All compact, complex, connected Lie groups are abelian.

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Abstract

We answer the last problem given in chapter 8 of [3]: 'Prove that all compact complex Lie groups are abelian'. The main focus in this short piece is on the proof that the adjoint map $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ is analytic, which is done by a more abstract, but ultimately cooler method than usual (i.e. as in [1] for example).

1 A commuting diagram lemma

We make heavy use of the following lemma (which can be found in [2])

Lemma 1.1. Suppose $\theta : G \times M \to M$ is a smooth group action of a Lie group G on a smooth manifold M. There is a natural way to define a map $\eta : T(G) \times T(M) \to T(M)$ such that the following diagram commutes:

$$\begin{tikzcd} T(G \times M) \arrow[r, "\pi_{1*} \times \pi_{2*}"] \arrow[dr, "\theta_*"'] & T(G) \times T(M) \arrow[d, "\eta"] \\ & T(M) \end{tikzcd} \tag{1.1}$$

Where $\pi_1 : G \times M \to G$ and $\pi_2 : G \times M \to M$ are the natural projections on to the first and second component of the product $G \times M$.

Proof. We define two smooth auxiliary maps θ^p and θ_g as follows:

$$\theta^p : G \to M \tag{1.2}$$
$$g \mapsto \theta(g, p) \tag{1.3}$$
$$\theta_g : M \to M \tag{1.4}$$
$$p \mapsto \theta(g, p) \tag{1.5}$$

We may define η as:

$$\eta : ((g, X), (p, Y)) \mapsto (\theta(g, p), \theta_{g*}Y + \theta^p_* X) \tag{1.6}$$

Observe that $\theta^p_* : T_g(G) \to T_{gp}(M)$ and $\theta_{g*} : T_p(M) \to T_{gp}(M)$ so this map is indeed well-defined. Smoothness should follow by observing that in the first component η is just the map $\theta : (g,p)\ mapsto \theta(g,p)$, and by the smoothness of θ the maps $g \mapsto \theta_g$ and $p \mapsto \theta^p$ are smooth.(Not quite sure how to check that this diagram does indeed commute), but [2] does this. $\square$

We specialise this to the case of a Lie group acting on itself via left multiplication. We shall denote left multiplication by g as L_g and right multiplication by g as R_g. η now becomes:

$$\eta : T(G) \times T(G) \to T(G) \tag{1.7}$$
$$\eta : ((p,X),(q,Y)) \to (pq, R_{q*}X + L_{p*}Y) \tag{1.8}$$
$$\tag{1.9}$$

which by lemma 1.1 is smooth. I claim that in the case where G is a complex manifold, and so the action of G on itself via left multiplication is an analytic action, η is in fact an analytic map. (I think this should follow by exactly the same reasoning that showed η was smooth, but I need to think about this a bit more). Let us now get to the point.

2 The adjoint map is analytic.

Consider the map:

$$\mathrm{Ad} : G \to \mathrm{Aut}\,(\mathfrak{g}) \tag{2.1}$$
$$g \mapsto \Phi_{g*}|_e \tag{2.2}$$
$$\Phi_g : h \mapsto g^{-1}hg \tag{2.3}$$

where Φ_g is analytic. $\mathrm{Aut}(\mathfrak{g}) \subset \mathbb{GL}(\mathfrak{g})$ has the natural complex structure induced by $\mathfrak{g}$. That is, if $\{X_i\}$ is a basis for $\mathfrak{g}$, then $\{X_i \otimes X^j\}$ is a basis for $\mathbb{GL}(\mathfrak{g})$. A map $\tau : M \to \mathbb{GL}(\mathfrak{g})$ will be analytic if and only if it is analytic in all of its coordinates. We can check this by considering the maps

$$\bar{\tau_i} : M \to \mathfrak{g} \tag{2.4}$$
$$m \mapsto \tau(m)X_i \tag{2.5}$$

and showing that they are analytic. This is equivalent to checking that

$$\bar{\tau_X} : M \to \mathfrak{g} \tag{2.6}$$
$$m \mapsto \tau(m)X \tag{2.7}$$

is analytic for arbitrary $X \in \mathfrak{g}$. Observe that we may rewrite Φ_g as $\Phi_g = L_{g^{-1}} \circ R_g$ and so

$$\mathrm{Ad}\, g = \Phi_{g*}|_e = L_{g^{-1}*}|_{R_g(e)} \circ R_{g*}|_e \tag{2.8}$$
$$= L_{g^{-1}*}|_g \circ R_{g*}|_e \tag{2.9}$$

So we need to show that the map $\bar{\mathrm{Ad}}: g \mapsto \Phi_{g*}|_e X$ is analytic for an arbitrary fixed X. We do this by introducing two auxiliary maps:

$$\gamma_R : G \to T(G) \times T(G) \tag{2.10}$$
$$\gamma_R : g \mapsto ((e, X), (g, 0)) \tag{2.11}$$
$$\gamma_L : G \to T(G) \times T(G) \tag{2.12}$$
$$\gamma_L : g \mapsto \big((g^{-1}, 0), (g, R_{g*}X)\big) \tag{2.13}$$

Since γ_R is basically the identity map, it should be analytic. Now, observe that $\eta \circ \gamma_R : g \mapsto (g, R_{g*}X)$ is a composition of analytic maps and hence is analytic. We may rewrite γ_L as:

$$\gamma_L : g \mapsto \big((g^{-1}, 0), \eta \circ \gamma_R(g)\big) \tag{2.14}$$

Which is analytic since it is analytic in both components (remember the inversion map is analytic). Finally observe that:

$$\eta \circ \gamma_L : g \mapsto (g^{-1}g, L_{g^{-1}*}R_{g*}X + R_{g*}0) = (e, L_{g^{-1}*}R_{g*}X) \tag{2.15}$$

is analytic since it is a composition of analytic maps, *and is precisely the map we need*: $\eta \circ \gamma_L := \Phi_{g*}|_e X$. Hence Ad is analytic.

3 All compact complex connected Lie groups are abelian

We now use the fact that Ad is analytic to prove the main result. First note that since $\mathrm{Ad} : G \to \mathrm{Aut}\,\mathfrak{g}$ is a holomorphic map on a compact set it must be constant. In addition, since

$$\Phi_e : h \mapsto ehe = h \tag{3.1}$$

is just the identity map, $\mathrm{Ad}\, e = \Phi_{e*}$ is also the identity map. We thus deduce that $\mathrm{Ad}(g)$ is the identity map (which we shall denote as I) for all g. Now Φ_g is a map of Lie groups (it is a map from G into itself) so it must commute with the exponential, exp. Thus:

$$\Phi_g(\exp(X)) = \exp(\Phi_{g*}X) = \exp X \tag{3.2}$$

that is, $\exp(X)$ commutes with g, for all $g \in G$, and all $X \in \mathfrak{g}$. But we know that exp is a diffeomorphism from some suitably small neighbourhood of $0 \in \mathfrak{g}$ to some neighbourhood $\mathcal{U}$ of $e \in G$. Hence for all $g, h \in \mathcal{U}$, $gh = hg$. We are almost done. Recall that if G is connected it is generated by any neighbourhood of e. So, if g and h are elements of G, we may write them as:

$$g = g^1 \dots g^n \tag{3.3}$$
$$h = h^1 \dots h^m \tag{3.4}$$

with all the g^i and h^j elements of $\mathcal{U}$,so $gh = g^1 \dots g^n h^1 \dots h^m$. We may now commute each of the g^i and h^j pairwise to obtain $gh = hg$ as required.

References

[1] J.M. Lee. *Introduction to Smooth Manifolds*. Springer-Verlag, New York, 2002.

[2] R.W. Sharpe. *Differential Geometry: Cartan's generalization of Klein's Erlangen Program*. Springer-Verlag, New York, 1996.

[3] Joe Harris William Fulton. *Representation Theory*. Springer-Verlag, New York, 1991.