The induced representation and Frobenius reciprocity

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Abstract

Given a subgroup $H \leq G$, and a representation W of H, we may construct a representation of G, known as the *induced* representation and denoted as $\operatorname{Ind} W$. Fulton and Harris give a constructive definition of this object; we shall however give a more theoretical one based on the idea of change the underlying ring of a module. As we shall see, this definiton is conceptually clearer and has the Frobenius reciprocity theoerem as an easy and transparent corrollary.

1 Tensor products over arbitrary rings

Suppose S, R and T are (possibly non-commutative) rings. Let A be a left S- and a right R- module. We denote this by ${}_SA_R$. In addition, suppose B is a left R- and a right T- module. As before, we shall denote this as ${}_RB_T$. We form the tensor product $A \otimes B$ as follows:

1. Take the free vector space $\mathcal{K}$ generated by pairs $(a, b) : \quad a \in A, b \in B$.

2. Quotient out by the relations:

   (a) $(a + c, b) \sim (a, b) + (c, b)$

   (b) $(a, b + d) \sim (a, b) + (a, d)$

   (c) $(ar, b) \sim (a, rb)$

3. Denote the class $[(a, b)]$ as $a \otimes b$

Notice how we used the fact that A is a right R-module and B is a left R-module in relation 3. The resulting vector space is now a left S-module and a right T-module: ${}_SA \otimes B_T$.

2 Adjoint Functors

The theory of adjoint functors is too complicated to explain here (that is, I don't really understand it yet) but I think the simplest and most approachable definition of adjoint functors is this one from Wikipedia which is in terms of hom-sets

Definition 1. A hom-set is the set of all morphisms between two objects in a category C:

$$\hom(a, b) = \{f : f \text{ in } C, f : a \to b\} \tag{2.1}$$

hom-sets can carry extra structure. For example $\hom(a, a)$ is always a monoid, for any a in any category C. Also, if C is an *Abelian Category*, then $\hom(a, b)$ is always an abelian group (although I'm still not entirely sure how the binary operation is defined. We can put a subscript: $\hom_C(a, b)$ when we wish to emphasise that a, b and all the morphisms in $\hom(a, b)$ live in the category C.

Definition 2. Given two functors $F : C \to D$ and $G : D \to C$, suppose that for each $c \in C$, $d \in D$ we have an isomorphism Φ_{cd}:

$$\Phi_{cd} : \hom_D(Fc, d) \to \hom_C(c, Gd) \tag{2.2}$$

F and G are adjoint functors if, for every $c \in C$, $d \in D$ such a Φ_{cd} exists and it satisfies certain naturality conditions (which I don't fully understand).

The important part is this: let ${}_R\mathbf{mod}$ denote the category of left R modules and ${}_S\mathbf{mod}$ denote the category of left S modules (R and S are (possibly noncommutative) rings). Using notation as in section 1, if we have a bimodule ${}_RX_S$ then we can define a functor:

$$F :_S \mathbf{mod} \to_R \mathbf{mod} \tag{2.3}$$
$${}_SY \mapsto X \otimes Y \tag{2.4}$$
$$\tag{2.5}$$

If we recall that $\hom_R({}_RX_S,_R Z)$ (this is a slight abuse of notation. We should write $\hom_{_R\mathbf{mod}}({}_RX_S,_R Z)$ but this becomes cumbersome) is a *left* S module we can define a second functor, G:

$$G :_R \mathbf{mod} \to_S \mathbf{mod} \tag{2.6}$$
$${}_RU \mapsto_S (\hom_R(X, Y)) \tag{2.7}$$
$$\tag{2.8}$$

The crux is that F and G are adjoint functors (although I have no idea how to prove this). This means that given any ${}_RY$ and ${}_SZ$ we have a natural isomorphism

$$\hom_R(FZ, Y) \cong \hom_S(Z, GY) \tag{2.9}$$

More transparently:

$$\hom_R(X \otimes Z, Y) \cong \hom_S(Z, (\hom_R(X, Y)) \tag{2.10}$$

3 The Induced representation

Let us now get to the point. If $H \leq G$ and W is an H-representation, this means we have a homomorphism $\rho : H \to \mathbb{GL}(W)$. Equivalently, we can say we have a ring homomorphism from the group ring $\mathbb{C}H$ into the ring of endomorphisms of W, $\mathrm{End}(W)$: $\rho : \mathbb{C}H \to \mathrm{End}\, W$. This makes $\mathrm{End}\, W$ a left $\mathbb{C}H$ module. $\mathbb{C}G$ is of course a left $\mathbb{C}G$-module, but it is also a right $\mathbb{C}H$ module, so we can take the tensor product $\mathbb{C}G \otimes W$. The resulting $\mathbb{C}G$-module, $\mathbb{C}G \otimes W$ (sometimes written $\mathrm{Ind}\, W$) is the *induced representation.* This is analogous to the process of changing the field of a vector space (from $\mathbb{R}$ to $\mathbb{C}$ say) by tensoring by a field which contains the original field as a sub-field. Suppose U is an arbitrary G representation - that is, U is a left $\mathbb{C}G$-module. Let us now apply (2.10):

$$\hom_{\mathbb{C}G}(\mathbb{C}G \otimes W, U) \cong \hom_{\mathbb{C}H}\left(W, \hom_{\mathbb{C}G}(\mathbb{C}G, U)\right) \tag{3.1}$$

We now use the fact that $\hom_{\mathbb{C}G}(\mathbb{C}G, U) \cong U$ via the isomorphism $f \mapsto f(1)$ to conclude:

$$\hom_{\mathbb{C}G}(\mathbb{C}G \otimes W, U) \cong \hom_{\mathbb{C}H}\left(W, U\right) \tag{3.2}$$

which is a statement of Frobenius reciprocity provided we consider U as an H-representation by restriction.

4 Minor holes

1. I am not entirely sure that I have the correct statement of the hom-tensor adjunction formula for left modules