Definition 1. *Types*

Let $Tp = \{p_1, p_2, p_3, \dots\}$ be the set of primitive types. The set $\mathbb{T}$ of typed is defined by following grammar:

$$\mathbb{T} ::= Tp \mid \mathbb{T} \bullet \mathbb{T} \mid \mathbb{T} \backslash \mathbb{T} \mid \mathbb{T}/\mathbb{T} \mid !\mathbb{T} \tag{1}$$

Definition 2. *Terms*

Let $\mathbb{V} = \{x, y, z, \dots\}$ be the set of variables. The set $\mathcal{T}$ of terms is defined by following grammar:

$$\mathcal{T} ::= \mathbb{V} \mid \lambda\mathbb{V}.\mathcal{T} \mid \kappa\mathbb{V}.\mathcal{T} \mid \mathcal{T}\mathcal{T} \mid \mathcal{T}\&\mathcal{T} \mid \mathcal{T} \otimes \mathcal{T} \mid \mathbf{let}\ \mathbb{V} = \mathbb{V} \otimes \mathbb{V}\ \mathbf{in}\ \mathcal{T} \mid !\mathcal{T} \tag{2}$$

Definition 3. *ND-style Lambek λ-calculus based on $L^*(\bullet, /, \backslash)$:*

$$\frac{}{x : A \vdash x : A}\ ax$$

$$\frac{x : A, \Pi \vdash M : B}{\Pi \vdash \lambda x.M : A \backslash B} \rightarrow \backslash \qquad \frac{\Gamma \vdash M : A \qquad \Pi \vdash N : A \backslash B}{\Gamma, \Pi \vdash N\$M : B}$$

$$\frac{\Pi, x : A \vdash M : B}{\Pi \vdash \kappa x.M : B/A} \rightarrow / \qquad \frac{\Gamma \vdash M : B/A \qquad \Pi \vdash N : A}{\Gamma, \Pi \vdash N\&M : B}$$

$$\frac{\Gamma \vdash M : A \qquad \Delta \vdash N : B}{\Gamma, \Delta \vdash M \bullet N : A \bullet B} \rightarrow \bullet \qquad \frac{\Gamma \vdash p : A \bullet B \qquad \Delta, x : A, y : B, \Pi \vdash N : C}{\Delta, \Gamma, \Pi \vdash \mathbf{let}\ p = x \bullet y\ \mathbf{in}\ N : C}$$

$$\frac{!\Gamma \vdash M : A}{!\Gamma \vdash !M : !A} \qquad \frac{\Gamma \vdash M : !A \qquad \Pi, x : A, \Delta \vdash N : B}{\Pi, \Gamma, \Delta \vdash \mathbf{let}\ !x = M\ \mathbf{in}\ N : A}$$

$$\frac{\Gamma \vdash M : !A \qquad \Delta \vdash N : B \qquad \Pi, x : !A, y : B, \Theta \vdash P : C}{\Pi, \Delta, \Gamma, \Theta \vdash ([x]M \smiley [y]N)\ \mathbf{in}\ P : C}$$

$$\frac{\Gamma \vdash M : A \qquad \Delta \vdash N : !B \qquad \Pi, x : A, y : !B, \Theta \vdash M : C}{\Pi, \Delta, \Gamma, \Theta \vdash ([y]N \blacksmiley [x]M)\ \mathbf{in}\ M : C}$$

$$\frac{\Gamma \vdash M : !A \qquad \Delta, x : !A, y : !A \vdash N : C}{\Delta, \Gamma \vdash \mathbf{let}\ x@y = M\ \mathbf{in}\ N : C}$$

$$\frac{\Pi \vdash M : B \qquad \Gamma, x : B, \Delta \vdash N : A}{\Gamma, \Pi, \Delta \vdash N[x := M] : A}\ \mathbf{subst}$$

Examples:

$$\dfrac{\dfrac{\dfrac{\dfrac{f : !(s/n) \vdash f : !(s/n) \qquad \dfrac{x : !n \vdash x : !n \qquad \dfrac{g : s/n \vdash g : s/n \qquad y : s \vdash y : s}{g : s/n, y : s \vdash y\&g : s}}{g : s/n, x : !n \vdash \mathbf{let}\ !y = x\ \mathbf{in}\ y\&g : s}}{f : !(s/n), x : !n \vdash \mathbf{let}\ !g = f\ \mathbf{in}\ (\mathbf{let}\ !y = x\ \mathbf{in}\ y\&g) : s}}{f : !(s/n), x : !n \vdash !(\mathbf{let}\ !g = f\ \mathbf{in}\ (\mathbf{let}\ !y = x\ \mathbf{in}\ y\&g)) : !s}}{x : !n \vdash \lambda f.!(\mathbf{let}\ !g = f\ \mathbf{in}\ (\mathbf{let}\ !y = x\ \mathbf{in}\ y\&g)) : !(s/n)\backslash !s}}{\vdash \kappa x.\lambda f.!(\mathbf{let}\ !g = f\ \mathbf{in}\ (\mathbf{let}\ !y = x\ \mathbf{in}\ y\&g)) :\ (!(s/n)\backslash !s)/!n}$$

$$\dfrac{\dfrac{x : !n \vdash x : !n \qquad y : n \vdash y : n}{x : !n \vdash \mathbf{let}\ !y = x\ \mathbf{in}\ y : n}}{\vdash \lambda x.\mathbf{let}\ !y = x\ \mathbf{in}\ \ : !n\backslash n}$$

$$\dfrac{\dfrac{x : !n \vdash x : !n}{x : !n \vdash !x : !!n}}{\vdash \kappa x.!x : !!n/!n}$$

$$
\dfrac{p : !A\bullet !B \vdash p : !A\bullet !B \qquad \dfrac{\dfrac{x : !A \vdash x : !A \qquad \dfrac{y : !B \vdash x : !B \qquad \dfrac{x_1 : A \vdash x_1 : A \qquad y_1 : B \vdash y_1 : B}{x_1 : A, y_1 : B \vdash x_1 \bullet y_1 : A \bullet B}}{x_1 :\ A, y : !B \vdash \mathbf{let}\ !y_1 = y\ \mathbf{in}\ x_1 \bullet y_1 : A \bullet B}}{x : !A, y : !B \vdash \mathbf{let}\ !x_1 = x\ \mathbf{in}\ (\mathbf{let}\ !y_1 = y\ \mathbf{in}\ x_1 \bullet y_1) : A \bullet B}}{x : !A, y : !B \vdash !(\mathbf{let}\ !x_1 = x\ \mathbf{in}\ (\mathbf{let}\ !y_1 = y\ \mathbf{in}\ x_1 \bullet y_1)) : !(A \bullet B)}}{\dfrac{p : !A\bullet !B \vdash\ \mathbf{let}\ x \bullet y = p\ \mathbf{in}\ !(\mathbf{let}\ !x_1 = x\ \mathbf{in}\ (\mathbf{let}\ !y_1 = y\ \mathbf{in}\ x_1 \bullet y_1)) : !(A \bullet B)}{\vdash \kappa p.\ \mathbf{let}\ x \bullet y = p\ \mathbf{in}\ !(\mathbf{let}\ !x_1 = x\ \mathbf{in}\ (\mathbf{let}\ !y_1 = y\ \mathbf{in}\ x_1 \bullet y_1)) : !(A \bullet B)/(!A\bullet !B)}}
$$

Definition 4. *Sequent-style Lambek λ-calculus based on $L^*(\bullet, /, \backslash, !)$:*

$$
\dfrac{}{x : A \Rightarrow x : A}\ ax
$$

$$
\dfrac{x : A, \Pi \Rightarrow M : B}{\Pi \Rightarrow \kappa x.M : A\backslash B} \rightarrow \backslash
\qquad
\dfrac{\Pi \Rightarrow M : A \qquad \Gamma, x : B, \Delta \Rightarrow N : C}{\Gamma, \Pi, f : A\backslash B, \Delta \Rightarrow N[x := f\& M] : C} \backslash \rightarrow
$$

$$
\dfrac{\Pi, x : A \Rightarrow M : A}{\Pi \Rightarrow \lambda x.M : B/A} \rightarrow /
\qquad
\dfrac{\Pi \Rightarrow M : A \qquad \Gamma, x : B, \Delta \Rightarrow N : C}{\Gamma, f : B/A, \Pi, \Delta \Rightarrow N[x := M\$ f] : C} / \rightarrow
$$

$$
\dfrac{\Gamma \Rightarrow M : A \qquad \Delta \Rightarrow N : B}{\Gamma, \Delta \Rightarrow M \bullet N : A \bullet B} \rightarrow \bullet
\qquad
\dfrac{\Gamma, x : A, y : B, \Delta \Rightarrow M : C}{\Gamma, p : A \bullet B, \Delta \Rightarrow \mathbf{let}\ p = x \bullet y\ \mathbf{in}\ M : C} \bullet \rightarrow
$$

$$
\dfrac{!\Gamma \vdash M : A}{!\Gamma \vdash !M : !A} \rightarrow !
\qquad
\dfrac{\Gamma, x : A, \Delta \Rightarrow M : B}{\Gamma, z : !A, \Delta \Rightarrow \mathbf{let}\ !x = z\ \mathbf{in}\ M : B}\ ! \rightarrow
$$

$$
\dfrac{\Gamma, x : !A, x_1 : A_1, \ldots, x_n : A_n, \Delta \Rightarrow M : B}{\Gamma, x_1 : A_1, \ldots, x_n : A_n, x : !A, \Delta \Rightarrow [x]x \smiley [x_n]x_n\ \mathbf{in}\ \ldots\ \mathbf{in}\ [x]x \smiley [x_1]x_1\ \mathbf{in}\ M : C}\ \mathbf{perm_1}
$$

$$
\dfrac{\Gamma, x_1 : A_1, \ldots, x_n : A_n, x : !A, \Delta \Rightarrow M : B}{\Gamma, x : !A, x_1 : A_1, \ldots, x_n : A_n, \Delta \Rightarrow [x_n]x_n \blacksmiley [x]x\ \mathbf{in}\ \ldots\ \mathbf{in}\ [x_1]x_1 \blacksmiley [x]x\ \mathbf{in}\ M : C}\ \mathbf{perm_2}
$$

$$
\dfrac{\Gamma, x : !A, y : !A, \Delta \Rightarrow M : B}{\Gamma, z : !A, \Delta \Rightarrow \mathbf{let}\ (x @ y) = z\ \mathbf{in}\ M : B}\ \mathbf{contr}
$$

$$
\dfrac{\Pi \vdash M : B \qquad \Gamma, x : B, \Delta \Rightarrow N : A}{\Gamma, \Pi, \Delta \Rightarrow N[x := M] : A}\ \mathbf{subst}
$$

Lemma 1. *Generation lemma*

Definition 5. *Reduction*

1. $(\lambda x.M)N \rightarrow_\beta M[x := N]$;
2. $N\&(\kappa x.M) \rightarrow_\beta M[x := N]$;
3. $\mathbf{let}\ u \bullet v = x \bullet y\ \mathbf{in}\ M \rightarrow_\beta M[x := u][y := v]$
4. $\lambda x.Mx \rightarrow_\eta M$;
5. $\kappa x.x\& M \rightarrow_\eta M$;
6. $\mathbf{let}\ !x = !M\ \mathbf{in}\ N \rightarrow_\eta N[x := M]$;
7. $[x]M \smiley [y]N\ \mathbf{in}\ ([y]N \blacksmiley [x]M\ \mathbf{in}\ P) \rightarrow_\eta P$;
8. $[y]N \blacksmiley [x]M\ \mathbf{in}\ ([x]M \smiley [y]N\ \mathbf{in}\ P) \rightarrow_\eta P$.

Lemma 2. *Equivalence between ND and S*
$\Gamma \vdash M : A \Leftrightarrow \Gamma \Rightarrow M : A$

Proof.

Only if part:

1) Let the derivation ends with

$$\frac{\Gamma \vdash M : A \qquad \Pi \vdash N : A\backslash B}{\Gamma, \Pi \vdash NM : B}$$

By IH $\Gamma \Rightarrow M : A$ and $\Pi \Rightarrow N : A\backslash B$.

$$\frac{\Pi \Rightarrow N : A\backslash B \qquad \dfrac{\Gamma \Rightarrow M : A \qquad y : B \Rightarrow y : B}{\Gamma, f : A\backslash B \Rightarrow fM}}{\Gamma, \Pi \Rightarrow NM : B}$$

2) Let the derivation ends with

$$\frac{\Gamma \vdash M : B/A \qquad \Pi \vdash N : A}{\Pi, \Gamma \vdash N\&M : B}$$

By IH $\Gamma \Rightarrow M : B/A$ and $\Pi \Rightarrow N : A$.

$$\frac{\Gamma \Rightarrow M : B/A \qquad \dfrac{\Pi \Rightarrow N : A \qquad y : B \Rightarrow y : B}{\Pi, f : B/A \Rightarrow N\&f : B}}{\Pi, \Gamma \vdash N\&M : B}$$

3) Let the derivation ends with:

$$\frac{\Gamma \vdash p : A \bullet B \qquad \Delta, x : A, y : B, \Pi \vdash N : C}{\Delta, \Gamma, \Pi \vdash \mathbf{let}\ p = x \bullet y\ \mathbf{in}\ N : C}$$

By IH $\Gamma \Rightarrow p : A \bullet B$ and $\Delta, x : A, y : B, \Pi \Rightarrow N : C$.

$$\frac{\Gamma \vdash p : A \bullet B \qquad \dfrac{\Delta, x : A, y : B, \Pi \Rightarrow N : C}{\Delta, q : A \bullet B, \Pi \Rightarrow \mathbf{let}\ x \bullet y = q\ \mathbf{in}\ N : C}\ \bullet\rightarrow}{\Delta, \Gamma, \Pi \Rightarrow x \bullet y = p\ \mathbf{in}\ N : C}\ \textbf{subst}$$

4) Let the derivation ends with:

$$\frac{\Gamma \vdash M : !A \qquad \Pi, x : A, \Delta \vdash N : B}{\Pi, \Gamma, \Delta \vdash \mathbf{let}\ !x = M\ \mathbf{in}\ N : A}$$

By IH $\Gamma \Rightarrow M : !A$ and $\Pi, x : A, \Delta \Rightarrow N : B$.

$$\frac{\Gamma \Rightarrow M : !A \qquad \dfrac{\Pi, x : A, \Delta \Rightarrow N : B}{\Pi, z : !A, \Delta \Rightarrow \mathbf{let}\ !x = z\ \mathbf{in}\ N : C}\ !\rightarrow}{\Pi, \Gamma, \Delta \Rightarrow \mathbf{let}\ !x = M\ \mathbf{in}\ N : C}\ \textbf{subst}$$

5) Let the derivation ends with

$$\frac{\Gamma \vdash M : !A \qquad \Delta, x : !A, y : !A, \Pi \vdash N : C}{\Delta, \Gamma, \Pi \vdash \mathbf{let}\ x@y = M\ \mathbf{in}\ N : C}$$

By IH $\Gamma \Rightarrow M : !A$ and $\Delta, x : !A, y : !A \Rightarrow N : C$.

$$\frac{\Gamma \Rightarrow M : !A \qquad \dfrac{\Delta, x : !A, y : !A, \Pi \Rightarrow N : C}{\Delta, z : !A, \Pi \Rightarrow let\ x@y = z\ \mathbf{in}\ N : C}\ \textbf{contr}}{\Delta, \Gamma, \Pi \Rightarrow let\ x@y = M\ \mathbf{in}\ N : C}\ \textbf{subst}$$

6) Let the derivation ends with

$$\frac{\Gamma \vdash M : !A \qquad \Delta \vdash N : B \qquad \Pi, x : !A, y : B, \Theta \vdash P : C}{\Pi, \Delta, \Gamma, \Theta \vdash ([x]M \text{☺} [y]N)\ \mathbf{in}\ P : C}$$

By IH $\Gamma \Rightarrow M : !A$, $\Delta \Rightarrow N : B$ and $\Pi, x : !A, y : B, \Theta \Rightarrow P : C$.

$$\frac{\Delta \Rightarrow N : B \qquad \dfrac{\Gamma \Rightarrow M : !A \qquad \dfrac{\Pi, x : !A, y : B, \Theta \Rightarrow P : C}{\Pi, y : B, x : !A, \Theta \Rightarrow [x]x\text{☺}[y]y \textbf{ in } P : C}\,\mathbf{perm_1}}{\Pi, y : B, \Gamma, \Theta \Rightarrow [x]M\text{☺}[y]y \textbf{ in } P : C}\,\mathbf{subst}}{\Pi, \Delta, \Gamma, \Theta \Rightarrow [x]M\text{☺}[y]N \textbf{ in } P : C}\,\mathbf{subst}$$

If part

1) Let the derivation end with

$$\frac{\Pi \Rightarrow M : A \qquad \Gamma, x : B, \Delta \Rightarrow N : C}{\Gamma, \Pi, f : A\backslash B, \Delta \Rightarrow N[x := fM] : C}\,\backslash \rightarrow$$

By IH $\Pi \vdash M : A$ and $\Gamma, x : B, \Delta \vdash N : C$.

$$\frac{\dfrac{\Pi \vdash M : A \qquad f : A\backslash B \vdash f : A\backslash B}{\Pi, f : A\backslash B \vdash fM : B} \qquad \Gamma, x : B, \Delta \vdash N : C}{\Gamma, \Pi, f : A\backslash B, \Delta \vdash N[x := fM] : C}$$

2) Let the derivation ends with

$$\frac{\Pi \Rightarrow M : A \qquad \Gamma, x : B, \Delta \Rightarrow N : C}{\Gamma, f : B/A, \Pi, \Delta \Rightarrow N[x := M\&f] : C}\,/ \rightarrow$$

By IH $\Pi \vdash M : A$ and $\Gamma, x : B, \Delta \vdash N : C$

$$\frac{\dfrac{f : B/A \vdash f : B/A \qquad \Pi \vdash M : A}{f : B/A, \Pi \vdash M\&f : B} \qquad \Gamma, x : B, \Delta \vdash N : C}{\Gamma, f : B/A, \Pi, \Delta \vdash N[x := M\&f] : C}$$

3) Let the derivation ends with

$$\frac{\Gamma, x : A, y : B, \Delta \Rightarrow M : C}{\Gamma, p : A \bullet B, \Delta \Rightarrow \textbf{let } p = x \bullet y \textbf{ in } M : C}\,\bullet \rightarrow$$

By IH $\Gamma, x : A, y : B, \Delta \vdash M : C$

$$\frac{p : A \bullet B \vdash A \bullet B \qquad \Gamma, x : A, y : B, \Delta \vdash M : C}{\Gamma, p : A \bullet B, \Delta \vdash \textbf{let } x \bullet y = p \textbf{ in } M : C}$$

4) Let the derivation ends with

$$\frac{\Gamma, x : A, \Delta \Rightarrow M : B}{\Gamma, z : !A, \Delta \Rightarrow \textbf{let } !x = z \textbf{ in } M : B}\,! \rightarrow$$

By IH $\Gamma, x : A, \Delta \vdash M : B$.

$$\frac{z : !A \vdash z : !A \qquad \Gamma, x : A, \Delta \vdash M : B}{\Gamma, z : !A, \Delta \vdash \textbf{let } !x = z \textbf{ in } M : B}$$

5) Let the derivation ends with

$$\frac{\Gamma, x : !A, y : !A, \Delta \Rightarrow M : B}{\Gamma, z : !A, \Delta \Rightarrow \textbf{let } (x@y) = z \textbf{ in } M : B}\,\mathbf{contr}$$

By IH $\Gamma, x : !A, y : !A, \Delta \vdash M : B$

$$\frac{z : !A \vdash z : !A \qquad \Gamma, x : !A, y : !A, \Delta \vdash M : B}{\Gamma, z : !A, \Delta \vdash \textbf{let } (x@y) = z \textbf{ in } M : B}$$

6) Let the derivation ends with

$$\frac{\Gamma, x : !A, x_1 : A_1, \ldots, x_n : A_n, \Delta \Rightarrow M : B}{\Gamma, x_1 : A_1, \ldots, x_n : A_n, x : !A, \Delta \Rightarrow [x]x\text{☺}[x_n]x_n \textbf{ in } \ldots \textbf{ in } [x]x\text{☺}[x_1]x_1 \textbf{ in } M : C}\,\mathbf{perm_1}$$

By IH $\Gamma, x : !A, x_1 : A_1, \dots, x_n : A_n, \Delta \vdash M : B$

$$\frac{x : !A \vdash x : !A \qquad x_1 : A_1 \vdash x_1 : A_1 \qquad \Gamma, x : !A, x_1 : A_1, \dots, x_n : A_n, \Delta \vdash M : B}{\Gamma, x_1 : A_1, x : !A, \dots, x_n : A_n, \Delta \vdash [x]x\smiley[x_1]x_1 \textbf{ in } M : B}$$

$$\dots$$

$$\frac{x : !A \vdash x : !A \qquad x_n : A_n \vdash x_n : A_n \qquad \Gamma, x_1 : A_1, \dots, x_{n-1} : A_{n-1}, x : !A, x_n : A_n, \Delta \vdash [x]x\smiley[x_{n-1}]x_{n-1} \textbf{ in } \dots \textbf{ in } [x]x\smiley[x_1]x_1 \textbf{ in } M : B}{\Gamma, x_1 : A_1, \dots, x_{n-1} : A_{n-1}, x_n : A_n, x : !A, \Delta \vdash [x]x\smiley[x_n]x_n \textbf{ in } \dots \textbf{ in } [x]x\smiley[x_1]x_1 \textbf{ in } M : B}$$

□

Theorem 1. *Curry-Howard Isomorphism*
$\Gamma \Rightarrow M : A \Leftrightarrow |\Gamma| \Rightarrow A$

Proof. 1) Basic case:
$x : A \Rightarrow x : A \Leftrightarrow A \Rightarrow A$ □

Theorem 2. *Subject reduction*
$\Gamma \vdash M : A$ and $M \twoheadrightarrow_\beta N$, then $\Gamma \vdash N : A$.

Proof. The general statement follows from transitivity of multi-step reduction.
1)

$$\frac{\Gamma \vdash N : A \qquad \Pi \vdash \lambda x.M : A \backslash B}{\Gamma, \Pi \vdash (\lambda x.M)N : B}$$

By generation, if $\Pi \vdash \lambda x.M : A \backslash B$, then $x : A, \Pi \vdash M : B$.
So $\Gamma, \Pi \vdash M[x := N]$ by **subst**-rule.

2)

$$\frac{\Gamma \vdash \kappa x.M : B/A \qquad \Pi \vdash N : A}{\Gamma, \Pi \vdash N \& (\kappa x.M)}$$

By generation, if $\Gamma \vdash \kappa x.M : B/A$, then $\Gamma, x : A \vdash M : B$.
So, $\Gamma, \Pi \vdash M[x := N]$ by **subst**-rule.

3)

$$\frac{\dfrac{u : A \vdash u : A \qquad v : B \vdash v : B}{u : A, v : B \vdash u \bullet v : A \bullet B} \qquad \dfrac{\Gamma, x : A, y : B \vdash M : C}{\Gamma, z : A \bullet B \vdash \textbf{let } x \bullet y = z \textbf{ in } M : C}}{\Gamma, u : A, v : B \vdash \textbf{let } x \bullet y = u \bullet v \textbf{ in } M : C}$$

$$\frac{v : B \vdash v : B \qquad \dfrac{u : A \vdash u : A \qquad \Gamma, x : A, y : B \vdash M : C}{\Gamma, u : A, y : B \vdash M[x := u] : C}}{\Gamma, u : A, v : B \vdash M[x := u][y := v] : C}$$

Cases with η follows from generation.

□

Theorem 3. *Strong normalization*
$\twoheadrightarrow$ is strongly normalizible.

Proof.

Definition 6. *The set of strongly computable terms*

- $SC_{p_i} = \{M : p_i \mid M$ *is strongly normalizible* $\}$;
- $SC_{A \backslash B} = \{M : A \backslash B \mid \forall N \in SC_A, MN \in SC_B\}$;
- $SC_{B/A} = \{M : B/A \mid \forall N \in SC_A, N \& M \in SC_B\}$;
- $SC_{A \bullet B} = \{M \bullet N \mid M \in SC_A$ *and* $N \in SC_B\}$;

- $SC_{!A} = \{M : !A \mid \forall N \in SC_B, \textbf{let } !x = M \textbf{ in } N \in SC_B, \textit{where } x \in FV(N) \wedge x \in SC_A \}$

Lemma 3. *If $M \in SC_A$, then M is strongly normalizible.*

Proof. Induction on the structure of A. ☐

Lemma 4. *If $M \to N$ and $M \in SC_A$, then $N \in SC_A$.*

Proof. ☐

Lemma 5. *If $M \to N$ and $N \in SC_A$, then $M \in SC_A$.*

Proof. ☐

Lemma 6. *Let $x_1 : A_1, \dots, x_n : A_n \vdash M : A$ and $\forall i \in \{1, \dots, n\}, x_i^{'} \in SC_{A_i}$, then $M[\vec{x} := \vec{x'}] \in SC_A$.*

Proof. ☐

☐

Theorem 4. *Church-Rosser property*

Proof. According to Newman's lemma, it is sufficient to establish local confluence. ☐

1 Sematics

Definition 7. *Monoidal comonad A monoidal comonad on some monoidal category $\mathcal{C}$ is a triple $\langle \mathcal{F}, \epsilon, \delta \rangle$, where $\mathcal{F}$ is a monoidal endofunctor and $\epsilon : \mathcal{F} \Rightarrow Id_{\mathcal{C}}$ (counit) and $\epsilon : \mathcal{F} \Rightarrow \mathcal{F}^2$ (comultiplication), such that the following diagrams commute:*

$$\begin{tikzcd}
\mathcal{F}A \otimes \mathcal{F}B \arrow[r, "\phi_{A,B}"] \arrow[dd, "\delta_A \otimes \delta_B"'] & \mathcal{F}(A \otimes B) \arrow[dr, "\delta_{A \otimes B}"] & \\
& & \mathcal{F}\mathcal{F}(A \otimes B) \\
\mathcal{F}\mathcal{F}A \otimes \mathcal{F}\mathcal{F}B \arrow[r, "\phi_{\mathcal{F}A,\mathcal{F}B}"'] & \mathcal{F}(\mathcal{F}A \otimes \mathcal{F}B) \arrow[ur, "\mathcal{F}(\phi_{A,B})"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}A \otimes \mathcal{F}B \arrow[rr, "\phi_{A,B}"] \arrow[dr, "\epsilon_A \otimes \epsilon_B"'] & & \mathcal{F}(A \otimes B) \arrow[dl, "\epsilon_{A \otimes B}"] \\
& A \otimes B &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "\phi"] \arrow[d, "\phi"'] & \mathcal{F}\mathbb{1} \arrow[d, "\delta_{\mathbb{1}}"] \\
\mathcal{F}\mathbb{1} \arrow[r, "\mathcal{F}(\phi)"'] & \mathcal{F}\mathcal{F}\mathbb{1}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{1} \arrow[rr, "id_{\mathbb{1}}"] \arrow[dr, "\phi"'] & & \mathbb{1} \\
& \mathcal{F}\mathbb{1} \arrow[ur, "\epsilon_{\mathbb{1}}"'] &
\end{tikzcd}$$

Definition 8. *Biclosed monoidal category*

Let $\mathcal{C}$ be the monoidal category. Biclosed monoidal category is a monoidal category with the following additional data:

1. *Bifunctors $\_\_ \Leftarrow \_\_, \_\_ \Rightarrow \_\_ : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$;*
2. *Natural isomophism $\textbf{curry}_{A,B,C} : Hom(A \otimes B, C) \cong (B, A \Rightarrow C)$;*
3. *Natural isomophism $\textbf{curry}^{'}_{A,B,C} : Hom(A \otimes B, C) \cong (A, C \Leftarrow B)$;*
4. *For each $A, B \in Ob_{\mathcal{C}}$, there are exist arrows $ev_{A,B} : A \otimes (A \Rightarrow B) \to B$ and $ev^{'}_{A,B} : (B \Leftarrow A) \otimes A \to B$, such that for all $f : A \otimes C \to B$:*
   (a) $ev_{A,B} \circ (id_A \otimes \textbf{curry}(f)) = f$;
   (b) $ev^{'}_{A,B} \circ (\textbf{curry}^{'}(f) \otimes id_A) = f$

Definition 9. *Linear biexponential comonad*

Let $\mathcal{C}$ is a monoidal category and $\mathcal{F}$ is monoidal endofunctor. A monoidal comonad $\langle \mathcal{F}, \epsilon, \delta, \phi \rangle$ is called a linear biexponential comonad, if $\mathcal{C}$ is a biclosed monoidal category and there exist additional natural transformations:

1. $\pi_1 : \mathcal{F}A \otimes B \to B \otimes \mathcal{F}A$;
2. $\pi_2 : B \otimes \mathcal{F}A \to \mathcal{F}A \otimes B$;
3. $\zeta : \mathcal{F}A \to \mathcal{F}A \otimes \mathcal{F}A$.

Such that:

1. *for each object A, tuple $\langle \mathcal{F}A, \zeta_A \rangle$ is a commutative cosemigroup:*

$$\begin{tikzcd}
\mathcal{F}A \arrow[r, "d_A"] \arrow[d, "\zeta_A"'] & \mathcal{F}A \otimes \mathcal{F}A \arrow[d, "id_{\mathcal{F}A} \otimes \zeta_A"] \\
\mathcal{F}A \otimes \mathcal{F}A \arrow[r, "\alpha_{\mathcal{F}A,\mathcal{F}A,\mathcal{F}A} \circ (\zeta_A \otimes id_{\mathcal{F}A})"'] & \mathcal{F}A \otimes (\mathcal{F}A \otimes \mathcal{F}A)
\end{tikzcd}$$

2. *for each object A, the following diagrams commute:*

$$\begin{tikzcd}
\mathcal{F}A \arrow[r, "\delta_A"] \arrow[d, "\zeta_A"'] & \mathcal{F}^2 A \arrow[dr, "\mathcal{F}\zeta_A"] & \\
\mathcal{F}A \otimes \mathcal{F}A \arrow[r, "\zeta_A \otimes \zeta_A"'] & \mathcal{F}^2 A \otimes \mathcal{F}^2 A \arrow[r, "\phi_{\mathcal{F}A,\mathcal{F}B}"'] & \mathcal{F}(\mathcal{F}A \otimes \mathcal{F}A)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}A \arrow[r, "\delta_A"] \arrow[d, "\zeta_A"'] & \mathcal{F}^2 A \arrow[d, "d_{\mathcal{F}A}"] \\
\mathcal{F}A \otimes \mathcal{F}A \arrow[r, "\delta_A \otimes \delta_A"'] & \mathcal{F}^2 A \otimes \mathcal{F}^2 A
\end{tikzcd}$$

Definition 10. *Interpretation*

1. *Types*
   (a) $[\![p_i]\!] = \hat{A}$;
   (b) $[\![A \backslash B]\!] = [\![A]\!] \Rightarrow [\![B]\!]$;
   (c) $[\![B/A]\!] = [\![B]\!] \Leftarrow [\![A]\!]$;
   (d) $[\![!A]\!] = \mathcal{F}[\![A]\!]$

2. *Typing rules*
   (a) $[\![x : A \vdash x : A]\!] = id_{[\![A]\!]} : [\![A]\!] \to [\![A]\!]$;
   (b)

$$\frac{[\![x : A, \Pi \vdash M : B]\!] = [\![M]\!] : [\![A]\!] \otimes [\![\Pi]\!] \to [\![B]\!]}{[\![\Pi \vdash \lambda x.M : A \backslash B]\!] = \mathbf{curry}_{[\![A]\!],[\![\Pi]\!],[\![B]\!]} : [\![\Pi]\!] \to [\![A]\!] \Rightarrow [\![B]\!]}$$

   (c)

$$\frac{[\![\Gamma \vdash M : A]\!] = [\![M]\!] : [\![\Gamma]\!] \to [\![A]\!] \qquad [\![\Pi \vdash N : A \backslash B]\!] = [\![N]\!] : [\![\Pi]\!] \to [\![A]\!] \Rightarrow [\![B]\!]}{[\![\Gamma, \Pi \vdash NM : B]\!] = ev_{[\![A]\!],[\![B]\!]} \circ [\![M]\!] \otimes [\![N]\!] : [\![\Gamma]\!] \otimes [\![\Pi]\!] \to [\![B]\!]}$$

   (d)

$$\frac{[\![\Pi, x : A \vdash M : B]\!] = [\![M]\!] : [\![\Pi]\!] \otimes [\![A]\!] \to [\![B]\!]}{[\![\Pi \vdash \kappa x.M : B/A]\!] = \mathbf{curry}'_{[\![\Pi]\!],[\![A]\!],[\![B]\!]} : [\![\Pi]\!] \to [\![B]\!] \Leftarrow [\![A]\!]}$$

   (e)

$$\frac{[\![\Gamma \vdash M : B/A]\!] = [\![M]\!] : [\![\Gamma]\!] \to [\![B]\!] \Leftarrow [\![A]\!] \qquad [\![\Pi \vdash N : A]\!] = [\![N]\!] : [\![\Pi]\!] \to [\![A]\!]}{[\![\Gamma, \Pi \vdash N \& M : B]\!] = ev'_{[\![A]\!],[\![B]\!]} \circ [\![M]\!] \otimes [\![N]\!] : [\![\Gamma]\!] \otimes [\![\Pi]\!] \to [\![B]\!]}$$

   (f)

$$\frac{[\![\Gamma \vdash M : A]\!] = [\![M]\!] : [\![\Gamma]\!] \to [\![A]\!] \qquad [\![\Delta \vdash N : B]\!] = [\![N]\!] : [\![\Pi]\!] \to [\![B]\!]}{[\![\Gamma, \Delta \vdash M \bullet N : A \bullet B]\!] = [\![M]\!] \otimes [\![N]\!] : [\![\Gamma]\!] \otimes [\![\Pi]\!] \to [\![A]\!] \otimes [\![B]\!]}$$

(g)

$$\frac{[\![\Gamma \vdash M : A \bullet B]\!] = [\![M]\!] : [\![\Gamma]\!] \to [\![A]\!] \otimes [\![B]\!] \qquad [\![\Delta, x : A, y : B, \Pi \vdash N : C]\!] = [\![N]\!] : [\![\Delta]\!] \otimes [\![A]\!] \otimes [\![B]\!] \otimes [\![\Pi]\!] \to [\![C]\!]}{[\![\Delta, \Gamma, \Pi \vdash \mathbf{let}\ M = x \bullet y\ \mathbf{in}\ N : C]\!] = (id_{[\![\Delta]\!]} \otimes [\![M]\!] \otimes id_{[\![\Pi]\!]}) \circ (\alpha_{[\![\Delta]\!],[\![A]\!],[\![B]\!]} \otimes id_{[\![\Pi]\!]}) \circ [\![N]\!] : [\![\Delta]\!] \otimes [\![\Gamma]\!] \otimes [\![\Pi]\!] \to [\![C]\!]}$$

(h)

$$\frac{[\![x_1 : !A_1, \ldots, x_n : !A_n \vdash M : A]\!] = [\![M]\!] : \mathcal{F}[\![A_1]\!] \otimes \cdots \otimes \mathcal{F}[\![A_n]\!] \to [\![A]\!]}{[\![x_1 : !A_1, \ldots, x_n : !A_n \vdash !M : !A]\!] = \mathcal{F}([\![M]\!]) \circ \phi_{[\![A_1]\!],\ldots,[\![A_n]\!]} \circ (\delta_{[\![A_1]\!]} \otimes \cdots \otimes \delta_{[\![A_n]\!]}) : \mathcal{F}[\![A_1]\!] \otimes \cdots \otimes \mathcal{F}[\![A_n]\!] \to \mathcal{F}[\![A]\!]}$$

(i)

$$\frac{[\![\Gamma \vdash M : !A]\!] = [\![M]\!] : [\![\Gamma]\!] \to \mathcal{F}[\![A]\!] \qquad [\![\Pi, x : A, \Delta \vdash N : B]\!] = [\![N]\!] : [\![\Pi]\!] \otimes [\![A]\!] \otimes [\![\Delta]\!] \to [\![B]\!]}{[\![\Pi, \Gamma, \Delta \vdash \mathbf{let}\ !x = M\ \mathbf{in}\ N : B]\!] = [\![N]\!] \circ (id_{[\![\Pi]\!]} \otimes (\epsilon_{[A]} \circ [\![M]\!]) \otimes id_{[\![\Delta]\!]}) : [\![\Pi]\!] \otimes [\![\Gamma]\!] \otimes [\![\Delta]\!] \to [\![B]\!]}$$

(j)

$$\frac{[\![\Gamma \vdash M : !A]\!] = [\![M]\!] : [\![\Gamma]\!] \to \mathcal{F}[\![A]\!] \qquad [\![\Delta \vdash N : B]\!] = [\![N]\!] : [\![\Delta]\!] \to [\![B]\!] \qquad [\![\Pi, x : !A, y : B, \Theta \vdash P : C]\!] = [\![\Pi]\!] \otimes \mathcal{F}[\![A]\!] \otimes [\![B]\!] \otimes [\![\Theta]\!] \to [\![C]\!]}{[\![\Pi, \Delta, \Gamma, \Theta \vdash ([x]M \odot [y]N)\ \mathbf{in}\ P : C]\!] = [\![\Pi]\!] \otimes [\![\Delta]\!] \otimes [\![\Gamma]\!] \otimes [\![\Theta]\!] \to [\![C]\!]}$$

Theorem 5. *Soundness*
If $\Gamma \vdash M : A$ *and* $M \twoheadrightarrow N$*, then* $[\![\Gamma \vdash M : A]\!] = [\![\Gamma \vdash N : A]\!]$

Proof. □

Theorem 6. *Completeness*

Proof. Term model □