha h(X^{'}, Z^{'}) + \beta h(Y^{'}, Z^{'}) \\ \text{and } h(X^{'}, Y^{'}) &= \overline{h(Y', X')} \end{aligned}$$

Frequently we shall only be concerned with the restriction of h to $T^{1,0}M$, which we shall also denote as h. Observe that h is a Hermitian metric on (M,J) in the usual sense. That is, it defines a positive definite Hermitian inner product on each holomorphic tangent space $T^{1,0}_xM$ and it does so in a smooth manner[2] Now suppose that $Z,W \in T^{1,0}M$. Using the map ξ introduced in (3.3) in the previous section we have that $Z = \frac{1}{\sqrt{2}}(X - iJX)$ and $W = \frac{1}{\sqrt{2}}(Y - iJY)$ for unique $X, Y \in TM$. Thus we have that:

$$\begin{aligned} h(Z,W) &= h(\frac{1}{\sqrt{2}}(X - iJX), \frac{1}{\sqrt{2}}(Y - iJY)) \\ &= \frac{1}{2}\left(g(X,Y) + g(JX,JY)\right) - \frac{i}{2}\left(g(JX,Y) - g(X,JY)\right) \\ &= g(X,Y) - ig(JX,Y) \end{aligned}$$

So the real part of h is our original Riemannian metric and the imaginary part is the alternating one-form ω, which is usually called the associated $(1,1)$ form. As an aside, we say that a $(1,1)$ form β is *real* if $\beta(X,Y) \in \mathbb{R}$ for all $X,Y \in TM \subset T^{\mathbb{C}}M$. A real $(1,1)$-form is said to *positive* (resp. *negative*) if $\beta(X,JX) > 0$ (respectively $\beta(X,JX) < 0$)

[2]This is how Griffiths and Harris define a Hermitian metric in [GH78], page 27, for example.

for all $X \in TM$. Equivalently (cf. pg. 188 of [Huy05]) β is positive (resp. negative) if $-i\beta(X, \bar{X}) > 0$ (resp, $-i\beta(X, \bar{X}) < 0$) for all $X \in T^{1,0}M$. It should be clear that the associated $(1, 1)$ form is a positive real form.

Definition 3.5 (Kähler manifold)**.** Given a Hermitian manifold (M, J, g), we say g is a Kähler metric, and (M, J, g) a Kähler manifold, if $d\omega = 0$. Frequently we shall drop the reference to the metric g and call a complex manifold (M, J) Kähler if there exists a Kähler metric on it. This is useful because the condition of having a Kähler metric places significant restrictions on the topology of (M, J).

Later we shall introduce several equivalent definitions of Kähler, and see that on a Kähler manifold there is a particularly nice relationship between the Riemannian geometry and the complex geometry.

3.4 The Chern and Levi-Civita connections

Suppose we have a vector bundle $E \to M$; we would like a way to differentiate its sections. The appropriate notion of 'differentiate' is given by the idea of a *connection*.

Definition 3.6. Suppose that M is a smooth manifold and $E \to M$ is a smooth real (respectively complex) vector bundle. A connection is a $\mathbb{R}$- (respectively $\mathbb{C}$-) linear map:

$$D : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$$

satisfying, for any $f \in \mathcal{A}^0_M$ and $\sigma \in \mathcal{A}^0(E)$

$$D(f\sigma) = (df)\sigma + fD\sigma$$

Frequently we shall apply the connection to a section X of E, and then evaluate this 'E-valued one form' on a vector field $Y \in \mathcal{A}^0(TM)$, which we write as $D_Y X$.
Given a frame $\{e_1, \dots, e_n\}$ for E we can give a useful, 'local' description of the connection D. We define $A \in \mathcal{A}^1(\mathrm{End}(E))$ as follows:

$$De_i = A_i^j e_j$$

We can think of A as a matrix of one-forms, or as a one-form taking values in the bundle $\mathrm{End}(E)$. Now for an arbitrary section σ of E we have that $\sigma = \sigma^i e_i$.

$$\begin{aligned} D(\sigma) = D(\sigma^i e_i) &= (d\sigma^i)e_i + \sigma^i A_i^j e_j \\ &= (d\sigma^j + A_i^j \sigma^i)e_j \end{aligned}$$

So following Huybrechts [Huy05] we shall frequently write the connection as $D = d + A$, and think of it locally as 'the exterior derivative plus a matrix of one forms'.
Given a connection on E, we can extend it to a connection on arbitrary tensor products of E:

Definition 3.7 (Definition 4.6 in [Lee97])**.** We first define D as it acts on $\mathcal{A}^0_M$. Given $f \in \mathcal{A}^0_M$, $Df = df$. Now we define a connection on E^*, also denoted as D by requiring that, for $\alpha \in \mathcal{A}^0(E^*)$ and $X \in \mathcal{A}^0(E)$:

$$(D\alpha)(X) = D\left(\alpha(X)\right) - \alpha(DX)$$

Finally we extend D to arbitrary tensor fields by defining, for $F \in \mathcal{A}^0(E^{\otimes k}) \otimes (E^*)^{\otimes l})$, $X_i \in \mathcal{A}^0(E^{\otimes k})$ and $\alpha^j \in \mathcal{A}^0(E^*)$,

$$\begin{aligned}
&(DF)(X_1, \dots, X_k, \alpha^1, \dots, \alpha^l) = D(F(X_1, \dots, X_k, \alpha^1, \dots, \alpha^l)) - \\
&\sum_{i=1}^{k} F(X_1, \dots, DX_i, \dots, X_k, \alpha^1, \dots, \alpha^l)) \\
&- \sum_{j=1}^{l} F(X_1, \dots, X_k, \alpha^1, \dots, D\alpha^j, \dots, \alpha^l)
\end{aligned}$$

where if $DX_i = \omega \otimes \tilde{X}_i$ then:

$$F(X_1, \dots, DX_i, \dots, X_k, \alpha^1, \dots, \alpha^l) = \omega \otimes F(X_1, \dots, \tilde{X}_i, \dots, X_k, \alpha^1, \dots, \alpha^l)$$

3.4.1 The $\bar{\partial}$ operator

Recall that for the bundle T^*M there exists an exterior derivative operator:

$$d : \mathcal{A}^k_M \to \mathcal{A}^{k+1}_M$$

but given an arbitrary smooth vector bundle $E \to M$ no such operator on $\mathcal{A}^k(E)$ exists. However, if $E \to M$ is a *holomorphic* vector bundle then we may define an operator $\bar{\partial}_E$ which is the analogue of $\bar{\partial}$:

$$\bar{\partial}_E : \mathcal{A}^{p,q}(E) \to \mathcal{A}^{p,q+1}(E)$$

To do this, start with an open cover $\{U_i\}$ of M such that $E|_{U_i}$ is trivial for all i. That is, for all i, there exists a map $\varphi_i : E|_{U_i} \to U_i \times \mathbb{C}^n$ and moreover the transition functions:

$$\begin{aligned} g_{ij} :& U \to Gl(n, \mathbb{C}) \\ & z \mapsto (\varphi_j \circ \varphi_i^{-1})|_z \end{aligned}$$

are holomorphic [3]. Now take any section $\sigma \in \mathcal{A}^{p,q}(E)(U)$; if rank$(E) = n$ then on the intersections $U \cap U_i$ we may view σ as an n-tuple of (p, q) forms:

$$\varphi_i(\sigma|_{U \cap U_i}) = (\sigma_i^1, \ldots, \sigma_i^n)$$

Now define:

$$\bar{\partial}_E(\sigma|_{U \cap U_i}) = \varphi_i^{-1}(\bar{\partial}(\sigma_i^1), \ldots, \bar{\partial}(\sigma_i^n))$$

or:

$$\bar{\partial}_E(\sigma|_{U \cap U_i}) = \varphi_i^{-1}(\bar{\partial}(\sigma_i))$$

for short. Similarly, on $U \cap U_j$

$$\varphi_j(\sigma|_{U \cap U_j}) = (\sigma_j^1, \ldots, \sigma_j^n) = \sigma_j$$

and so we define:

$$\bar{\partial}_E(\sigma|_{U \cap U_j}) = \varphi_j^{-1}(\bar{\partial}(\sigma_j))$$

In order to show that $\bar{\partial}_E$ is well defined we must show that if $U_i \cap U_j \neq \emptyset$ then these two definitions coincide. By definition on $U \cap U_i \cap U_j$ we have:

$$\sigma_j(z) = \varphi_j \circ \varphi_i^{-1}(z)(\sigma_i(z)) = g_{ij}(z)(\sigma_i(z))$$

Thus:

$$\begin{aligned} \bar{\partial}_E(\sigma) &= \varphi_j^{-1}(\bar{\partial}(\sigma_j)) \\ &= \varphi_j^{-1}(\bar{\partial}(g_{ij}(\sigma_i))) \\ &= \varphi_j^{-1}\left(g_{ij}(\bar{\partial}(\sigma_i)) + \bar{\partial}(g_{ij})(\sigma_i)\right) \\ &= \varphi_j^{-1} \circ g_{ij}(\bar{\partial}(\sigma_i)) \text{ since } g_{ij} \text{ is holomorphic implies } \bar{\partial} g_{ij} = 0. \\ &= \varphi_i^{-1}(\bar{\partial}(\sigma_i)) \end{aligned}$$

hence $\bar{\partial}_E$ is well defined.

In general, a vector bundle $E \to M$, whether it is holomorphic or merely smooth, supports infinitely many connections.

[3] The existence of such a cover is guaranteed by the definition of a holomorphic vector bundle. See [GH78] page 66

3.4.2 The Chern connection

Suppose that (M, J) is a complex manifold and $E \to M$ is a holomorphic vector bundle endowed with a hermitian metric h (That is, a Hermitian inner product h_x on each fibre E_x such that the map $x \mapsto h_x$ is smooth). One might reasonably ask for a connection on E that agrees with the metric somehow, and indeed we say that D is *metric* if:

$$dg(X, Y) = g(DX, Y) + g(X, DY)$$

or equivalently if $D(g) = 0$. Since (M, J) is a complex manifold, we get a decomposition:

$$\mathcal{A}^1_M(E) = \mathcal{A}^{1,0}_M(E) \oplus \mathcal{A}^{0,1}_M(E)$$

and so, in a similar way to the exterior derivative (see (3.2)), we can split D as $D = D^{'} + D^{''}$ where:

$$\begin{aligned} D^{'} &: \mathcal{A}^0_M(E) \to \mathcal{A}^{1,0}_M(E) \\ D^{''} &: \mathcal{A}^0_M(E) \to \mathcal{A}^{0,1}_M(E) \end{aligned}$$

We say that D *agrees with the complex structure* if $D^{''} = \bar{\partial}_E$. In terms of the 'local description' of the connection given earlier we note that:

$$D^{''} = \bar{\partial} + A^{0,1}$$

with $A^{0,1} \in \mathcal{A}^{0,1}(End(E))$. So, as pointed out in the proof of proposition 4.A.7 in [Huy05], D agrees with the complex structure if and only if $A^{0,1} = 0$. If D is metric and agrees with the complex structure then we call D the *Chern connection* of E. Showing that the Chern connection is unique, as well as that it exists, is a standard calculation, and may be found on page 73 of [GH78].

3.4.3 The Levi-Civita connection

If (M, J, g) is a Hermitian manifold $T^{1,0}M$ is a holomorphic vector bundle and so it has a unique Chern connection D. In this section we discuss a useful way to specify a unique connection ∇ on the real tangent bundle of the Riemannian manifold (M, g). Here, and elsewhere, we shall consistently use the notation D to refer to connections on holomorphic vector bundles, and ∇ to refer to connections on real, smooth vector bundles.

Again we say that ∇ is metric if:

$$dg(X,Y) = g(\nabla X, Y) + g(X, \nabla Y) \tag{3.4}$$

or equivalently if $\nabla g = 0$. If the torsion tensor[4]:

$$T_\nabla = \nabla_X Y - \nabla_Y X - [X,Y]$$

vanishes we say that ∇ is *torsion-free.*

Theorem 3.8. *If (M,g) is a Riemannian manifold there exists a unique metric, torsion free connection on TM called the* Levi-Civita *connection.*

Proof. see Theorem 5.4 on page 68 of [Lee97]. □

3.4.4 The Chern and Levi-Civita connections on a Kähler manifold

If (M,g,J) is a Kähler manifold, there is a surprising relationship between the Chern connection on $T^{'}M$ (which we shall denote by D) and the Levi-Civita connection on TM (which we shall denote by ∇). To see this, first observe that to D we may associate a connection on TM, which for now we shall denote as $\hat{D}$, via the map ξ introduced in (3.3):

$$\begin{aligned}\hat{D} &: \mathcal{A}^0(TM) \to \mathcal{A}^1(TM)\\ \hat{D}(X) &= \mathbf{Re}\,(D \circ \xi(X))\end{aligned}$$

We define the torsion of D to be the torsion of the (real) connection $\hat{D}$:

$$T_D = T_{\hat{D}} = \hat{D}_X Y - \hat{D}_Y X - [X,Y]$$

Now we have the following theorem connecting the Kähler and Riemannian geometry of a Kähler manifold:

Theorem 3.9. *Suppose that (M,J,g) is a complex manifold. With ∇ and D as above, the following are equivalent:*

1. *(M,J,g) is Kähler.*

2. *$\nabla\omega = 0$*

[4]Observe that although ∇ is not a tensor, T_∇ is.

3. $\nabla J = 0$

4. The induced real connection $\hat{D}$ *coincides with* ∇

5. $T_D = 0$

Proof. We show the equivalence of the first three properties. The proof of the other equivalences may be found on pg. 208 of [Huy05]. Recall that the associated $(1,1)$ form (cf. The definition of Kähler in 3.5) is given by $\omega(X,Y) = g(JX,Y)$ we have (cf. the discussion on extending connections to tensor fields in section 3.4):

$$\begin{aligned}
(\nabla\omega)(X,Y) &= d(\omega(X,Y)) - \omega(\nabla X, Y) - \omega(X, \nabla Y) \\
&= d\left(g((JX),Y)\right) - g(J(\nabla X),Y) - g(JX,\nabla Y) \\
&= g(\nabla(JX),Y) + g(JX,\nabla(Y)) - g(J(\nabla X),Y) - g(JX,\nabla Y) \\
&= g(\nabla(JX) - J(\nabla X),Y) \\
&= g\left(\nabla J(X),Y\right)
\end{aligned}$$

where the third line follows from the fact that the Levi-Civita connection is metric. Thus $\nabla J = 0$ if and only if $\nabla\omega = 0$. But $\nabla\omega = 0$ implies $d\omega = 0$ as $d\omega$ is the anti-symmetrization of $\nabla\omega$. Conversely, if (M,J,g) is Kähler we have the following identity (cf. pg. 164 of [Zil10]):

$$4g((\nabla_X J)Y,Z) = 6d\omega(X,JY,JZ) - 6d\omega(X,Y,Z) - g(N_J(Y,Z),JX)$$

for any $X,Y,Z \in TM$ where N_J is the Nijenhuis tensor introduced at the beginning of this chapter. Because $N_J = 0$, if $d\omega = 0$ it follows that $\nabla J = 0$ (as X was arbitrary). ☐

Henceforth whenever D is the Chern connection of (M,g,J), instead of using $\hat{D}$ to denote the induced Riemannian connection, we can use ∇, which until now has been used to denote the Levi-Civita connection of (M,g), without ambiguity.

3.5 Holonomy

Suppose we are given a Riemannian manifold (M,g) and a curve $\gamma : [0,1] \to M$. The Levi-Civita connection ∇ gives us the notion of *parallel transport*

Definition 3.10 (Parallel Transport)**.** For any curve $\gamma : [0,1] \to M$ we have the linear ordinary differential equation:

$$\nabla_{\dot{\gamma}(t)} V(t) = 0$$

Because this equation is linear, we know that for any initial value $X \in T_{\gamma(0)}M$ the solution to the initial value problem:

$$\begin{aligned} \nabla_{\dot{\gamma}(t)} V(t) &= 0 \\ V(0) &= X \end{aligned} \tag{3.5}$$

is defined for all $t \in [0,1]$ (This is Theorem 4.12 on page 60 of [Lee97]). Now for any vector $X \in T_\gamma(0)M$ we define the *parallel transport* of X along γ as:

$$P_\gamma X = V(1)$$

This map is a linear isomorphism from $T_{\gamma(0)}M$ to $T_{\gamma(1)}M$.

In addition, if we denote by $V_1(t)$ and $V_2(t)$ the solutions to (3.5) with initial data X_1 and X_2 respectively observe that, since ∇ is metric (cf. 3.4):

$$\dot{\gamma} g(V_1(t), V_2(t)) = g(\nabla_{\dot{\gamma}} V_1(t), V_2(t)) + g(V_1(t), \nabla_{\dot{\gamma}} V_2(t)) = 0$$

Hence:

$$g(X_1, X_2) = g(V_1(0), V_2(0)) = g(V_1(1), V_2(1)) = g(P_\gamma X_1, P_\gamma X_2)$$

and so P_γ is an isometry. If we choose γ to be a closed curve centred at x ,that is, $\gamma(0) = \gamma(1) = x$, P_γ becomes a linear isometry of the vector space $T_x M$:

$$P_\gamma \in O(T_x M) \cong O(m, \mathbb{R})$$

Thus we may define:

Definition 3.11 (The Holonomy group)**.** The holonomy group of (M, g) at x is the group of all such P_γ, where γ is a curve starting and ending at x. We denote this group by $Hol_x(M, g)$. It is a subgroup of $O(T_x M)$ and is in fact a *Lie group* (see [Zil10] page 133). If M is connected, the holonomy groups at x and y are conjugate as subgroups of $O(m, \mathbb{R})$ and hence we shall frequently drop the index x and just talk about the holonomy group of (M, g), $Hol(M, g)$.

A few remarks are in order:

Remark 3.12. 1. If $Hol(M, g)_0$ denotes the identity component of $Hol(M, g)$, it is not too difficult to show that $Hol(M, g)_0$ is the group of all parallel transports around null-homotopic curves. Hence $Hol(M, g)_0$ is the holonomy group of the universal cover $\tilde{M}$ of M (see page 133 of [Zil10]).

2. Since P_γ is always an isometry $Hol_x(M,g) \subset O(T_xM)$. It is not always true that $Hol_x(M,g)$ is closed, but $Hol(M,g)_0$ will always be. Thus $Hol(M,g)_0$ is always a compact Lie group.(Again, see page 133 of [Zil10])

3. We have an obvious representation of $Hol(M,g)_0$ on T_xM, given by its very definition. Since $Hol(M,g)_0$ is compact, by theorem 2.7 we expect this representation to decompose into a direct sum of irreducible representations.

4. We can define parallel transport of elements of T^*M using the transpose of P_γ:

$$P_\gamma^{-t} : T_p^*M \to T_p^*M \tag{3.6}$$

and so we may parallel transport a tensor $A \in (T_p^*M)^k \otimes (T_pM)^l$ by defining:

$$(P_\gamma A)(X_1, \ldots, X_k, \omega^1, \ldots, \omega^l) = A(P_\gamma X_1, \ldots, P_\gamma X_k, P_\gamma^{-t}\omega^1, \ldots, P_\gamma^{-t}\omega^l) \tag{3.7}$$

There are three major theorems relating to holonomy that we shall make heavy use of in this thesis. The first of these is:

Theorem 3.13 (The Berger Holonomy theorem)**.** *Suppose that (M,g) is a simply connected Riemannian manifold of dimension m, and that $Hol_x(M,g)$ acts irreducibly on T_xM. Then, if M is not a symmetric space, $Hol(M,g)$ is one of the following:*

1. $SO(m)$
2. $U(m/2)$ *when m is even*
3. $SU(m/2)$
4. $Sp(m/4) \cdot Sp(1)$ *if m is divisible by 4*
5. $Sp(m/4)$
6. G_2

7. *Spin*(7)

Proof. See [Ber55] or [Olm05]. □

The second result is the so-called holonomy principle mentioned in the introduction:

Theorem 3.14 (The Holonomy Principle)**.** *Let $A \in \mathcal{A}^0_M\left((TM)^{\otimes k} \otimes (T^*M)^{\otimes l}\right)$ be a tensor field on a connected Riemannian manifold (M, g). A is parallel (that is, $\nabla A = 0$) if and only if for any point $x \in M$, $A_x \in (T_xM)^k \otimes (T^*_xM)^l$ is invariant under $Hol_x(M, g)$.*

Proof. Suppose that A_p is invariant under $Hol_p(M)$. Then for any $q \in M$, choose a path γ_1 such that $\gamma_1(0) = p$ and $\gamma_1(1) = q$. Now define

$$A_q = P_\gamma A_p \tag{3.8}$$

This is well defined since if we choose another path γ_2 satisfying $\gamma_2(0) = p$ and $\gamma_2(1) = q$ then we have that $\gamma_2^{-1}\gamma_1$ is a closed curve centred at p. Then:

$$P_{\gamma_2^{-1}\gamma_1} A_p = A_p \tag{3.9}$$
$$\Rightarrow P^{-1}_{\gamma_2} P_{\gamma_1}(A_p) = A_p \tag{3.10}$$
$$P_{\gamma_1}(A_p) = P_{\gamma_2}(A_p) \tag{3.11}$$
$$\tag{3.12}$$

To show that A is a parallel tensor field it suffices to show that, for all $x \in M$, $\nabla_{\dot\gamma} A = 0$ for arbitrary path γ through x. But this is true since by definition $A_{\gamma(t)}$ satisfies the differential equation:

$$\nabla_{\dot\gamma} A_{\gamma(t)} = 0 \tag{3.13}$$

For the converse, observe that for any path γ with $\gamma(0) = p$ and $\gamma(1) = q$, $A_{\gamma(t)}$ is a solution to (3.5) with initial data A_p since:

$$\nabla_{\dot\gamma} A_{\gamma(t)} = 0 \tag{3.14}$$
$$A_{\gamma(0)} = A_p \tag{3.15}$$

By the uniqueness of solutions to ODE's it is *the* solution to (3.5), and hence:

$$P_\gamma A_p = A_{\gamma(1)} = A_q \tag{3.16}$$

So if we take γ to be a closed curve with $\gamma(0) = \gamma(1) = p$ then:

$$P_\gamma A_p = A_p \tag{3.17}$$

That is, A is invariant under the holonomy group. See also Theorem 2.3 pg.8 of [GHJ03]. □

Using holonomy we can give yet another equivalent definition of Kähler manifold.

Proposition 3.15 (Proposition 4.1 on pg.15 in [GHJ03]). *$Hol(M, g) \subset U(n)$ if and only if there exists a complex structure J on M such that (M, g, J) is a Kähler manifold. Note we are implicitly assuming that (M, g) is a $2n$ dimensional real manifold.*

Proof. If (M, g, J) is Kähler, by theorem 3.9 we have that $\nabla J = 0$ so by the holonomy principle J_p is holonomy invariant. This means that:

$$P_\gamma J_x = J_x P_\gamma \quad \forall\ P_\gamma \in Hol_x(M, g) \tag{3.18}$$

In the parlance of §2.3.2 the holonomy representation is thus a *complex representation* and so $Hol(M, g) \subset Gl(n, \mathbb{C})$. But of course $Hol(M, g) \subset O(2n)$ and so $Hol_p(M, g) \subset O(2n) \cap Gl(n, \mathbb{C}) = U(n)$.

Conversely, assume that $Hol_x(M, g) \subset U(n)$. Then there exists a complex structure $J_x \in \mathrm{End}(T_x M)$ left invariant by $Hol_x(M, g)$. Observe that $1 = \det(J_x^4) = (\det(J_x))^4$ and the fact that J_x is a matrix with real entries implies that $\det(J_x) = \pm 1$ and so $J_x \in O(T_x M, g_x)$. Thus J_x is compatible with the metric:

$$g_x(J_x X, J_x Y) = g_x(X, Y) \quad \forall\ X, Y \in T_x M \tag{3.19}$$

Again by the holonomy principle J_x extends to a parallel almost complex structure J on M. If we can show that this is a complex structure (i.e. that it is *integrable*) by 3.9 (M, J, g) will be Kähler. But J is indeed integrable, and to show this one uses the fact that the torsion of ∇ vanishes:

$$\tau_\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = 0 \tag{3.20}$$

to replace all the terms in Nijenhuis tensor N_J (cf. 3.3) with terms involving ∇ and J. For example:

$$[JX, Y] = \nabla_{JX}(Y) - \nabla_Y(JX) \tag{3.21}$$

since $0 = (\nabla J)(X) = \nabla(JX) - J\nabla(X)$ we can commute J and ∇, and one can check that all the terms in N_J cancel out pairwise. □

3.6 Decompositions of Riemannian manifolds and De Rham's theorem

Given two smooth manifolds M_1 and M_2, their Cartesian product $M_1 \times M_2$ is also a smooth manifold, and we have natural smooth projection maps:

$$\pi_i : M_1 \times M_2 \to M_i \tag{3.22}$$

If M_1 and M_2 are in addition equipped with Riemannian metrics g_1 and g_2 respectively, then we can pull these back by π_1 and π_2 respectively two get two symmetric bilinear forms $\pi_1^* g_1$ and $\pi_2^* g_2$ on $M_1 \times M_2$. Unless one of the M_i is zero dimensional, these will both be degenerate. However their sum will be non-degenerate, since

$$\begin{aligned}
&g_1(\pi_{1,*}(X), \pi_{1,*}(X)) + g_2(\pi_{2,*}(X), \pi_{2,*}(X)) = 0 \\
\Rightarrow &g_1(\pi_{1,*}(X), \pi_{1,*}(X)) = 0 \text{ and } g_2(\pi_{2,*}(X), \pi_{2,*}(X)) = 0 \text{ since both are non-negative} \\
\Rightarrow &\pi_{1,*}(X) = \pi_{2,*}(X) = 0 \\
\Rightarrow &X = 0
\end{aligned}$$

Thus $\pi_1^* g_1 + \pi_2^* g_2$ gives a Riemannian metric on $M_1 \times M_2$ which we shall denote as $g_1 \oplus g_2$. We shall call $(M_1 \times M_2, g_1 \oplus g_2)$ the *Riemannian product*. Frequently we shall write a Riemannian product as $(M_1, g_1) \times (M_2, g_2)$.

Conversely, given a Riemannian manifold (M, g), one might ask whether it is isometric[5] to a product of lower dimensional Riemannian manifolds? We shall say that (M, g) is *decomposable* if it is, and *indecomposable* if it is not. The question of when a Riemannian manifold is decomposable leads us to the third and final holonomy-related theorem:

Theorem 3.16 (The De Rham decomposition theorem)**.** *Let (M, g) be a simply connected and complete Riemannian manifold and x any point in M. The representation of $Hol_x(M, g)$ on $T_x M$ will, by the discussion in remark 3.12, decompose into the direct sum of irreducible representations:*

$$T_x M = V_0 \oplus \ldots \oplus V_r$$

with

$$V_0 = \{X \in T_x M : P_\gamma X = X \,\forall P_\gamma \in Hol_x(M, g)\}$$

Then:

$$(M, g) \cong (M_0, g_0) \times \ldots \times (M_n, g_n) \tag{3.23}$$

[5]By isometric we mean isometrically isomorphic. If (M_1, g_1) and (M_2, g_2) are isometric we shall generally write $(M_1, g_1) \cong (M_2, g_2)$

with each (M_i, g_i) simply-connected and indecomposable and:

1. *If we write $x = (x_0, \dots, x_n)$ then $T_{x_i} \cong V_i$ and so $dim(M_i) = dim(V_i)$.*

2. *(M_0, g_0) is isometric to $\mathbb{R}^n$ equipped with a flat metric*[6].

3. *The Levi-Civita connection of M also splits as a direct sum:*

$$\nabla = \nabla_1 \oplus \dots \oplus \nabla_r \tag{3.24}$$

where, as with the metric, this direct sum is to be interpreted as:

$$\nabla_X Y = \sum_i (\nabla_i)_{\pi_i(X)} \pi_i(Y) \tag{3.25}$$

4. *the holonomy group splits as a product:*

$$Hol_x = Hol_{x_1} \times \dots \times Hol_{x_r} \tag{3.26}$$

We are omitting the holonomy group of M_0, since, being isometric to $\mathbb{R}^n$, its holonomy is trivial.

5. *This decomposition is unique up to reordering.*

Proof. This is theorem 6.2 in [KN63] □

We shall refer to an isometry such as (3.23) as the *De Rham decomposition* of (M, g), and to the factors (M_i, g_i) as *De Rham factors.* The uniqueness of this decomposition has very important consequences for the group of isometries of M (which we shall denote, here and elsewhere, as $I(M)$) when M is simply-connected.

Corollary 3.17. *Suppose that $(M, g) \cong (M_1, g_1) \times \dots \times (M_r, g_r)$ is the De Rham decomposition of a simply connected manifold M. It is possible that some of the factors are isomorphic, and by reordering if necessary we can write:*

$$(M, g) \cong (M_1, g_1)^{k_1} \times \dots \times (M_m, g_m)^{k_m}$$

[6]That is, a metric such that the curvature of the Levi-Civita connection vanishes

where $(M_i, g_i)^{k_i}$ is to be understood as a product of k_i isometric copies of (M_i, g_i). Then:

$$I(M) = \left(I(M_1)^{k_1} \ltimes S_{k_1}\right) \times \ldots \times \left(I(M_m)^{k_m} \ltimes S_{k_m}\right)$$

where S_{k_i} denotes the symmetric group on k_i letters.

Proof. See pg. 118 of [Mok89] and the references cited therein. □

Note that we can extend theorem (3.16) to cover Kähler manifolds:

Theorem 3.18 (De Rham's theorem for Kähler manifolds)**.** *Suppose that (M, J, g) is a simply connected and complete Kähler manifold. Then (M, g) decomposes as a Riemannian product:*

$$(M.g) = (M_0, g_0) \times \ldots \times (M_r, g_r)$$

Let $pr_i : M \to M_i$ be the projection on to the i-th factor. In addition to all the properties mentioned in (3.16), we also have that:

1. *For each i $J(pr_i^* TM_i) \subset pr_i^* TM_i$ and thus J induces a parallel complex structure $J_i : TM_i \to TM_i$*
2. *$T_x^{1,0}M = pr_1^* T_{x_1}^{1,0} M_1 \oplus \ldots \oplus pr_r^* T_{x_r}^{1,0} M_r \cong T_{x_1}^{1,0} M_1 \oplus \ldots \oplus T_{x_r}^{1,0} M_r$ where the pr_i are extended by $\mathbb{C}$-linearity.*
3. *$\omega = \omega_0 \oplus \ldots \oplus \omega_r$*
4. *(M_i, J_i, g_i) is a Kähler manifold for all i.*

Proof. A proof of (1) may be found on page 57 of [Bal06]. In particular, each J_i is parallel because $\nabla_i(J_i) = pr_i(\nabla J) = 0$. (2) follows since $J(X_1, \ldots, X_r) = (J_1 X_1, \ldots, J_r X_r)$ for $X_i \in pr_i^* T^{\mathbb{C}} M_i$. (3) is now an easy consequence of 1 and the splitting of the metric since if we take $X_i, Y_i \in T_{p_i}(M_i)$ we have:

$$\omega(X_i, Y_i) = g(JX_i, Y_i) = g_i(J_i X_i, Y_i) = \omega_i(X_i, Y_i) \tag{3.27}$$

where the second equality follows from the fact that $JX_i \in T_{p_i}(M_i)$. □

Suppose (M, J, g) is a Kähler manifold and $\varphi : M \to M$ is a diffeomorphism. As before, we say that φ is an isometry if $\varphi^* g = g$ and that φ is a biholomorphism if $\varphi J = J\varphi$. Denote by $I(M, g)$ the group of isometries of (M, g) and $\mathcal{H}(M, J)$ the group of biholomorphisms. It is natural to consider the group of biholomorphic isometries, as this will preserve both structures on M. We shall denote this group as:

$$\mathrm{Aut}(M, J, g) = I(M, g) \cap \mathcal{H}(M, J)$$

3.7 Curvature

Suppose we have a connection D on a real or complex vector bundle $E \to M$. As discussed on pages 74 and 75 of [GH78], we may extend D to a map from $\mathcal{A}^k(E)$ to $\mathcal{A}^{k+1}(E)$[7] for arbitrary p as follows. Any $\sigma \in \mathcal{A}^k(E)(U)$ may be written locally as a linear combination of terms of the form $\alpha \otimes s$ with $\alpha \in \mathcal{A}^k_M(U)$ and $s \in \mathcal{A}^0(E)(U)$. Then:

$$D(\alpha \otimes s) := (d\alpha) \otimes s + (-)^k \alpha \wedge (D\tau)$$

and then extend to $D\sigma$ by $\mathbb{R}$- or $\mathbb{C}$-linearity. Note that from this definition we get that D satisfies the graded Liebniz law. That is, for $\sigma = \alpha \otimes s \in \mathcal{A}^k(E)(U)$ and $\beta \in \mathcal{A}^l_M(U)$ we have:

$$\begin{aligned} D(\beta \wedge \sigma) &= D(\beta \wedge \alpha \otimes s) \\ &= d(\beta \wedge \alpha) \otimes s + (-1)^{k+l} \beta \wedge \alpha \otimes D(s) \\ &= (d\beta) \wedge (\alpha \otimes s) + (-1)^l \beta \wedge \Big(d\alpha \otimes s + (-1)^k \alpha \otimes D(s) \Big) \\ &= d\beta \wedge \sigma + (-1)^l \beta \wedge D(\sigma) \end{aligned}$$

This calculation can be found on page 182 of [Huy05]. Composing $D : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ with $D : \mathcal{A}^1(E) \to \mathcal{A}^2(E)$ we get:

$$\begin{aligned} &F_D : \mathcal{A}^0(E) \to \mathcal{A}^2(E) \\ &F_D := D \circ D = D^2 \end{aligned}$$

F_D is called the *curvature* of D. We note that, although D is not tensorial, F_D is:

Lemma 3.19. *F_D is $\mathcal{A}^0_M$ linear, and hence is a tensor field*

[7] We shall abuse notation a little bit and use the same symbol, D, for the extension of D to any $\mathcal{A}^k(E)$

Proof. For $f \in \mathcal{A}^0_M(U)$ and $\sigma \in \mathcal{A}^0(E)(U)$ we have:

$$\begin{aligned} F_D(f\sigma) &= D(df \wedge \sigma + fD(\sigma)) \\ &= d^2 f \wedge \sigma - df \wedge D(\sigma) + df \wedge D(\sigma) + fD^2(\sigma) \\ &= fD^2(\sigma) \end{aligned}$$

□

Thus F_D is a section of the sheaf $\mathcal{A}^2(\mathrm{End}(E))$ and can be thought of as a matrix of 2-forms. We shall frequently use the following, 'local form' of the curvature tensor. Recall that with respect to some frame $\{e_1, \ldots, e_n\}$ we can write the connection operator D as

$$D = d + A$$

where $A \in \mathcal{A}^1(\mathrm{End}(E))$. For any $\sigma \in \mathcal{A}^0(E)$ write $\sigma = s^i e_i$.

$$\begin{aligned} F_D(\sigma) &= (d + A)(d + A)(s^i e_i) \\ &= (d + A)(ds^i \otimes e_i + s^i A^j_i e_j) \\ &= d^2 s^i \otimes e_i + ds^i \wedge A^j_i \otimes e_j + s^i(dA^j_i)e_j + A^j_i \wedge ds^i \otimes e_j + s^i A^j_k \wedge A^k_i e_j \\ &= ds^i \wedge A^j_i \otimes e_j + s^i(dA^j_i)e_j - ds^i \wedge A^j_i \otimes e_j + s^i A^j_k \wedge A^k_i e_j \\ &= s^i \left(dA^j_i + A^j_k \wedge A^k_i \right) e_j \\ &= (DA + A \wedge A)(\sigma) \end{aligned}$$

and so we shall frequently write $F_D = dA + A \wedge A$. We note for future reference the following result:

Lemma 3.20. *If $E \to M$ is a holomorphic, hermitian vector bundle and D is a metric connection which agrees with the complex structure, such as the Chern connection on $T^{1,0}M$ (cf, §3.4.2) then locally F_D is a skew-hermitian matrix of $(1,1)$-forms*

Proof. See Proposition 4.3.8 on pg. 184 of [Huy05] □

Remark 3.21. Note that as per the discussion in 3.4.4 if (M, J, g) is a Kähler manifold with Levi-Civita connection ∇ and Chern Connection D, we have that:

$$\nabla = \xi^{-1} \circ D \circ \xi$$

From this we easily get that:

$$F_\nabla = \nabla \circ \nabla = \xi^{-1} \circ D^2 \circ \xi = \xi^{-1} \circ F_D \circ \xi$$

Or informally, since $\xi^{-1} = \mathbf{Re}$, 'the Riemannian curvature is the real part of the Chern curvature'.

Since $F_D \in \mathcal{A}^2(\mathrm{End}(E)) = \mathcal{A}^2(E^* \otimes E)$ a natural thing to do is to take the trace of the endomorphism part of F_D:

Definition 3.22 (The Ricci form)**.** Consider the holomorphic tangent bundle $T^{1,0}M$ of the Kähler manifold (M, J, g), equipped with the Chern connection D.The *Ricci form* of (M, J, g) is defined, for any $X, Y \in T^{\mathbb{C}}M$, as:

$$\begin{aligned}\rho(X, Y) &= i\mathrm{tr}(F_D) \\ &= i\mathrm{tr}(Z \mapsto F_D(X, Y)Z)\end{aligned}$$

informally, the trace contracts the endomorphism part of F_D and we are left with a $(1, 1)$-form. (See [Huy05], pg.211 for an alternate definition).

Observe that ρ is a *real* $(1, 1)$-form, since:

$$\bar{\rho} = (-i)\overline{\mathrm{tr}(F_D)} = -i\mathrm{tr}(\bar{F_D}) = -i\mathrm{tr}(\bar{F_D}^T)) = i\mathrm{tr}(F_D) = \rho$$

where F_D^T denotes transpose and the final equality follows from lemma 3.20. This means that if $X, Y \in TM \hookrightarrow T^{\mathbb{C}}M$ then $\rho(X, Y) \in \mathbb{R}$. In Riemannian geometry it is more common to contract the 4-tensor F_∇ in a different way so as to end up with a *symmetric* 2-tensor:

Definition 3.23 (The Ricci tensor)**.** Consider the real tangent bundle TM of the Riemannian manifold (M, g), equipped with the Levi-Civita connection ∇. The *Ricci tensor* is defined as:

$$\mathrm{Ric}(X, Y) = \mathrm{tr}(Z \mapsto F_\nabla(Z, X)Y)$$

Note that here we *are not* taking the trace over the endomorphism part of the curvature. In fact since F_∇ is the real part of F_D, which is skew-hermitian, F_∇ is skew-symmetric. Thus tracing over the endomorphism part of F_∇ would give us zero!
So the Ricci form and the Ricci tensor are not the same (one is anti-symmetric while one is symmetric) but they are closely related, as the following theorem shows:

Theorem 3.24. *If $X, Y \in TM$ then:*

$$\rho(X, Y) = Ric(JX, Y))$$

Proof. For a proof see Proposition 4.A.11 on pg.211 of [Huy05]. We caution though that there the author uses Ric instead of ρ to denote the Ricci form, and r instead of Ric to denote the Ricci tensor. □

Since ρ is a real, $(1,1)$ form, it makes sense to compare it to the other real $(1,1)$ form that we have around, the Kähler form ω:

Definition 3.25. (M, J, g) is *Kähler-Einstein* if $\rho = \lambda\omega$ for some $\lambda \in \mathbb{R}$.

Observe that since $\rho(X,Y) = \mathrm{Ric}(J(X),Y)$ and $\omega(X,Y) = g(J(X),Y)$, (M,J,g) being Kähler-Einstein is equivalent to:

$$\mathrm{Ric} = \lambda g \quad \text{for some } \lambda \in \mathbb{R}$$

That is, (M,J,g) being Kähler-Einstein is equivalent to the underlying Riemannian manifold being Einstein.

3.8 Chern classes

We shall introduce Chern classes in the same manner as Huybrechts in [Huy05]. Let $R = \bigoplus_i R_i$ be a commutative, graded ring and denote by $Mat_n(R)$ the $n \times n$ matrices with coefficients in R. Consider the homogeneous polynomials P_k defined on by:

$$\det(Id + B) = 1 + P_1(B) + P_2(B) + \ldots + P_r(B) + \ldots$$

where $P_i \in R_i$. We note two things about them. Firstly, if $R_i = 0$ for $i > k$ then obviously $P_i = 0$ for $i > k$. Secondly, for any $g \in Gl(R)$ we have that:

$$\det(Id + gBg^{-1}) = \det\left(g(Id + B)g^{-1}\right) = \det(Id + B)$$

and so by comparing terms of the same degree we have that:

$$P_k(gBg^{-1}) = P_k(B)$$

As mentioned in §3.7, if D is the Chern connection of $T^{1,0}M$ we can think of its curvature F_D locally as being a matrix of 2-forms. So, for a sufficiently refined open cover (U_α) of M if $F_D|_{U_\alpha} = (F_D)_\alpha$ and

$$S_\alpha = \bigoplus_{i=0}^{n} \mathcal{A}_M^{2i}(U_\alpha)$$

then $(F_D)_\alpha \in Mat_n(S_\alpha)$ and hence let us consider $P_k((F_D)_\alpha)$. Note that S_α consists of even forms only!

Lemma 3.26. *The Čech co-chain $(P_k(F_D)_\alpha))$ is closed. That is, it patches together to form a globally defined $2k$-form*

Proof. Let $g_{\alpha\beta} : U_\alpha \cap U_\beta \to Gl(n, \mathbb{C})$ denote the transition functions for $T^{1,0}M$ with respect to the open cover (U_α). On $U_\alpha \cap U_\beta$ $(F_D)_\alpha$ and $(F_D)_\beta$ are related by [8]:

$$(F_D)_\alpha = g_{\alpha\beta}(F_D)_\beta g_{\alpha\beta}^{-1}$$

and so:

$$\begin{aligned} P_k((F_D)_\alpha)|_{U_\alpha\cap U_\beta} &= P_k(g_{\alpha\beta}(F_D)_\beta g_{\alpha\beta}^{-1})|_{U_\alpha\cap U_\beta} \\ &= P_k((F_D)_\beta)|_{U_\alpha\cap U_\beta} \end{aligned}$$

Thus

$$P_k(F_D) = P_k((F_D)_\alpha) \text{ if } x \in U_\alpha$$

is indeed a globally defined $2k$-form. □

Definition 3.27 (Chern Classes)**.** The k-th Chern class of (M, J) is defined as:

$$c_k(M, J) = [P_k(\frac{i}{2\pi}F_D)]$$

where D is a Chern connection on $T^{(1,0)}M$ with respect to some Hermitian metric g and the square brackets denote the (de Rham) cohomology class.

It is a non-trivial fact that $c_k(M, J)$ is even well defined; *a priori* if we considered a different Hermitian metric g' and hence a different Chern connection D' we might get a cohomology class $[P_k(\frac{i}{2\pi}F_{D'})]$. However Lemma 4.4.6 on pg.195 of [Huy05] guarantees that

$$P_k(\frac{i}{2\pi}F_D) - P_k(\frac{i}{2\pi}F_{D'})$$

is an exact $2k$-form, hence both connections do indeed define the same class in cohomology. Let us calculate formulas for the first two Chern classes. We shall be working with F_D considered locally as a 2-form valued matrix, and we shall denote its (i, j)-th entry as Ω_i^j.

Lemma 3.28.

$$c_1(M, J) = [\frac{i}{2\pi}\Omega_j^j] = [\frac{1}{2\pi}\rho]$$

[8] cf. [GH78] pg. 75

and:

$$c_2(M,J) = [\frac{-1}{8\pi^2}(\Omega_j^j \wedge \Omega_k^k - \Omega_j^k \wedge \Omega_k^j)]$$

Proof. By the definition of the determinant, we have that:

$$\det(Id + \frac{i}{2\pi}\Omega_i^j) = \sum_{\sigma \in S_n} \epsilon(\sigma)(\delta_{1\sigma(1)} + \frac{i}{2\pi}\Omega_1^{\sigma(1)}) \dots (\delta_{n\sigma(n)} + \frac{i}{2\pi}\Omega_n^{\sigma(n)}) \tag{3.28}$$

Where S_n is the group of permutations on n letters, $\epsilon(\sigma)$ denotes the sign of the permutation and:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

We now break this sum up into sums over equivalence classes in S_n. That is, we consider:

$$\sum_{\sigma \in S_n} f(\sigma) = f(id) - \sum_{\sigma \text{ a 2-cycle}} f(\sigma) + \sum_{\sigma \text{ a 3-cycle}} f(\sigma) + \dots$$

and so (3.28) becomes:

$$\begin{aligned}\det(Id + \Omega_i^j) =& (1 + \frac{i}{2\pi}\Omega_1^1) \dots (1 + \frac{i}{2\pi}\Omega_n^n) \\ & - \sum_{j<k} \big((1 + \frac{i}{2\pi}\Omega_1^1) \dots (\frac{i}{2\pi}\Omega_k^j) \dots (\frac{i}{2\pi}\Omega_j^k) \dots (1 + \frac{i}{2\pi}\Omega_n^n)\big) \\ & + (\text{terms containing } 2n\text{-forms for } n \geq 3)\end{aligned}$$

expanding we get:

$$\begin{aligned}\det(Id + \Omega_i^j) =& 1 + (\frac{i}{2\pi})\sum_j (\Omega_j^j) + (\frac{-1}{4\pi^2})\sum_{j<k}(\Omega_j^j \wedge \Omega_k^k) \\ & - (\frac{-1}{4\pi^2})\sum_{j<k}(\Omega_j^k \wedge \Omega_k^j) \\ & + (\text{terms containing } 2n\text{-forms for } n \geq 3)\end{aligned}$$

Thus:

$$\begin{aligned}c_1(M,J) &= [(\frac{i}{2\pi})\sum_j (\Omega_j^j)] \\ c_2(M,J) &= [\frac{-1}{4\pi^2}\sum_{j<k}(\Omega_j^j \wedge \Omega_k^k - \Omega_j^k \wedge \Omega_k^j)]\end{aligned}$$

Now consider the general term of the sum in the expression for $c_2(M,J)$. Let us denote it by a_{jk} for the moment:

$$a_{jk} = \Omega_j^j \wedge \Omega_k^k - \Omega_j^k \wedge \Omega_k^j$$

Observe that a_{jk} is symmetric in its indices, since each Ω^l_n is a sum of two forms:

$$a_{kj} = \Omega^k_k \wedge \Omega^j_j - \Omega^j_k \wedge \Omega^k_j = \Omega^j_j \wedge \Omega^k_k - \Omega^k_j \wedge \Omega^j_k = a_{jk}$$

Furthermore:

$$a_{jj} = \Omega^j_j \wedge \Omega^j_j - \Omega^j_j \wedge \Omega^j_j = 0$$

Thus we may replace the restricted double sum $\sum_{j<k} a_{jk}$ with a *bone fide* double sum:

$$\sum_{j<k} a_{jk} = \frac{1}{2}\sum_j \sum_k a_{jk} = \frac{1}{2}\sum_{j,k} a_{jk}$$

where the factor of a half comes in because we are double counting (since $a_{jk} = a_{kj}$).The upshot of this is that we may write:

$$c_2(M,J) = [\frac{-1}{8\pi^2}\sum_{j,k}\left(\Omega^j_j \wedge \Omega^k_k - \Omega^k_j \wedge \Omega^j_k\right)]$$

as required. The fact that $c_1(M,J) = [\frac{1}{2\pi}\rho]$ now follows from the definition of ρ as the trace of the endomorphism part of F_D (see definition 3.22). □

Recalling the definition of a Kähler-Einstein metric:

Theorem 3.29. *Suppose that* (M,J,g) *is a Kähler-Einstein manifold of negative scalar curvature* λ*. Then* $c_1(M,J)$ *is a negative definite* $(1,1)$*-form.*

Proof. By assumption, we have that $\rho = \lambda\omega$ for some $\lambda < 0$. By lemma 3.28 we have that:

$$c_1(M,J) = [\frac{1}{2\pi}\rho] = [\frac{\lambda}{2\pi}\omega]$$

Since ω is a positive definite $(1,1)$-form, $c_1(M,J)$ is negative definite. □

Since $c_1(M,J)$ depends only on the topology of M and the (homotopy class of) the complex structure J, the above theorem gives a necessary condition on (M,J) for it to allow a Kähler-Einstein metric. An extremely powerful (and deep) theorem that we shall use repeatedly in the sequel, is that the above necessary condition is in fact sufficient.

Theorem 3.30. *Suppose that* $c_1(M,J)$ *is negative definite; then there exists a Kähler-Einstein metric* g *on* M *such that its associated* $(1,1)$*-form* ω *satisfies:*

$$c_1(M,J) = [\frac{1}{2\pi}\rho] = [\frac{\lambda}{2\pi}\omega]$$

If we require the Scalar curvature to be -1 *then this metric is unique.*

Proof. This is (part of) theorem 1 in [Yau77]. See also [Yau78] and [Aub78] □

Before we move on, we combine the results of this section, theorem 3.15 with the Berger holonomy theorem (theorem 3.13) to get:

Theorem 3.31. *Suppose that (M, J) is an irreducible Kähler manifold with negative definite first Chern class: $c_1(M, J) < 0$. Then for any Kähler metric g on (M, J) either $Hol(M, g) = U(m)$ or (M, J, g) is a uniformised by a bounded symmetric domain.*

Proof. Consider the list of possible non-symmetric holonomy groups given in theorem 3.13, and suppose that (M, J, g) is non-symmetric. We may eliminate the possibilities:

1. $Hol(M, g) = SO(2m)$
2. $Hol(M, g) = Sp(m/2) \cdot Sp(1)$
3. $Hol(M, g) = G_2$
4. $Hol(M, g) = Spin(7)$

as none of these are contained in $U(m)$, contradicting theorem 3.15. Furthermore we can eliminate the possibilities:

1. $Hol(M, g) = SU(m)$
2. $Hol(M, g) = Sp(m/4)$

as manifolds with these holonomy groups are *Ricci flat*. That is, if ρ is the Ricci form associated to g, then $\rho = 0$. But then, by lemma 3.28, $c_1 = [\frac{1}{2\pi}\rho] = 0$, contradicting the fact that $c_1(M, J) < 0$. Hence the only possibility left is:

$$Hol(M, g) = U(m)$$

□

Chapter 4

Hermitian Symmetric Spaces and Bounded Symmetric Domains

In this section we aim to give a very brief overview of the theory of Symmetric Spaces, extracting just enough theory to build the uniformization results contained in chapters 5 and 6. We shall assume a working knowledge of Lie algebras. For the sake of brevity we shall omit the proofs of several key results, and shall refer the reader to the comprehensive [Hel78] or the very readable [Zil10] for further details. Loosely speaking, an (irreducible) symmetric space M arises by taking the quotient of a simple Lie group G by a maximal compact subgroup K. Since the maximal compact subgroup is unique up to inner automorphism, this means we can classify all possible symmetric spaces using the classification of Lie groups. Moreover, through its close relation with G, M picks up several remarkable properties:

1. M is a complete, homogeneous Riemannian manifold.

2. The Riemannian data on M (metric, Levi-Civita connection, curvature) is determined by Lie algebraic data (the Lie bracket and the Killing form) on $\mathfrak{g}$[1].

3. Symmetric spaces have large groups of isometries (namely G) and provide examples of Riemannian manifolds with holonomy not covered by the Berger holonomy theorem. (We shall see shortly that, in all cases of interest in this thesis, $Hol(M) = K$.)

[1] From here on we shall implicitly be using the convention that if a Lie group is denoted by a particular capital letter then its Lie algebra is denoted by the same, lowercase letter in Gothic script.

4. Since we have a fibration:
$$K \to G \to M$$
it is easy to compute the homotopy groups of M, given the homotopy groups of K and G, using the long exact sequence in homotopy. In all cases of interest in this thesis, this calculation shows that M is simply connected.

The last two points in particular, point to why one might ask about manifolds uniformised by a symmetric space, as we should expect a simply connected manifold with a large group of isometries to have many quotient manifolds. But before we get too ahead of ourselves, let us clarify what we mean by a symmetric space.

4.1 Elementary properties of Riemannian symmetric spaces

Definition 4.1 (Definition 6.1, pg. 129 in [Zil10])**.** Suppose that (M, g) is a Riemannian manifold. Then (M, g) is a *symmetric space* if, for all $x \in M$, there is an isometry $s_x : M \to M$ with $s_x(x) = x$ and $d(s_x)|_x = -Id$

We shall call s_x the *symmetry at x*.

Remark 4.2.

1. Note that the definition of 'symmetric space' depends on the pair (M, g). It is possible to define two metrics g and $g^{'}$ on M such that (M, g) is a symmetric space but $(M, g^{'})$ is not.

2. We may define *locally symmetric spaces* as Riemannian manifolds (M, g) such that for each point $x \in M$ there exists an $r > 0$ and a *local* isometry $s_x : B_r(x) \to B_r(x)$ satisfying $d(s_x)|_x = -Id$.

Before continuing we should mention a remarkable observation about isometries of a Riemannian manifold, which we shall use frequently in this chapter.

Theorem 4.3. *Suppose that (M, g) is a complete, connected Riemannian manifold, and that $f_1, f_2 : M \to M$ are isometries of M such that there exists a $x \in M$ satisfying:*
$$f_1(x) = f_2(x) \text{ and } df_1|_x = df_2|_x$$
then $f_1 = f_2$

Proof. We need to show that, given any $y \in M$, $y \neq x$, we have that $f_1(y) = f_2(y)$. Since (M, g) is complete it is geodesically complete [2], and so there exists a length-minimizing

[2]This is the content of the Hopf-Rinow theorem

geodesic γ such that $\gamma(0) = x$ and $\gamma(1) = y$. We know that isometries map geodesics to geodesics, hence $f_1 \circ \gamma(t)$ and $f_2 \circ \gamma(t)$ are both geodesics and

$$f_1 \circ \gamma(0) = f_1(x) = f_2(x) = f_2 \circ \gamma(0)$$

Now observe that:

$$\begin{aligned} (f_1 \circ \gamma)'(0) &= df_1|_{\gamma(0)}(\gamma'(0)) = df_1|_x(\gamma'(0)) \\ \text{and } (f_2 \circ \gamma)'(0) &= df_2|_{\gamma(0)}(\gamma'(0)) = df_2|_x(\gamma'(0)) \\ \text{and by assumption } df_1|_x = df_2|_x &\text{ so } (f_1 \circ \gamma)'(0) = (f_2 \circ \gamma)'(0) \end{aligned}$$

but we know that geodesics are uniquely determined by their initial data[3] hence

$$f_1 \circ \gamma(t) = f_2 \circ \gamma(t)$$

and so in particular $f_1(y) = f_2(y)$. □

There is a second, equivalent definition of symmetric space that is worth mentioning:

Theorem 4.4. *(M, g) is a symmetric space if and only if (M, g) is complete and for all $x \in M$, there exists a non-trivial involutive isometry $s_x : M \to M$ having x as an isolated fixed point.*

Proof. We need the following fact from Riemannian geometry (cf. Prop. 5.11 in [Lee97]). Given any $x \in M$ and an orthonormal basis $\{e_1, \ldots e_n\}$ for (T_xM, g_x) there exists a neighbourhood $U \ni x$ and coordinates $\{x_1, \ldots, x_n\}$ on U such that for any geodesic $\gamma_X(t)$ satisfying $\gamma_X(0) = x$ and $\gamma_X'(0) = X_i e^i$, with respect to the coordinates x^i:

$$\gamma_X(t) = (tX_1, \ldots, tX_n)$$

such a neighbourhood is called a *normal neighbourhood* and such coordinates are called *normal coordinates.*

Now suppose that for all $x \in M$ there exists an isometry s_x such that $s_x(x) = x$ and $s_x^2 = id$. Then

$$\begin{aligned} id = d(s_x^2)|_x &= ds_x|_{s_x(x)} \circ ds_x|_x \\ &= ds_x|_x \circ ds_x|_x \end{aligned}$$

thus $ds_x|_x = \pm id$. If $ds_x|x = Id$ then by theorem 4.3 $s_x = Id$ contradicting the assumption that s_x was non-trivial. Hence $ds_x|_x = -Id$ and so (M, g) is a symmetric

[3]That is, if $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1'(0) = \gamma_2'(0)$ then $\gamma_1(t) = \gamma_2(t)$ for all $t \in \mathbb{R}$

space.

Conversely if (M, g) is a symmetric space then for every x we have an isometry s_x such that x is a fixed point of s_x and $ds_x|_x = -Id$. Choosing an orthonormal basis e^i for (T_xM, g_x) we get a normal neighbourhood U and normal coordinates x^i. For any $y \in U$ by completeness there exists a geodesic $\gamma_X(t)$ such that $\gamma_X(0) = x$ and $\gamma_X(1) = y$. We shall write $\gamma_X^{'}(0) = X_i e^i$, then y is given in normal coordinates as $y = (X_1, \dots, X_n)$. Now observe that:

$$s_x \circ \gamma_X(0) = \gamma_X(0)$$
$$\text{and } ds_x(\gamma^{'}(0)) = -\gamma^{'}(0) = -X_i e^i$$

Hence s_x carries γ_X to the geodesic γ_{-X}[4]. In particular:

$$s_x(y) = s_x(\gamma_X(1)) = \gamma_{-X}(1) \neq y$$

so there are no fixed points of s_x other than x in U. Finally since $s_x^2(x) = x$ and $d(s_x^2) = ds_x|_x \circ ds_x|_x = Id$, by theorem 4.3 $s_x^2 = Id$. □

Let us list a few basic properties of symmetric spaces:

Theorem 4.5. *Suppose that (M, g) is a symmetric space. Then:*

1. *M is complete.*

2. *$I(M)$ is a Lie group and it acts transitively on M. In fact, $I(M)_0$, the identity component of $I(M)$, acts transitively on M.*

3. *If $x \in M$, denote by $K_x < I(M)$ the stabilizer of x. Then K_x is compact. Since $I(M)$ acts homogeneously on M, for any other $y \in M$, K_x and K_y are conjugate in $I(M)$.*

4. *If $I(M)_0$ is simply-connected and K_x is connected for any $x \in M$ then M is simply connected. Conversely, if M is simply connected then K_x is connected.*

5. *If ∇ and F_∇ denote the Levi-Civita connection and curvature tensor associated to the metric g, then $\nabla(F_\nabla) = 0$*

[4] s_x is locally the geodesic reversal map

Proof. The fact that M is complete is proposition 6.2 on pg. 130 in [Zil10]. The fact that $I(M)$ is a Lie group and K_x is compact is true for any Riemannian manifold M, see Theorem 2.5 on page 204 of [Hel78]).The fact that $I(M)_0$ acts transitively is Corollary 6.5 (pg.132) in [Zil10]. For any $x, y \in M$ there then exists a $\varphi \in I(M)$ such that $\varphi(x) = y$, and so $K_y = \varphi K_x \varphi^{-1}$. Since we now have that $M \cong I(M)_0/K_x$ for any $x \in M$, the third item follows from writing out the long exact sequence in homotopy associated to the fibration $K \to I(M)_0 \to M$ and observing that we get the following

$$\ldots \to \pi_1(I(M)_0) \to \pi_1(M) \to \pi_0(K_x) \to \pi_0(I(M)_0) \ldots$$

Hence $\pi_1(I(M)_0) = \pi_0(K_x) = 0 \Rightarrow \pi_1(M) = 0$ and $\pi_1(M) = 0 \Rightarrow \pi_0(K_x) = 0$
To prove the fourth item, we observe something more general. Suppose that $A \in \Gamma((T^*M)^{\otimes 2k+1})$ is an odd-order, covariant tensor field on M which is isometry invariant. That is, for any $\varphi \in I(M)$ and $X_1, \ldots, X_{2k+1} \in T_xM$ we have that

$$(\varphi^* A)(X_1, \ldots, X_{2k+1}) = A_{\varphi(x)}(\varphi_* X_1, \ldots, \varphi_* X_{2k+1}) = A_x(X_1, \ldots, X_{2k+1})$$

Then in fact $A = 0$, since for any $x \in M$, taking $\varphi = s_x$ (and so $\varphi(x) = x$ and $\varphi_*|_x = -id$), the above implies that:

$$\begin{aligned} A_x(X_1, \ldots, X_{2k+1}) &= A_x(-X_1, \ldots, -X_{2k+1}) \\ &= (-1)^{2k+1} A_x(X_1, \ldots, X_{2k+1}) \text{ By multilinearity} \\ &= -A_x(X_1, \ldots, X_{2k+1}) \end{aligned}$$

for all $X_1, \ldots, X_{2k+1} \in T_xM$. Applying this to the situation at hand, since $F_\nabla \in \Gamma((T^*M)^{\otimes 4})$ is isometry invariant, we have that $\nabla F_\nabla \in \Gamma((T^*M)^{\otimes 5})$ is also isometry invariant and hence $\nabla F_\nabla = 0$. □

When no confusion can arise as to which symmetric space M we are referring to, we shall usually denote $I(M)_0$ as G, and its Lie algebra as $\mathfrak{g}$. In addition, we shall denote the stabilizer of a point $x \in M$ as K_x (or sometimes just K, when the particular point x is not important), and its Lie algebra as $\mathfrak{k}$.

Remark 4.6. Although there exist non-simply-connected symmetric spaces (for example $\mathbb{R}P^n$, which has S^n as a double-cover) it is a theorem (cf. Proposition 6.53 in [Zil10]) that all bounded symmetric domains are simply connected. Since it is frequently much simpler, and we are ultimately interested only in bounded symmetric domains [5], from here on we shall only consider simply connected symmetric spaces.

[5] And their compact duals, which also happen to be simply connected, but more on that later

4.1.1 Some examples

Let us give a few examples of symmetric spaces. Observe that to show (M, g) is a symmetric space is suffices to produce a Lie group G acting transitively on M via isometries, and a symmetry s_x at a single point, as then for any other point $y \in M$ we get s_y by choosing $\varphi \in G$ such that $\varphi(x) = y$ and defining $s_y = \varphi s_x \varphi^{-1}$

1. If g_E denotes the usual Euclidean metric on $\mathbb{C}^n$ then $(\mathbb{C}^n, g_E)$ is a symmetric space. To define the symmetry at a point x, we first note that any $y \in \mathbb{C}^n$ can be written as $y = x + (y - x)$. Then s_x is defined as:

$$s_x(y) = x - (y - x)$$

   Observe that the Levi-Civita connection associated to g_E is flat[6].

2. Consider $\mathbb{CP}^n$ equipped with the Fubini-Study metric g_{FS}. We know that $U(n+1)$ acts transitively via isometries. For a given point $x \in \mathbb{CP}^n$, choose homogeneous coordinates such that $x = [1, 0, \ldots, 0]$. Let $D = \mathrm{diag}(1, -1, \ldots, -1) \in U(n+1)$ and define $s_x([z_0, \ldots, z_n]) = [D \cdot (z_0, \ldots, z_n)] = [z_0, -z_1, \ldots, -z_n]$. $\mathbb{CP}^n$ is a compact symmetric space.

3. Let $\mathcal{D}_1$ be the unit disk equipped with the Poincaré metric:

$$g_z(\zeta_1, \zeta_2) = \frac{4\zeta_1\bar{\zeta}_2}{(1 - |z|^2)^2}$$

   (where the factor of 4 is included to ensure that the curvature comes out to be -1). Denote by $I_{1,1}$ the 2×2 matrix:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.1}$$

   Then the group:

$$SU(1,1) = \{A \in Gl(2, \mathbb{C}) : \; A^H I_{1,1} A = I_{1,1} \text{ and } \det(A) = 1\}$$

   acts via fractional linear maps:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}$$

[6] That is, has vanishing curvature tensor

and this action is both transitive and isometric. The symmetry at 0 is given by:

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \cdot z = -z$$

$\mathcal{D}_1$ is a non-compact which, unlike $(\mathbb{C}^n, g_E)$, is not flat. The unit disk is our prototypical example of a *Bounded Symmetric Domain*. That is, a symmetric space which can be realized as a bounded domain in a complex vector space.

4. Given a square matrix A we write $A > 0$ if A is positive definite. Consider the set:

$$\mathcal{D}^I_{n,n} = \left\{ Z \in Mat(n, \mathbb{C}) : I_n - Z^H Z > 0 \right\}$$

where I_n is the $n \times n$ identity matrix. $\mathcal{D}^I_{n,n}$ is a bounded domain of the vector space $Mat(n, \mathbb{C})$. Let $I_{n,n}$ denote the matrix

$$\begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix}$$

and define the Lie group:

$$SU(n, n) = \left\{ X \in Gl(2n, \mathbb{C}) : X^H I_{n,n} X = I_{n,n} \right\}$$

We shall usually write $X \in U(n, n)$ as a block matrix:

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Where each block is an $n \times n$ matrix.

Lemma 4.7. *$U(n, n)$ acts holomorphically on $\mathcal{D}^I_{n,n}$ via fractional linear transformations:*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ + B)(CZ + D)^{-1} \tag{4.2}$$

Proof. Firstly, observe that:

$$I - Z^H Z = (Z^H \; I) \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} Z \\ I \end{pmatrix}$$

and

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} Z \\ I \end{pmatrix} = \begin{pmatrix} AZ + B \\ CZ + D \end{pmatrix}$$

For brevity we write $E = AZ + B$ and $F = CZ + D$. Now:

$$\begin{aligned}
-E^H E + F^H F &= (E^H \ F^H) \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} E \\ F \end{pmatrix} \\
&= (Z^H \ I) \begin{pmatrix} A & B \\ C & D \end{pmatrix}^H \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} Z \\ I \end{pmatrix} \\
&= (Z^H \ I) \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} Z \\ I \end{pmatrix} \\
&> 0
\end{aligned}$$

So, suppose that $Fv = 0$ for some $v \in \mathbb{C}^n$. Then:

$$v^H(-E^H E + F^H F)v = -(v^H E^H)Ev = -(Ev)^H Ev \leq 0$$

But $-E^H E + F^H F$ is positive definite, so this is only possible if $v = 0$, hence F is invertible. We know that if P is a positive definite matrix, then $X^H P X$ is positive definite for all invertible matrices X, and so:

$$F^{-H}(-E^H E + F^H F)F^{-1} = -F^{-H}E^H E F^{-1} + I = I - (AZ + B)(CZ + D)^{-1}$$

is positive definite, thus $(AZ + B)(CZ + D)^{-1} \in \mathcal{D}^I_{n,n}$ and this action is well defined. to see that the map:

$$Z \mapsto (AZ + B)(CZ + D)^{-1}$$

is holomorphic it suffices to observe that this map is given by rational functions in each coordinate:

$$(AZ + B)(CZ + D)^{-1} = \frac{1}{\det(CZ + D)}(AZ + B)\mathrm{Adj}(CZ + D)$$

and that $\det(CZ + D) \neq 0$ for all $Z \in \mathcal{D}^I_{n,n}$. This proof is a variation of an argument given in [Fre99] pg. 10-11, amongst other places. □

There is a natural metric on $\mathcal{D}^I_{n,n}$ given by:

$$g_Z(X, Y) = \mathrm{tr}((I - Z^H Z)^{-1} X (I - Z^H Z)^{-1} Y)$$

and one can check that in fact $SU(n, n)$ acts transitively and via isomorphisms. The map $Z \mapsto -Z$, which is given by the action of $\begin{pmatrix} -iI & 0 \\ 0 & iI \end{pmatrix}$, gives an involution of $\mathcal{D}^I_{n,n}$ fixing 0. $\mathcal{D}^I_{n,n}$ is another example of a bounded symmetric domain.

Notice how the metric, the transitive action and indeed the very definition of $\mathcal{D}^I_{n,n}$ are all formally very similar to that of the unit disk.

5. Let $Gr(k, \mathbb{K}^n)$ be the Grassmannian of k-planes in $\mathbb{K}^n$, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. Given any $V \in Gr(k, \mathbb{K}^n)$ we may define an involution having V as an isolated fixed point geometrically as follows. Given any k-plane V', choose a basis $\{e_1, \ldots, e_k\}$ for V'. Define $\sigma_V(e_i)$ to be the reflection of e_i in V, and $\sigma_V(V')$ to be the k-plane spanned by $\{\sigma_V(e_1), \ldots, \sigma_V(e_k)\}$. Then it is obvious that σ_V is an involution fixing V and perhaps less obvious, but still intuitive that σ_V does not fix any V' 'near' V. It can be shown more rigorously that V is indeed an isolated fixed point and moreover that there exists a metric g on $Gr(k, \mathbb{K}^n)$ with respect to which σ_V is an isometry, making $(Gr(k, \mathbb{K}^n), g)$ a symmetric space, but the details of this do not concern us right now (see [Zil10] pg. 144-145). $Gr(k, \mathbb{K}^n)$ is another example of a compact symmetric space.

We can create more examples of symmetric spaces by taking the Riemannian product of two given symmetric spaces.

4.2 The Isotropy representation

Given any $\varphi \in K_x$, since $\varphi(x) = x$ we have that $d\varphi|_x \in Gl(T_xM)$. Thus we have a representation:

$$\chi : K_x \to Gl(T_xM, \mathbb{R})$$

If in addition $d\varphi_1|_x = d\varphi_2|_x$ by theorem 4.3 we have that $\varphi_1 = \varphi_2$. Hence χ, which we shall refer to as the *isotropy representation*, is faithful! Furthermore, since $\varphi \in K_x$ is an isometry, $d\varphi$ preserves the inner product g_x on T_xM:

$$g_x(d\varphi_x X, d\varphi|_x Y) = g_x(X, Y)$$

so $\chi(K_x) \subset O(T_xM)$. Since K_x is compact (cf. theorem 4.5) and χ is continuous, $\chi(K_x)$ is a closed subgroup of $O(T_xM)$. If (M, g) is simply connected, we say that it is *irreducible* if χ is an irreducible representation. A second remarkable property of χ is the following:

Theorem 4.8. *If (M, g) be a simply-connected symmetric space then*

$$Hol_x(M, g) \subset K_x$$

Proof. See Corollary 6.6 on pg. 133 in [Zil10] □

4.2.1 Relating the isotropy and Adjoint representations

From theorem 4.1 we see that a symmetric space (M, g) gives us a real, connected Lie group $G = I(M)_0$, together with a compact subgroup $K = K_x < G$. In addition we have an involutive automorphism defined on G:

$$\begin{aligned} \sigma &: G \to G \\ &: \varphi \mapsto s_x \varphi s_x^{-1} = s_x \varphi s_x \end{aligned}$$

note that σ does indeed map G (which is the identity component of $I(M)$) into G as it maps e to e and is continuous, so it must map connected components to connected components. We now claim that:

Theorem 4.9 (Theorem 3.3 pg. 208 of [Hel78])**.** *If K_σ denotes the fixed point set of σ then:*

$$(K_\sigma)_0 \subset K \subset K_\sigma$$

so in particular if K_σ is connected then $K_\sigma = K$.

Proof. suppose that $k \in K$. Then:

$$s_x k s_x(x) = s_x k(x) = s_x(x) = x$$

and:

$$d(s_x k s_x)|_x = ds_x|_{ks_x(x)} \circ dk|_{s_x(x)} \circ s_x|_x = -Id \circ dk|_x \circ (-Id) = dk|_x$$

Hence by theorem 4.3 $\sigma(k) = k$ and so $K \subset K_\sigma$. To show the other containment we show that K and K_σ have the same Lie algebra. Since from the above it follows that, denoting the Lie algebra of K_σ as $\mathfrak{k}_\sigma$, $\mathfrak{k} \subset \mathfrak{k}_\sigma$, it suffices to show the opposite containment. Recall that (cf. theorem 2.5)

$$\sigma(\exp(X)) = \exp(d\sigma(X)) \quad \forall X \in \mathfrak{g}$$

If $X \in \mathfrak{k}_\sigma$; then $d\sigma(X) = X$ and so

$$s_x \exp(tX)(x) = s_x \exp(tX) s_x(x) = \exp(td\sigma(X))(x) = \exp(tX)(x) \quad \forall t \in \mathbb{R}$$

That is, for all t, $\exp(tX)$ is a fixed point for s_x. But by assumption x is an isolated fixed point of s_x, so, for a small open neighbourhood $\mathcal{N}$ of x, $\{\exp(tX)(x) : t \in \mathbb{R}\} \cap \mathcal{N} = \{x\}$. The map $\gamma : t \mapsto \exp(tX)(x)$ is continuous, so $\gamma^{-1}(\mathcal{N})$ is open. But $\gamma^{-1}(\mathcal{N}) = \gamma^{-1}(\{x\})$ so it is also closed. Since $0 \in \gamma^{-1}(\mathcal{N})$, we conclude that $\gamma^{-1}(\mathcal{N}) = \mathbb{R}$ and so $\exp(tX)(x) = x$ $\forall t \in \mathbb{R}$. Thus $X \in \mathfrak{k}$. ∎

By theorem 4.1, if (M, g) is simply connected then K is connected and so $K = K_\sigma$. Denote by $\mathfrak{p}$ the -1-eigenspace of $d\sigma$. Then:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

If we identify $M \cong G/K_x$, then G acts on the cosets by left-translation:

$$\varphi(\psi K) = (\varphi\psi)K$$

and we have the following:

Theorem 4.10. *The map*

$$\tau : \mathfrak{p} \to T_x M \tag{4.3}$$

$$X \mapsto \frac{d}{dt}|_{t=0}(\exp(tX) \cdot p) \tag{4.4}$$

is a K_x-equivariant isomorphism.

Proof. Observe that τ is linear, and that $dim_{\mathbb{R}}(T_p M) = dim_{\mathbb{R}}(G/K) = dim_{\mathbb{R}}(\mathfrak{p})$. Moreover, τ is injective, since:

$$\tau(X) = 0 \Rightarrow \frac{d}{dt}|_{t=0}(\exp(tX) \cdot p) = 0 \Rightarrow \exp(tX) \cdot p = p \quad t \in (-\epsilon, \epsilon) \tag{4.5}$$

$$\Rightarrow \exp(tX) \in K \quad t \in (-\epsilon, \epsilon) \tag{4.6}$$

$$\Rightarrow X \in \mathfrak{k} \tag{4.7}$$

and $\mathfrak{k} \cap \mathfrak{p} = 0$. Hence τ is a vector space isomorphism. For $k \in K$ and $X \in \mathfrak{p}$ recall that

$$\chi(k)(\tau(X)) = dk|_x(\tau(X)) = \frac{d}{dt}|_{t=0}(k \cdot \gamma(t))$$

where $\gamma(t)$ is a smooth curve in M satisfying $\gamma(0) = p$ and $\gamma'(0) = X$. Using the identification $M \cong G/K$ we can take $\gamma(t) = \exp(tX)K$. So

$$\begin{aligned}
\chi(k)(\tau(X)) &= \frac{d}{dt}|_{t=0}(k\exp(tX)K) \\
&= \frac{d}{dt}|_{t=0}(k\exp(tX)k^{-1}kK) \\
&= \frac{d}{dt}|_{t=0}(k\exp(tX)k^{-1}K) \text{ since } k \in K \\
&= \frac{d}{dt}|_{t=0}\left(\exp(Ad_k(tX))K\right) \text{ by theorem 2.5} \\
&= \frac{d}{dt}|_{t=0}(\exp\left(tAd_k(X)\right)K) \\
&= \frac{d}{dt}|_{t=0}(\exp\left(tAd_k(X)\right)\cdot x) \\
&= \tau(Ad_k(X))
\end{aligned}$$

□

4.3 Type

If (M, g) is an irreducible symmetric space, we say that it is of *compact type* if $B|_{\mathfrak{p}} > 0$[7], of *non-compact type* if $B|_{\mathfrak{p}} < 0$ and of *Euclidean type* if $\mathfrak{p}$ is an abelian subalgebra.

Lemma 4.11. *Every irreducible symmetric space, (M, g), is of one of these three types.*

Proof. Because g_x is a χ invariant inner product on T_xM, τ^*g_x is an Ad-invariant inner product on $\mathfrak{p}$ (cf. theorem 4.10). But $B|_{\mathfrak{p}}$ is an Ad-invariant symmetric bilinear form on $\mathfrak{p}$, so by theorem 2.13 $B|_{\mathfrak{p}} = ag_x$ for some $a \in \mathbb{R}$. So (M, g) is of compact type if $a < 0$, of non-compact type if $a > 0$ and of Euclidean type if $a = 0$. This is a variant of the proof of Prop. 6.33 on pg. 154 of [Zil10]. □

We shall shortly show that symmetric spaces of compact (resp. non-compact) type are indeed compact (resp. non-compact). But for now we note the following:

Theorem 4.12. *Let (M, g) be an irreducible simply connected symmetric space not of Euclidean type. Then $\mathfrak{g}$ is simple if and only if (M, g) is irreducible*

Proof. If (M, g) is irreducible and simply connected by definition the isotropy action χ is irreducible. It is also of course faithful (cf. discussion at the beginning of §4.2). Hence

[7] As in Chapter §2.4 B denotes the Killing form of $\mathfrak{g}$

$Ad: K \to Gl(\mathfrak{p})$ is irreducible and faithful and thus so is $ad: \mathfrak{k} \to \mathfrak{gl}(\mathfrak{p})$. If $X \in \mathfrak{k} \cap \mathfrak{z}(\mathfrak{g})$ we see that $ad_X = 0$, so by faithfulness we must have

$$\mathfrak{k} \cap \mathfrak{z}(\mathfrak{g}) = \{0\}$$

Applying proposition 2.19 we see that $B|_{\mathfrak{k}} < 0$.
Recall that $\mathfrak{k}$ (respectively $\mathfrak{p}$) was defined as the $+1$ (respectively -1) eigenspace of the Lie algebra automorphism $d\sigma$. So, if $X \in \mathfrak{k}$, $Y \in \mathfrak{p}$ we have that:

$$B(X,Y) = B(d\sigma X, d\sigma Y) = B(X, -Y) = -B(X,Y)$$

hence $B(\mathfrak{k}, \mathfrak{p}) = 0$. Thus if $B|_{\mathfrak{p}} > 0$ (or $B_{\mathfrak{p}} < 0$) for any $X \in \mathfrak{g}$, writing $X = X_1 + X_2$ with $X_1 \in \mathfrak{k}$ and $X_2 \in \mathfrak{p}$ we see that if $X \neq 0$ one of:

$$B(X, X_1) = B(X_1, X_1)$$
$$B(X, X_2) = B(X_2, X_2)$$

is non-zero. Hence B is non-degenerate, and so by Cartan's second criterion (cf. theorem 2.15) $\mathfrak{g}$ is semi-simple. Moreover observe that:

$$d\sigma([X_1, Y_1]) = [d\sigma(X_1), d\sigma(Y_1)] = [X_1, -Y_1] = -[X_1, Y_1] \Rightarrow [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$$
$$d\sigma([X_1, X_2]) = [d\sigma(X_1), d\sigma(X_2)] = [X_1, X_2] \Rightarrow [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$$
$$d\sigma([Y_1, Y_2]) = [d\sigma(Y_1), d\sigma(Y_2)] = [-Y_1, -Y_2] = [Y_1, Y_2] \Rightarrow [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$$

thus if $B|_{\mathfrak{p}} > 0$ observe that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition (cf. the definition given in §2.5). The fact that $\mathfrak{g}$ is simple then follows by theorem 2.20. We defer the proof that $\mathfrak{g}$ is simple in the compact case until we have developed the idea of duality. □

Observe that g is determined up to a scalar multiple by B, since $g_x = (\tau^{-1})^* B|_{\mathfrak{p}}$ and for any other $y \in M$, choosing $\varphi \in G$ such that $\varphi(y) = x$ we have that:

$$g_y = \varphi^* g_x$$

In fact all the Riemannian data of (M, g) (that is, metric, connection and curvature) is determined neatly by the Lie theoretic data of $\mathfrak{g}$:

Proposition 4.13. *For any vector fields $X, Y, Z \in \Gamma(TM)$ we have that:*

$$(\nabla_X Y)(x) = [\tau^{-1}(X(x)), \tau^{-1}(Y(x))]$$
$$F_\nabla(X,Y)(Z)(x) = -\tau\big([[\tau^{-1}(X(x)), \tau^{-1}(Y(x))], \tau^{-1}(Z(x))\big]$$

Proof. See proposition 6.30 on page 153 of [Zil10] □

Observe that if (M, g) is of compact type then $B < 0$, and so by Theorem 2.17 G, and hence M, is compact. If (M, g) is of non-compact type then by Proposition **??** (and see also Proposition 6.34 of [Zil10]) all the sectional curvatures of (M, g) are non-positive. But then by the Cartan-Hadamard theorem (Theorem 11.5 on pg. 196 of [Lee97]) M is non-compact.

4.4 The Decomposition theorem

In general a symmetric space need not be irreducible. For example if (M_1, g_1) and (M_2, g_2) are both symmetric spaces with isotropy representations:

$$\chi_1 : K_1 \to Gl(T_{x_1} M_1)$$
$$\chi_2 : K_2 \to Gl(T_{x_2} M_2)$$

then one can easily check that the isotropy representation of $(M_1, g_1) \times (M_2, g_2)$[8] is $\chi_1 \boxtimes \chi_2$. So, as one would expect, $T_{(x_1,x_2)}(M_1 \times M_2) \cong T_{x_1} M_1 \oplus T_{x_2} M_2$ has two invariant subspaces, namely $T_{x_1} M_1$ and $T_{x_2} M_2$. Just as in §3.6 where we expressed an arbitrary simply connected Riemannian manifold as a Riemannian product of irreducible Riemannian manifolds, so we have:

Theorem 4.14. *Let (M, g) be a simply connected symmetric space. Then (M, g) is isometric to a Riemannian product of irreducible symmetric spaces:*

$$(M, g) \cong (M_0, g_0) \times \ldots \times (M_r, g_r)$$

Moreover

$$I(M)_0 = I(M_0)_0 \times \ldots \times I(M_r)_0$$

Proof. This is a simple application of 3.16. See Corollary 6.10 on pg. 134 of [Zil10]. □

For non-irreducible symmetric spaces (M, g) we say that (M, g) is of compact (resp. non-compact) type if all of its irreducible factors are of compact (resp. non-compact) type. If none of the (M_i, g_i) are of Euclidean type, then all of the $I(M_i)_0$ are simple (cf. Theorem 4.12) and so $I(M)_0$ is semi-simple. So we say that (M, g) is *semi-simple* if its decomposition into irreducible factors contains no Euclidean factors. If (M, g) is semi-simple we have the following:

[8]That is, the Riemannian product of (M_1, g_1) and (M_2, g_2). cf §3.6

Theorem 4.15. *If (M, g) is a semi-simple, simply-connected symmetric space, then:*

$$Hol(M, g) = K$$

Proof. This is proposition 6.36 on pg. 156 of [Zil10] coupled with the fact that is (M, g) is simply connected then both $Hol(M, g)$ and K are connected. $\square$

Thus for a semi-simple simply connected symmetric space the decomposition given in theorem (4.14) coincides with the de Rham decomposition given in theorem 3.16.

4.5 Symmetric pairs, symmetric Lie algebras and Cartan decompositions

4.5.1 Riemann symmetric pairs

From the discussion in §4.2.1 we see that a symmetric space (M, g) gives us the data (G, K, σ) where $G = I(M)_0$ is connected, σ is an involutive automorphism of G and $(K_\sigma)_0 < K < K_\sigma$.

Definition 4.16. A *Riemann Symmetric Pair* is a pair of Lie groups (G, K) such that G is connected,$K < G$ is compact and moreover there exists an involutive automorphism σ on G such that if K^σ denotes the fixed point set of σ then $K_0^\sigma \subset K \subset K^\sigma$ (see also [Hel78] pg. 209 where a more general definition is provided).

The above discussion shows that any symmetric space gives us a symmetric pair of Lie groups. Conversely, given a symmetric pair (G, K) we may construct a Symmetric space as follows:

1. Let $M = G/K$. Then M is a smooth manifold and G acts transitively on M by left-translation:
$$\psi \cdot (\varphi K) = (\psi\varphi)K$$
As in §4.2.1 we have a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ into $+1$ and -1 eigenspaces of $d\sigma$. The stabilizer of the coset K is obviously K, and as in theorem 4.10 we have a K-equivariant isomorphism $\tau : \mathfrak{p} \to T_K M$.

2. Pick any inner product $< \cdot, \cdot >_0$ on $T_K M$. By 'averaging' we get a K-invariant inner product:

$$g_K(X,Y) = \int_K < \chi(k)(X), \chi(k)(Y) >_0 d\mu(k)$$

where μ is the Haar measure on K. Note that this integral converges because K is compact. For any $\varphi K \in M$ define

$$g_{\varphi K} = (\varphi^{-1})^* g_K$$

this is well defined because if $\varphi' K - \varphi K$ then $\varphi' = \varphi k$ for some $k \in K$; so

$$g_{\varphi' K} = (\varphi')^{-1} g_K = (k^{1-})^* \varphi^* g_K = \varphi^* g_K$$

since g_K is K-invariant. Note that if $B|_{\mathfrak{p}} > 0$ or $B|_{\mathfrak{p}} < 0$ (i.e. G/K is of non-compact or compact type) then we may use $\pm B_{\mathfrak{p}}$ as an $Ad(K)$-invariant inner product.

3. Define

$$s_K(\varphi K) = \sigma(\varphi)K$$

We can check that this is well-defined in a very similar way to checking $g_{\varphi K}$ is well-defined. If $X \in T_K M$ then $\exp(\tau^{-1}(tX))K$ is a path on M satisfying

$$\frac{d}{dt}|_{t=0} \exp(\tau^{-1}(tX))K = \tau(\tau^{-1}(X)) = X$$

by the definition of τ. So:

$$\begin{aligned} ds_K|_K(X) &= \frac{d}{dt}|_{t=0} s_K(\exp(\tau^{-1}(tX))K) \\ &= \frac{d}{dt}|_{t=0} \sigma(\exp(\tau^{-1}(tX)))K \\ &= \frac{d}{dt}|_{t=0} \exp(d\sigma(\tau^{-1}(tX)))K \\ &= \frac{d}{dt}|_{t=0} \exp(-\tau^{-1}(tX))K \text{ since } \tau^{-1}(tX) \in \mathfrak{p} \\ &= \tau(-\tau^{-1}(X)) = -X \end{aligned}$$

Thus s_K is involutive. This also shows that s_K is an isometry. Hence s_K is the symmetry at K, and so (M, g) is a symmetric space with $s_{\varphi K} = \varphi \circ s_K \circ \varphi^{-1}$.

4.5.2 Orthogonal Symmetric Lie Algebras

We may profitably phrase this data in terms of Lie algebras as follows:

Definition 4.17 (See [Hel78] pg. 229)**.** An orthogonal symmetric Lie algebra is a pair $(\mathfrak{g}, s)$ where:

1. $\mathfrak{g}$ is a real Lie algebra.

2. s is an involutive automorphism of $\mathfrak{g}$.

3. The fixed point set of s, which we shall denote by $\mathfrak{k}$, is a compact subalgebra.

We say this pair is effective if: $\mathfrak{k} \cap \mathfrak{z}(\mathfrak{g}) = \{0\}$

We may decompose $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and -1 eigenspaces of s and we say that $(\mathfrak{g}, s)$ is of compact, non-compact or Euclidean type if $B|_{\mathfrak{p}} < 0$, $B_{\mathfrak{p}} > 0$ or $B_{\mathfrak{p}} = 0$ respectively. The similarity between this notation and the notation introduced in §4.2.1 is not coincidental:

Lemma 4.18. *If $(\mathfrak{g}, s)$ is an effective orthogonal symmetric Lie algebra of non-compact type then $\mathfrak{g}$ is semi-simple and the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ into $+1$ and -1 eigenspaces of s is a Cartan decomposition.*

Proof. Since $(\mathfrak{g}, s)$ is of non-compact type we have that $B|_{\mathfrak{p}} > 0$. Since $\mathfrak{k}$ is a compactly imbedded subalgebra and $\mathfrak{k} \cap \mathfrak{z}(\mathfrak{g}) = \{0\}$ we have (cf. theorem 2.19) that $B_{\mathfrak{k}} < 0$. The rest of the proof is identical to that of theorem 4.12. □

Lemma 4.19. *Suppose (G, K) is a Riemann symmetric pair. Then $(\mathfrak{g}, d\sigma)$ is an effective orthogonal symmetric Lie algebra.*

Proof. Since σ is involutive so is $d\sigma$ and its fixed point set $\mathfrak{k}$ is indeed a compact subalgebra. Now suppose $X \in \mathfrak{k} \cap \mathfrak{z}(\mathfrak{g})$. Since $\exp(tX) \in K$ for all $t \in \mathbb{R}$, we have $\exp(tX)(x) = x$. Moreover for any $Y \in \mathfrak{p}$, if τ is as in theorem 4.10 then $\tau(Y) \in T_x M$ and:

$$\begin{aligned} Ad(\exp(tX))(Y) &= e^{ad_{tX}}(Y) = Y + ad_{tX}(Y) + \frac{1}{2!}ad_{tX}^2(Y) + \dots \\ &= Y \text{ since } X \in \mathfrak{z}(\mathfrak{g}) \end{aligned}$$

implies that $\chi(\exp(tX) = id$ for all $t \in \mathbb{R}$. Thus by 4.3:

$$\exp(tX) = id \; \forall t \in \mathbb{R}$$

implying that $X = 0$. $\square$

Conversely given an effective orthogonal symmetric Lie algebra $(\mathfrak{g}, s)$, we say that a Riemann symmetric pair (G, K) is *associated* to $(\mathfrak{g}, s)$ if the Lie algebra of G is $\mathfrak{g}$ and the Lie algebra of K is $\mathfrak{k}$. Obviously the involutive automorphism σ of G then satisfies $d\sigma = s$. There will generally be several symmetric pairs associated to $(\mathfrak{g}, s)$, and thus there appear to be many different ways to create a symmetric space given the data of an effective orthogonal symmetric Lie algebra.

4.5.3 Classifying non-compact Symmetric spaces

Let $\mathfrak{g}$ be a non-compact semi-simple real Lie algebra and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$. We know that $\mathfrak{k}$ is compactly imbedded (cf. §2.5). since any $X \in \mathfrak{g}$ may be written uniquely as $X = X_1 + X_2$ with $X_1 \in \mathfrak{k}$ and $X_2 \in \mathfrak{p}$ we may define the *Cartan involution*:

$$s : X_1 + X_2 \mapsto X_1 - X_2$$

This can easily be checked to be an automorphism, but for a proof of this we refer the reader to pg. 360 of [Kna96]. By construction the fixed point set of s is $\mathfrak{k}$. Since $\mathfrak{g}$ is semi-simple, $\mathfrak{z}(\mathfrak{g}) = \{0\}$ and so $(\mathfrak{g}, s)$ gives an effective orthogonal symmetric Lie algebra of non-compact type. Because $\mathfrak{g}$ is semi-simple, $ad_{\mathfrak{g}}(\mathfrak{g}) \cong \mathfrak{g}$ and so a natural choice of associated Riemannian symmetric pair is $(\mathrm{Int}(\mathfrak{g}), K)$, where K is the (connected) subgroup corresponding of $\mathrm{Int}(\mathfrak{g})$ corresponding to $ad_{\mathfrak{g}}(\mathfrak{k}) \cong \mathfrak{k}$. We know K is compact as $\mathfrak{k}$ is compactly imbedded. Thus we get a simply connected symmetric space of non-compact type: $(\mathrm{Int}(\mathfrak{g})/K, g)$, constructed as in §4.5.1. The following theorem tells us that it doesn't matter which Cartan decomposition, nor which associated Riemannian symmetric pair, we choose; they all give isometric symmetric spaces.

Theorem 4.20. *Suppose that (M, g) and (M', g') are non-compact symmetric spaces such that $I(M)$ and $I(M')$ have the same Lie algebra. Then they are isometric.*

Proof. See Corollary 1.3 on pg. 255 of [Hel78]. $\square$

From this we can conclude that all non-compact symmetric spaces are simply-connected (although there are other ways to prove this, cf. Prop. 6.40 on pg. 160 of [Zil10]). This also gives a complete classification of non-compact symmetric spaces, since semi-simple real Lie algebras are direct sums of simple real Lie algebras (cf. §2.2) and these (which are all real forms of simple complex Lie algebras) are completely classified.

4.6 Duality

There is a very useful duality between Effective Orthogonal Symmetric Lie algebras of compact and non-compact type constructed as follows. Let (g, s) be a non-compact Effective Orthogonal Symmetric Lie algebra and write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the decomposition into $+1$ and -1 eigenspaces of s, as in §4.5.2. Consider the real subspace $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p} \subset \mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$. Since $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition:

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$$
$$[\mathfrak{k}, i\mathfrak{p}] \subset i\mathfrak{p}$$
$$[i\mathfrak{p}, i\mathfrak{p}] \subset -\mathfrak{k} = \mathfrak{k}$$

so $\mathfrak{u}$ is in fact a real Lie algebra. For any $X \in \mathfrak{u}$, we may write it uniquely as $X = X_1 + iX_2$ with $X_1 \in \mathfrak{k}$ and $X_2 \in \mathfrak{p}$; define s^* as

$$s^*(X) = X_1 - iX_2$$

For any $X_1 \in \mathfrak{k}$, $iX_2 \in \mathfrak{p}$

$$B(X_1, X_1) < 0$$
$$B(X_1, iX_2) = 0$$
$$B(iX_2, iX_2) = -B(X_2, X_2) < 0$$

Finally we can easily check that $\mathfrak{z}(\mathfrak{u}) = \{0\}$. We conclude that $(\mathfrak{u}, s)$ is an Effective Orthogonal Symmetric Lie algebra of compact type. If (M, g) and (M^*, g^*) are symmetric spaces associated to $(\mathfrak{g}, s)$ and $(\mathfrak{u}, s^*)$ respectively, we say that they are *dual*. This establishes a one-to one correspondence between non-compact symmetric spaces and simply connected symmetric spaces of compact type, which allows us to use the classify compact symmetric spaces by using the classification of their duals (cf. theorem 4.20). This also allows us to fill in the gap left in theorem 4.12 since if (M^*, g^*) is a compact and irreducible symmetric space with $U = I(M^*)_0$, we see that its non-compact dual (M, g) is also irreducible. If $G = I(M)_0$ we have established that (M, g) irreducible implies that $\mathfrak{g}$ is simple, hence $\mathfrak{g}^{\mathbb{C}}$ is simple. But $\mathfrak{u}$ is also a real form of $\mathfrak{g}^{\mathbb{C}}$ hence it is also simple.

For a symmetric space (M, g) we define its *rank* to be the maximal dimension of a totally geodesic submanifold $S \subset M$ such that if ∇ is the Levi-Civita connection on M $\nabla|_S = 0$. Equivalently, it is the dimension of a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$. We remark that the rank of a symmetric space is equal to the rank of its dual, and refer the reader to pg. 245-248 of [Hel78] for further details.

4.7 Hermitian symmetric spaces

Let us now consider symmetric spaces that are in addition Hermitian manifolds.

Definition 4.21. Let (M, g) be a symmetric space. We say that it is an *Hermitian symmetric space* if (M, g) is a Hermitian manifold and for all $x \in M$ the symmetry s_x is holomorphic.

One immediate consequence of this definition is the following:

Theorem 4.22. *If (M, J, g) is a Hermitian symmetric space then it is Kähler.*

Proof. Let s_x be the symmetry at any point $x \in M$. Since s_x is, by assumption, a holomorphic map, we have that J is s_x invariant, and so ∇J is s_x invariant. Thus ∇J is an s_x invariant tensor of odd rank (it is of rank 3) and so it must vanish (cf. the proof of item 5 in theorem 4.1). □

Corollary 4.23. *If (M, J, g) is a semi-simple Hermitian symmetric space then χ is a unitary representation.*

Proof. By Proposition 3.15 we know that (M, J, g) Kähler implies the holonomy representation is unitary. But by theorem 4.15 $Hol(M, g) = K_x$ and the holonomy representation is given by χ □

Extending χ by complex linearity we get $\chi : K_x \to Gl(T_xM \otimes \mathbb{C}, \mathbb{C})$ and in fact this restricts to a representation on the holomorphic tangent space $\chi : K_x \to Gl(T_xM^{1,0}, \mathbb{C})$ [9] since if $X \in T_xM^{1,0}$ then:

$$J_x(\chi(k)(X)) = \chi(k)(J_x(X)) = \chi(k)(iX) = i\chi(X) \tag{4.8}$$

Henceforth when we talk of the isotropy representation we shall usually mean the representation on $T_x^{1,0}M$. Amongst Riemann symmetric spaces, we may recognise Hermitian

[9] cf. the definition of holomorphic tangent space in §3.2

symmetric spaces as those coming from Riemmanian symmetric pairs (G, K) such that K has non-trivial centre.

Theorem 4.24 (Theorem 6.1 and Proposition 6.2, page 381-382 [Hel78])**.** *1. The non-compact irreducible Hermitian symmetric spaces are exactly the manifolds of the form $\Omega = G/K$ where G is a connected, simple, non-compact real Lie group with $Z(G) = \{e\}$ and K is a maximal connected subgroup with non-discrete centre.*

2. The compact irreducible Hermitian symmetric spaces are exactly the manifolds of the form $B = U/K$ where U is a connected, compact simple Lie group with $Z(U) = \{e\}$ and K is a maximal, connected proper subgroup of U.

Moreover, in both cases $Z(K) \cong U(1)$, or equivalently $\mathfrak{z}(\mathfrak{k}) \cong \mathbb{R}$

Proof. For a full proof see chapter 8 of [Hel78]. The main point is that in both the compact and non-compact cases, if $Z(K) \cong U(1)$ there is a $j \in Z(K)$ of order 4. Because Ad is a faithful representation, $Ad(j)|_{\mathfrak{p}}$ has order 4 and $Ad(j^2)|_{\mathfrak{p}}$ has order 2. Moreover

$$Ad(j)Ad(k) = Ad(jk) = Ad(kj) = Ad(k)Ad(j)$$

so $Ad(j)|_{\mathfrak{p}}$ and $Ad(j^2)|_{\mathfrak{p}}$ are Ad-equivariant maps. Because $Ad(j^2)|_{\mathfrak{p}}$ has order two it has eigenvalues ± 1. Since Ad is assumed to be irreducible, by Schur's lemma (cf. §2.3.3) $Ad(j^2)|_{\mathfrak{p}} = id$ or $Ad(j^2)|_{\mathfrak{p}} = -id$. But $Ad(j^2)|_{\mathfrak{k}} = id$, so if $Ad(j^2)|_{\mathfrak{p}} = id$ then $j^2 \in Z(K) = \{e\}$ contradicting our assumption that j has order 4. Thus $Ad(j)$ is a complex structure on $\mathfrak{p}$. Since $\tau : \mathfrak{p} \to T_x\Omega$ (or $\tau : \mathfrak{p} \to T_xB$) is a K-equivariant isomorphism (cf. 4.10), $J = \chi(j)$ is a χ-equivariant complex structure on $T_x\Omega$ (or T_xB). Because Ω (respectively B) is a symmetric space $Hol_x(\Omega) = K$ (respectively $Hol_x(B) = K$), by theorem 4.15. Thus J is holonomy invariant. By Proposition 3.15 (M, J, g) is a Kähler manifold. □

Recall that in §4.1.1 we gave two examples of symmetric spaces that were bounded symmetric domains. More precisely:

Definition 4.25. A bounded domain $\Omega \subset \mathbb{C}^n$ is called symmetric if every $x \in \Omega$ is an isolated fixed point of an involutive biholomorphism s_x.[10]

On any bounded symmetric domain Ω there exists a unique hermitian metric g_B with respect to which every biholomorphism of Ω is an isometry. This metric is called the

[10] By 'biholomorphism' we mean a holomorphic diffeomorphism with a holomorphic inverse. Since s_x is involutive and hence is its own inverse, this is equivalent to requiring s_x to be a holomorphic diffeomorphism.

Bergmann metric (cf. [Hel78] Chapter 8 §3 where this metric is constructed). A deep theorem of Harish-Chandra's tells us the following:

Theorem 4.26 (Theorem 7.1, Chpt.8 of [Hel78])**.** *1. Every bounded symmetric domain $\Omega \subset \mathbb{C}^N$ equipped with its Bergmann metric and the complex structure induced fomr $\mathbb{C}^N$ is a Hermitian symmetric space of non-compact type.*

2. Every Hermitian symmetric space (M, J, g) can be realized as a bounded symmetric domain equipped with the Bergmann metric (Ω, J, g_B) where J is restriction of the complex structure of $\mathbb{C}^N$ to $\Omega \subset \mathbb{C}^N$.

Proof. See [Hel78] pg. 383 to 393 □

Because of this we shall frequently use the phrase 'bounded symmetric domain' synonymously with 'Hermitan symmetric space of non-compact type'. Another deep theorem about Hermitian symmetric spaces which we shall use, but not prove is the following:

Theorem 4.27 (The Borel embedding theorem)**.** *If (Ω, g, J) is a bounded symmetric domain and $(B, J^{'}, g^{'})$ its compact dual, then there exists a holomorphic, isometric embedding of Ω into B as an open set.*

Proof. This is Prop 7.14 Chpt. 8 of [Hel78] □

4.7.1 Groups of biholomorphisms and groups of isometries

If (M, J, g) is a Hermitian symmetric space, denote by $\mathcal{H}(M, J)$ the group of all biholomorphic maps from (M, J) to itself. For a bounded symmetric domain (Ω, g, J), we emphasise that the Bergmann metric construction guarantees that $I(M, g) = \mathcal{H}(M, J) = \mathrm{Aut}(M, J, g)$. For a compact Hermitian symmetric space (B, J, g) however, $\mathrm{Aut}(M, J, g) = U$ is strictly contained in the group of biholomorphisms. In fact $\mathcal{H}$ is the simply-connected complex Lie group corresponding to the complex Lie algebra $\mathfrak{u} \otimes \mathbb{C}$.

4.7.2 Classification of Hermitian Symmetric Spaces

We end this section by discussing the classification of bounded symmetric domains as well as related data such as dimension and rank that we shall use later. For a more comprehensive, and highly readable description of the classical bounded symmetric domains, we refer the reader to [Gar]. Note that all dimensions referred to in this section are dimensions over $\mathbb{C}$

The first family is $\mathcal{D}^I_{p,q} = \{Z \in \mathrm{Mat}_{p,q}(\mathbb{C}) : I_q - Z^H Z > 0\}$. It should be obvious that $\mathcal{D}^I_{p,q} \cong \mathcal{D}^I_{q,p}$, so we shall always assume that $p \geq q$. As a homogeneous space we may write it as $SU(p,q)/S(U(p) \times U(q))$ and one can easily see that it is of dimension pq. Moreover, it has rank $\min(p,q)$. Its compact dual is the Grassmannian of complex p-dimensional subspaces of $\mathbb{C}^{p+q}$, and can be written as either $SU(p+q)/S(U(p)\times U(q))$ or $Sl(p+q,\mathbb{C})/S(Gl(p,\mathbb{C}) \times Gl(q,\mathbb{C}))$. $\mathcal{D}^I_{p,q}$ is of tube type if and only if $p = q$, in which case its upper half plane representation is $\{Z \in \mathrm{Mat}_{p,p}(\mathbb{C}) : \frac{1}{2i}(Z - Z^H) > 0\}$. Note that $\mathcal{D}^I_{p,1}$ is the open unit ball in $\mathbb{C}^p$ and has isotropy (or equivalently holonomy cf. 4.15) group $S(U(p) \times U(1)) \cong U(p)$ We say these bounded symmetric domains are of *ball type*. The existence of bounded symmetric domains with holonomy $U(p)$ is problematic, as much of the machinery we have developed for detecting when the universal cover of a given Kähler manifold (M, J) is a bounded symmetric domain relies on showing that $Hol(M, g) \neq U(m)$ (cf. theorem 3.31). We return to this problem in Chapter 7 where the possibility of (M, J) having a factor in its universal cover isometric to a bounded symmetric domain of ball type necessitates the introduction of the Miyaoka-Yau inequality.

The second family is $\mathcal{D}^{II}_n = \{Z \in \mathrm{Mat}_n(\mathbb{C}) : Z^T = -Z \text{ and } I_n - Z^H Z > 0\}$, which can be written as $O^*(2n)/U(n)$ where

$$O^*(2n) = \{A \in Gl(n, \mathbb{H}) : A^*(iI_n)A = iI_n\}$$

Note that i in the above definition is the quaternion i, and hence will not commute with the quaternionic matrix A. It has dimension $\frac{n}{2}(n-1)$ and rank $[\frac{n}{2}]$[11]. Its compact dual is the space of orthogonal (with respect to any fixed inner product) complex structures on $\mathbb{R}^{2n}$ and can be written as either $SO(2n,\mathbb{R})/U(n)$ or $SO(2n,\mathbb{C})/Gl(n,\mathbb{C})$. $\mathcal{D}^{II}_n$ is always of tube type, and its upper half plane representation is $\{Z \in \mathrm{Mat}_n(\mathbb{C}) : Z^T = -Z \text{ and } \frac{1}{2i}(Z - Z^H)\}$.

The third family is $\mathcal{D}^{III}_n = \{Z \in \mathrm{Mat}_{n,n}(\mathbb{C}) : Z^T = Z \text{ and } I_n - \bar{Z}Z > 0\}$. Classically, these are referred to as *Siegel spaces*, and were first studied by Siegel in [Sie43]. We have that $\mathcal{D}^{III}_n = Sp(n,\mathbb{R})/U(n)$[12] and $\mathcal{D}^{III}_n$ has dimension $\frac{1}{2}n(n+1)$ and rank n. These are also always of tube type, and their upper half plane representation is $\{Z \in \mathrm{Mat}_n(\mathbb{C}) : Z = Z^T \text{ and } \frac{1}{2i}(Z - \bar{Z}) > 0\}$ Its compact dual is the space of complex

[11] $[a]$ denotes the integral part of a

[12] We are using the convention $Sp(n,\mathbb{R})$ consists of $2n \times 2n$ real matrices preserving a symplectic form. Likewise, $Sp(n,\mathbb{C})$ consists of $2n \times 2n$ complex matrices preserving a $\mathbb{C}$ linear symplectic form

Table 4.1: Classification of Bounded Symmetric Domains

BSD	as Homogeneous space	of tube type?	rank	$dim_{\mathbb{C}}$
$\mathcal{D}^I_{p,q}$	$SU(p,q)/S(U(p)\times U(q))$	only if $p=q$	$\min(p,q)$	pq
$\mathcal{D}^{II}_n$	$O^*(2n)/U(n)$	yes	$[\frac{n}{2}]$	$\frac{n}{2}(n-1)$
$\mathcal{D}^{III}_{n,n}$	$Sp(n,\mathbb{R})/U(n)$	yes	n	$\frac{n(n+1)}{2}$
$\mathcal{D}^{IV}_n$	$SO(n,2)_0/S(O(n)\times O(2))_0$	yes	2	n
$\mathcal{D}^V$	$E_6^{-14}/(SO(2)\times Spin(10))$	no	2	16
$\mathcal{D}^{VI}$	$E_7^{-25}/(SO(2)\times E_6)$	yes	3	27

Lagrangian subspaces of $\mathbb{C}^{2n}$, and can be written as $Sp(n)/U(n)$ or $Sp(n,\mathbb{C})/Gl(n,\mathbb{C})$.

As should be apparent from the above definitions, $\mathcal{D}^{II}_n$ and $\mathcal{D}^{III}_n$ are contained in $\mathcal{D}^I_{n,n}$ The fourth family is slightly different however: $\mathcal{D}^{IV}_n = \{z \in \mathbb{C}^n : z_1^2 + z_2^2 + \ldots + z_n^2 < \frac{1}{2}(1 + |z_1 + z_2 + \ldots + z_n|) < 1\}$. Bounded symmetric domains of this type are often called *Lie spheres* and are also given by $SO(n,2)_0/S(O(n)\times O(2))_0$[13]. They have dimension n, and rank 2, regardless of the dimension. They are also always of tube type, and their upper half plane representations are more satisfying than their bounded domain representations, as they resemble light cones: $\{(z,w) \in \mathbb{C}\times\mathbb{C}^n : \frac{1}{2i}(z-\bar{z}) - |w| > 0\}$.

There are also two exceptional bounded symmetric domains, associated to the exceptional Lie groups E_6 and E_7. They are $\mathcal{D}^V = E_6^{-14}/(SO(2)\times Spin(10))$ and $\mathcal{D}^{VI} = E_7^{-25}/(SO(2)\times E_6)$ where the superscripts -14 and -25 denote which real form of E_6 and E_7 we are considering. $\mathcal{D}^V$) has dimension 16 and rank 2 while $\mathcal{D}^{VI}$ has dimension 27 and rank 3. Both are of tube type.
The above data is summarised in table 4.1.

4.8 Mok Characteristic varieties

In this section we aim to prove the following theorem.

Theorem 4.28. *Let (Ω, J, g) be a bounded symmetric domain not of ball type. Then there exists a nested family of projective varieties contained in the projectivized tangent bundle,*

$$\mathcal{S}_1 \subset \ldots \subset \mathcal{S}_r \subset \mathbb{P}T^{1,0}M$$

satisfying the following properties:

[13] recall that $So(n,2)$ is not connected.

1. *Each* $\mathcal{S}_i$ *is* G*-invariant, and hence its fibre over any point* $x \in M$*,* $\mathcal{S}_{i,x}$ *is isotropy, or* χ*, -invariant. Moreover,* these are the only χ-invariant subvarieties of $\mathbb{P}(T_x^{1,0}M)$.

2. $\mathcal{S}_i$ *is non-singular if and only if* $i = 1$*.*

Following Catanese and Di Scala in [CDS], we call these varieties *Mok Characteristic Varieties*, in honour of Ngaiming Mok, who first introduced such objects. See [Mok89] and the bibliography contained therein. We begin by constructing the Mok characteristic varieties for Bounded symmetric domains of type $I_{p,q}$ with $p, q > 1$.

4.8.1 The $\mathcal{D}^I_{p,q}$ case

Recall that:

$$I_{p,q} = SU(p,q)/S(U(p) \times U(q))$$

and that its compact dual is the Grassmannian of q planes in $\mathbb{C}^{p+q}$:

$$Gr(p+q,q) = SU(p+q)/S(U(p) \times U(q)) = Sl(p+q)/P$$

Where:

$$P = \left\{ \begin{pmatrix} A_{pp} & 0 \\ B_{qp} & C_{qq} \end{pmatrix} \in Sl(p+q,\mathbb{C}) \right\} \tag{4.9}$$

For brevity, let us fix p and q, and write $\Omega = I_{p,q}$, $\mathcal{B} = Gr(p+q,q)$, $G = SU(p,q)$, $K = S(U(p) \times U(q))$, $H = SU(p+q)$ and $G_{\mathbb{C}} = Sl(p+q,\mathbb{C})$. By the Borel embedding theorem (cf. Theorem 4.27) we have an embedding $\Omega \hookrightarrow \mathcal{B}$ identifying $T_x\Omega$ and $T_x\mathcal{B}$. We shall exploit this identification by first classifying P-invariant varieties in $T_x\mathcal{B}$, and then relating them to K-invariant varieties in $T_x\mathcal{B}$. As usual, we shall denote the Lie algebra corresponding to a Lie group by the same letter in gothic script. Thus:

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ -B^H & D \end{pmatrix} : A \in \mathfrak{u}(p),\ D \in \mathfrak{u}(q), B \in Mat(p,q,\mathbb{C}),\ \mathrm{trace}(A) + \mathrm{trace}(D) = 0 \right\}$$

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : A \in \mathfrak{u}(p), D \in \mathfrak{u}(q),\ \mathrm{trace}(A) + \mathrm{trace}(D) = 0 \right\} \text{ and hence}$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & B \\ -B^H & 0 \end{pmatrix} : B \in Mat(p,q,\mathbb{C}) \right\}$$

Observe that as asserted in theorem 4.24:

$$\mathfrak{z}(\mathfrak{k}) = \left\{ \begin{pmatrix} \frac{ai}{p} I_p & 0 \\ 0 & \frac{-ai}{q} I_q \end{pmatrix} : a \in \mathbb{R} \right\} \cong \mathbb{R}$$

As an aside, let us show that, as claimed in the previous section, the unique 4-torsion element of $Z(K)$ does indeed give a complex structure on $\mathfrak{p}$. The exact form of this element j depends on the parities of p and q, so let us assume that q is odd and p is even. In this case,

$$\begin{aligned} j &= \exp(\begin{pmatrix} \frac{qp\pi i}{2p} I_p & 0 \\ 0 & \frac{-qp\pi i}{2q} I_q \end{pmatrix}) = \begin{pmatrix} e^{\frac{q\pi i}{2}} I_p & 0 \\ 0 & e^{-\frac{p\pi i}{2}} I_q \end{pmatrix} \\ &= \begin{pmatrix} \pm i I_p & 0 \\ 0 & \pm I_q \end{pmatrix} \end{aligned}$$

where the sign depends on the residue classes of p and q modulo 4. So $J = Ad(j)$ is the complex structure on $\mathfrak{p}$, and indeed:

$$\begin{aligned} Ad(j) \begin{pmatrix} 0 & B \\ -B^H & 0 \end{pmatrix} &= \begin{pmatrix} \pm i I_p & 0 \\ 0 & \pm I_q \end{pmatrix} \begin{pmatrix} 0 & B \\ -B^H & 0 \end{pmatrix} \begin{pmatrix} \mp i I_p & 0 \\ 0 & \pm I_q \end{pmatrix} \\ &= \begin{pmatrix} 0 & iB \\ iB^H & 0 \end{pmatrix} = \begin{pmatrix} 0 & (iB) \\ -(iB)^H & 0 \end{pmatrix} \end{aligned}$$

Thus as we expect, $Ad(j)$ preserves $\mathfrak{p}$ and satisfies $Ad(j)^2 = -Id$.

Complexifying $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{p}$ we get:

$$\begin{aligned} \mathfrak{g}_{\mathbb{C}} &= \mathfrak{sl}(p+q, \mathbb{C}) \\ \mathfrak{k}_{\mathbb{C}} &= \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : A \in \mathfrak{gl}(p, \mathbb{C}), D \in \mathfrak{gl}(q, \mathbb{C}) : \ \text{trace}(A) + \text{trace}(D) = 0 \right\} \\ \mathfrak{p}_{\mathbb{C}} &= \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} : B \in Mat(p, q, \mathbb{C}), \ C \in Mat(q, p, \mathbb{C}) \right\} \end{aligned}$$

Note that every element of $\mathfrak{p}_{\mathbb{C}}$ now contains two independent matrices! Of course we still have:

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}$$

Extending J to $\mathfrak{p}_{\mathbb{C}}$ by $\mathbb{C}$-linearity, we may write:

$$\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$$

where $\mathfrak{p}^+$ (respectively $\mathfrak{p}^-$) is the $+i$ (respectively $-i$) eigenspace of J. One can easily check (cf. the example of p odd, q even given above) that:

$$\mathfrak{p}^+ = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} : B \in Mat(p,q,\mathbb{C}) \right\} \cong Mat(p,q,\mathbb{C})$$
$$\mathfrak{p}^- = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} : C \in Mat(q,p,\mathbb{C}) \right\} \cong Mat(q,p,\mathbb{C})$$

Moreover since $\mathfrak{p}$ and $T_x\mathcal{B}$ are equivariantly isomorphic by theorem 4.10, $\mathfrak{p}^+$ is identified with $T_x^{(1,0)}\mathcal{B}$. One can easily check that $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are abelian subalgebras, so denote by $P^- = \exp(\mathfrak{p}^-)$ the subgroup associated to $\mathfrak{p}^-$. We see that P (cf. (**??**)) is the semi-direct product of P^- and $K_{\mathbb{C}}$, the Lie group associated to $\mathfrak{k}_{\mathbb{C}}$. As in the non-compact case, we have an isotropy action:

$$\rho : P \to Gl(T_x^{1,0}\mathcal{B}, \mathbb{C})$$

however unlike in the noncompact case, since P does not act via isometries, *this action is not effective*. It can be shown (cf. [KO81]) that $\rho(P) \cong K_{\mathbb{C}}$ and that, as in the noncompact case, we may identify ρ with the Adjoint action of $K_{\mathbb{C}}$ on $\mathfrak{p}^+$. Note that:

$$K_{\mathbb{C}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : A \in Gl(p,\mathbb{C}),\ D \in Gl(q,\mathbb{C}),\ \det(A)\det(D) = 1 \right\}$$

and:

$$\begin{aligned} Ad(\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix})(\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}) &= \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix} \\ &= \begin{pmatrix} ABD^{-1} & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

But if $B \in T_x^{1,0}\mathcal{B} \cong Mat(p,q,\mathbb{C})$ is of rank k we may always choose $A \in Gl(p,\mathbb{C})$ and $D \in Gl(q,\mathbb{C})$ satisfying $\det(A)\det(D) = 1$ such that:

$$\rho(\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix})(B) = \mathrm{diag}(\underbrace{1,1,\ldots,1}_{\text{first } k \text{ entries } 1},0,\ldots,0)$$

Furthermore, since multiplication by invertible matrices preserves rank, two matrices of different rank cannot be conjugate to each other under ρ. Without loss of generality, suppose that $\min(p,q) = q$, then the maximum rank $B \in Mat(p,q,\mathbb{C})$ can have is q, which is the rank of Ω as a symmetric space. Thus we get q distinct ρ-orbits, $\mathcal{O}_1,\ldots,\mathcal{O}_q$, where $\mathcal{O}_k$ is the set of all rank k matrices in $T_x^{1,0}\mathcal{B}$.

Let $C\mathcal{S}_{k,x} \subset T_x^{1,0}\mathcal{B}$ denote the $k-th$ generic determinantal variety, defined by the vanishing of all $(k+1)\times(k+1)$ minors. Then it is clear that:

$$C\mathcal{S}_{k,x} = \mathcal{O}_1 \cup \ldots \cup \mathcal{O}_k$$

So $C\mathcal{S}_{k,x}$ is ρ-invariant. Moreover:

$$\operatorname{rank}(B) \leq k \Rightarrow \operatorname{rank}(\alpha B) \leq k \ \forall \alpha \in \mathbb{C}^*$$

so the equations defining $C\mathcal{S}_{k,x}$ are homogeneous and hence $C\mathcal{S}_{k,x}$ is a cone over a projective variety $\mathcal{S}_{k,x} \subset \mathbb{P}T_x^{1,0}\mathcal{B}$.

Theorem 4.29. *1. The singular locus of $\mathcal{S}_{k,x}$, for $k \geq 2$ is precisely $\mathcal{S}_{k-1,x}$.*

2. $\mathcal{S}_{1,x}$ is smooth.

3. $\mathcal{S}_{k,x}$ is the Zariski-closure of $\mathcal{O}_k$.

4. $\mathcal{S}_{q,x} = \mathbb{P}T_x^{1,0}\mathcal{B}$

Proof. These are all proved in Chapter 2 of [ACGH85]. For example 1) is a proposition on pg. 69 □

These are the *only* invariant subvarieties of $\mathbb{P}T_x^{1,0}\mathcal{B}$, since any such variety must be a union of orbits.
Now we extend these varieties to bundles of projective varieties by defining:

$$\mathcal{S}_{i,y} = \varphi_* \mathcal{S}_{i,x}$$

for any $\varphi \in G_{\mathbb{C}}$ such that $\varphi(x) = y$, where the map $\varphi_* : \mathbb{P}T_x^{1,0}\mathcal{B} \to \mathbb{P}T_y^{1,0}\mathcal{B}$ is the obvious one induced by $\varphi_* : T_x^{1,0}\mathcal{B} \to T_y^{1,0}\mathcal{B}$. Note that the invariance of $\mathcal{S}_{i,x}$ under the isotropy action at x, ρ, ensures that our definition does not depend on the choice of φ.
Restricting the bundle $\mathcal{S}_i$ to $\Omega \subset \mathcal{B}$, we indeed obtain a family of projective varieties $\mathcal{S}_1 \subset \ldots \subset \mathcal{S}_r = \mathbb{P}T^{1,0}\Omega$ with $\mathcal{S}_i$ smooth if and only if $i = 1$. Moreover since $K \subset K_{\mathbb{C}}$ the fact that these varieties are $K_{\mathbb{C}}$ invariant implies that they are K-invariant. However, of vital importance in the sequel is that these are the only K-invariant subvarieties, and in particular that $\mathcal{S}_{1,x}$ is the unique smooth, K-invariant subvariety of $\mathbb{P}T_x^{1,0}\Omega$. To show this, we need the following lemma:

Lemma 4.30. *Suppose that G is an algebraic group acting on a projective space $\mathbb{P}W$. If $H < G$ is a Zariski-dense subgroup, and $V \subset \mathbb{P}W$ is an H-invariant subvariety, then V is G-invariant.*

Proof. Let the action of G on $\mathbb{P}W$ be given by $\mu : G \times \mathbb{P}W \to \mathbb{P}W$. This map is a morphism of algebraic varieties (cf. [Bri] pg. 4), thus it is Zariski continuous. Because V is Zariski closed $\mu^{-1}(V) \subset G \times \mathbb{P}W$ is closed. By the invariance assumption, $H \times V \subset \mu^{-1}(V)$, and so $G \times V = \overline{H \times V} \subset \overline{\mu^{-1}(V)} = \mu^{-1}(V)$. Hence $\mu(g, v) \in V$ for all $g \in G$ and $v \in V$, and so V is G-invariant. □

So we argue as follows; suppose $V \subset \mathbb{P}T_x^{1,0}\Omega \cong \mathbb{P}T_x^{1,0}\mathcal{B}$ is smooth and K-invariant. Since K is Zariski-dense in $K_{\mathbb{C}}$, by Lemma 4.30 V is $K_{\mathbb{C}}$-invariant. But $\mathcal{S}_{1,x}$ is the unique smooth $K_{\mathbb{C}}$-invariant subvariety of $\mathbb{P}T_x^{1,0}\mathcal{B}$, thus $V = \mathcal{S}_{1,x}$. Note that we can identify $\mathcal{S}_{1,x}$, the locus of rank one matrices, with the Segre embedding of $\mathbb{P}(\mathbb{C}^p) \times \mathbb{P}(\mathbb{C}^q) \hookrightarrow \mathbb{P}(\mathbb{C}^{pq}) \cong \mathbb{P}(T_x^{1,0}\Omega)$ (cf. [Mok89] pg. 249) hence $dim(\mathcal{S}_{1,x}) = p + q - 2$.

4.8.2 Constructing Characteristic varieties for general bounded symmetric domains

We now outline the construction of the Mok characteristic varieties for an arbitrary bounded symmetric domain. As before, given a bounded symmetric domain Ω, we denote by $\mathcal{B}$ its compact dual, and by the Borel embedding theorem $\Omega \hookrightarrow \mathcal{B}$. Let G denote the group of isometries of Ω, H the group of isometries of $\mathcal{B}$ and $G_{\mathbb{C}}$ the complexification of G, which is the full group of holomorphisms of $\mathcal{B}$. As before, P will denote the stabilizer of some point $p \in \mathcal{B}$ under the action of $G_{\mathbb{C}}$ on $\mathcal{B}$, thus $\mathcal{B} \cong G_{\mathbb{C}}/P$. If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition, then:

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}$$
$$\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}^{+} \oplus \mathfrak{p}^{-}$$

where $\mathfrak{p}^{+}$ (respectively $\mathfrak{p}^{-}$) is the $+i$ (respectively $-i$) eigenspace of the complex structure J. As in the previous case (and cf. [Mok02] pg.4 or [KO81] pg 210 for more details), $P = P^{-} \ltimes K_{\mathbb{C}}$, we have an isotropy action:

$$\rho : P \to Gl(T_x^{1,0}\mathcal{B}, \mathbb{C})$$

and the Adjoint representation:

$$Ad : K_{\mathbb{C}} \to Gl(\mathfrak{p}^+)$$

The map $\tau : T_x^{1,0}\mathcal{B} \to \mathfrak{p}^+$ is $K_{\mathbb{C}}$-equivariant (cf. theorem 4.10). We have the following:

Theorem 4.31. *If r is the rank of Ω (which is the same as the rank of $\mathcal{B}$), then there are precisely r orbits $\mathcal{O}_1, \ldots, \mathcal{O}_r$ of the action of $K_{\mathbb{C}}$ on $\mathbb{P}(\mathfrak{p}^+)$ such that for each k, the Zariski closure of $\mathcal{O}_k$ is a projective variety and is in fact a union of the preceding orbits: $\overline{\mathcal{O}_k} = \mathcal{O}_k \cup \mathcal{O}_{k-1} \cup \ldots \cup \mathcal{O}_1$. Finally $\mathcal{O}_1$ is smooth and closed, and $\overline{\mathcal{O}_r} = \mathbb{P}(\mathfrak{p}^+)$.*

Proof. These orbits are constructed in [Mok02], pg. 4-5, where it is also shown that their closures yield an increasing sequence of projective varieties:

$$\mathcal{O}_1 \subset \overline{\mathcal{O}_2} \subset \ldots \subset \overline{\mathcal{O}_r} \tag{4.10}$$

and that $\overline{\mathcal{O}_r} = \mathbb{P}(\mathfrak{p}^+)$. The fact that $\mathcal{O}_1$ is smooth and closed is shown proposition 1 and preceding definitions on page 101 of [Mok89] □

Remark 4.32. Observe that once one has constructed the r $K_{\mathbb{C}}$ orbits in $\mathbb{P}(\mathfrak{p}^+)$, and verified that they have strictly increasing dimensions:

$$dim(\mathcal{O}_1) < \ldots < dim(\mathcal{O}_r)$$

The rest of the proposition follows from the general theory of algebraic groups acting on varieties, as in this case G is acting on $\mathbb{P}(\mathfrak{p}^+)$ algebraically. For example, we have:

Proposition 4.33 (see Proposition 1.11 in [Bri]). *Let X be a projective variety on which the algebraic group G is acting. Then each orbit $G \cdot x$ is a smooth, quasi-projective variety, every component of which has dimension $dim(G) - dim(G_x)$ where G_x is the stabilizer of x. Moreover, the (Zariski) closure, $\overline{G \cdot x}$ is the union of $G \cdot x$ with orbits of strictly smaller dimension, and any orbit of minimal dimension is closed.*

Define $\mathcal{S}_{k,x} = \tau(\overline{\mathcal{O}_k})$ then $\mathcal{S}_{k,x}$ is a ρ-invariant subvariety of $\mathbb{P}T^{1,0}\mathcal{B}_x$. Taking the $G_{\mathbb{C}}$ orbit of $\overline{\mathcal{O}_k}$ we get a $G_{\mathbb{C}}$-invariant bundle of subvarieties of $\mathbb{P}T^{1,0}\mathcal{B}$. Restricting to $\Omega \hookrightarrow \mathcal{B}$ we get the k-th Mok characteristic bundle, which we denote as $\mathcal{S}_k$. In [Mok89] pg. 249-251, Mok describes $S_{1,x}$ for all possible bounded symmetric domains. This is summarised in table 4.2. The first characteristic varieties possess several remarkable properties, namely:

Theorem 4.34. *If Ω be a bounded symmetric domain and $\mathcal{S}_{1,x} \subset \mathbb{P}(T_x^{1,0}\Omega)$ be the first characteristic variety at some point $x \in \Omega$. Then:*

BSD, Ω	$dim_{\mathbb{C}}(\Omega)$	description of $\mathcal{S}_{1,x}(\Omega)$ as an HSS	$dim_{\mathbb{C}}(\mathcal{S}_{1,x}(\Omega))$
$I_{p,q}$	pq	$\mathbb{CP}^{p-1} \times \mathbb{CP}^{q-1}$	p+q-2
$II_{n,n}$	$\frac{n(n-1)}{2}$	$Gr(2,n)$	2(n-2)
$III_{n,n}$	$\frac{n(n+1)}{2}$	$\mathbb{CP}^{n-1}$	n-1
IV_n	n	Q_{n-2} (compact dual of IV_{n-2})	n-2
V	16	$SO(10)/U(5)$	10
VI	27	V	16

TABLE 4.2: First Characteristic Varieties of Bounded Symmetric Domains

1. *The embedding* $\mathcal{S}_{1,x} \hookrightarrow \mathbb{P}T_x^{1,0}\Omega$ *is* full. *That is,* $\mathcal{S}_{1,x}$ *is not contained in any hyperplane.*

2. *If* $\mathbb{P}T_x^{1,0}\Omega$ *is endowed with the Fubini-Study metric of constant holomorphic sectional curvature* 1 *then* $\mathcal{S}_{1,x}$ *is a totally geodesic submanifold.*

3. $\mathcal{S}_{1,x}$ *considered is itself a Hermitian symmetric space, of compact type and of rank 1 or 2.*

Proof. See [Mok89] pg. 245 - 251. □

Since $G \hookrightarrow G_{\mathbb{C}}$ we get that $\mathcal{S}_k$ is G invariant for all k. So, to prove theorem 4.8 we now need to show that amongst the varieties $\overline{\mathcal{O}_i}$, only $\overline{\mathcal{O}_1} = \mathcal{O}_1$ is smooth. To do this we need to develop a few ideas about submanifolds.

4.8.3 Normal Holonomy

Suppose that $(\bar{M}, \bar{g})$ is a Riemannian manifold with Levi-Civita connection $\bar{\nabla}$ and $M \subset \bar{M}$ is an embedded submanifold, with the induced metric $g = \bar{g}|_M$. Then the tangent bundle of $\bar{M}$, restricted to M, splits into the direct sum of the tangent bundle of M and the *normal bundle* of M:

$$T\bar{M}|_M = TM \oplus NM$$

If $\pi^T : T\bar{M}|_M \to TM$ and $\pi^{\perp} : T\bar{M}|_M \to NM$ are the tangential and orthogonal projections, one can check that the Levi-Civita connection of M is given by:

$$\nabla_X Y = \pi^T(\bar{\nabla}_{\bar{X}}\bar{Y}) \quad X, Y \in \mathcal{A}^0(TM)$$

(cf.[Lee97] theorem 8.2 pg.135) where $\bar{X}$ and $\bar{Y}$ are arbitrary extensions of the vector fields on M to $\bar{M}$ (and it does not matter which extension we choose). More interestingly, we may define a connection on NM, the *normal connection*, using the formula (cf. [BCO03]):

$$\nabla^{\perp}_{X}\zeta = \pi^{\perp}(\bar{\nabla}_{\bar{X}}\bar{\zeta}) \quad X \in \mathcal{A}^{0}(TM), \zeta \in \mathcal{A}^{0}(NM)$$

where as before $\bar{X}$ (resp. $\bar{\zeta}$) denotes an arbitrary extension of X (resp. ζ) to M. As we did for the Levi-Civita connection in section 3.5, for any closed curve $\gamma : [0,1] \to M$ based at $x \in M$ we may define a parallel transport operator:

$$P_{\gamma} : N_{x}M \to N_{x}M$$

by defining $P_{\gamma}X = V(1)$ where $V(t)$ is the solution to the linear, first order initial value problem:

$$\begin{aligned} &\nabla_{\dot{\gamma}(t)}V(t) = 0 \\ &V(0) = X \end{aligned}$$

and hence a holonomy group:

$$Hol^{\perp}_{x}(M) = \{P_{\gamma} : \gamma(0) = \gamma(1) = x\}$$

which we shall refer to as the *normal holonomy group*. Just as the usual holonomy group carries a lot of intrinsic information about M, so the normal holonomy group tells us a lot about the *extrinsic* geometry of M as a submanifold of $\bar{M}$. Of use to us is the following theorem of Console and Di Scala:

Theorem 4.35. *Suppose that $M \subset \mathbb{CP}^{n}$ is an Hermitian symmetric manifold embedded into $\mathbb{CP}^{n}$ as a full*[14]*, totally geodesic submanifold. Then there exists an irreducible Hermitian symmetric space H/S not of Euclidean type such that $S \cong Hol^{\perp}_{x}(M)$, $T_{[S]}(H/S) \cong N_{x}M$ and the normal holonomy representation of $Hol^{\perp}_{x}(M)$ on $N_{x}M$ may be identified with the isotropy representation of S on $T_{[S]}(H/S)$.*

Proof. See [CDS09], theorem 2.5 pg. 5. □

Note that in particular this means that $Hol^{\perp}_{x}(M)$ acts irreducibly on $N_{x}M$. Now suppose that K is a compact, semi-simple Lie group acting on $(\mathbb{CP}^{n}, g_{FS})$ via isometries (where g_{FS} is the Fubini-Study metric of constant holomorphic sectional curvature 1). Let $M \subset \mathbb{CP}^{n}$ be a K-invariant submanifold. For any $x \in M$, let $H \leq K$ denote the stabilizer of x. Since $x \in \mathbb{CP}^{n}$, as before we have an isotropy action:

$$\chi : H \to Gl(T^{1,0}_{x}\mathbb{CP}^{n})$$

[14] recall that full means that M is not contained in any hyperplane

Because $M \subset \mathbb{CP}^n$ is K-invariant, $T_x^{1,0}(M) \subset T_x^{1,0}\mathbb{PC}^n$ is H-invariant. Moreover, since $\chi(\varphi)$ is an isometry for all $\varphi \in H$, for any $\zeta \in N_xM$;

$$g_{FS}(\chi(\varphi)(\zeta), Y) = g_{FS}(\zeta, \chi(\varphi^{-1})(Y)) = 0 \quad \forall Y \in T_x^{1,0}M$$

hence N_xM is also H-invariant. We call the restriction of χ to N_xM (denoted $\chi|_{N_xM}$) the *slice representation* (cf. [BCO03] page 38). If M is in addition an Hermitian Symmetric Space, then as is shown in the proof of proposition 2.3 in [CDS09], we may identify the slice representation with the normal holonomy representation (compare this to theorem 4.15 where we identify the usual isotropy representation with the usual holonomy representation). It follows from 4.35 that χ_{N_xM} is an *irreducible representation.*

Theorem 4.36. *Let K be a compact, semi-simple Lie group acting on $\mathbb{PC}^n$ via isometries. Suppose that $M_1 \subset \mathbb{PC}^n$ is a Hermitian symmetric space embedded as a full, totally geodesic, K-invariant submanifold of $\mathbb{PC}^n$. If $M_2 \subset \mathbb{PC}^n$ is a K-invariant projective variety such that:*

$$M_1 \subsetneq M_2 \subsetneq \mathbb{PC}^n$$

Then M_2 is singular at every point $x \in M_1$.

Proof. Suppose M_2 is non-singular at $x \in M_1$. Then:

$$T_x^{1,0}M_1 \subsetneq T_x^{1,0}M_2 \subsetneq T_x^{1,0}\mathbb{PC}^n$$

If $H \leq K$ is the stabilizer of x and $\chi : H \to Gl(T_x^{1,0}\mathbb{PC}^n)$ the isotropy representation then $T_x^{1,0}M_1$, N_xM_1 and $T_x^{1,0}M_2$ are all χ-invariant proper subspaces. It follows that $T_x^{1,0}M_2 \cap N_xM_1 \subset N_xM_1$ is $\chi_{N_xM_1}$-invariant. But this is a contradiction as under the hypotheses of this theorem $\chi_{N_xM_1}$ is an irreducible representation. □

Now by Theorem 4.34 $\mathcal{S}_1$ is a Hermitian Symmetric space embedded as a full, totally geodesic submanifold of $\mathbb{CP}^n$. We know that for any $k > 1$ $\mathcal{S}_1 \subset \mathcal{S}_k$. But then by Theorem 4.36 $mathcalS_k$ is singular along $\mathcal{S}_1$. Hence $\mathcal{S}_1$ is the only smooth Characteristic variety.

4.8.4 Characteristic Varieties of Quotients of Bounded Symmetric Domains

Let (Ω, J, g) be a bounded symmetric domain. Suppose (M, J', g') is a compact Kähler manifold having Ω as its universal cover. We may assume that the covering map

$p : \Omega \to M$ is locally a biholomorphism and that $p^* g' = g$.
Given any covering space $\hat{p} : (\hat{M}, \hat{J}, \hat{g}) \to (M, J, g)$ an automorphism $\varphi \in \mathrm{Aut}(\hat{M})$ is called a deck transformation if $\hat{p} \circ \varphi = \hat{p}$. We denote the group of all deck transformations as $D_M(\hat{M})$. If $\hat{M}$ is in fact the universal cover $\tilde{M}$, then $D_M(\hat{M}) = \pi_1(M)$ (cf. Proposition 1.39 in [Hat02]) and thus $\pi_1(M)$ can be considered as a subgroup of $\mathrm{Aut}(M)$. Moreover (cf. Proposition 1.40 in [Hat02]) $M \cong \tilde{M}/\pi_1(M)$.
Returning to the case at hand, we know that $M = \Omega/\Gamma$ with Γ a torsion-free discrete subgroup of $\mathrm{Aut}(\Omega)$, so define $\mathcal{S}_k(M) = \mathcal{S}_k(\Omega)/\Gamma \subset \mathbb{P}TM$. Because Γ acts without fixed points on Ω, for any $x \in \Omega$, $\mathcal{S}_k(\Omega)_x \cong \mathcal{S}_k(M)_{p(x)}$. So $\dim(\mathcal{S}_k(M)) = \dim(\mathcal{S}_k(\Omega))$ and $\mathcal{S}_1(M)$ is smooth. Recall that the $\mathcal{S}_k(\Omega)_x$ are χ-invariant (cf. theorem 4.28) and that by Theorem 4.15 we can identify χ with the action of $Hol_x(\Omega, g)$ on $\mathbb{P}T_x^{1,0}\Omega$. Because $Hol_x(\Omega, g) = Hol_{p(x)}(M, g')_0$ (cf. The first remark in 3.12), we conclude that $\mathcal{S}_1(M)_{p(x)}$ is the unique smooth, $Hol_{p(x)}(M, g')_0$-invariant subvariety of $\mathbb{P}T_{p(x)}M$.

Lemma 4.37. *$\mathcal{S}_1(M)_{p(x)}$ is $Hol_{p(x)}(M, g')$-invariant.*

Proof. Suppose this is not the case. Then there exists a $\varphi \in Hol(M, g)$ such that $\varphi(\mathcal{S}_1(M)_{p(x)}) \neq \mathcal{S}_1(M)_{p(x)}$. We claim that $\varphi(\mathcal{S}_1(M)_{p(x)})$ is a $Hol(M, g)_0$ invariant subvariety. That it is a subvariety is obvious since the action of $Hol(M, g)$ on $\mathbb{P}T_xM$ is given by elements of $PGl(T_xM, \mathbb{C})$. That it is invariant follows from the fact that $Hol(M, g')_0$ is a normal subgroup, so for any $\psi \in Hol(M, g')_0$:

$$\psi\varphi(\mathcal{S}_1(M)_{p(x)}) = \varphi(\psi'(\mathcal{S}_1(M)_{p(x)})) \subset \varphi(\mathcal{S}_1(M)_{p(x)}) \text{ as } \psi' \in Hol(M, g')_0$$

But this is a contradiction since by assumption $\mathcal{S}_1(M)_{p(x)}$ is the unique $Hol(M, g)_0$ invariant subvariety. □

To summarise, if the universal cover of a compact Kähler manifold (M, J, g) is an irreducible bounded symmetric domain not of ball type, then there is a unique smooth, $Hol(M, g)_y$-invariant variety contained in $\mathbb{P}T_yM$ for any $y \in M$. The dimension of this variety is determined by Ω (cf. table 4.2).

Chapter 5

A Necessary Condition for $\tilde{M}$ to be a bounded symmetric domain

If $E \to M$ is a holomorphic vector bundle, we shall denote by $\Gamma(E)$ the vector space of global holomorphic sections of E. For brevity we shall write

$$(T^{1,0})^r_s M = (T^{1,0}M)^{\otimes r} \otimes ((T^{1,0})^* M)^{\otimes s}$$

This is obviously a holomorphic vector bundle. Recall that the *canonical bundle* of M, K_M is defined as:

$$K_M = \bigwedge^m (T^{1,0})^* \quad m = \dim_{\mathbb{C}}(M)$$

its dual bundle, K_M^{-1} is given by:

$$K_M^{-1} = \bigwedge^m (T^{1,0}) \quad m = \dim_{\mathbb{C}}(M)$$

In this section we discuss the relevant parts of a paper by Kobayashi [Kob80] which shows that if there exists a non-zero $\sigma \in \Gamma((T^{1,0})^r_r M)$ then it is necessarily parallel. We then use this result to prove our first uniformisation theorem:

Theorem 5.1 (Theorem B in [Kob80]). *Let (M, J, g) be a compact Kähler manifold with $c_1(M, J) < 0$ and $\dim_{\mathbb{C}}(M) = m$. Then:*

$$\Gamma(S^{mq} T^{1,0} M \otimes K_M^q) = \Gamma(S^{mq} (T^{1,0})^* M \otimes K_M^{-q}) = 0$$

unless the universal covering space $\tilde{M}$ of M is biholomorphic to a product $D \times N$ of a bounded symmetric domain D and a complex manifold N with dim$(D) > 0$ and dim$(N) \geq 0$.

Actually we prove something stronger:

Theorem 5.2. *With hypotheses as in 5.1, if $\Gamma(S^{mq}TM \otimes K_M^q) \neq 0$ or $\Gamma(S^{mq}T^*M \otimes K_M^{-q}) \neq 0$ then $\tilde{M}$ is biholomorphic to a product of bounded symmetric domains.*

In order to prove this we first prove:

Theorem 5.3 (part two of theorem 1 in [Kob80])**.** *Let (M, J) be a compact Kähler manifold with $c_1(M) < 0$. Then any $\xi \in \Gamma((T^{1,0})^r_r M) = 0$ is parallel.*

Proof. Because $c_1(M, J) < 0$, by theorem 3.30 there exists a Kähler-Einstein metric g on (M, J) such that $\rho = c\omega$, with $c < 0$. Equivalently $\mathrm{Ric} = cg$. Let $\Delta = dd^* + d^*d$ be the d-Laplacian on (M, J, g). For any $\xi \in \Gamma(((T^{1,0})^r_s M))$ consider the $\mathcal{C}^\infty$ function $f = g(\xi, \xi)$ [1]. Because g is an Einstein metric, Theorem 8.1 on pg. 142 of [YB53] gives:

$$\Delta f = g(\nabla\xi, \nabla\xi) - c(r - s)g(\xi, \xi)$$

If $r = s$ we have that:

$$\Delta f = g(\nabla\xi, \nabla\xi) \geq 0$$

thus f is sub-harmonic. Since M is compact, we may use the maximum principle for sub-harmonic functions to conclude that f is constant, so $\Delta f = 0$ But then:

$$g(\nabla\xi, \nabla\xi) = 0$$

thus proving the theorem. □

Before we tackle theorem 5.2, we need to collect several elementary lemmas from representation theory.

5.1 Some Lemmas about representations

Lemma 5.4. *Let $\rho : G \to Gl(V)$ be a complex representation of a reductive[2] Lie group G. Thus V admits a decomposition as a direct sum of irreducible representations*

$$V = \bigoplus_{i=1}^{m} V_i \tag{5.1}$$

[1]Here g denotes the metric on $((T^{1,0})^r_s M$ induced by g

[2]In particular, all compact Lie groups are reductive

Then G-invariant elements $v \in V$ correspond to copies of the trivial representation

$$\rho_{triv} : G \to Gl(\mathbb{C})$$
$$g \mapsto 1$$

In fact, if we have k linearly independent G-invariant vectors in V then we have k copies of ρ_{triv} in V.

Proof. Suppose $v \in V$ is G-invariant, that is $g \cdot v = v \ \forall g \in G$. Then $\mathbb{C}v$ is a one dimensional invariant subspace of V. It is irreducible since it is one dimensional, and thus contains no proper subspaces. If $v_1, \ldots, v_k$ are all G-invariant, then $\mathbb{C}v_1, \ldots, \mathbb{C}v_k$ are all copies of the trivial representation inside V.
Conversely, suppose that in (5.1) $V_1, \ldots, V_k$ are all distinct copies of the trivial representation. Then by definition of the trivial representation, any $v_i \in V_i$ is G-invariant, and so we may choose k linearly independent G-invariant vectors. □

If $\rho_1 : G \to Gl(V)$ and $\rho_2 : H \to Gl(W)$ are representations then we can create a representation:

$$\rho_1 \boxtimes \rho_2 : G \times H \to Gl(V \otimes W)$$
$$\rho_1 \boxtimes \rho_2(g, h) = \rho_1(g) \otimes \rho_2(h)$$

called the *exterior product* of ρ_1 and ρ_2.[3] This behaves differently to the tensor product of two representations of the same Lie group (cf. §2.3.1). For example:

Lemma 5.5. *$V \boxtimes W$ is irreducible if V and W are irreducible* [4] *as G and H representations respectively. Moreover, if V and W are not irreducible, but decompose into irreducible representations as:*

$$V = V_1 \oplus \ldots \oplus V_l$$
$$W = W_1 \oplus \ldots \oplus W_m$$

then the decomposition:

$$V \boxtimes W = \bigoplus_{i,j} V_i \boxtimes W_j$$

is an irreducible decomposition of $V \boxtimes W$.

[3] Occasionally we will write $V \boxtimes W$ instead of $V \otimes W$ to indicate that we are considering $V \otimes W$ as the representation space of an exterior product of representations.

[4] actually this is an if and only if, though the converse direction takes a bit of work and we don't need it

Proof. The first part of this lemma is standard, see for example Lemma 3.1 in [CDS], but we reproduce it here for completeness. Given a representation

$$\rho : G \to Gl(V) \tag{5.2}$$

we define the *character* of ρ as:

$$\begin{aligned} \chi &: G \to \mathbb{C} \\ \chi_\rho &: g \mapsto \operatorname{tr}(\rho(g)) \end{aligned}$$

then ρ is an irreducible representation if and only if $\int_G |\chi_\rho|^2 d\mu = 1$, where $d\mu$ is the Haar measure of G, normalised such that $\int_G d\mu = 1$. If we denote the character of V (respectively W) by ρ_V (respectively ρ_W), then the character of $V \boxtimes W$, $X_{V\times W}$ is $\chi_V \cdot \chi_W$ (This is a standard property of characters, see proposition 2.1 in [FH91]). Thus:

$$\begin{aligned} \int_{G\times H} \chi_{V\times W} d\mu_{G\times W} &= \int_G \chi_V d\mu_G \int_H \chi_W d\mu_W \\ &= 1 \cdot 1 = 1 \end{aligned}$$

For the second part, observe that:

$$\begin{aligned} V \boxtimes W &= (V_1 \oplus \ldots \oplus V_l) \boxtimes (W_1 \oplus \ldots \oplus W_m) && (5.3) \\ &= \oplus_{i,j} (V_i \boxtimes W_j) && (5.4) \end{aligned}$$

by elementary linear algebra. Each $V_i \boxtimes W_j$ is irreducible, by the first part of this lemma. Thus this is a decomposition into irreducible representations. □

We want to consider the situation where there exists an element $w \in V \boxtimes W$ which is $G \times H$-invariant. First we need a lemma from multilinear algebra.

Lemma 5.6. *Suppose that V and W are both vector spaces, and consider their tensor product $V \otimes W$. If $\{w_1, \ldots w_n\} \subset W$ is a linearly independent subset, and $\{v_1, \ldots, v_n\} \subset V$ is an arbitrary subset, then:*

$$\sum_{i=1} v_i \otimes w_i = 0 \Leftrightarrow v_i = 0 \, \forall i$$

Proof. This follows from a standard calculation. □

Lemma 5.7. *For any $u \in V \boxtimes W$ we know we may write u as:*

$$u = \sum_{j=1}^{m} v_j \otimes w_j$$

then u is $G \times H$-invariant if and only if v_i is invariant under G and w_i is invariant under H for all i.

Proof. If v_i is invariant under G and w_i is invariant under H for all i then it is obvious that u is invariant under $G \times H$. Now, suppose that $u = \sum_j v_j \otimes w_j$ is invariant under the action of $G_1 \otimes G_2$. We may assume that $\{w_1, \ldots, w_n\}$ is a linearly independent set, since if $w_1 = \sum_{k=2}^n \lambda_k w_k$ we can write:

$$\sum_j v_j \otimes w_j = \sum_{j=2}^n \lambda_j v_1 \otimes w_j + \sum_{j=2}^n v_j \otimes w_j = \sum_{j=2}^n (\lambda_j v_1 + v_j) \otimes w_j$$

and if $\{w_2, \ldots, w_n\}$ is still linearly dependent we repeat the process. Now if $(g,h) \cdot u = u$ for all $(g,h) \in G \times H$ it must be true that $(g, id) \cdot u = u$ for all $g \in G$. So:

$$\begin{aligned} (g, id) \cdot u = \sum_{j=1}^n (g \cdot v_i) \otimes w_i = \sum_{j=1}^n v_i \otimes w_i \\ \Rightarrow \sum_{j=1}^n (g \cdot v_i - v_i) \otimes w_i = 0 \\ \Rightarrow g_1 \cdot v_i - v_i = 0 \, \forall i \end{aligned}$$

where the last line follows from lemma 5.6. An identical argument shows that $g_2 \cdot w_i = w_i$ for all i. □

from this lemma we get:

Corollary 5.8. *Suppose that V_i is a representation of H_i and consider the exterior product $V_1 \boxtimes \ldots \boxtimes V_r$ as a representation of $H_1 \times \ldots \times H_r$. Then any $u \in V_1 \boxtimes \ldots \boxtimes V_r$ may be written as:*

$$u = \sum_j v_j^{(1)} \otimes \ldots \otimes v_j^{(r)}$$

and we have that u is $H_1 \times \ldots \times H_r$ invariant if and only if $v^{(i)}$ is H_i invariant for all i.

Proof. Consider $V_1 \boxtimes (V_2 \boxtimes \ldots \boxtimes V_r)$ as the exterior product of the representation V_1 of H_1 and $V_2 \boxtimes \ldots \boxtimes V_r$ of $H_2 \times \ldots \times H_r$. Then

$$u = \sum_j v_j^{(1)} \otimes (v_j^{(2)} \otimes \ldots \otimes v_j^{(r)})$$

so lemma 5.7 implies that $v_j^{(1)}$ is H_1 invariant for all j, and $v_j^{(2)} \otimes \ldots \otimes v_j^{(r)}$ is $H_2 \times \ldots \times H_r$ invariant for all j. Thus the corollary follows by induction. The converse is similiar. □

Corollary 5.9. *With notation as above, if $V_1 \boxtimes \ldots \boxtimes V_r$ contains an invariant element then each V_i must contain at least one copy of the trivial representation of H_i. In particular, for each i, V_i cannot be an irreducible H_i representation.*

Proof. By corollary 5.8 the existence of an invariant $u \in V_1 \boxtimes \ldots \boxtimes V_r$ implies the existence, for each i, of at least one $v_i \in V_i$ which is H_i-invariant. Applying lemma 5.4 we see that each V_i has a proper invariant subspace, and so is not irreducible. □

One final lemma:

Lemma 5.10. *Now suppose that $V_1, \ldots, V_r$ are all representations of the same group G. We may take the direct sum $V_1 \oplus \ldots \oplus V_r$ which has a natural representation of G:*

$$g \cdot (v_1, \ldots, v_r) = (g \cdot v_1, \ldots, g \cdot v_r)$$

Then $(v_1, \ldots, v_r)$ is invariant and non-zero if and only if for all $v_i \neq 0$ v_i is G-invariant.

Proof.

$$g \cdot (v_1, \ldots, v_r) = (g \cdot v_1, \ldots, g \cdot v_r) = (v_1, \ldots, v_r)$$

implies that $g \cdot v_i = v_i$ for all i □

5.2 The proof of theorem 5.2

Let us return to complex geometry and prove theorem (5.2).

Proof. (Of theorem (5.2)) Let $\xi \in \Gamma(S^{mq}(T^{1,0})^* M \otimes K_M^{-q})$. Since $S^{mq}(T^{1,0})^* M \otimes K_M^{-q}$ is a sub-bundle of $(T^{1,0})^{mq}_{mq} M$ by theorem (5.3) ξ is parallel with respect to the Levi-Civita connection associated to a Kähler-Einstein metric g. By the holonomy principle (cf. theorem 3.14), ξ_x is $Hol_x(M, g)$-invariant for any $x \in M$ and also lifts to a holonomy invariant element of $S^{mq}(T^{1,0})^*_{\tilde{x}} \tilde{M} \otimes K_{\tilde{M}}^{-q}$. By the De Rham decomposition theorem for Kähler manifolds (cf. 3.18) we know that:

$$\begin{aligned}
&\tilde{M} \cong (M_1, g_1) \times \ldots \times (M_r, g_r) \\
&T_x \tilde{M} \cong T_{x_1} M_1 \oplus \ldots \oplus T_{x_r} M_r \\
&Hol(\tilde{M}, g) \cong Hol(M_1, g_1) \times \ldots \times Hol(M_r, g_r) \\
&\forall h \in Hol_x(\tilde{M}),\ \forall X = (X_1, \ldots, X_r) \in T_x \tilde{M}\ \ h \cdot X = (h_1 \cdot X_1, \ldots, h_r \cdot X_r)
\end{aligned}$$

Now $S^{mq}(T^{1,0})^*\tilde{M} = S^{mq}((T^{1,0})^*M_1 \oplus \ldots \oplus (T^{1,0})^*M_r)$ and some elementary linear algebra (see also pg. 473 of [FH91]) shows:

$$S^{mq}(T^{1,0})^*\tilde{M} = \bigoplus_{\sum l_i = mq, l_i \geq 0} S^{l_1}(T^{1,0})^*M_1 \otimes \ldots \otimes S^{l_r}(T^{1,0})^*M_r$$

and similarly:

$$\begin{aligned} K_{\tilde{M}}^{-1} = \bigwedge^{m} T^{1,0}\tilde{M} &= \bigwedge^{m}(T^{1,0}M_1 \oplus \ldots \oplus T^{1,0}M_r) \\ &\cong \bigoplus_{\sum l_i = m, l_i \geq 0} (\bigwedge^{l_1} T^{1,0}M_1) \otimes \ldots \otimes (\bigwedge^{l_r} T^{1,0}M_r) \end{aligned}$$

Now observe that if in any summand in (**??**) there is an $l_i > m_i = \dim_{\mathbb{C}}(M_i)$, then $\bigwedge^{l_i} TM_i = 0$ and so this summand vanishes. On the other hand, if in a summand there exists an $l_i < m_i$ then there must exist another index in the same summand l_j such that $l_j > m_j$ (since $\sum l_i = m$). Thus $\bigwedge^{l_j} TM_j = 0$ and again this summand vanishes. Thus the only non-vanishing summand in (**??**) is where $l_i = m_i$ for all i. Hence:

$$\begin{aligned} K_{\tilde{M}}^{-1} &\cong (\bigwedge^{m_1} T^{1,0}M_1) \otimes \ldots \otimes (\bigwedge^{m_r} T^{1,0}M_r) \\ &= K_{M_1}^{-1} \otimes \ldots \otimes K_{M_r}^{-1} \end{aligned}$$

and so:

$$K_{\tilde{M}}^{-q} \cong K_{M_1}^{-q} \otimes \ldots \otimes K_{M_r}^{-q}$$

and:

$$S^{mq}(T^{1,0})^*\tilde{M} \otimes K_{\tilde{M}}^{-q} \cong \bigoplus_{\sum l_i = mq, l_i \geq 0} (S^{l_1}(T^{1,0})^*M_1 \otimes K_{M_1}^{-q}) \otimes \ldots \otimes (S^{l_r}(T^{1,0})^*M_r \otimes K_{M_r}^{-q})$$

and at the point $x \in \tilde{M}$:

$$(S^{mq}(T^{1,0})^*\tilde{M} \otimes K_{\tilde{M}}^{-q})_x \cong \bigoplus_{\sum l_i = mq, l_i \geq 0} (S^{l_1}(T^{1,0})^* \otimes K_{M_1}^{-q})_{x_1} \otimes \ldots \otimes (S^{l_r}(T^{1,0})_r^* \otimes K_{M_r}^{-q})_{x_r} \tag{5.5}$$

and let us point out that in (5.5) $\cong$ denotes a $Hol(\tilde{M}, \tilde{g})$-equivariant isomorphism. Furthermore, on the right hand side of (5.5) we should be using our exterior product notation, $\boxtimes$ instead of $\otimes$ as each summand above is an exterior product of the $Hol(M_i, g_i)$ representations $S^{l_i}(T^{1,0})_i^* \otimes K_{M_i}^{-q}$. Note that $S^{l_i}(T^{1,0})_i^* \otimes K_{M_i}^{-q}$ is *not* an exterior product however, as its factors $S^{l_i}(T^{1,0})_i^*$ and $K_{M_i}^{-q}$ are representations of the same group $Hol(M_i, g_i)$.

The existence of a non-zero $Hol_x(\tilde{M}, \tilde{g})$-invariant $\xi_x \in (S^{mq}(T^{1,0})^*\tilde{M} \otimes K_{\tilde{M}}^{-q})_x$ implies, by lemma (5.10), that at least one of the summands in (5.5), call it $(S^{l_1}T_1^* \otimes K_{M_1}^{-q})_{x_1} \otimes \ldots \otimes (S^{l_r}T_r^* \otimes K_{M_r}^{-q})_{x_r}$, has a non-zero, $Hol_x(\tilde{M}, \tilde{g})$-invariant element, which we shall denote as ζ. By corollary (5.9) each factor $S^{l_i}T_i^* \otimes K_{M_i}^{-q}$ contains a $Hol(M_i, g_i)$-invariant element, and so by lemma 5.4 either $S^{l_i}T_i^* \otimes K_{M_i}^{-q}$ is reducible or $S^{l_i}T_i^* \otimes K_{M_i}^{-q}$ is the trivial representation of $Hol(M_i, g_i)$.

In the first case observe that as a $Hol(M_i, g_i)$ representation, K_{M_i} is in fact the determinental representation of $Hol(M_i, g_i)$ (cf. §2.3.1). So if $S^{l_i}T_i^* \otimes K_{M_i}^{-q}$ is reducible by theorem (2.8) $\neq U(m_i)$. We conclude by theorem 3.31 that (M, J, g) is a bounded symmetric domain.

In the second case, M_i must be one-dimensional. Because $\tilde{M}_i$ is simply-connected and $c_1(\tilde{M}) < 0$ we conclude that (M_i, g_i) is the unit disk with the Poincaré metric (cf. the description given in §4.1.1) which is a bounded symmetric domain. So $(\tilde{M}, \tilde{g})$ is indeed a product of bounded symmetric domains. □

Remark 5.11. Note that this theorem says nothing about Kähler manifolds M which have a copy of an n-dimensional ball B_n in their universal cover, as the ball B_n is a symmetric space with holonomy $U(n)$.

Chapter 6

A Necessary and Sufficient Condition for $\tilde{M}$ to be a Bounded Symmetric Domain

We have proved a sufficient condition for a Kähler manifold to be uniformised by a bounded symmetric domain, but we have said nothing about whether or not this condition is necessary. In this section we discuss theorem 2.1 in [CDS12]), which is a variation of theorem 5.2 that does give a necessary and sufficient condition and moreover allows us to identify which bounded symmetric domains (cf. the classification in table ??) occur in $(\tilde{M}, g)$. It will turn out that this condition is sufficient only for bounded symmetric domains of tube type whose rank divides their dimension. Before we discuss the main result, let us first introduce some terminology:

Definition 6.1. Recall that the set of all holomorphic line bundles on a complex manifold M forms a group, the *Picard* group $(Pic(M))$ with tensor product as the group operation and the trivial line bundle $\mathcal{O}_M$ as identity. A line bundle η is said to be 2-torsion if $\eta \otimes \eta = \mathcal{O}_M$.

Definition 6.2. Let $m = \dim_{\mathbb{C}}(M)$. A *semi-special tensor* ξ is a global section of the vector bundle $S^m(T^{1,0})^*M \otimes K_M^{-1} \otimes \eta$ where η is a 2-torsion line bundle:

$$\xi \in \Gamma(S^m(T^{1,0})^*M \otimes K_M^{-1} \otimes \eta)$$

Theorem 6.3 (Theorem 1.2 in [CDS12])**.** *Let (M, J) be a compact Kähler manifold of dimension m. Then*

1. $c_1(M) < 0$

2. M admits a non-zero semi-special tensor ξ

hold if and only if the universal cover of M, $\tilde{M}$ is biholomorphic to a product of bounded symmetric domains $\tilde{M} = \Omega_1 \times \ldots \Omega_k$ where:

1. Ω_i is of tube type for all i

2. For all i the rank of Ω_i, r_i, divides the dimension m_i of Ω_i

Let us discuss the proof of the 'only if' part of this theorem as given in [CDS12]. We call a tensor $\gamma \in \Gamma(S^m(T^{1,0})^*M \otimes K_M^{-1})$ *special*. Note that k-torsion line bundles have the nice property that, if L is a k-torsion line bundle on M, then there exists a regular k-fold cover:

$$p^{(k)} : M^{(k)} \to M$$

such that $(p^k)^*L \cong \mathcal{O}_{M^{(k)}}$. If we denote by $\tilde{p} : \tilde{M} \to M$ the universal cover of M and by $\hat{p} : \tilde{M} \to M^{(k)}$ the universal cover of $M^{(k)}$ [1] then $\tilde{p}$ factors as:

$$\tilde{p} = p^{(k)} \circ \hat{p}$$

So, if M admits a semi-special tensor ξ there exists a double cover $p^{(2)} : M^{(2)} \to M$ such that

$$(p^{(2)})^*(\xi) \in \Gamma(S^m(T^{1,0})^*M^{(2)} \otimes K_{M^{(2)}}^{-1})$$

and thus $\tilde{\xi} = \tilde{p}^*(\xi) = \hat{p}^* \circ (p^{(2)})^*(\xi)$ is a special tensor. It follows from theorem 5.2 that $(\tilde{M}, \tilde{J}, \tilde{g})$ is a bounded symmetric domain. However more can be said in this case, as we are fixing the q of theorem 5.2 to be 1.
As in the proof of theorem 5.2, there exists a Kähler-Einstein metric $\tilde{g}$ on $\tilde{M}$ and a splitting into a product of irreducible factors:

$$(\tilde{M}, \tilde{g}, \tilde{J}) = (M_1, g_1, J_1) \times \ldots \times (M_k, g_k, J_k) \tag{6.1}$$

which are all bounded symmetric domains. Using the notation of section 5.2, a decomposition:

$$(S^m(T^{1,0})^*\tilde{M} \otimes K_{\tilde{M}}^{-1})_x \cong \bigoplus_{\sum l_i = m, l_i \geq 0} (S^{l_1}(T^{1,0})^*M_1 \otimes K_{M_1}^{-1})_{x_1} \ldots \otimes (S^{l_r}(T^{1,0})^*M_r \otimes K_{M_r}^{-1})_{x_r} \tag{6.2}$$

[1] Obviously M and $M^{(k)}$ have the same universal cover.

and a $Hol(\tilde{M}, \tilde{g}, \tilde{J})$-invariant ζ:

$$\zeta \in (S^{l_1}T^* \otimes (K_{M_1}^{-1})_p) \ldots \otimes (S^{l_k}T_k^* \otimes (K_{M_k}^{-1})_p)$$

for some partition $\sum_j l_j = m$. As before, this means that each factor in (6) contains a an H_i-invariant element where $H_i = Hol(M_i, g_i)$ or equivalently (cf. theorem 4.15) H_i is the isotropy subgroup of M_i at x. Since all the M_i are bounded symmetric domains they contain the origin. Thus we can, and will, assume that $x_i = 0$ for all i in (5.5).

Let $\{z_{i,1}, \ldots, z_{i,m_i}\}$ be a coordinate system on M_i such that at 0 the Kähler-Einstein metric on M_i is equal to the identity matrix in coordinates. Then we have that:

$$K_{M_i}^{-1} = \mathbb{C} \cdot (dz_{i,1} \wedge \ldots \wedge dz_{i,m_i})^{-1}$$

Since $K_{M_i}^{-1}$ is one-dimensional, any element of $S^{l_i}T_i^* \otimes (K_{M_i}^{-1})$ can be written as:

$$f \otimes (dz_{i,1} \wedge \ldots \wedge dz_{i,m_i})^{-1}$$

Where $f \in S^{l_i}T_i^*$ can be thought of as a homogeneous polynomial of degree l_i on T_i. Now:

$$h \cdot \left(f \otimes (dz_{i,1} \wedge \ldots \wedge dz_{i,m_i})^{-1}\right) = (h \cdot f) \otimes (\det(h)^{-1}(dz_{i,1} \wedge \ldots \wedge dz_{i,m_i})^{-1})$$

so in order for this element to be invariant, we must have:

$$h \cdot f = \hat{\chi}(h) f \text{ where } \hat{\chi}(h) \in \mathbb{C}^* \tag{6.3}$$

$$\hat{\chi}(h) = \det(h) \quad \forall h \in H_i \tag{6.4}$$

Because $(M_i, g_i J_i)$ is Kähler, $H_i \subset U(m_i)$ we have $|\hat{\chi}(h)| = 1$ for all $h \in H_i$. We shall call an f satisfying (6.3)[2] an *H_i-semi-invariant polynomial*. We shall show that (6.3) and (6.4) can be satisfied only if M_i is in fact of tube type, and its rank divides its dimension. Before we can do this however, we need to collect a few results from the theory of bounded symmetric domains. This is done in section 2.1 of [CDS12] and we recall it here (albeit in a slightly different form).

6.0.1 The Shilov boundary and inner polynomial functions

Definition 6.4. For a bounded domain Ω, let us denote by $\mathcal{H}(\Omega)$ the algebra of all holomorphic functions on $\bar{\Omega}$. The *Shilov boundary*, S, of Ω is a closed subset of the

[2]That is, a polynomial such that H_i acts on it by multiplication by a character

topological boundary $S \subset \partial\Omega$ such that, for every $f \in \mathcal{H}(\Omega)$ we have:

$$|f(z)| \leq \max_{s \in S}\{|f(s)|\} \quad \forall\, z \in \bar{\Omega}$$

(see [Sto]).

For our purposes, it suffices to note three things about S:

1. In the simplest possible case $\Omega = \mathbb{D}^1$, the unit disk, $S = \partial\Omega \cong \mathcal{S}^1$

2. In general however, S is a proper subset of $\partial\Omega$.

3. If Ω is a bounded symmetric domain and K is the isotropy group of $0 \in \Omega$ then K acts transitively on S. For example, consider again the case of the unit disk. In this case $K = U(1)$ and it obviously acts transitively on $S = \mathcal{S}^1$.

Definition 6.5. Given a bounded symmetric domain Ω, an inner function is a bounded holomorphic function on Ω such that if we denote by f^* its radial limit:

$$f^*(z) = \lim_{r \nearrow 1} f(rz) \quad z \in \partial\Omega$$

then this limit exists and has modulus 1 ($|f^*(z)| = 1$) for almost all $z \in S$ where as above S is the Shilov boundary.

Remark 6.6. Observe that in the case where $\Omega = \mathbb{D}^1$ this definition coincides with the usual definition of inner function in Hardy Space theory.

Lemma 6.7 (Corollary 2.2 in [CDS12]). *Let Ω be a bounded symmetric domain. If $f \in S^n T_0^*\Omega$ is K- semi-invariant then it is, up to a scalar multiple, an inner polynomial function on Ω.*

Proof. Suppose that $\Omega \subset \mathbb{C}^m$. As before we can consider $f \in S^n(T_0^*\Omega)$ to be a polynomial of degree n on $T_0 \cong \mathbb{C}^m$ and so by restricting to Ω we get a polynomial on Ω. Obviously $f^*(z)$ exists for all $z \in \partial\Omega$. Fixing a point $z_0 \in S$, we have that:

$$\begin{aligned} |f(z)|^2 &= |f(g \cdot z_0)|^2 \quad \text{for some } g \in K \\ &= |\chi(g) f(z_0)|^2 \\ &= |f(z_0)|^2 \quad \text{since } |\chi(g)| = 1 \end{aligned}$$

□

Now we use the following result of Korànyi and Vàgi:

Lemma 6.8 (See lemma 2.5 in [KV79] or theorem 2.1 in [CDS12]). *Suppose that Ω is an irreducible bounded symmetric domain. If Ω is of tube type there exists a unique polynomial N_Ω such that if f is any polynomial inner function on Ω then:*

$$f = cN_\Omega^k$$

where $|c| = 1$ and $k \in \mathbb{Z}^+$. Moreover, the degree of f is equal to the rank of Ω. If Ω is not of tube type the only polynomial inner functions on Ω are constants (of modulus 1).

Proof. A proof may be found in [KV79] and in fact they construct the functions explicitly. □

In order for $S^{l_j}(T_j^*) \otimes K_{M_j}^{-1}$ to contain an element satisfying (6.3) there are two possibilities:

1. M_j is a bounded symmetric domain of tube type and $f = N_j^{n_j}$ for some positive integer n_j, where N_j is the unique polynomial discussed in lemma 6.8.

2. M_j is an arbitrary bounded symmetric domain $l_j = 0$ and the semi invariant polynomial is a constant $\alpha \in \mathbb{C}$. In this case, $\hat{\chi}(h) = \det(h) = 1$ for all $h \in H_i$ (cf. (6.4)).

If $\det(h) = 1$ for all $h \in H_i$ then $H_i \subset SU(m_i)$ and hence (M_i, g_i, J_i) is Ricci-flat. But this contradicts our assumption that $c_1(M_i, g_i) < 0$ (cf. the line of reasoning used in the proof of 3.31). We conclude that the only possibility is the first and so all M_i must be of tube type. Since (M_i, J_i, g_i) is a bounded symmetric domain and H_i is the isotropy group of 0, by theorem 4.24:

$$U(1) \cong Z(H_i) = \{\mathrm{diag}(e^{i\theta}, \ldots, e^{i\theta}) : \theta \in \mathbb{R}\}$$

If we restrict the action of H_i on $S^{l_i}(T_i^*)$ to $Z(H_i)$, we get that:

$$\det(\mathrm{diag}(e^{i\theta}, \ldots, e^{i\theta})) = e^{im_i\theta}$$

where as usual $m_i = \dim_{\mathbb{C}}(M_i)$ and:

$$\begin{aligned}\mathrm{diag}(e^{i\theta}, \ldots, e^{i\theta}) \cdot N_j^{n_j}(z_{j,1}, \ldots, z_{j,m_j}) &= N_j^{n_j}(e^{i\theta}z_{j,1}, \ldots, e^{i\theta}z_{j,m_j}) \\ &= e^{i\theta n_j r_j} N_j^{n_j}(z_{j,1}, \ldots, z_{j,m_j})\end{aligned}$$

Table 6.1: Bounded Symmetric Domains of Tube Type whose rank divides their dimension

BSD	rank	$dim_{\mathbb{C}}$
$\mathcal{D}^{I}_{n,n}$ for $n \geq 1$	n	n^2
$\mathcal{D}^{II}_{2n}$ for $n \geq 1$	n	$n(2n-1)$
$\mathcal{D}^{III}_{2n+1}$ for $n \geq 0$	$2n+1$	$(2n+1)(n+1)$
$\mathcal{D}^{IV}_{2n}$ for for $n \geq 1$	2	$2n$
$\mathcal{D}^{VI}$	3	27

Since N_j is homogeneous of degree r_j. Thus:

$$\hat{\chi}(\mathrm{diag}(e^{i\theta}, \dots, e^{i\theta})) = e^{i\theta n_j r_j}$$

so in order to satisfy (6.4) we must have:

$$n_j r_j = m_j$$

So each M_j is a bounded symmetric domain of tube type whose rank divides its dimension. From our classification of bounded symmetric domains in table **??**, let us extract a list of the domains of tube type whose dimension divides their rank (see also Theorem 2.3 of [CDS12]).

This table is actually slightly redundant, as we have the low dimensional isomorphisms:

$$\begin{aligned}
&\mathcal{D}^{I}_{1,1} \cong \mathcal{D}^{II}_{2} \cong \mathcal{D}^{III}_{1} \\
&\mathcal{D}^{I}_{2,2} \cong \mathcal{D}^{IV}_{4} \\
&\mathcal{D}^{II}_{4} \cong \mathcal{D}^{IV}_{6}
\end{aligned}$$

which follow from the 'accidental isomorphisms' of Low dimensional Lie groups:

$$\begin{aligned}
&SU(1,1) \cong Sp(2,\mathbb{R}) \cong O^*(2n) \\
&SU(2,2) \cong SO(4,2,\mathbb{R})
\end{aligned}$$

accounting for these redundancies, we see that the bounded symmetric domains of tube type whose rank divides their dimension are *uniquely determined by their rank and dimension*.

So, not only do we know that all the factors in 6.1 are bounded symmetric domains, we can also determine *which* domain they are knowing only their dimensions.

We do not discuss the proof of 'if' part of theorem 6.3 here, as it involves a detour through Jordan algebra theory, but refer the interested reader to §4.2 and 4.3 of [CDS12]

Chapter 7

Extending a uniformisation result of Catanese and Di Scala to include ball factors

In this chapter we prove a theorem very similar to Theorem 1.2 in [CDS], which uses the Mok characteristic varieties introduced in §4.8 to provide necessary and sufficient conditions for a Kähler manifold (M, J) with $c_1(M, J) < 0$ to have as its universal cover a bounded symmetric domain:

$$\tilde{M} \cong \tilde{M}_1 \times \ldots \times \tilde{M}_r$$

with each M_i a bounded symmetric domain which is not a ball factor and moreover allows tells us which bounded symmetric domain each M_i is. By *ball factor* we mean an M_i which is isometric to the unit ball $B_n \subset \mathbb{C}^n$ considered with the induced complex structure and its Bergmann metric, so that it is a bounded symmetric domain (cf. §4.7.2)). Our theorem extends this to include the possibility of ball factors, thus providing a complete, although not entirely satisfactory answer to the two questions raised in the introduction.

This chapter has three sections. In the first, we discuss some properties of Chern classes and introduce the Miyaoka-Yau inequality, which is the necessary tool for dealing with ball factors. In the second we discuss some projective algebraic geometry, particularly an operation called the 'join' of two projective varieties. In the final section we assemble the tools and definitions developed in this chapter and previously to state and prove our main theorem.

7.1 Chern Classes and the Miyaoka-Yau inequality

Recall that (and cf. §3.8) that the $k-th$ Chern class of a complex manifold (M, J) (denoted $c_k(M, J)$ or simply c_k if it is clear which complex manifold we are referring to) is a (de Rham) cohomology class $c_k(M) \in H^{2k}(M, \mathbb{R})$ which depends only on the topology of M and the homotopy class of J. There is a binary operation defined on cohomology classes called the cup product, which given $\alpha \in H^p(M, \mathbb{R})$ and $\beta \in H^q(M, \mathbb{R})$ gives a cohomology class (written $\alpha \smile \beta$) in $H^{p+q}(M, \mathbb{R})$. In practice we can choose closed forms $\hat{\alpha}$ and $\hat{\beta}$ representing α and β respectively, in which case the cup product is given by the exterior product: $\alpha \smile \beta = [\hat{\alpha} \wedge \hat{\beta}]$

Theorem 7.1. *Let (M, J, g) be a Kähler manifold with $\dim_{\mathbb{C}}(M) = m$ and $c_1(M, J) < 0$. Then*

$$\frac{n}{2(n+1)} c_1^n - c_2 \smile c_1^{n-2} \leq 0 \tag{7.1}$$

with equality holding if and only if the universal cover of (M, J) is a bounded symmetric domain of ball type, B_m.

Proof. A thorough and accessible proof of this theorem may be found in [Cho08]. See also Theorem 4 in [Yau77] for a proof in the two dimensional case. We remark that the existence of a Kähler-Einstein metric is essential in the proof of this theorem for dimension greater than two. □

The inequality 7.1 makes sense because $(c_1^n - \frac{2(n+1)}{n} c_2 \smile c_1^{n-2}) \in H^{2n}(M, \mathbb{R}) \cong \mathbb{R}$, where the identification with $\mathbb{R}$ is given by mapping classes $\gamma \in H^{2n}(M, \mathbb{R})$ to their value when paired with the fundamental homology class $[M] \in H_{2n}(M, \mathbb{R})$. More plainly, for any Hermitian metric g on (M, J) we get a $2k$-form $c_k(M, J, g)$ (or simply $c_k(g)$ when it is clear which (M, J) we are referring to) representing c_k as constructed in §3.8. Then:

$$\begin{aligned} c_1^n &= [c_1^{\wedge n}] \\ c_2 \smile c_1^{n-2} &= [c_2(g) \wedge c_1(g)^{\wedge n-2}] \\ \text{and so } \frac{n}{2(n+1)} c_1^n - c_2 \smile c_1^{n-2} &= [\frac{n}{2(n+1)} c_1(g)^{\wedge n} - c_2 \wedge c_1^{\wedge n-2}] \end{aligned}$$

Thus 7.1 becomes:

$$\int_M \left(\frac{n}{2(n+1)} c_1(g)^{\wedge n} - c_2 \wedge c_1^{\wedge n-2} \right) \leq 0$$

with equality holding if and only if $(\tilde{M}, J) \cong B_m$

We are interested in proving a version of 7.1 when M is uniformized not merely by a ball, but by the product of a ball and some other simply connected Kähler manifold (with negative first Chern class):

$$\tilde{M} = B_{n_1} \times M_2 \tag{7.2}$$

Loosely speaking, we do this by looking for a splitting $T^{1,0}M = T_1 \oplus T_2$ and applying 7.1 to the sub-bundles T_1 and T_2. We shall need the following theorem on splittings of the holomorphic tangent bundle originally proven by Yau in [Yau93] and rediscovered by Beauville ([Bea]).

Theorem 7.2. *Suppose that (M, J) is a Kähler manifold admitting a Kähler-Einstein metric g. Then $T^{1,0}M$ splits as a direct sum of sub-bundles:*

$$T^{1,0}M = T_1 \oplus \ldots \oplus T_r$$

if and only if the universal cover $\tilde{p} : \tilde{M} \to M$ splits as a product of Kähler manifolds:

$$\tilde{M} = M_1 \times \ldots \times M_r$$

such that $\tilde{p}^* T_i = T^{1,0}\tilde{M}_i$ [1] *and the fundamental group $\pi_1(M)$ acts on $\tilde{M}$ 'diagonally':*

$$\begin{aligned} &x = (x_1, \ldots, x_r) \in \tilde{M} \text{ and } \gamma \in \pi_1(M) \\ &\gamma \cdot x = (\gamma \cdot x_1, \ldots, \gamma \cdot x_r) \end{aligned}$$

Remark 7.3. Note that we are not claiming that the M_i are irreducible factors!

Proof. See [Bea]. We remark that the existence of a Kähler-Einstein metric g on (M, J) is an important part of the proof. □

We also need two results relating the Chern classes of vector bundles on M to those of vector bundles on $\tilde{M}$ which are collected below in theorem 7.4. Recall, as per the discussion in §4.8.4, we may regard $\pi_1(M)$ as a subgroup of $\mathrm{Aut}(\tilde{M})$.

Theorem 7.4. *Let $p : (\tilde{M}, \tilde{J}) \to (M, J)$ be the universal cover of (M, J). Then*

1. *Pullback of differential forms induces an injective map of complexes:*

$$p^* : H^\bullet(M, \mathbb{R}) \to H^\bullet(\tilde{M}, \mathbb{R})$$

[1] Technically we should write $\tilde{p}^* T_i = \mathrm{pr}_i^* T^{1,0} M_i$ where $\mathrm{pr}_i : \tilde{M} \to M_i$ is the projection on to the i-th factor

2. *Chern classes 'behave well' under pullback. That is, for any holomorphic bundle $E \to M$ we have that:*
$$p^*(c_k(E)) = c_k(p^*(E))$$

3. *Suppose that $c_1(M) < 0$ and that $(\tilde{M}, \tilde{J})$ splits as a direct product:*
$$(\tilde{M}, \tilde{J}) \cong (M_1, J_1) \times \ldots \times (M_s, J_s)$$
Then $c_1(M_i) < 0$ for all i.

Proof. The first item follows from the fact that $p^* : H^\bullet(M, \mathbb{R}) \to H^\bullet(\tilde{M}, \mathbb{R})^{\pi_1(M)}$ is an isomorphism. The second item is a standard property of Chern classes, and is explained on pg. 197 of [Huy05] for example. For the third item, observe that for $i = 1, \ldots, s$ we have projections:
$$pr_i : \tilde{M} \to M_i$$
and that
$$T^{1,0}\tilde{M} = pr_1^* T^{1,0} M_1 \oplus \ldots \oplus pr_s^* T^{1,0} M_s$$
We know (cf. the top of pg. 197 in [Huy05]) that:
$$\begin{aligned} c_1(\tilde{M}) = c_1(T^{1,0}\tilde{M}) &= c_1(pr_1^* T^{1,0} M_1) + \ldots c_1(pr_s^* T^{1,0} M_s) \\ &= pr_1^* c_1(M_1) + \ldots pr_s^* c_s(M_s) \end{aligned}$$
where the final equality follows from the first item of this theorem. Now suppose there exists j and an $X_j \in T^{1,0}_{x_j} M_j$ such that $c_1(M_j)(X_j, \bar{X}_j) \geq 0$. Choose $X \in T^{1,0}_x \tilde{M}$ such that $(pr_i)_*(X) = 0$ for $i \neq j$ and $(pr_j)_*(X) = X_j$. Such an X certainly exists. Then:
$$c_1(\tilde{M})(X, \bar{X}) = c_1(M_j)(X_j, \bar{X}_j) \geq 0$$
contradicting our assumption that $c_1(\tilde{M}) < 0$. □

Returning to equation 7.2:

Theorem 7.5. *Suppose that (M, J) is a Kähler manifold with $c_1(M) < 0$. Denote by $p : \tilde{M} \to M$ the universal cover of M. If $\tilde{M} \cong B_{n_1} \times M_2$ where B_{n_1} is the n_1-th dimensional ball and M_2 has no de Rham factors isomorphic to B_{n_1} then $T^{1,0}M = T_1 \oplus T_2$ with:*
$$c_1(T_1)^{n_1} - \frac{2(n_1+1)}{n_1} c_2(T_1) \smile c_1(T_1)^{n_1-2} = 0$$

Conversely, $\tilde{M}$ contains a ball factor if there is a splitting into holomorphic sub-bundles:

$$T^{1,0}M = T_1 \oplus T_2$$

such that:

$$c_1(T_1)^{n_1} - \frac{2(n_1+1)}{n_1} c_2(T_1) \smile c_1(T_1)^{n_1-2} = 0$$

Proof. Because $c_1(M) < 0$, by Theorem 3.30 there exists a Kähler-Einstein metric g on (M, J). Because M_2 contains no de Rham factors isomorphic to B_{n_1}, by Theorem 3.17 the automorphism group of $\tilde{M}$ splits as:

$$\mathrm{Aut}(\tilde{M}) = \mathrm{Aut}(B_{n_1}) \times \mathrm{Aut}(M_2)$$

Thus $\pi_1(M) \subset \mathrm{Aut}(\tilde{M})$ is a direct product: $\pi_1(M) = \Gamma_1 \times \Gamma_2$, and so it acts diagonally on $\tilde{M}$. By 7.2 we have that $TM = T_1 \oplus T_2$ with $p^*(T_1) \cong T^{1,0}B_{n_1}$. Item 2 of 7.4 then implies that:

$$c_k(B_{n_1}) = c_k(T^{1,0}B_{n_1}) = c_k(p^*T_1) = p^*(c_k(T_1))$$

The Miyaoka-Yau inequality (Theorem 7.1) tells us that:

$$\begin{aligned} 0 &= c_1(B_{n_1})^{n_1} - \frac{2(n_1+1)}{n_1} c_2(B_{n_1}) \smile c_1(B_{n_1})^{n_1-2} \\ &= p^*\left(c_1(T_1)^{n_1} - \frac{2(n_1+1)}{n_1} c_2(T_1) \smile c_1(T_1)^{n_1-2}\right) \end{aligned}$$

because p^* gives a map of complexes, and so is $\mathbb{R}$-linear and commutes with $\smile$. By item 1 of 7.4 p^* is injective, so:

$$c_1(T_1)^{n_1} - \frac{2(n_1+1)}{n_1} c_2(T_1) \smile c_1(T_1)^{n_1-2} = 0$$

Conversely, if $T^{1,0}M = T_1 \oplus T_2$ with $\mathrm{rank}(T_i) = n_i$ for $i = 1, 2$ and:

$$c_1(T_1)^{n_1} - \frac{2(n_1+1)}{n_1} c_2(T_1) \smile c_1(T_1)^{n_1-2} = 0$$

By theorem (7.2) $\tilde{M} = M_1 \times M_2$ with $p^*(T_i) = T^{1,0}M_i$ (for $i = 1, 2$). Moreover:

$$\begin{aligned} c_1(M_1)^{n_1} - \frac{2(n_1+1)}{n_1} c_2(M_1) \smile c_1(M_1)^{n_1-2} &= p^*\left(c_1(T_1)^{n_1} - \frac{2(n_1+1)}{n_1} c_2(T_1) \smile c_1(T_1)^{n_1-2}\right) \\ &= p^*(0) = 0 \end{aligned}$$

So again by theorem 7.1 we have that $M_1 \cong B_{n_1}$. $\square$

We would like to extend theorem 7.5 to the case where $\tilde{M}$ can have any number of ball factors, some of which may be isomorphic. The main obstruction to this is that if there are two copies of B_n in $\tilde{M}$ then $\pi_1(M)$ could fail to act diagonally. To see this, lets observe the simplest possible case, where $\tilde{M} = B_n \times B_n$. By 3.17 we have:

$$\mathrm{Aut}(\tilde{M}) = \mathrm{Aut}(B_{n_1}) \ltimes S_2$$

thus there exist automorphisms which 'permute' the ball factors. For example, if $(x_1, x_2) \in \tilde{M}$ then:

$$(g_1, g_2, (12)) \cdot (x_1, x_2) = (g_1(x_2), g_2(x_1)$$

Since $\pi_1(M) < \mathrm{Aut}(\tilde{M})$ we cannot guarantee that it acts diagonally. So theorem 7.2 need not apply and $T^{1,0}M$ may not split [2]. However, we can salvage a useful result by passing to a finite cover if necessary:

Theorem 7.6. *Suppose that M is a Kähler manifold supporting a Kähler-Einstein metric (for example, we could require that $c_1(M) < 0$). Then there exists a k-fold cover $p^{(k)} : M^{(k)} \to M$ such that:*

$$(p^{(k)})^*(T^{1,0}M) = T^{1,0}M^{(k)} = T_1 \oplus \ldots \oplus T_s \oplus T^{'} \tag{7.3}$$

and, if m_i denotes the rank of T_i,

$$c_1(T_i)^{m_i} - \frac{2(m_i+1)}{m_i} c_2(T_i) \smile c_1(T_i)^{m_i-2} = 0 \tag{7.4}$$

for all $1 \leq i \leq s$ and

$$c_1(T^{'})^{m^{'}} - \frac{2(m^{'}+1)}{m^{'}} c_2(T^{'}) \smile c_1(T^{'})^{m^{'}-2} = 0$$

if and only if

$$\tilde{M} = B_{m_1} \times \ldots \times B_{m_s} \times M^{'}$$

where $M^{'}$ is some simply-connected Kähler manifold with $c_1(M^{'}) < 0$. (We are omitting the g_i's and J_i's, but this is meant to be read as a Riemannian product of Kähler manifolds)

Proof. Note that $M^{(k)}$ and M have the same universal cover, and if $\hat{p} : \tilde{M} \to M^{(k)}$ and $p : \tilde{M} \to M$ are the respective covering maps, then $p = p^{(k)} \circ \hat{p}$. If (7.3) holds, because $M^{(k)}$ is also Kähler-Einstein, by 7.2:

$$\tilde{M} = M_1 \times \ldots \times M_s \times M^{'}$$

[2] Can I cite some examples of Bidisk quotients whose tangent bundle does not split?

with $\hat{p}^*(T_i) \cong T^{1,0}M_i$ and $\hat{p}^*(T') \cong T^{1,0}M'$. By item 3 of 7.4 we have that $c_1(M_i) < 0$ for all i and $c_1(M') < 0$. As in the proof of 7.5, from (7.4) we have:

$$\begin{aligned} c_1(M_i)^{m_i} - \frac{2(m_i+1)}{m_i} c_2(M_i) \smile c_1(M_i)^{m_i-2} &= \hat{p}^* \left(c_1(T_i)^{m_i} - \frac{2(m_i+1)}{m_i} c_2(T_i) \smile c_1(T_i)^{m_i-2} \right) \\ &= \hat{p}^*(0) = 0 \end{aligned}$$

Thus $M_i \cong B_{m_i}$ for $1 \le i \le s$. The other direction is trickier. Suppose that

$$\tilde{M} = B_{m_1} \times \ldots \times B_{m_s} \times M'$$

It could be that some of the m_i are repeated, thus we have isomorphic factors in the above decomposition. So let us rather write:

$$\tilde{M} = (B_{m_1})^{k_1} \times \ldots \times (B_{m_r})^{k_r} \times \tilde{M}_s$$

where of course $\sum_{i=1}^{r} k_i = s$. Then we have by the corollary to the De Rham decomposition theorem 3.17 that

$$\mathrm{Aut}(\tilde{M}) = \left(\mathrm{Aut}(B_{m_1})^{k_1} \ltimes S_{k_1} \right) \times \ldots \times \left(\mathrm{Aut}(B_{m_r})^{k_r} \ltimes S_{k_r} \right) \times \mathrm{Aut}(M')$$

Considering $\pi_1(M)$ as a subgroup of $\mathrm{Aut}(\tilde{M})$, if $g \in \pi_1(M)$ then we can write

$$g = \left((g_{1,1}, \ldots, g_{1,k_1}, \sigma_1), \ldots, (g_{r,1}, \ldots, g_{r,k_r}, \sigma_r), g' \right)$$

and if $x = \left((x_{1,1}, \ldots, x_{1,k_1}), \ldots, (x_{r,1}, \ldots, x_{r,k_r}), x' \right) \in \tilde{M}$ then

$$g \cdot x = \left((g_{1,1}(x_{1,\sigma_1(1)}), \ldots, g_{1,k_1}(x_{1,\sigma(k_1)})), \ldots, (g_{r,1}(x_{r,\sigma_r(1)}, \ldots, g_{r,k_r}(x_{r,\sigma_r(k_r)}), g'(x') \right) \tag{7.5}$$

and so we see that g does not act diagonally unless $\sigma_1 = \ldots = \sigma_r = id$. To solve this problem we restrict to the identity component of $\mathrm{Aut}(\tilde{M})$

$$\mathrm{Aut}(\tilde{M})_0 = \mathrm{Aut}(B_{m_1})^{k_1} \times \ldots \times \mathrm{Aut}(B_{m_r})^{k_r} \times \mathrm{Aut}(M')_0$$

and define:

$$\Gamma_0 = \pi_1(M) \cap \mathrm{Aut}(\tilde{M})_0$$

Now:

1. Because conjugation by any $g \in \mathrm{Aut}(\tilde{M})$ is a continuous map sending the identity to itself, it maps the identity component $\mathrm{Aut}(\tilde{M})_0$ to itself. Hence $\mathrm{Aut}(\tilde{M})_0$ is a normal subgroup. So Γ_0 is a normal subgroup of $\pi_1(M)$

2. Moreover, because $\mathrm{Aut}(\tilde{M})_0$ is of finite index in $\mathrm{Aut}(\tilde{M})$ (as each S_{k_i} is finite) Γ_0 is a finite index subgroup of $\pi_1(M)$.

3. Because $\Gamma_0 < \pi_1(M)$, theorem 1.36 of [Hat02], states that there exists a cover $p^{(k)} : M^{(k)} \to M$ such that $p_*^{(k)}(\pi_1(M^{(k)})) = \Gamma_0$. Since Γ_0 is of finite index, this cover is finite. Since Γ_0 is normal, Proposition 1.39 of [Hat02] tells us this cover is normal, or regular. That is, the deck transformations act transitively on each fibre.

4. Because $p_*^{(k)} : \pi_1(M^k) \to \pi_1(M)$ is injective (cf. Proposition 1.31 in [Hat02]):

$$\pi_1(M^{(k)}) \cong p_*^{(k)}(\pi_1(M^{(k)})) \cong \Gamma_0$$

It follows that $M^{(k)} \cong \tilde{M}/\Gamma_0$.

Finally, observe that, for any $g \in \mathrm{Aut}(\tilde{M})$ written as:

$$g = \left((g_{1,1}, \ldots, g_{1,k_1}, \sigma_1), \ldots, (g_{r,1}, \ldots, g_{r,k_r}, \sigma_r), g^{'}\right)$$

$g \in \mathrm{Aut}(\tilde{M})_0$ if and only if $\sigma_1 = \ldots = \sigma_r = id$. Thus by (7.5) and the discussion following it we have that all $g \in \Gamma_0 < \mathrm{Aut}_0(\tilde{M})$ do indeed act diagonally.

Now, we may apply theorem 7.2 to $M^{(k)}$ to conclude that:

$$T^{1,0}M^{(k)} = T_{1,1} \oplus \ldots \oplus T_{1,k_1} \oplus \ldots \oplus T_{r,1} \oplus \ldots \oplus T_{r,k_r} \oplus T^{'}$$

with $\hat{p}^*T^{'} \cong T^{1,0}M'$ and $\hat{p}^*T_{i,j}$ corresponding to the tangent bundle of one of the k_i copies of B_{m_i} in $\tilde{M}$, lets denote this as:

$$\hat{p}^*T_{i,j} = T^{1,0}B_{m_i,j}$$

Using lemma 7.4 and arguing as we did in the proof of theorem **??**:

$$\begin{aligned}
0 &= c_1(B_{m_i,j})^{m_i} - \frac{2(m_i+1)}{m_i}c_2(B_{m_i,j}) \smile c_1(B_{m_i,j})^{m_i-2} \\
&= \hat{p}^*\left(c_1(T_{i,j})^{m_i} - \frac{2(m_i+1)}{m_i}c_2(T_{i,j}) \smile c_1(T_{i,j})^{m_i-2}\right) \\
\Rightarrow & c_1(T_{i,j})^{m_i} - \frac{2(m_i+1)}{m_i}c_2(T_{i,j}) \smile c_1(T_{i,j})^{m_i-2} = 0
\end{aligned}$$

as $\hat{p}$ is injective. □

7.2 The Join operation and Invariant Subvarieties

Suppose we have two projective varieties $V_1 \subset \mathbb{CP}^{n_1}$ and $V_2 \subset \mathbb{CP}^{n_2}$ with ideals $I_1 = (f_1, \dots, f_r) \subset \mathbb{C}[x_0, \dots, x_{n_1}]$ and $I_2 = (g_1, \dots, g_s) \subset \mathbb{C}[y_0, \dots, y_{n_2}]$. As usual we denote by $CV_1 \subset \mathbb{C}^n_1 + 1$ and $CV_2 \subset \mathbb{C}^{n_2+1}$ the cones over V_1 and V_2 respectively. A natural operation on these cones is to consider their direct sum:

$$CV_1 \oplus CV_2 = \{v_1 + v_2 : v_1 \in CV_1 \text{ and } v_2 \in CV_2\} \subset \mathbb{C}^{n_1+n_2+2}$$

and one can see that the ideal of $CV_1 \oplus CV_2$ is generated by $(f_1, \dots, f_r, g_1, \dots, g_s)$ in $\mathbb{C}[x_0, \dots, x_{n_1}, [y_0, \dots, y_{n_2}]$ (cf. the math.stackexchange thread : [Ele]). Since all of the f_i and g_j are homogeneous, this is a homogeoneous ideal, and so we get a projective variety (*a priori* it is not necessarily irreducible) $\mathbb{P}(CV_1 \oplus CV_2) \subset \mathbb{CP}^{n_1+n_2+1}$. We call this variety the *join* of V_1 and V_2, and denote this as:

$$V_1 * V_2 = \mathbb{P}(CV_1 \oplus CV_2)$$

Alternatively if we consider $V_1 \subset \mathbb{CP}^{n_1} \subset \mathbb{CP}^{n_1+n_2+1}$ and $V_2 \subset \mathbb{CP}^{n_2} \subset \mathbb{CP}^{n_1+n_2+1}$ we can define $V_1 * V_2$ as the closure of the union of all lines in $\mathbb{CP}^{n_1+n_2+1}$ intersecting V_1 and V_2, thus we can see that $V_1 * V_2$ is irreducible if V_1 and V_2 are. In general given any finite collection of vector spaces $T_1, \dots T_r$ and projective varieties $V_1 \subset \mathbb{P}T_1, \dots V_r \subset \mathbb{P}T_r$ we may extend the definition of join to get:

$$V_1 * V_2 * \dots * V_r \subset \mathbb{P}(T_1 \oplus T_2 \oplus \dots \oplus T_r)$$

by first joining V_1 and V_2, and then joining $V_1 * V_2$ and V_3, and so on. We shall say that a projective variety $V \subset \mathbb{P}T$ is *non-degenerate* if it is not contained in any hyperplane.

Lemma 7.7. *Suppose that $V = V_1 * V_2 \subset \mathbb{P}T = \mathbb{P}(T_1 \oplus T_2)$ is non-degenerate and smooth. Then V_1 and V_2 are both smooth.*

Proof. Recall that if $f_1, \dots, f_r$ generate $\mathcal{I}(V_1)$ and $g_1, \dots, g_s$ generate $\mathcal{I}(V_2)$ then $f_1, \dots, f_r, g_1, \dots, g_s$ generate $\mathcal{I}(V)$. Take coordinates $(x_{j,0}, \dots, x_{j,n_j})$ on T_i where $n_j + 1 = \dim_{\mathbb{C}}(T_j)$; then $(\mathbf{x}_1, \mathbf{x}_2) = (x_{1,0}, \dots, x_{1,n_1}, x_{2,0}, \dots, x_{2,n_2})$ give homogeneous coordinates on $\mathbb{P}T$. Suppose one of V_1 or V_2 is singular, we may as well assume it is V_1. If $\mathbf{v}_1 = (v_{1,0}, \dots, v_{1,n_1}) \in V_1$ is a singular point, for any f_i we must have

$$\frac{\partial f_i}{\partial x_{1,k}}(\mathbf{v}_1) = 0 \quad \forall k: \; 0 \leq k \leq n_1$$

Now $(\mathbf{v}_1, \mathbf{0}) \in V_1 * V_2 = V$ and considering f as an element of $\mathcal{I}(V)$ we have:

$$\frac{\partial f_i}{\partial x_{1,k}}(\mathbf{v}_1, \mathbf{0}) = 0 \quad \forall k: \; 0 \leq k \leq n_1$$

and

$$\frac{\partial f_i}{\partial x_{2,k}}(\mathbf{v}_1, \mathbf{0}) = 0 \quad \forall k : 0 \leq k \leq n_2$$

since the f_i have no $\mathbf{x}_2$ dependence. Similarly

$$\frac{\partial g_j}{\partial x_{1,k}}(\mathbf{v}_1, \mathbf{0}) = 0 \quad \forall k : 0 \leq k \leq n_1$$

for $1 \leq j \leq s$ as the g_j have no $\mathbf{x}_1$ dependence. So, the only way that V can be non-singular is if for some j,

$$\frac{\partial g_j}{\partial x_{2,k}}(\mathbf{v}_1, \mathbf{0}) \neq 0 \text{ for some } k : 0 \leq k \leq n_2$$

but this means that g_j is linear, and thus $V_2 \subset \mathcal{V}(g_j)$ and $V \subset \mathbb{PT}_1 * \mathcal{V}(g_j)$ contradicting the assumption that V is non-degenerate. $\square$

Corollary 7.8. *If $V = V_1 * V_2 * \ldots * V_r$ is smooth and non-degenerate, then V_1 is smooth for all i.*

Proof. This follows from Lemma 7.7 by induction: if the statement holds for $r = k$, if $V_1 * V_2 * \ldots * V_k * V_{k+1}$ is smooth and non-degenerate so are $V_1 * \ldots * V_k$ and V_{k+1}. $\square$

In a similar vein we have:

Lemma 7.9 (Lemma 7.1 in [CDS]). *Let $V \subset \mathbb{P}T$ be a smooth projective variety. Then V cannot be written non-trivially as a join $V_1 * \mathbb{P}T_2$ where $T = T_1 \oplus T_2$ and $V_1 \subset T_1$ is non-degenerate.*

Proof. Suppose that V is smooth and $V = V_1 * \mathbb{P}T_2$. As before, choose generators $f_1, \ldots, f_r$ for $\mathcal{I}(V_1)$. Because $\mathcal{I}(\mathbb{P}T_2) = \{0\}$, we have that $\mathcal{I}(V)$ is also generated by $f_1, \ldots, f_r$ over the homogeneous coordinate ring of $\mathbb{P}T$. As in the proof of lemma if $\mathbf{x}_j = (x_{j,0}, \ldots, x_{j,n_j})$ are coordinates on T_j for $j = 1, 2$ then

$$\frac{\partial f_i}{\partial x_{2,k}}(\mathbf{0}, \mathbf{v}_2) = 0 \quad \forall k : 0 \leq k \leq n_2$$

for all $\mathbf{v}_2 \in T_2$ since the f_i have no $\mathbf{x}_2$ dependence. Because V_2 is assumed to be non-degenerate, the f_i are all of degree 2 or greater, and so:

$$\frac{\partial f_i}{\partial x_{1,k}}(\mathbf{0}, \mathbf{v}_2) = 0 \quad \forall k : 0 \leq k \leq n_2$$

for all $\mathbf{v}_2 \in T_2$. Thus V is singular along $\{(0, \mathbf{v}_2) : v_2 \in T_2\} \cong \mathbb{P}T_2$. But V was assumed to be smooth, thus $T_2 = 0$. □

Now consider the following scenario, which will arise naturally in the next section when we are considering the holonomy group acting on the tangent space to a complex manifold. Suppose a Lie group H acts linearly on a vector space T. Since H acts linearly its action descends to $\mathbb{P}T$, so suppose that $V \subset \mathbb{P}T$ is an H-invariant, irreducible projective variety. If in addition $H = H_1 \times \ldots \times H_r$ and T splits into H-invariant subspaces

$$T = T_1 \oplus \ldots \oplus T_r$$

such that for all i, H_j acts trivially on T_i unless $i = j$, in which case T_i is an irreducible H_i representation, then we have the following:

Lemma 7.10. *With hypotheses as above, we have:*

$$V = V_1 * \ldots * V_r$$

where for all i $V_i \subset \mathbb{P}T_i$ is an irreducible H_i invariant projective variety. Moreover, if V is smooth, then so is V_i for all i.

Proof. Since V is H invariant it is the union of H-orbits. The assumption that V is irreducible implies that it is in fact the closure of a single H-orbit. To see this, recall that H-orbits are disjoint, and suppose that

$$V = O_1 \cup O_2$$

where O_1 and O_2 are distinct H orbits. Then $\overline{O_1} \cup \overline{O_2} = \overline{V} = V$. By the irreducibility of V either $\overline{O}_1 =\subset \overline{O}_2$ or vice versa.
Thus the cone over V, CV will be the closure of an H orbit of a line $\{\lambda(\mathbf{v}_1, \ldots, \mathbf{v}_r) : \lambda \in \mathbb{C}\}$:

$$CV = \cup_{\lambda \in \mathbb{C}} \{h \cdot (\lambda \mathbf{v}_1, \ldots, \lambda \mathbf{v}_r) : \ \forall h \in H\}$$

where:

$$\mathbf{v}_i = (v_{i,0}, \ldots, v_{i,n_i}) \in T_i \ \ \dim_{\mathbb{C}}(T_i) = n_i + 1$$

Consider the H orbit of a point $(\mathbf{v}_1, \dots, \mathbf{v}_r)$:

$$
\begin{aligned}
H \cdot (\mathbf{v}_1, \dots, \mathbf{v}_r) &= \{h \cdot (v_1, \dots, v_r) : \quad \forall h \in H\} \\
&= \{(h_1 \cdot \mathbf{v}_1, \dots, h_r \cdot \mathbf{v}_r) : \quad \forall h_1 \in H_1, \cdots, h_r \in H_r\} \\
&= \{(h_1 \cdot \mathbf{v}_1, 0, \dots, 0) : \quad h_1 \in H_1\} \oplus \dots \oplus \{(0, \dots, h_r \cdot \mathbf{v}_r) : \quad h_r \in H_r\}
\end{aligned}
$$

Thus:

$$
\begin{aligned}
CV &= \cup_{\lambda \in \mathbb{C}} \left(\{(h_1 \cdot \lambda \mathbf{v}_1, 0, \dots, 0) : \quad h_1 \in H_1\} \oplus \dots \oplus \{(0, \dots, 0, h_r \cdot \lambda \mathbf{v}_r) : \quad h_r \in H_r\}\right) \\
&= (\cup_{\lambda \in \mathbb{C}} \{(h_1 \cdot \lambda \mathbf{v}_1, 0, \dots, 0) : \quad h_1 \in H_1\}) \oplus (\cup_{\lambda \in \mathbb{C}} \{(0, \dots, 0, h_r \cdot \lambda \mathbf{v}_r) : \quad h_r \in H_r\}) \\
&= CV_1 \oplus \dots \oplus CV_r
\end{aligned}
$$

We see that each CV_i is a cone over a projective variety, as it is an affine variety invariant under the $\mathbb{C}^*$-action $z \mapsto \lambda z$. Hence:

$$V = V_1 * \dots * V_r$$

If V is smooth then each V_i is smooth by Corollary 7.8 $\square$

7.3 Statement and Proof of the Main Result

Without further ado, let us state the main theorem.

Theorem 7.11. *Suppose that (M, J) is a complex manifold with $c_1(M, J) < 0$. Let g be the Kähler-Einstein metric on (M, J) whose existence is guaranteed by Theorem 3.30. Then the following are equivalent.*

1. *There exists a smooth holonomy invariant projective variety $V \subset \mathbb{P}T_x^{1,0}M$ such that:*
   (a) *If $T_x^{'}$ is the smallest subspace of $T_x^{1,0}M$ such that $V \subset \mathbb{P}T_x^{'}$ and $T_x^{''}$ is its g-orthogonal complement, then $T^{''}$ extends to a holomorphic vector bundle $T^{''} \to M$.*
   (b) *There is a k-fold cover $p^{(k)} : M^{(k)} \to M$ such that*

   $$(p^{(k)})^* T^{''} = T_1 \oplus \dots \oplus T_s$$

   with each T_i a holomorphic sub-bundle.

(c) If m_i denotes the rank of T_i:

$$c_1(T_i)^{m_i} - \frac{2(m_i+1)}{m_i} c_2(T_i) \smile c_1(T_i)^{m_i-2} = 0 \tag{7.6}$$

2. The universal cover of M, $\tilde{M}$ has de Rham decomposition:

$$\tilde{M} \cong B_{m_1} \times \ldots \times B_{m_s} \times M_{s+1} \times M_{s+r} \tag{7.7}$$

Where each M_i is a bounded symmetric domain not of ball type determined uniquely by V. (We are omitting the g_i's and J_i's, but this is meant to be read as a Riemannian product of Kähler manifolds)

Proof. Let $\hat{p} : \tilde{M} \to M^{(k)}$ and $p : \tilde{M} \to M$ be the respective covering maps. Obviously $p = \hat{p} \circ p^{(k)}$. The metric g will pull-back to Kähler-Einstein metrics $g^{(k)}$ on $(M^{(k)}, J^{(k)})$ and $\tilde{g}$ on $(\tilde{M}, \tilde{J})$. Because $T^{1,0}_{\tilde{x}}\tilde{M} \cong T^{1,0}_{p(\tilde{x})}M$ for any $\tilde{x} \in \tilde{M}$ satisfying $p(\tilde{x}) = x$, we have a decomposition $T^{1,0}_{\tilde{x}}\tilde{M} = \tilde{T}^{'}_{\tilde{x}} \oplus \tilde{T}^{''}_{\tilde{x}}$ with $\tilde{T}^{'}_{\tilde{x}} \cong T^{'}_x$, $\tilde{T}^{''}_{\tilde{x}} \cong T^{''}_x$ and $\tilde{V} \subset \tilde{T}'$ such that $\tilde{V} \cong V$. Moreover, as $Hol_{\tilde{x}}(\tilde{M}, \tilde{g}) = Hol_x(M, g)_0$ (cf. the first remark of 3.12) $\tilde{V}$ is $Hol_{\tilde{x}}(\tilde{M}, \tilde{g})$-invariant, so $\tilde{T}^{'}_{\tilde{x}} = \mathrm{span}_{\mathbb{C}}(\tilde{V})$ is a $Hol_{\tilde{x}}(\tilde{M}, \tilde{g})$-invariant subspace. But then $\tilde{T}^{''}_{\tilde{x}}$, the orthogonal complement of $\tilde{T}^{'}_{\tilde{x}}$, is also $Hol_{\tilde{x}}(\tilde{M}, \tilde{g})$-invariant, and so by the de Rham decomposition theorem (Theorem 3.16) we have that[3]:

$$(\tilde{M}, \tilde{J}, \tilde{g}) \cong (M^{'}, J^{'}, g^{'}) \times (M^{''}, J^{''}, g^{''})$$

In addition, we know, by the third item of Theorem 7.4, that $c_1(M^{'}) < 0$ and $c_1(M^{''}) < 0$. Using the assumption that $(p^{(k)})^*T^{''}$ splits, we have that:

$$T^{1,0}M^{''} \cong \tilde{T}^{''} = \hat{p}^*(p^{(k)})^*T^{''} = \hat{p}^*T_1 \oplus \ldots \oplus \hat{p}^*T_s$$

and so by 7.2:

$$(M^{''}, J^{''}, g^{''}) \cong (M_1, J_1, g_1) \times \ldots \times (M_s, J_s, g_s)$$

and $\hat{p}^*T_i \cong T^{1,0}M_i$ for all i. Again by Theorem 7.4 we have that $c_1(M_i) < 0$ for all i. Using the assumption 7.6 and arguing as we did in the proof of Theorem 7.5, we see that $M_i = B_{m_i}$ for $i = 1, \ldots, s$ with g_i the Bergmann metric and J_i the induced complex structure.

Now consider $\tilde{T}^{''}_{\tilde{x}}$. Let

$$\tilde{T}^{''} = T_{s+1,\tilde{x}} \oplus \ldots \oplus T_{s+r,\tilde{x}}$$

[3]Because $\tilde{T}^{'}_{\tilde{x}}$ and $\tilde{T}^{''}_{\tilde{x}}$ are not necessarily irreducible $Hol_{\tilde{x}}(\tilde{M}, \tilde{g})$ representations, $M^{'}$ and $M^{''}$ need not be irreducible

be the decomposition into irreducible $Hol_{\tilde{x}}(\tilde{M}, \tilde{g})$ representations and:

$$(M^{''}, J^{''}, g^{''}) \cong (M_{s+1}, J_{s+1}, g_{s+1}) \times \ldots \times (M_{s+r}, J_{s+r}, g_{s+r})$$

The corresponding de Rham decomposition. By lemma 7.10 we have that:

$$V = V_{s+1} * \ldots * V_{s+r}$$

with each of the $V_{s+i} \subset \mathbb{P}T_{s+i,\tilde{x}} \cong \mathbb{P}T^{1,0}_{x_{s+i}} M_{s+i}$ irreducible, proper and $Hol(M_i, g_i)$-invariant. Thus $Hol(M_i, g_i) \neq U(m_i)$ and so by theorem 3.31 each $(M_{s+i}, J_{s+i}, g_{s+i})$ is a bounded symmetric domain, and because V_i is smooth it is the first Mok Characteristic Variety. Moreover if we know $\dim(M_{s+i}) = \dim(T_{s+i,\tilde{x}})$ and $\dim(V_{s+i})$ then we can determine which bounded symmetric domain M_i is.
Conversely, if

$$\tilde{M} \cong B_{m_1} \times \ldots \times B_{m_s} \times M_{s+1} \times M_{s+r}$$

Applying Theorem 7.5 with $M^{'} = M_{s+1} \times M_{s+r}$ we get the existence of $M^{(k)}$ such that:

$$T^{1,0}M^{(k)} = T_1 \oplus \ldots \oplus T_s \oplus T^{'}$$

as required. Since each M_{s+i} is not of ball type, let V_{s+i} denote its first Mok characteristic variety. Then $V = V_{s+1} * \ldots * V_{s+r} \subset \mathbb{P}T^{1,0}_{\tilde{x}} M^{'}$ descends to a smooth, proper $Hol(M, g)$-invariant subvariety of $\mathbb{P}T^{'}$ □

Remark 7.12. We would like to be able to detect the existence of the holonomy invariant variety $V \subset \mathbb{P}T^{1,0}_x M$ indirectly. That is, without knowing what the holonomy group is. This is done in [CDS] by using a global section σ of $T^{1,0}M \otimes T^{1,0}M \otimes (T^{1,0})^*M \otimes (T^{1,0})^*M \cong \mathrm{End}(T^{1,0}M \otimes (T^{1,0})^*M)$. By Theorem 5.3if such a global section exists it must be parallel. So σ_x is holonomy invariant. By considering the intersection $\ker(\sigma) \cap \{t \otimes t^* \in T^{1,0}M \otimes (T^{1,0})^*M$ and take the closure of the projection of this set onto $T^{1,0}M$ we obtain a cone over a holonomy invariant projective variety.

Appendix A

The Miyaoka-Yau inequality

In this appendix we prove the following relationship between the Chern classes of a Kähler manifold with negative first Chern class.

Theorem A.1. *Let (M, J, g) be a Kähler manifold of dimension n with $c_1(M, J) < 0$. Then, denoting for brevity $c_j = c_j(M, J)$:*

$$\frac{n}{2(n+1)} c_1^n - c_2 \smile c_1^{n-2} \leq 0 \tag{A.1}$$

with equality holding if and only if M is uniformised by the n-dimensional ball.

Our proof shall follow closely that of [Cho08].

Remark A.2. For two cohomology classes $\alpha \in H^p(M, \mathbb{R})$ and $\beta \in H^q(M, \mathbb{R})$ the cup product $\alpha \smile \beta$ gives a cohomology class in $H^{p+q}(M, \mathbb{R})$. In practice however, we shall represent these cohomology classes as forms, in which case the cup product is given by the exterior product:

$$c_2 \smile c_1^{n-2} = [c_2(M, J, g) \wedge c_1(M, J, g)^{n-2}] \tag{A.2}$$

where $c_j(M, J, g)$ denotes the j-th Chern form calculated using the Chern connection of (M, J, g) (see section *which section*). Of course c_1^{n-2} is the $(n-2)$-fold cup product of c_1 with itself, or equivalently the class of the $(n-2)$-fold wedge product of $c_1(M, J, g)$ with itself. The inequality (A.1) makes sense because $(c_1^n - \frac{2(n+1)}{n} c_2 \smile c_1^{n-2}) \in H^{2n}(M, \mathbb{R}) \cong \mathbb{R}$, where the identification with $\mathbb{R}$ is given by mapping classes $\gamma \in H^{2n}(M, \mathbb{R})$ to their value when paired with the fundamental homology class $[M] \in H_{2n}(M, \mathbb{R})$. More concretely, (A.1) is equivalent to the statement:

$$\int_M \left(c_1(M, J, g)^n - \frac{2(n+1)}{n} c_2(M, J, g) \wedge c_1(M, J, g)^{n-2} \right) \leq 0 \tag{A.3}$$

where $c_i(M, J, g)$ is the i-th Chern form computed using the Chern connection associated to g, for any Kähler metric g on (M, J).

We shall perform this calculation in coordinates, and use the fact that by (*eqref the theorem containing Yau and Aubin's theorem*) A Kähler manifold with negative first Chern class possesses a Kähler-Einstein metric, which henceforth shall be denoted as g. Our approach to the proof shall follow that of [Cho08].

Firstly, recall that since (M, J, g) is Kähler, F_D is the complexification of F_∇, where as before, ∇ denotes the Levi-Civita connection. When working in coordinates, it is useful (and indeed much more common in Riemannian geometry) to work not with the curvature endomorphism F_∇ but with a 4-tensor R defined as:

$$R(X, Y, Z, W) = g(F_\nabla(X, Y)Z, W) \tag{A.4}$$

Suppose that $\{X_1, \dots, X_n, X_{n+1} = JX_1, \dots, X_{2n} = JX_n\}$ is an orthonormal basis for TM, then:

$$R_{\alpha\beta\gamma\delta} = R(X_\alpha, X_\beta, X_\gamma, X_\delta) \tag{A.5}$$

where here Greek indices take values between 1 and $2n$. We now extend this tensor by $\mathbb{C}$-linearity to $T^{\mathbb{C}}M$. Now, as before, we choose the basis for $T^{\mathbb{C}}M$ $\{Z_1, \dots, Z_n, \bar{Z}_1, \dots, \bar{Z}_n\}$ where:

$$Z_j = \frac{1}{\sqrt{2}}(X_j - iJX_j) \tag{A.6}$$

With respect to this basis we denote components of (the extension of) R as R_{ijkl} or $R_{\bar{i}jk\bar{l}}$ and so on. From here on we shall assume that Latin indices take values between 1 and n, while Greek indices, unless otherwise stated, will denote an index between 1 and n that may be either barred or unbarred. Note that the extension of R to $T^{\mathbb{C}}M$ possesses all of the original symmetries of R as given in *Reference the correct section here.* For completeness, we record them here in index notation:

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} \tag{A.7}$$

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} \tag{A.8}$$

$$R_{\alpha\beta\gamma\delta} + R_{\gamma\alpha\beta\delta} + R_{\beta\gamma\alpha\delta} = 0 \tag{A.9}$$

Lemma A.3. *If R is the extension of the Riemann curvature tensor to $T^{\mathbb{C}}M$ then:*

$$R_{\mu\nu kl} = R_{\mu\nu\bar{k}\bar{l}} = 0 \tag{A.10}$$

Proof. We use the fact that since $\nabla J = J\nabla$, $F_\nabla(X, Y)JZ = JF_\nabla(X, Y)Z$. Then taking two $(1, 0)$ vectors such as Z_k and Z_l , and two arbitrary basis vectors Z_μ, Z_ν, we have that:

$$
\begin{aligned}
R_{\mu\nu kl} &= g(R(Z_\mu, Z_\nu)Z_k, Z_l) && \text{(A.11)}\\
&= g(JR(Z_\mu, Z_\nu)Z_k, JZ_l) \text{ Since } g \text{ is Hermitian} && \text{(A.12)}\\
&= g(R(Z_\mu, Z_\nu)JZ_k, JZ_l) && \text{(A.13)}\\
&= g(R(Z_\mu, Z_\nu)(iZ_k), (iZ_l)) \text{ By the definition of a } (1,0) \text{ vector} && \text{(A.14)}\\
&= i^2 g(R(Z_\mu, Z_\nu)Z_k, Z_l) \text{since we have extended by } \mathbb{C}\text{-linearity} && \text{(A.15)}\\
&= -R_{\mu\nu kl} && \text{(A.16)}
\end{aligned}
$$

Thus $R_{\mu\nu kl} = 0$. The proof that $R_{\mu\nu\bar{k}\bar{l}} = 0$ is identical. □

Since $R_{\mu\nu kl} = R_{kl\mu\nu}$ we have also that:

$$R_{kl\mu\nu} = R_{\bar{k}\bar{l}\mu\nu} = 0 \tag{A.17}$$

Putting this together, we get that the only independent, non-zero components of R on a Kähler manifold are of the form $R_{i\bar{j}k\bar{l}}$, since for a component like $R_{\bar{i}\bar{j}\bar{k}l}$ we have that:

$$R_{\bar{i}j\bar{k}l} = R_{j\bar{i}l\bar{k}} \tag{A.18}$$

Note that if we apply this to (A.9), we get:

$$
\begin{aligned}
R_{i\bar{j}k\bar{l}} &= -R_{ki\bar{j}\bar{l}} - R_{\bar{j}ki\bar{l}}\\
&= -R_{\bar{j}ki\bar{l}}\\
&= R_{k\bar{j}i\bar{l}}
\end{aligned}
$$

This is the *Bianchi identity for Kähler manifolds.* As in section (*reference initial bit about hermitian metrics*) we denote by h the extension of g to $T^{\mathbb{C}}M$ by $\mathbb{C}$-sesquilinearity. Thus h is an hermitian metric on M in the usual sense of giving a positive definite, sesquilinear form on $T_p^{\mathbb{C}}M$ for all p which varies smoothly with p, and

$$h(X, Y) = g(X, \bar{Y}) \tag{A.19}$$

or in indices:

$$h(X, Y) = g_{i\bar{j}} X^i \bar{Y}^{\bar{j}} \tag{A.20}$$

Now in Riemannian geometry we use the identification between TM and T^*M given by g (and its inverse) to 'raise' and 'lower' indices, writing:

$$R^{\alpha}_{\beta\gamma\delta} = g^{\alpha\epsilon} R_{\epsilon\beta\gamma\delta} \tag{A.21}$$

Here the Greek indices take values between 1 and $2n$, as in (A.5). We may do the same thing in the complex case, but there is a slight difference. $h \in \mathcal{A}^0((T^{1,0})^* \otimes (T^{0,1})^*)$ identifies $T^{(1,0)}M$ (respectively $T^{(0,1)}M$) with the *conjugate* of its dual bundle, $(T^{0,1})^*$ (respectively $(T^{1,0})^*$). Thus, when raising and lowering indices, we must remember to change from barred to unbarred indices and vice versa. For example:

$$g^{\bar{i}n} R_{n\bar{k}l\bar{m}} = R^{\bar{i}}{}_{\bar{k}l\bar{m}} \tag{A.22}$$

Extending h to arbitrary tensor bundles This extension may most easily be seen in coordinates. Recall that for $\xi = \xi^i Z_i$ and $\zeta = \zeta^i Z_i$ in $T^{1,0}M$, we define:

$$(\xi, \zeta) = h(\xi, \bar{\zeta}) = h_{i\bar{j}} \xi^i \bar{\zeta}^{\bar{j}} \tag{A.23}$$

Thus we extend to tensors with k components by contracting with k copies of h, with indices up or down as appropriate. For example:

$$||R||^2 = h^{i_1\bar{i}_2} h^{\bar{j}_1 j_2} h^{k_1\bar{k}_2} h^{\bar{l}_1 l_2} R_{i_1\bar{j}_1 k_1\bar{l}_1} (\bar{R})_{\bar{i}_2 j_2\bar{k}_2 l_2} \tag{A.24}$$

Later, we shall use that, for tensors like R, which are extensions of real tensors, we have that:

$$(\bar{R})_{\bar{i}j\bar{k}l} = \bar{R}(\bar{Z}_i, Z_j, \bar{Z}_k Z_l) = \overline{R(\bar{Z}_i, Z_j, \bar{Z}_k Z_l)} = R(Z_i, \bar{Z}_j, Z_k, \bar{Z}_l) = R_{i\bar{j}k\bar{l}} \tag{A.25}$$

Let us denote by $\{\varphi^1, \ldots, \varphi^n\}$ the unitary co-frame dual to $\{Z_1, \ldots, Z_n\}$. With respect to this frame:

$$h = \delta_{i\bar{j}} \varphi^i \bar{\varphi}^j \tag{A.26}$$

$$\omega = i\delta_{i\bar{j}} \varphi^i \wedge \bar{\varphi}^j \tag{A.27}$$

By the earlier discussion the curvature of the Chern connection, F_D, is given as:

$$F_D = R^i_{jk\bar{l}} \varphi^k \wedge \bar{\varphi}^l \tag{A.28}$$

We express three objects introduced in Chapter one, the Ricci form and the first two Chern forms in terms of this frame too:

$$\rho = i\text{trace}(F_D) = iR^j{}_{jk\bar{l}} \varphi^k \wedge \bar{\varphi}^l \tag{A.29}$$

Thus by (3.28)

$$c_1(M) = [\frac{i}{2\pi} R^j{}_{jk\bar{l}} \varphi^k \wedge \bar{\varphi}^l] \tag{A.30}$$

and:

$$c_2(M) = [\frac{-1}{8\pi^2}\left(R^i{}_{ik\bar{l}}R^j{}_{jm\bar{n}} - R^j{}_{ik\bar{l}}R^i{}_{jm\bar{n}}\right)\varphi^k \wedge \bar{\varphi}^l \wedge \varphi^m \wedge \bar{\varphi}^n] \tag{A.31}$$

We also introduce a quantity that should be familiar from Riemannian geometry, the scalar curvature:

$$\sigma = R^i{}_{ij}{}^j \tag{A.32}$$

Again following [Cho08] we do two preliminary calculations and collect them in the following lemma:

Lemma A.4. *Let (M, J, g) be a Kähler manifold. We denote by c_1 and c_2 the Chern forms $c_1(M, J, g)$ and $c_2(M, J, g)$ respectively, and of course by ω_g the Kähler form. Then:*

$$c_1^2 \wedge \omega_g^{n-2} = \frac{1}{4\pi^2 n(n-1)}(\sigma^2 - ||\rho||^2)\omega_g^n \tag{A.33}$$

and

$$c_2 \wedge \omega_g^{n-2} = \frac{1}{8\pi^2 n(n-1)}(\sigma^2 - 2||\rho||^2 + ||R||^2)\omega_g^n \tag{A.34}$$

Proof. As before, we take $\{\varphi^1, \dots, \varphi^n\}$ to be an orthonormal frame. Then:

$$\omega_g^{n-2} = (i)^{n-2}(n-2)!\sum_{p,q} \varphi^1 \wedge \bar{\varphi}^1 \wedge \dots \widehat{\varphi^p \wedge \bar{\varphi}^p} \wedge \dots \wedge \widehat{\varphi^q \wedge \bar{\varphi}^q} \wedge \dots \wedge \varphi^n \bar{\varphi}^n \tag{A.35}$$

where the hat denotes that a term is omitted, and:

$$\omega_g^n = (i)^n(n)! \wedge \bar{\varphi}^1 \wedge \dots \wedge \varphi^n \bar{\varphi}^n \tag{A.36}$$

Now using the coordinate form of c_1 given in (A.29) we get that:

$$c_1^2 \wedge \omega_g^{n-2} = (\frac{-1}{4\pi^2})(i)^{n-2}(n-2)!\sum_{p,q}\Big(R^j{}_{jk\bar{l}}R^i{}_{im\bar{n}}\varphi^k \wedge \bar{\varphi}^l \wedge \varphi^m \wedge \bar{\varphi}^n \wedge \varphi^1 \wedge \bar{\varphi}^1 \dots$$
$$\dots \widehat{\varphi^p \wedge \bar{\varphi}^p} \dots \wedge \dots \wedge \widehat{\varphi^q \wedge \bar{\varphi}^q} \wedge \dots \wedge \varphi^n \bar{\varphi}^n\Big)$$

If we examine this rather imposing sum of wedge products more closely we see that there are effectively three possibilities:

1. $k = l = p$ and $m = n = q$ in which case

$$\varphi^k \wedge \bar{\varphi}^l \wedge \varphi^m \wedge \bar{\varphi}^n = \varphi^p \wedge \bar{\varphi}^p \wedge \varphi^q \wedge \bar{\varphi}^q \tag{A.37}$$

and the total wedge product becomes:

$$\varphi^1 \wedge \bar{\varphi}^1 \ldots \wedge \varphi^n \wedge \bar{\varphi}^n = \frac{1}{n!}\omega_g^n \tag{A.38}$$

2. $k = n = p$ and $l = m = q$ in which case:

$$\varphi^k \wedge \bar{\varphi}^l \wedge \varphi^m \wedge \bar{\varphi}^n = -\varphi^p \wedge \bar{\varphi}^p \wedge \varphi^q \wedge \bar{\varphi}^q \tag{A.39}$$

and the total wedge product becomes:

$$\varphi^1 \wedge \bar{\varphi}^1 \ldots \wedge \varphi^n \wedge \bar{\varphi}^n = -\frac{1}{n!}\omega_g^n \tag{A.40}$$

3. In any other case we have a repeated factor of φ^j or $\bar{\varphi}^j$ for some index j and thus the total wedge product is equal to zero.

Thus:

$$c_1^2 \wedge \omega_g^{n-2} = (\frac{-1}{4\pi^2})(i)^{-2}\frac{1}{n(n-1)}\left(R^j{}_{jp\bar{p}}R^i{}_{iq\bar{q}} - R^j{}_{jp\bar{q}}R^i{}_{iq\bar{p}}\right)\omega_g^n \tag{A.41}$$

and now we use that, since $h^{i\bar{j}} = \delta^{i\bar{j}}$ and $R_{i\bar{j}k\bar{l}} = \bar{R}_{\bar{i}j\bar{k}l}$:

$$\sigma = R^j{}_{jp}{}^p = \delta^{p\bar{r}}R^j{}_{jp\bar{r}} \tag{A.42}$$

$$= R^j{}_{jp\bar{p}} \tag{A.43}$$

and similarly for the other terms on the right hand side of (A.41) to get:

$$\begin{aligned} c_1^2 \wedge \omega_g^{n-2} &= (\frac{-1}{4\pi^2})(i)^{-2}\frac{1}{n(n-1)}\left(||\sigma||^2 - ||\rho||^2\right)\omega_g^n \\ &= \frac{1}{4\pi^2 n(n-1)}\left(||\sigma||^2 - ||\rho||^2\right)\omega_g^n \end{aligned}$$

And so we have the first equality. For the second equality we perform a similar calculation. As before we get:

$$\begin{aligned} c_2 \wedge \omega_g^{n-2} = \frac{-1}{8\pi^2}\sum_{p,q}\left(R^i{}_{ik\bar{l}}R^j{}_{jm\bar{n}} - R^j{}_{ik\bar{l}}R^i{}_{jm\bar{n}}\right)\varphi^k \wedge \bar{\varphi}^l \wedge \varphi^m \wedge \bar{\varphi}^n \wedge \ldots \\ \wedge \varphi^1 \wedge \bar{\varphi}^1 \ldots \wedge \widehat{\varphi^p \wedge \bar{\varphi}^p} \ldots \wedge \ldots \wedge \widehat{\varphi^q \wedge \bar{\varphi}^q} \wedge \ldots \wedge \varphi^n \bar{\varphi}^n \end{aligned}$$

and as before the only possibilities are $k = l = p$ and $m = n = q$ or $k = n = p$ and $l = m = q$, and so we get:

$$c_2 \wedge \omega_g^{n-2} = \frac{-1}{8\pi^2}(i)^{-2}\frac{1}{n(n-1)}\left(R^i{}_{ip\bar{p}}R^j{}_{jq\bar{q}} - R^i{}_{ip\bar{q}}R^j{}_{jq\bar{p}} - R^j{}_{ip\bar{p}}R^i{}_{jq\bar{q}} + R^j{}_{ip\bar{q}}R^i{}_{jq\bar{p}}\right)\omega_g^n$$

Again using the fact that $h^{i\bar{j}} = \delta^{i\bar{j}}$ and $\bar{R}_{\bar{i}j\bar{k}l} = R^{i\bar{j}k\bar{l}}$ we get *also need something like* $R^{j}_{ip\bar{p}} = R^{p}_{pi\bar{j}}$:

$$\begin{aligned} c_2 \wedge \omega_g^{n-2} &= \frac{-1}{8\pi^2}(i)^{-2}\frac{1}{n(n-1)}\left(||\sigma||^2 - ||\rho||^2 - ||\rho||^2 + ||R||^2\right)\omega_g^n \\ &= \frac{1}{8\pi^2}\frac{1}{n(n-1)}\left(||\sigma||^2 - 2||\rho||^2 + ||R||^2\right)\omega_g^n \end{aligned}$$

□

Before we prove the main result, we need one more lemma.

Lemma A.5 (Lemma 1.38 in [Cho08])**.** *Suppose that* (M, J, g) *is an* n*-dimensional Kähler-Einstein manifold, with* $\rho = \frac{\sigma}{n}\omega$*. Then:*

$$\frac{2\sigma^2}{n(n+1)} - ||R||^2 \leq 0 \tag{A.44}$$

with equality holding if and only if (M, J, g) *has constant holomorphic sectional curvature.*

Note that in the proof we shall use the following result, which we state here without proof:

Lemma A.6. *A Kähler-Einstein manifold* (M, J, g)*, with scalar curvature* σ*, is of constant sectional curvature if and only if:*

$$R_{i\bar{j}k\bar{l}} = \frac{\sigma}{n(n+1)}\left(\delta_{i\bar{j}}\delta_{k\bar{l}} + \delta_{i\bar{l}}\delta_{k\bar{j}}\right) \tag{A.45}$$

Proof. A proof may be found in [**?**], proposition 7.6. Note however that the constant of proportionality used there is related to the sectional curvature, c, and not the scalar curvature σ as we have here. □

Proof. (Of lemma A.5) We consider the trace-free part of the curvature tensor:

$$T_{i\bar{j}k\bar{l}} = R_{i\bar{j}k\bar{l}} - \frac{\sigma}{n(n+1)}\left(\delta_{i\bar{j}}\delta_{k\bar{l}} + \delta_{i\bar{l}}\delta_{k\bar{j}}\right) \tag{A.46}$$

we take the magnitude of this tensor. First, note that:

$$\begin{aligned}
||\frac{\sigma}{n(n+1)}\left(\delta_{i\bar{j}}\delta_{k\bar{l}}+\delta_{i\bar{l}}\delta_{k\bar{j}}\right)||^2 &= \frac{\sigma^2}{n^2(n+1)^2}\left(\delta^{\bar{i}j}\delta^{\bar{k}l}+\delta^{\bar{i}l}\delta^{\bar{k}j}\right)\left(\delta_{j\bar{i}}\delta_{l\bar{k}}+\delta_{l\bar{i}}\delta_{j\bar{k}}\right) \\
&= \frac{\sigma^2}{n^2(n+1)^2}\left(\delta^j_j\delta^l_l+\delta^j_l\delta^l_j+\delta^l_j\delta^j_l+\delta^l_l\delta^j_j\right) \\
&= \frac{\sigma^2}{n^2(n+1)^2}\left(n^2+\delta^j_j+\delta^l_l+n^2\right) \\
&= \frac{\sigma^2}{n^2(n+1)^2}\left(2n^2+2n\right) \\
&= \frac{2\sigma^2}{n(n+1)}
\end{aligned}$$

Secondly:

$$\begin{aligned}
\frac{-\sigma}{n(n+1)}R^{\bar{i}j\bar{k}l}(\delta_{j\bar{i}}\delta_{l\bar{k}}+\delta_{l\bar{i}}\delta_{j\bar{k}}) &= \frac{-\sigma}{n(n+1)}\left(R^{\bar{i}j\bar{k}l}\delta_{j\bar{i}}\delta_{l\bar{k}}+R^{\bar{k}j\bar{i}l}\delta_{l\bar{i}}\delta_{j\bar{k}}\right) \\
&= \frac{-\sigma}{n(n+1)}\left(R^{\bar{i}}{}_{\bar{i}}{}^{\bar{k}}{}_{\bar{k}}+R^{\bar{k}}{}_{\bar{k}}{}^{\bar{i}}{}_{\bar{i}}\right) \\
&= \frac{-2\sigma}{n(n+1)}\left(R^{\bar{i}}{}_{\bar{i}}{}^{\bar{k}}{}_{\bar{k}}\right) \\
&= \frac{-2\sigma^2}{n(n+1)}
\end{aligned}$$

Where in the first line we have used the Bianchi identity for Kähler manifolds, and in the second line we have used the identity (*should probably reference the identity used here*). Finally note that since this is a real quantity:

$$(\delta\delta, T) = \overline{\frac{-2\sigma^2}{n(n+1)}} = \frac{-2\sigma^2}{n(n+1)}$$

and probably need to explain this better as well. Putting this all together, we have:

$$\begin{aligned}
0 \geq ||T_{i\bar{j}k\bar{l}}||^2 = R^{\bar{i}j\bar{k}l}R_{i\bar{j}k\bar{l}} - \frac{4\sigma^2}{n(n+1)} + \frac{2\sigma^2}{n(n+1)} \\
= ||R||^2 - \frac{2\sigma^2}{n(n+1)}
\end{aligned}$$

Giving the required inequality. Now if equality holds we have that $||T||^2 = 0$ and so $T = 0$. Thus:

$$R_{i\bar{j}k\bar{l}} = \frac{\sigma}{n(n+1)}\left(\delta_{i\bar{j}}\delta_{k\bar{l}}+\delta_{i\bar{l}}\delta_{k\bar{j}}\right) \tag{A.47}$$

and so by lemma A.6 we have that (M, J, g) has constant sectional curvature. □

Now we can put these pieces together to prove the main result of this section:

Proof. (Of theorem A.1) Suppose that $M, J)$ has negative first Chern class. Then by *reference Aubin/Yau theorem here* it possesses a unique Kähler-Einstein metric g. With respect to this metric, we get that:

$$\rho = \frac{\sigma}{n}\omega$$

thus:

$$||\rho||^2 = \frac{\sigma^2}{n^2(n)} = \frac{\sigma^2}{n}$$

From lemma A.4, equation (A.33) we get that:

$$c_1^2 \wedge \omega_g^{n-2} = \frac{1}{4\pi^2 n(n-1)}\left(\sigma^2 - \frac{\sigma^2}{n}\right)\omega_g^n = \frac{\sigma^2}{4\pi^2 n^2}\omega_g^n \tag{A.48}$$

Thus:

$$\frac{n}{2(n+1)}c_1^2 \wedge \omega_g^{n-2} = \frac{\sigma^2}{8\pi^2 n(n+1)}\omega_g^n \tag{A.49}$$

From (A.34) we get:

$$\begin{aligned} c_2 \wedge \omega_g^{n-2} &= \frac{1}{8\pi^2 n(n-1)}\left(\sigma^2 - 2(\frac{\sigma}{n}) + ||R||^2\right)\omega_g^n \\ &= \frac{\sigma^2(n-2)}{8\pi^2 n^2(n-1)}\omega_g^n + \frac{1}{8\pi^2 n(n-1)}||R||^2\omega_g^n \end{aligned}$$

So:

$$\begin{aligned} \frac{n}{2(n+1)}c_1^2 \wedge \omega_g^{n-2} - c_2 \wedge \omega_g^{n-2} &= \left(\frac{\sigma^2}{8\pi^2 n(n+1)} - \frac{\sigma^2(n-2)}{8\pi^2 n^2(n-1)}\right)\omega_g^n - \frac{1}{8\pi^2 n(n-1)}||R||^2\omega_g^n \\ &= \frac{\sigma^2}{8\pi^2 n}\left(\frac{1}{n+1} - \frac{n-2}{n(n-1)}\right)\omega_g^n - \frac{1}{8\pi^2 n(n-1)}||R||^2\omega_g^n \\ &= \frac{\sigma^2}{8\pi^2 n}\left(\frac{2}{n(n+1)(n-1)}\right) - \frac{1}{8\pi^2 n(n-1)}||R||^2\omega_g^n \\ &= \frac{1}{8\pi^2 n(n-1)}\left(\frac{2\sigma^2}{n(n+1)} - ||R||^2\right)\omega_g^n \\ &\leq 0 \end{aligned}$$

where the final inequality follows from lemma A.5. Now we have from *reference this lemma* that $c_1(M, J, g) = \frac{1}{2\pi}\rho = \frac{\sigma}{2\pi n}\omega$ so we can rewrite the above as:

$$\frac{n}{2(n+1)}c_1^2 \wedge \left((\frac{2\pi n}{\sigma})^{n-2}c_1^{n-2}\right) - c_2 \wedge \left((\frac{2\pi n}{\sigma})^{n-2}c_1^{n-2}\right) \leq 0 \tag{A.50}$$

thus of course:

$$\frac{n}{2(n+1)}c_1(M, J, g)^n - c_2(M, J, g) \wedge c_1^{n-2}(M, J, g) \leq 0 \tag{A.51}$$

since any other representative of the Chern class $c_i(M, J)$ can differ from $c_i(M, J, g)$ by at most a closed form, we get that:

$$\int_M \left(\frac{n}{2(n+1)} c_1(M, J, \tilde{g})^n - c_2(M, J, \tilde{g}) \wedge c_1^{n-2}(M, J, \tilde{g}) \right) \leq 0 \tag{A.52}$$

for all Kähler metrics $\tilde{g}$ on (M, J), giving the inequality part of theorem A.1. Now suppose that equality holds in (A.52) and denote by $\tilde{p} : \tilde{M} \to M$ the universal cover of M. Then $\tilde{p}^*(TM) = T\tilde{M}$ *check this* and moreover, we know that Chern classes 'commute' with the pullback operation. That is:

$$p^*(c_i(TM)) = c_i(p^*(TM)) \tag{A.53}$$

Of course pullback also commutes with cup products, so:

$$\begin{aligned} p^*(c_1(TM)^n) &= (p^*(c_1(TM)))^n \\ &= (c_1(p^*(TM)))^n \\ &= c_1(T\tilde{M})^n \end{aligned}$$

and similarly:

$$p^*(c_2(TM) \cup c_1(TM)^{n-2}) = c_2(T\tilde{M}) \cup c_1(T\tilde{M}) \tag{A.54}$$

Thus:

$$0 = p^* \left(\frac{n}{2(n+1)} c_1(TM)^n - c_2(TM) \smile c_1(TM)^{n-2} \right) \tag{A.55}$$

$$= \frac{n}{2(n+1)} c_1(T\tilde{M})^n - c_2(TM) \smile c_1(T\tilde{M})^{n-2} \tag{A.56}$$

and in particular for the unique Kähler-Einstein metric $\tilde{g}$ on $\tilde{M}$:

$$\int_M \left(\frac{n}{2(n+1)} c_1(\tilde{M}, \tilde{J}, \tilde{g})^n - c_2(\tilde{M}, \tilde{J}, \tilde{g}) \wedge c_1^{n-2}(\tilde{M}, \tilde{J}, \tilde{g}) \right) = 0 \tag{A.57}$$

But we know from (A.51) that the integrand is less than or equal to zero at every point. Thus we must have:

$$\frac{n}{2(n+1)} c_1(\tilde{M}, \tilde{J}, \tilde{g})^n - c_2(\tilde{M}, \tilde{J}, \tilde{g}) \wedge c_1^{n-2}(\tilde{M}, \tilde{J}, \tilde{g}) = 0 \tag{A.58}$$

Appealing to lemma A.5 we have that $(\tilde{M}, \tilde{J}, \tilde{g})$ has constant holomorphic sectional curvature, which is negative since:

$$c_1(M) = [\frac{\sigma}{2\pi n} \omega_g] \tag{A.59}$$

is negative. Thus applying theorem **??** we get that$(\tilde{M}, \tilde{J}, \tilde{g}) \cong (B^n, J, g_c)$. $\square$

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