Chapter 14

1. The magnitude of the force of one particle on the other is given by $F = Gm_1m_2/r^2$, where m_1 and m_2 are the masses, r is their separation, and G is the universal gravitational constant. We solve for r:

$$r = \sqrt{\frac{Gm_1m_2}{F}} = \sqrt{\frac{(6.67 \times 10^{-11}\,\mathrm{N\cdot m^2/kg^2})(5.2\,\mathrm{kg})(2.4\,\mathrm{kg})}{2.3 \times 10^{-12}\,\mathrm{N}}} = 19\ \mathrm{m}\ .$$

2. (a) The gravitational force exerted on the baby (denoted with subscript b) by the obstetrician (denoted with subscript o) is given by

$$F_{bo} = \frac{Gm_om_b}{r_{bo}^2} = \frac{(6.67 \times 10^{-11}\,\mathrm{N\cdot m^2/kg^2})(70\,\mathrm{kg})(3\,\mathrm{kg})}{(1\,\mathrm{m})^2} = 1 \times 10^{-8}\ \mathrm{N}\ .$$

(b) The maximum (minimum) forces exerted by Jupiter on the baby occur when it is separated from the Earth by the shortest (longest) distance $r_{\min}$ ($r_{\max}$), respectively. Thus

$$F_{bJ}^{\max} = \frac{Gm_Jm_b}{r_{\min}^2} = \frac{(6.67 \times 10^{-11}\,\mathrm{N\cdot m^2/kg^2})(2 \times 10^{27}\,\mathrm{kg})(3\,\mathrm{kg})}{(6 \times 10^{11}\,\mathrm{m})^2} = 1 \times 10^{-6}\ \mathrm{N}\ .$$

(c) And we obtain

$$F_{bJ}^{\min} = \frac{Gm_Jm_b}{r_{\max}^2} = \frac{(6.67 \times 10^{-11}\,\mathrm{N\cdot m^2/kg^2})(2 \times 10^{27}\,\mathrm{kg})(3\mathrm{kg})}{(9 \times 10^{11}\,\mathrm{m})^2} = 5 \times 10^{-7}\ \mathrm{N}\ .$$

(d) No. The gravitational force exerted by Jupiter on the baby is greater than that by the obstetrician by a factor of up to $1 \times 10^{-6}\,\mathrm{N}/1 \times 10^{-8}\,\mathrm{N} = 100$.

3. We use $F = Gm_sm_m/r^2$, where m_s is the mass of the satellite, m_m is the mass of the meteor, and r is the distance between their centers. The distance between centers is $r = R + d = 15\,\mathrm{m} + 3\,\mathrm{m} = 18\,\mathrm{m}$. Here R is the radius of the satellite and d is the distance from its surface to the center of the meteor. Thus,

$$F = \frac{(6.67 \times 10^{-11}\,\mathrm{N\cdot m^2/kg^2})(20\,\mathrm{kg})(7.0\,\mathrm{kg})}{(18\,\mathrm{m})^2} = 2.9 \times 10^{-11}\ \mathrm{N}\ .$$

4. We use subscripts s, e, and m for the Sun, Earth and Moon, respectively.

$$\frac{F_{sm}}{F_{em}} = \frac{\frac{Gm_sm_m}{r_{sm}^2}}{\frac{Gm_em_m}{r_{em}^2}} = \frac{m_s}{m_e}\left(\frac{r_{em}}{r_{sm}}\right)^2$$

Plugging in the numerical values (say, from Appendix C) we find

$$\frac{1.99 \times 10^{30}}{5.98 \times 10^{24}}\left(\frac{3.82 \times 10^{8}}{1.50 \times 10^{11}}\right)^2 = 2.16\ .$$

5. The gravitational force between the two parts is

$$F = \frac{Gm(M-m)}{r^2} = \frac{G}{r^2}\left(mM - m^2\right)$$

which we differentiate with respect to m and set equal to zero:

$$\frac{dF}{dm} = 0 = \frac{G}{r^2}(M-2m) \implies M = 2m$$

which leads to the result $m/M = 1/2$.

6. Let the distance from Earth to the spaceship be r. $R_{em} = 3.82 \times 10^8$ m is the distance from Earth to the moon. Thus,

$$F_m = \frac{GM_m m}{(R_{em}-r)^2} = F_E = \frac{GM_e m}{r^2} ,$$

where m is the mass of the spaceship. Solving for r, we obtain

$$\begin{aligned} r &= \frac{R_{em}}{\sqrt{M_m/M_e}+1} \\ &= \frac{3.82 \times 10^8\,\text{m}}{\sqrt{(7.36 \times 10^{22}\,\text{kg})/(5.98 \times 10^{24}\,\text{kg})}+1} = 3.44 \times 10^8\,\text{m} . \end{aligned}$$

7. At the point where the forces balance $GM_e m/r_1^2 = GM_s m/r_2^2$, where M_e is the mass of Earth, M_s is the mass of the Sun, m is the mass of the space probe, r_1 is the distance from the center of Earth to the probe, and r_2 is the distance from the center of the Sun to the probe. We substitute $r_2 = d - r_1$, where d is the distance from the center of Earth to the center of the Sun, to find

$$\frac{M_e}{r_1^2} = \frac{M_s}{(d-r_1)^2} .$$

Taking the positive square root of both sides, we solve for r_1. A little algebra yields

$$r_1 = \frac{d\sqrt{M_e}}{\sqrt{M_s}+\sqrt{M_e}} = \frac{(150 \times 10^9\,\text{m})\sqrt{5.98 \times 10^{24}\,\text{kg}}}{\sqrt{1.99 \times 10^{30}\,\text{kg}}+\sqrt{5.98 \times 10^{24}\,\text{kg}}} = 2.6 \times 10^8\ \text{m} .$$

Values for M_e, M_s, and d can be found in Appendix C.

8. Using $F = GmM/r^2$, we find that the topmost mass pulls upward on the one at the origin with 1.9×10^{-8} N, and the rightmost mass pulls rightward on the one at the origin with 1.0×10^{-8} N. Thus, the (x, y) components of the net force, which can be converted to polar components (here we use magnitude-angle notation), are

$$\vec{F}_{\text{net}} = \left(1.0 \times 10^{-8}, 1.9 \times 10^{-8}\right) \implies \left(2.1 \times 10^{-8} \angle\ 61°\right) .$$

The magnitude of the force is 2.1×10^{-8} N.

9. The gravitational forces on m_5 from the two 500-kg masses cancel each other. Contributions to the net force on m_5 come from the remaining two masses:

$$F_{\text{net}} = \frac{(6.67 \times 10^{-11}\,\text{N}\cdot\text{m}^2/\text{kg}^2)(250\,\text{kg})(300\,\text{kg} - 100\,\text{kg})}{(\sqrt{2} \times 10^{-2}\,\text{m})^2} = 0.017\ \text{N} .$$

The force is directed along the diagonal between the 300 kg and 100 kg masses, towards the 300-kg mass.

10. (a) The distance between any of the spheres at the corners and the sphere at the center is $r = \ell/2\cos 30° = \ell/\sqrt{3}$ where ℓ is the length of one side of the equilateral triangle. The net (downward) contribution caused by the two bottom-most spheres (each of mass m) to the total force on m_4 has magnitude

$$2F_y = 2\left(\frac{Gm_4m}{r^2}\right)\sin 30° = 3\frac{Gm_4m}{\ell^2} \ .$$

This must equal the magnitude of the pull from M, so

$$3\frac{Gm_4m}{\ell^2} = \frac{Gm_4M}{(\ell/\sqrt{3})^2}$$

which readily yields $m = M$.

(b) Since m_4 cancels in that last step, then the amount of mass in the center sphere is not relevant to the problem. The net force is still zero.

11. We use m_1 for the 20 kg of the sphere at $(x_1, y_1) = (0.5, 1.0)$ (SI units understood), m_2 for the 40 kg of the sphere at $(x_2, y_2) = (-1.0, -1.0)$, and m_3 for the 60 kg of the sphere at $(x_3, y_3) = (0, -0.5)$. The mass of the 20 kg object at the origin is simply denoted m. We note that $r_1 = \sqrt{1.25}$, $r_2 = \sqrt{2}$, and $r_3 = 0.5$ (again, with SI units understood). The force $\vec{F}_n$ that the n^{th} sphere exerts on m has magnitude Gm_nm/r_n^2 and is directed from the origin towards m_n , so that it is conveniently written as

$$\vec{F}_n = \frac{Gm_nm}{r_n^2}\left(\frac{x_n}{r_n}\,\hat{\imath} + \frac{y_n}{r_n}\,\hat{\jmath}\right) = \frac{Gm_nm}{r_n^3}\left(x_n\hat{\imath} + y_n\hat{\jmath}\right) \ .$$

Consequently, the vector addition to obtain the net force on m becomes

$$\begin{aligned}
\vec{F}_{\text{net}} &= \sum_{n=1}^{3}\vec{F}_n \\
&= Gm\left(\left(\sum_{n=1}^{3}\frac{m_nx_n}{r_n^3}\right)\hat{\imath} + \left(\sum_{n=1}^{3}\frac{m_ny_n}{r_n^3}\right)\hat{\jmath}\right) \\
&= -9.3\times 10^{-9}\,\hat{\imath} - 3.2\times 10^{-7}\,\hat{\jmath}
\end{aligned}$$

in SI units. Therefore, we find the net force magnitude is $|\vec{F}_{\text{net}}| = 3.2\times 10^{-7}$ N.

12. We note that r_A (the distance from the origin to sphere A, which is the same as the separation between A and B) is 0.5, $r_C = 0.8$, and $r_D = 0.4$ (with SI units understood). The force $\vec{F}_k$ that the k^{th} sphere exerts on m_B has magnitude Gm_km_B/r_k^2 and is directed from the origin towards m_k so that it is conveniently written as

$$\vec{F}_k = \frac{Gm_km_B}{r_k^2}\left(\frac{x_k}{r_k}\,\hat{\imath} + \frac{y_k}{r_k}\,\hat{\jmath}\right) = \frac{Gm_km_B}{r_k^3}\left(x_k\hat{\imath} + y_k\hat{\jmath}\right) \ .$$

Consequently, the vector addition (where k equals A, B and D) to obtain the net force on m_B becomes

$$\begin{aligned}
\vec{F}_{\text{net}} &= \sum_{k}\vec{F}_k \\
&= Gm_B\left(\left(\sum_{k}\frac{m_kx_k}{r_k^3}\right)\hat{\imath} + \left(\sum_{k}\frac{m_ky_k}{r_k^3}\right)\hat{\jmath}\right) \\
&= 3.7\times 10^{-5}\,\hat{\jmath}\ \text{N} \ .
\end{aligned}$$

13. If the lead sphere were not hollowed the magnitude of the force it exerts on m would be $F_1 = GMm/d^2$. Part of this force is due to material that is removed. We calculate the force exerted on m by a sphere that just fills the cavity, at the position of the cavity, and subtract it from the force of the solid sphere.

The cavity has a radius $r = R/2$. The material that fills it has the same density (mass to volume ratio) as the solid sphere. That is $M_c/r^3 = M/R^3$, where M_c is the mass that fills the cavity. The common factor $4\pi/3$ has been canceled. Thus,

$$M_c = \left(\frac{r^3}{R^3}\right) M = \left(\frac{R^3}{8R^3}\right) M = \frac{M}{8} \ .$$

The center of the cavity is $d - r = d - R/2$ from m, so the force it exerts on m is

$$F_2 = \frac{G(M/8)m}{(d - R/2)^2} \ .$$

The force of the hollowed sphere on m is

$$F = F_1 - F_2 = GMm\left(\frac{1}{d^2} - \frac{1}{8(d - R/2)^2}\right) = \frac{GMm}{d^2}\left(1 - \frac{1}{8(1 - R/2d)^2}\right) \ .$$

14. We follow the method shown in Sample Problem 14-3. Thus,

$$a_g = \frac{GM_E}{r^2} \implies da_g = -2\frac{GM_E}{r^3}dr$$

which implies that the change in weight is

$$W_{\text{top}} - W_{\text{bottom}} \approx m\,(da_g) \ .$$

But since $W_{\text{bottom}} = GmM_E/R^2$ (where R is Earth's mean radius), we have

$$m da_g = -2\frac{GmM_E}{R^3}dr = -2W_{\text{bottom}}\frac{dr}{R} = -2(530\,\text{N})\frac{410\,\text{m}}{6.37 \times 10^6\,\text{m}}$$

which yields -0.068 N for the weight change (the minus sign indicating that it is a decrease in W). We are not including any effects due to the Earth's rotation (as treated in Eq. 14-12).

15. The acceleration due to gravity is given by $a_g = GM/r^2$, where M is the mass of Earth and r is the distance from Earth's center. We substitute $r = R + h$, where R is the radius of Earth and h is the altitude, to obtain $a_g = GM/(R + h)^2$. We solve for h and obtain $h = \sqrt{GM/a_g} - R$. According to Appendix C, $R = 6.37 \times 10^6$ m and $M = 5.98 \times 10^{24}$ kg, so

$$h = \sqrt{\frac{(6.67 \times 10^{-11}\,\text{m}^3/\text{s}^2\cdot\text{kg})(5.98 \times 10^{24}\,\text{kg})}{4.9\,\text{m/s}^2}} - 6.37 \times 10^6\,\text{m} = 2.6 \times 10^6\ \text{m} \ .$$

16. (a) The gravitational acceleration at the surface of the Moon is $g_{\text{moon}} = 1.67$ m/s^2 (see Appendix C). The ratio of weights (for a given mass) is the ratio of g-values, so $W_{\text{moon}} = (100\,\text{N})(1.67/9.8) = 17\,\text{N}$.

(b) For the force on that object caused by Earth's gravity to equal 17 N, then the free-fall acceleration at its location must be $a_g = 1.67$ m/s^2. Thus,

$$a_g = \frac{GM_E}{r^2} \implies r = \sqrt{\frac{GM_E}{a_g}} = 1.5 \times 10^7\,\text{m}$$

so the object would need to be a distance of $r/R_E = 2.4$ "radii" from Earth's center.

17. If the angular velocity were any greater, loose objects on the surface would not go around with the planet but would travel out into space.

(a) The magnitude of the gravitational force exerted by the planet on an object of mass m at its surface is given by $F = GmM/R^2$, where M is the mass of the planet and R is its radius. According to Newton's second law this must equal mv^2/R, where v is the speed of the object. Thus,

$$\frac{GM}{R^2} = \frac{v^2}{R} \ .$$

Replacing M with $(4\pi/3)\rho R^3$ (where ρ is the density of the planet) and v with $2\pi R/T$ (where T is the period of revolution), we find

$$\frac{4\pi}{3} G\rho R = \frac{4\pi^2 R}{T^2} \ .$$

We solve for T and obtain

$$T = \sqrt{\frac{3\pi}{G\rho}} \ .$$

(b) The density is $3.0 \times 10^3 \text{ kg/m}^3$. We evaluate the equation for T:

$$T = \sqrt{\frac{3\pi}{(6.67 \times 10^{-11} \text{ m}^3/\text{s}^2{\cdot}\text{kg})(3.0 \times 10^3 \text{ kg/m}^3)}} = 6.86 \times 10^3 \text{ s} = 1.9 \text{ h} \ .$$

18. (a) The gravitational acceleration is

$$a_g = \frac{GM}{R^2} = 7.6 \text{ m/s}^2 \ .$$

(b) Note that the total mass is $5M$. Thus,

$$a_g = \frac{G(5M)}{(3R)^2} = 4.2 \text{ m/s}^2 \ .$$

19. (a) The forces acting on an object being weighed are the downward force of gravity and the upward force of the spring balance. Let F_g be the magnitude of the force of Earth's gravity and let W be the magnitude of the force exerted by the spring balance. The reading on the balance gives the value of W. The object is traveling around a circle of radius R and so has a centripetal acceleration. Newton's second law becomes $F_g - W = mV^2/R$, where V is the speed of the object as measured in an inertial frame and m is the mass of the object. Now $V = R\omega \pm v$, where ω is the angular velocity of Earth as it rotates and v is the speed of the ship relative to Earth. We note that the first term gives the speed of a point fixed to the rotating Earth. The plus sign is used if the ship is traveling in the same direction as the portion of Earth under it (west to east) and the negative sign is used if the ship is traveling in the opposite direction (east to west).

Newton's second law is now $F_g - W = m(R\omega \pm v)^2/R$. When we expand the parentheses we may neglect the term v^2 since v is much smaller than $R\omega$. Thus, $F_g - W = m(R^2\omega^2 \pm 2R\omega v)/R$ and $W = F_g - mR\omega^2 \mp 2m\omega v$. When $v = 0$ the scale reading is $W_0 = F_g - mR\omega^2$, so $W = W_0 \mp 2m\omega v$. We replace m with W_0/g to obtain $W = W_0(1 \mp 2\omega v/g)$.

(b) The upper sign ($-$) is used if the ship is sailing eastward and the lower sign ($+$) is used if the ship is sailing westward.

20. (a) Plugging $R_h = 2GM_h/c^2$ into the indicated expression, we find

$$a_g = \frac{GM_h}{(1.001R_h)^2} = \frac{GM_h}{(1.001)^2\left(2GM_h/c^2\right)^2} = \frac{c^4}{(2.002)^2 G}\,\frac{1}{M_h}$$

which yields $a_g = \left(3.02 \times 10^{43} \text{ kg}{\cdot}\text{m/s}^2\right)/M_h$.

(b) Since M_h is in the denominator of the above result, a_g decreases as M_h increases.

(c) With $M_h = \left(1.55 \times 10^{12}\right)\left(1.99 \times 10^{30} \text{ kg}\right)$, we obtain $a_g = 9.8 \text{ m/s}^2$.

(d) This part refers specifically to the very large black hole treated in the previous part. With that mass for M in Eq. 14-15, and $r = 2.002GM/c^2$, we obtain

$$da_g = -2\frac{GM}{(2.002GM/c^2)^3}\,dr = -\frac{2c^6}{(2.002)^3(GM)^2}\,dr$$

where $dr \rightarrow 1.70$ m as in the Sample Problem. This yields (in absolute value) an acceleration difference of 7.3×10^{-15} m/s^2.

(e) The miniscule result of the previous part implies that, in this case, any effects due to the differences of gravitational forces on the body are negligible.

21. From Eq. 14-13, we see the extreme case is when "g" becomes zero, and plugging in Eq. 14-14 leads to

$$0 = \frac{GM}{R^2} - R\omega^2 \implies M = \frac{R^3\omega^2}{G} \ .$$

Thus, with $R = 20000$ m and $\omega = 2\pi$ rad/s, we find $M = 4.7 \times 10^{24}$ kg.

22. (a) What contributes to the GmM/r^2 force on m is the (spherically distributed) mass M contained within r (where r is measured from the center of M). At point A we see that $M_1 + M_2$ is at a smaller radius than $r = a$ and thus contributes to the force:

$$|F_{\text{on}\,m}| = \frac{G\,(M_1 + M_2)\,m}{a^2} \quad .$$

(b) In the case $r = b$, only M_1 is contained within that radius, so the force on m becomes GM_1m/b^2.

(c) If the particle is at C, then no other mass is at smaller radius and the gravitational force on it is zero.

23. Using the fact that the volume of a sphere is $4\pi R^3/3$, we find the density of the sphere:

$$\rho = \frac{M_{\text{total}}}{\frac{4}{3}\pi R^3} = \frac{1.0 \times 10^4\,\text{kg}}{\frac{4}{3}\pi(1.0\,\text{m})^3} = 2.4 \times 10^3 \text{ kg/m}^3 \ .$$

When the particle of mass m (upon which the sphere, or parts of it, are exerting a gravitational force) is at radius r (measured from the center of the sphere), then whatever mass M is at a radius less than r must contribute to the magnitude of that force (GMm/r^2).

(a) At $r = 1.5$ m, all of M_{total} is at a smaller radius and thus all contributes to the force:

$$|F_{\text{on}\,m}| = \frac{GmM_{\text{total}}}{r^2} = m\left(3.0 \times 10^{-7}\,\text{N/kg}\right) \ .$$

(b) At $r = 0.50$ m, the portion of the sphere at radius smaller than that is

$$M = \rho\left(\frac{4}{3}\pi r^3\right) = 1.3 \times 10^3 \text{ kg} \ .$$

Thus, the force on m has magnitude $GMm/r^2 = m\left(3.3 \times 10^{-7}\,\text{N/kg}\right)$.

(c) Pursuing the calculation of part (b) algebraically, we find

$$|F_{\text{on}\,m}| = \frac{Gm\rho\left(\frac{4}{3}\pi r^3\right)}{r^2} = mr\left(6.7 \times 10^{-7}\,\frac{\text{N}}{\text{kg}\cdot\text{m}}\right) \ .$$

24. Since the volume of a sphere is $4\pi R^3/3$, the density is

$$\rho = \frac{M_{\text{total}}}{\frac{4}{3}\pi R^3} = \frac{3M_{\text{total}}}{4\pi R^3} .$$

When we test for gravitational acceleration (caused by the sphere, or by parts of it) at radius r (measured from the center of the sphere), the mass M which is at radius less than r is what contributes to the reading (GM/r^2). Since $M = \rho(4\pi r^3/3)$ for $r \le R$ then we can write this result as

$$\frac{G\left(\frac{3M_{\text{total}}}{4\pi R^3}\right)\left(\frac{4\pi r^3}{3}\right)}{r^2} = \frac{GM_{\text{total}}\, r}{R^3}$$

when we are considering points on or inside the sphere. Thus, the value a_g referred to in the problem is the case where $r = R$:

$$a_g = \frac{GM_{\text{total}}}{R^2} ,$$

and we solve for the case where the acceleration equals $a_g/3$:

$$\frac{GM_{\text{total}}}{3R^2} = \frac{GM_{\text{total}}\, r}{R^3} \quad \Longrightarrow \quad r = \frac{R}{3} .$$

Now we treat the case of an external test point. For points with $r > R$ the acceleration is GM_{total}/r^2, so the requirement that it equal $a_g/3$ leads to

$$\frac{GM_{\text{total}}}{3R^2} = \frac{GM_{\text{total}}}{r^2} \quad \Longrightarrow \quad r = R\sqrt{3} .$$

25. (a) The magnitude of the force on a particle with mass m at the surface of Earth is given by $F = GMm/R^2$, where M is the total mass of Earth and R is Earth's radius. The acceleration due to gravity is

$$a_g = \frac{F}{m} = \frac{GM}{R^2} = \frac{(6.67 \times 10^{-11}\,\text{m}^3/\text{s}^2\cdot\text{kg})(5.98 \times 10^{24}\,\text{kg})}{(6.37 \times 10^6\,\text{m})^2} = 9.83\ \text{m/s}^2 .$$

(b) Now $a_g = GM/R^2$, where M is the total mass contained in the core and mantle together and R is the outer radius of the mantle (6.345×10^6 m, according to Fig. 14–36). The total mass is $M = 1.93 \times 10^{24}\,\text{kg} + 4.01 \times 10^{24}\,\text{kg} = 5.94 \times 10^{24}\,\text{kg}$. The first term is the mass of the core and the second is the mass of the mantle. Thus,

$$a_g = \frac{(6.67 \times 10^{-11}\,\text{m}^3/\text{s}^2\cdot\text{kg})(5.94 \times 10^{24}\,\text{kg})}{(6.345 \times 10^6\,\text{m})^2} = 9.84\ \text{m/s}^2 .$$

(c) A point 25 km below the surface is at the mantle-crust interface and is on the surface of a sphere with a radius of $R = 6.345 \times 10^6$ m. Since the mass is now assumed to be uniformly distributed the mass within this sphere can be found by multiplying the mass per unit volume by the volume of the sphere: $M = (R^3/R_e^3)M_e$, where M_e is the total mass of Earth and R_e is the radius of Earth. Thus,

$$M = \left(\frac{6.345 \times 10^6\,\text{m}}{6.37 \times 10^6\,\text{m}}\right)^3 (5.98 \times 10^{24}\,\text{kg}) = 5.91 \times 10^{24}\,\text{kg} .$$

The acceleration due to gravity is

$$a_g = \frac{GM}{R^2} = \frac{(6.67 \times 10^{-11}\,\text{m}^3/\text{s}^2\cdot\text{kg})(5.91 \times 10^{24}\,\text{kg})}{(6.345 \times 10^6\,\text{m})^2} = 9.79\ \text{m/s}^2 .$$

26. (a) The gravitational potential energy is

$$U = -\frac{GMm}{r} = -\frac{\left(6.67 \times 10^{-11}\,\mathrm{m^3/kg\cdot s^2}\right)(5.2\,\mathrm{kg})(2.4\,\mathrm{kg})}{19\,\mathrm{m}} = -4.4 \times 10^{-11}\ \mathrm{J}\ .$$

(b) Since the change in potential energy is

$$\Delta U = -\frac{GMm}{3r} - \left(-\frac{GMm}{r}\right) = -\frac{2}{3}(-4.4 \times 10^{-11}\,\mathrm{J}) = 2.9 \times 10^{-11}\ \mathrm{J}\ ,$$

the work done by the gravitational force is $W = -\Delta U = -2.9 \times 10^{-11}$ J.

(c) The work done by you is $W' = \Delta U = 2.9 \times 10^{-11}$ J.

27. (a) We note that r_C (the distance from the origin to sphere C, which is the same as the separation between C and B) is 0.8, $r_D = 0.4$, and the separation between spheres C and D is $r_{CD} = 1.2$ (with SI units understood). The total potential energy is therefore

$$-\frac{GM_BM_C}{r_C^2} - \frac{GM_BM_D}{r_D^2} - \frac{GM_CM_D}{r_{CD}^2} = -1.3 \times 10^{-4}\ \mathrm{J}$$

using the mass-values given in problem 12.

(b) Since any gravitational potential energy term (of the sort considered in this chapter) is necessarily negative ($-GmM/r^2$ where all variables are positive) then having another mass to include in the computation can only lower the result (that is, make the result more negative).

(c) The observation in the previous part implies that the work I do in removing sphere A (to obtain the case considered in part (a)) must lead to an increase in the system energy; thus, I do positive work.

(d) To put sphere A back in, I do negative work, since I am causing the system energy to become more negative.

28. The gravitational potential energy is

$$U = -\frac{Gm(M-m)}{r} = -\frac{G}{r}\left(Mm - m^2\right)$$

which we differentiate with respect to m and set equal to zero (in order to minimize). Thus, we find $M - 2m = 0$ which leads to the ratio $m/M = 1/2$ to obtain the least potential energy. (Note that a second derivative of U with respect to m would lead to a positive result regardless of the value of m – which means its graph is everywhere concave upward and thus its extremum is indeed a minimum).

29. (a) The density of a uniform sphere is given by $\rho = 3M/4\pi R^3$, where M is its mass and R is its radius. The ratio of the density of Mars to the density of Earth is

$$\frac{\rho_M}{\rho_E} = \frac{M_M}{M_E}\,\frac{R_E^3}{R_M^3} = 0.11\left(\frac{0.65 \times 10^4\,\mathrm{km}}{3.45 \times 10^3\,\mathrm{km}}\right)^3 = 0.74\ .$$

(b) The value of a_g at the surface of a planet is given by $a_g = GM/R^2$, so the value for Mars is

$$a_{gM} = \frac{M_M}{M_E}\,\frac{R_E^2}{R_M^2}\,a_{gE} = 0.11\left(\frac{0.65 \times 10^4\,\mathrm{km}}{3.45 \times 10^3\,\mathrm{km}}\right)^2 (9.8\,\mathrm{m/s^2}) = 3.8\ \mathrm{m/s^2}\ .$$

(c) If v is the escape speed, then, for a particle of mass m

$$\tfrac{1}{2}mv^2 = G\frac{mM}{R}$$

and

$$v = \sqrt{\frac{2GM}{R}} \ .$$

For Mars

$$v = \sqrt{\frac{2(6.67 \times 10^{-11}\,\mathrm{m^3/s^2\!\cdot\!kg})(0.11)(5.98 \times 10^{24}\,\mathrm{kg})}{3.45 \times 10^6\,\mathrm{m}}} = 5.0 \times 10^3 \ \mathrm{m/s} \ .$$

30. The amount of (kinetic) energy needed to escape is the same as the (absolute value of the) gravitational potential energy at its original position. Thus, an object of mass m on a planet of mass M and radius R needs $K = GmM/R$ in order to (barely) escape.

    (a) Setting up the ratio, we find

    $$\frac{K_m}{K_E} = \frac{M_m}{M_E}\frac{R_E}{R_m} = 0.045$$

    using the values found in Appendix C.

    (b) Similarly, for the Jupiter escape energy (divided by that for Earth) we obtain

    $$\frac{K_J}{K_E} = \frac{M_J}{M_E}\frac{R_E}{R_J} = 28 \ .$$

31. (a) The work done by you in moving the sphere of mass m_2 equals the change in the potential energy of the three-sphere system. The initial potential energy is

    $$U_i = -\frac{Gm_1m_2}{d} - \frac{Gm_1m_3}{L} - \frac{Gm_2m_3}{L-d}$$

    and the final potential energy is

    $$U_f = -\frac{Gm_1m_2}{L-d} - \frac{Gm_1m_3}{L} - \frac{Gm_2m_3}{d} \ .$$

    The work done is

    $$\begin{aligned} W &= U_f - U_i = Gm_2\left(m_1\left(\frac{1}{d} - \frac{1}{L-d}\right) + m_3\left(\frac{1}{L-d} - \frac{1}{d}\right)\right) \\ &= (6.67 \times 10^{-11}\,\mathrm{m^3/s^2\!\cdot\!kg})(0.10\,\mathrm{kg})\left[(0.80\,\mathrm{kg})\left(\frac{1}{0.040\,\mathrm{m}} - \frac{1}{0.080\,\mathrm{m}}\right)\right. \\ &\qquad \left. +(0.20\,\mathrm{kg})\left(\frac{1}{0.080\,\mathrm{m}} - \frac{1}{0.040\,\mathrm{m}}\right)\right] \\ &= +5.0 \times 10^{-11}\ \mathrm{J} \ . \end{aligned}$$

    (b) The work done by the force of gravity is $-(U_f - U_i) = -5.0 \times 10^{-11}\,\mathrm{J}$.

32. Energy conservation for this situation may be expressed as follows:

    $$\begin{aligned} K_1 + U_1 &= K_2 + U_2 \\ K_1 - \frac{GmM}{r_1} &= K_2 - \frac{GmM}{r_2} \end{aligned}$$

    where $M = 5.0 \times 10^{23}\,\mathrm{kg}$, $r_1 = R = 3.0 \times 10^6\,\mathrm{m}$ and $m = 10\,\mathrm{kg}$.

    (a) If $K_1 = 5.0 \times 10^7\,\mathrm{J}$ and $r_2 = 4.0 \times 10^6\,\mathrm{m}$, then the above equation leads to

    $$K_2 = K_1 + GmM\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = 2.2 \times 10^7 \ \mathrm{J} \ .$$

(b) In this case, we require $K_2 = 0$ and $r_2 = 8.0 \times 10^6$ m, and solve for K_1:

$$K_1 = K_2 + GmM\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = 6.9 \times 10^7 \text{ J} .$$

33. (a) We use the principle of conservation of energy. Initially the rocket is at Earth's surface and the potential energy is $U_i = -GMm/R_e = -mgR_e$, where M is the mass of Earth, m is the mass of the rocket, and R_e is the radius of Earth. The relationship $g = GM/R_e^2$ was used. The initial kinetic energy is $\frac{1}{2}mv^2 = 2mgR_e$, where the substitution $v = 2\sqrt{gR_e}$ was made. If the rocket can escape then conservation of energy must lead to a positive kinetic energy no matter how far from Earth it gets. We take the final potential energy to be zero and let K_f be the final kinetic energy. Then, $U_i + K_i = U_f + K_f$ leads to $K_f = U_i + K_i = -mgR_e + 2mgR_e = mgR_e$. The result is positive and the rocket has enough kinetic energy to escape the gravitational pull of Earth.

(b) We write $\frac{1}{2}mv_f^2$ for the final kinetic energy. Then, $\frac{1}{2}mv_f^2 = mgR_e$ and $v_f = \sqrt{2gR_e}$.

34. Energy conservation for this situation may be expressed as follows:

$$\begin{aligned} K_1 + U_1 &= K_2 + U_2 \\ \frac{1}{2}mv_1^2 - \frac{GmM}{r_1} &= \frac{1}{2}mv_2^2 - \frac{GmM}{r_2} \end{aligned}$$

where $M = 7.0 \times 10^{24}$ kg, $r_2 = R = 1.6 \times 10^6$ m and $r_1 = \infty$ (which means that $U_1 = 0$). We are told to assume the meteor starts at rest, so $v_1 = 0$. Thus, $K_1 + U_1 = 0$ and the above equation is rewritten as

$$\frac{1}{2}mv_2^2 = \frac{GmM}{r_2} \implies v_2 = \sqrt{\frac{2GM}{R}} = 2.4 \times 10^4 \text{ m/s} .$$

35. (a) We use the principle of conservation of energy. Initially the particle is at the surface of the asteroid and has potential energy $U_i = -GMm/R$, where M is the mass of the asteroid, R is its radius, and m is the mass of the particle being fired upward. The initial kinetic energy is $\frac{1}{2}mv^2$. The particle just escapes if its kinetic energy is zero when it is infinitely far from the asteroid. The final potential and kinetic energies are both zero. Conservation of energy yields $-GMm/R + \frac{1}{2}mv^2 = 0$. We replace GM/R with a_gR, where a_g is the acceleration due to gravity at the surface. Then, the energy equation becomes $-a_gR + \frac{1}{2}v^2 = 0$. We solve for v:

$$v = \sqrt{2a_gR} = \sqrt{2(3.0\,\text{m/s}^2)(500 \times 10^3\,\text{m})} = 1.7 \times 10^3 \text{ m/s} .$$

(b) Initially the particle is at the surface; the potential energy is $U_i = -GMm/R$ and the kinetic energy is $K_i = \frac{1}{2}mv^2$. Suppose the particle is a distance h above the surface when it momentarily comes to rest. The final potential energy is $U_f = -GMm/(R+h)$ and the final kinetic energy is $K_f = 0$. Conservation of energy yields

$$-\frac{GMm}{R} + \frac{1}{2}mv^2 = -\frac{GMm}{R+h} .$$

We replace GM with a_gR^2 and cancel m in the energy equation to obtain

$$-a_gR + \frac{1}{2}v^2 = -\frac{a_gR^2}{(R+h)} .$$

The solution for h is

$$\begin{aligned} h &= \frac{2a_gR^2}{2a_gR - v^2} - R \\ &= \frac{2(3.0\,\text{m/s}^2)(500 \times 10^3\,\text{m})^2}{2(3.0\,\text{m/s}^2)(500 \times 10^3\,\text{m}) - (1000\,\text{m/s})^2} - (500 \times 10^3\,\text{m}) \\ &= 2.5 \times 10^5 \text{ m} . \end{aligned}$$

(c) Initially the particle is a distance h above the surface and is at rest. Its potential energy is $U_i = -GMm/(R+h)$ and its initial kinetic energy is $K_i = 0$. Just before it hits the asteroid its potential energy is $U_f = -GMm/R$. Write $\frac{1}{2}mv_f^2$ for the final kinetic energy. Conservation of energy yields

$$-\frac{GMm}{R+h} = -\frac{GMm}{R} + \frac{1}{2}mv^2 \ .$$

We substitute $a_g R^2$ for GM and cancel m, obtaining

$$-\frac{a_g R^2}{R+h} = -a_g R + \frac{1}{2}v^2 \ .$$

The solution for v is

$$\begin{aligned} v &= \sqrt{2a_g R - \frac{2a_g R^2}{R+h}} \\ &= \sqrt{2(3.0\,\mathrm{m/s^2})(500 \times 10^3\,\mathrm{m}) - \frac{2(3.0\,\mathrm{m/s^2})(500 \times 10^3\,\mathrm{m})^2}{500 \times 10^3\,\mathrm{m} + 1000 \times 10^3\,\mathrm{m}}} \\ &= 1.4 \times 10^3\ \mathrm{m/s}\ . \end{aligned}$$

36. (a) We note that *height* $= R - R_{\mathrm{Earth}}$ where $R_{\mathrm{Earth}} = 6.37 \times 10^6$ m. With $M = 5.98 \times 10^{24}$ kg, $R_0 = 6.57 \times 10^6$ m and $R = 7.37 \times 10^6$ m, we have

$$K_i + U_i = K + U \quad \Longrightarrow \quad \frac{1}{2}m\left(3.7 \times 10^3\right)^2 - \frac{GmM}{R_0} = K - \frac{GmM}{R}$$

Solving, we find $K = 3.8 \times 10^7$ J.

(b) Again, we use energy conservation.

$$K_i + U_i = K_f + U_f \quad \Longrightarrow \quad \frac{1}{2}m\left(3.7 \times 10^3\right)^2 - \frac{GmM}{R_0} = 0 - \frac{GmM}{R_f}$$

Therefore, we find $R_f = 7.40 \times 10^6$ m. This corresponds to a distance of $1034.9 \approx 1.03 \times 10^3$ km above the earth's surface.

37. (a) The momentum of the two-star system is conserved, and since the stars have the same mass, their speeds and kinetic energies are the same. We use the principle of conservation of energy. The initial potential energy is $U_i = -GM^2/r_i$, where M is the mass of either star and r_i is their initial center-to-center separation. The initial kinetic energy is zero since the stars are at rest. The final potential energy is $U_f = -2GM^2/r_i$ since the final separation is $r_i/2$. We write Mv^2 for the final kinetic energy of the system. This is the sum of two terms, each of which is $\frac{1}{2}Mv^2$. Conservation of energy yields

$$-\frac{GM^2}{r_i} = -\frac{2GM^2}{r_i} + Mv^2 \ .$$

The solution for v is

$$v = \sqrt{\frac{GM}{r_i}} = \sqrt{\frac{(6.67 \times 10^{-11}\,\mathrm{m^3/s^2 \cdot kg})(10^{30}\,\mathrm{kg})}{10^{10}\,\mathrm{m}}} = 8.2 \times 10^4\,\mathrm{m/s}\ .$$

(b) Now the final separation of the centers is $r_f = 2R = 2 \times 10^5$ m, where R is the radius of either of the stars. The final potential energy is given by $U_f = -GM^2/r_f$ and the energy equation becomes

$-GM^2/r_i = -GM^2/r_f + Mv^2$. The solution for v is

$$\begin{aligned} v &= \sqrt{GM\left(\frac{1}{r_f} - \frac{1}{r_i}\right)} \\ &= \sqrt{(6.67 \times 10^{-11}\,\mathrm{m^3/s^2\cdot kg})(10^{30}\,\mathrm{kg})\left(\frac{1}{2 \times 10^5\,\mathrm{m}} - \frac{1}{10^{10}\,\mathrm{m}}\right)} \\ &= 1.8 \times 10^7 \text{ m/s} . \end{aligned}$$

38. (a) The initial gravitational potential energy is

$$U_i = -\frac{GM_AM_B}{r_i} = -\frac{\left(6.67 \times 10^{-11}\right)(20)(10)}{0.80} = -1.67 \times 10^{-8} \text{ J} .$$

(b) We use conservation of energy (with $K_i = 0$):

$$\begin{aligned} U_i &= K + U \\ -1.67 \times 10^{-8} &= K - \frac{\left(6.67 \times 10^{-11}\right)(20)(10)}{0.60} \end{aligned}$$

which yields $K = 5.6 \times 10^{-9}$ J. Note that the value of r is the difference between 0.80 m and 0.20 m.

39. Energy conservation for this situation may be expressed as follows:

$$\begin{aligned} K_1 + U_1 &= K_2 + U_2 \\ \frac{1}{2}mv_1^2 - \frac{GmM}{r_1} &= \frac{1}{2}mv_2^2 - \frac{GmM}{r_2} \end{aligned}$$

where $M = 5.98 \times 10^{24}$ kg, $r_1 = R = 6.37 \times 10^6$ m and $v_1 = 10000$ m/s. Setting $v_2 = 0$ to find the maximum of its trajectory, we solve the above equation (noting that m cancels in the process) and obtain $r_2 = 3.2 \times 10^7$ m. This implies that its *altitude* is $r_2 - R = 2.5 \times 10^7$ m.

40. Kepler's law of periods, expressed as a ratio, is

$$\left(\frac{a_M}{a_E}\right)^3 = \left(\frac{T_M}{T_E}\right)^2 \implies 1.52^3 = \left(\frac{T_M}{1\,\mathrm{y}}\right)^2$$

where we have substituted the mean-distance (from Sun) ratio for the semimajor axis ratio. This yields $T_M = 1.87$ y. The value in Appendix C (1.88 y) is quite close, and the small apparent discrepancy is not significant, since a more precise value for the semimajor axis ratio is $a_M/a_E = 1.523$ which does lead to $T_M = 1.88$ y using Kepler's law. A question can be raised regarding the use of a ratio of mean distances for the ratio of semimajor axes, but this requires a more lengthy discussion of what is meant by a "mean distance" than is appropriate here.

41. The period T and orbit radius r are related by the law of periods: $T^2 = (4\pi^2/GM)r^3$, where M is the mass of Mars. The period is 7 h 39 min, which is 2.754×10^4 s. We solve for M:

$$\begin{aligned} M &= \frac{4\pi^2 r^3}{GT^2} \\ &= \frac{4\pi^2(9.4 \times 10^6\,\mathrm{m})^3}{(6.67 \times 10^{-11}\,\mathrm{m^3/s^2\cdot kg})(2.754 \times 10^4\,\mathrm{s})^2} = 6.5 \times 10^{23} \text{ kg} . \end{aligned}$$

42. With $T = 27.3(86400) = 2.36 \times 10^6$ s, Kepler's law of periods becomes

$$T^2 = \left(\frac{4\pi^2}{GM_E}\right)r^3 \implies M_E = \frac{4\pi^2\left(3.82 \times 10^8\right)^3}{\left(6.67 \times 10^{-11}\right)\left(2.36 \times 10^6\right)^2}$$

which yields $M_E = 5.93 \times 10^{24}$ kg for the mass of Earth.

43. Let N be the number of stars in the galaxy, M be the mass of the Sun, and r be the radius of the galaxy. The total mass in the galaxy is NM and the magnitude of the gravitational force acting on the Sun is $F = GNM^2/r^2$. The force points toward the galactic center. The magnitude of the Sun's acceleration is $a = v^2/R$, where v is its speed. If T is the period of the Sun's motion around the galactic center then $v = 2\pi R/T$ and $a = 4\pi^2 R/T^2$. Newton's second law yields $GNM^2/R^2 = 4\pi^2 MR/T^2$. The solution for N is

$$N = \frac{4\pi^2 R^3}{GT^2 M} \; .$$

The period is 2.5×10^8 y, which is 7.88×10^{15} s, so

$$N = \frac{4\pi^2 (2.2 \times 10^{20}\,\mathrm{m})^3}{(6.67 \times 10^{-11}\,\mathrm{m^3/s^2 \cdot kg})(7.88 \times 10^{15}\,\mathrm{s})^2 (2.0 \times 10^{30}\,\mathrm{kg})} = 5.1 \times 10^{10} \; .$$

44. Kepler's law of periods, expressed as a ratio, is

$$\left(\frac{r_s}{r_m}\right)^3 = \left(\frac{T_s}{T_m}\right)^2 \implies \left(\frac{1}{2}\right)^3 = \left(\frac{T_s}{1\,\text{lunar month}}\right)^2$$

which yields $T_s = 0.35$ lunar month for the period of the satellite.

45. (a) If r is the radius of the orbit then the magnitude of the gravitational force acting on the satellite is given by GMm/r^2, where M is the mass of Earth and m is the mass of the satellite. The magnitude of the acceleration of the satellite is given by v^2/r, where v is its speed. Newton's second law yields $GMm/r^2 = mv^2/r$. Since the radius of Earth is 6.37×10^6 m the orbit radius is $r = 6.37 \times 10^6\,\mathrm{m} + 160 \times 10^3\,\mathrm{m} = 6.53 \times 10^6\,\mathrm{m}$. The solution for v is

$$v = \sqrt{\frac{GM}{r}} = \sqrt{\frac{(6.67 \times 10^{-11}\,\mathrm{m^3/s^2 \cdot kg})(5.98 \times 10^{24}\,\mathrm{kg})}{6.53 \times 10^6\,\mathrm{m}}} = 7.82 \times 10^3\ \mathrm{m/s} \; .$$

(b) Since the circumference of the circular orbit is $2\pi r$, the period is

$$T = \frac{2\pi r}{v} = \frac{2\pi (6.53 \times 10^6\,\mathrm{m})}{7.82 \times 10^3\,\mathrm{m/s}} = 5.25 \times 10^3\ \mathrm{s} \; .$$

This is equivalent to 87.4 min.

46. (a) The distance from the center of an ellipse to a focus is ae where a is the semimajor axis and e is the eccentricity. Thus, the separation of the foci (in the case of Earth's orbit) is

$$2ae = 2\left(1.50 \times 10^{11}\,\mathrm{m}\right)(0.0167) = 5.01 \times 10^9\ \mathrm{m} \; .$$

(b) To express this in terms of solar radii (see Appendix C), we set up a ratio:

$$\frac{5.01 \times 10^9\,\mathrm{m}}{6.96 \times 10^8\,\mathrm{m}} = 7.2 \; .$$

47. (a) The greatest distance between the satellite and Earth's center (the apogee distance) is $R_a = 6.37 \times 10^6\,\mathrm{m} + 360 \times 10^3\,\mathrm{m} = 6.73 \times 10^6\,\mathrm{m}$. The least distance (perigee distance) is $R_p = 6.37 \times 10^6\,\mathrm{m} + 180 \times 10^3\,\mathrm{m} = 6.55 \times 10^6\,\mathrm{m}$. Here 6.37×10^6 m is the radius of Earth. From Fig. 14-13, we see that the semimajor axis is

$$a = \frac{R_a + R_p}{2} = \frac{6.73 \times 10^6\,\mathrm{m} + 6.55 \times 10^6\,\mathrm{m}}{2} = 6.64 \times 10^6\ \mathrm{m} \; .$$

(b) The apogee and perigee distances are related to the eccentricity e by $R_a = a(1+e)$ and $R_p = a(1-e)$. Add to obtain $R_a + R_p = 2a$ and $a = (R_a + R_p)/2$. Subtract to obtain $R_a - R_p = 2ae$. Thus,

$$e = \frac{R_a - R_p}{2a} = \frac{R_a - R_p}{R_a + R_p} = \frac{6.73 \times 10^6\,\text{m} - 6.55 \times 10^6\,\text{m}}{6.73 \times 10^6\,\text{m} + 6.55 \times 10^6\,\text{m}} = 0.0136\ .$$

48. To "hover" above Earth ($M_E = 5.98 \times 10^{24}$ kg) means that it has a period of 24 hours (86400 s). By Kepler's law of periods,

$$86400^2 = \left(\frac{4\pi^2}{GM_E}\right) r^3 \implies r = 4.225 \times 10^7\ \text{m}\ .$$

Its altitude is therefore $r - R_E$ (where $R_E = 6.37 \times 10^6$ m) which yields 3.59×10^7 m.

49. (a) The period of the comet is 1420 years (and one month), which we convert to $T = 4.48 \times 10^{10}$ s. Since the mass of the Sun is 1.99×10^{30} kg, then Kepler's law of periods gives

$$\left(4.48 \times 10^{10}\right)^2 = \left(\frac{4\pi^2}{\left(6.67 \times 10^{-11}\right)\left(1.99 \times 10^{30}\right)}\right) a^3 \implies a = 1.89 \times 10^{13}\ \text{m}\ .$$

(b) Since the distance from the focus (of an ellipse) to its center is ea and the distance from center to the aphelion is a, then the comet is at a distance of

$$ea + a = (0.11 + 1)\left(1.89 \times 10^{13}\,\text{m}\right) = 2.1 \times 10^{13}\ \text{m}$$

when it is farthest from the Sun. To express this in terms of Pluto's orbital radius (found in Appendix C), we set up a ratio:

$$\left(\frac{2.1 \times 10^{13}}{5.9 \times 10^{12}}\right) R_P = 3.6 R_P\ .$$

50. (a) The period is $T = 27(3600) = 97200$ s, and we are asked to assume that the orbit is circular (of radius $r = 100000$ m). Kepler's law of periods provides us with an approximation to the asteroid's mass:

$$(97200)^2 = \left(\frac{4\pi^2}{GM}\right)(100000)^3 \implies M = 6.3 \times 10^{16}\ \text{kg}\ .$$

(b) Dividing the mass M by the given volume yields an average density equal to $6.3 \times 10^{16}/1.41 \times 10^{13} = 4.4 \times 10^3\,\text{kg/m}^3$, which is about 20% less dense than Earth (the average density of Earth is given in a Table in Chapter 15).

51. (a) If we take the logarithm of Kepler's law of periods, we obtain

$$2\log(T) = \log\left(4\pi^2/GM\right) + 3\log(a) \implies \log(a) = \frac{2}{3}\log(T) - \frac{1}{3}\log\left(4\pi^2/GM\right)$$

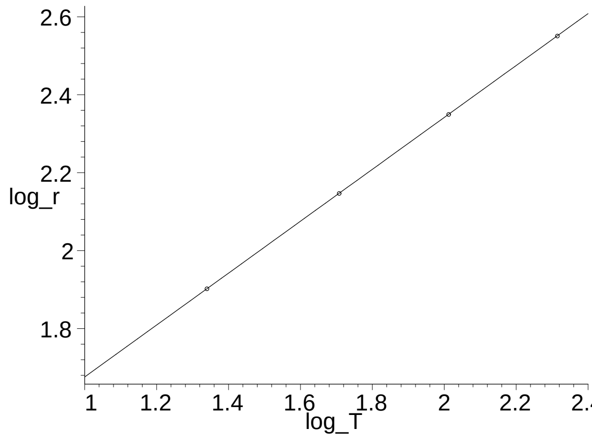
where we are ignoring an important subtlety about units (the arguments of logarithms cannot have units, since they are transcendental functions). Although the problem can be continued in this way, we prefer to set it up without units, which requires taking a ratio. If we divide Kepler's law (applied to the Jupiter-moon system, where M is mass of Jupiter) by the law applied to Earth orbiting the Sun (of mass M_o), we obtain

$$(T/T_E)^2 = \left(\frac{M_\text{o}}{M}\right)\left(\frac{a}{r_E}\right)^3$$

where $T_E = 365.25$ days is Earth's orbital period and $r_E = 1.50 \times 10^{11}$ m is its mean distance from the Sun. In this case, it is perfectly legitimate to take logarithms and obtain

$$\log\left(\frac{r_E}{a}\right) = \frac{2}{3}\log\left(\frac{T_E}{T}\right) + \frac{1}{3}\log\left(\frac{M_\text{o}}{M}\right)$$

(written to make each term positive) which is the way we plot the data ($\log(r_E/a)$ on the vertical axis and $\log(T_E/T)$ on the horizontal axis).



(b) When we perform a least-squares fit to the data, we obtain $\log(r_E/a) = 0.666\log(T_E/T) + 1.01$, which confirms the expectation of slope $= 2/3$ based on the above equation.

(c) And the 1.01 intercept corresponds to the term $\frac{1}{3}\log\left(\frac{M_o}{M}\right)$ which implies

$$\frac{M_o}{M} = 10^{3.03} \implies M = \frac{M_o}{1.07 \times 10^3} .$$

Plugging in $M_o = 1.99 \times 10^{30}$ kg (see Appendix C), we obtain $M = 1.86 \times 10^{27}$ kg for Jupiter's mass. This is reasonably consistent with the value 1.90×10^{27} kg found in Appendix C.

52. From Kepler's law of periods (where $T = 2.4(3600) = 8640$ s), we find the planet's mass M:

$$(8640\,\text{s})^2 = \left(\frac{4\pi^2}{GM}\right)\left(8.0 \times 10^6\,\text{m}\right)^3 \implies M = 4.06 \times 10^{24}\ \text{kg} .$$

But we also know $a_g = GM/R^2 = 8.0\,\text{m/s}^2$ so that we are able to solve for the planet's radius:

$$R = \sqrt{\frac{GM}{a_g}} = 5.8 \times 10^6\ \text{m} .$$

53. We follow the approach shown in Sample Problem 14-7. In our system, we have $m_1 = m_2 = M$ (the mass of our Sun, 1.99×10^{30} kg). From Eq. 14-37, we see that $r = 2r_1$ in this system (so r_1 is one-half the Earth-to-Sun distance r). And Eq. 14-39 gives $v = \pi r/T$ for the speed. Plugging these observations into Eq. 14-35 leads to

$$\frac{Gm_1m_2}{r^2} = m_1\frac{(\pi r/T)^2}{r/2} \implies T = \sqrt{\frac{2\pi^2 r^3}{GM}} .$$

With $r = 1.5 \times 10^{11}$ m, we obtain $T = 2.2 \times 10^7$ s. We can express this in terms of Earth-years, by setting up a ratio:

$$T = \left(\frac{T}{1\,\text{y}}\right)(1\,\text{y}) = \left(\frac{2.2 \times 10^7\,\text{s}}{3.156 \times 10^7\,\text{s}}\right)(1\,\text{y}) = 0.71\ \text{y} .$$

54. The magnitude of the net gravitational force on one of the smaller stars (of mass m) is

$$\frac{GMm}{r^2} + \frac{Gmm}{(2r)^2} = \frac{Gm}{r^2}\left(M + \frac{m}{4}\right) .$$

This supplies the centripetal force needed for the motion of the star:

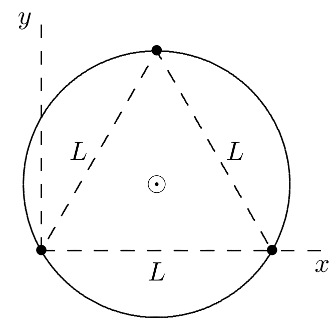
$$\frac{Gm}{r^2}\left(M+\frac{m}{4}\right)=m\frac{v^2}{r} \qquad \text{where } v=\frac{2\pi r}{T}\ .$$

Plugging in for speed v, we arrive at an equation for period T:

$$T=\frac{2\pi r^{3/2}}{\sqrt{G(M+m/4)}}\ .$$

55. Each star is attracted toward each of the other two by a force of magnitude GM^2/L^2, along the line that joins the stars. The net force on each star has magnitude $2(GM^2/L^2)\cos 30°$ and is directed toward the center of the triangle. This is a centripetal force and keeps the stars on the same circular orbit if their speeds are appropriate. If R is the radius of the orbit, Newton's second law yields $(GM^2/L^2)\cos 30° = Mv^2/R$.

The stars rotate about their center of mass (marked by $\odot$ on the diagram to the right) at the intersection of the perpendicular bisectors of the triangle sides, and the radius of the orbit is the distance from a star to the center of mass of the three-star system. We take the coordinate system to be as shown in the diagram, with its origin at the left-most star. The altitude of an equilateral triangle is $(\sqrt{3}/2)L$, so the stars are located at $x=0$, $y=0$; $x=L$, $y=0$; and $x=L/2$, $y=\sqrt{3}L/2$. The x coordinate of the center of mass is $x_c=(L+L/2)/3=L/2$ and the y coordinate is $y_c=(\sqrt{3}L/2)/3=L/2\sqrt{3}$. The distance from a star to the center of mass is $R=\sqrt{x_c^2+y_c^2}=\sqrt{(L^2/4)+(L^2/12)}=L/\sqrt{3}$.



Once the substitution for R is made Newton's second law becomes $(2GM^2/L^2)\cos 30° = \sqrt{3}Mv^2/L$. This can be simplified somewhat by recognizing that $\cos 30° = \sqrt{3}/2$, and we divide the equation by M. Then, $GM/L^2 = v^2/L$ and $v=\sqrt{GM/L}$.

56. (a) From Eq. 14-44, we see that the energy of each satellite is $-GM_Em/2r$. The total energy of the two satellites is twice that result; $-GM_Em/r$.

(b) We note that the speed of the wreckage will be zero (immediately after the collision), so it has no kinetic energy at that moment. Replacing m with $2m$ in the potential energy expression, we therefore find the total energy of the wreckage at that instant is $-2GM_Em/r$.

(c) An object with zero speed at that distance from Earth will simply fall towards the Earth, its trajectory being toward the center of the planet.

57. (a) We use the law of periods: $T^2=(4\pi^2/GM)r^3$, where M is the mass of the Sun $(1.99\times10^{30}\,\text{kg})$ and r is the radius of the orbit. The radius of the orbit is twice the radius of Earth's orbit: $r=2r_e=2(150\times10^9\,\text{m})=300\times10^9\,\text{m}$. Thus,

$$\begin{aligned} T &= \sqrt{\frac{4\pi^2r^3}{GM}} \\ &= \sqrt{\frac{4\pi^2(300\times10^9\,\text{m})^3}{(6.67\times10^{-11}\,\text{m}^3/\text{s}^2\cdot\text{kg})(1.99\times10^{30}\,\text{kg})}} = 8.96\times10^7\,\text{s}\ . \end{aligned}$$

Dividing by $(365\,\text{d/y})(24\,\text{h/d})(60\,\text{min/h})(60\,\text{s/min})$, we obtain $T=2.8\,\text{y}$.

(b) The kinetic energy of any asteroid or planet in a circular orbit of radius r is given by $K=GMm/2r$, where m is the mass of the asteroid or planet. We note that it is proportional to m and inversely proportional to r. The ratio of the kinetic energy of the asteroid to the kinetic energy of Earth is $K/K_e=(m/m_e)(r_e/r)$. We substitute $m=2.0\times10^{-4}m_e$ and $r=2r_e$ to obtain $K/K_e=1.0\times10^{-4}$.

58. Although altitudes are given, it is the orbital radii which enter the equations. Thus, $r_A = 6370 + 6370 = 12740$ km, and $r_B = 19110 + 6370 = 25480$ km

(a) The ratio of potential energies is

$$\frac{U_B}{U_A} = \frac{-\frac{GmM}{r_B}}{-\frac{GmM}{r_A}} = \frac{r_A}{r_B} = \frac{1}{2} \; .$$

(b) Using Eq. 14-42, the ratio of kinetic energies is

$$\frac{K_B}{K_A} = \frac{\frac{GmM}{2r_B}}{\frac{GmM}{2r_A}} = \frac{r_A}{r_B} = \frac{1}{2} \; .$$

(c) From Eq. 14-44, it is clear that the satellite with the largest value of r has the smallest value of $|E|$ (since r is in the denominator). And since the values of E are negative, then the smallest value of $|E|$ corresponds to the largest energy E. Thus, satellite B has the largest energy, by an amount

$$\Delta E = E_B - E_A = -\frac{GmM}{2}\left(\frac{1}{r_B} - \frac{1}{r_A}\right) \; .$$

Being careful to convert the r values to meters, we obtain $\Delta E = 1.1 \times 10^8$ J. The mass M of Earth is found in Appendix C.

59. The total energy is given by $E = -GMm/2a$, where M is the mass of the central attracting body (the Sun, for example), m is the mass of the object (a planet, for example), and a is the semimajor axis of the orbit. If the object is a distance r from the central body the potential energy is $U = -GMm/r$. We write $\frac{1}{2}mv^2$ for the kinetic energy. Then, $E = K + U$ becomes $-GMm/2a = \frac{1}{2}mv^2 - GMm/r$. We solve for v^2. The result is

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right) \; .$$

60. (a) For $r = R_p$,

$$\begin{aligned} v_p^2 &= GM_s\left(\frac{2}{R_p} - \frac{1}{a}\right) \\ &= \left(6.67 \times 10^{-11}\,\mathrm{m^3/s^2 \cdot kg}\right)\left(1.99 \times 10^{30}\,\mathrm{kg}\right)\left(\frac{2}{8.9 \times 10^{10}\,\mathrm{m}} - \frac{1}{2.7 \times 10^{12}\,\mathrm{m}}\right) \\ v_p &= 5.4 \times 10^4 \text{ m/s} \; . \end{aligned}$$

(b) For $r = R_a$,

$$\begin{aligned} v_a^2 &= GM_s\left(\frac{2}{R_a} - \frac{1}{a}\right) \\ &= \left(6.67 \times 10^{-11}\,\mathrm{m^3/s^2 \cdot kg}\right)\left(1.99 \times 10^{30}\,\mathrm{kg}\right)\left(\frac{2}{5.3 \times 10^{12}\,\mathrm{m}} - \frac{1}{2.7 \times 10^{12}\,\mathrm{m}}\right) \\ v_a &= 9.6 \times 10^2 \text{ m/s} \; . \end{aligned}$$

(c) We appeal to angular momentum conservation:

$$L = mvr = mv_aR_a = mv_pR_p = \text{constant} \qquad \Longrightarrow \quad \frac{v_a}{v_p} = \frac{R_p}{R_a} \; .$$

61. The energy required to raise a satellite of mass m to an altitude h (at rest) is given by

$$E_1 = \Delta U = GM_E m \left(\frac{1}{R_E} - \frac{1}{R_E + h} \right) ,$$

and the energy required to put it in circular orbit once it is there is

$$E_2 = \frac{1}{2} m v_{\text{orb}}^2 = \frac{GM_E m}{2(R_E + h)} .$$

Consequently, the energy difference is

$$\Delta E = E_1 - E_2 = GM_E m \left[\frac{1}{R_E} - \frac{3}{2(R_E + h)} \right] .$$

(a) Since

$$\frac{1}{R_E} - \frac{3}{2(R_E + h)} = \frac{1}{6370\,\text{km}} - \frac{3}{2(6370\,\text{km} + 1500\,\text{km})} < 0$$

the answer is no ($E_1 < E_2$).

(b) Since

$$\frac{1}{R_E} - \frac{3}{2(R_E + h)} = \frac{1}{6370\,\text{km}} - \frac{3}{2(6370\,\text{km} + 3185\,\text{km})} = 0$$

we have $E_1 = E_2$.

(c) Since

$$\frac{1}{R_E} - \frac{3}{2(R_E + h)} = \frac{1}{6370\,\text{km}} - \frac{3}{2(6370\,\text{km} + 4500\,\text{km})} > 0$$

the answer is yes ($E_1 > E_2$).

62. (a) The pellets will have the same speed v but opposite direction of motion, so the *relative speed* between the pellets and satellite is $2v$. Replacing v with $2v$ in Eq. 14-42 is equivalent to multiplying it by a factor of 4. Thus,

$$K_{\text{rel}} = 4 \left(\frac{GM_E m}{2r} \right) = \frac{2 \left(6.67 \times 10^{-11}\,\text{m}^3/\text{kg}{\cdot}\text{s}^2\right) \left(5.98 \times 10^{24}\,\text{kg}\right) (0.0040\,\text{kg})}{(6370 + 500) \times 10^3\,\text{m}} = 4.6 \times 10^5 \text{ J} .$$

(b) We set up the ratio of kinetic energies:

$$\frac{K_{\text{rel}}}{K_{\text{bullet}}} = \frac{4.6 \times 10^5\,\text{J}}{\frac{1}{2}(0.0040\text{kg})(950\,\text{m/s})^2} = 2.6 \times 10^2 .$$

63. (a) The force acting on the satellite has magnitude GMm/r^2, where M is the mass of Earth, m is the mass of the satellite, and r is the radius of the orbit. The force points toward the center of the orbit. Since the acceleration of the satellite is v^2/r, where v is its speed, Newton's second law yields $GMm/r^2 = mv^2/r$ and the speed is given by $v = \sqrt{GM/r}$. The radius of the orbit is the sum of Earth's radius and the altitude of the satellite: $r = 6.37 \times 10^6 + 640 \times 10^3 = 7.01 \times 10^6$ m. Thus,

$$v = \sqrt{\frac{(6.67 \times 10^{-11}\,\text{m}^3/\text{s}^2{\cdot}\text{kg})(5.98 \times 10^{24}\,\text{kg})}{7.01 \times 10^6\,\text{m}}} = 7.54 \times 10^3 \text{ m/s} .$$

(b) The period is $T = 2\pi r/v = 2\pi(7.01 \times 10^6\,\text{m})/(7.54 \times 10^3\,\text{m/s}) = 5.84 \times 10^3\,\text{s}$. This is 97 min.

(c) If E_0 is the initial energy then the energy after n orbits is $E = E_0 - nC$, where $C = 1.4 \times 10^5$ J/orbit. For a circular orbit the energy and orbit radius are related by $E = -GMm/2r$, so the radius after n orbits is given by $r = -GMm/2E$.
The initial energy is

$$E_0 = -\frac{(6.67 \times 10^{-11}\,\mathrm{m^3/s^2\cdot kg})(5.98 \times 10^{24}\,\mathrm{kg})(220\,\mathrm{kg})}{2(7.01 \times 10^6\,\mathrm{m})} = -6.26 \times 10^9\ \mathrm{J}\ ,$$

the energy after 1500 orbits is

$$E = E_0 - nC = -6.26 \times 10^9\ \mathrm{J} - (1500\,\mathrm{orbit})(1.4 \times 10^5\ \mathrm{J/orbit}) = -6.47 \times 10^9\ \mathrm{J}\ ,$$

and the orbit radius after 1500 orbits is

$$r = -\frac{(6.67 \times 10^{-11}\,\mathrm{m^3/s^2\cdot kg})(5.98 \times 10^{24}\,\mathrm{kg})(220\,\mathrm{kg})}{2\,(-6.47 \times 10^9\,\mathrm{J})} = 6.78 \times 10^6\ \mathrm{m}\ .$$

The altitude is $h = r - R = 6.78 \times 10^6\,\mathrm{m} - 6.37 \times 10^6\,\mathrm{m} = 4.1 \times 10^5\,\mathrm{m}$. Here R is the radius of Earth. This torque is internal to the satellite-Earth system, so the angular momentum of that system is conserved.

(d) The speed is

$$v = \sqrt{\frac{GM}{r}} = \sqrt{\frac{(6.67 \times 10^{-11}\,\mathrm{m^3/s^2\cdot kg})(5.98 \times 10^{24}\,\mathrm{kg})}{6.78 \times 10^6\,\mathrm{m}}} = 7.67 \times 10^3\,\mathrm{m/s}\ .$$

(e) The period is

$$T = \frac{2\pi r}{v} = \frac{2\pi(6.78 \times 10^6\,\mathrm{m})}{7.67 \times 10^3\,\mathrm{m/s}} = 5.6 \times 10^3\ \mathrm{s}\ .$$

This is equivalent to 93 min.

(f) Let F be the magnitude of the average force and s be the distance traveled by the satellite. Then, the work done by the force is $W = -Fs$. This is the change in energy: $-Fs = \Delta E$. Thus, $F = -\Delta E/s$. We evaluate this expression for the first orbit. For a complete orbit $s = 2\pi r = 2\pi(7.01 \times 10^6\,\mathrm{m}) = 4.40 \times 10^7\,\mathrm{m}$, and $\Delta E = -1.4 \times 10^5$ J. Thus,

$$F = -\frac{\Delta E}{s} = \frac{1.4 \times 10^5\ \mathrm{J}}{4.40 \times 10^7\,\mathrm{m}} = 3.2 \times 10^{-3}\ \mathrm{N}\ .$$

(g) The resistive force exerts a torque on the satellite, so its angular momentum is not conserved.

(h) The satellite-Earth system is essentially isolated, so its momentum is very nearly conserved.

64. We define the "effective gravity" in his environment as $g = 220/60 = 3.67\,\mathrm{m/s^2}$. Thus, using equations from Chapter 2 (and selecting downwards as the positive direction), we find the the "fall-time" to be

$$\Delta y = v_0 t + \frac{1}{2} a t^2 \implies t = \sqrt{\frac{2(2.1)}{3.67}} = 1.1\ \mathrm{s}\ .$$

65. We estimate the planet to have radius $r = 10$ m. To estimate the mass m of the planet, we require its density equal that of Earth (and use the fact that the volume of a sphere is $4\pi r^3/3$).

$$\frac{m}{4\pi r^3/3} = \frac{M_E}{4\pi R_E^3/3} \implies m = M_E\left(\frac{r}{R_E}\right)^3$$

which yields (with $M_E \approx 6 \times 10^{24}$ kg and $R_E \approx 6.4 \times 10^6$ m) $m = 2.3 \times 10^7$ kg.

(a) With the above assumptions, the acceleration due to gravity is

$$a_g = \frac{Gm}{r^2} = \frac{\left(6.7 \times 10^{-11}\right)\left(2.3 \times 10^{7}\right)}{10^2} = 1.5 \times 10^{-5} \text{ m/s}^2 .$$

(b) Eq. 14-27 gives the escape speed:

$$v = \sqrt{\frac{2Gm}{r}} \approx 0.02 \text{ m/s} .$$

66. From Eq. 14-41, we obtain $v = \sqrt{GM/r}$ for the speed of an object in circular orbit (of radius r) around a planet of mass M. In this case, $M = 5.98 \times 10^{24}$ kg and $r = 700 + 6370 = 7070\,\text{km} = 7.07 \times 10^6\,\text{m}$. The speed is found to be $v = 7.51 \times 10^3$ m/s. After multiplying by 3600 s/h and dividing by 1000 m/km this becomes $v = 2.7 \times 10^4$ km/h.

(a) For a head-on collision, the relative speed of the two objects must be $2v = 5.4 \times 10^4$ km/h.

(b) A perpendicular collision is possible if one satellite is, say, orbiting above the equator and the other is following a longitudinal line. In this case, the relative speed is given by the Pythagorean theorem: $\sqrt{v^2 + v^2} = 3.8 \times 10^4$ km/h.

67. (a) It is possible to use $v^2 = v_0^2 + 2a\Delta y$ as we did for free-fall problems in Chapter 2 because the acceleration can be considered approximately constant over this interval. However, our approach will not assume constant acceleration; we use energy conservation:

$$\frac{1}{2}mv_0^2 - \frac{GMm}{r_0} = \frac{1}{2}mv^2 - \frac{GMm}{r} \implies v = \sqrt{\frac{2GM(r_0 - r)}{r_0\, r}}$$

which yields $v = 1.4 \times 10^6$ m/s.

(b) We estimate the height of the apple to be $h = 7\,\text{cm} = 0.07\,\text{m}$. We may find the answer by evaluating Eq. 14-10 at the surface (radius r in part (a)) and at radius $r + h$, being careful not to round off, and then taking the difference of the two values, or we may take the differential of that equation – setting dr equal to h. We illustrate the latter procedure:

$$|da_g| = \left|-2\frac{GM}{r^3}\,dr\right| \approx 2\frac{GM}{r^3}\,h = 3 \times 10^6 \text{ m/s}^2 .$$

68. (a) We partition the full range into arcs of 3° each: 360°/3° = 120. Thus, the maximum number of geosynchronous satellites is 120.

(b) Kepler's law of periods, applied to a satellite around Earth, gives

$$T^2 = \left(\frac{4\pi^2}{GM_E}\right) r^3$$

where $T = 24\,\text{h} = 86400\,\text{s}$ for the geosynchronous case. Thus, we obtain $r = 4.23 \times 10^7$ m.

(c) Arclength s is related to angle of arc θ (in radians) by $s = r\theta$. Thus, with $theta = 3(\pi/180) = 0.052$ rad, we find $s = 2.2 \times 10^6$ m.

(d) Points on the surface (which, of course, is not in orbit) are moving toward the east with a period of 24 h. If the satellite is found to be east of its expected position (above some point on the surface for which it used to stay directly overhead), then its period must now be *smaller* than 24 h.

(e) From Kepler's law of periods, it is evident that smaller T requires smaller r. The storm moved the satellite towards Earth.

69. (a) Their initial potential energy is $-Gm^2/R_i$ and they started from rest, so energy conservation leads to

$$-\frac{Gm^2}{R_i} = K_{\text{total}} - \frac{Gm^2}{0.5R_i} \implies K_{\text{total}} = \frac{Gm^2}{R_i} .$$

(b) The have equal mass, and this is being viewed in the center-of-mass frame, so their speeds are identical and their kinetic energies are the same. Thus,

$$K = \frac{1}{2}K_{\text{total}} = \frac{Gm^2}{2R_i} .$$

(c) With $K = \frac{1}{2}mv^2$, we solve the above equation and find $v = \sqrt{Gm/R_i}$.

(d) Their relative speed is $2v = 2\sqrt{Gm/R_i}$. This is the (instantaneous) rate at which the gap between them is closing.

(e) The premise of this part is that we assume we are not moving (that is, that body A acquires no kinetic energy in the process). Thus, $K_{\text{total}} = K_B$ and the logic of part (a) leads to $K_B = Gm^2/R_i$.

(f) And $\frac{1}{2}mv_B^2 = K_B$ yields $v_B = \sqrt{2Gm/R_i}$.

(g) The answer to part (f) is incorrect, due to having ignored the accelerated motion of "our" frame (that of body A). Our computations were therefore carried out in a noninertial frame of reference, for which the energy equations of Chapter 8 are not directly applicable.

70. (a) The equation preceding Eq. 14-40 is adapted as follows:

$$\frac{m_2^3}{(m_1 + m_2)^2} = \frac{v^3T}{2\pi G}$$

where $m_1 = 0.9M_{\text{Sun}}$ is the estimated mass of the star. With $v = 70\,\text{m/s}$ and $T = 1500$ days (or $1500 \times 86400 = 1.3 \times 10^8$ s), we find

$$\frac{m_2^3}{(0.9M_{\text{Sun}} + m_2)^2} = 1.06 \times 10^{23} \text{ kg} .$$

Since $M_{\text{Sun}} \approx 2 \times 10^{30}$ kg, we find $m_2 \approx 7 \times 10^{27}$ kg. This solution may be reached in several ways (see discussion in the Sample Problem). Dividing by the mass of Jupiter (see Appendix C), we obtain $m \approx 3.7m_J$.

(b) Since $v = 2\pi r_1/T$ is the speed of the star, we find

$$r_1 = \frac{vT}{2\pi} = \frac{(70\,\text{m/s})\left(1.3 \times 10^8\,\text{s}\right)}{2\pi} = 1.4 \times 10^9 \text{ m}$$

for the star's orbital radius. If r is the distance between the star and the planet, then $r_2 = r - r_1$ is the orbital radius of the planet. And r can be figured from Eq. 14-37, which leads to

$$r_2 = r_1\left(\frac{m_1 + m_2}{m_2} - 1\right) = r_1\frac{m_1}{m_2} = 3.7 \times 10^{11} \text{ m} .$$

Dividing this by 1.5×10^{11} m (Earth's orbital radius, r_E) gives $r_2 = 2.5r_E$.

71. (a) From Ch. 2, we have $v^2 = v_0^2 + 2a\Delta x$, where a may be interpreted as an average acceleration in cases where the acceleration is not uniform. With $v_0 = 0$, $v = 11000$ m/s and $\Delta x = 220$ m, we find $a = 2.75 \times 10^5\,\text{m/s}^2$. Therefore,

$$a = \left(\frac{2.75 \times 10^5\,\text{m/s}^2}{9.8\,\text{m/s}^2}\right)g = 2.8 \times 10^4 g$$

which is certainly enough to kill the passengers.

(b) Again using $v^2 = v_0^2 + 2a\Delta x$, we find

$$a = \frac{7000^2}{2(3500)} = 7000\,\mathrm{m/s^2} = 714g\ .$$

(c) Energy conservation gives the craft's speed v (in the absence of friction and other dissipative effects) at altitude $h = 700\,\mathrm{km}$ after being launched from $R = 6.37 \times 10^6\,\mathrm{m}$ (the surface of Earth) with speed $v_0 = 7000$ m/s. That altitude corresponds to a distance from Earth's center of $r = R + h = 7.07 \times 10^6\,\mathrm{m}$.

$$\frac{1}{2}mv_0^2 - \frac{GMm}{R} = \frac{1}{2}mv^2 - \frac{GMm}{r}\ .$$

With $M = 5.98 \times 10^{24}\,\mathrm{kg}$ (the mass of Earth) we find $v = 6.05 \times 10^3\,\mathrm{m/s}$. But to orbit at that radius requires (by Eq. 14-41) $v' = \sqrt{GM/r} = 7.51 \times 10^3\,\mathrm{m/s}$. The difference between these is $v' - v = 1.46 \times 10^3$ m/s, which presumably is accounted for by the action of the rocket engine.

72. We apply the work-energy theorem to the object in question. It starts from a point at the surface of the Earth with zero initial speed and arrives at the center of the Earth with final speed v_f. The corresponding increase in its kinetic energy, $\frac{1}{2}mv_f^2$, is equal to the work done on it by Earth's gravity: $\int F\,dr = \int(-Kr)dr$ (using the notation of that Sample Problem referred to in the problem statement). Thus,

$$\frac{1}{2}mv_f^2 = \int_R^0 F\,dr = \int_R^0 (-Kr)\,dr = \frac{1}{2}KR^2$$

where R is the radius of Earth. Solving for the final speed, we obtain $v_f = R\sqrt{K/m}$. We note that the acceleration of gravity $a_g = g = 9.8\,\mathrm{m/s^2}$ on the surface of Earth is given by $a_g = GM/R^2 = G(4\pi R^3/3)\rho/R^2$, where ρ is Earth's average density. This permits us to write $K/m = 4\pi G\rho/3 = g/R$. Consequently,

$$\begin{aligned} v_f &= R\sqrt{\frac{K}{m}} = R\sqrt{\frac{g}{R}} = \sqrt{gR} \\ &= \sqrt{(9.8\,\mathrm{m/s^2})\,(6.37 \times 10^6\,\mathrm{m})} = 7.9 \times 10^3\ \mathrm{m/s}\ . \end{aligned}$$

73. Equating Eq. 14-18 with Eq. 14-10, we find

$$a_{gs} - a_g = \frac{4\pi G\rho R}{3} - \frac{4\pi G\rho r}{3} = \frac{4\pi G\rho(R - r)}{3}$$

which yields $a_{gs} - a_g = 4\pi G\rho D/3$. Since $4\pi G\rho/3 = a_{gs}/R$ this is equivalent to

$$a_{gs} - a_g = a_{gs}\frac{D}{R} \quad\Longrightarrow\quad a_g = a_{gs}\left(1 - \frac{D}{R}\right)\ .$$

74. Let v and V be the speeds of particles m and M, respectively. These are measured in the frame of reference described in the problem (where the particles are seen as initially at rest). Now, momentum conservation demands

$$mv = MV \quad\Longrightarrow\quad v + V = v\left(1 + \frac{m}{M}\right)$$

where $v + V$ is their relative speed (the instantaneous rate at which the gap between them is shrinking). Energy conservation applied to the two-particle system leads to

$$\begin{aligned} K_i + U_i &= K + U \\ 0 - \frac{GmM}{r} &= \frac{1}{2}mv^2 + \frac{1}{2}MV^2 - \frac{GmM}{d} \\ -\frac{GmM}{r} &= \frac{1}{2}mv^2\left(1 + \frac{m}{M}\right) - \frac{GmM}{d}\ . \end{aligned}$$

If we take the initial separation r to be large enough that GmM/r is approximately zero, then this yields a solution for the speed of particle m:

$$v = \sqrt{\frac{2GM}{d\left(1+\frac{m}{M}\right)}} \ .$$

Therefore, the relative speed is

$$v + V = \sqrt{\frac{2GM}{d\left(1+\frac{m}{M}\right)}}\left(1+\frac{m}{M}\right) = \sqrt{\frac{2G(M+m)}{d}} \ .$$

75. The initial distance from each fixed sphere to the ball is $r_0 = \infty$, which implies the initial gravitational potential energy is zero. The distance from each fixed sphere to the ball when it is at $x = 0.30$ m is $r = 0.50$ m, by the Pythagorean theorem.

(a) With $M = 20$ kg and $m = 10$ kg, energy conservation leads to

$$K_i + U_i = K + U \implies 0 + 0 = K - 2\frac{GmM}{r}$$

which yields $K = 2GmM/r = 5.3 \times 10^{-8}$ J.

(b) Since the y-component of each force will cancel, the force will be $-2F_x = -2\left(GmM/r^2\right)\cos\theta$, where $\theta = \tan^{-1} 4/3 = 53°$. Thus, the result (in Newtons – and using unit-vector notation) is $\vec{F}_{\text{net}} = -6.4 \times 10^{-8}\,\hat{\imath}$.

76. Energy conservation leads to

$$K_i + U_i = K + U \implies \frac{1}{2}m\left(\sqrt{\frac{GM}{r}}\right)^2 - \frac{GmM}{R} = 0 - \frac{GmM}{R_{\text{max}}}$$

Consequently, we find $R_{\text{max}} = 2R$.

77. Consider that the leftmost rod is made of point-like particles (*mass elements*) of infinitesimal mass $dm = (M/L)dx$. The force on each of these, adapting the result of Sample Problem 14-9, is

$$\frac{G(dm)M}{x(L+x)} = \frac{G(M/L)(dx)M}{x(L+x)}$$

where x is the distance from the leftmost edge of the rightmost rod to a particular mass element of the leftmost rod. We take $+x$ to be leftward in this calculation. The magnitude of the net gravitational force exerted by the rightmost rod on the leftmost rod is therefore

$$\left|\vec{F}_{\text{net}}\right| = \frac{GM^2}{L}\int_d^{d+L}\frac{dx}{x(L+x)}$$

and is the same (by Newton's third law) as that exerted by the leftmost rod on the rightmost one. The integral can be evaluated (though the problem does not require us to do this), and the result is

$$\left|\vec{F}_{\text{net}}\right| = \frac{GM^2}{L^2}\ln\left(\frac{(d+L)^2}{d(d+2L)}\right) \ .$$

78. See Appendix C. We note that, since $v = 2\pi r/T$, the centripetal acceleration may be written as $a = 4\pi^2 r/T^2$. To express the result in terms of g, we divide by $9.8\,\text{m/s}^2$.

(a) The acceleration associated with Earth's spin ($T = 24\,\text{h} = 86400\,\text{s}$) is

$$a = g\,\frac{4\pi^2\left(6.37\times10^6\,\text{m}\right)}{(86400\,\text{s})^2\,(9.8\,\text{m/s}^2)} = 0.0034g \ .$$

(b) The acceleration associated with Earth's motion around the Sun ($T = 1\,\text{y} = 3.156 \times 10^7\,\text{s}$) is

$$a = g\,\frac{4\pi^2\left(1.5 \times 10^{11}\,\text{m}\right)}{\left(3.156 \times 10^7\,\text{s}\right)^2\left(9.8\,\text{m/s}^2\right)} = 0.00061g\ .$$

(c) The acceleration associated with the Solar System's motion around the galactic center ($T = 2.5 \times 10^8\,\text{y} = 7.9 \times 10^{15}\,\text{s}$) is

$$a = g\,\frac{4\pi^2\left(2.2 \times 10^{20}\,\text{m}\right)}{\left(7.9 \times 10^{15}\,\text{s}\right)^2\left(9.8\,\text{m/s}^2\right)} = 1.4 \times 10^{-11}g\ .$$

79. (a) We convert distances to meters, and use $v = \sqrt{GM/r}$ for speed when the probe is in circular orbit (this equation is readily obtained from Eq. 14-41). Our notations for the speeds are: v_o for the original speed of the probe when it is in a circular Venus-like orbit (of radius r_o); v_p for the speed when the rockets have fired and it is at the perihelion ($r_p = r_\text{o}$) of its subsequent elliptical orbit; and, v_f for its final speed once it is in a circular Earth-like orbit (of radius r_f which coincides with the aphelion distance r_a of the aforementioned ellipse). We find

$$v_\text{o} = \sqrt{\frac{GM}{r_\text{o}}} = \sqrt{\frac{\left(6.67 \times 10^{-11}\right)\left(1.99 \times 10^{30}\right)}{1.08 \times 10^{11}}} = 3.51 \times 10^4\ \text{m/s}\ .$$

With $m = 6000$ kg, the original energy is given by Eq. 14-44:

$$E_\text{o} = -\frac{GMm}{2r_\text{o}} = -3.69 \times 10^{12}\ \text{J}\ .$$

Once the rockets have fired, the probe starts on an elliptical path with semimajor axis

$$a = \frac{r_p + r_a}{2} = \frac{r_\text{o} + r_f}{2} = 1.29 \times 10^{11}\ \text{m}$$

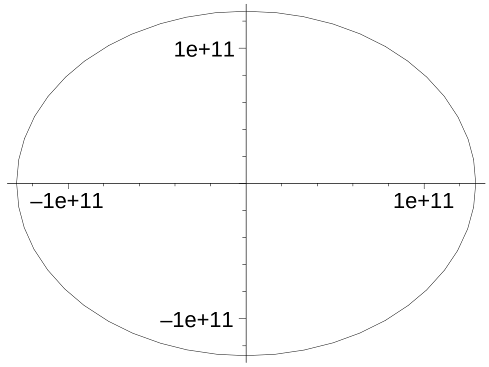
where $r_f = 1.5 \times 10^{11}$ m. By Eq. 14-46, its energy is now

$$E_\text{ellipse} = -\frac{GMm}{2a} = -3.09 \times 10^{12}\ \text{J}\ .$$

The energy "boost" required when the probe is at r_o is therefore $E_\text{ellipse} - E_\text{o} = 6.0 \times 10^{11}$ J. The speed of the probe at the moment it has received this boost is figured from the kinetic energy ($v_p = \sqrt{2K/m}$) where $K = E_\text{ellipse} - U$. Thus,

$$v_p = \sqrt{\frac{2}{m}\left(-\frac{GMm}{2a} + \frac{GMm}{r_p}\right)} = 3.78 \times 10^4\ \text{m/s}$$

which means the speed increase is $v_p - v_\text{o} = 2.75 \times 10^3$ m/s. The orbit (if it were allowed to complete one full revolution) is plotted below. The Sun is not shown; it is not exactly at the center but rather 2.1×10^{10} m to the right of origin (if we are assuming the perihelion is the rightmost point shown and the aphelion is the leftmost point shown).



(b) When the probe reaches $r_f = r_a$ it still has energy E_{ellipse} but now has speed

$$v_a = \frac{r_p v_p}{r_a} = \frac{\left(1.08 \times 10^{11}\right)\left(3.78 \times 10^4\right)}{1.5 \times 10^{11}} = 2.722 \times 10^4 \text{ m/s}$$

as a result of angular momentum conservation (see discussion of Kepler's law of areas), though this could also be figured similarly to the way we found v_p in the previous part. To be in circular motion at that radius, the speed must be

$$v_f = \sqrt{\frac{GM}{r_f}} = 2.975 \times 10^4 \text{ m/s} .$$

Thus, the speed increase needed at this stage must be $v_f - v_a = 2.53 \times 10^3$ m/s. Thus, using Eq. 14-44 again, the necessary energy increase here is

$$-\frac{GMm}{2\,r_f} - E_{\text{ellipse}} = 4.3 \times 10^{11} \text{ J} .$$

80. (a) Taking the differential of $F = GmM/r^2$ and approximating dF and dr as ΔW and $-h$, respectively, we arrive at

$$\Delta W = \frac{2GMmh}{r^3} = \frac{2G\left(4\pi\rho r^3/3\right)mh}{r^3}$$

where in the last step we have used the definition of average density ($\rho = M/V$ where $V_{\text{sphere}} = 4\pi r^3/3$). The above expression is easily simplified to yield the desired expression.

(b) We divide the previous result by $W = mg$ and obtain

$$\frac{\Delta W}{W} = \frac{8\pi G\rho h}{3g} .$$

We replace the lefthand side with 1×10^{-6} and set $\rho = 5500\,\text{kg/m}^3$, and obtain $h = 3.2$ m.

81. He knew that some force F must point toward the center of the orbit in order to hold the Moon in orbit around Earth, and that the approximation of a circular orbit with constant speed means the acceleration must be

$$a = \frac{v^2}{r} = \frac{(2\pi r/T)^2}{r} = \frac{4\pi^2 r^2}{T^2\,r} .$$

Plugging in $T^2 = Cr^3$ (where C is some constant) this leads to

$$F = ma = m\,\frac{4\pi^2 r^2}{Cr^4} = \frac{4\pi^2 m}{C\,r^2}$$

which indicates a force inversely proportional to the square of r.

82. (a) Kepler's law of periods is

$$T^2 = \left(\frac{4\pi^2}{GM}\right) r^3 .$$

Thus, with $M = 6.0 \times 10^{30}$ kg and $T = 300(86400) = 2.6 \times 10^7$ s, we obtain $r = 1.9 \times 10^{11}$ m.

(b) That its orbit is circular suggests that its speed is constant, so

$$v = \frac{2\pi r}{T} = 4.6 \times 10^4 \text{ m/s} .$$

83. (a) Using Kepler's law of periods, we obtain

$$T = \sqrt{\left(\frac{4\pi^2}{GM}\right) r^3} = 2.15 \times 10^4 \text{ s} .$$

(b) The speed is constant (before she fires the thrusters), so $v_\text{o} = 2\pi r/T = 1.23 \times 10^4$ m/s.

(c) A two percent reduction in the previous value gives $v = 0.98 v_\text{o} = 1.20 \times 10^4$ m/s.

(d) The kinetic energy is $K = \frac{1}{2}mv^2 = 2.17 \times 10^{11}$ J.

(e) The potential energy is $U = -GmM/r = -4.53 \times 10^{11}$ J.

(f) Adding these two results gives $E = K + U = -2.35 \times 10^{11}$ J.

(g) Using Eq. 14-46, we find the semimajor axis to be

$$a = \frac{-GMm}{2E} = 4.04 \times 10^7 \text{ m} .$$

(h) Using Kepler's law of periods for elliptical orbits (using a instead of r) we find the new period is

$$T' = \sqrt{\left(\frac{4\pi^2}{GM}\right) a^3} = 2.03 \times 10^4 \text{ s} .$$

This is smaller than our result for part (a) by $T - T' = 1.22 \times 10^3$ s.

84. (a) With $M = 2.0 \times 10^{30}$ kg and $r = 10000$ m, we find

$$a_g = \frac{GM}{r^2} = 1.3 \times 10^{12} \text{ m/s}^2 .$$

(b) Although a close answer may be gotten by using the constant acceleration equations of Chapter 2, we show the more general approach (using energy conservation):

$$K_\text{o} + U_\text{o} = K + U$$

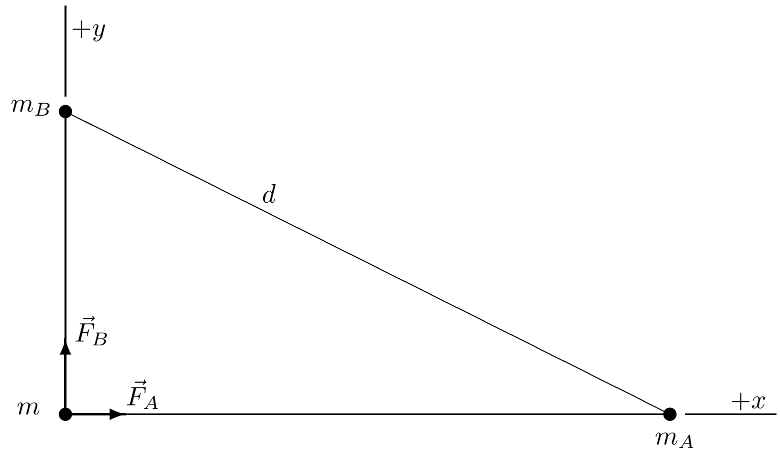
where $K_\text{o} = 0$, $K = \frac{1}{2}mv^2$ and U given by Eq. 14-20. Thus, with $r_\text{o} = 10001$ m, we find

$$v = \sqrt{2GM\left(\frac{1}{r} - \frac{1}{r_\text{o}}\right)} = 1.6 \times 10^6 \text{ m/s} .$$

85. It is clear from the given data that the $m = 2.0$ kg sphere cannot be along the line between m_A and m_B (that is, it is "off-axis"). The magnitudes of the individual forces (acting on m, exerted by m_A and m_B respectively) are

$$F_A = \frac{Gm_A m}{r_A^2} = 2.7 \times 10^{-6} \text{ N} \quad \text{and} \quad F_B = \frac{Gm_B m}{r_B^2} = 3.6 \times 10^{-6} \text{ N}$$

where $r_A = 0.20$ m and $r_B = 0.15$ m. Letting d stand for the distance between m_A and m_B then we note that $d^2 = r_A^2 + r_B^2$ (that is, the line between m_A and m_B forms the hypotenuse of a right triangle with m at the right-angle corner, as illustrated in the figure below).



Choosing x and y axes as shown above, then (in Newtons) $\vec{F}_A = 2.7 \times 10^{-6}\,\hat{\imath}$ and $\vec{F}_B = 3.6 \times 10^{-6}\,\hat{\jmath}$, which makes the vector addition very straightforward: we find

$$F_{\text{net}} = \sqrt{F_A^2 + F_B^2} = 4.4 \times 10^{-6}\ \text{N}$$

and (as measured counterclockwise from the x axis) $\theta = 53°$. It is not difficult to check that the direction of $\vec{F}_{\text{net}}$ (given by θ) is along a line that is perpendicular to the segment d.

86. (a) We use Eq. 14-27:

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{2\,(6.67 \times 10^{-11})\,(1.99 \times 10^{30})}{1.50 \times 10^{11}}} = 4.21 \times 10^4\ \text{m/s}\ .$$

(b) Earth's orbital speed is gotten by solving Eq. 14-41:

$$v_{\text{orb}} = \sqrt{\frac{GM}{R}} = \sqrt{\frac{(6.67 \times 10^{-11})\,(1.99 \times 10^{30})}{1.50 \times 10^{11}}} = 2.97 \times 10^4\ \text{m/s}\ .$$

The difference is therefore $v_{\text{esc}} - v_{\text{orb}} = 1.23 \times 10^4$ m/s.

(c) To obtain the speed (relative to Earth) mentioned above, the object must be launched with initial speed

$$v_0 = \sqrt{\left(1.23 \times 10^4\right)^2 + 2\frac{GM_E}{R_E}} = 1.66 \times 10^4\ \text{m/s}\ .$$

However, this is not precisely the same as the speed it would need to be launched at if it is desired that the object be just able to escape the solar system. The computation needed for that is shown below.

Including the Sun's gravitational influence as well as that of Earth (and accounting for the fact that Earth is moving around the Sun) the object at moment of launch has energy

$$K + U_E + U_S = \frac{1}{2}m\left(v_{\text{launch}} + v_{\text{orb}}\right)^2 - \frac{GmM_E}{R_E} - \frac{GmM_S}{R}$$

which must equate to zero for the object to (barely) escape the solar system. Consequently,

$$v_{\text{launch}} = \sqrt{2G\left(\frac{M_E}{R_E} + \frac{M_S}{R}\right)} - v_{\text{orb}} = \sqrt{2\,(6.67 \times 10^{-11})\left(\frac{5.98 \times 10^{24}}{6.37 \times 10^6} + \frac{1.99 \times 10^{30}}{1.50 \times 10^{11}}\right)} - 2.97 \times 10^4$$

which yields $v_{\text{launch}} = 1.38 \times 10^4$ m/s.

87. (a) Converting T to seconds (by multiplying by 3.156×10^7) we do a linear fit of T^2 versus a^3 by the method of least squares. We obtain (with SI units understood)

$$T^2 = -7.4 \times 10^{15} + 2.982 \times 10^{-19}\, a^3 \ .$$

The coefficient of a^3 should be $4\pi^2/GM$ so that this result gives the mass of the Sun as

$$M = \frac{4\pi^2}{(6.67 \times 10^{-11}\ \mathrm{m^3/kg{\cdot}s^2})\,(2.982 \times 10^{-19}\ \mathrm{s^2/m^3})} = 1.98 \times 10^{30}\ \mathrm{kg}\ .$$

(b) Since $\log T^2 = 2\log T$ and $\log a^3 = 3\log a$ then the coefficient of $\log a$ in this next fit should be close to $3/2$, and indeed we find

$$\log T = -9.264 + 1.50007 \log a \ .$$

In order to compute the mass, we recall the property $\log AB = \log A + \log B$, which when applied to Eq. 14-33 leads us to identify

$$-9.264 = \frac{1}{2}\log\left(\frac{4\pi^2}{GM}\right) \implies M = 1.996 \times 10^{30} \approx 2.00 \times 10^{30}\ \mathrm{kg}\ .$$

88. (a) We write the centripetal acceleration (which is the same for each, since they have identical mass) as $r\omega^2$ where ω is the unknown angular speed. Thus,

$$\frac{G(M)(M)}{(2r)^2} = \frac{GM^2}{4r^2} = Mr\omega^2$$

which gives $\omega = \frac{1}{2}\sqrt{MG/r^3} = 2.2 \times 10^{-7}$ rad/s.

(b) To barely escape means to have total energy equal to zero (see discussion prior to Eq. 14-27). If m is the mass of the meteoroid, then

$$\frac{1}{2}mv^2 - \frac{GmM}{r} - \frac{GmM}{r} = 0 \implies v = \sqrt{\frac{4GM}{r}} = 8.9 \times 10^4\ \mathrm{m/s}\ .$$

89. (a) Circular motion requires that the force in Newton's second law provide the necessary centripetal acceleration:

$$\frac{GmM}{r^2} = m\frac{v^2}{r}$$

which is identical to Eq. 14-39 in the textbook. Since the left-hand side of this equation is the force given as 80 N, then we can solve for the combination mv^2 by multiplying both sides by $r = 2.0 \times 10^7$ m. Thus, $mv^2 = \left(2.0 \times 10^7\right)(80) = 1.6 \times 10^9$ J. Therefore,

$$K = \frac{1}{2}mv^2 = \frac{1}{2}\left(1.6 \times 10^9\right) = 8.0 \times 10^8\ \mathrm{J}\ .$$

(b) Since the gravitational force is inversely proportional to the square of the radius, then

$$\frac{F'}{F} = \left(\frac{r}{r'}\right)^2 \ .$$

Thus, $F' = (80)(2/3)^2 = 36$ N.

90. (a) Because it is moving in a circular orbit, F/m must equal the centripetal acceleration:

$$\frac{80\ \mathrm{N}}{50\ \mathrm{kg}} = \frac{v^2}{r}$$

But $v = 2\pi r/T$, where $T = 21600$ s, so we are led to

$$1.6\ \mathrm{m/s^2} = \frac{4\pi^2}{T^2}\, r$$

which yields $r = 1.9 \times 10^7$ m.

(b) From the above calculation, we infer $v^2 = (1.6 \text{ m/s}^2)r$ which leads to $v^2 = 3.0 \times 10^7 \text{ m}^2/\text{s}^2$. Thus, $K = \frac{1}{2}mv^2 = 7.6 \times 10^8$ J.

(c) As discussed in §14-4, F/m also tells us the gravitational acceleration:

$$a_g = 1.6 \text{ m/s}^2 = \frac{GM}{r^2}$$

We therefore find $M = 8.6 \times 10^{24}$ kg.

91. (a) The total energy is conserved, so there is no difference between its values at aphelion and perihelion.

(b) Since the change is small, we use differentials:

$$dU = \left(\frac{GM_E M_S}{r^2}\right) dr \approx \left(\frac{\left(6.67 \times 10^{-11}\right)\left(1.99 \times 10^{30}\right)\left(5.98 \times 10^{24}\right)}{\left(1.5 \times 10^{11}\right)^2}\right)\left(5 \times 10^9\right)$$

which yields $\Delta U \approx 1.8 \times 10^{32}$ J. A more direct subtraction of the values of the potential energies leads to the same result.

(c) and (d) From the previous two parts, we see that the variation in the kinetic energy ΔK must also equal 1.8×10^{32} J. So, with $\Delta K \approx dK = mv\,dv$, where $v \approx 2\pi R/T$, we have

$$1.8 \times 10^{32} \approx \left(5.98 \times 10^{24}\right)\left(\frac{2\pi\left(1.5 \times 10^{11}\right)}{3.156 \times 10^7}\right)\Delta v$$

which yields a difference of $\Delta v \approx 1$ km/s in Earth's speed (relative to the Sun) between aphelion and perihelion.

92. (a) From Kepler's law of periods, we see that T is proportional to $r^{3/2}$.

(b) Eq. 14-42 shows that K is inversely proportional to r.

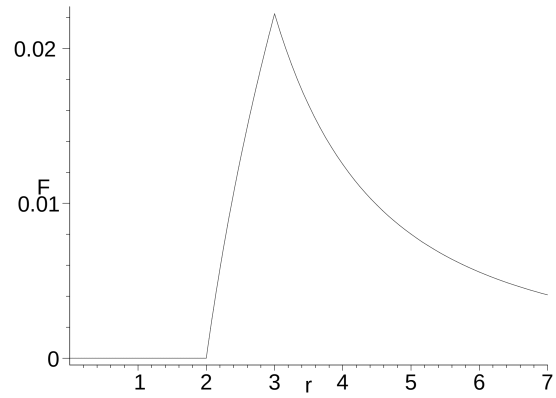
(c) and (d) From the previous part, knowing that K is proportional to v^2, we find that v is proportional to $1/\sqrt{r}$. Thus, by Eq. 14-30, the angular momentum (which depends on the product rv) is proportional to $r/\sqrt{r} = \sqrt{r}$.

93. The orbital speed is readily found from Eq. 14-41 to be $v_{\text{orb}} = \sqrt{GM/r}$. Comparing this with the expression for the escape velocity, Eq. 14-27, we immediately obtain the desired result.

94. (a) When testing for a gravitational force at $r < b$, none is registered. But at points within the shell $b \le r \le a$, the force will increase according to how much mass M' of the shell is at smaller radius. Specifically, for $b \le r \le a$, we find

$$F = \frac{GmM'}{r^2} = \frac{GmM\left(\frac{r^3 - b^3}{a^3 - b^3}\right)}{r^2} .$$

Once $r = a$ is reached, the force takes the familiar form GmM/r^2 and continues to have this form for $r > a$. We have chosen $m = 1$ kg, $M = 3 \times 10^9$ kg, $b = 2$ m and $a = 3$ m in order to produce the following graph of F versus r (in SI units).



(b) Starting with the large r formula for force, we integrate and obtain the expected $U = -GmM/r$ (for $r \geq a$). Integrating the force formula indicated above for $b \leq r \leq a$ produces

$$U = \frac{GmM\left(r^3 + 2b^3\right)}{2r\left(a^3 - b^3\right)} + C$$

where C is an integration constant that we determine to be

$$C = -\frac{3GmMa^2}{2a\left(a^3 - b^3\right)}$$

so that this U and the large r formula for U agree at $r = a$. Finally, the $r < a$ formula for U is a constant (since the corresponding force vanishes), and we determine its value by evaluating the previous U at $r = b$. The resulting graph is shown below.