Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

Faculty of Engineering, Mathematics and Science

School of Computer Science and Statistics

SF Integrated Computer Science **Trinity Term 2020**
SF CSL

MAU22C00: Discrete Mathematics

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Instructions that apply to all take-home exams

1. This is an open-book exam. You are allowed to use your class notes, textbooks and any material that is available through the internet. However, you are not allowed to seek help from others and you are not allowed to post questions on online forums such as Stack Exchange.

2. If you have any questions about the content of this exam, you may seek clarification from Prof. John Stalker for questions 1-3 and Prof. Andreea Nicoara for questions 4-6 using the e-mail addresses provided. You are not allowed to discuss this exam with others.

3. Solutions must be submitted through Blackboard by 12pm noon on Friday, May 8th. You must submit a single pdf file for each exam separately and sign the following declaration in each case.

Plagiarism declaration: I have read and I understand the plagiarism provisions in the General Regulations of the University Calendar which are available through https://www.tcd.ie/calendar.

Signature: ______________________________

Additional instructions for this particular exam

Attempt all questions. All questions have equal weight.

1.

(a) Let P be the set of polynomials in one real variable. Consider $p \in P$ given by

$$p(t) = t^3 + 2t^2 + 3t + 5.$$

Let $\sim$ be the relation on P defined by saying that $a \sim b$ if $a = b + pq$ for some $q \in P$.

(i) Show that $\sim$ is an equivalence relation.

(ii) Show that each equivalence class has a unique representative of degree less than 3.

(iii) Find that representative for the equivalence class of t^5.

(b) Suppose that m and n are non-negative integers, S is a set with m elements, T is a set with n elements and φ is a function from S to T, Prove, by induction or otherwise, that

(a) if $m < n$ then there is a $w \in T$ such that there is no $u \in S$ with $\varphi(u) = w$,

(b) if $m > n$ then there are $u, v \in S$ such that $u \neq v$ but $\varphi(u) = \varphi(v)$.

2. Let $(A, *)$ be the semigroup

$$A = \{0, 1, 2, 3\}$$

and

$$\begin{array}{llll} 0*0=0 & 0*1=0 & 0*2=0 & 0*3=0 \\ 1*0=0 & 1*1=1 & 1*2=2 & 1*3=3 \\ 2*0=0 & 2*1=2 & 2*2=0 & 2*3=2 \\ 3*0=0 & 3*1=3 & 3*2=2 & 3*3=1 \end{array}$$

(a) Is $(A, *)$ a semigroup? Is it a monoid? Is it a group? Justify your answers.

(b) Consider the subset $\{1, 3\}$, still with $*$ as the operation. Is it a semigroup? Is it a monoid? Is it a group? Justify your answers.

(c) Consider the subset $\{0, 1, 3\}$, still with $*$ as the operation. Is it a semigroup? Is it a monoid? Is it a group?

(d) In general, a homomorphism from a semigroup $(A, *)$ to a semigroup $(B, \star)$ is a function $f\colon A \to B$ such that

$$f(a * b) = f(a) \star f(b).$$

There are exactly six homomorphisms from $(A, *)$ to itself. Find at least four of them.

(e) Suppose that $(B, \star)$ is a semigroup. Let C be the set of invertible homomorphisms from $(B, \star)$ to itelf. Let $\circ$ be composition of functions. Show that $(C, \circ)$ is a group.

3. Consider a language L with alphabet

$$A = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}, \mathtt{f}, \mathtt{g}, \mathtt{h}, \mathtt{i}, \mathtt{l}, \mathtt{m}, \mathtt{n}, \mathtt{o}, \mathtt{p}, \mathtt{r}, \mathtt{s}, \mathtt{t}, \mathtt{u}\}$$

consisting of all non-empty strings in A *except* those where an `a`, `o` or `u` is followed by a non-empty string of consonants followed by an `e` or `i`, or those where an `e` or `i` is followed by a non-empty string of consonants followed by an `a`, `i` or `o`. Consonants are, of course, all symbols other than `a`, `e`, `i`, `o` and `u`.

(a) Draw the diagram for a finite state automaton recognising this language.

Note: You don't need a separate arrow for each symbol that causes a transition from one state to another. A single arrow labeled with all the symbols which cause that state transition suffices. Also, if your diagram has a large number of states then you are almost certainly doing the problem wrong.

(b) Is the language finite? Is it regular? Is it context-free? Justify your answers.

(c) The language L has the property that if a string belongs to L then so does every non-empty substring. In other words, "is a non-empty substring of" is an order relation on L. Find all minimal elements, if any, and all least elements, if any.

(d) Apply the Myhill-Nerode Theorem to this example. To be more precise, the Myhill-Nerode Theorem says that a language is regular if and only if the set of equivalence classes of strings with respect to a certain equivalence relation defined in terms of the language is finite. If you've said above that L is regular then identify all the equivalence classes. If you've said above that L is not regular then show that there are infinitely many equivalence classes.

4. Let (V, E) be the graph with vertices A, B, C, D, E, F, G, H, I, and J, and edges AB, AC, AD, AE, BC, BD, BE, CD, CE, DE, AF, AJ, FJ, BJ, BI, IJ, CI, CH, HI, DG, DH, GH, EF, EG, and FG.

(a) (i) Draw this graph.

(ii) How many edges would you need to add to this graph in order to make it complete? Justify your answer.

(iii) What is the minimum number of edges you would need to remove from this graph in order to make it regular? Justify your answer.

(iv) What is the minimum number of edges you would need to remove from this graph in order to make it disconnected? Justify your answer.

(v) What is the minimum number of colours you need to colour this graph? Recall that colouring a graph means that distinct colours are assigned to vertices joined by an edge. Justify your answer.

(vi) Does this graph have an Eulerian circuit? Justify your answer.

(vii) Does this graph have a Hamiltonian circuit? Justify your answer.

(b) How many distinct isomorphisms φ from the graph (V, E) to itself are there? Justify your answer. Write down three of them.

(c) Let a cost function be given on (V, E) according to the following table:

HI	GH	AD	BE	BC	DE	CH	EF	AJ	FJ	CD	DH	EG
1	1	1	2	2	3	4	4	5	5	5	6	6
DG	BI	FG	CE	AB	IJ	BD	AC	BJ	CI	AF	AE	
6	7	8	9	9	10	11	12	13	14	14	15	

Determine the minimum spanning tree generated by Prim's Algorithm, starting from the vertex C, where that algorithm is applied with the queue specified in the table above. For each step of the algorithm, write down the edge that is added.

5.

(a) Is $\{(x, y) \in \mathbb{R}^2 \mid y = x^3 + 2\}$ finite, countably infinite, or uncountably infinite? Justify your answer.

(b) Is

$$\{x \in \mathbb{R} \mid a_0x^3 + a_1x^2 + a_2x + a_3 = 0;\ a_0, a_1, a_2, a_3 \in \mathbb{Z},\ \exists\, a_i \neq 0\}$$

finite, countably infinite, or uncountably infinite? This is the set of all real numbers that satisfy a polynomial equation of degree up to 3 with integer coefficients. Justify your answer.

(c) Let $L = \{0^m 1^n \mid m, n \in \mathbb{N},\ m + n = 15\}$. Is the language L over the binary alphabet finite, countably infinite, or uncountably infinite? Justify your answer.

(d) Is the language $\tilde{L}$ given by the regular expression $(0 \cup \epsilon) \circ (1 \cup \epsilon)^*$ over the binary alphabet finite, countably infinite, or uncountably infinite? Justify your answer.

(e) In lecture, we defined the language

$$E_{DFA} = \{\langle B \rangle \mid B \text{ is a DFA and } L(B) = \emptyset\}$$

when we examined whether the emptiness testing problem for deterministic finite state acceptors was a Turing-decidable language. Is E_{DFA} finite, countably infinite, or uncountably infinite? Justify your answer.

6.

(a) Let $\tilde{L} = \{0^m 1^{m-2} \mid m \in \mathbb{N},\ m \geq 2\}$. Using the Pumping Lemma, show that L is not a regular language.

(b) Write down the algorithm of a Turing machine that recognises

$$\tilde{L} = \{0^m 1^{m-2} \mid m \in \mathbb{N},\ m \geq 2\}.$$

Process the following strings according to your algorithm: 0, 1, 00, 000, 0001, 00001101.

(c) Let L be a language over a finite alphabet A. A language L' over the same alphabet A is called a *sublanguage* of L if $L' \subset L$. Assume that L is Turing-decidable. Does it follow that all of its sublanguages L' are likewise Turing-decidable? Provide a proof, if you believe that the statement is true, or a counterexample, if you believe that the statement is false.

(d) Let L_1 and L_2 be two Turing-recognisable languages over the same finite alphabet A. Construct an enumerator that outputs $L_1 \cap L_2$.