Why the exponential map of a nilpotent Lie Algebra is surjective

Daniel Mckenzie

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1 A more constructive definition of nilpotency

Of the several ways to define what it means for a Lie algebra $\mathfrak{g}$ to be nilpotent, the most common seems to be this:

Definition 1 (Nilpotent Lie algebra, first attempt)**.** Define the *lower central series* of a Lie algebra $\mathfrak{g}$ as follows:

$$\mathcal{D}_1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \tag{1.1}$$
$$\mathcal{D}_k\mathfrak{g} = [\mathfrak{g}, \mathcal{D}_{k-1}\mathfrak{g}] \tag{1.2}$$

$\mathfrak{g}$ is said to be *nilpotent* if $\mathcal{D}_k = 0$ for some k.

It is not too difficult (c.f. [2]) to show that this is equivalent to:

Definition 2 (Nilpotent Lie algebra, second attempt)**.** A Lie algebra $\mathfrak{g}$ is *nilpotent* if there is a chain of ideals $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \ldots \mathfrak{g}_n = 0$ such that $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ is in the centre of $\mathfrak{g}/\mathfrak{g}_{i+1}$, which we shall denote as $Z(\mathfrak{g}/\mathfrak{g}_{i+1})$.

Observe that by the third isomorphism theorem:

$$(\mathfrak{g}/\mathfrak{g}_{i+1})/(\mathfrak{g}_i/\mathfrak{g}_{i+1}) \cong \mathfrak{g}/\mathfrak{g}_i \tag{1.3}$$

Although this is an *isomorphism of vector spaces, not of Lie algebras*, since the isomorphism theorems hold only in abelian categories, and the category of Lie algebras is not abelian. Nevertheless it does suggest the following short exact sequence:

$$0 \to \mathfrak{g}_i/\mathfrak{g}_{i+1} \to \mathfrak{g}/\mathfrak{g}_{i+1} \to \mathfrak{g}/\mathfrak{g}_i \to 0 \tag{1.4}$$

which is the same as saying that $\mathfrak{g}/\mathfrak{g}_{i+1}$ is a central extension of $\mathfrak{g}/\mathfrak{g}_i$ by $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ since by definition 2 $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ is in the centre of $\mathfrak{g}/\mathfrak{g}_{i+1}$. Setting $i = 0$ we get the following sequence:

$$0 \to \mathfrak{g}_0/\mathfrak{g}_1 = \mathfrak{g}/\mathfrak{g}_1 \to \mathfrak{g}/\mathfrak{g}_1 \to \mathfrak{g}/\mathfrak{g}_0 = 0 \to 0 \tag{1.5}$$

which says that $\mathfrak{g}/\mathfrak{g}_1 \subset Z(\mathfrak{g}/\mathfrak{g}_1)$ and so $\mathfrak{g}/\mathfrak{g}_1$ is abelian. Setting $i = n-1$ we get:

$$0 \to \mathfrak{g}_{n-1}/\mathfrak{g}_n = \mathfrak{g}_{n-1}/0 = \mathfrak{g}_{n-1} \to \mathfrak{g}/\mathfrak{g}_n = \mathfrak{g} \to \mathfrak{g}/\mathfrak{g}_{n-1} \to 0 \tag{1.6}$$

In other words our nilpotent lie algebra $\mathfrak{g}$ is obtained by n successive central extensions of an abelian Lie algebra. We take this as the definition of a nilpotent Lie Algebra:

Definition 3 (Nilpotent Lie algebra, third attempt)**.** A Nilpotent Lie algebra $\mathfrak{g}$ (of order n) is constructed by starting with an abelian Lie algebra $\mathfrak{g}_0$ and performing n successive central extensions:

$$0 \to \mathfrak{h}_i \xrightarrow{\tau_i} \mathfrak{g}_i \to \mathfrak{g}_{i-1} \to 0 \tag{1.7}$$

where $\mathfrak{h}_i$ is an abelian Lie algebra and τ_i places it in the centre of $\mathfrak{g}_i$. [Note that our notation has changed, now $\mathfrak{g} = \mathfrak{g}_n$. We shall use this notation consistently from here on]. We also have not defined what we mean by the order of the Lie algebra. We shall make this precise in the next section.

Two final comments are in order. The first is that we can give a similar definition for solvable Lie algebras, in terms of extensions:

Definition 4 (Solvable Lie algebras)**.** A solvable Lie algebra $\mathfrak{g}$ is constructed by starting with an abelian Lie algebra $\mathfrak{g}_0$ and performing n successive (*not necessarily central*, this will make a big difference later on) extensions such that:

$$0 \to \mathfrak{h}_i \xrightarrow{\tau_i} \mathfrak{g}_i \to \mathfrak{g}_{i-1} \to 0 \tag{1.8}$$

where $\mathfrak{h}_i$ is an abelian Lie algebra and $\tau_i(\mathfrak{h}_i)$ an ideal in $\mathfrak{g}_i$.

The second is that if we consider the exponential map exp as a functor which associates to a Lie algebra $\mathfrak{g}$ the unique simply-connected Lie group G having $\mathfrak{g}$ as its Lie algebra, it should be exact (I haven't been able to find a proof of this, but it should be true). This means that a short exact sequence like:

$$0 \to \mathfrak{h}_i \xrightarrow{\tau_i} \mathfrak{g}_i \to \mathfrak{g}_{i-1} \to 0 \tag{1.9}$$

will give a short exact sequence of groups:

$$0 \to H_i \xrightarrow{\tau_i} G_i \to G_{i-1} \to 0 \tag{1.10}$$

with $\tau_i(H_i) \subset Z(G_i)$. This suggests the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{h}_i \arrow[r] \arrow[d, "\exp_{\mathfrak{h}_i}"] & \mathfrak{g}_i \arrow[r] \arrow[d, "\exp_{\mathfrak{g}_i}"] & \mathfrak{g}_{i-1} \arrow[r] \arrow[d, "\exp_{\mathfrak{g}_{i-1}}"] & 0 \\
0 \arrow[r] & H_i \arrow[r] & G_i \arrow[r] & G_{i-1} \arrow[r] & 0
\end{tikzcd} \tag{1.11}$$

and a method for proving the surjectivity of $\exp_{\mathfrak{g}_i}$. We know that $\exp_{\mathfrak{h}_i}$ is surjective since $\mathfrak{h}_i$ is abelian (c.f. [2]).If we procede by induction, assuming $\exp_{\mathfrak{g}_{i-1}}$ is surjective, the following lemma appears to finish the proof:

Lemma 1.1 (The short five lemma)**.** Suppose A, B, C, D, E, F are objects in an abelian category (or in the category of groups) and we have the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & D \arrow[r] & E \arrow[r] & F \arrow[r] & 0 \end{tikzcd} \tag{1.12}$$

with the rows being exact. If f and h are epimorphisms (that is, in set-theoretic terms, they are surjective), then so is g. [c.f. [1]]

However, as usual, the devil is in the details. The objects in the sequence (1.11) do not lie in the same category, so we may not naïvely apply lemma (1.1). In the next section we shall show that, in the nilpotent case, we may change our perspective slightly and consider the relevant Lie algebras as Lie groups under a different binary operation.

2 Thinking of Lie algebras as groups

On an arbitrary Lie algebra $\mathfrak{h}$ we may define a new binary operation $*$ as follows:

$$X * Y = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [Y, X]] \ldots \tag{2.1}$$

where the sum continues indefinitely, and each new term is a nested bracket of a higher order. This is known as the *Baker-Campbell-Hausdorff formula*.This operation has the nice property (when we place suitable restrictions on X and Y)

$$\exp(X * Y) = \exp X \exp Y \tag{2.2}$$

which we shall use later. It is a fact that if $\mathfrak{h}$ is nilpotent the series in (2.1) truncates. We can also deduce this inductively from the third definition of a nilpotent Lie algebra (3) given earlier: given the short exact sequence of Lie algebras:

$$0 \to \mathfrak{h}_i \overset{\tau_i}{\to} \mathfrak{g}_i \overset{\tau_{i-1}}{\to} \mathfrak{g}_{i-1} \to 0 \tag{2.3}$$

where $\tau_i(\mathfrak{h}_i) \subset Z(\mathfrak{g}_i)$ suppose that all terms involving k nested brackets (which we shall write somewhat loosely as $[X, [Y, [X, \ldots]]]$) in $\mathfrak{g}_{i-1}$ vanish. Then given a term involving k nested commutators in $\mathfrak{g}_i$, $[X, [Y, [X, \ldots]]]$, we have $\tau_{i-1}([X, [Y, [X, \ldots]]]) = [\tau_{i-1}(X), [\tau_{i-1}(Y), [\tau_{i-1}(X), [\ldots]]] = 0$ so $[X, [Y, [X, \ldots]]] \in \tau_i(\mathfrak{h}_i) \subset Z(\mathfrak{g}_i)$ which implies that any term involving $k{+}1$ nested commutators $[Y, [X, Y, [X, \ldots]]]]$ will vanish (we may actually consider more general terms of k nested commutators involving $k+1$ distinct vectors and show that these too will vanish, but this

is not necessary for our purposes). The fact that (2.1) truncates makes $(\mathfrak{h}, *)$ into a *Lie group*. This is a standard result for nilpotent Lie algebras. the zero vector 0 is the identity for $*$ and the inverse of a vector X is $-X$. The only thing that needs checking (and which is true only if the underlying Lie algebra is nilpotent) is the associativity. Observe that by property (2.2) exp is now a *group homomorphism* [Note: I think this involves some checking. We know (2.2) will hold for a suitably small neighbourhood of the origin in $\mathfrak{h}$, but I think that because (2.1) truncates this will eliminate the problems with convergence that prevent us extending (2.1) to the entire Lie group. However I haven't proved this]. Also, any map $\tau : \mathfrak{g}_1 \to \mathfrak{g}_2$ of Lie algebras becomes a map of Lie groups since $\tau(X*Y) = \tau(X+Y+\frac{1}{2}[X,Y]+\ldots) = \tau(X)+\tau(Y)+[\tau(X),\tau(Y)]+\ldots = \tau(X)*\tau(Y)$
Returning to (1.11), in the nilpotent case (but *not* in the solvable case) we now have a commutative diagram *in the category of groups.* We may now apply lemma (1.1) and obtain the main result:

Theorem 2.1 (The exponential map of a nilpotent Lie algebra is surjective)**.** *Given a Lie group G with a lie algebra $\mathfrak{g}$, if $\mathfrak{g}$ is nilpotent then the map* $\exp : \mathfrak{g} \to G$ *is surjective.*

Proof. The proof is a summary of the above discussion. If $\mathfrak{g}$ is nilpotent we may construct it as a sequence of central extensions:

$$0 \to \mathfrak{h}_i \xrightarrow{\tau_i} \mathfrak{g}_i \to \mathfrak{g}_{i-1} \to 0 \qquad \mathfrak{h}_i \text{ abelian and } \tau_i(\mathfrak{h}_i) \subset Z(\mathfrak{g}_i) \tag{2.4}$$

where $\mathfrak{g}_0$ is an abelian Lie algebra and $\mathfrak{g} = \mathfrak{g}_n$. Each of the above short exact sequences (2.4) will give us the following commutative diagram in the category of groups:

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{h}_i \arrow[r] \arrow[d, "\exp_{\mathfrak{h}_i}"] & \mathfrak{g}_i \arrow[r] \arrow[d, "\exp_{\mathfrak{g}_i}"] & \mathfrak{g}_{i-1} \arrow[r] \arrow[d, "\exp_{\mathfrak{g}_{i-1}}"] & 0 \\ 0 \arrow[r] & H_i \arrow[r] & G_i \arrow[r] & G_{i-1} \arrow[r] & 0 \end{tikzcd} \tag{2.5}$$

where the map $\exp_{\mathfrak{h}_i}$ is surjective since $\mathfrak{h}_i$ is abelian. setting $i = 1$ we have:

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{h}_1 \arrow[r] \arrow[d, "\exp_{\mathfrak{h}_1}"] & \mathfrak{g}_1 \arrow[r] \arrow[d, "\exp_{\mathfrak{g}_1}"] & \mathfrak{g}_0 \arrow[r] \arrow[d, "\exp_{\mathfrak{g}_0}"] & 0 \\ 0 \arrow[r] & H_1 \arrow[r] & G_1 \arrow[r] & G_0 \arrow[r] & 0 \end{tikzcd} \tag{2.6}$$

Now $\exp_{\mathfrak{g}_0}$ is surjective since $\mathfrak{g}_0$ is abelian, therefore by the short five lemma (1.1), $\exp_{\mathfrak{g}_1}$. We may now proceed inductively, at each stage using the fact that $\exp_{\mathfrak{h}_i}$ is surjective and the hypothesis that $\exp_{\mathfrak{g}_{i-1}}$ is surjective, as well as the short five lemma (1.1) to conclude that $\exp_{\mathfrak{g}_i}$ is surjective.This implies that the map $\exp_{\mathfrak{g}_n} : \mathfrak{g} = \mathfrak{g}_n \to G_n$ is surjective, where G_n is simply connected. Now we know that G_n will be the universal covering group of our original group G so we have a covering group homomorphism $\pi : G_n \to G$. We can use the fact that the exponential $\exp : \mathfrak{g} \to G$ is defined as the unique map satisfying (c.f. [2], prop. 8.33)

1. $0 \in \mathfrak{g}$ should be carried to $e \in G$

2. the differential at 0 should be the identity

3. lines through the origin in $\mathfrak{g}$ should be carried to one parameter subgroups of G

to show that $\exp : \mathfrak{g} \to G$ is equal to the composition $\pi \circ \exp_{\mathfrak{g}_n}$ and so will be surjective (this may also require a bit of checking). □

References

[1] Saunders Mac Lane. *Categories for the Working Mathematician*. Springer-Verlag, New York, 1971.

[2] Joe Harris William Fulton. *Representation Theory*. Springer-Verlag, New York, 1991.