Floating point

Computer Systems
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Based on slides by:

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Today: Integer arithmetic and floating point

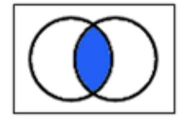
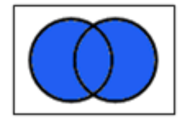
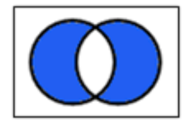
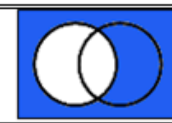
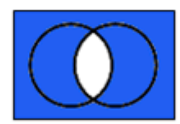
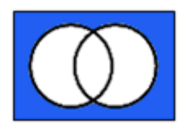
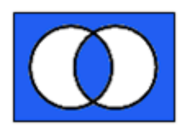
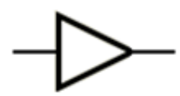
- **Recap**
  - Representing information as bits
  - Bit-level manipulations
  - Integers
- **Floating Points**

Everything is bits!

- Why bits? Why no decimals?
- What can we do with bits?
- How do we make integral values? Unsigned/ signed?

- **Do-it-yourself recap**

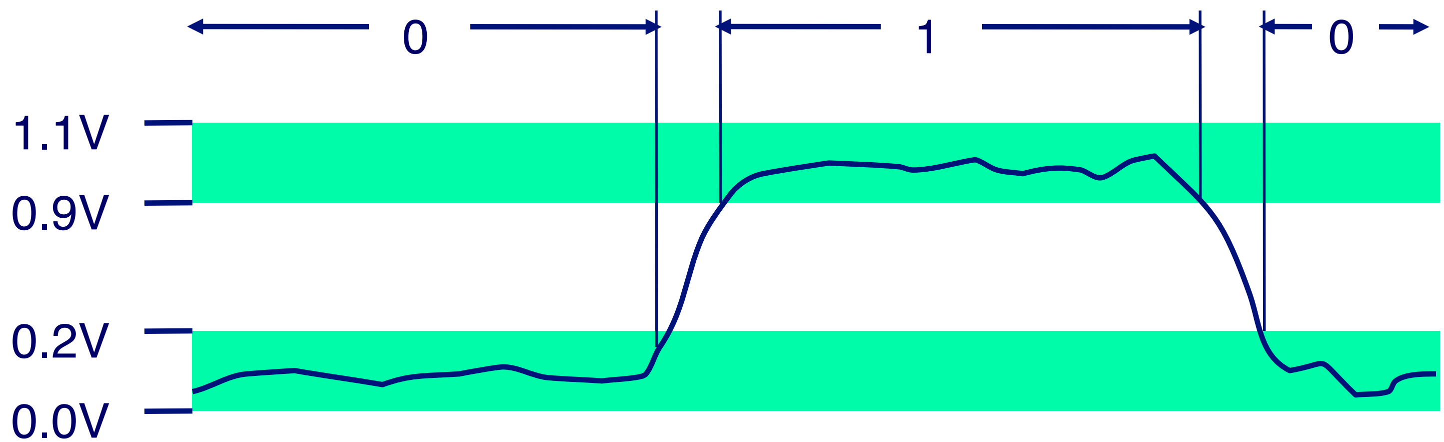
  5 minutes discussions!

Expression	Symbol	Venn diagram	Boolean algebra	Values
AND			$A \cdot B$	A B Output 0 0 0 0 1 0 1 0 0 1 1 1
OR			$A + B$	A B Output 0 0 0 0 1 1 1 0 1 1 1 1
XOR			$A \oplus B$	A B Output 0 0 0 0 1 1 1 0 1 1 1 0
NOT			$\overline{A}$	A Output 0 1 1 0
NAND			$\overline{A \cdot B}$	A B Output 0 0 1 0 1 1 1 0 1 1 1 0
NOR			$\overline{A + B}$	A B Output 0 0 1 0 1 0 1 0 0 1 1 0
XNOR			$\overline{A \oplus B}$	A B Output 0 0 1 0 1 0 1 0 0 1 1 1
BUF			A	IN Output 0 0 1 1

Venn Diagram for logic gates is a schematic representation of A and B overlapping each other inside a rectangle area, the diagram shows the relation of the boolean operators.

Everything is bits

- **Why bits? Electronic Implementation**
  - Easy to store with bistable elements
  - Reliably transmitted on noisy and inaccurate wires



- **... But there exist many models that are not**
  - E.g. Ternary (3-state) logic, analog computers, quantum computers

Encoding Byte Values

- **Byte = 8 bits**
  - Binary 00000000_2 to 11111111_2
  - Decimal: 0_{10} to 255_{10}
  - Hexadecimal 00_{16} to FF_{16}
    - Base 16 number representation
    - Use characters ‘0’ to ‘9’ and ‘A’ to ‘F’
    - Write $FA1D37B_{16}$ in C as
      – 0xFA1D37B
      – 0xfa1d37b

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

Example Data Representations

C Data Type	Typical 32-bit	Typical 64-bit	x86-64
`char`	1	1	1
`short`	2	2	2
`int`	4	4	4
`long`	4	8	8
`int32_t`	4	4	4
`int64_t`	8	8	8
`float`	4	4	4
`double`	8	8	8
`long double`	–	–	10/16
pointer	4	8	8

Boolean Algebra

- **Developed by George Boole in 19th Century**
  - Algebraic representation of logic
    - Encode "True" as 1 and "False" as 0

And

- A&B = 1 when both A=1 and B=1

&	0	1
0	0	0
1	0	1

Or

- A|B = 1 when either A=1 or B=1

|	0	1
0	0	1
1	1	1

Not

- ~A = 1 when A=0

~	
0	1
1	0

Exclusive-Or (Xor)

- A^B = 1 when either A=1 or B=1, but not both

^	0	1
0	0	1
1	1	0

Shift Operations

- **Left Shift: `x << y`**
  - Shift bit-vector **x** left **y** positions
    - – Throw away extra bits on left
    - Fill with 0's on right
- **Right Shift: `x >> y`**
  - Shift bit-vector **x** right **y** positions
    - Throw away extra bits on right
  - Logical shift
    - Fill with 0's on left
  - Arithmetic shift
    - Replicate most significant bit on left
- **Undefined Behavior**
  - Shift amount < 0 or ≥ word size

Argument x	01100010
<< 3	00010000
Log. >> 2	00011000
Arith. >> 2	00011000

Argument x	10100010
<< 3	00010000
Log. >> 2	00101000
Arith. >> 2	11101000

Encoding Integers

Unsigned

$$B2U(X) = \sum_{i=0}^{w-1} x_i \cdot 2^i$$

Two's Complement

$$B2T(X) = -x_{w-1} \cdot 2^{w-1} + \sum_{i=0}^{w-2} x_i \cdot 2^i$$

Sign Bit

```
short int x =  15213;
short int y = -15213;
```

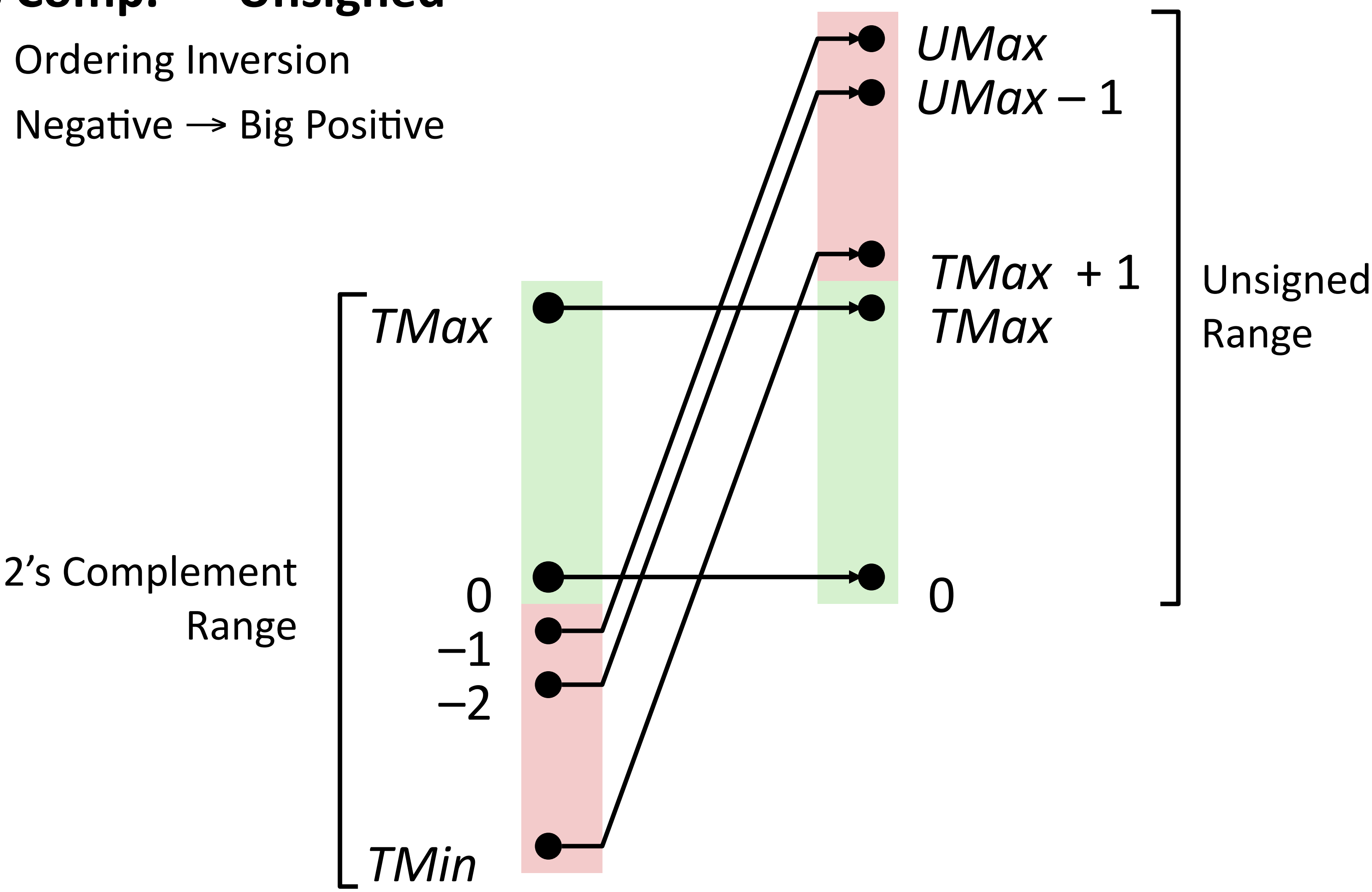
- **C `short` 2 bytes long**

	Decimal	Hex	Binary
`x`	15213	`3B 6D`	`00111011 01101101`
`y`	-15213	`C4 93`	`11000100 10010011`

- **Sign Bit**
  - For 2's complement, most significant bit indicates sign
    - 0 for nonnegative
    - 1 for negative

Conversion Visualized

- **2's Comp. → Unsigned**
  - Ordering Inversion
  - Negative → Big Positive



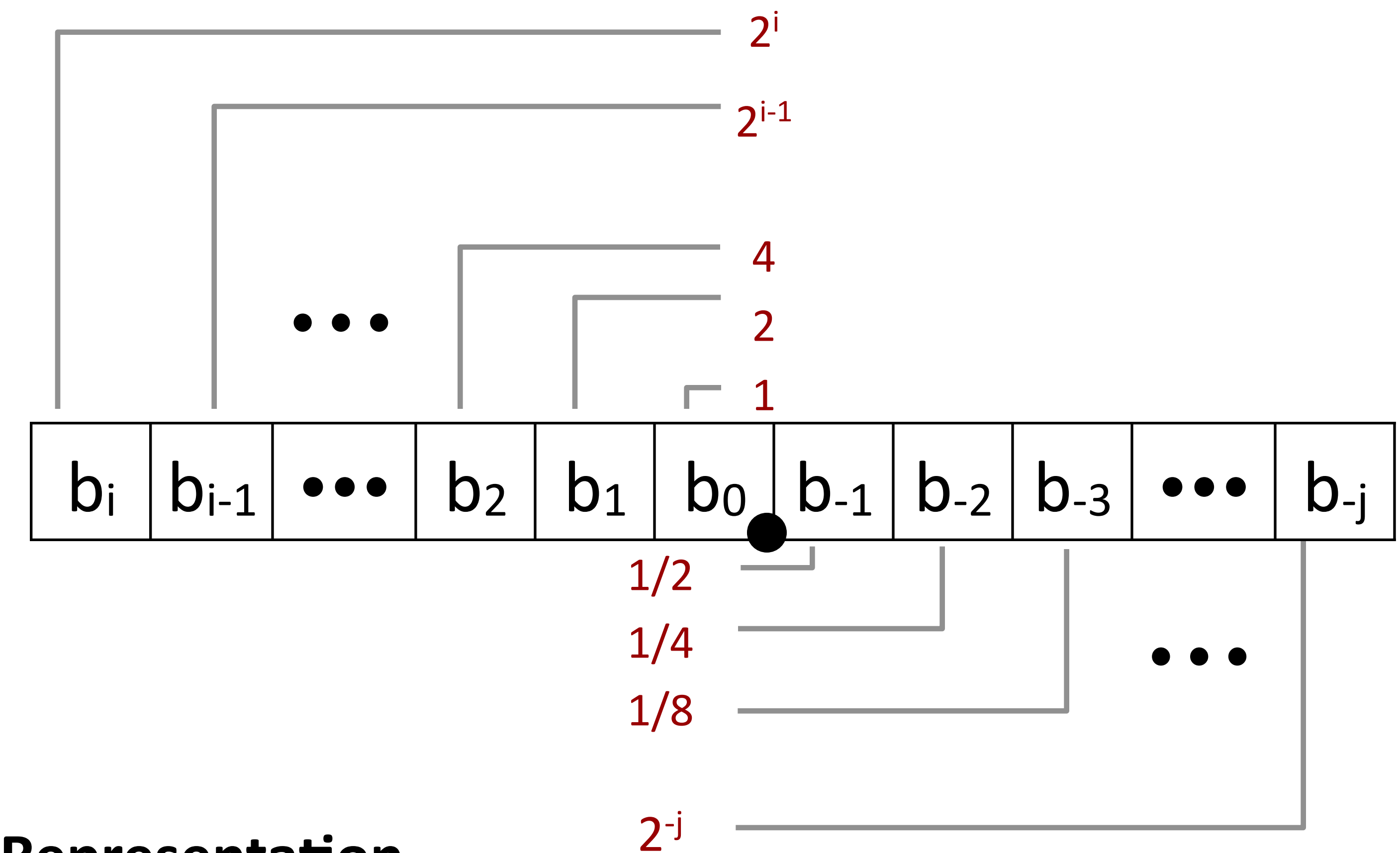
Today: Integer arithmetic and floating point

- Recap
- **Floating Points**
  - Background: Fractional binary numbers
  - IEEE floating point standard: Definition
  - Example and properties
  - Rounding, addition, multiplication
  - Floating point in C
  - Summary

Fractional binary numbers

- What is 1011.101_2?

Fractional Binary Numbers



- **Representation**
  - Bits to right of "binary point" represent fractional powers of 2
  - Represents rational number:

$$\sum_{k=-j}^{i} b_k \times 2^k$$

Fractional Binary Numbers: Examples

- **Value / Representation**

Value	Representation
5 3/4	101.11_2
2 7/8	10.111_2
1 7/16	1.0111_2

- **Observations**
  - Divide by 2 by shifting right (unsigned)
  - Multiply by 2 by shifting left
  - Numbers of form $0.111111\ldots_2$ are just below 1.0
    - $1/2 + 1/4 + 1/8 + \ldots + 1/2^i + \ldots \rightarrow 1.0$
    - Use notation $1.0 - \varepsilon$

Representable Numbers

- **Limitation #1**
  - Can only exactly represent numbers of the form $x/2^k$
    - Other rational numbers have repeating bit representations
  - Value Representation
    - 1/3 `0.0101010101[01]`...$_2$
    - 1/5 `0.001100110011[0011]`...$_2$
    - 1/10 `0.0001100110011[0011]`...$_2$

- **Limitation #2**
  - Just one setting of binary point within the *w* bits
    - Limited range of numbers (very small values? very large?)

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IEEE Floating Point

- **IEEE Standard 754**
  - Established in 1985 as uniform standard for floating point arithmetic
    - Before that, many idiosyncratic formats
  - Supported by all major CPUs

- **Driven by numerical concerns**
  - Standards for rounding, overflow, underflow
  - Hard to make fast in hardware
    - Numerical analysts predominated over hardware designers in defining standard

Floating Point Representation

- **Numerical Form:**

$$(-1)^{s}\ M\ \ 2^{E}$$

  - Sign bit s determines whether number is negative or positive
  - Significand M normally a fractional value in range [1.0,2.0).
  - Exponent E weights value by power of two

- **Encoding**
  - MSB s is sign bit s
  - exp field encodes E (but is not equal to E)
  - frac field encodes M (but is not equal to M)

s	exp	frac

Precision options

- **Single precision: 32 bits**

s	exp	frac
1	8-bits	23-bits

- **Double precision: 64 bits**

s	exp	frac
1	11-bits	52-bits

- **Extended precision: 80 bits (Intel only)**

s	exp	frac
1	15-bits	63 or 64-bits

"Normalized" Values

$v = (-1)^s M 2^E$

- **When: exp ≠ 000…0 and exp ≠ 111…1**

- **Exponent coded as a biased value: E = Exp – Bias**
  - Exp: unsigned value of exp field
  - Bias = 2^{k-1} - 1, where k is number of exponent bits
    - Single precision: 127 (Exp: 1…254, E: -126…127)
    - Double precision: 1023 (Exp: 1…2046, E: -1022…1023)

- **Significand coded with implied leading 1: M = $1.xxx…x_2$**
  - xxx…x: bits of frac field
  - Minimum when frac=000…0 (M = 1.0)
  - Maximum when frac=111…1 (M = 2.0 – ε)
  - Get extra leading bit for "free"

Normalized Encoding Example

$v = (-1)^s \, M \, 2^E$
$E = \text{Exp} - \text{Bias}$

- **Value:** `float F = 15213.0;`
  - $15213_{10} = 11101101101101_2$
    $= 1.1101101101101_2 \times 2^{13}$

- **Significand**

  M = `1.`$\underline{\texttt{1101101101101}}_2$

  `frac` = $\underline{\texttt{1101101101101}}\texttt{0000000000}_2$

- **Exponent**

  E = 13

  $Bias$ = 127

  Exp = 140 = `10001100`$_2$

- **Result:**

s	exp	frac
0	10001100	11011011011010000000000

Denormalized Values

$v = (-1)^s \, M \, 2^E$
$E = 1 - \text{Bias}$

- **Condition: exp = 000…0**

- **Exponent value: E = 1 – Bias (instead of E = 0 – Bias)**
- **Significand coded with implied leading 0: $M = 0.xxx...x_2$**
  - `xxx…x`: bits of `frac`
- **Cases**
  - `exp` = **000…0**, `frac` = **000…0**
    - Represents zero value
    - Note distinct values: +0 and –0 (why?)
  - `exp` = **000…0**, `frac` ≠ **000…0**
    - Numbers closest to 0.0
    - Equispaced

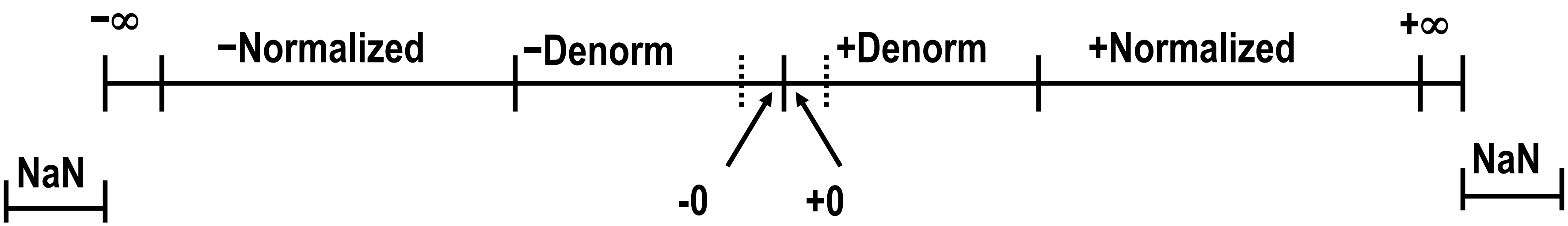
Special Values

- **Condition: `exp` = `111`…`1`**

- **Case: `exp` = `111`…`1`, `frac` = `000`…`0`**
  - Represents value ∞ (infinity)
  - Operation that overflows
  - Both positive and negative
  - E.g., 1.0/0.0 = −1.0/−0.0 = +∞, 1.0/−0.0 = −∞

- **Case: `exp` = `111`…`1`, `frac` ≠ `000`…`0`**
  - Not-a-Number (NaN)
  - Represents case when no numeric value can be determined
  - E.g., sqrt(–1), ∞ − ∞, ∞ × 0

Visualization: Floating Point Encodings



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Play the game

- https://topps.diku.dk/compsys/floating-point.html

Tiny Floating Point Example

s	exp	frac
1	4-bits	3-bits

- **8-bit Floating Point Representation**
  - the sign bit is in the most significant bit
  - the next four bits are the exponent, with a bias of 7
  - the last three bits are the `frac`

- **Same general form as IEEE Format**
  - normalized, denormalized
  - representation of 0, NaN, infinity

Dynamic Range (Positive Only)

$v = (-1)^s M 2^E$

n: E = Exp – Bias

d: E = 1 – Bias

```
s exp  frac E Value
0 0000 000 -6 0
0 0000 001 -6 1/8*1/64 = 1/512
0 0000 010 -6 2/8*1/64 = 2/512
…
0 0000 110 -6 6/8*1/64 = 6/512
0 0000 111 -6 7/8*1/64 = 7/512
0 0001 000 -6 8/8*1/64 = 8/512
0 0001 001 -6 9/8*1/64 = 9/512
…
0 0110 110 -1 14/8*1/2 = 14/16
0 0110 111 -1 15/8*1/2 = 15/16
0 0111 000 0 8/8*1    = 1
0 0111 001 0 9/8*1    = 9/8
0 0111 010 0 10/8*1   = 10/8
…
0 1110 110 7 14/8*128 = 224
0 1110 111 7 15/8*128 = 240
0 1111 000 n/a inf
```

Denormalized numbers: rows from 0 0000 000 to 0 0000 111 (closest to zero: 0 0000 001; largest denorm: 0 0000 111)

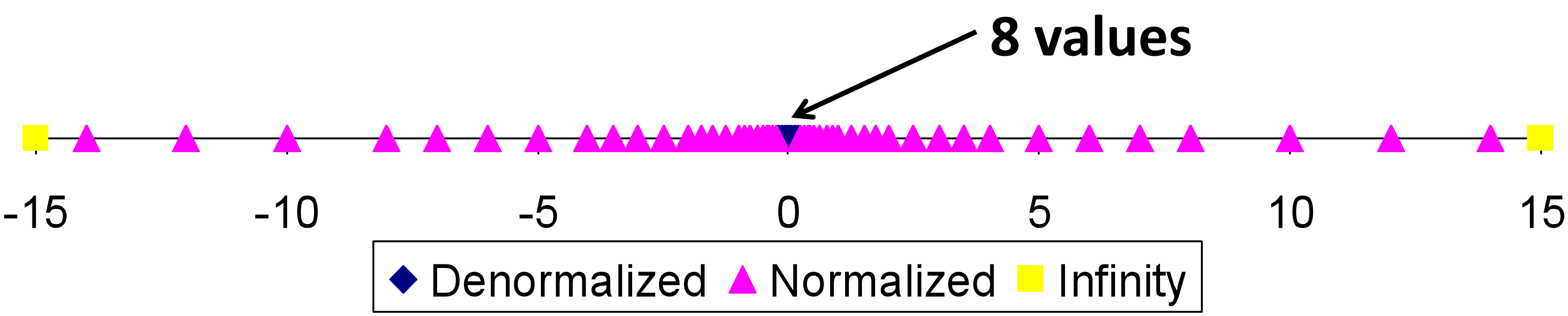
Normalized numbers: rows from 0 0001 000 to 0 1110 111 (smallest norm: 0 0001 000; closest to 1 below: 0 0110 111; closest to 1 above: 0 0111 001; largest norm: 0 1110 111)

Distribution of Values

- **6-bit IEEE-like format**
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is $2^{3-1}-1 = 3$

s	exp	frac
1	3-bits	2-bits

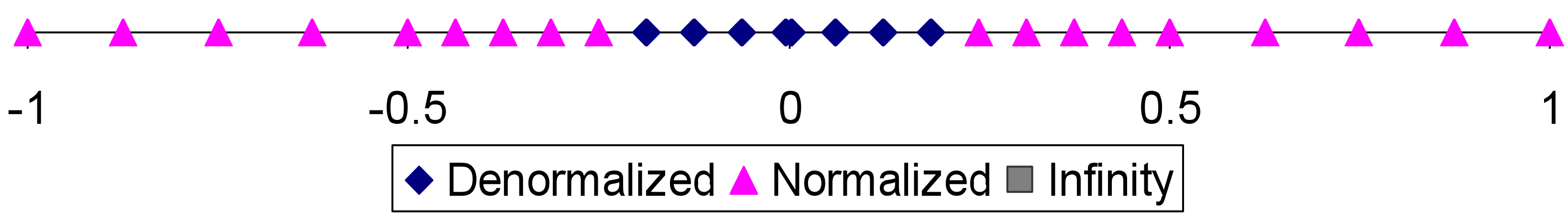
- **Notice how the distribution gets denser toward zero.**



Distribution of Values (close-up view)

- **6-bit IEEE-like format**
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is 3

s	exp	frac
1	3-bits	2-bits



Special Properties of the IEEE Encoding

- **FP Zero Same as Integer Zero**
  - All bits = 0
- **Can (Almost) Use Unsigned Integer Comparison**
  - Must first compare sign bits
  - Must consider −0 = 0
  - NaNs problematic
    - Will be greater than any other values
    - What should comparison yield?
  - Otherwise OK
    - Denorm vs. normalized
    - Normalized vs. infinity

For more details: https://www.gnu.org/software/libc/manual/html_node/Infinity-and-NaN.html

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Floating Point Operations: Basic Idea

- $x +_f y = \mathrm{Round}(x + y)$

- $x \times_f y = \mathrm{Round}(x \times y)$

- **Basic idea**
  - First compute exact result
  - Make it fit into desired precision
    - Possibly overflow if exponent too large
    - Possibly round to fit into `frac`

Rounding

- **Rounding Modes (illustrate with $ rounding)**

	$1.40	$1.60	$1.50	$2.50	–$1.50
Towards zero	$1	$1	$1	$2	–$1
Round down (−∞)	$1	$1	$1	$2	–$2
Round up (+∞)	$2	$2	$2	$3	–$1
Nearest Even (default)	$1	$2	$2	$2	–$2

Closer Look at Round-To-Even

- **Default Rounding Mode**
  - Hard to get any other kind without dropping into assembly
  - All others are statistically biased
    - Sum of set of positive numbers will consistently be over- or under-estimated

- **Applying to Other Decimal Places / Bit Positions**
  - When exactly halfway between two possible values
    - Round so that least significant digit is even
  - E.g., round to nearest hundredth

7.8949999	7.89	(Less than half way)
7.8950001	7.90	(Greater than half way)
7.8950000	7.90	(Half way—round up)
7.8850000	7.88	(Half way—round down)

Rounding Binary Numbers

- **Binary Fractional Numbers**
  - "Even" when least significant bit is `0`
  - "Half way" when bits to right of rounding position = $100\ldots_2$

- **Examples**
  - Round to nearest 1/4 (2 bits right of binary point)

Value	Binary	Rounded	Action	Rounded Value
2 3/32	10.00011_2	10.00_2	(<1/2—down)	2
2 3/16	10.00110_2	10.01_2	(>1/2—up)	2 1/4
2 7/8	10.11100_2	11.00_2	(1/2—up)	3
2 5/8	10.10100_2	10.10_2	(1/2—down)	2 1/2

FP Multiplication

- **$(-1)^{s1}$ M1 2^{E1} x $(-1)^{s2}$ M2 2^{E2}**
- **Exact Result: $(-1)^{s}$ M 2^{E}**
  - Sign s: s1 ^ s2
  - Significand M: M1 x M2
  - Exponent E: E1 + E2

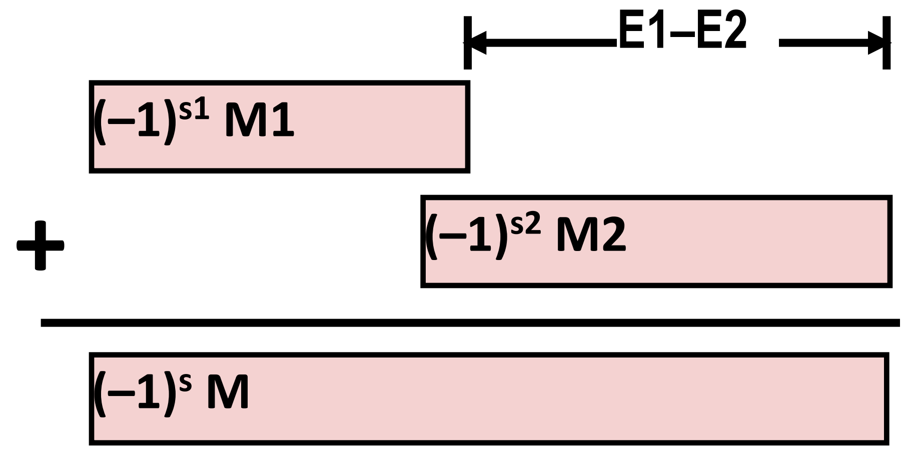
- **Fixing**
  - If M ≥ 2, shift M right, increment E
  - If E out of range, overflow
  - Round M to fit `frac` precision

- **Implementation**
  - Biggest chore is multiplying significands

Floating Point Addition

- $(-1)^{s1}\,M1\ \ 2^{E1}\ \ +\ \ (-1)^{s2}\,M2\ \ 2^{E2}$
  - Assume E1 > E2

Get binary points lined up



- **Exact Result:** $(-1)^{s}\,M\ \ 2^{E}$
  - Sign s, significand M:
    - Result of signed align & add
  - Exponent E: E1

- **Fixing**
  - If M ≥ 2, shift M right, increment E
  - if M < 1, shift M left k positions, decrement E by k
  - Overflow if E out of range
  - Round M to fit `frac` precision

Mathematical Properties of FP Add

- **Compare to those of Abelian Group**
  - Closed under addition? **Yes**
    - But may generate infinity or NaN
  - Commutative? **Yes**
  - Associative? **No**
    - Overflow and inexactness of rounding
    - `(3.14+1e10)-1e10 = 0, 3.14+(1e10-1e10) = 3.14`
  - 0 is additive identity? **Yes**
  - Every element has additive inverse? **Almost**
    - Yes, except for infinities & NaNs
- **Monotonicity** **Almost**
  - $a \geq b \Rightarrow a+c \geq b+c$?
    - Except for infinities & NaNs

Mathematical Properties of FP Mult

- **Compare to Commutative Ring**
  - Closed under multiplication? **Yes**
    - But may generate infinity or NaN
  - Multiplication Commutative? **Yes**
  - Multiplication is Associative? **No**
    - Possibility of overflow, inexactness of rounding
    - Ex: `(1e20*1e20)*1e-20`= `inf`, `1e20*(1e20*1e-20)`= `1e20`
  - 1 is multiplicative identity? **Yes**
  - Multiplication distributes over addition? **No**
    - Possibility of overflow, inexactness of rounding
    - `1e20*(1e20-1e20)`= `0.0`, `1e20*1e20 - 1e20*1e20` = `NaN`
- **Monotonicity**
  - $a \geq b$ & $c \geq 0 \Rightarrow a * c \geq b * c$? **Almost**
    - Except for infinities & NaNs

Today: Integer arithmetic and floating point

- Recap
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  - Background: Fractional binary numbers
  - IEEE floating point standard: Definition
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Floating Point in C

- **C Guarantees Two Levels**
  - `float`   single precision
  - `double`   double precision

- **Conversions/Casting**
  - Casting between `int`, `float`, and `double` changes bit representation
  - `double`/`float` → `int`
    - Truncates fractional part
    - Like rounding toward zero
    - Not defined when out of range or NaN: Generally sets to TMin
  - `int` → `double`
    - Exact conversion, as long as `int` has ≤ 53 bit word size
  - `int` → `float`
    - Will round according to rounding mode

Floating Point Puzzles

- **For each of the following C expressions, either:**
  - Argue that it is true for all argument values
  - Explain why not true

```
int x = …;
float f = …;
double d = …;
```

Assume neither `d` nor `f` is NaN

- `x == (int)(float) x`
- `x == (int)(double) x`
- `f == (float)(double) f`
- `d == (double)(float) d`
- `f == -(-f);`
- `2/3 == 2/3.0`
- `d < 0.0` ⇒ `((d*2) < 0.0)`
- `d > f` ⇒ `-f > -d`
- `d * d >= 0.0`
- `(d+f)-d == f`

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Summary

- **IEEE Floating Point has clear mathematical properties**
- **Represents numbers of form M x 2^E**
- **One can reason about operations independent of implementation**
  - As if computed with perfect precision and then rounded
- **Not the same as real arithmetic**
  - Violates associativity/distributivity
  - Makes life difficult for compilers & serious numerical applications programmers

Creating Floating Point Number

s	exp	frac
1	4-bits	3-bits

- **Steps**
  - Normalize to have leading 1
  - Round to fit within fraction
  - Postnormalize to deal with effects of rounding

- **Case Study**
  - Convert 8-bit unsigned numbers to tiny floating point format

Example Numbers

`128`	`10000000`
`15`	`00001101`
`33`	`00010001`
`35`	`00010011`
`138`	`10001010`
`63`	`00111111`

Normalize

s	exp	frac
1	4-bits	3-bits

- **Requirement**
  - Set binary point so that numbers of form 1.xxxxx
  - Adjust all to have leading one
    - Decrement exponent as shift left

Value	Binary	Fraction	Exponent
128	10000000	1.0000000	7
15	00001101	1.1010000	3
17	00010001	1.0001000	4
19	00010011	1.0011000	4
138	10001010	1.0001010	7
63	00111111	1.1111100	5

Rounding

1.BBGRXXX

Guard bit: LSB of result

Sticky bit: OR of remaining bits

Round bit: 1st bit removed

- **Round up conditions**
  - Round = 1, Sticky = 1 ➙ > 0.5
  - Guard = 1, Round = 1, Sticky = 0 ➙ Round to even

Value	Fraction	GRS	Incr?	Rounded
`128`	`1.0000000`	`000`	`N`	`1.000`
`15`	`1.1010000`	`100`	`N`	`1.101`
`17`	`1.0001000`	`010`	`N`	`1.000`
`19`	`1.0011000`	`110`	`Y`	`1.010`
`138`	`1.0001010`	`011`	`Y`	`1.001`
`63`	`1.1111100`	`111`	`Y`	`10.000`

Postnormalize

- **Issue**
  - Rounding may have caused overflow
  - Handle by shifting right once & incrementing exponent

Value	Rounded	Exp	Adjusted	Result
`128`	`1.000`	`7`		`128`
`15`	`1.101`	`3`		`15`
`17`	`1.000`	`4`		`16`
`19`	`1.010`	`4`		`20`
`138`	`1.001`	`7`		`134`
`63`	`10.000`	`5`	`1.000/6`	`64`

Interesting Numbers `{single,double}`

Description	*exp*	*frac*	*Numeric Value*
■ **Zero**	**00…00**	**00…00**	**0.0**
■ **Smallest Pos. Denorm.**	**00…00**	**00…01**	$2^{-\{23,52\}} \times 2^{-\{126,1022\}}$
▪ Single ≈ 1.4×10^{-45}			
▪ Double ≈ 4.9×10^{-324}			
■ **Largest Denormalized**	**00…00**	**11…11**	$(1.0 - \varepsilon) \times 2^{-\{126,1022\}}$
▪ Single ≈ 1.18×10^{-38}			
▪ Double ≈ 2.2×10^{-308}			
■ **Smallest Pos. Normalized**	**00…01**	**00…00**	$1.0 \times 2^{-\{126,1022\}}$
▪ Just larger than largest denormalized			
■ **One**	**01…11**	**00…00**	**1.0**
■ **Largest Normalized**	**11…10**	**11…11**	$(2.0 - \varepsilon) \times 2^{\{127,1023\}}$
▪ Single ≈ 3.4×10^{38}			
▪ Double ≈ 1.8×10^{308}			