Chapter 40

1. (a) This is computed in part (a) of Sample Problem 40-1.

   (b) With $m_p = 1.67 \times 10^{-27}$ kg, we obtain

$$E_1 = \left(\frac{h^2}{8mL^2}\right) n^2 = \left(\frac{(6.63 \times 10^{-34}\,\text{J}\cdot\text{s})^2}{8m_p(100 \ \times 10^{12}\,\text{m})^2}\right)(1)^2 = 3.29 \times 10^{-21}\,\text{J} = 0.0206 \text{ eV} .$$

2. According to Eq. 40-4 $E_n \propto L^{-2}$. As a consequence, the new energy level E'_n satisfies

$$\frac{E'_n}{E_n} = \left(\frac{L'}{L}\right)^{-2} = \left(\frac{L}{L'}\right)^2 = \frac{1}{2} ,$$

   which gives $L' = \sqrt{2}L$. Thus, the width of the potential well must be multiplied by a factor of $\sqrt{2}$.

3. To estimate the energy, we use Eq. 40-4, with $n = 1$, L equal to the atomic diameter, and m equal to the mass of an electron:

$$E = n^2 \frac{h^2}{8mL^2} = \frac{(1)^2(6.63 \times 10^{-34}\,\text{J}\cdot\text{s})^2}{8(9.11 \times 10^{-31}\,\text{kg})(1.4 \times 10^{-14}\,\text{m})^2} = 3.07 \times 10^{-10}\,\text{J} = 1920 \text{ MeV} .$$

4. We can use the mc^2 value for an electron from Table 38-3 (511×10^3 eV) and the hc value developed in problem 3 of Chapter 39 by writing Eq. 40-4 as

$$E_n = \frac{n^2h^2}{8mL^2} = \frac{n^2(hc)^2}{8(mc^2)L^2} .$$

   For $n = 3$, we set this expression equal to 4.7 eV and solve for L:

$$L = \frac{n(hc)}{\sqrt{8(mc^2)E_n}} = \frac{3(1240\,\text{eV}\cdot\text{nm})}{\sqrt{8(511 \times 10^3\,\text{eV})(4.7\,\text{eV})}} = 0.85 \text{ nm} .$$

5. With $m_p = 1.67 \times 10^{-27}$ kg, we obtain

$$E_1 = \left(\frac{h^2}{8mL^2}\right) n^2 = \left(\frac{(6.63 \times 10^{-34}\,\text{J}\cdot\text{s})^2}{8m_p(100 \ \times 10^{12}\,\text{m})^2}\right)(1)^2 = 3.29 \times 10^{-21}\,\text{J} = 0.0206 \text{ eV} .$$

   Alternatively, we can use the mc^2 value for a proton from Table 38-3 (938×10^6 eV) and the $hc = 1240\,\text{eV}\cdot\text{nm}$ value developed in problem 3 of Chapter 39 by writing Eq. 40-4 as

$$E_n = \frac{n^2h^2}{8mL^2} = \frac{n^2(hc)^2}{8(m_pc^2)L^2} .$$

   This alternative approach is perhaps easier to plug into, but it is recommended that both approaches be tried to find which is most convenient.

6. Since $E_n \propto L^{-2}$ in Eq. 40-4, we see that if L is doubled, then E_1 becomes $(2.6\,\text{eV})(2)^{-2} = 0.65\,\text{eV}$.

7. We can use the mc^2 value for an electron from Table 38-3 (511×10^3 eV) and the hc value developed in problem 3 of Chapter 39 by writing Eq. 40-4 as

$$E_n = \frac{n^2h^2}{8mL^2} = \frac{n^2(hc)^2}{8(mc^2)L^2} \; .$$

The energy to be absorbed is therefore

$$\begin{aligned} \Delta E &= E_4 - E_1 = \frac{(4^2-1^2)h^2}{8m_eL^2} = \frac{15(hc)^2}{8(m_ec^2)L^2} \\ &= \frac{15(1240\,\text{eV}\cdot\text{nm})^2}{8(511\times10^3\,\text{eV})(0.250\,\text{nm})^2} = 90.3\text{ eV} \; . \end{aligned}$$

8. (a) Let the quantum numbers of the pair in question be n and $n+1$, respectively. We note that

$$E_{n+1} - E_n = \frac{(n+1)^2h^2}{8mL^2} - \frac{n^2h^2}{8mL^2} = \frac{(2n+1)h^2}{8mL^2}$$

Therefore, $E_{n+1} - E_n = (2n+1)E_1$. Now

$$E_{n+1} - E_n = E_5 = 5^2E_1 = 25E_1 = (2n+1)E_1 \; ,$$

which leads to $2n+1 = 25$, or $n = 12$.

(b) Now let

$$E_{n+1} - E_n = E_6 = 6^2E_1 = 36E_1 = (2n+1)E_1 \; ,$$

which gives $2n+1 = 36$, or $n = 17.5$. This is not an integer, so it is impossible to find the pair that fits the requirement.

9. From Eq. 40-4

$$E_{n+2} - E_n = \left(\frac{h^2}{8mL^2}\right)(n+2)^2 - \left(\frac{h^2}{8mL^2}\right)n^2 = \left(\frac{h^2}{2mL^2}\right)(n+1) \; .$$

10. (a) Let the quantum numbers of the pair in question be n and $n+1$, respectively. Then $E_{n+1} - E_n = E_1(n+1)^2 - E_1n^2 = (2n+1)E_1$. Letting

$$E_{n+1} - E_n = (2n+1)E_1 = 3(E_4 - E_3) = 3(4^2E_1 - 3^2E_1) = 21E_1 \; ,$$

we get $2n+1 = 21$, or $n = 10$.

(b) Now letting

$$E_{n+1} - E_n = (2n+1)E_1 = 2(E_4 - E_3) = 2(4^2E_1 - 3^2E_1) = 14E_1 \; ,$$

we get $2n+1 = 14$, which does not have an integer-valued solution. So it is impossible to find the pair of energy levels that fits the requirement.

11. The energy levels are given by $E_n = n^2h^2/8mL^2$, where h is the Planck constant, m is the mass of an electron, and L is the width of the well. The frequency of the light that will excite the electron from the state with quantum number n_i to the state with quantum number n_f is $f = \Delta E/h = (h/8mL^2)(n_f^2 - n_i^2)$ and the wavelength of the light is

$$\lambda = \frac{c}{f} = \frac{8mL^2c}{h(n_f^2 - n_i^2)} \; .$$

We evaluate this expression for $n_i = 1$ and $n_f = 2$, 3, 4, and 5, in turn. We use $h = 6.626 \times 10^{-34}$ J·s, $m = 9.109 \times 10^{-31}$ kg, and $L = 250 \times 10^{-12}$ m, and obtain 6.87×10^{-8} m for $n_f = 2$, 2.58×10^{-8} m for $n_f = 3$, 1.37×10^{-8} m for $n_f = 4$, and 8.59×10^{-9} m for $n_f = 5$.

12. We can use the mc^2 value for an electron from Table 38-3 (511×10^3 eV) and the hc value developed in problem 3 of Chapter 39 by rewriting Eq. 40-4 as

$$E_n = \frac{n^2h^2}{8mL^2} = \frac{n^2(hc)^2}{8(mc^2)L^2} \ .$$

(a) The first excited state is characterized by $n = 2$, and the third by $n' = 4$. Thus,

$$\begin{aligned} \Delta E &= \frac{(hc)^2}{8(mc^2)L^2}\left(n'^2 - n^2\right) \\ &= \frac{(1240\,\text{eV}\cdot\text{nm})^2}{8(511 \times 10^3\,\text{eV})(0.250\,\text{nm})^2}\left(4^2 - 2^2\right) \\ &= (6.02\,\text{eV})(16 - 4) \end{aligned}$$

which yields $\Delta E = 72.2\,\text{eV}$.

(b) Now that the electron is in the $n' = 4$ level, it can "drop" to a lower level (n'') in a variety of ways. Each of these drops is presumed to cause a photon to be emitted of wavelength

$$\lambda = \frac{hc}{E_{n'} - E_{n''}} = \frac{8(mc^2)L^2}{hc\left(n'^2 - n''^2\right)} \ .$$

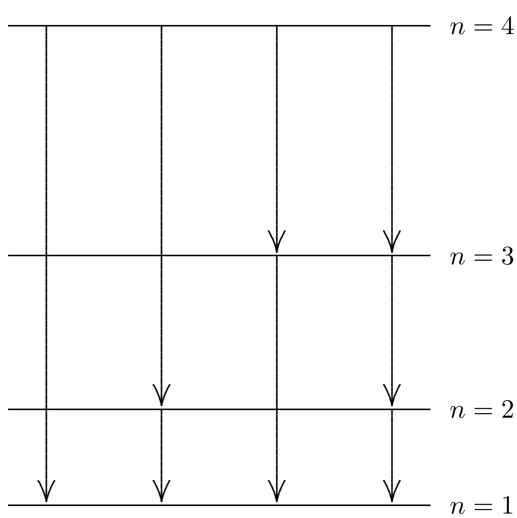
For example, for the transition $n' = 4$ to $n'' = 3$, the photon emitted would have wavelength

$$\lambda = \frac{8(511 \times 10^3\,\text{eV})(0.250\,\text{nm})^2}{(1240\,\text{eV}\cdot\text{nm})\left(4^2 - 3^2\right)} = 29.4 \text{ nm} \ ,$$

and once it is then in level $n'' = 3$ it might fall to level $n''' = 2$ emitting another photon. Calculating in this way all the possible photons emitted during the de-excitation of this system, we find $\lambda_{4\to1} = 13.7\,\text{nm}$, $\lambda_{4\to2} = 17.2\,\text{nm}$, $\lambda_{3\to1} = 25.8\,\text{nm}$, $\lambda_{4\to3} = 29.4\,\text{nm}$, $\lambda_{3\to2} = 41.2\,\text{nm}$, and $\lambda_{2\to1} = 68.7\,\text{nm}$.

(c) A system making the $4 \to 1$ transition will make no further transitions unless it is re-excited. If it makes the $4 \to 2$ transition, then that is likely to followed by the $2 \to 1$ transition. However, if it makes the $4 \to 3$ transition, then it could make either the $3 \to 1$ transition or the pair of transitions: $3 \to 2$ and $2 \to 1$.

(d) The possible transitions are shown below. The energy levels are not drawn to scale.



13. We can use the mc^2 value for an electron from Table 38-3 (511×10^3 eV) and the hc value developed in problem 3 of Chapter 39 by writing Eq. 40-4 as

$$E_n = \frac{n^2h^2}{8mL^2} = \frac{n^2(hc)^2}{8(mc^2)L^2} .$$

(a) With $L = 3.0 \times 10^9$ nm, the energy difference is

$$E_2 - E_1 = \frac{1240^2}{8(511 \times 10^3)(3.0 \times 10^9)^2}\left(2^2 - 1^2\right) = 1.3 \times 10^{-19} \text{ eV} .$$

(b) Since $(n+1)^2 - n^2 = 2n+1$, we have

$$\Delta E = E_{n+1} - E_n = \frac{h^2}{8mL^2}(2n+1) = \frac{(hc)^2}{8(mc^2)L^2}(2n+1) .$$

Setting this equal to 1.0 eV, we solve for n:

$$\begin{aligned} n &= \frac{4(mc^2)L^2\Delta E}{(hc)^2} - \frac{1}{2} \\ &= \frac{4(511 \times 10^3 \text{ eV})(3.0 \times 10^9 \text{ nm})^2(1.0 \text{ eV})}{(1240 \text{ eV}\cdot\text{nm})^2} - \frac{1}{2} \\ &\approx 12 \times 10^{18} . \end{aligned}$$

(c) At this value of n, the energy is

$$E_n = \frac{1240^2}{8(511 \times 10^3)(3.0 \times 10^9)^2}\left(6 \times 10^{18}\right)^2 \approx 6 \times 10^{18} \text{ eV} .$$

(d) Since

$$\frac{E_n}{mc^2} = \frac{6 \times 10^{18} \text{ eV}}{511 \times 10^3 \text{ eV}} \gg 1 ,$$

the energy is indeed in the relativistic range.

14. (a) With Eq. 40-11, we compare the ψ_1^2 and ψ_2^2 graphs in Fig. 40-6. The former has a maximum at the center and the latter is zero there. Thus, the excitation of the system described in this problem implies the electron has become much less likely to be detected near the middle of the well.

(b) Examining the $0 \le x \le 25$ pm regions of those two graphs, we conclude that the excited state electron is somewhat more likely to be "near" (not "at") a well wall. Eq. 40-13 supports this conclusion in the sense that there is more "area" under the curve of ψ_2^2 in the $0 \le x \le 25$ pm region than under the ψ_1^2 curve for that region.

15. (a) The allowed energy values are given by $E_n = n^2h^2/8mL^2$. The difference in energy between the state n and the state $n+1$ is

$$\Delta E_{\text{adj}} = E_{n+1} - E_n = \left[(n+1)^2 - n^2\right]\frac{h^2}{8mL^2} = \frac{(2n+1)h^2}{8mL^2}$$

and

$$\frac{\Delta E_{\text{adj}}}{E} = \left[\frac{(2n+1)h^2}{8mL^2}\right]\left(\frac{8mL^2}{n^2h^2}\right) = \frac{2n+1}{n^2} .$$

As n becomes large, $2n+1 \longrightarrow 2n$ and $(2n+1)/n^2 \longrightarrow 2n/n^2 = 2/n$.

(b) As $n \longrightarrow \infty$, ΔE_{adj} and E do not approach 0, but $\Delta E_{\text{adj}}/E$ does.

(c) See part (b).

(d) See part (b).

(e) $\Delta E_{\text{adj}}/E$ is a better measure than either ΔE_{adj} or E alone of the extent to which the quantum result is approximated by the classical result.

16. We follow Sample Problem 40-3 in the presentation of this solution. The integration result quoted below is discussed in a little more detail in that Sample Problem. We note that the arguments of the sine functions used below are in radians.

(a) The probability of detecting the particle in the region $0 \le x \le \frac{L}{4}$ is

$$\left(\frac{2}{L}\right)\left(\frac{L}{\pi}\right)\int_0^{\pi/4} \sin^2 y\, dy = \frac{2}{\pi}\left(\frac{y}{2} - \frac{\sin 2y}{4}\right)_0^{\pi/4} = 0.091 \ .$$

(b) As expected from symmetry,

$$\left(\frac{2}{L}\right)\left(\frac{L}{\pi}\right)\int_{\pi/4}^{\pi} \sin^2 y\, dy = \frac{2}{\pi}\left(\frac{y}{2} - \frac{\sin 2y}{4}\right)_{\pi/4}^{\pi} = 0.091 \ .$$

(c) For the region $\frac{L}{4} \le x \le \frac{3L}{4}$, we obtain

$$\left(\frac{2}{L}\right)\left(\frac{L}{\pi}\right)\int_{\pi/4}^{3\pi/4} \sin^2 y\, dy = \frac{2}{\pi}\left(\frac{y}{2} - \frac{\sin 2y}{4}\right)_{\pi/4}^{3\pi/4} = 0.82$$

which we could also have gotten by subtracting the results of part (a) and (b) from 1; that is, $1 - 2(0.091) = 0.82$.

17. The probability that the electron is found in any interval is given by $P = \int |\psi|^2\, dx$, where the integral is over the interval. If the interval width Δx is small, the probability can be approximated by $P = |\psi|^2\, \Delta x$, where the wave function is evaluated for the center of the interval, say. For an electron trapped in an infinite well of width L, the ground state probability density is

$$|\psi|^2 = \frac{2}{L}\sin^2\left(\frac{\pi x}{L}\right) \ ,$$

so

$$P = \left(\frac{2\,\Delta x}{L}\right)\sin^2\left(\frac{\pi x}{L}\right) \ .$$

(a) We take $L = 100\,\text{pm}$, $x = 25\,\text{pm}$, and $\Delta x = 5.0\,\text{pm}$. Then,

$$P = \left[\frac{2(5.0\,\text{pm})}{100\,\text{pm}}\right]\sin^2\left[\frac{\pi(25\,\text{pm})}{100\,\text{pm}}\right] = 0.050 \ .$$

(b) We take $L = 100\,\text{pm}$, $x = 50\,\text{pm}$, and $\Delta x = 5.0\,\text{pm}$. Then,

$$P = \left[\frac{2(5.0\,\text{pm})}{100\,\text{pm}}\right]\sin^2\left[\frac{\pi(50\,\text{pm})}{100\,\text{pm}}\right] = 0.10 \ .$$

(c) We take $L = 100\,\text{pm}$, $x = 90\,\text{pm}$, and $\Delta x = 5.0\,\text{pm}$. Then,

$$P = \left[\frac{2(5.0\,\text{pm})}{100\,\text{pm}}\right]\sin^2\left[\frac{\pi(90\,\text{pm})}{100\,\text{pm}}\right] = 0.0095 \ .$$

18. (a) We recall that a derivative with respect to a dimensional quantity carries the (reciprocal) units of that quantity. Thus, the first term in Eq. 40-18 has dimensions of ψ multiplied by dimensions of x^{-2}. The second term contains no derivatives, does contain ψ, and involves several other factors that (as we show below) turn out to have dimensions of x^{-2}:

$$\frac{8\pi^2 m}{h^2}\,[E - U(x)] \quad\Longrightarrow\quad \frac{\text{kg}}{(\text{J}\cdot\text{s})^2}[\text{J}]$$

assuming SI units. Recalling from Eq. 7-9 that $\text{J} = \text{kg}\cdot\text{m}^2/\text{s}^2$, then we see the above is indeed in units of m^{-2} (which means dimensions of x^{-2}).

(b) In one-dimensional Quantum Physics, the wavefunction has units of $\text{m}^{-1/2}$ as Sample Problem 40-2 shows. Thus, since each term in Eq. 40-18 has units of ψ multiplied by units of x^{-2}, then those units are $\text{m}^{-1/2}\cdot\text{m}^{-2} = \text{m}^{-2.5}$.

19. According to Fig. 40-9, the electron's initial energy is 109 eV. After the additional energy is absorbed, the total energy of the electron is $109\,\text{eV} + 400\,\text{eV} = 509\,\text{eV}$. Since it is in the region $x > L$, its potential energy is 450 eV (see Section 40-5), so its kinetic energy must be $509\,\text{eV} - 450\,\text{eV} = 59\,\text{eV}$.

20. From Fig. 40-9, we see that the sum of the kinetic and potential energies in that particular finite well is 280 eV. The potential energy is zero in the region $0 < x < L$. If the kinetic energy of the electron is detected while it is in that region (which is the only region where this is likely to happen), we should find $K = 280\,\text{eV}$.

21. (a) and (b) Schrödinger's equation for the region $x > L$ is

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2}\,[E - U_0]\,\psi = 0 \ ,$$

where $E - U_0 < 0$. If $\psi^2(x) = Ce^{-2kx}$, then $\psi(x) = C'e^{-kx}$, where C' is another constant satisfying $C'^2 = C$. Thus $d^2\psi/dx^2 = 4k^2C'e^{-kx} = 4k^2\psi$ and

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2}\,[E - U_0]\,\psi = k^2\psi + \frac{8\pi^2 m}{h^2}\,[E - U_0]\,\psi \ .$$

This is zero provided that

$$k^2 = \frac{8\pi^2 m}{h^2}\,[U_0 - E] \ .$$

The quantity on the right-hand side is positive, so k is real and the proposed function satisfies Schrödinger's equation. If k is negative, however, the proposed function would be physically unrealistic. It would increase exponentially with x. Since the integral of the probability density over the entire x axis must be finite, ψ diverging as $x \to \infty$ would be unacceptable. Therefore, we choose

$$k = \frac{2\pi}{h}\sqrt{2m\,(U_0 - E)} > 0 \ .$$

22. (a) and (b) In the region $0 < x < L$, $U_0 = 0$, so Schrödinger's equation for the region is

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2}E\psi = 0$$

where $E > 0$. If $\psi^2(x) = B\sin^2 kx$, then $\psi(x) = B'\sin kx$, where B' is another constant satisfying $B'^2 = B$. Thus $d^2\psi/dx^2 = -k^2B'\sin kx = -k^2\psi(x)$ and

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2}E\psi = -k^2\psi + \frac{8\pi^2 m}{h^2}E\psi \ .$$

This is zero provided that

$$k^2 = \frac{8\pi^2 mE}{h^2} \, .$$

The quantity on the right-hand side is positive, so k is real and the proposed function satisfies Schrödinger's equation. In this case, there exists no physical restriction as to the sign of k. It can assume either positive or negative values. Thus

$$k = \pm \frac{2\pi}{h}\sqrt{2mE} \, .$$

23. Schrödinger's equation for the region $x > L$ is

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2}\left[E - U_0\right]\psi = 0 \, .$$

If $\psi = De^{2kx}$, then $d^2\psi/dx^2 = 4k^2 De^{2kx} = 4k^2\psi$ and

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2}\left[E - U_0\right]\psi = 4k^2\psi + \frac{8\pi^2 m}{h^2}\left[E - U_0\right]\psi \, .$$

This is zero provided

$$k = \frac{\pi}{h}\sqrt{2m\left(U_0 - E\right)} \, .$$

The proposed function satisfies Schrödinger's equation provided k has this value. Since U_0 is greater than E in the region $x > L$, the quantity under the radical is positive. This means k is real. If k is positive, however, the proposed function is physically unrealistic. It increases exponentially with x and becomes large without bound. The integral of the probability density over the entire x axis must be unity. This is impossible if ψ is the proposed function.

24. We can use the mc^2 value for an electron from Table 38-3 (511×10^3 eV) and the hc value developed in problem 3 of Chapter 39 by writing Eq. 40-20 as

$$E_{nx,ny} = \frac{2h^2}{8m}\left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2}\right) = \frac{(hc)^2}{8(mc^2)}\left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2}\right) \, .$$

For $n_x = n_y = 1$, we obtain

$$E_{1,1} = \frac{(1240\,\text{eV}\cdot\text{nm})^2}{8(511\times 10^3\,\text{eV})}\left(\frac{1}{(0.800\,\text{nm})^2} + \frac{1}{(1.600\,\text{nm})^2}\right) = 0.73 \text{ eV} \, .$$

25. We can use the mc^2 value for an electron from Table 38-3 (511×10^3 eV) and the hc value developed in problem 3 of Chapter 39 by writing Eq. 40-21 as

$$E_{nx,ny,nz} = \frac{2h^2}{8m}\left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} + \frac{n_z^2}{L_z^2}\right) = \frac{(hc)^2}{8(mc^2)}\left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} + \frac{n_z^2}{L_z^2}\right) \, .$$

For $n_x = n_y = n_z = 1$, we obtain

$$E_{1,1} = \frac{(1240\,\text{eV}\cdot\text{nm})^2}{8(511\times 10^3\,\text{eV})}\left(\frac{1}{(0.800\,\text{nm})^2} + \frac{1}{(1.600\,\text{nm})^2} + \frac{1}{(0.400\,\text{nm})^2}\right) = 3.1 \text{ eV} \, .$$

26. We are looking for the values of the ratio

$$\frac{E_{nx,ny}}{h^2/8mL^2} = L^2\left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2}\right) = \left(n_x^2 + \frac{1}{4}\,n_y^2\right)$$

and the corresponding differences.

(a) For $n_x = n_y = 1$, the ratio becomes $1 + \frac{1}{4} = 1.25$.

(b) For $n_x = 1$ and $n_y = 2$, the ratio becomes $1 + \frac{1}{4}(4) = 2.00$. One can check (by computing other (n_x, n_y) values) that this is the next to lowest energy in the system.

(c) The lowest set of states that are degenerate are $(n_x, n_y) = (1, 4)$ and $(2, 2)$. Both of these states have that ratio equal to $1 + \frac{1}{4}(16) = 5.00$.

(d) For $n_x = 1$ and $n_y = 3$, the ratio becomes $1 + \frac{1}{4}(9) = 3.25$. One can check (by computing other (n_x, n_y) values) that this is the lowest energy greater than that computed in part (b). The next higher energy comes from $(n_x, n_y) = (2, 1)$ for which the ratio is $4 + \frac{1}{4}(1) = 4.25$. The difference between these two values is $4.25 - 3.25 = 1.00$.

27. The energy levels are given by

$$E_{n_x,n_y} = \frac{h^2}{8m}\left[\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2}\right] = \frac{h^2}{8mL^2}\left[n_x^2 + \frac{n_y^2}{4}\right]$$

where the substitutions $L_x = L$ and $L_y = 2L$ were made. In units of $h^2/8mL^2$, the energy levels are given by $n_x^2 + n_y^2/4$. The lowest five levels are $E_{1,1} = 1.25$, $E_{1,2} = 2.00$, $E_{1,3} = 3.25$, $E_{2,1} = 4.25$, and $E_{2,2} = E_{1,4} = 5.00$. It is clear that there are no other possible values for the energy less than 5. The frequency of the light emitted or absorbed when the electron goes from an initial state i to a final state f is $f = (E_f - E_i)/h$, and in units of $h/8mL^2$ is simply the difference in the values of $n_x^2 + n_y^2/4$ for the two states. The possible frequencies are 0.75 (1,2 $\longrightarrow$ 1,1), 2.00 (1,3 $\longrightarrow$ 1,1), 3.00 (2,1 $\longrightarrow$ 1,1), 3.75 (2,2 $\longrightarrow$ 1,1), 1.25 (1,3 $\longrightarrow$ 1,2), 2.25 (2,1 $\longrightarrow$ 1,2), 3.00 (2,2 $\longrightarrow$ 1,2), 1.00 (2,1 $\longrightarrow$ 1,3), 1.75 (2,2 $\longrightarrow$ 1,3), 0.75 (2,2 $\longrightarrow$ 2,1), all in units of $h/8mL^2$.

28. We are looking for the values of the ratio

$$\frac{E_{n_x,n_y,n_z}}{h^2/8mL^2} = L^2\left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} + \frac{n_z^2}{L_z^2}\right) = \left(n_x^2 + n_y^2 + n_z^2\right)$$

and the corresponding differences.

(a) For $n_x = n_y = n_z = 1$, the ratio becomes $1 + 1 + 1 = 3.00$.

(b) For $n_x = n_y = 2$ and $n_z = 1$, the ratio becomes $4 + 4 + 1 = 9.00$. One can check (by computing other (n_x, n_y, n_z) values) that this is the third lowest energy in the system. One can also check that this same ratio is obtained for $(n_x, n_y, n_z) = (2, 1, 2)$ and $(1, 2, 2)$.

(c) For $n_x = n_y = 1$ and $n_z = 3$, the ratio becomes $1 + 1 + 9 = 11.00$. One can check (by computing other (n_x, n_y, n_z) values) that this is three "steps" up from the lowest energy in the system. One can also check that this same ratio is obtained for $(n_x, n_y, n_z) = (1, 3, 1)$ and $(3, 1, 1)$. If we take the difference between this and the result of part (b), we obtain $11.00 - 9.00 = 2.00$.

(d) For $n_x = n_y = 1$ and $n_z = 2$, the ratio becomes $1 + 1 + 4 = 6.00$. One can check (by computing other (n_x, n_y, n_z) values) that this is the next to the lowest energy in the system. One can also check that this same ratio is obtained for $(n_x, n_y, n_z) = (2, 1, 1)$ and $(1, 2, 1)$. Thus, three states (three arrangements of (n_x, n_y, n_z) values) have this energy.

(e) For $n_x = 1$, $n_y = 2$ and $n_z = 3$, the ratio becomes $1 + 4 + 9 = 14.00$. One can check (by computing other (n_x, n_y, n_z) values) that this is five "steps" up from the lowest energy in the system. One can also check that this same ratio is obtained for $(n_x, n_y, n_z) = (1, 3, 2), (2, 3, 1), (2, 1, 3), (3, 1, 2)$ and $(3, 2, 1)$. Thus, six states (six arrangements of (n_x, n_y, n_z) values) have this energy.

29. The ratios computed in problem 28 can be related to the frequencies emitted using $f = \Delta E/h$, where each level E is equal to one of those ratios multiplied by $h^2/8mL^2$. This effectively involves no more

than a cancellation of one of the factors of h. Thus, for a transition from the second excited state (see part (b) of problem 28) to the ground state (treated in part (a) of that problem), we find

$$f = (9.00 - 3.00)\left(\frac{h}{8mL^2}\right) = (6.00)\left(\frac{h}{8mL^2}\right) .$$

In the following, we omit the $h/8mL^2$ factors. For a transition between the fourth excited state and the ground state, we have $f = 12.00 - 3.00 = 9.00$. For a transition between the third excited state and the ground state, we have $f = 11.00 - 3.00 = 8.00$. For a transition between the third excited state and the first excited state, we have $f = 11.00 - 6.00 = 5.00$. For a transition between the fourth excited state and the third excited state, we have $f = 12.00 - 11.00 = 1.00$. For a transition between the third excited state and the second excited state, we have $f = 11.00 - 9.00 = 2.00$. For a transition between the second excited state and the first excited state, we have $f = 9.00 - 6.00 = 3.00$, which also results from some other transitions.

30. For $n = 1$

$$\begin{aligned} E_1 &= -\frac{m_e e^4}{8\varepsilon_0^2 h^2} \\ &= -\frac{(9.11 \times 10^{-31}\,\text{kg})(1.6 \times 10^{-19}\,\text{C})^4}{8(8.85 \times 10^{-12}\,\text{F/m})^2(6.63 \times 10^{-34}\,\text{J}\cdot\text{s})^2(1.60 \times 10^{-19}\,\text{J/eV})} \\ &= -13.6\ \text{eV} . \end{aligned}$$

31. From Eq. 40-6,

$$\Delta E = hf = (4.14 \times 10^{-15}\,\text{eV}\cdot\text{s})(6.2 \times 10^{14}\,\text{Hz}) = 2.6\ \text{eV} .$$

32. The difference between the energy absorbed and the energy emitted is

$$E_{\text{photon absorbed}} - E_{\text{photon emitted}} = \frac{hc}{\lambda_{\text{absorbed}}} - \frac{hc}{\lambda_{\text{emitted}}} .$$

Thus, using the result of problem 3 in Chapter 39, the net energy absorbed is

$$hc\Delta\left(\frac{1}{\lambda}\right) = (1240\,\text{eV}\cdot\text{nm})\left(\frac{1}{375\,\text{nm}} - \frac{1}{580\,\text{nm}}\right) = 1.17\ \text{eV} .$$

33. The energy E of the photon emitted when a hydrogen atom jumps from a state with principal quantum number u to a state with principal quantum number ℓ is given by

$$E = A\left(\frac{1}{\ell^2} - \frac{1}{u^2}\right)$$

where $A = 13.6\,\text{eV}$. The frequency f of the electromagnetic wave is given by $f = E/h$ and the wavelength is given by $\lambda = c/f$. Thus,

$$\frac{1}{\lambda} = \frac{f}{c} = \frac{E}{hc} = \frac{A}{hc}\left(\frac{1}{\ell^2} - \frac{1}{u^2}\right) .$$

The shortest wavelength occurs at the series limit, for which $u = \infty$. For the Balmer series, $\ell = 2$ and the shortest wavelength is $\lambda_B = 4hc/A$. For the Lyman series, $\ell = 1$ and the shortest wavelength is $\lambda_L = hc/A$. The ratio is $\lambda_B/\lambda_L = 4$.

34. (a) The energy level corresponding to the probability density distribution shown in Fig. 40-20 is the $n = 2$ level. Its energy is given by

$$E_2 = -\frac{13.6\,\text{eV}}{2^2} = -3.4\ \text{eV} .$$

(b) As the electron is removed from the hydrogen atom the final energy of the proton-electron system is zero. Therefore, one needs to supply at least 3.4 eV of energy to the system in order to bring its energy up from $E_2 = -3.4\,\text{eV}$ to zero. (If more energy is supplied, then the electron will retain some kinetic energy after it is removed from the atom.)

35. (a) Since energy is conserved, the energy E of the photon is given by $E = E_i - E_f$, where E_i is the initial energy of the hydrogen atom and E_f is the final energy. The electron energy is given by $(-13.6\,\text{eV})/n^2$, where n is the principal quantum number. Thus,

$$E = E_i - E_f = \frac{-13.6\,\text{eV}}{(3)^2} - \frac{-13.6\,\text{eV}}{(1)^2} = 12.1 \text{ eV} .$$

(b) The photon momentum is given by

$$p = \frac{E}{c} = \frac{(12.1\,\text{eV})(1.60 \times 10^{-19}\,\text{J/eV})}{3.00 \times 10^{8}\,\text{m/s}} = 6.45 \times 10^{-27} \text{ kg}\cdot\text{m/s} .$$

(c) Using the result of problem 3 in Chapter 39, the wavelength is

$$\lambda = \frac{1240\,\text{eV}\cdot\text{nm}}{12.1\,\text{eV}} = 102 \text{ nm} .$$

36. (a) The "home-base" energy level for the Balmer series is $n = 2$. Thus the transition with the least energetic photon is the one from the $n = 3$ level to the $n = 2$ level. The energy difference for this transition is

$$\Delta E = E_3 - E_2 = -(13.6\,\text{eV})\left(\frac{1}{3^2} - \frac{1}{2^2}\right) = 1.889\,\text{eV} .$$

Using the result of problem 3 in Chapter 39, the corresponding wavelength is

$$\lambda = \frac{hc}{\Delta E} = \frac{1240\,\text{eV}\cdot\text{nm}}{1.889\,\text{eV}} = 658 \text{ nm} .$$

(b) For the series limit, the energy difference is

$$\Delta E = E_\infty - E_2 = -(13.6\,\text{eV})\left(\frac{1}{\infty^2} - \frac{1}{2^2}\right) = 3.40 \text{ eV} .$$

The corresponding wavelength is then

$$\lambda = \frac{hc}{\Delta E} = \frac{1240\,\text{eV}\cdot\text{nm}}{3.40\,\text{eV}} = 366 \text{ nm} .$$

37. If kinetic energy is not conserved, some of the neutron's initial kinetic energy is used to excite the hydrogen atom. The least energy that the hydrogen atom can accept is the difference between the first excited state ($n = 2$) and the ground state ($n = 1$). Since the energy of a state with principal quantum number n is $-(13.6\,\text{eV})/n^2$, the smallest excitation energy is $13.6\,\text{eV} - (13.6\,\text{eV})/(2)^2 = 10.2\,\text{eV}$. The neutron does not have sufficient kinetic energy to excite the hydrogen atom, so the hydrogen atom is left in its ground state and all the initial kinetic energy of the neutron ends up as the final kinetic energies of the neutron and atom. The collision must be elastic.

38. (a) We use Eq. 40-25. At $r = a$

$$\psi^2(r) = \left(\frac{1}{\sqrt{\pi}a^{3/2}}\, e^{-a/a}\right)^2 = \frac{1}{\pi a^3}\, e^{-2} = \frac{1}{\pi(5.29 \times 10^{-2}\,\text{nm})^3}\, e^{-2} = 291 \text{ nm}^{-3} .$$

(b) We use Eq. 40-31. At $r = a$

$$P(r) = \frac{4}{a^3}\,a^2 e^{-2a/a} = \frac{4e^{-2}}{a} = \frac{4e^{-2}}{5.29\times10^{-2}\,\text{nm}} = 10.2\ \text{nm}^{-1}\ .$$

39. (a) We use Eq. 40-31. At $r = 0$, $P(r) \propto r^2 = 0$.

(b) At $r = a$

$$P(r) = \frac{4}{a^3}\,a^2 e^{-2a/a} = \frac{4e^{-2}}{a} = \frac{4e^{-2}}{5.29\times10^{-2}\,\text{nm}} = 10.2\ \text{nm}^{-1}\ .$$

(c) At $r = 2a$

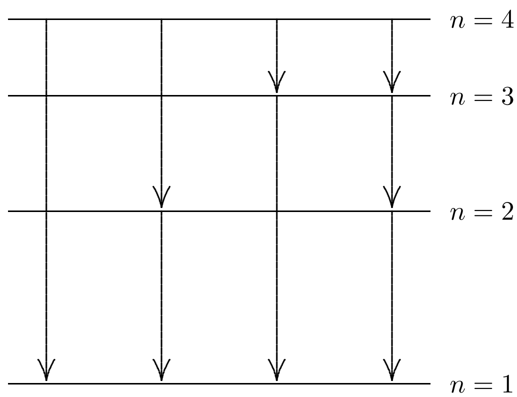
$$P(r) = \frac{4}{a^3}\,(2a)^2 e^{-4a/a} = \frac{16e^{-4}}{a} = \frac{16e^{-4}}{5.29\times10^{-2}\,\text{nm}} = 5.54\ \text{nm}^{-1}\ .$$

40. (a) $\Delta E = -(13.6\,\text{eV})(4^{-2} - 1^{-2}) = 12.8\,\text{eV}$.

(b) The values of the photon energies are:

$$\begin{aligned}
E_{4\to1} &= \Delta E_{\text{part (a)}} = 12.8\ \text{eV}\\
E_{3\to1} &= -(13.6\,\text{eV})(3^{-2} - 1^{-2}) = 12.1\ \text{eV}\\
E_{2\to1} &= -(13.6\,\text{eV})(2^{-2} - 1^{-2}) = 10.2\ \text{eV}\\
E_{4\to2} &= -(13.6\,\text{eV})(4^{-2} - 2^{-2}) = 2.55\ \text{eV}\\
E_{3\to2} &= -(13.6\,\text{eV})(3^{-2} - 2^{-2}) = 1.89\ \text{eV}\\
E_{4\to3} &= -(13.6\,\text{eV})(4^{-2} - 3^{-2}) = 0.66\ \text{eV}
\end{aligned}$$

The various photon energies correspond to the transitions between energy levels indicated below. The levels are not drawn to scale.



41. (a) We take the electrostatic potential energy to be zero when the electron and proton are far removed from each other. Then, the final energy of the atom is zero and the work done in pulling it apart is $W = -E_i$, where E_i is the energy of the initial state. The energy of the initial state is given by $E_i = (-13.6\,\text{eV})/n^2$, where n is the principal quantum number of the state. For the ground state, $n = 1$ and $W = 13.6\,\text{eV}$.

(b) For the state with $n = 2$, $W = (13.6\,\text{eV})/(2)^2 = 3.40\,\text{eV}$.

42. Conservation of linear momentum of the atom-photon system requires that

$$p_{\text{recoil}} = p_{\text{photon}} \implies m_p v_{\text{recoil}} = \frac{hf}{c}$$

where we use Eq. 39-7 for the photon and use the classical momentum formula for the atom (since we expect its speed to be much less than c). Thus, from Eq. 40-6 and Table 38-3,

$$\begin{aligned} v_{\text{recoil}} &= \frac{\Delta E}{m_p c} = \frac{E_4 - E_1}{(m_p c^2)/c} \\ &= \frac{(-13.6\,\text{eV})\left(4^{-2} - 1^{-2}\right)}{(938 \times 10^6\,\text{eV})/(2.998 \times 10^8\,\text{m/s})} \\ &= 4.1 \text{ m/s} . \end{aligned}$$

43. (a) and (b) Using Eq. 40-6 and the result of problem 3 in Chapter 39, we find

$$\Delta E = E_{\text{photon}} = \frac{hc}{\lambda} = \frac{1240\,\text{eV}\cdot\text{nm}}{486.1\,\text{nm}} = 2.55 \text{ eV} .$$

Referring to Fig. 40-16, we see that this must be one of the Balmer series transitions (this fact could also be found from Fig. 40-17). Therefore, $n_{\text{low}} = 2$, but what precisely is n_{high}?

$$\begin{aligned} E_{\text{high}} &= E_{\text{low}} + \Delta E \\ -\frac{13.6\,\text{eV}}{n^2} &= -\frac{13.6\,\text{eV}}{2^2} + 2.55\,\text{eV} \end{aligned}$$

which yields $n = 4$. Thus, the transition is from the $n = 4$ to the $n = 2$ state.

44. (a) The calculation is shown in Sample Problem 40-6. The difference in the values obtained in parts (a) and (b) of that Sample Problem is $122\,\text{nm} - 91.4\,\text{nm} \approx 31\,\text{nm}$.

(b) Fig. 40-17 shows that the width of the Balmer series is $656.3\,\text{nm} - 364.6\,\text{nm} \approx 292\,\text{nm}$. This can be confirmed with a calculation very much like the one shown in Sample Problem 40-6, but with the longest wavelength arising from the $3 \to 2$ transition, and the series limit obtained from the $\infty \to 2$ transition.

(c) We use Eq. 39-1. For the Lyman series,

$$\Delta f = \frac{2.998 \times 10^8\,\text{m/s}}{91.4 \times 10^{-9}\,\text{m}} - \frac{2.998 \times 10^8\,\text{m/s}}{122 \times 10^{-9}\,\text{m}} = 8.2 \times 10^{14} \text{ Hz}$$

or 8.2×10^2 THz. For the Balmer series,

$$\Delta f = \frac{2.998 \times 10^8\,\text{m/s}}{364.6 \times 10^{-9}\,\text{m}} - \frac{2.998 \times 10^8\,\text{m/s}}{656.3 \times 10^{-9}\,\text{m}} = 3.65 \times 10^{14} \text{ Hz}$$

which is equivalent to 365 THz.

45. Letting $a = 5.292 \times 10^{-11}$ m be the Bohr radius, the potential energy becomes

$$U = -\frac{e^2}{4\pi\varepsilon_0 a} = \frac{(8.99 \times 10^9\,\text{N}\cdot\text{m}^2/\text{C}^2)(1.602 \times 10^{-19}\,\text{C})^2}{5.292 \times 10^{-11}\,\text{m}} = -4.36 \times 10^{-18}\,\text{J} = -27.2 \text{ eV} .$$

The kinetic energy is $K = E - U = (-13.6\,\text{eV}) - (-27.2\,\text{eV}) = 13.6\,\text{eV}$.

46. (a) and (b) Using Eq. 40-6 and the result of problem 3 in Chapter 39, we find

$$\Delta E = E_{\text{photon}} = \frac{hc}{\lambda} = \frac{1240\,\text{eV}\cdot\text{nm}}{121.6\,\text{nm}} = 10.2 \text{ eV} .$$

Referring to Fig. 40-16, we see that this must be one of the Lyman series transitions. Therefore, $n_{\text{low}} = 1$, but what precisely is n_{high}?

$$\begin{aligned} E_{\text{high}} &= E_{\text{low}} + \Delta E \\ -\frac{13.6\,\text{eV}}{n^2} &= -\frac{13.6\,\text{eV}}{1^2} + 10.2\,\text{eV} \end{aligned}$$

which yields $n = 2$ (this is confirmed by the calculation found from Sample Problem 40-6). Thus, the transition is from the $n = 2$ to the $n = 1$ state.

47. (a) Since $E_2 = -0.85\,\text{eV}$ and $E_1 = -13.6\,\text{eV} + 10.2\,\text{eV} = -3.4\,\text{eV}$, the photon energy is $E_{\text{photon}} = E_2 - E_1 = -0.85\,\text{eV} - (-3.4\,\text{eV}) = 2.6\,\text{eV}$.

(b) From

$$E_2 - E_1 = (-13.6\,\text{eV})\left(\frac{1}{n_2^2} - \frac{1}{n_1^2}\right) = 2.6\text{ eV}$$

we obtain

$$\frac{1}{n_2^2} - \frac{1}{n_1^2} = -\frac{2.6\,\text{eV}}{13.6\,\text{eV}} \approx -\frac{3}{16} = \frac{1}{4^2} - \frac{1}{2^2}\,.$$

Thus, $n_2 = 4$ and $n_1 = 2$. So the transition is from the $n = 4$ state to the $n = 2$ state. One can easily verify this by inspecting the energy level diagram of Fig. 40-16.

48. The wavelength λ of the photon emitted in a transition belonging to the Balmer series satisfies

$$E_{\text{ph}} = \frac{hc}{\lambda} = E_n - E_2 = -(13.6\,\text{eV})\left(\frac{1}{n^2} - \frac{1}{2^2}\right) \quad \text{where } n = 3,\, 4,\, 5,\, \ldots$$

Using the result of problem 3 in Chapter 39, we find

$$\lambda = \frac{4hcn^2}{(13.6\,\text{eV})(n^2-4)} = \frac{4(1240\,\text{eV}\cdot\text{nm})}{13.6\,\text{eV}}\left(\frac{n^2}{n^2-4}\right)\,.$$

Plugging in the various values of n, we obtain these values of the wavelength: $\lambda = 656\,\text{nm}$ (for $n = 3$), $\lambda = 486\,\text{nm}$ (for $n = 4$), $\lambda = 434\,\text{nm}$ (for $n = 5$), $\lambda = 410\,\text{nm}$ (for $n = 6$), $\lambda = 397\,\text{nm}$ (for $n = 7$), $\lambda = 389\,\text{nm}$ (for $n = 8$), etc. Finally for $n = \infty$, $\lambda = 365\,\text{nm}$. These values agree well with the data found in Fig. 40-17. [One can also find λ beyond three significant figures by using the more accurate values for m_e, e and h listed in Appendix B when calculating E_n in Eq. 40-24. Another factor that contributes to the error is the motion of the atomic nucleus. It can be shown that this effect can be accounted for by replacing the mass of the electron m_e by $m_e m_p/(m_p + m_e)$ in Eq. 40-24, where m_p is the mass of the proton. Since $m_p \gg m_e$, this is not a major effect.]

49. According to Sample Problem 40-8, the probability the electron in the ground state of a hydrogen atom can be found inside a sphere of radius r is given by

$$p(r) = 1 - e^{-2x}\left(1 + 2x + 2x^2\right)$$

where $x = r/a$ and a is the Bohr radius. We want $r = a$, so $x = 1$ and

$$p(a) = 1 - e^{-2}\,(1 + 2 + 2) = 1 - 5e^{-2} = 0.323\,.$$

The probability that the electron can be found outside this sphere is $1 - 0.323 = 0.677$. It can be found outside about 68% of the time.

50. Using Eq. 40-6 and the result of problem 3 in Chapter 39, we find

$$\Delta E = E_{\text{photon}} = \frac{hc}{\lambda} = \frac{1240\,\text{eV}\cdot\text{nm}}{102.6\,\text{nm}} = 12.09\text{ eV}\,.$$

Referring to Fig. 40-16, we see that this must be one of the Lyman series transitions. Therefore, $n_{\text{low}} = 1$, but what precisely is n_{high}?

$$\begin{aligned} E_{\text{high}} &= E_{\text{low}} + \Delta E \\ -\frac{13.6\,\text{eV}}{n^2} &= -\frac{13.6\,\text{eV}}{1^2} + 12.09\,\text{eV} \end{aligned}$$

which yields $n = 3$. Thus, the transition is from the $n = 3$ to the $n = 1$ state.

51. The proposed wave function is

$$\psi = \frac{1}{\sqrt{\pi}a^{3/2}}e^{-r/a}$$

where a is the Bohr radius. Substituting this into the right side of Schrödinger's equation, our goal is to show that the result is zero. The derivative is

$$\frac{d\psi}{dr} = -\frac{1}{\sqrt{\pi}a^{5/2}}e^{-r/a}$$

so

$$r^2\frac{d\psi}{dr} = -\frac{r^2}{\sqrt{\pi}a^{5/2}}e^{-r/a}$$

and

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) = \frac{1}{\sqrt{\pi}a^{5/2}}\left[-\frac{2}{r}+\frac{1}{a}\right]e^{-r/a} = \frac{1}{a}\left[-\frac{2}{r}+\frac{1}{a}\right]\psi \ .$$

The energy of the ground state is given by $E = -me^4/8\varepsilon_0^2h^2$, and the Bohr radius is given by $a = h^2\varepsilon_0/\pi me^2$, so $E = -e^2/8\pi\varepsilon_0 a$. The potential energy is given by $U = -e^2/4\pi\varepsilon_0 r$, so

$$\begin{aligned}\frac{8\pi^2 m}{h^2}[E-U]\psi &= \frac{8\pi^2 m}{h^2}\left[-\frac{e^2}{8\pi\varepsilon_0 a}+\frac{e^2}{4\pi\varepsilon_0 r}\right]\psi = \frac{8\pi^2 m}{h^2}\frac{e^2}{8\pi\varepsilon_0}\left[-\frac{1}{a}+\frac{2}{r}\right]\psi \\ &= \frac{\pi m e^2}{h^2\varepsilon_0}\left[-\frac{1}{a}+\frac{2}{r}\right]\psi = \frac{1}{a}\left[-\frac{1}{a}+\frac{2}{r}\right]\psi \ .\end{aligned}$$

The two terms in Schrödinger's equation cancel, and the proposed function ψ satisfies that equation.

52. From Sample Problem 40-8, we know that the probability of finding the electron in the ground state of the hydrogen atom inside a sphere of radius r is given by

$$p(r) = 1 - e^{-2x}\left(1+2x+2x^2\right)$$

where $x = r/a$. Thus the probability of finding the electron between the two shells indicated in this problem is given by

$$\begin{aligned}p(a<r<2a) &= p(2a)-p(a) \\ &= \left[1-e^{-2x}\left(1+2x+2x^2\right)\right]_{x=2} - \left[1-e^{-2x}\left(1+2x+2x^2\right)\right]_{x=1} \\ &= 0.44 \ .\end{aligned}$$

53. The radial probability function for the ground state of hydrogen is $P(r) = (4r^2/a^3)e^{-2r/a}$, where a is the Bohr radius. (See Eq. 40-31.) We want to evaluate the integral $\int_0^\infty P(r)\,dr$. Eq. 15 in the integral table of Appendix E is an integral of this form. We set $n = 2$ and replace a in the given formula with $2/a$ and x with r. Then

$$\int_0^\infty P(r)\,dr = \frac{4}{a^3}\int_0^\infty r^2 e^{-2r/a}\,dr = \frac{4}{a^3}\frac{2}{(2/a)^3} = 1 \ .$$

54. (a) The allowed values of l for a given n are $0, 1, 2, \dots, n-1$. Thus there are n different values of l.

(b) The allowed values of m_l for a given l are $-l, -l+1, \cdots, l$. Thus there are $2l+1$ different values of m_l.

(c) According to part (a) above, for a given n there are n different values of l. Also, each of these l's can have $2l+1$ different values of m_l [see part (b) above]. Thus, the total number of m_l's is

$$\sum_{l=0}^{n-1}(2l+1) = n^2 \ .$$

55. Since Δr is small, we may calculate the probability using $p = P(r)\,\Delta r$, where $P(r)$ is the radial probability density. The radial probability density for the ground state of hydrogen is given by Eq. 40-31:

$$P(r) = \left(\frac{4r^2}{a^3}\right) e^{-2r/a}$$

where a is the Bohr radius.

(a) Here, $r = 0.500a$ and $\Delta r = 0.010a$. Then,

$$p = \left(\frac{4r^2\,\Delta r}{a^3}\right) e^{-2r/a} = 4(0.500)^2(0.010)\,e^{-1} = 3.68 \times 10^{-3} \ .$$

(b) We set $r = 1.00a$ and $\Delta r = 0.010a$. Then,

$$p = \left(\frac{4r^2\,\Delta r}{a^3}\right) e^{-2r/a} = 4(1.00)^2(0.010)\,e^{-2} = 5.41 \times 10^{-3} \ .$$

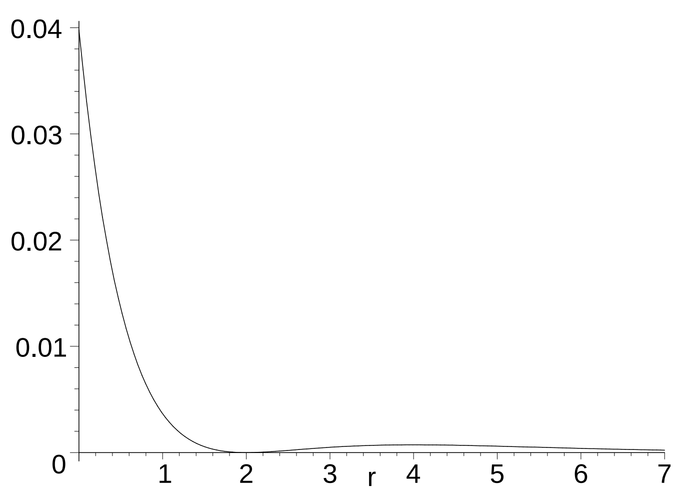
56. According to Fig. 40-23, the quantum number n in question satisfies $r = n^2 a$. Letting $r = 1.0\,\text{mm}$, we solve for n:

$$n = \sqrt{\frac{r}{a}} = \sqrt{\frac{1.0 \times 10^{-3}\,\text{m}}{5.29 \times 10^{-11}\,\text{m}}} \approx 4.3 \times 10^3 \ .$$

57. The radial probability function for the ground state of hydrogen is $P(r) = (4r^2/a^3)e^{-2r/a}$, where a is the Bohr radius. (See Eq. 40-31.) The integral table of Appendix E may be used to evaluate the integral $r_{\text{avg}} = \int_0^\infty rP(r)\,dr$. Setting $n = 3$ and replacing a in the given formula with $2/a$ (and x with r), we obtain

$$r_{\text{avg}} = \int_0^\infty rP(r)\,dr = \frac{4}{a^3}\int_0^\infty r^3 e^{-2r/a}\,dr = \frac{4}{a^3}\,\frac{6}{(2/a)^4} = 1.5a \ .$$

58. (a) The plot shown below for $|\psi_{200}(r)|^2$ is to be compared with the dot plot of Fig. 40-20. We note that the horizontal axis of our graph is labeled "r," but it is actually r/a (that is, it is in units of the parameter a). Now, in the plot below there is a high central peak between $r = 0$ and $r \sim 2a$, corresponding to the densely dotted region around the center of the dot plot of Fig. 40-20. Outside this peak is a region of near-zero values centered at $r = 2a$, where $\psi_{200} = 0$. This is represented in the dot plot by the empty ring surrounding the central peak. Further outside is a broader, flatter, low peak which reaches its maximum value at $r = 4a$. This corresponds to the outer ring with near-uniform dot density which is lower than that of the central peak.



(b) The extrema of $\psi^2(r)$ for $0 < r < \infty$ may be found by squaring the given function, differentiating with respect to r, and setting the result equal to zero:

$$-\frac{1}{32}\,\frac{(r-2a)(r-4a)}{a^6\pi}\,e^{-r/a} = 0$$

which has roots at $r = 2a$ and $r = 4a$. We can verify directly from the plot above that $r = 4a$ is indeed a local maximum of $\psi^2_{200}(r)$. As discussed in part (a), the other root ($r = 2a$) is a local minimum.

(c) Using Eq. 40-30 and Eq. 40-28, the radial probability is

$$P_{200}(r) = 4\pi r^2\psi^2_{200}(r) = \frac{r^2}{8a^3}\left(2-\frac{r}{a}\right)^2 e^{-r/a}\ .$$

(d) Let $x = r/a$. Then

$$\begin{aligned}
\int_0^\infty P_{200}(r)\,dr &= \int_0^\infty \frac{r^2}{8a^3}\left(2-\frac{r}{a}\right)^2 e^{-r/a}\,dr \\
&= \frac{1}{8}\int_0^\infty x^2(2-x)^2e^{-x}\,dx \\
&= \int_0^\infty (x^4-4x^3+4x^2)e^{-x}\,dx \\
&= \frac{1}{8}\left[4!-4(3!)+4(2!)\right] \\
&= 1
\end{aligned}$$

where the integral formula

$$\int_0^\infty x^n\,e^{-x}\,dx = n!$$

is used.

59. (a) ψ_{210} is real. Squaring it, we obtain the probability density:

$$|\psi_{210}|^2 = \frac{r^2}{32\pi a^5}e^{-r/a}\cos^2\theta\ .$$

Each of the other functions is multiplied by its complex conjugate, obtained by replacing i with $-i$ in the function. Since $e^{i\phi}e^{-i\phi} = e^0 = 1$, the result is the square of the function without the exponential factor:

$$|\psi_{21+1}|^2 = \frac{r^2}{64\pi a^5}e^{-r/a}\sin^2\theta$$

and

$$|\psi_{21-1}|^2 = \frac{r^2}{64\pi a^5}e^{-r/a}\sin^2\theta\ .$$

The last two functions lead to the same probability density.

(b) The total probability density for the three states is the sum:

$$\begin{aligned}
|\psi_{210}|^2+|\psi_{21+1}|^2+|\psi_{21-1}|^2 &= \frac{r^2}{32\pi a^5}e^{-r/a}\left[\cos^2\theta+\frac{1}{2}\sin^2\theta+\frac{1}{2}\sin^2\theta\right] \\
&= \frac{r^2}{32\pi a^5}e^{-r/a}\ .
\end{aligned}$$

The trigonometric identity $\cos^2\theta+\sin^2\theta = 1$ is used. We note that the total probability density does not depend on θ or ϕ; it is spherically symmetric.