Quantale model of noncommutative linear logic with subexponentials

Definition 1. *A subexponential signature is an ordered quintuple:*
$\Sigma = \langle I, \preceq, W, C, E \rangle$,

where $I = \{s_1, \ldots, s_n\}$, $\langle I, \preceq \rangle$ is a preorder. W, C, E are subsets of I and $W \cup C \subseteq E$.

Definition 2. *Noncommutative linear logic with subexponentials ($SMALC_\Sigma$), where Σ is a subexponential signature.*

$$\frac{}{A \Rightarrow A}\ ax$$

$$\frac{\Gamma \Rightarrow A \qquad \Delta, B, \Theta \Rightarrow C}{\Delta, \Gamma, A\backslash B, \Theta \Rightarrow C}\ \backslash \rightarrow \qquad \frac{A, \Pi \Rightarrow B}{\Pi \Rightarrow A\backslash B}\ \rightarrow \backslash$$

$$\frac{\Gamma \Rightarrow A \qquad \Delta, B, \Theta \Rightarrow C}{\Delta, B/A, \Gamma, \Theta \Rightarrow C}\ / \rightarrow \qquad \frac{\Pi, A \Rightarrow B}{\Pi \Rightarrow B/A}\ \rightarrow /$$

$$\frac{\Gamma, A, B, \Delta \Rightarrow C}{\Gamma, A \bullet B, \Delta \Rightarrow C}\ \bullet \rightarrow \qquad \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \bullet B}\ \rightarrow \bullet$$

$$\frac{\Gamma, A_i, \Delta \Rightarrow B}{\Gamma, A_1 \& A_2, \Delta \Rightarrow B}\ \&, i = 1, 2 \rightarrow \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \& B}\ \rightarrow \&$$

$$\frac{\Gamma, A, \Delta \Rightarrow C \qquad \Gamma, B, \Delta \Rightarrow C}{\Gamma, A \vee B, \Delta \Rightarrow C}\ \vee \rightarrow \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}\ \rightarrow \vee, i = 1, 2$$

$$\frac{\Gamma, \Delta \Rightarrow A}{\Gamma, \mathbf{1}, \Delta \Rightarrow A}\ \mathbf{1} \rightarrow \qquad \frac{}{\Rightarrow \mathbf{1}}\ \rightarrow \mathbf{1}$$

$$\frac{\Gamma, A, \Delta \Rightarrow C}{\Gamma, !^s A, \Delta \Rightarrow C}\ ! \rightarrow \qquad \frac{!^{s_1} A_1, \ldots, !^{s_n} A_n \Rightarrow A}{!^{s_1} A_1, \ldots, !^{s_n} A_n \Rightarrow !^s A}\ \rightarrow !, \forall j, s_j \succeq s$$

$$\frac{\Gamma, \Delta \Rightarrow B}{\Gamma, !^s A, \Delta \Rightarrow B}\ \mathbf{weak}_!, s \in C$$

$$\frac{\Gamma, !^s A, \Delta, !^s A, \Theta \Rightarrow B}{\Gamma, !^s A, \Delta, \Theta \Rightarrow B}\ \mathbf{ncontr}_1, s \in C$$

$$\frac{\Gamma, !^s A, \Delta, !^s A, \Theta \Rightarrow B}{\Gamma, \Delta, !^s A, \Theta \Rightarrow B}\ \mathbf{ncontr}_2, s \in C$$

$$\frac{\Gamma, \Delta, !^s A, \Theta \Rightarrow B}{\Gamma, !^s A, \Delta, \Theta \Rightarrow A}\ \mathbf{ex}_1, s \in E$$

$$\frac{\Gamma, !^s A, \Delta, \Theta \Rightarrow B}{\Gamma, \Delta, !^s A, \Theta \Rightarrow A}\ \mathbf{ex}_1, s \in E$$

Lemma 1. *Let $A \Leftrightarrow B$, then $C[p_i := A] \Leftrightarrow C[p_i := B]$*

Proof. By induction on C. □

Lemma 2.

- $!_{s_i}\Gamma \to A$ *iff* $!_{s_i}\Gamma \to !_{s_i}A$.
- $!_{s_i}A \leftrightarrow !_{s_i}(!_{s_i}A)$

Proof.

1. $!_{s_i}\Gamma \to A$ iff $!_{s_i}\Gamma \to !_{s_i}A$;

$$\frac{!_{s_i}\Gamma \to A}{!_{s_i}\Gamma \to !_{s_i}A}\ \to !_{s_i}$$

$$\dfrac{!_{s_i}\Gamma \to !_{s_i}A \qquad \dfrac{A \to A}{!_{s_i}A \to A}\ !_{s_i}\to}{!_{s_i}\Gamma \to A}\ \text{cut}$$

2. $!_{s_i}A \leftrightarrow !_{s_i}!_{s_i}A$

$$\dfrac{\dfrac{\dfrac{A \to A}{!_{s_i}A \to A}}{!_{s_i}A \to !_{s_i}A}}{!_{s_i}!_{s_i}A \to !_{s_i}A}$$

□

Definition 3. *Monoidal comonad*

A monoidal comonad on some monoidal category $\mathcal{C}$ is a triple $\langle \mathcal{F}, \epsilon, \delta\rangle$, where $\mathcal{F}$ is a monoidal endofunctor and $\epsilon : \mathcal{F} \Rightarrow Id_{\mathcal{C}}$ (counit) and $\epsilon : \mathcal{F} \Rightarrow \mathcal{F}^2$ (comultiplication), such that the following diagrams commute:

$$\begin{tikzcd}
\mathcal{F}A \otimes \mathcal{F}B \arrow[r, "\phi_{A,B}"] \arrow[dd, "\delta_A \otimes \delta_B"'] & \mathcal{F}(A \otimes B) \arrow[dr, "\delta_{A\otimes B}"] & \\
& & \mathcal{F}\mathcal{F}(A \otimes B) \\
\mathcal{F}\mathcal{F}A \otimes \mathcal{F}\mathcal{F}B \arrow[r, "\phi_{\mathcal{F}A,\mathcal{F}B}"'] & \mathcal{F}(\mathcal{F}A \otimes \mathcal{F}B) \arrow[ur, "\mathcal{F}(\phi_{A,B})"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}A \otimes \mathcal{F}B \arrow[rr, "\phi_{A,B}"] \arrow[dr, "\epsilon_A \otimes \epsilon_B"'] & & \mathcal{F}(A \otimes B) \arrow[dl, "\epsilon_{A\otimes B}"] \\
& A \otimes B &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "\phi"] \arrow[d, "\phi"'] & \mathcal{F}\mathbb{1} \arrow[d, "\delta_{\mathbb{1}}"] \\
\mathcal{F}\mathbb{1} \arrow[r, "\mathcal{F}(\phi)"'] & \mathcal{F}\mathcal{F}\mathbb{1}
\end{tikzcd}$$

$$\begin{tikzcd} \mathbb{1} \arrow[rr, "id_{\mathbb{1}}"] \arrow[dr, "\phi"'] & & \mathbb{1} \\ & \mathcal{F}\mathbb{1} \arrow[ur, "\epsilon_{\mathbb{1}}"'] & \end{tikzcd}$$

Definition 4. *Biclosed monoidal category*

Let $\mathcal{C}$ be a monoidal category. Biclosed monoidal category is a monoidal category with the following additional data:

1. *Bifunctors* $\_\_ \multimapinv \_\_, \_\_ \multimap \_\_ : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$*;*
2. *Natural isomophism* $\mathbf{curry}_{A,B,C} : Hom(A \otimes B, C) \cong (B, A \multimap C)$*;*
3. *Natural isomophism* $\mathbf{curry}^{'}_{A,B,C} : Hom(A \otimes B, C) \cong (A, C \multimap B)$*;*
4. *For each* $A, B \in Ob_{\mathcal{C}}$*, there are exist arrows* $ev_{A,B} : A \otimes (A \Rightarrow B) \to B$ *and* $ev^{'}_{A,B} : (B \Leftarrow A) \otimes A \to B$*, such that for all* $f : A \otimes C \to B$*:*
   (a) $\Lambda_l \circ (id_A \otimes \mathbf{curry}(f)) = f$*;*
   (b) $\Lambda_r \circ (\mathbf{curry}^{'}(f) \otimes id_A) = f$

Definition 5. *Let F be endofunctor and $A \in Ob\mathcal{C}$, then a coalgebra of F is a tuple $\langle A, \theta \rangle$, where $\theta : A \to FA$.*

Given coalgebras $\langle A, \theta \rangle$ and $\langle A, \psi \rangle$, a homomorphism is a morphism $f : A \to B$, s.t. the diagram below commutes:

$$\begin{tikzcd} A \arrow[r, "\theta"] \arrow[d, "f"'] & FA \arrow[d, "Ff"] \\ B \arrow[r, "\psi"'] & FB \end{tikzcd}$$

that is, $Ff \circ \theta = \psi \circ f$

Definition 6. *Subexponential model structure*

Let $\Sigma = \langle I, \preceq, W, C, E \rangle$ be a subexponential signature and $\mathcal{C}$ be a biclosed monoidal category, then a subexponential model structure is $\langle \mathcal{C}, \{\mathcal{F}_s\}_{s \in I} \rangle$ with the following additional data:

- *for all $s \in I$, $\mathcal{F}_s$ is a monoidal comonad;*
- *if $s \in W$, then for all $A \in Ob(\mathcal{C})$, there exists a morphism $w_{A_s} : F_s A \to \mathbb{1}$;*
- *if $s \in C$, then for all $A \in Ob(\mathcal{C})$, there exists morphisms $w_{A_l} : F_s A \otimes A \otimes F_s A \to F_s A \otimes B$ and $w_{A_r} : F_s A \otimes A \otimes F_s A \to B \otimes F_s A$;*
- *if $s \in E$, then for all $A \in Ob(\mathcal{C})$, there is an isomorpism, $e_A : F_s A \otimes B \cong B \otimes F_s A$;*
- *if $s_1 \in W$, $s_2 \in I$ and $s_1 \preceq s_2$, then there is a morphism $w_{A_{s_2}} : F_{s_2} A \to \mathbb{1}$ for all $A \in Ob(\mathcal{C})$ and ditto for E and C;*
- *Let $\bigotimes^n_{s \in J, i=0} F_s A$, where $J \subset I$, and $s^{'} \in I$, s.t. $s \geq s^{'}$ for all $s \in I'$; Then there exists morphism a morphism $\theta_{\bigotimes^n_{s \in J, i=1} F_{s_j} A_i} : \bigotimes^n_{s \in J, i=0} F_s A \to F_{s^{'}}(\bigotimes^n_{s \in J, i=0} F_s A)$, such that $\langle \bigotimes^n_{s \in J, i=1} F_{s_j} A_i, \theta_{\bigotimes^n_{s \in J, i=1} F_{s_j} A_i} \rangle$ is a coalgebra on F_s.*

Definition 7. *Let $\langle \mathcal{C}, \{\mathcal{F}_s\}_{s\in I}\rangle$ be a subexponential model structure for subexponential signature $\Sigma = \langle I, \preceq, W, C, E\rangle$. Let $v : Tp \to Ob(\mathcal{C})$ be a valuation map. Then the interpretation function $[\![.]\!]$ is defined as follows:*

(1) $[\![\mathbf{1}]\!] = \mathbb{1}$
(2) $[\![A\backslash B]\!] = [\![A]\!] \multimap [\![B]\!]$
(3) $[\![A/B]\!] = [\![A]\!] \circ\!\!- [\![B]\!]$
(4) $[\![A \bullet B]\!] = [\![A]\!] \otimes [\![B]\!]$
(5) $[\![!_s A]\!] = F_s[\![A]\!]$

Theorem 1. *The following statements are equivalent:*

- $SMLC_\Sigma + (cut) \vdash \Gamma \Rightarrow A$
- $SMLC_\Sigma \vdash \Gamma \Rightarrow A$
- $\exists f, f : [\![\Gamma]\!] \to [\![A]\!]$

  Proof.
  (1) $\Rightarrow$ (2): cut elimination.

- (2) $\Rightarrow$ (3): Soundness:

$$\frac{}{id_A : A \to A}$$

$$\frac{f : \Gamma \to A \qquad g : \Delta \otimes B \otimes \Theta \to C}{g \circ (id_\Delta \otimes (ev_{A,B_l} \circ (f \otimes id_{A \multimap B})) \otimes id_\Theta) : \Delta \otimes (\Gamma \otimes A \multimap B) \otimes \Theta \to C}$$

$$\frac{f : A \otimes \Pi \to B}{\Lambda_l(f) : \Pi \to A \multimap B}$$

$$\frac{f : \Gamma \to A \qquad g : \Delta \otimes B \otimes \Theta \to C}{g \circ (id_\Delta \otimes (ev_{A,B_l} \circ (id_{B \circ\!- A} \otimes f)) \otimes id_\Theta) : \Delta \otimes (B \circ\!- A \otimes \Gamma) \otimes \Theta \to C}$$

$$\frac{f : \Pi \otimes A \to B}{\Lambda_r(f) : \Pi \to B \circ\!- A}$$

$$\frac{f : \Gamma \otimes A \otimes B \otimes \Delta \to C}{f \circ (\alpha_{\Gamma,A,B} \otimes id_\Delta) : \Gamma \otimes (A \otimes B) \otimes \Delta \to C}$$

$$\frac{f : \Gamma \to A \qquad g : \Delta \to B}{f \otimes g : \Gamma \otimes \Delta \to A \otimes B}$$

$$\frac{f : \Gamma \otimes A_i \otimes \Delta \to B}{f \circ (id_\Gamma \otimes \pi_i id_\Delta) : \Gamma \otimes (A_1 \times A_2) \otimes \Delta \to B}$$

$$\frac{f : \Gamma \to A \qquad g : \Gamma \to B}{\langle f, g\rangle : \Gamma \to A \times B}$$

$$\frac{f : \Gamma \otimes A \otimes \Delta \to C \qquad g : \Gamma \otimes B \otimes \Delta \to C}{id_\Gamma \otimes [f, g] \otimes id_\Delta : \Gamma \otimes (A + B) \otimes \Delta \to C}$$

$$\frac{}{id_{\mathbb{1}} : \mathbb{1} \to \mathbb{1}}$$

$$\frac{f : \Gamma \otimes \Delta \to A}{f \circ (\rho_\Gamma \otimes id_\Delta) : (\Gamma \otimes \mathbb{1}) \otimes \Delta \to A}$$

$$\frac{f : \Gamma \otimes A \otimes \Delta \to B}{f \circ (id_\Gamma \otimes \delta_s^A \otimes id_\Delta) : \Gamma \otimes F_s A \otimes \Delta \to B}$$

$$\frac{\dfrac{f : F_{s_1} A_1 \otimes \cdots \otimes F_{s_n} A_n \to B}{F_s(f) : F_s(F_{s_1} A_1 \otimes \cdots \otimes F_{s_n} A_n) \to F_s B}}{F_s(f) \circ \theta_{\bigotimes_{s \in J, i=1}^{n} F_{s\,j} A_i} : F_{s_1} A_1 \otimes \cdots \otimes F_{s_n} A_n \to F_s B}$$

$$\frac{\dfrac{f : \Gamma \otimes \Delta \to A}{f \circ (\rho_\Gamma \otimes id_\Delta) : (\Gamma \otimes \mathbb{1}) \otimes \Delta \to A}}{f \circ (\rho_\Gamma \otimes id_\Delta) \circ (id_\Gamma \otimes w_{A_s}) \otimes id_\Delta : (\Gamma \otimes F_s A) \otimes \Delta \to A}$$

$$\frac{f : \Gamma \otimes (F_s A \otimes B \otimes F_s A) \otimes \Delta \to C}{f \circ (id_\Gamma \otimes c_{A_s}^{\ l} \otimes id_\Delta) : \Gamma \otimes (F_s A \otimes B) \otimes \Delta \to C}$$

$$\frac{f : \Gamma \otimes (F_s A \otimes B \otimes F_s A) \otimes \Delta \to C}{(id_\Gamma \otimes c_{A_s}^{\ r} \otimes id_\Delta) \circ f : \Gamma \otimes (B \otimes F_s A) \otimes \Delta \to C}$$

$$\frac{f : \Gamma \otimes (\Delta \otimes F_s A) \otimes \Theta \to B}{(id_\Gamma \otimes (id_\Delta \otimes e_{A_s}) \otimes id_\Theta) \circ f : \Gamma \otimes (F_s A \otimes \Delta) \otimes \Theta \to B}$$

$$\frac{f : \Gamma \otimes (F_s A \otimes \Delta) \otimes \Theta \to B}{(id_\Gamma \otimes (id_\Delta \otimes e_{A_s}^{\ -1}) \otimes id_\Theta) \circ f : \Gamma \otimes (\Delta \otimes F_s A) \otimes \Theta \to B}$$

- Completeness:

  Definition 8.

$\square$

1 Concrete model

Definition 9. *Quantale A quantale is a triple $\langle A, \bigvee, \cdot \rangle$, such that $\langle A, \bigvee \rangle$ is a complete lattice and $\langle A, \cdot \rangle$ is a semigroup. A quanlate is called unital, if $\langle A, \cdot \rangle$ is a monoid.*

It is easy to see, that any (unital) quantale is a residual (monoid) semigroup. We define divisions as follows:

1. $a \backslash b = \bigvee \{c \mid a \cdot c \leqslant b\}$
2. $b / a = \bigvee \{c \mid c \cdot a \leqslant b\}$

Definition 10. *Let $\langle A, \bigvee, \cdot \rangle$ be a quantale. The center of a quantale is the set $Z(Q) = \{a \in Q \mid \forall b \in Q, a \cdot b = b \cdot a\}$*

Definition 11. *An open modality on quantale Q is a map $I : Q \to Q$, such that*

1. *$I(x) \leqslant x$;*
2. *$I(x) = I(I(x))$;*
3. *$x \leqslant y \Rightarrow I(x) \leqslant I(y)$;*
4. *$I(x) \cdot I(y) = I(I(x) \cdot I(y))$.*

Lemma 3.
Let $\langle A, \bigvee, \cdot \rangle$ be a quantale and $I : Q \to Q$ is an open modality on Q, then $I(x) \cdot I(y) \leqslant I(x \cdot y)$.

Proof.
$I(x) \cdot I(y) \leqslant x \cdot y$, then $I(I(x) \cdot I(y)) \leqslant I(x \cdot y)$, but $I(x) \cdot I(y) \leqslant I(I(x) \cdot I(y))$. Thus, $I(x) \cdot I(y) \leqslant I(x \cdot y)$. □

Definition 12. *An open modality is called central, if $\forall a, b \in Q, I(a) \cdot b = b \cdot I(a)$.*

Definition 13. *An open modality is called weak idempotent, if $\forall a, b \in Q, I(a) \cdot b \leqslant I(a) \cdot b \cdot I(a)$ and $b \cdot I(a) \leqslant I(a) \cdot b \cdot I(a)$.*

Definition 14. *An open modality is called unital, if $\forall a \in Q, I(a) \leqslant e$.*

Lemma 4. *Let I be an interior on some unital quantale $\langle Q, \bigvee, \cdot, e \rangle$. Then, if I is unital and weak idempotent, then I is central.*

Proof.
$b \cdot I(a) \leqslant$
Right weak idempotence
$I(a) \cdot b \cdot I(a) \leqslant$
Unitality
$I(a) \cdot b \cdot I(e) \leqslant$
Identity
$I(a) \cdot b \leqslant$
Left weak idempotence
$I(a) \cdot b \cdot I(a) \leqslant$
Unitality
$e \cdot b \cdot I(a) \leqslant$
Identity
$b \cdot I(a)$
Hence, $b \cdot I(a) = I(a) \cdot b$

□

Proposition 1.

Let Q be a quantale and $S \subseteq Q$ a subquantale, then $I : Q \to Q$, such that $I(a) = \bigvee\{s \in S \mid x \leqslant a\}$, is an open modality. Moreover, $\{x \in Q \mid I(x) = x\} = S$.

Proof. See □

Proposition 2.

Let Q be a quantale and $S_1, S_2 \subseteq Q$, such that $S_1 \subseteq S_2$.
Then $I_1(a) \leqslant I_2(a)$.

Proof.

Let $a \in Q$, so $\{s \in S_1 \mid s \leqslant a\} \subseteq \{s \in S_2 \mid s \leqslant a\}$, so $\bigvee\{s \in S_1 \mid s \leqslant a\} \subseteq \bigvee\{s \in S_2 \mid s \leqslant a\}$. Thus, $I_1(a) \leqslant I_2(a)$. □

Proposition 3.

Let Q be a quantale and $S \subseteq Q$ a subquantale, then the following operations are open modalities:

1. $I_z(a) = \bigvee\{s \in S \,| s \leqslant a, s \in Z(Q)\}$;
2. $I_{\mathbb{1}}(a) = \bigvee\{s \in S \,| s \leqslant a, s \leqslant \mathbb{1}\}$;
3. $I_{idem}(a) = \bigvee\{s \in S \,| s \leqslant a, \forall b \in Q, b \cdot s \vee s \cdot b \leqslant s \cdot b \cdot s\}$;
4. $I_{z,\mathbb{1}}, I_{z,idem}, I_{\mathbb{1},idem}, I_{z,\mathbb{1},idem}$.

Proof. Immediatly. □

Proposition 4.

1. $\forall a \in Q, I_{\mathbb{1},idem}(a) \leqslant I_z(a)$.
2. $\forall a \in Q, I_{z,\mathbb{1},idem} = I_{\mathbb{1},idem}(a)$

Proof. Follows from Lemma 3. □

Proposition 5.

1. $I_z(a) \vee I_{\mathbb{1}}(a) \vee I_{idem}(a) \leqslant I(a)$
2. $I_{z,\mathbb{1},idem} \leqslant I_{z,\mathbb{1}}(a) \wedge I_{z,idem}(a)$

Lemma 5. $\forall a \in Q, I_1(a) \leqslant I_2(I_1(a))$*, if* $I_1(a) \leqslant I_2(a)$.

Proof. $I_1(a) \leqslant I_1(I_1(a)) \leqslant I_2(I_1(a))$ □

Lemma 6. $I_1(a_1) \cdot I_2(a_2) \leqslant I^{'}(I_1(a_1) \cdot I_2(a_2))$*, where* $I_i \leqslant I^{'}, i = 1, 2$.

Proof.

$$\begin{aligned}
&I_1(a_1) \cdot I_2(a_2) \leqslant \\
&I_1(I_1(a_1)) \cdot I_2(I_2(a_2)) \leqslant \\
&I^{'}(I_1(a_1)) \cdot I^{'}(I_2(a_2)) \leqslant \\
&I^{'}(I_1(a_1) \cdot I_2(a_2))
\end{aligned}$$

□

Definition 15. *Interpretation of subexponential signature*

Let $\Sigma = \langle I, \preceq, W, C, E\rangle$ be a subexponential signature, where $|I| = n$ and $\mathcal{S} = \{\Box_1, \ldots, \Box_n\}$ be a set of open modalities on quantale Q. Subexponential interpretation is a contravariant map $\sigma : I \to \mathcal{S}$ defined as follows:

$$\sigma(s_i) = \begin{cases} \Box_i : Q \to Q,\ s.t. \forall a \in Q, \Box_i(a) = \{s \in S_i \mid s \leqslant a\}, \\ \quad \text{if } s_i \notin W \cap C \cap E \\ \Box_i : Q \to Q,\ s.t. \forall a \in Q, \Box_i(a) = \{s \in S_i \mid s \leqslant a, \leqslant \mathbb{1}\}, \\ \quad \text{if } s_i \in W \\ \Box_i : Q \to Q,\ s.t. \forall a \in Q, \Box_i(a) = \{s \in S_i \mid s \leqslant a, \in Z(Q)\}, \\ \quad \text{if } s_i \in E \\ \Box_i : Q \to Q,\ s.t. \forall a \in Q, \Box_i(a) = \{s \in S_i \mid s \leqslant a, \forall b, b \cdot s \vee s \cdot b \leqslant s \cdot b \cdot s\}, \\ \quad \text{if } s_i \in E \\ \text{otherwise, if } s_i \text{ belongs to some intersection of subsets, then we combine the relevant conditions} \end{cases}$$

Definition 16. *Let Q be a quantale, $f : Tp \to Q$ a valuation and $\sigma : I \to \mathcal{S}$ a subexponential interpretation, then interpretation is defined inductively:*

$$\begin{aligned} &[\![p_i]\!] = f(p_i) \\ &[\![\mathbb{1}]\!] = e \\ &[\![A \bullet B]\!] = [\![A]\!] \cdot [\![B]\!] \\ &[\![A \backslash B]\!] = [\![A]\!] \backslash [\![B]\!] \\ &[\![A / B]\!] = [\![A]\!] / [\![B]\!] \\ &[\![A \& B]\!] = [\![A]\!] \wedge [\![B]\!] \\ &[\![A \vee B]\!] = [\![A]\!] \vee [\![B]\!] \\ &[\![!_{s_i} A]\!] = \sigma(s_i)[\![A]\!] \end{aligned}$$

Theorem 2. $\Gamma \to A \Rightarrow [\![\Gamma]\!] \leqslant [\![A]\!]$

Proof. We consider the case with polymodal promotion rule.

Let $!_{s_1} A_1, \ldots, !_{s_n} A_n \to A$ and $\forall i, s \preceq s_i$. Then $\forall a \in Q, \sigma(s_i)(a) \leqslant \sigma(s)(a)$.

By IH, $\sigma(s_1)[\![A_1]\!] \cdot \cdots \cdot \sigma(s_n)[\![A_n]\!] \leqslant [\![A]\!]$.

Thus, $\sigma(s)(\sigma(s_1)[\![A_1]\!] \cdot \cdots \cdot \sigma(s_n)[\![A_n]\!]) \leqslant \sigma(s)([\![A]\!])$.

By Lemma 5, $\sigma(s_1)[\![A_1]\!] \cdot \cdots \cdot \sigma(s_n)[\![A_n]\!] \leqslant \sigma(s)(\sigma(s_1)[\![A_1]\!] \cdot \cdots \cdot \sigma(s_n)[\![A_n]\!])$.

So, $\sigma(s_1)[\![A_1]\!] \cdot \cdots \cdot \sigma(s_n)[\![A_n]\!] \leqslant \sigma(s)([\![A]\!])$. $\Box$

2 Quantale completeness

Definition 17.

Let $\mathcal{F} \subseteq Fm$, an ideal is a subset $\mathcal{I} \subseteq \mathcal{F}$, such that:

- *If $B \in \mathcal{I}$ and $A \to B$, then $A \in \mathcal{I}$;*
- *If $A, B \in \mathcal{I}$, then $A \vee B \in \mathcal{I}$.*

Definition 18.

Let $S \subseteq \mathcal{F} \subseteq Fm$, then $\bigvee S = \bigcap\{\mathcal{I} \subseteq \mathcal{F} \mid S \subseteq \mathcal{I}\}$

Proposition 6. *$\bigvee S$ is an ideal.*

Lemma 7. *$A \in \mathcal{F}$, then $\{B \mid B \to A\} = \bigvee\{A\}$.*

Proof.

Let $A \in \mathcal{F}$. Then $\{B \mid B \to A\} \subseteq \bigvee\{A\}$.

On the other hand, $\{B \mid B \to A\}$ is an ideal, hence, $\{A\} \subseteq \{B \mid B \to A\}$. $\square$

Lemma 8. *$\bigvee\{A\} \subseteq \bigvee\{B\}$ iff $A \to B$.*

Proof. Let $\bigvee\{A\} \subseteq \bigvee\{B\}$, then $\{C|C \to A\} \subseteq \{C \mid C \to B\}$.

$A \in \{C|C \to A\}$, so far as $A \to A$, but $\{C \mid C \to B\}$, hence $A \to B$.

On the other hand, let $A \to B$ and $C \in \bigvee\{A\}$. Then $C \in \bigvee\{C' \mid C' \to A\}$, hence, $C \to A$, thus $C \to B$ by cut. So, $C \in \bigvee\{A\}$.

$\square$

Lemma 9. *Let $\mathcal{Q} = \{\bigvee S | S \subseteq Fm\}$ and $\bigvee \mathcal{A} \cdot \bigvee \mathcal{B} = \{A \bullet B | A \in \mathcal{A}, B \in \mathcal{B}\}$. Then $\langle \mathcal{Q}, \subseteq, \cdot, \bigvee \mathbf{1}\rangle$ is a quantale.*

Proof. See $\square$

Lemma 10. *Interior lemma.*

Let $Q_1 \subseteq \mathcal{Q}$, define a map $\square : Q \to Q$, such that $\square(A) = \{Q \in Q_1 \mid Q \subseteq A\}$. Then $\square$ is a quantic conucleus.

Lemma 11. *Let $!_s \in I$, $I \notin W \cap E \cap C$ and $Q \subseteq \mathcal{Q}$. Then there exist a subset $Q \subseteq \mathcal{Q}$ and a quantic conucleus $\square_s(\bigvee\{A\}) = \{\bigvee Q \in \mathcal{Q} \mid \}$*

Proof. $\square$

Proof. See $\square$

Lemma 12. *Let $Q \subseteq \mathcal{Q}$, then the following operators are quantic conuclei:*

1. $\square_z(A) = \bigvee\{\bigvee\{W\} \in Q \mid \bigvee\{W\} \subseteq \bigvee\{A\}, \bigvee\{W\} \in Z(Q)\}$;
2. $\square_{\mathbf{1}}(A) = \bigvee\{\bigvee\{W\} \in Q \mid \bigvee\{W\} \subseteq \bigvee\{A\}, \bigvee\{W\} \subseteq \bigvee\{\mathbf{1}\}\}$;
3. $\square_{idem}(A) = \bigvee\{\bigvee\{W\} \in Q \mid \bigvee\{W\} \subseteq \bigvee\{A\}, \forall B \in Fm, (\bigvee\{B\} \cdot \bigvee\{W\}) \cup (\bigvee\{W\} \cdot \bigvee\{B\}) \subseteq \bigvee\{W\} \cdot \bigvee\{A\} \cdot \bigvee\{W\}\}$;
4. $\square_{z,\mathbb{1}}, \square_{z,idem}, \square_{\mathbb{1},idem}, \square_{z,\mathbb{1},idem}$.

Proof. Follow from one of lemmas above. $\square$

Lemma 13. *Let $!_s \in I$, $I \notin W \cap E \cap C$, then $\square_s(\bigvee A) = \bigvee\{\bigvee(!_s W) \mid \bigvee !_s W \subseteq \bigvee A\}$ is a quantic conucleus.*

Proof.

1. $\square_s(\bigvee A) \subseteq \bigvee A$;

   $\bigvee(!_s W) \in \square_s(\bigvee A)$, then $\bigvee(!_s W) \in \bigvee\{\bigvee(!_s W) | \bigvee !_s W \subseteq \bigvee A\}$. Hence, $\bigvee(!_s F) \subseteq \bigvee\{A\}$, so, $!_s F \to A$. Therefore, $!_s F \in \bigvee\{A\}$.

2. $\square_s(\square_s(\bigvee A)) = \bigvee \square_s(\bigvee A)$;

   $\square_s(\square_s(\bigvee A)) = \{\bigvee(!_s !_s F) \mid \bigvee(!_s !_s F) \subseteq \bigvee A\}$.

   Let $\bigvee(!_s !_s F) \in \square_s(\square_s(\bigvee A))$, then $!_s !_s F \to A$, hence $!_s F \to A$ by equivalence, so $\bigvee(!_s !_s F) \in \square_s(\bigvee A)$

3. $\bigvee A \subseteq \bigvee B \Rightarrow \Box_s(\bigvee A) \subseteq \Box_s(\bigvee B)$;

4. $\Box_s \bigvee A \cdot \Box_s \bigvee A = \Box_s(\Box_s \bigvee A \cdot \Box_s \bigvee A)$.

$\Box$

Lemma 14. *Let Q be a quantale constructed above and $\Box_1, \ldots, \Box_n$ be a family of quantic conuclei on Q. Then there exist a model $\langle Q, [\![.]\!]\rangle$, such that $[\![A]\!] = \bigvee\{A\}$, $A \in Fm$.*

Proof. $\Box$

Theorem 3. $\Gamma \models A \Rightarrow \Gamma \rightarrow A$.