Introduction to FEM

2nd Computer Exercise: 2D Stationary Heat Conduction

Contents

1 Governing Equations

Energy balance:

$$-q_{x,x} - q_{y,y} + q_G = 0$$

where $q_x\,,\ q_y$: heat flux

q_G : heat source

Fourier's law:

$$q_x + \lambda_x \cdot T_{,x} = 0$$

$$q_y + \lambda_y \cdot T_{,y} = 0$$

where λ_x, λ_y : conductivity

2 Differential Equation

Inserting Fourier's law into the energy balance:

$$(\lambda_x \cdot T_{,x})_{,x} + (\lambda_y \cdot T_{,y})_{,y} + q_G = 0$$

for constant isotropic conductivities: $\lambda_x = \lambda_y = \lambda =$ constant

$$\lambda \cdot T_{,xx} + \lambda \cdot T_{,yy} + q_G = 0$$

$$\lambda \cdot [T_{,xx} + T_{,yy}] + q_G = 0$$

3 Boundary Condition

Dirichlet b.c.:

$T - T_R = 0$ where $T =$ Temperature

Neumann b.c.:

$q - q_R = 0$ where $q =$ Heat flux

4 Weak Formulation

Principle of weighted residuals:

$$-\delta R = \int (\delta T_{,x} \cdot \lambda \cdot T_{,x} + \delta T_{,y} \cdot \lambda \cdot T_{,y} - q_G) \, dA - \delta T \cdot q_R = 0$$

in matrix notation

$$-\delta R = \delta \mathbf{v}^T \cdot \left\{ \int_A \mathbf{\Omega}^T \cdot \mathbf{D}^T \cdot \mathbf{E} \cdot \mathbf{D} \cdot \mathbf{\Omega} \, dA \cdot \mathbf{v} - \int_A \mathbf{\Omega}^T \cdot \mathbf{p} \, dA \right\}$$

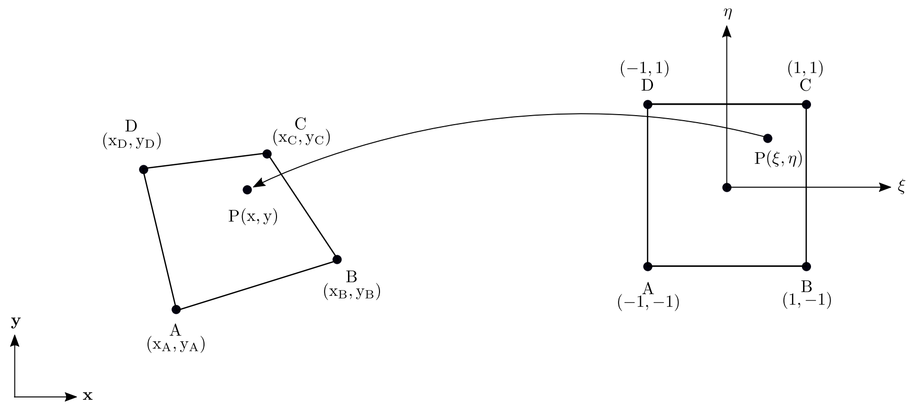
$$= \delta \mathbf{v}^T \cdot \left\{ \underbrace{\int_A \mathbf{H}^T \cdot \mathbf{E} \cdot \mathbf{H} \, dA \cdot}_{\text{Element matrix}} \mathbf{v} - \underbrace{\int_A \mathbf{\Omega}^T \cdot \mathbf{p} \, dA}_{\text{Element load vector}} \right\}$$

5 Discretisation

5.1 Geometry (Isoparametric elements)

5.1.1 Transformation of coordinates

Given the local coordinates of a point P(ξ, η) and the local shape function ϕ, the corresponding global coordinates of the point P(x,y) are found with the transformation



$$
\begin{aligned}
x &= \phi_A \cdot x_A + \phi_B \cdot x_B + \phi_C \cdot x_C + \phi_D \cdot x_D \\
&= \begin{bmatrix} \phi_A & \phi_B & \phi_C & \phi_D \end{bmatrix} \cdot \begin{bmatrix} x_A \\ x_B \\ x_C \\ x_D \end{bmatrix} \\
&= \boldsymbol{\phi} \cdot \tilde{\mathbf{x}}
\end{aligned}
$$

where $\phi_A \dots \phi_D$ are the bilinear shape functions and are defined as follows

$$
\phi_A = \frac{1}{4}(1-\xi)(1-\eta) \qquad \phi_B = \frac{1}{4}(1+\xi)(1-\eta)
$$

$$
\phi_C = \frac{1}{4}(1+\xi)(1+\eta) \qquad \phi_D = \frac{1}{4}(1-\xi)(1+\eta).
$$

Similarly $y = \boldsymbol{\phi} \cdot \tilde{\mathbf{y}}$. This can be summarised in matrix-vector notaion as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \boldsymbol{\phi} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\phi} \end{bmatrix} \cdot \begin{bmatrix} \tilde{\mathbf{x}} \\ \tilde{\mathbf{y}} \end{bmatrix}$$

$$\mathbf{x} = \ \boldsymbol{\Omega}(\xi, \eta) \ \cdot \ \hat{\mathbf{x}} \, .$$

5.1.2 Transformation of Partial derivatives

Since the coordinates x and y are now a function of ξ and η, the following rule applies for the transformation of partial derivatives

$$\begin{bmatrix} \partial_\xi \\ \partial_\eta \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}}_{\mathbf{J}} \cdot \begin{bmatrix} \partial_x \\ \partial_y \end{bmatrix} .$$

Substituting the transformation for x and y defined in Section 5.1.1 leads to the following definition for the components of Jacobian matrix $\mathbf{J}$

$$\frac{\partial x}{\partial \xi} = \boldsymbol{\phi}_{,\xi} \cdot \tilde{\mathbf{x}} \qquad , \qquad \frac{\partial y}{\partial \xi} = \boldsymbol{\phi}_{,\xi} \cdot \tilde{\mathbf{y}}$$

$$\frac{\partial x}{\partial \eta} = \boldsymbol{\phi}_{,\eta} \cdot \tilde{\mathbf{x}} \qquad , \qquad \frac{\partial y}{\partial \eta} = \boldsymbol{\phi}_{,\eta} \cdot \tilde{\mathbf{y}}$$

$$\mathbf{J} = \begin{bmatrix} \boldsymbol{\phi}_{,\xi} \cdot \tilde{\mathbf{x}} & \boldsymbol{\phi}_{,\xi} \cdot \tilde{\mathbf{y}} \\ \boldsymbol{\phi}_{,\eta} \cdot \tilde{\mathbf{x}} & \boldsymbol{\phi}_{,\eta} \cdot \tilde{\mathbf{y}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\phi}_{,\xi} \\ \boldsymbol{\phi}_{,\eta} \end{bmatrix} \ \cdot \ \begin{bmatrix} \tilde{\mathbf{x}} & \tilde{\mathbf{y}} \end{bmatrix} \ .$$

Similarly the reverse transformation is defined by

$$\begin{bmatrix} \partial_x \\ \partial_y \end{bmatrix} = \underbrace{\frac{1}{\det \mathbf{J}} \begin{bmatrix} \boldsymbol{\phi}_{,\eta} \cdot \tilde{\mathbf{y}} & -\boldsymbol{\phi}_{,\eta} \cdot \tilde{\mathbf{x}} \\ -\boldsymbol{\phi}_{,\xi} \cdot \tilde{\mathbf{y}} & \boldsymbol{\phi}_{,\xi} \cdot \tilde{\mathbf{x}} \end{bmatrix}}_{\mathbf{J}^{-1}} \cdot \begin{bmatrix} \partial_\xi \\ \partial_\eta \end{bmatrix} \ .$$

5.1.3 Transformation of domain of integration

By means of the Jacobian matrix $\mathbf{J}$, more precisely its determinant det $\mathbf{J}$, the domain of integration dA is transformed from the global coordinate system (x, y) to the local coordinate system (ξ, η) in which the shape functions are defined as

$$\int\limits_y \int\limits_x \ldots dx\ dy = \int\limits_{-1}^{1} \int\limits_{-1}^{1} \ldots \det\mathbf{J}\ d\xi\ d\eta\ .$$

5.2 Physics

5.2.1 Primary unknown (Temperature)

The temperature field is interpolated using the bi-linear ansatz functions $\mathbf{\Omega}(\xi, \eta)$ and the nodal unknowns as follows

$$\begin{aligned} T(\xi,\eta) &= \phi_A(\xi,\eta) \cdot T_A + \phi_B(\xi,\eta) \cdot T_B + \phi_C(\xi,\eta) \cdot T_C + \phi_D(\xi,\eta) \cdot T_D \\ &= \begin{bmatrix} \phi_A & \phi_B & \phi_C & \phi_D \end{bmatrix} \cdot \begin{bmatrix} T_A \\ T_B \\ T_C \\ T_D \end{bmatrix} \\ \mathbf{u} &= \mathbf{\Omega} \cdot \mathbf{v} \end{aligned}$$

where $\phi_A, \ldots \phi_D$ are the ansatz functions and were defined in section 5.1.1 . Similarly the virtual temperatures are defined as

$$\delta\mathbf{u} = \mathbf{\Omega} \cdot \delta\mathbf{v}\ .$$

5.2.2 Derivative of primary unknown

$$\begin{aligned} \mathbf{H} &= \mathbf{D} \cdot \mathbf{\Omega} \\ &= \begin{bmatrix} -\partial_x \\ -\partial_y \end{bmatrix} \cdot \begin{bmatrix} \phi_A & \phi_B & \phi_C & \phi_D \end{bmatrix} \end{aligned}$$

Since the shape functions are functions of ξ and η (local coordinates), the partial derivatives needs to be transformed as introduced in section 5.1.2 as follows

$$\mathbf{H} = \frac{1}{\det\mathbf{J}} \begin{bmatrix} \boldsymbol{\phi},_\eta \cdot\tilde{\mathbf{y}} & -\boldsymbol{\phi},_\eta \cdot\tilde{\mathbf{x}} \\ -\boldsymbol{\phi},_\xi \cdot\tilde{\mathbf{y}} & \boldsymbol{\phi},_\xi \cdot\tilde{\mathbf{x}} \end{bmatrix} \cdot \begin{bmatrix} -\partial_\xi \\ -\partial_\eta \end{bmatrix} \begin{bmatrix} \phi_A & \phi_B & \phi_C & \phi_D \end{bmatrix}$$

6 Derivation of Element Matrix and Vector

6.1 Element Matrix

As defined in Section 4, the element matrix is given by

$$\int_A \mathbf{H}^T \cdot \mathbf{E} \cdot \mathbf{H} \, dA.$$

Substituting the respective matrix definitions and transforming the domain of integration results in

$$\int_{-1}^{1}\int_{-1}^{1} \begin{bmatrix} \phi_A \\ \phi_B \\ \phi_C \\ \phi_D \end{bmatrix} \cdot \begin{bmatrix} -\partial_\xi & -\partial_\eta \end{bmatrix} \frac{1}{\det\mathbf{J}} \cdot \begin{bmatrix} \boldsymbol{\phi}_\eta \cdot \tilde{\mathbf{y}} & -\boldsymbol{\phi}_\xi \cdot \tilde{\mathbf{y}} \\ -\boldsymbol{\phi}_\eta \cdot \tilde{\mathbf{x}} & \boldsymbol{\phi}_\xi \cdot \tilde{\mathbf{x}} \end{bmatrix} \cdot \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \cdot$$
$$\frac{1}{\det\mathbf{J}} \cdot \begin{bmatrix} \boldsymbol{\phi}_\eta \cdot \tilde{\mathbf{y}} & -\boldsymbol{\phi}_\eta \cdot \tilde{\mathbf{x}} \\ -\boldsymbol{\phi}_\xi \cdot \tilde{\mathbf{y}} & \boldsymbol{\phi}_\xi \cdot \tilde{\mathbf{x}} \end{bmatrix} \cdot \begin{bmatrix} -\partial_\xi \\ -\partial_\eta \end{bmatrix} \cdot \begin{bmatrix} \phi_A & \phi_B & \phi_C & \phi_D \end{bmatrix} \cdot \det\mathbf{J} \, d\xi \, d\eta.$$

6.2 Element Vector

As defined in Section 4, the element load vector is given by

$$\int_A \boldsymbol{\Omega}^T \cdot \mathbf{p} \, dA.$$

Assuming a constant heat source in space $\mathbf{p} = q_G$ and transferring the domain of integration results in

$$\int_{-1}^{1}\int_{-1}^{1} \begin{bmatrix} \phi_A \\ \phi_B \\ \phi_C \\ \phi_D \end{bmatrix} \cdot q_G \det\mathbf{J} \, d\xi \, d\eta.$$

7 Numerical Integration

Since an analytical integration of the element matrix and element vector are either impossible or tedious, a numerical integration (Gaussian Quadrature) is performed.

7.1 Element Matrix

$$\int_{-1}^{1}\int_{-1}^{1} \mathbf{H}^T(\xi,\eta)\cdot\mathbf{E}\cdot\mathbf{H}(\xi,\eta)\,\det\mathbf{J}(\xi,\eta)\,d\xi\,d\eta \approx \underbrace{\sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{H}^T(\xi_i,\eta_j)\cdot\mathbf{E}\cdot\mathbf{H}(\xi_i,\eta_j)\cdot\det\mathbf{J}(\xi_i,\eta_j)\cdot w_i\cdot w_j}_{\mathbf{K}}$$

where, n - number of quadrature points along ξ and η and
w_i,w_j - weights at quadrature point (ξ_i, η_j).

7.2 Element load vector

$$\int_{-1}^{1}\int_{-1}^{1} \mathbf{\Omega}^T(\xi,\eta)\cdot\mathbf{p}\cdot\det\mathbf{J}(\xi,\eta)\,d\xi\,d\eta \approx \underbrace{\sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{\Omega}^T(\xi_i,\eta_j)\cdot\mathbf{p}\cdot\det\mathbf{J}(\xi_i,\eta_j)\cdot w_i\cdot w_j}_{\mathbf{f}}$$

8 Assembly, application of boundary condition and solution

The discretised weak form of stationary heat equation over all finite element results as

$$\sum \delta\mathbf{v}^T\cdot\Big\{\mathbf{K}\cdot\mathbf{v}-\mathbf{f}\Big\} = \mathbf{0}$$

Since the above equation must hold for arbitrary $\delta\mathbf{v}^T$, it reduces to

$$\mathbf{K}\cdot\mathbf{v}-\mathbf{f} = \mathbf{0}.$$

By incorporating the appropriate Dirichlet boundary conditions into the above system of equations, a unique solution for the temperature can be obtained.

9 Postprocessing

Once the temperature has been obtained, the fluxes at a node can be calculated using the Fourier's law as

$$\underbrace{\begin{bmatrix} q_x \\ q_y \end{bmatrix}}_{\boldsymbol{\sigma}} = \underbrace{\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}}_{\mathbf{E}} \cdot \underbrace{\begin{bmatrix} -\partial_x \\ -\partial_y \end{bmatrix} \cdot \mathbf{u}}_{\boldsymbol{\epsilon}}.$$

Substituting the definition of temperature into the above equation results in

$$\underbrace{\begin{bmatrix} q_x \\ q_y \end{bmatrix}}_{\boldsymbol{\sigma}} = \underbrace{\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}}_{\mathbf{E}} \cdot \underbrace{\begin{bmatrix} -\partial_x \\ -\partial_y \end{bmatrix}}_{\mathbf{D}} \cdot \underbrace{\begin{bmatrix} \phi_A & \phi_B & \phi_C & \phi_D \end{bmatrix}}_{\boldsymbol{\Omega}} \cdot \underbrace{\begin{bmatrix} T_A \\ T_B \\ T_C \\ T_D \end{bmatrix}}_{\mathbf{v}}.$$

Since the ansatz functions $\phi_A \dots \phi_D$ are functions of local coordinates (ξ, η), the partial derivatives needs to be transformed as described in Section 5.1.2. This results in

$$\underbrace{\begin{bmatrix} q_x \\ q_y \end{bmatrix}}_{\boldsymbol{\sigma}} = \underbrace{\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}}_{\mathbf{E}} \cdot \underbrace{\frac{1}{\det \mathbf{J}} \begin{bmatrix} \boldsymbol{\phi}_\eta \cdot \tilde{\mathbf{y}} & -\boldsymbol{\phi}_\eta \cdot \tilde{\mathbf{x}} \\ -\boldsymbol{\phi}_\xi \cdot \tilde{\mathbf{y}} & \boldsymbol{\phi}_\xi \cdot \tilde{\mathbf{x}} \end{bmatrix} \cdot \begin{bmatrix} -\partial_\xi \\ -\partial_\eta \end{bmatrix} \cdot \begin{bmatrix} \phi_A & \phi_B & \phi_C & \phi_D \end{bmatrix}}_{\mathbf{H}} \cdot \underbrace{\begin{bmatrix} T_A \\ T_B \\ T_C \\ T_D \end{bmatrix}}_{\mathbf{v}}.$$

In matrix notation, the above equation can be simplified as

$$\boldsymbol{\sigma} = \mathbf{E} \cdot \mathbf{H} \cdot \mathbf{v}$$
$$\boldsymbol{\sigma} = \mathbf{S} \cdot \mathbf{v}.$$

The calculation of fluxes at all nodes of a finite element can be summarised as follows

$$\underbrace{\begin{bmatrix} \boldsymbol{\sigma}(\xi = -1; \eta = -1) \\ \boldsymbol{\sigma}(\xi = \ \ 1; \eta = -1) \\ \boldsymbol{\sigma}(\xi = \ \ 1; \eta = \ \ 1) \\ \boldsymbol{\sigma}(\xi = -1; \eta = \ \ 1) \end{bmatrix}}_{\tilde{\boldsymbol{\sigma}}} = \underbrace{\begin{bmatrix} \mathbf{S}(\xi = -1; \eta = -1) \\ \mathbf{S}(\xi = \ \ 1; \eta = -1) \\ \mathbf{S}(\xi = \ \ 1; \eta = \ \ 1) \\ \mathbf{S}(\xi = -1; \eta = \ \ 1) \end{bmatrix}}_{\tilde{\mathbf{S}}} \cdot \underbrace{\begin{bmatrix} T_A \\ T_B \\ T_C \\ T_D \end{bmatrix}}_{\mathbf{v}}.$$