2MAE004 - Mission Analysis and Orbital Mechanics Assignment (Academic Year: 22-23): *The Two-Body Problem*

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Introduction

This report aims to illustrate the methodology behind the answers to the questions given in the assignment report. The following are the assumptions made for answering said questions:

- The mass of the smaller body is negligible compared to the central body.
- The coordinate system is inertial.
- The two bodies, Earth and satellite, are spherically symmetric with uniform density, enabling us to treat the bodies as point masses.
- No other forces act on the system, apart from the gravitational forces along the line joining the centres of mass.

All relevant MATLAB files and images have been submitted on LMS.

1. Solving Kepler's Equation

To solve Kepler's equation, a MATLAB script was used to numerically compute the solution by using the following parameters:

- **Radius of the Earth:** $R_E = 6378km$
- **The gravitational parameter of the Earth:** $\mu_E = 3.986 \times 10^5 km^3/s{-}2$

1 a) MATLAB Function using the Newton-Raphson Method

Kepler's equation is defined as the following:

$$E - e\sin(E) = M = n(t - t_0) \tag{1}$$

Where E is the eccentric anomaly, e the eccentricity, M the mean anomaly, n the mean angular motion, t the time and lastly t_0 is the time of periapsis/perigee passage.

As can be seen in equation (1), the variable E appears both as a number and as part of a sine function; thus this can only be numerically solve by using the Newton-Raphson method. To solve it, the Kepler's equation must be solved for when it is 0, such as to obtain the eccentric anomaly from a given mean anomaly; this will allow the estimation of the spacecraft's position. Thus $f(E)$ is the Kepler's equation equal to 0, and $f'(E)$ its derivative, which are defined as the following:

$$f(E) = E - e\sin(E) - M \tag{2}$$

$$f'(E) = 1 - e\cos(E) \tag{3}$$

The algorithm implemented must have a limit due to computational constraints, as such a maximum number of iterations is defined such that even if an acceptable tolerance is not reached, the algorithm will stop.

```
269  function [Ek] = Kepler(M, e, tol, ite_max)
270 -    Ek = M;
271 -    f_E = @(E)(E - e * sin(E) - M);
272 -    df_E = @(E)(1 - e * cos(E));
273
274 -    tol_E = tol + 1;
275 -    ite = 0;
276
277 -    while (tol_E > tol && ite < ite_max)
278 -        ite = ite + 1;
279 -        Ek1 = Ek;
280 -        delta_E =  (f_E(Ek)) / (df_E(Ek));
281 -        ite;
282 -        Ek = Ek - delta_E;
283 -        tol_E = abs(Ek - Ek1);
284 -    end
285 -  end
```

Figure 1: MATLAB function for the Newton-Raphson method.

Shown in figure 1, is the MATLAB function which takes in 4 inputs: M the mean anomaly, e the eccentricity of the orbit, tol the tolerance of the answer and lastly ite_max the maximum number of iterations which will stop the algorithm in case a solution lower than the tolerance cannot be found. This algorithm completes the following steps:

1. Define the first guess as the mean anomaly: $E_0 = M$ (MATLAB script line 270).
2. Begin an iteration process that will stop the algorithm once the tolerance of the answer has been reached (MATLAB script lineS 274-277).
3. Compute the following: $\Delta E = f(E)/f'(E)$ (MATLAB script line 280).
4. Calculate the next guess/iteration: $E_1 = E_0 - \Delta E$ (MATLAB script line 282).
5. Repeat steps 1-4 until the tolerance has been reached: $|\Delta E| < tol$ (MATLAB script line 283).

1 b) Execution of MATLAB Function

The following are the parameters used for this exercise: mean anomaly ($M = 21°$), eccentricity ($e = 0.25$), and maximum iterations ($ite_{max} = 100$).

Using the function "[E] = Kepler(M,e,tol,ite_{max})" with those parameters, results in the following:

- The eccentric anomaly after 1 iterations is: $E_1 = 0.4833\, rad = 27.69°$.
- The number of iterations for a tolerance of 10^{-6} is 3, giving an eccentric anomaly of $E_3 = 0.4825\, rad = 27.64°$.

The eccentric anomaly will help us in calculating the true anomaly θ, and the radial distance r. Where now a new variable, the semi-major axis a, is required to calculate the current position of the spacecraft. When using semi-major axis of $a = 24,000km$ and a tolerance of $tol = 10^{-12}$, the following two equations are used and they result in the following:

$$\theta = 2 \arctan \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2} = 0.6151\, rad = 35.25° \tag{4}$$

$$r = \frac{p}{1 + e \cos\theta} = 18,685km \tag{5}$$

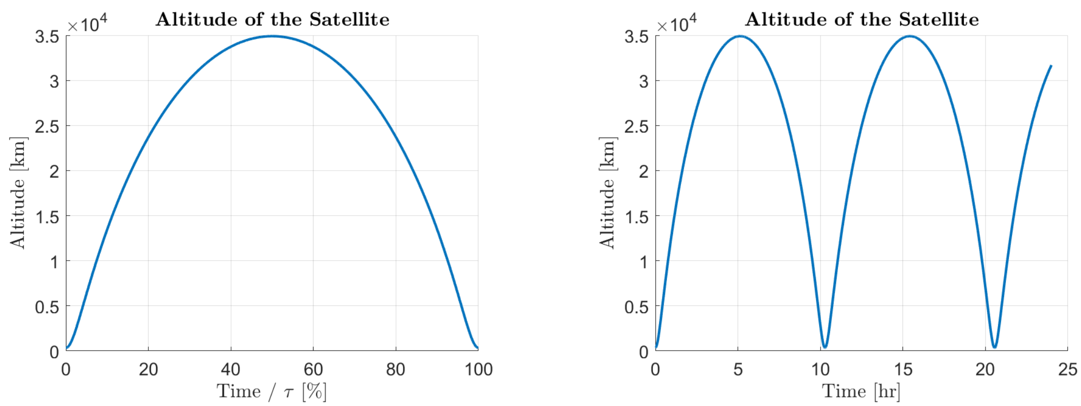
Where p is the semi-latus rectum of the ellpipse that is defined as $p = a * (1 - e^{-2})$.

If the calculations are repeated for an $M = 180°$, the eccentric anomaly will now be calculated for when it is at the apoapsis/apogee of the orbit. As such, the radial distance is now $r = 30,000km$ which is equal to the radius of apogee defined as $r_a = a(1 + e)$.

1 c) Computation of MEO Satellite Orbit

Using the previous algorithm, it is always assumed that the initial time of simulation is the time when the spacecraft is found to be at the perigee of the orbit. The following are the parameters used for this exercise: time at perigee ($t_0 = 0\ s$), semi-major axis ($a = 24,000km$), and eccentricity ($e = 0.72$).

It is of interest to know what the altitude of the spacecraft is, which is defined as the distance from the earth to the satellite as the following equation: $h = r - R_E$. Where h is the altitude, r the radial distance of the orbit that uses the equation defined in equation (5) and R_E the radius of the Earth. The result of plotting the results as a function of time results in the following two figures:



(a) Altitude of the orbit for 1 orbital period.

(b) Altitude of the orbit for 1 day.

Figure 2: Altitude of the MEO satellite orbit

As can be seen on figure 2 (a), the altitude at perigee and apogee are equal to $h_p = 342km$ and $h_a = 34,902km$ respectively. These is obtained from radius of perigee and apogee which are equal to $r_p = 6,720\ km$ and $r_a = 41,280\ km$ respectively; these are defined by $r_p = a(1-e)$ and $r_a = a(1+e)$. On the other hand, figure 2 (b) shows us the periodic nature of the orbit, where every $10.27hr$ the orbit repeats. This time is exactly equal to the orbit period defined by $\tau = 2\pi\sqrt{a^3/\mu_E}$.

1 d) Analysis of the Eclipse Times

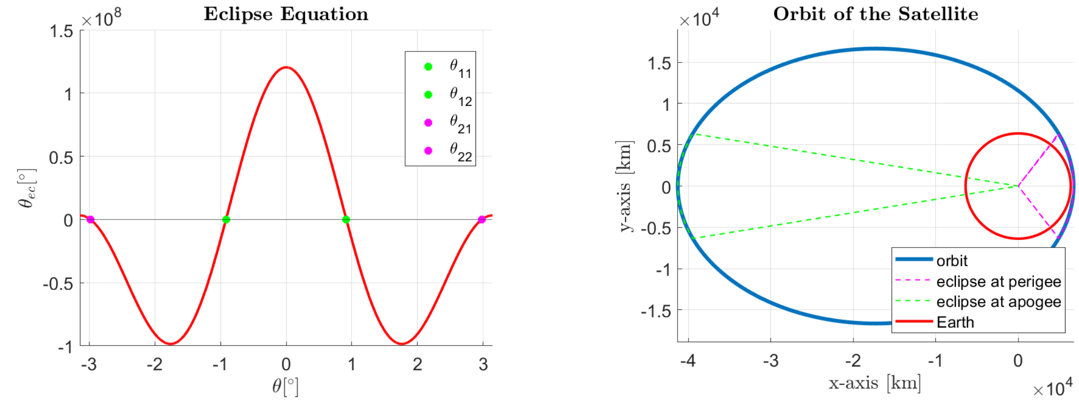
Analyzing the eclipse time, the duration for which the satellite is covered by the shadow of the Earth from the sunlight, is essential for mission analysis. This can be defined by the following eclipse equation:

$$\theta_{ec} = f(\theta) \rightarrow \alpha\cos^2(\theta) + \beta\cos(\theta) + \gamma = 0 \tag{6}$$

Where $\alpha = R_E^2 e^2 + p^2$, $\beta = 2R_E^2 e$, and $\gamma = R_E^2 - p^2$. The solutions for equation (6) when $\theta_{ec} = 0$, define the angles for which the satellite is in the eclipse at either the perigee or apogee of the orbit. The following conditions determine whether the satellite is in the perigee of the apogee of the orbit:

- If $\theta \in [-2\pi, -\pi]$ or $\theta \in [+\pi, +2\pi]$, the spacecraft is in the half of the orbit closest to its perigee.
- If $\theta \in [-\pi, 0]$ or $\theta \in [0, +\pi]$, the spacecraft is in the half of the orbit closest to its apogee.

It is defined that the positive elements of solution of equation (6) determines how much of the orbit is in the eclipse. The variables θ_{11} and θ_{12} defines the apogee eclipse, and θ_{21} and $\theta 22$ defines the perigee eclipse. All of the previous concepts can be seen in the following figure:



(a) Solution to the eclipse equation.

(b) Orbit of the spacecraft in the Earth eclipse.

Figure 3: Analysis of the spacecraft's orbit in the Earth eclipse.

As can be seen in figure 3 (a), the positive elements determine the angles that correspond to the perigee or apogee of the orbit. Visualising the equation helps illustrate when the eclipse occurs, and by using its results the figure 3 (b) is obtained. Where even though a small segment of the solution to the eclipse equation is in the perigee compared to the apogee section, due to the high velocity found in the perigee an equal segment is found to that of the apogee. Nonetheless, there is an equal coverage in terms of degrees for both regions.

2. Numerical Integration of the Equations of Motion Using MATLAB Differential Equation Solvers

This section aims to show the results of the computation to the solution of the ordinary differential equation that defines the motion of an orbit.

2 a) Two-Body Equations of Motion in 3D

To define an orbit's motion, the following ordinary differential equation (ODE) is used:

$$\ddot{\vec{r}} + \frac{\mu_E}{|r|^3}\vec{r} = 0 \rightarrow \ddot{\vec{r}} = -\frac{\mu_E}{|r|^3}\vec{r} \tag{7}$$

Where $\dot{\vec{r}}$ is the 3D velocity vector of the spacecraft, $\vec{r}$ is the 3D position vector of the radial distance from the foci to the spacecraft where its magnitude is $|r|$. The solution to equation (7) leads to the following state vectors that define the two-body equations of motion in 3D:

$$X = \begin{bmatrix} r_x \cdot \hat{x} \\ r_y \cdot \hat{y} \\ r_z \cdot \hat{z} \\ \dot{r}_x \cdot \hat{x} \\ \dot{r}_y \cdot \hat{y} \\ \dot{r}_z \cdot \hat{z} \end{bmatrix} = \begin{bmatrix} r_x \cdot \hat{x} \\ r_y \cdot \hat{y} \\ r_z \cdot \hat{z} \\ v_x \cdot \hat{x} \\ v_y \cdot \hat{y} \\ v_z \cdot \hat{z} \end{bmatrix} \qquad \dot{X} = \begin{bmatrix} \dot{r}_x \cdot \hat{x} \\ \dot{r}_y \cdot \hat{y} \\ \dot{r}_z \cdot \hat{z} \\ -\frac{\mu_E}{|r|^3} r_x \cdot \hat{x} \\ -\frac{\mu_E}{|r|^3} r_y \cdot \hat{y} \\ -\frac{\mu_E}{|r|^3} r_z \cdot \hat{z} \end{bmatrix} = \begin{bmatrix} v_x \cdot \hat{x} \\ v_y \cdot \hat{y} \\ v_z \cdot \hat{z} \\ a_x \cdot \hat{x} \\ a_y \cdot \hat{y} \\ a_z \cdot \hat{z} \end{bmatrix} \tag{8}$$

2 b) MATLAB Function for Equations of Motion in 3D

```
255 function orb_acc = twobody(t, X)
256 % Gravitational parameter of the Earth [km^3 / s^2]
257 miu = 3.986e5;
258 
259 orb_acc = zeros(6, 1);
260 
261 orb_acc(1) = X(4);
262 orb_acc(2) = X(5);
263 orb_acc(3) = X(6);
264 
265 % Magnitude of the radial elements [km]
266 mag = sqrt(X(1)^2 + X(2)^2 + X(3)^2);
267 orb_acc(4) = -(miu / mag^3) * X(1);
268 orb_acc(5) = -(miu / mag^3) * X(2);
269 orb_acc(6) = -(miu / mag^3) * X(3);
270 end
```

Figure 4: MATLAB function for the derivative in time of the Cartesian element state vector of an orbit.

The state vector X found in equation (8) are the Cartesian element vectors used to help define an orbit. The MATLAB function, shown in figure 4, has a discretized time and the initial Cartesian element vector as its inputs such as to solve the ordinary differential equation.

The MATLAB lines 261-263 define the velocity components as the first 3 elements of the state vector $\dot{X}$, while the MATLAB lines 266-269 define the acceleration components of the final 3 elements.

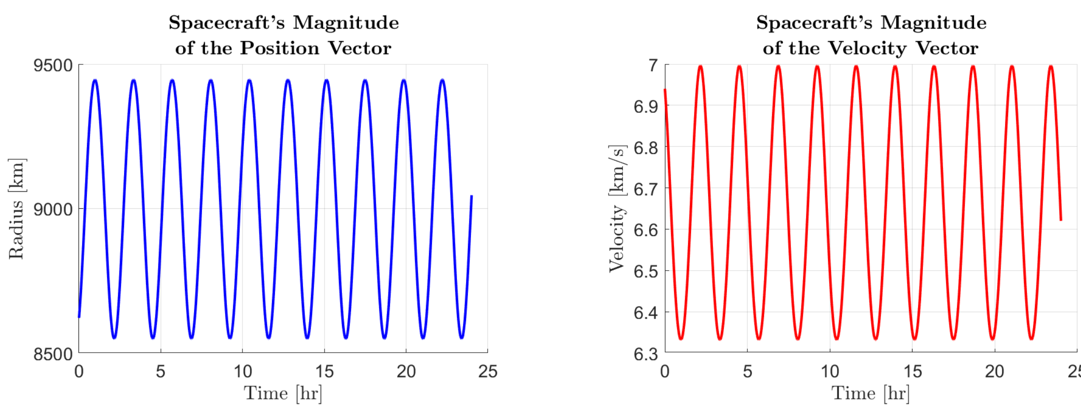
2 c) Computational Solution to the ODE

The following initial state vector, $X_0 = [\overrightarrow{r}, \overrightarrow{v}]^T$, is used for this exercise, where:

$$\overrightarrow{r} = [7,115.804 \cdot \hat{x};\ 3,391.696 \cdot \hat{y};\ 3,492.221 \cdot \hat{z}] \cdot km \rightarrow |r| = 8,621.71\ km$$
$$\overrightarrow{v} = [-3.762 \cdot \hat{x};\ 4.063 \cdot \hat{y};\ 4.181 \cdot \hat{z}] \cdot km/s \rightarrow |v| = 6.940\ km/s$$

Additionally, the following settings are used to define the computational limits of this ODE solution: options = odeset('RelTol',1e-12,'AbsTol', 1e-12).

This results in followings solutions for the spacecraft's position vector and velocity vector:



(a) Magnitude of the position vector. **(b)** Magnitude of the velocity vector.

Figure 5: Magnitude of vectors from the solution of the spacecraft's orbit equations of motion.

As can be seen in both components of 5, the velocity and subsequently the position are sinusoidal functions, which have a period equal to that of the orbit's period which is $\tau \approx 2\ hr$. It can be noted, that the positive increase of the position magnitude is directly correlated to the decrease in velocity magnitude; this occurs approaching the apogee, and the opposite of this correlation occurs at the perigee. From the position magnitude it can be seen, that the radius of perigee is of $r_p \approx 8,600\ km$ and the radius of apogee is $r_a \approx 9,600\ km$. When plotting the results of the ODE in 3D, the following figure is obtained:

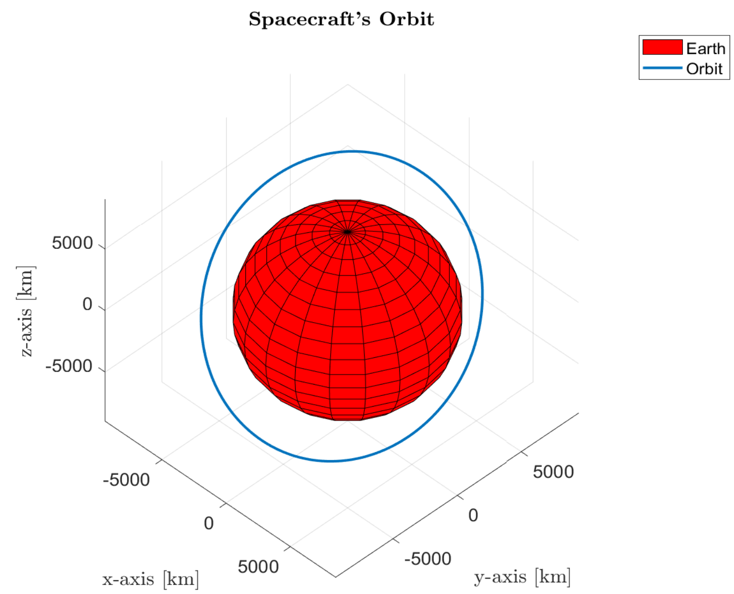


Figure 6: 3D plot of the spacecraft's orbit with a suitably dimensioned Earth.

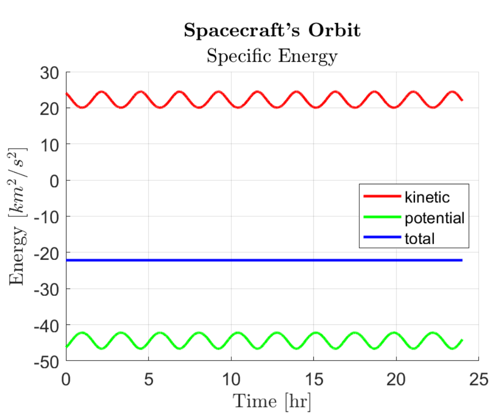
2 d) Variation of the Energy and Angular Momentum of the Orbit

The solution found from the previous sub-section, can be used in the following equations to determine the spacecraft's orbit specific energies, ϵ and specific angular momentum h respectively:

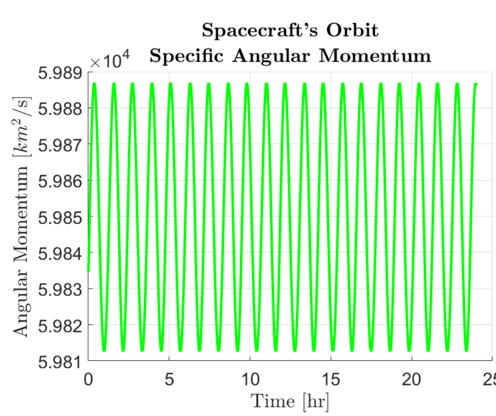
$$\epsilon = \epsilon_k + \epsilon_p \rightarrow -\frac{\mu_E}{2a} = \frac{|v|^2}{2} - \frac{\mu_E}{|r|} \tag{9}$$

$$\overrightarrow{h} = \overrightarrow{r} \times \overrightarrow{v} \tag{10}$$

Where $\epsilon_k = |v|^2/2$ is the specific kinetic energy and $\epsilon_p = -\mu_E/|r|$ is the specific potential energy. When plotting out the results in times for equations (9) and (10), the following figures are obtained:



(a) Specific energies.



(b) Magnitude of specific angular momentum.

Figure 7: Analysis of the energy of the spacecraft's orbit.

The figure 7 shows that even though the spacecraft travels through thousands of kilometers in distance in the span of a few hours, the energy of the orbit stays the same. This is to be expected as the angular momentum, although varying over time, has small changes of magnitude in the scale of 10^{-4}; what this refers to is that the angular momentum of spacecraft is conserved throughout its motion. Lastly, the total specific energy of the orbit is negative and practically constant signifying an elliptical orbit.

2 e) Determining Orbit from a Cartesian Vector

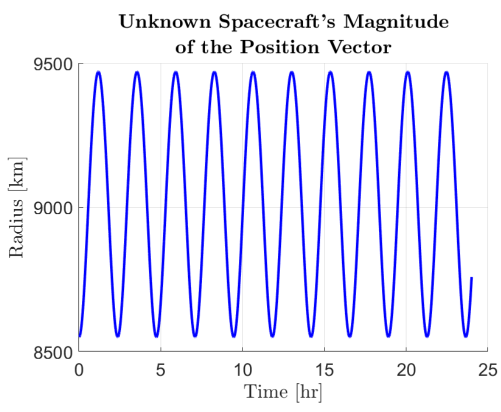
The following initial state vector, $X_0 = [\overrightarrow{r}, \overrightarrow{v}]^T$, is used for this exercise, where:

$$\overrightarrow{r} = [0 \cdot \hat{x};\ 0 \cdot \hat{y};\ 8,550 \cdot \hat{z}] \cdot km \rightarrow |r| = 8,550\ km$$
$$\overrightarrow{v} = [0 \cdot \hat{x};\ -7.0 \cdot \hat{y};\ 0 \cdot \hat{z}] \cdot km/s \rightarrow |v| = 7.0\ km/s$$

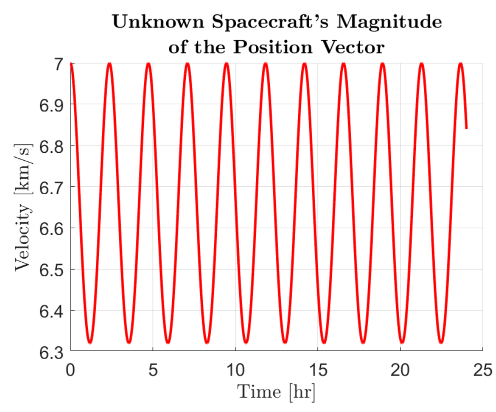
This Cartesian vector is the only given parameter of the orbit, however even with only this the Keplerian elements can be deduced by using the following formulas:

$$a = \frac{\mu_E}{2}\left(\frac{\mu_E}{|r|} - \frac{|v|^2}{2}\right)^{-1} = 9,010km$$
$$\overrightarrow{e} = \frac{\overrightarrow{v} \times \overrightarrow{h}}{\mu_E} - \frac{\overrightarrow{r}}{|r|} \rightarrow |e| = 0.051$$
$$i = \cos\left(\frac{h_z}{|h|}\right) = 90°$$
$$tau = 2\pi\sqrt{\frac{a^3}{\mu_E}} = 8,511\ s = 2.36\ hr$$

By translating the Cartesian vector into Keplerian elements, the orbit's nature can be narrowed down. By plotting the orbit's magnitude for its position and velocity vector, the following figures are obtained:



(a) Magnitude of position vector.



(b) Magnitude of velocity vector.

Figure 8: Magnitudes of the spacecraft's orbit state vector.

From the previous figures, figure 8 (a) and (b), it can be determined that the spacecraft begins at its perigee at $r_p = 8,550\ km$, the given Cartesian vector's z-axis component. The spacecraft's apogee is determined to be at $r_a = 9,460\ km$. When plotting in 3D the orbit more information is obtained, which is shown in the following figure:

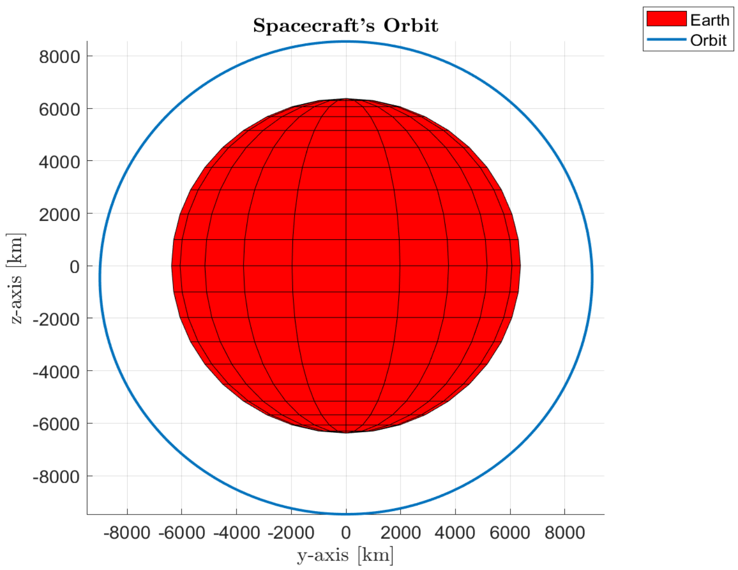


Figure 9: 3D orbit of the unknown spacecraft.

By analyzing all the previous figures and the Keplerian elements of the unknown spacecraft, it can be concluded that it is a polar orbit as both its perigee and apogee are located in the north and south pole respectively. This indicates an orbit that spends less time in the northern hemisphere and slightly more in the southern one. Additionally, it is an Medium Earth Orbit (MEO) due to it's altitude being higher than 2,000 km. Lastly, it is slightly elliptical due to its lower eccentricity.

3. Orbit Phasing and Rendezvous

To determine the optimal parameters for a chaser orbit, the ISS will be defined to have a circular orbit at an altitude of 404 km in the equatorial plane. Additionally, the ISS must begin at $\Delta\Theta = 100°$ from the chaser orbit.

3 a) Initial State Vector of the ISS and its ODE Solution

To solve this optimisation problem, the orbit of the ISS must first be computed, as such the Keplerian elements of this orbit can be deduced from the description given in the given of the exercise. This results is the following parameters: $a = 6782\ km$ and e, i, Ω, ω, and $\theta = 0$. With this, the orbit period of the ISS is determined to be $\tau = 92.63\ min$. However, the parameter of $\Delta\Theta$ is also included as this determines the initial position of the spacecraft. Thus, to solve this problem the following MATLAB line is used:

$$[r_{ISS}, v_{ISS}] = keplerian2ijk(a_{ISS}, 0, 0, 0, 0, 0, 'truelon', \Delta\Theta);$$

The function "keplerian2ijk" takes in the Keplerian elements of the orbit and transform them into the position and velocity vectors. As such, the following is initial state vector of the ISS given the previously described input parameters, $X_{0,ISS} = [\overrightarrow{r}, \overrightarrow{v}]^T$, where:

$$\overrightarrow{r} = [-1.177 \cdot \hat{x};\ 6,679 \cdot \hat{y};\ 0 \cdot \hat{z}] \cdot km \rightarrow |r| = 6,782\ km$$
$$\overrightarrow{v} = [-7.550 \cdot \hat{x};\ -1.331 \cdot \hat{y};\ 0 \cdot \hat{z}] \cdot km/s \rightarrow |v| = 7.666\ km/s$$

The orbit of the ISS is plotted along side the orbit of the chaser in the following sub-section.

3 b) Initial State Vector of the chaser and its ODE Solution

The objective is for the chaser to reach the ISS after 12 orbits, thus $N_{rev} = 12$. Using the following equations leads to the calculation of the Keplerian elements of the orbit of the chaser:

$$N_{rev}\tau_{cha}(n_{cha} - n_{ISS}) = \Delta\Omega \tag{11}$$

$$a_{cha} = \left(1 - \frac{\Delta\Omega}{2\pi N_{rev}}\right)^{\frac{2}{3}} a_{ISS} = 6,677\ km \rightarrow \tau_{cha} = 90.50\ min \tag{12}$$

$$e_{cha} = \frac{a_{ISS}}{a_{cha}} - 1 = 0.016 \tag{13}$$

By plugging in the previous parameters, as well as an argument of perigee of $\omega = 180°$ such that the chaser begins its orbit in its apogee such as to intersect the ISS, the following MALTAB line can be used to calculate the initial state vector of the spacecraft:

$$[r_{cha}, v_{cha}] = kepler ian2ijk(a_{cha}, ecc_{cha}, 0, 0, 0, 180, 'lonper', 180);$$

This calculates the initial state vector of the chaser spacecraft, which is equal to $X_{0,cha} = [\overrightarrow{r}, \overrightarrow{v}]^T$, where:

$$\overrightarrow{r} = [6,782 \cdot \hat{x};\ 0 \cdot \hat{y};\ 0 \cdot \hat{z}] \cdot km \rightarrow |r| = 6,782\ km$$
$$\overrightarrow{v} = [0 \cdot \hat{x};\ +7.606 \cdot \hat{y};\ 0 \cdot \hat{z}] \cdot km/s \rightarrow |v| = 7.606\ km/s$$

Plugging in this initial state vector into the ODE solver of MATLAB and propagating the orbit along time, results in the following figure:

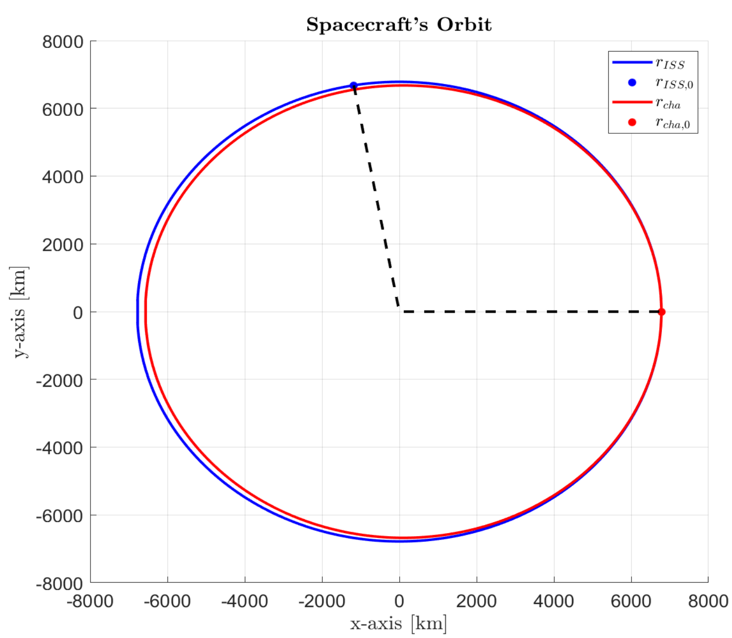


Figure 10: Orbit of the ISS and the chaser spacecraft.

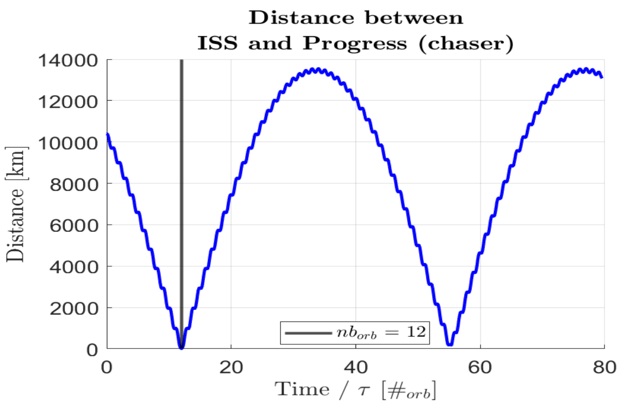


Figure 11: Distance magnitude between the ISS and the chaser.

As can be seen in figure 11, by placing the beginning of both the ISS orbit and the chaser orbit, an optimal number of 12 orbits periods is achieved such that both spacecraft's intersect.

It can be noted that this is the optimal parameter, as by tuning the variables of the $\Delta\Omega$ and the position of the chaser's orbit beginning completely change the distances achieved over time. Thus, it can be concluded that it is not only important to create an adequate orbit, but to also place the spacecraft at a specific time such as to obtain the optimal number of orbit periods before intersection.

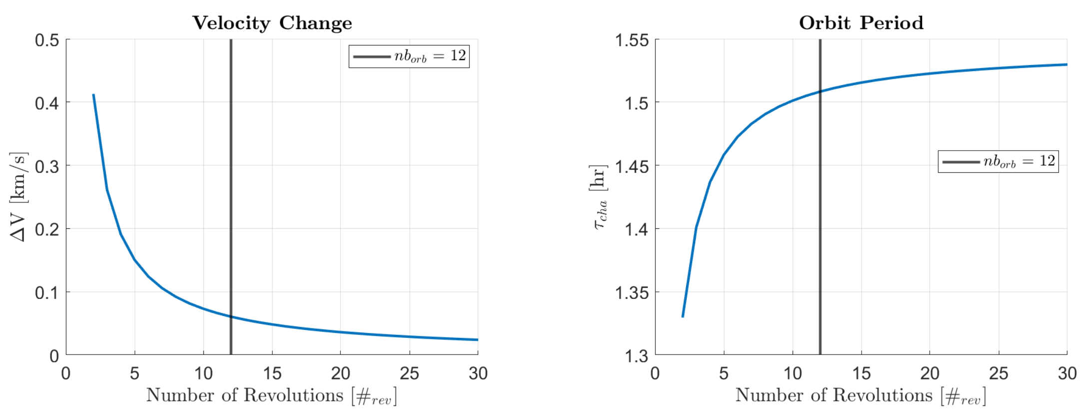
3 c) The Study of the ΔV of the Chaser

By studying the change of the ΔV, the change in velocity from orbit to another, of the chaser spacecraft, we can obtain more information about the mission parameters.

To study the effects of a varying number of orbit periods before intersection, where $N_{rev} \in [2,\ 30]$, the equation 12 varies as it is a function of N_{rev} and thus equation 13 changes as well. As the radius of apogee is the section where the change of velocity will occur, it will play a key role in the following calculations. The following is the equation used to determine the required ΔV as a function of N_{rev}:

$$\Delta V = f(N_{rev}) = v_{ISS} - v_{cha} = \sqrt{\frac{\mu_E}{r_{ISS}}} - \sqrt{\mu_E\left(\frac{2}{r_{a,cha}} - \frac{1}{a_{cha}}\right)} \tag{14}$$

This gives the following 2 figures, where on the left the velocity change $\Delta\Omega$ and the orbit period τ are plotted as a function of the number of revolutions N_{rev}:



(a) Velocity Change. **(b)** Orbit Period.

Figure 12: Parameters changes according to the number of revolutions N_{rev}.

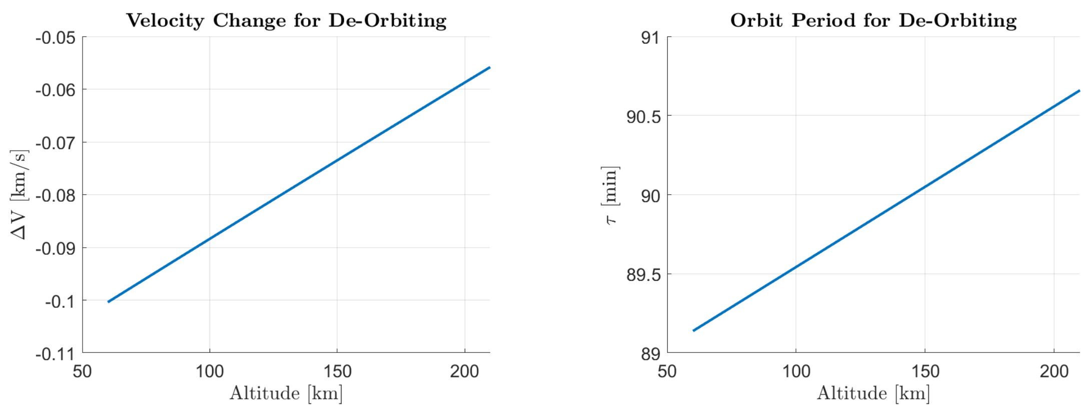
From the previous 2 figures, shown in figure 12 (a) and (b), we can see that an a number of revolutions of $N_{rev} = 12$ gives an optimal $\Delta\Omega = 0.06\ km/s$ and $\tau = 1.51\ hr$. Any additional number of revolutions will lead to a higher orbit period, however the velocity change would be lower. This shows the beginning of an optimisation algorithm to most appropiately determine the schedule and the orbit speed of the chaser spacecraft.

3 d) The ΔV for Different Perigees of the Chaser

This study has a similar methodology to the previous exercise, where now instead the chaser spacecraft starts at the orbit of the ISS and must now de-orbit such that it re-enters the Earth's atmosphere at a certain altitude. Depending on how much negative ΔV is applied, as in how much the spacecraft is slowed down, different altitudes will be obtained. These different altitudes will primarily change the radius of perigee of the de-orbiting orbit, $r_{p,cha}$ and thus its semi-major, $a_{p,cha}$. Thus, the following equation can be used to determine the velocity change ΔV as a function of the altitude of perigee or reentry $r_{p,cha}$:

$$\Delta V = f(r_{p,cha}) = v_{p,cha} - v_{ISS} = \sqrt{\mu_E \left(\frac{2}{r_{p,cha}} - \frac{1}{a_{p,cha}} \right)} - \sqrt{\frac{\mu_E}{r_{ISS}}} \tag{15}$$

This equation and the orbit period result in the following figures:



(a) Velocity Change. **(b)** Orbit Period.

Figure 13: Parameters changes according to the number of revolutions N_{rev}.

As seen in figure 13 (a), the lower the altitude desired, the more the spacecraft has to slow down. For an altitude of $r_{p,cha} = 210\ km$ a velocity change of $\Delta V = -0.056\ km/s$ is required. On the other hand, an altitude of $r_{p,cha} = 60\ km$ requires a velocity change of $\Delta V = -0.100\ km/s$. As can be seen in figure 13 (b), the orbit period changes by a minute depending on the desired orbit and thus has no real consequence as compared to the previous sub-sections orbit period calculation.