

On R -models

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Definition 1.

$$\begin{array}{c}
 \overline{A \rightarrow A} \text{ }^{ax} \\
 \\
 \frac{\Gamma \rightarrow A \quad \Delta, B, \Theta \rightarrow C}{\Delta, \Gamma, A \setminus B, \Theta \rightarrow C} \setminus \rightarrow \qquad \frac{A, \Pi \rightarrow B}{\Pi \rightarrow A \setminus B} \rightarrow \setminus \\
 \\
 \frac{\Gamma \rightarrow A \quad \Delta, B, \Theta \rightarrow C}{\Delta, B/A, \Gamma, \Theta \rightarrow C} / \rightarrow \qquad \frac{\Pi, A \rightarrow B}{\Pi \rightarrow B/A} \rightarrow / \\
 \\
 \frac{\Gamma, A, B, \Delta \rightarrow C}{\Gamma, A \bullet B, \Delta \rightarrow C} \bullet \rightarrow \qquad \frac{\Gamma \rightarrow A \quad \Delta \rightarrow B}{\Gamma, \Delta \rightarrow A \bullet B} \rightarrow \bullet
 \end{array}$$

Definition 2.

R -model is a pair $\mathcal{M} = \langle W, R, v \rangle$, where R is a transitive relation on W and $v : Tp \rightarrow 2^R$ is a valuation, such that:

1. $\mathcal{M}, w \models p_i \Leftrightarrow w \in v(p_i)$;
2. $\mathcal{M}, \langle a, b \rangle \models A \bullet B \Leftrightarrow$ there exists $c \in W$, $\mathcal{M}, a \models A$ and $\mathcal{M}, b \models B$;
3. $\mathcal{M}, \langle a, b \rangle \models A \setminus B \Leftrightarrow$ for all $c \in R^{-1}(a)$, $\mathcal{M}, \langle c, a \rangle \models A$ implies $\mathcal{M}, \langle c, b \rangle \models B$;
4. $\mathcal{M}, \langle a, b \rangle \models B/A \Leftrightarrow$ for all $c \in R(a)$, $\mathcal{M}, \langle a, c \rangle \models A$ implies $\mathcal{M}, \langle b, c \rangle \models B$;
5. $\mathcal{M}, \langle a, b \rangle \models \Gamma \rightarrow A \Leftrightarrow \mathcal{M}, \langle a, b \rangle \models \Gamma$ implies $\mathcal{M}, \langle a, b \rangle \models A$

where $\mathcal{M}, \langle a, b \rangle \models \Gamma$ denotes $\mathcal{M}, \langle a, b \rangle \models A_1 \bullet \dots \bullet A_n$, or, equivalently, there exist c_1, \dots, c_{n-1} , such that $\mathcal{M}, \langle a, c_1 \rangle \models A_1, \mathcal{M}, \langle c_1, c_2 \rangle \models A_2, \dots, \mathcal{M}, \langle c_{n-1}, b \rangle \models A_n$ implies that $\mathcal{M}, \langle a, b \rangle \models B$.

Theorem 1. Let \mathbb{F} be a R -frame, then $\mathbb{F} \models L$.

Definition 3.

Let $\mathcal{F}_1, \mathcal{F}_2$ be R -frames and $\mathcal{M}_1 = \langle \mathcal{F}_1, v_1 \rangle, \mathcal{M}_2 = \langle \mathcal{F}_2, v_2 \rangle$ be R -models.

A map $f : W_1 \rightarrow W_2$ is said to be a R -frame p -morphism if the following conditions hold:

1. f is onto;
2. $\forall a, b \in W_1 (aR_1b \Rightarrow f(a)R_2f(b))$ (monotonicity);

3. $\forall d \in W_1, c \in W_2, f(d)R_2c \Rightarrow \exists c' \in W_1, f(c') = c \ \& \ dR_1c'$ (left lift property);
4. $\forall d \in W_1, c \in W_2, cR_2f(d) \Rightarrow \exists c' \in W_1, f(c') = c \ \& \ c'R_1d$ (right lift property);

A map $f : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ is R -model p -moprhism, iff:

$$\mathcal{M}_1, \langle a, b \rangle \models p_i \Leftrightarrow \mathcal{M}_2, \langle f(a), f(b) \rangle \models p_i$$

Lemma 1. Let $f : \mathcal{M}_1 \rightarrow \mathcal{M}_2$, then $\mathcal{M}_1, \langle a, b \rangle \models A \Leftrightarrow \mathcal{M}_2, \langle f(a), f(b) \rangle \models A$, for all $a, b \in W_1$ and for all $A \in Fm$

Proof.

1. \Rightarrow

- (a) Basic case: follows from the definition.
- (b) Let $A = B \bullet C$ and $\mathcal{M}_1, \langle a, b \rangle \models B \bullet C$, then there exists $c \in W_1$, such that $\mathcal{M}_1, \langle a, c \rangle \models B$ and $\mathcal{M}_1, \langle c, b \rangle \models C$.
Then, aR_1c and cR_1b , so $f(a)R_2f(c)$ and $f(c)R_2f(b)$.
Thus, by IH, $\mathcal{M}_2, \langle f(a), f(c) \rangle \models B$ and $\mathcal{M}_2, \langle f(c), f(b) \rangle \models C$, so $\mathcal{M}_2, \langle f(a), f(b) \rangle \models B \bullet C$.
- (c) Let $A = B \setminus C$ and $\mathcal{M}_1, \langle a, b \rangle \models B \setminus C$. Let $c \in W_2$, such that $cR_2f(a)$.
Then, by left lift property, there exist $d \in W_1$, such that $f(d) = c$ and dR_1a .
Thus, $\mathcal{M}_1, \langle d, a \rangle \models A$ implies $\mathcal{M}_1, \langle d, b \rangle \models B$.
By IH, $\mathcal{M}_2, \langle c, f(a) \rangle \models A$ implies $\mathcal{M}_2, \langle c, f(b) \rangle \models B$, then $\mathcal{M}_2, \langle f(a), f(b) \rangle \models A \setminus B$.
- (d) Similarly to (c), but via right lift property.

2. \Leftarrow

- (a) Basic case: follows from the definition.
- (b) Let $A = B \bullet C$. Let $\mathcal{M}_2, \langle f(a), f(b) \rangle \models B \bullet C$. Then there exists $c \in W_2$, such that $\mathcal{M}_2, \langle f(a), c \rangle \models B$ and $\mathcal{M}_2, \langle c, f(b) \rangle \models C$.
So far as f is surjection, then there exists $d \in W_1$, such that $c = f(d)$, then $\mathcal{M}_2, \langle f(a), f(d) \rangle \models B$ and $\mathcal{M}_2, \langle f(d), f(b) \rangle \models C$, and, by IH, $\mathcal{M}_1, \langle a, d \rangle \models B$ and $\mathcal{M}_1, \langle d, b \rangle \models C$, then $\mathcal{M}_1, \langle a, b \rangle \models B \bullet C$.
- (c) Let $A = B \setminus C$ and $\mathcal{M}_2, \langle f(a), f(b) \rangle \models B \setminus C$.
Let $c \in W_1$ and cR_1a , then $f(c)R_1f(a)$ by monotonicity, so $\mathcal{M}_2, \langle f(c), f(a) \rangle \models A$ implies $\mathcal{M}_2, \langle f(c), f(b) \rangle \models B$.
By IH, $\mathcal{M}_1, \langle c, a \rangle \models A$ implies $\mathcal{M}_1, \langle c, b \rangle \models B$. Thus, $\mathcal{M}_1, \langle c, a \rangle \models A \setminus B$.
- (d) Similarly to (c).

□

Lemma 2.

1. Let \mathcal{M}_1 and \mathcal{M}_2 be R -models and $f : \mathcal{M}_1 \rightarrow \mathcal{M}_2$. Then $\mathcal{M}_1 \models A$ iff $\mathcal{M}_2 \models A$.
2. Let \mathcal{F}_1 and \mathcal{F}_2 be R -frames and $f : \mathcal{F}_1 \rightarrow \mathcal{F}_2$, then $\mathcal{F}_1 \models A$ implies $\mathcal{F}_2 \models A$.

Proof.

1.
 - Only if:
Let $\mathcal{M}_1 \models A$. Let $c, d \in W_2$. So far as f is onto, then there exists $a, b \in W_1$, such that $f(a) = c$ and $f(b) = d$.
Then $\mathcal{M}_1, \langle a, b \rangle \models A$, thus $\mathcal{M}_2, \langle f(a), f(b) \rangle \models A$. That is, $\mathcal{M}_2, \langle c, d \rangle \models A$
 - If:
Follows from the previous lemma.
2. Let $f : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ and $\mathcal{F}_1 \models A$.
Let $\mathcal{M}_1 = \langle \mathcal{F}_1, v_1 \rangle$ and $\mathcal{M}_2 = \langle \mathcal{F}_2, v_2 \rangle$, such that for all $p \in Tp$, $\mathcal{M}_1, \langle a, b \rangle \models p \Leftrightarrow_{def} \mathcal{M}_1, \langle f(a), f(b) \rangle \models p$. Thus, $\mathcal{M}_1 \rightarrow \mathcal{M}_2$ and $\mathcal{M}_1 \models A$. Thus, $\mathcal{M}_2 \models A$.

□

Definition 4.

Definition 5.

1. Let $\mathcal{F} = \langle W, R \rangle$ be a transitive frame and $V \subseteq W$, such that $V \neq \emptyset$ and V is downward and upward closed with respect to R . Then a frame $\mathcal{F} \upharpoonright V = \langle V, R \cap V \times V \rangle$ is a generated subframe.
2. If, $\mathcal{M} = \langle W, R, \theta \rangle$ and $V \subseteq W$, such that $\mathcal{F} \upharpoonright V$ is a generated subframe. Then $\mathcal{M} \upharpoonright V = \langle V, R \cap V \times V, \theta' \rangle$ is a generated submodel, where $\theta'(p) = \theta(p) \cap V$.

Lemma 3. Let $\mathcal{M} = \langle W, R, \theta \rangle$ and V is R -closed subset of W , then:

1. If $a, b \in V$, then $\mathcal{M}, \langle a, b \rangle \models A$ iff $\mathcal{M} \upharpoonright V, \langle a, b \rangle \models A$;
2. $\text{Log}(\mathcal{F}) \subseteq \text{Log}(\mathcal{F} \upharpoonright V)$

Proof.

1. (a) Let $a, b \in V$ and $\langle a, b \rangle \in \theta(p)$, then $\mathcal{M} \upharpoonright V, \langle a, b \rangle \models p$, for $p \in Tp$.
- (b) Let $A = B \bullet C$ and $\mathcal{M}, \langle a, b \rangle \models B \bullet C$, then there exists $c \in W$, such that aRc and $\mathcal{M}, \langle a, c \rangle \models B$, cRb and $\mathcal{M}, \langle c, b \rangle \models C$. So far as V is upwardly and downwardly closed, then $c \in V$. Then, by IH, $\mathcal{M} \upharpoonright V, \langle a, c \rangle \models B$, cRb and $\mathcal{M} \upharpoonright V, \langle c, b \rangle \models C$. So $\mathcal{M} \upharpoonright V, \langle a, b \rangle \models B \bullet C$.
- (c) Let $A = B \setminus C$ and $\mathcal{M}, \langle a, b \rangle \models B \setminus C$. Then for all $c \in R^{-1}(a)$, $\mathcal{M}, \langle c, a \rangle \models B$ implies $\mathcal{M}, \langle c, b \rangle \models C$. Let $\mathcal{M}, \langle c, a \rangle \models B$. By IH, $\mathcal{M} \upharpoonright V, \langle c, a \rangle \models B$.
 $c \in V$, because V is R -closed upwardly and downwardly. On the other hand, cRb , so $b \in V$. Hence, $\langle c, b \rangle \in \mathcal{F} \upharpoonright V$. By IH, $\mathcal{M} \upharpoonright V, \langle c, b \rangle \models C$. Hence, $\mathcal{M} \upharpoonright V, \langle a, b \rangle \models A \setminus B$.

2. Let $\mathcal{F} \upharpoonright V \models \Gamma \rightarrow A$. Then there exist a valuation v and $\langle a, b \rangle \in R \cap V \times V$, such that $\mathcal{M} \upharpoonright V, \langle a, b \rangle \models \Gamma$ and $\mathcal{M} \upharpoonright V, \langle a, b \rangle \not\models A$.

Let us define a valuation v' for the frame \mathcal{F} , such that $v(p) = v'(p)$. Then, by the first part, $\mathcal{M}, \langle a, b \rangle \models \Gamma$ and $\mathcal{M}, \langle a, b \rangle \not\models A$. So, $\mathcal{F} \not\models \Gamma \rightarrow A$.

□

Corollary 1.

Let \mathcal{F} be a frame and $\mathcal{F} \upharpoonright V$ is a generated subframe of \mathcal{F} . Then $\text{Log}(\mathcal{F}) = \text{Log}(\mathcal{F} \upharpoonright V)$.

Let \mathbb{Z}^* be the set of all finite sequences of integers. Let us define the following relation R :

1. $\alpha R \alpha n$, for $n \geq 0$;
2. $\alpha n R \alpha$, for $n < 0$.