Categorical model of noncommutative linear logic with subexponentials

Definition 1. *A subexponential signature is an ordered quintuple:*
$\Sigma = \langle I, \preceq, W, C, E \rangle$,

where $I = \{s_1, \ldots, s_n\}$, $\langle I, \preceq \rangle$ is a preorder. W, C, E are subsets of I and $W \cup C \subseteq E$.

Definition 2. *Noncommutative linear logic with subexponentials ($SMALC_\Sigma$), where Σ is a subexponential signature.*

$$\frac{}{A \Rightarrow A}\, ax$$

$$\frac{\Gamma \Rightarrow A \qquad \Delta, B, \Theta \Rightarrow C}{\Delta, \Gamma, A\backslash B, \Theta \Rightarrow C} \backslash \to \qquad \frac{A, \Pi \Rightarrow B}{\Pi \Rightarrow A\backslash B} \to \backslash$$

$$\frac{\Gamma \Rightarrow A \qquad \Delta, B, \Theta \Rightarrow C}{\Delta, B/A, \Gamma, \Theta \Rightarrow C} / \to \qquad \frac{\Pi, A \Rightarrow B}{\Pi \Rightarrow B/A} \to /$$

$$\frac{\Gamma, A, B, \Delta \Rightarrow C}{\Gamma, A \bullet B, \Delta \Rightarrow C} \bullet \to \qquad \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \bullet B} \to \bullet$$

$$\frac{\Gamma, A_i, \Delta \Rightarrow B}{\Gamma, A_1 \& A_2, \Delta \Rightarrow B} \&, i = 1, 2 \to \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \& B} \to \&$$

$$\frac{\Gamma, A, \Delta \Rightarrow C \qquad \Gamma, B, \Delta \Rightarrow C}{\Gamma, A \vee B, \Delta \Rightarrow C} \vee \to \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2} \to \vee, i = 1, 2$$

$$\frac{\Gamma, \Delta \Rightarrow A}{\Gamma, \mathbf{1}, \Delta \Rightarrow A} \mathbf{1} \to \qquad \frac{}{\Rightarrow \mathbf{1}} \to \mathbf{1}$$

$$\frac{\Gamma, A, \Delta \Rightarrow C}{\Gamma, !^s A, \Delta \Rightarrow C} ! \to \qquad \frac{!^{s_1} A_1, \ldots, !^{s_n} A_n \Rightarrow A}{!^{s_1} A_1, \ldots, !^{s_n} A_n \Rightarrow !^s A} \to !, \forall j, s_j \succeq s$$

$$\frac{\Gamma, \Delta \Rightarrow B}{\Gamma, !^s A, \Delta \Rightarrow B} \mathbf{weak}_!, s \in C$$

$$\frac{\Gamma, !^s A, \Delta, !^s A, \Theta \Rightarrow B}{\Gamma, !^s A, \Delta, \Theta \Rightarrow B} \mathbf{ncontr}_1, s \in C$$

$$\frac{\Gamma, !^s A, \Delta, !^s A, \Theta \Rightarrow B}{\Gamma, \Delta, !^s A, \Theta \Rightarrow B} \mathbf{ncontr}_2, s \in C$$

$$\frac{\Gamma, \Delta, !^s A, \Theta \Rightarrow B}{\Gamma, !^s A, \Delta, \Theta \Rightarrow A} \, \mathbf{ex}_1, s \in E$$

$$\frac{\Gamma, !^s A, \Delta, \Theta \Rightarrow B}{\Gamma, \Delta, !^s A, \Theta \Rightarrow A} \, \mathbf{ex}_1, s \in E$$

Definition 3. *Monoidal comonad*

A monoidal comonad on some monoidal category $\mathcal{C}$ is a triple $\langle \mathcal{F}, \epsilon, \delta \rangle$, where $\mathcal{F}$ is a monoidal endofunctor and $\epsilon : \mathcal{F} \Rightarrow Id_{\mathcal{C}}$ (counit) and $\epsilon : \mathcal{F} \Rightarrow \mathcal{F}^2$ (comultiplication), such that the following diagrams commute:

$$\begin{tikzcd}
\mathcal{F}A \otimes \mathcal{F}B \arrow[r, "\phi_{A,B}"] \arrow[dd, "\delta_A \otimes \delta_B"'] & \mathcal{F}(A \otimes B) \arrow[dr, "\delta_{A \otimes B}"] & \\
 & & \mathcal{F}\mathcal{F}(A \otimes B) \\
\mathcal{F}\mathcal{F}A \otimes \mathcal{F}\mathcal{F}B \arrow[r, "\phi_{\mathcal{F}A,\mathcal{F}B}"'] & \mathcal{F}(\mathcal{F}A \otimes \mathcal{F}B) \arrow[ur, "\mathcal{F}(\phi_{A,B})"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}A \otimes \mathcal{F}B \arrow[rr, "\phi_{A,B}"] \arrow[dr, "\epsilon_A \otimes \epsilon_B"'] & & \mathcal{F}(A \otimes B) \arrow[dl, "\epsilon_{A \otimes B}"] \\
 & A \otimes B &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "\phi"] \arrow[d, "\phi"'] & \mathcal{F}\mathbb{1} \arrow[d, "\delta_{\mathbb{1}}"] \\
\mathcal{F}\mathbb{1} \arrow[r, "\mathcal{F}(\phi)"'] & \mathcal{F}\mathcal{F}\mathbb{1}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{1} \arrow[rr, "id_{\mathbb{1}}"] \arrow[dr, "\phi"'] & & \mathbb{1} \\
 & \mathcal{F}\mathbb{1} \arrow[ur, "\epsilon_{\mathbb{1}}"'] &
\end{tikzcd}$$

Definition 4. *Biclosed monoidal category*

Let $\mathcal{C}$ be a monoidal category. Biclosed monoidal category is a monoidal category with the following additional data:

1. *Bifunctors $\_\_ \multimapinv \_\_, \_\_ \multimap \_\_ : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$;*
2. *Natural isomophism $\mathbf{curry}_{A,B,C} : Hom(A \otimes B, C) \cong (B, A \multimap C)$;*
3. *Natural isomophism $\mathbf{curry}'_{A,B,C} : Hom(A \otimes B, C) \cong (A, C \multimap B)$;*
4. *For each $A, B \in Ob_{\mathcal{C}}$, there are exist arrows $ev_{A,B} : A \otimes (A \Rightarrow B) \to B$ and $ev'_{A,B} : (B \Leftarrow A) \otimes A \to B$, such that for all $f : A \otimes C \to B$:*
   (a) $\Lambda_l \circ (id_A \otimes \mathbf{curry}(f)) = f$;
   (b) $\Lambda_r \circ (\mathbf{curry}'(f) \otimes id_A) = f$

Definition 5. *Let F be endofunctor and $A \in Ob\mathcal{C}$, then a coalgebra of F is a tuple $\langle A, \theta \rangle$, where $\theta : A \to FA$.*

Given coalgebras $\langle A, \theta \rangle$ and $\langle A, \psi \rangle$, a homomorphism is a morphism $f : A \to B$, s.t. the diagram below commutes:

$$\begin{tikzcd} A \arrow[r, "\theta"] \arrow[d, "f"'] & FA \arrow[d, "Ff"] \\ B \arrow[r, "\psi"'] & FB \end{tikzcd}$$

that is, $Ff \circ \theta = \psi \circ f$

Definition 6. *Subexponential model structure*

Let $\Sigma = \langle I, \preceq, W, C, E \rangle$ be a subexponential model structure and $\mathcal{C}$ be a biclosed monoidal category, then a subexponential model structure is $\langle \mathcal{C}, \{\mathcal{F}_s\}_{s \in I} \rangle$ with the following additional data:

- *for all $s \in I$, $\mathcal{F}_s$ is a monoidal comonad;*
- *if $s \in W$, then for all $A \in Ob(\mathcal{C})$, there exists a morphism $w_{A_s} : F_s A \to \mathbb{1}$;*
- *if $s \in C$, then for all $A \in Ob(\mathcal{C})$, there exists morphisms $w_{A_l} : F_s A \otimes B \to F_s A \otimes A \otimes F_s A$ and $w_{A_r} : B \otimes F_s A \to F_s A \otimes A \otimes F_s A$;*
- *if $s \in E$, then for all $A \in Ob(\mathcal{C})$, there is an isomorpism, $e_A : F_s A \otimes B \cong B \otimes F_s A$;*
- *if $s_1 \in W$, $s_2 \in I$ and $s_1 \preceq s_2$, then there is a morphism $w_{A_{s_2}} : F_{s_2} A \to \mathbb{1}$ for all $A \in Ob(\mathcal{C})$ and ditto for E and C;*
- *Let $\bigotimes_{s \in J, i=0}^{n} F_s A$, where $J \subset I$, and $s^{'} \in I$, s.t. $s \succeq s^{'}$ for all $s \in I'$; Then there exists morphism a morphism $\theta : \bigotimes_{s \in J, i=0}^{n} F_s A \to F_{s'} (\bigotimes_{s \in J, i=0}^{n} F_s A)$, such that $\langle \bigotimes_{s \in J, i=0}^{n} F_s A, \theta \rangle$ is a coalgebra on F_s.*

Theorem 1. *The following statements are equivalent:*

- *$SMLC_{\Sigma} + (cut) \vdash \Gamma \Rightarrow A$*
- *$SMLC_{\Sigma} \vdash \Gamma \Rightarrow A$*
- *$\exists f, f : [\![\Gamma]\!] \to [\![A]\!]$*

Proof.
- (1) $\Rightarrow$ (2): cut elimination.
- (2) $\Rightarrow$ (3): Soundness:

$$\frac{}{id_A : A \to A}$$

$$\frac{f : \Gamma \to A \qquad g : \Delta \otimes B \otimes \Theta \to C}{g \circ (id_{\Delta} \otimes (ev_{A,B_l} \circ (f \otimes id_{A \multimap B})) \otimes id_{\Theta}) : \Delta \otimes (\Gamma \otimes A \multimap B) \otimes \Theta \to C}$$

$$\frac{f : A \otimes \Pi \to B}{\Lambda_l(f) : \Pi \to A \multimap B}$$

$$\frac{f : \Gamma \to A \qquad g : \Delta \otimes B \otimes \Theta \to C}{g \circ (id_{\Delta} \otimes (ev_{A,B_l} \circ (id_{B \multimapinv A} \otimes f)) \otimes id_{\Theta}) : \Delta \otimes (B \multimapinv A \otimes \Gamma) \otimes \Theta \to C}$$

$$\frac{f : \Pi \otimes A \to B}{\Lambda_r(f) : \Pi \to B \multimapinv A}$$

$$\frac{f : \Gamma \otimes A \otimes B \otimes \Delta \to C}{f \circ (\alpha_{\Gamma,A,B} \otimes id_\Delta) : \Gamma \otimes (A \otimes B) \otimes \Delta \to C}$$

$$\frac{f : \Gamma \to A \qquad g : \Delta \to B}{f \otimes g : \Gamma \otimes \Delta \to A \otimes B}$$

$$\frac{}{id_{\mathbb{1}} : \mathbb{1} \to \mathbb{1}}$$

$$\frac{f : \Gamma \otimes \Delta \to A}{f \circ (\rho_\Gamma \otimes id_\Delta) : (\Gamma \otimes \mathbb{1}) \otimes \Delta \to A}$$

- Completeness:

$\square$

1 Concrete model

Definition 7. *Quantale*