

Models of Lambek calculus enriched with subexponentials

Daniel Rogozin^{1,2}

¹Lomonosov Moscow State University

²Serokell OÜ

Abstract

1 The Lambek Calculus with subexponentials

Definition 1. A subexponential signature is an ordered quintuple:

$$\Sigma = \langle I, \leq, W, C, E \rangle,$$

where $I = \{s_1, \dots, s_n\}$, $\langle I, \leq \rangle$ is a preorder. W, C, E are upwardly closed subsets of I and $W \cap C \subseteq E$.

Definition 2.

$$\mathcal{F}_\Sigma ::= Tp \mid (\mathcal{F}_\Sigma / \mathcal{F}_\Sigma) \mid (\mathcal{F}_\Sigma \backslash \mathcal{F}_\Sigma) \mid (\mathcal{F}_\Sigma \bullet \mathcal{F}_\Sigma) \mid (\mathcal{F}_\Sigma \vee \mathcal{F}_\Sigma) \mid (\mathcal{F}_\Sigma \wedge \mathcal{F}_\Sigma) \mid !_s \mathcal{F}_\Sigma$$

Definition 3. Noncommutative linear logic with subexponentials ($SMALC_\Sigma$), where Σ is a subexponential signature.

$$\begin{array}{c} \overline{A \rightarrow A} \text{ }^{ax} \\ \\ \frac{\Gamma \rightarrow A \quad \Delta, B, \Theta \rightarrow C}{\Delta, \Gamma, A \backslash B, \Theta \rightarrow C} \backslash \rightarrow \qquad \frac{A, \Pi \rightarrow B}{\Pi \rightarrow A \backslash B} \rightarrow \backslash \\ \\ \frac{\Gamma \rightarrow A \quad \Delta, B, \Theta \rightarrow C}{\Delta, B / A, \Gamma, \Theta \rightarrow C} / \rightarrow \qquad \frac{\Pi, A \rightarrow B}{\Pi \rightarrow B / A} \rightarrow / \\ \\ \frac{\Gamma, A, B, \Delta \rightarrow C}{\Gamma, A \bullet B, \Delta \rightarrow C} \bullet \rightarrow \qquad \frac{\Gamma \rightarrow A \quad \Delta \rightarrow B}{\Gamma, \Delta \rightarrow A \bullet B} \rightarrow \bullet \\ \\ \frac{\Gamma, A_i, \Delta \rightarrow B}{\Gamma, A_1 \& A_2, \Delta \rightarrow B} \&, i = 1, 2 \rightarrow \qquad \frac{\Gamma \rightarrow A \quad \Gamma \rightarrow B}{\Gamma \rightarrow A \& B} \rightarrow \& \\ \\ \frac{\Gamma, A, \Delta \rightarrow C \quad \Gamma, B, \Delta \rightarrow C}{\Gamma, A \vee B, \Delta \rightarrow C} \vee \rightarrow \qquad \frac{\Gamma \rightarrow A_i}{\Gamma \rightarrow A_1 \vee A_2} \rightarrow \vee, i = 1, 2 \\ \\ \frac{\Gamma, \Delta \rightarrow A}{\Gamma, 1, \Delta \rightarrow A} 1 \rightarrow \qquad \overline{\rightarrow 1} \rightarrow 1 \\ \\ \frac{\Gamma, A, \Delta \rightarrow C}{\Gamma, !^s A, \Delta \rightarrow C} ! \rightarrow \qquad \frac{!^{s_1} A_1, \dots, !^{s_n} A_n \rightarrow A}{!^{s_1} A_1, \dots, !^{s_n} A_n \rightarrow !^s A} \rightarrow !, \forall j, s_j \geq s \end{array}$$

Structural rules:

$$\begin{array}{c}
\frac{\Gamma, !^s A, \Delta, !^s A, \Theta \rightarrow B}{\Gamma, !^s A, \Delta, \Theta \rightarrow B} \text{ncontr}_1, s \in C \qquad \frac{\Gamma, !^s A, \Delta, !^s A, \Theta \rightarrow B}{\Gamma, \Delta, !^s A, \Theta \rightarrow B} \text{ncontr}_2, s \in C \\
\\
\frac{\Gamma, \Delta, !^s A, \Theta \rightarrow B}{\Gamma, !^s A, \Delta, \Theta \rightarrow A} \text{ex}_1, s \in E \qquad \frac{\Gamma, !^s A, \Delta, \Theta \rightarrow B}{\Gamma, \Delta, !^s A, \Theta \rightarrow A} \text{ex}_2, s \in E \\
\\
\frac{\Gamma, \Delta \rightarrow B}{\Gamma, !^s A, \Delta \rightarrow B} \text{weak}_1, s \in C \qquad \frac{\Gamma \rightarrow A \quad \Pi, A, \Delta \rightarrow B}{\Gamma, \Pi, \Delta \rightarrow B} \text{cut}
\end{array}$$

Theorem 1.

1. Cut-rule is admissible;
2. $SMALC_\Sigma$ is undecidable, if $C \neq \emptyset$;
3. If C is empty, then the decidability problem of $SMALC_\Sigma$ belongs to PSPACE.

2 Semantics

Definition 4. *Quantale*

A quantale is a triple $\langle A, \vee, \cdot \rangle$, such that $\langle A, \vee \rangle$ is a complete lattice and $\langle A, \cdot \rangle$ is a semi-group. A quantale is called unital, if $\langle A, \cdot \rangle$ is a monoid.

Some example of quantales:

- Let A be a semigroup (monoid), then $\langle \mathcal{P}(A), \cdot, \subseteq \rangle$ is a free (unital) quantale.
- Let R be a ring and $Sub(R)$ be a set of additive subgroups of R . We define $A \cdot B$ as an additive subgroup generated by finite sums of products ab and order is defined by inclusion.
- Any locale is a quantale with $\cdot = \wedge$.

It is easy to see, that any (unital) quantale is a residual (monoid) semigroup. We define divisions as follows:

1. $a \backslash b = \bigvee \{c \mid a \cdot c \leq b\}$
2. $b / a = \bigvee \{c \mid c \cdot a \leq b\}$

Definition 5.

Let $\mathcal{Q} = \langle A, \vee, \cdot \rangle$ be a quantale. The center of a quantale is the set $\mathcal{Z}(\mathcal{Q}) = \{a \in A \mid \forall b \in A, a \cdot b = b \cdot a\}$

Definition 6. An open modality (or quantic conucleus) on quantale \mathcal{Q} is a map $\Box : \mathcal{Q} \rightarrow \mathcal{Q}$, such that

1. $\Box x \leq x$;
2. $\Box x = \Box \Box x$;
3. $x \leq y \Rightarrow \Box x \leq \Box y$;
4. $\Box x \cdot \Box y = \Box(\Box x \cdot \Box y)$.

For unital quantale, we require that $\Box e = e$.

Note that, we may replace the last condition on equivalent condition $\Box(x) \cdot \Box(y) \leq \Box(x \cdot y)$.

Definition 7. We define a partial order on open modalities on \mathcal{Q} as $\Box_1 \leq \Box_2 \Leftrightarrow \forall a \in \mathcal{Q}, \Box_1(a) \leq \Box_2(a)$.

Lemma 1. Let \mathcal{Q} be a quantale and $\Box_{\mathcal{Q}}$ be a set of all open modalities on \mathcal{Q} . Then $\Box_{\mathcal{Q}}$ is a small category.

Proof. $\langle \Box_{\mathcal{Q}}, \leq \rangle$ form a partial order, so $\langle \Box_{\mathcal{Q}}, \leq \rangle$ is a small category. \square

Definition 8.

1. An open modality is called *central*, if $\forall a, b \in \mathcal{Q}, \Box(a) \cdot b = b \cdot \Box(a)$.
2. An open modality is called *pseudo-idempotent*, if $\forall a, b \in \mathcal{Q}, \Box(a) \cdot b \leq \Box(a) \cdot b \cdot \Box(a)$ and $b \cdot \Box(a) \leq \Box(a) \cdot b \cdot \Box(a)$.
3. An open modality is called *unital*, if $\forall a \in \mathcal{Q}, \Box(a) \leq e$.

Lemma 2. Let \Box be an open modality on some unital quantale $\mathcal{Q} = \langle A, \vee, \cdot, e \rangle$. Then, if \Box is unital and weak idempotent, then \Box is central.

Proof.

$$\begin{aligned}
& b \cdot \Box(a) \leq \\
& \quad \text{Right weak idempotence} \\
& \Box(a) \cdot b \cdot \Box(a) \leq \\
& \quad \text{Unitality} \\
& \Box(a) \cdot b \cdot e \leq \\
& \quad \text{Identity} \\
& \Box(a) \cdot b \leq \\
& \quad \text{Left weak idempotence} \\
& \Box(a) \cdot b \cdot \Box(a) \leq \\
& \quad \text{Unitality} \\
& e \cdot b \cdot \Box(a) \leq \\
& \quad \text{Identity} \\
& b \cdot \Box(a)
\end{aligned}$$

Hence, $b \cdot \Box(a) = \Box(a) \cdot b$, so $\forall a \in A, \Box(a) \in \mathcal{Z}(\mathcal{Q})$. \square

Proposition 1.

Let \mathcal{Q} be a quantale and $S \subseteq \mathcal{Q}$ a subquantale, then $\Box : \mathcal{Q} \rightarrow \mathcal{Q}$, such that $\Box(a) = \bigvee \{s \in S \mid s \leq a\}$, is an open modality.

Proof. See \square

Proposition 2.

Let \mathcal{Q} be a quantale and $S_1 \subseteq S_2 \subseteq \mathcal{Q}$.

Then $\Box_1(a) \leq \Box_2(a)$.

Proof. Immediatly. \square

Proposition 3.

Let \mathcal{Q} be a quantale and $S \subseteq \mathcal{Q}$ a subquantale, then the following operations are open modalities:

1. $\Box_z(a) = \bigvee \{s \in S \mid s \leq a, s \in \mathcal{Z}(\mathcal{Q})\};$
2. $\Box_1(a) = \bigvee \{s \in S \mid s \leq a, s \leq 1\};$
3. $\Box_{idem}(a) = \bigvee \{s \in S \mid s \leq a, \forall b \in \mathcal{Q}, b \cdot s \vee s \cdot b \leq s \cdot b \cdot s\};$
4. $\Box_{z,1}, I_{z,idem}, I_{1,idem}, I_{z,1,idem}.$

Proof. Immediately. □

Proposition 4.

1. $\forall a \in \mathcal{Q}, \Box_{1,idem}(a) \leq \Box_z(a).$
2. $\forall a \in \mathcal{Q}, \Box_{z,1,idem} = \Box_{1,idem}(a)$

Proof. Follows from Lemma 3. □

Lemma 3. $\Box_1(a_1) \cdot \Box_2(a_2) \leq \Box'_i(\Box_1(a_1) \cdot \Box_2(a_2)),$ where $\Box_i \leq \Box'_i, i = 1, 2.$

Proof.

$$\begin{aligned}
& \Box_1(a_1) \cdot \Box_2(a_2) \leq \\
& \Box_1(\Box_1(a_1)) \cdot \Box_2(\Box_2(a_2)) \leq \\
& \Box'_i(\Box_1(a_1)) \cdot \Box'_i(\Box_2(a_2)) \leq \\
& \Box'_i(\Box_1(a_1) \cdot \Box_2(a_2))
\end{aligned}$$
□

Definition 9. *Interpretation of subexponential signature*

Let $\Sigma = \langle I, \leq, W, C, E \rangle$ be a subexponential signature, where $|I| = n$ and $\Box_{\mathcal{Q}}$ is a category of open modalities on a quantale \mathcal{Q} . Subexponential interpretation is a contravariant functor $\sigma : I \rightarrow \Box_{\mathcal{Q}}$ defined as follows:

$$\sigma(s_i) = \begin{cases} \Box_i : \mathcal{Q} \rightarrow \mathcal{Q}, s.t. \forall a \in \mathcal{Q}, \Box_i(a) = \{s \in S_i \mid s \leq a\}, \\ \quad \text{if } s_i \notin W \cap C \cap E \\ \Box_i : \mathcal{Q} \rightarrow \mathcal{Q}, s.t. \forall a \in \mathcal{Q}, \Box_i(a) = \{s \in S_i \mid s \leq a, s \leq 1\}, \\ \quad \text{if } s_i \in W \\ \Box_i : \mathcal{Q} \rightarrow \mathcal{Q}, s.t. \forall a \in \mathcal{Q}, \Box_i(a) = \{s \in S_i \mid s \leq a, s \in \mathcal{Z}(\mathcal{Q})\}, \\ \quad \text{if } s_i \in E \\ \Box_i : \mathcal{Q} \rightarrow \mathcal{Q}, s.t. \forall a \in \mathcal{Q}, \Box_i(a) = \{s \in S_i \mid s \leq a, \forall b, b \cdot s \vee s \cdot b \leq s \cdot b \cdot s\}, \\ \quad \text{if } s_i \in E \\ \text{otherwise, if } s_i \text{ belongs to some intersection of subsets, then we combine the relevant conditions} \end{cases}$$

Definition 10. Let \mathcal{Q} be an unital quantale, $f : Tp \rightarrow \mathcal{Q}$ a valuation and $\sigma : I \rightarrow \Box_{\mathcal{Q}}$ a subexponential interpretation, then interpretation is defined inductively:

$$\begin{aligned}
\llbracket p_i \rrbracket &= f(p_i) \\
\llbracket 1 \rrbracket &= e \\
\llbracket A \bullet B \rrbracket &= \llbracket A \rrbracket \cdot \llbracket B \rrbracket \\
\llbracket A \setminus B \rrbracket &= \llbracket A \rrbracket \setminus \llbracket B \rrbracket \\
\llbracket A / B \rrbracket &= \llbracket A \rrbracket / \llbracket B \rrbracket \\
\llbracket A \& B \rrbracket &= \llbracket A \rrbracket \wedge \llbracket B \rrbracket \\
\llbracket A \vee B \rrbracket &= \llbracket A \rrbracket \vee \llbracket B \rrbracket \\
\llbracket !_{s_i} A \rrbracket &= \sigma(s_i) \llbracket A \rrbracket
\end{aligned}$$

Definition 11. $\Gamma \models A \Leftrightarrow \forall f, \forall \sigma, \llbracket \Gamma \rrbracket \leq \llbracket A \rrbracket$

Theorem 2. $\Gamma \rightarrow A \Rightarrow \llbracket \Gamma \rrbracket \leq \llbracket A \rrbracket$

Proof. We consider the promotion case, the rest modal cases are immediatly shown.

Let $!_{s_1}A_1, \dots, !_{s_n}A_n \rightarrow A$ and $\forall i, s \leq s_i$.

Then $\forall a \in Q, \sigma(s_i)(a) \leq \sigma(s)(a)$.

By IH, $\sigma(s_1)\llbracket A_1 \rrbracket \cdots \sigma(s_n)\llbracket A_n \rrbracket \leq \llbracket A \rrbracket$.

Thus, $\sigma(s)(\sigma(s_1)\llbracket A_1 \rrbracket \cdots \sigma(s_n)\llbracket A_n \rrbracket) \leq \sigma(s)(\llbracket A \rrbracket)$.

By Lemma 5, $\sigma(s_1)\llbracket A_1 \rrbracket \cdots \sigma(s_n)\llbracket A_n \rrbracket \leq \sigma(s)(\sigma(s_1)\llbracket A_1 \rrbracket \cdots \sigma(s_n)\llbracket A_n \rrbracket)$.

So, $\sigma(s_1)\llbracket A_1 \rrbracket \cdots \sigma(s_n)\llbracket A_n \rrbracket \leq \sigma(s)(\llbracket A \rrbracket)$. □

3 Quantale completeness

Definition 12.

Let $\mathcal{F} \subseteq Fm$, an ideal is a subset $\mathcal{I} \subseteq \mathcal{F}$, such that:

- If $B \in \mathcal{I}$ and $A \rightarrow B$, then $A \in \mathcal{I}$;
- If $A, B \in \mathcal{I}$, then $A \vee B \in \mathcal{I}$.

Definition 13.

Let $S \subseteq \mathcal{F} \subseteq Fm$, then $\bigvee S = \bigcap \{\mathcal{I} \subseteq \mathcal{F} \mid S \subseteq \mathcal{I}\}$

Proposition 5. $\bigvee S$ is an ideal.

Lemma 4. $A \subseteq Fm$, then $\{B \mid B \rightarrow A\} = \bigvee \{A\}$.

Proof.

Let $A \in Fm$. Then $\{B \mid B \rightarrow A', A' \in A\} \subseteq \bigvee \{A\}$, so far as $\bigvee A$ is an ideal.

On the other hand, $\{B \mid B \rightarrow A\}$ is an ideal, it is easy to see that this set is closed under \vee .
So, $\bigvee A \subseteq \{B \mid B \rightarrow A\}$. □

Lemma 5. $\bigvee \{A\} \subseteq \bigvee \{B\}$ iff $A \rightarrow B$.

Proof. Let $\bigvee \{A\} \subseteq \bigvee \{B\}$, then $\{C \mid C \rightarrow A\} \subseteq \{D \mid D \rightarrow B\}$.

Thus, $A \in \{C \mid C \rightarrow A\}$, then $A \in \{D \mid D \rightarrow B\}$, hence $A \rightarrow B$.

On the other hand, let $A \rightarrow B$ and $C \in \bigvee \{A\}$.

Thus, $C \rightarrow A$, then $C \rightarrow B$ by cut. □

Lemma 6. Let $\mathcal{Q} = \{\bigvee S \mid S \subseteq Fm\}$ and $\bigvee \mathcal{A} \cdot \bigvee \mathcal{B} = \bigvee \{A \bullet B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$. Then $\langle \mathcal{Q}, \subseteq, \cdot, \bigvee 1 \rangle$ is a quantale.

Proof. See □

Lemma 7. Let $!_s \in I$ and A be an arbitrary formula, then $\Box_s(\bigvee \{A\}) = \bigvee \{B \mid !_s B \rightarrow A\}$ is a quantic conucleus.

Proof.

See Yetter. □

Lemma 8. Let A be a formula, then $\Box_s \bigvee \{A\} = \bigvee \{!_s A\}$, for each $s \in I$.

Proof. Let $A \in Fm$ and $s \in \mathcal{I}$.

Let $!_s B \in \Box_s \bigvee \{A\}$, then $!_s B \rightarrow A$, then $!_s B \rightarrow !_s A$ by promotion. So, $!_s B \in \bigvee \{!_s A\}$.

Let $C \in \bigvee \{!_s A\}$, then $C \rightarrow !_s A$, so $!_s C \rightarrow !_s A$ by dereliction, but $!_s A \rightarrow A$, hence $!_s C \rightarrow A$ by cut. So, $!_s C \in \Box_s \bigvee \{A\}$. \square

Lemma 9.

Let $i, j \in I$ and $i \leq j$, then for all $A \in Fm$, $\Box_j(\bigvee \{A\}) \subseteq \Box_i(\bigvee \{A\})$.

Proof.

Let $i, j \in I$ and $i \leq j$. Then for all $A \in Fm$, $!_j A \rightarrow !_i A$ by promotion. Then $\bigvee \{!_j A\} \subseteq \bigvee \{!_i A\}$, so $\Box_j(\bigvee \{A\}) \subseteq \Box_i(\bigvee \{A\})$. \square

Lemma 10.

1. Let $s \in W$, then for all $A \subseteq Fm$, $\Box_s \{A\} \subseteq \{1\}$;
2. Let $s \in E$, then $\Box_s(\bigvee \{A\}) \cdot \bigvee \{B\} = \bigvee \{B\} \cdot \Box_s(\bigvee \{A\})$.
3. Let $s \in C$, then $(\Box_s \bigvee A \cdot \bigvee B) \cup (\bigvee B \cdot \Box_s \bigvee A) \subseteq \Box_s \bigvee A \cdot \bigvee B \cdot \Box_s \bigvee A$, for all $B \subseteq Fm$.

Proof.

Follows from $!_s A \rightarrow 1$, so $s \in W$;

Follows from $!_s A \bullet B \leftrightarrow B \bullet !_s A$;

Follows from $!_s A \bullet B \rightarrow !_s A \bullet B \bullet !_s A$ and similarly for $B \bullet !_s A$. \square

Definition 14.

Let Q be a syntactic quantale as proposed above and $\mathcal{I} = \langle I, \leq, W, C, E \rangle$ be a subexponential signature.

We define a map $\Box : \mathcal{I} \rightarrow Mod_Q$ as follows:

$$\Box(i)(\bigvee \{A\}) = \{!_i B \mid !_i B \rightarrow A\}.$$

Lemma 11. \Box is a subexponential interpretation.

Proof. Follows from lemmas above. \square

Lemma 12.

Let Q be a quantale constructed above and \Box_1, \dots, \Box_n be a family of quantic conuclei on Q . Then there exist a model $\langle Q, \llbracket \cdot \rrbracket \rangle$, such that $\llbracket A \rrbracket = \bigvee \{A\}$, $A \in Fm$.

Proof.

We define an interpretation as follows:

1. $\llbracket p_i \rrbracket = \bigvee \{p_i\}$
2. $\llbracket 1 \rrbracket = \bigvee \{1\}$
3. $\llbracket A \bullet B \rrbracket = \bigvee \{A \bullet B\}$
4. $\llbracket A/B \rrbracket = \bigvee \{A/B\}$
5. $\llbracket B \setminus A \rrbracket = \bigvee \{B \setminus A\}$
6. $\llbracket A \& B \rrbracket = \bigvee \{A \& B\}$
7. $\llbracket A \vee B \rrbracket = \bigvee \{A \vee B\}$

$$8. \llbracket !_s A \rrbracket = \Box(s)(\bigvee \{A\}) = \bigvee \{!_s A\}.$$

□

Theorem 3. $\Gamma \models A \Rightarrow \Gamma \rightarrow A$.

Proof. Follows from lemmas above.

□

4 Relational semantics

Definition 15.

Let A be a set. Then relational quantale on A is a tuple $\mathcal{Q} = \langle \mathcal{A}, \mathcal{I} \rangle$ where $\mathcal{A} \subseteq 2^{A \times A}$:

- $\langle \mathcal{A}, \subseteq \rangle$ is a complete semi-lattice;
- Multiplication is defined as $R \circ S = \{ \langle a, c \rangle \mid \exists b \in A, \langle a, b \rangle \in R \text{ and } \langle b, c \rangle \in S \}$
- $\langle \mathcal{A}, \circ, \mathcal{I} \rangle$ is a monoid;
- For each indexing set J , $R \circ \bigvee_{j \in J} S_j = \bigvee_{j \in J} (R \circ S_j)$ and $\bigvee_{j \in J} R_j \circ S = \bigvee_{j \in J} (R_j \circ S)$.

Theorem 4. Every unital quantale $\mathcal{Q} = \langle A, \leq, \cdot, \bigvee \rangle$ with quantic conucleus \Box is isomorphic to some relation quantale $\hat{\Box}$

Proof. Consider a relational quantale by Brown and Gurr $\hat{\mathcal{Q}} = \langle \mathcal{R}, \subseteq, \circ, \bigvee \rangle$ where:

1. $\hat{a} = \{ \langle b, c \rangle \mid b \cdot a \leq c \}$;
2. $\widehat{a \cdot b} = \hat{a} \circ \hat{b}$;
3. $\widehat{\bigvee a} = \bigvee \hat{a}$.

Consider $\hat{\Box} \hat{a} = \bigvee$

1. $\hat{\Box} \hat{a} \subseteq \hat{a}$;
2. $\hat{\Box} \hat{a} \subseteq \hat{\Box} \hat{\Box} \hat{a}$;
3. $\hat{a} \subseteq \hat{b} \Rightarrow \hat{\Box} \hat{a} \subseteq \hat{\Box} \hat{b}$;
4. $\hat{\Box} \hat{a} \circ \hat{\Box} \hat{b} \subseteq \hat{\Box}(\hat{a} \circ \hat{b})$;
5. $\hat{\Box} \hat{\varepsilon} = \hat{\varepsilon}$.

Let us show that $\hat{\Box} \hat{a} = \widehat{\Box a} = \{ \langle b, c \rangle \mid b \cdot \Box a \leq c \}$

□

Definition 16. Let W be a set. A Kripke frame is a pair $\mathcal{W} = \langle W, R, I \rangle$, where R is a ternary relation on W and I is an unary relation on W .

A Kripke model is a triple $\mathcal{M} = \langle \mathcal{W}, v \rangle$, where \mathcal{W} is a Kripke frame and $v : Tp \rightarrow 2^W$.

A forcing relation is defined as follows:

1. $\mathcal{W}, w \Vdash p_i \Leftrightarrow w \in v(p_i)$;
2. $\mathcal{W}, w \Vdash \mathbf{1} \Leftrightarrow I(w)$;
3. $\mathcal{W}, w \Vdash A \bullet B \Leftrightarrow \exists u, v \in W, R(u, v, w) \text{ and } \mathcal{W}, u \Vdash A \text{ and } \mathcal{W}, v \Vdash B$;

4. $\mathcal{W}, w \Vdash A \setminus B \Leftrightarrow \forall u, v \in W, R(u, w, v) \text{ and } \mathcal{W}, u \Vdash A \text{ implies } \mathcal{W}, v \Vdash B$;
5. $\mathcal{W}, w \Vdash B/A \Leftrightarrow \forall u, v \in W, R(w, u, v) \text{ and } \mathcal{W}, u \Vdash A \text{ implies } \mathcal{W}, v \Vdash B$;
6. $\mathcal{W}, w \Vdash A \vee B \Leftrightarrow \forall u, v \in W, R(u, v, w) \text{ either } \mathcal{W}, u \Vdash A \text{ or } \mathcal{W}, v \Vdash B$;
7. $\mathcal{W}, w \Vdash A \wedge B \Leftrightarrow \mathcal{W}, w \Vdash A \text{ and } \mathcal{W}, w \Vdash B$;
8. $\mathcal{W}, w \Vdash \Gamma \rightarrow A \Leftrightarrow \mathcal{W}, w \Vdash \bullet\Gamma \text{ implies } \mathcal{W}, w \Vdash A$.

Theorem 5. *Soundness*

Let \mathbb{F} be a class of Kripke frames, then $\text{Log}(\mathbb{F}) = L_1^* \wedge \vee$.

Proof.

1. Let $\Gamma \rightarrow A$. By IH, for each $\mathcal{W}, w \Vdash \Gamma \rightarrow A$.
2. Let $\Gamma, A, \Delta \rightarrow C$ and $\Gamma, B, \Delta \rightarrow C$,
3. Let $\Gamma \rightarrow A$ and $\Gamma \rightarrow B$
4. Let $\Gamma \rightarrow A_1 \wedge A_2$, so $\mathcal{W}, w \Vdash \Gamma \rightarrow A_i, i = 1, 2$.

□

Theorem 6. *Strong completeness*

$\Gamma \models A \text{ implies that } L_1^* \wedge \vee \vdash \Gamma \rightarrow A$

Definition 17. By $L_1^* \wedge \vee_{\mathbf{S4}}$ we mean SMALC_Σ , where $\Sigma = \{s\}$ and $C = W = E = \emptyset$.