Note on equidistribution for block sparsity

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Suppose that $\mathbf{g} \in \mathbb{R}^d$ satisfies $\|\mathbf{g}\|_0 \leq s$. Suppose further that we divide $\{1, \ldots, d\}$ into k blocks, $\mathcal{I}_1, \ldots, \mathcal{I}_C$. For simplicity, we assume that $|\mathcal{I}_c| = d/C$ for all c. Let $g_{i_1}, \ldots, g_{i_j}, \ldots, g_{i_s}$ denote the non-zero entries of $\mathbf{g}$. Define the random variables $X_1, \ldots, X_j, \ldots, X_s$ as follows:

$$X_j = \begin{cases} 1 & \text{if } i_j \in \mathcal{I}_1 \\ \vdots & \vdots \\ c & \text{if } i_j \in \mathcal{I}_c \\ \vdots & \vdots \\ k & \text{if } i_j \in \mathcal{I}_C \end{cases}$$

We may assume that, for all j and all c, $\mathbb{P}[X_j = c] = 1/C$. Now, define the random variables $Y_c = \#\{X_j : \ X_j = c\}$ for $c = 1, \ldots, C$. Finally, define the **random vector** $\mathbf{Y} = (Y_1, \ldots, Y_C) \in \mathbb{R}^C$. Then $\mathbf{Y}$ satisfies the **multinomial distribution**. Observe that $\sum_{c=1}^{C} Y_c = s$.

1 The $C = 2$ case

When $C = 2$, observe that $Y_2 = s - Y_1$, *i.e.* Y_2 is a function of Y_1, so we can ignore it. In this case Y_1 is a binomial r.v.. Observe that $\mathbb{E}[Y_1] = 0.5s$ Now use the Chernoff Bound:

$$\mathbb{P}\left[|Y_1 - 0.5s| \leq \delta(0.5s)\right] \leq 2\exp\left(-(0.5s)\delta^2/3\right)$$

Observe that we now get, for free:

$$\mathbb{P}\left[|Y_2 - 0.5s| \leq \delta(0.5s)\right] \leq 2\exp\left(-(0.5s)\delta^2/3\right)$$

from the fact that $Y_2 = s - Y_1$.

2 The general case

The general case will be similar. Observe that for each Y_c individually we may derive bounds of the form:

$$\mathbb{P}\left[|Y_c - s/C| \leq \delta(s/C)\right] \leq \epsilon$$

We then use a neat trick for dealing with possibly dependent random variables called the **union bound**:

$$\mathbb{P}\left[\exists\ c \text{ s.t. } |Y_c - s/C| \geq \delta(s/C)\right] \leq \sum_{c=1}^{C} \mathbb{P}\left[|Y_c - s/C| \geq \delta(s/C)\right] \tag{1}$$

$$\leq C\epsilon \tag{2}$$

and so:

$$\mathbb{P}\left[|Y_c - s/C| \leq \delta(s/C) \text{ for all } c\right] = 1 - \mathbb{P}\left[\exists\ c \text{ s.t. } |Y_c - s/C| \geq \delta(s/C)\right]$$

$$\geq 1 - C\epsilon$$

To Do:

1. Work out what ϵ needs to be, so that the bound holds with probability $1 - 1/s$.

2. We can probably improve the probability $1-C\epsilon$ to $1-(C-1)\epsilon$ by exploiting the fact that $Y_C = s - \sum_{c=1}^{C-1} Y_c$.