CSC236 – Tutorial 1: Simple Induction

Exercise 1: Binary Palindromes

Consider strings made up only of the characters 0 and 1; these are **binary strings**. A **binary palindrome** is a palindrome that is also a binary string.
Let $f(n)$ be the number of binary palindromes of length $2n$, for $n \geq 0$. We claim that $f(n) = 2^n$.
Evaluate each of the following proof attempts. Is each proof acceptable? If not, why not?

Attempt 1

The only length-0 string is the empty string, and that is a palindrome. So, we have $1 = 2^0$ palindrome of length 0, as required.
Now for length $2n$. Assume $2n \to 2^n$ palindromes. We want to prove this for $n+1$, which is length $2n+2$. A string of length $2n+2$ has a 0 at the beginning and end, or a 1 at the beginning and end, with $2n$ symbols in between. Each of these imply 2^n palindromes. $2^n + 2^n = 2^{n+1}$.

Attempt 2

The predicate here is $P(n) : f(n) = 2^n$. We must prove that $\forall n \geq 0, P(n)$.
First, the **base case**. When $n = 0$, the length of our binary strings is $2n = 2 * 0 = 0$, so we must show that there are $2^0 = 1$ palindromes of length 0. The only binary string of length 0 is the empty string, and the empty string is a palindrome. So, there is 1 palindrome of length 0 as required.
Now, the **inductive step**. Let $n \geq 0$, and assume $P(n)$ holds.
Our **inductive hypothesis** is as follows: $f(n) = 2^n$.
And what we are trying to prove is: $f(n+1) = 2^{n+1}$.
$f(n+1)$ is the number of binary palindromes of length $2(n+1) = 2n+2$. $f(n)$ is the number of binary palindromes of length $2n$, and by the inductive hypothesis, we know that there are 2^n binary palindromes of length $2n$. For each of those $2n$ smaller palindromes, we can add a 0 to the beginning and the end, or a 1 to the beginning and the end, to get a palindrome of length $2n+2$. So the total number of binary palindromes of length $2n+2$ is $2 * 2^n = 2^{n+1}$.
Therefore, $P(n+1)$ holds. This completes the inductive step.

Exercise 2: Stamps

Assume that you have an unlimited supply of 3c and 4c stamps (c stands for cents). Use simple induction to prove that every amount greater than or equal to 6c can be made using 3c and 4c stamps.

Exercise 3: Subsets

Evaluate the following proof attempt that, for all $n \in \mathbb{N}$, every set of size n has exactly 2^n subsets.
A set S of size $n = 1$ has two subsets (the empty set, and S itself), and $2^1 = 2$ as required.
For the inductive case, suppose that $n \geq 1$ and that every set of size $n - 1$ has exactly 2^{n-1} subsets. Consider set $T = \{a_1, a_2, \ldots, a_n\}$. Some subsets of T include a_n, and, by the IH, there are 2^{n-1} of these, one for each subset of $\{a_1, a_2, \ldots, a_{n-1}\}$. Other subsets of T do **NOT** include a_n; again, by the IH, there are 2^{n-1} of these. These are all of the subsets of T, totaling $2^{n-1} + 2^{n-1} = 2^n$, as required.