

## III.5 Interpolation and quadrature

*Polynomial interpolation* is the process of finding a polynomial that equals data at a precise set of points. *Quadrature* is the act of approximating an integral by a weighted sum:

$$\int_a^b f(x)w(x)dx \approx \sum_{j=1}^n w_j f(x_j).$$

In these notes we see that the two concepts are intrinsically linked: interpolation leads naturally to quadrature rules.

1. Polynomial Interpolation: we describe how to interpolate a function by a polynomial and a set of points.
2. Interpolatory quadrature rule: polynomial interpolation leads naturally to ways to integrate

functions, but only realisable in the simplest cases.

### 1. Polynomial Interpolation

We already saw a special case of polynomial interpolation, where we saw that the polynomial

$$f(z) \approx \sum_{k=0}^{n-1} f_k^n z^k$$

equaled  $f$  at evenly spaced points on the unit circle:  $e^{i2\pi j/n}$ . But here we consider the following:

**Definition 1 (interpolatory polynomial)** Given  $n$  distinct points  $x_1, \dots, x_n \in \mathbb{R}$  and  $n$  samples  $f_1, \dots, f_n \in \mathbb{R}$ , a degree  $n - 1$  interpolatory polynomial  $p(x)$  satisfies

$$p(x_j) = f_j$$

The easiest way to solve this problem is to invert the Vandermonde system:

**Definition 2 (Vandermonde)** The *Vandermonde matrix* associated with  $n$  distinct points  $x_1, \dots, x_n \in \mathbb{R}$  is the matrix

$$V := \begin{bmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} \end{bmatrix}$$

**Proposition 1 (interpolatory polynomial uniqueness)** The interpolatory polynomial is unique, and the Vandermonde matrix is invertible.

**Proof** Suppose  $p$  and  $\tilde{p}$  are both interpolatory polynomials. Then  $p(x) - \tilde{p}(x)$  vanishes at  $n$  distinct points  $x_j$ . By the fundamental theorem of algebra it must be zero, i.e.,  $p = \tilde{p}$ .

For the second part, if  $V\mathbf{c} = 0$  for  $\mathbf{c} \in \mathbb{R}$  then for  $q(x) = c_1 + \cdots + c_n x^{n-1}$  we have

$$q(x_j) = \mathbf{e}_j^\top V\mathbf{c} = 0$$

hence  $q$  vanishes at  $n$  distinct points and is therefore 0, i.e.,  $\mathbf{c} = 0$ .

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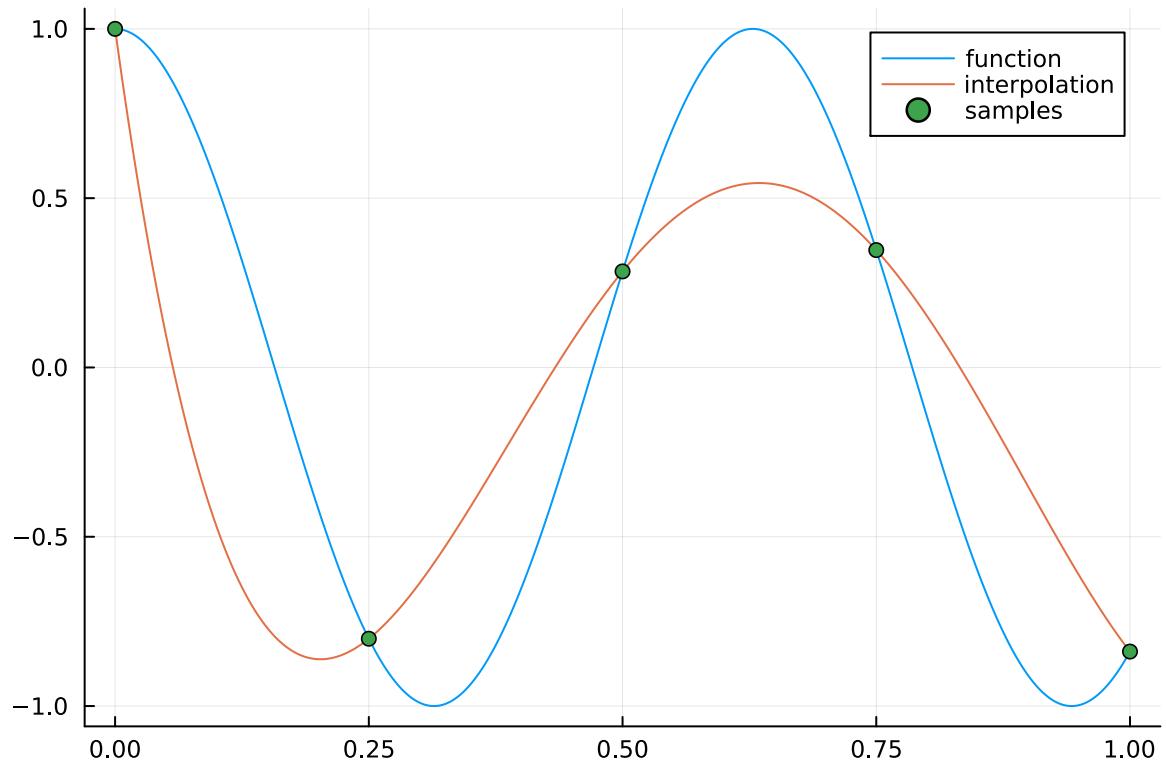
Thus a quick-and-dirty way to do interpolation is to invert the Vandermonde matrix (which we saw in the least squares setting with more samples than coefficients):

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In [1]: using Plots, LinearAlgebra
f = x -> cos(10x)
n = 5

x = range(0, 1; length=n) # evenly spaced points (BAD for interpolation)
V = x .^ (0:n-1)' # Vandermonde matrix
c = V \ f.(x) # coefficients of interpolatory polynomial
p = x -> dot(c, x .^ (0:n-1))

g = range(0,1; length=1000) # plotting grid
plot(g, f.(g); label="function")
plot!(g, p.(g); label="interpolation")
scatter!(x, f.(x); label="samples")
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Out[1]:



But it turns out we can also construct the interpolatory polynomial directly. We will use the following which equal 1 at one grid point and zero at the others:

**Definition 3 (Lagrange basis polynomial)** The *Lagrange basis polynomial* is defined as

$$\ell_k(x) := \prod_{j \neq k} \frac{x - x_j}{x_k - x_j} = \frac{(x - x_1) \cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_n)}{(x_k - x_1) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_n)}$$

Plugging in the grid points verifies the following:

**Proposition 2 (delta interpolation)**

$$\ell_k(x_j) = \delta_{kj}$$

We can use these to construct the interpolatory polynomial:

**Theorem 1 (Lagrange interpolation)** The unique polynomial of degree at most  $n - 1$  that interpolates  $f$  at  $n$  distinct points  $x_j$  is:

$$p(x) = f(x_1)\ell_1(x) + \cdots + f(x_n)\ell_n(x)$$

**Proof** Note that

$$p(x_j) = \sum_{k=1}^n f(x_k)\ell_k(x_j) = f(x_j)$$

so we just need to show it is unique. Suppose  $\tilde{p}(x)$  is a polynomial of degree at most  $n - 1$  that also interpolates  $f$ . Then  $\tilde{p} - p$  vanishes at  $n$  distinct points. Thus by the fundamental theorem of algebra it must be zero.

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**Example 1** We can interpolate  $\exp(x)$  at the points 0, 1, 2:

$$\begin{aligned} p(x) &= \ell_1(x) + e\ell_2(x) + e^2\ell_3(x) = \frac{(x-1)(x-2)}{(-1)(-2)} + e\frac{x(x-2)}{(-1)} + e^2\frac{x(x-1)}{2} \\ &= (1/2 - e + e^2/2)x^2 + (-3/2 + 2e - e^2/2)x + 1 \end{aligned}$$

**Remark** Interpolating at evenly spaced points is a really **bad** idea: interpolation is inherently ill-conditioned. The labs have explored this issue experimentally.

## 2. Interpolatory quadrature rules

By integrating an interpolant exactly we get a simple approach to approximating integrals. Using the Lagrange basis we can rewrite this procedure as a simple weighted sum:

**Definition 4 (interpolatory quadrature rule)** Given a set of points  $\mathbf{x} = [x_1, \dots, x_n]$  the interpolatory quadrature rule is:

$$\Sigma_n^{w,\mathbf{x}}[f] := \sum_{j=1}^n w_j f(x_j)$$

where

$$w_j := \int_a^b \ell_j(x) w(x) dx$$

**Proposition 3 (interpolatory quadrature is exact for polynomials)** Interpolatory quadrature is exact for all degree  $n - 1$  polynomials  $p$ :

$$\int_a^b p(x) w(x) dx = \Sigma_n^{w,\mathbf{x}}[f]$$

**Proof** The result follows since, by uniqueness of interpolatory polynomial:

$$p(x) = \sum_{j=1}^n p(x_j) \ell_j(x)$$

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**Example 2 (arbitrary points)** Find the interpolatory quadrature rule for  $w(x) = 1$  on  $[0, 1]$  with points  $[x_1, x_2, x_3] = [0, 1/4, 1]$ ? We have:

$$\begin{aligned}
w_1 &= \int_0^1 w(x)\ell_1(x)dx = \int_0^1 \frac{(x-1/4)(x-1)}{(-1/4)(-1)}dx = -1/6 \\
w_2 &= \int_0^1 w(x)\ell_2(x)dx = \int_0^1 \frac{x(x-1)}{(1/4)(-3/4)}dx = 8/9 \\
w_3 &= \int_0^1 w(x)\ell_3(x)dx = \int_0^1 \frac{x(x-1/4)}{3/4}dx = 5/18
\end{aligned}$$

That is we have

$$\Sigma_n^{w,x}[f] = -\frac{f(0)}{6} + \frac{8f(1/4)}{9} + \frac{5f(1)}{18}$$

This is indeed exact for polynomials up to degree 2 (and no more):

$$\Sigma_n^{w,x}[1] = 1, \Sigma_n^{w,x}[x] = 1/2, \Sigma_n^{w,x}[x^2] = 1/3, \Sigma_n^{w,x}[x^3] = 7/24 \neq 1/4.$$

**Example 3 (Chebyshev roots)** Find the interpolatory quadrature rule for

$w(x) = 1/\sqrt{1-x^2}$  on  $[-1, 1]$  with points equal to the roots of  $T_3(x)$ . This is a special case of Gaussian quadrature which we will approach in another way below. We use:

$$\int_{-1}^1 w(x)dx = \pi, \int_{-1}^1 xw(x)dx = 0, \int_{-1}^1 x^2w(x)dx = \pi/2$$

Recall from before that  $x_1, x_2, x_3 = \sqrt{3}/2, 0, -\sqrt{3}/2$ . Thus we have:

$$\begin{aligned}
w_1 &= \int_{-1}^1 w(x)\ell_1(x)dx = \int_{-1}^1 \frac{x(x+\sqrt{3}/2)}{(\sqrt{3}/2)\sqrt{3}\sqrt{1-x^2}}dx = \frac{\pi}{3} \\
w_2 &= \int_{-1}^1 w(x)\ell_2(x)dx = \int_{-1}^1 \frac{(x-\sqrt{3}/2)(x+\sqrt{3}/2)}{(-3/4)\sqrt{1-x^2}}dx = \frac{\pi}{3} \\
w_3 &= \int_{-1}^1 w(x)\ell_3(x)dx = \int_{-1}^1 \frac{(x-\sqrt{3}/2)x}{(-\sqrt{3})(-\sqrt{3}/2)\sqrt{1-x^2}}dx = \frac{\pi}{3}
\end{aligned}$$

(It's not a coincidence that they are all the same but this will differ for roots of other OPs.) That is we have

$$\Sigma_n^{w,x}[f] = \frac{\pi}{3}(f(\sqrt{3}/2) + f(0) + f(-\sqrt{3}/2))$$

This is indeed exact for polynomials up to degree  $n-1=2$ , but it goes all the way up to  $2n-1=5$ :

$$\begin{aligned}
\Sigma_n^{w,x}[1] &= \pi, \Sigma_n^{w,x}[x] = 0, \Sigma_n^{w,x}[x^2] = \frac{\pi}{2}, \\
\Sigma_n^{w,x}[x^3] &= 0, \Sigma_n^{w,x}[x^4] = \frac{3\pi}{8}, \Sigma_n^{w,x}[x^5] = 0 \\
\Sigma_n^{w,x}[x^6] &= \frac{9\pi}{32} \neq \frac{5\pi}{16}
\end{aligned}$$

We shall explain this miracle in the next chapter.