# Categorical model of noncommutative linear logic with subexponentials

 $\textbf{Definition 1.} \ \textit{A subexponential signature is an ordered quintuple:}$ 

$$\Sigma = \langle I, \leq, W, C, E \rangle,$$

where  $I = \{s_1, \ldots, s_n\}, \langle I, \leq \rangle$  is a preorder. W, C, E are subsets of I and  $W \cup C \subseteq E$ .

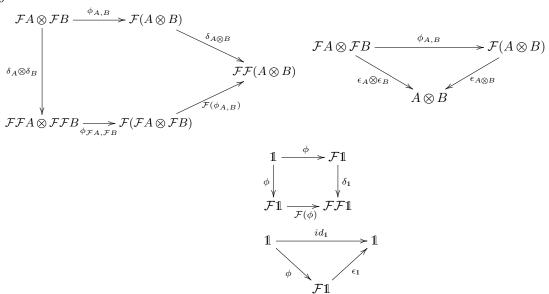
**Definition 2.** Noncommutative linear logic with subexponentials  $(SMALC_{\Sigma})$ , where  $\Sigma$  is a subexponential signature.

$$\frac{\Gamma, \Delta, !^{s}A, \Theta \Rightarrow B}{\Gamma, !^{s}A, \Delta, \Theta \Rightarrow A} \mathbf{ex}_{1}, s \in E$$

$$\frac{\Gamma, !^s A, \Delta, \Theta \Rightarrow B}{\Gamma, \Delta, !^s A, \Theta \Rightarrow A} \mathbf{ex}_1, s \in E$$

#### **Definition 3.** Monoidal comonad

A monoidal comonad on some monoidal category C is a triple  $\langle \mathcal{F}, \epsilon, \delta \rangle$ , where  $\mathcal{F}$  is a monoidal endofunctor and  $\epsilon : \mathcal{F} \Rightarrow Id_{\mathcal{C}}$  (counit) and  $\epsilon : \mathcal{F} \Rightarrow \mathcal{F}^2$  (comultiplication), such that the following diagrams commute:



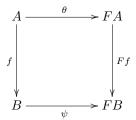
### **Definition 4.** Biclosed monoidal category

Let C be a monoidal category. Biclosed monoidal category is a monoidal category with the following additional data:

- 1. Bifunctors  $\_ \circ \_ , \_ \multimap \_ : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C};$
- 2. Natural isomophism  $\mathbf{curry}_{A,B,C} : Hom(A \otimes B, C) \cong (B, A \multimap C);$
- 3. Natural isomorphism  $\mathbf{curry}'_{A,B,C} : Hom(A \otimes B, C) \cong (A, C \multimap B);$
- 4. For each  $A, B \in Ob_{\mathcal{C}}$ , there are exist arrows  $ev_{A,B} : A \otimes (A \Rightarrow B) \to B$  and  $ev'_{A,B} : (B \Leftarrow A) \otimes A \to B$ , such that for all  $f : A \otimes C \to B$ :
  - (a)  $\Lambda_l \circ (id_A \otimes \mathbf{curry}(f)) = f$ ;
  - (b)  $\Lambda_r \circ (\mathbf{curry}'(f) \otimes id_A) = f$

**Definition 5.** Let F be endofunctor and  $A \in Ob\mathcal{C}$ , then a coalgebra of F is a tuple  $\langle A, \theta \rangle$ , where  $\theta : A \to FA$ .

Given coalgebras  $\langle A, \theta \rangle$  and  $\langle A, \psi \rangle$ , a homomorphism is a morphism  $f: A \to B$ , s.t. the diagram below commutes:



that is,  $Ff \circ \theta = \psi \circ f$ 

## **Definition 6.** Subexponential model structure

Let  $\Sigma = \langle I, \leq, W, C, E \rangle$  be a subexponential model structure and  $\mathcal{C}$  be a biclosed monoidal category, then a subexponential model structure is  $\langle \mathcal{C}, \{\mathcal{F}_s\}_{s \in I} \rangle$  with the following additional data:

• for all  $s \in I$ ,  $\mathcal{F}_s$  is a monoidal comonad;

• if  $s \in W$ , then for all  $A \in Ob(\mathcal{C})$ , there exists a morphism  $w_{A_s} : F_s A \to 1$ ;

• if  $s \in C$ , then for all  $A \in Ob(C)$ , there exists morphisms  $w_{Al} : F_sA \otimes B \to F_sA \otimes A \otimes F_sA$ and  $w_{Ar} : B \otimes F_sA \to F_sA \otimes A \otimes F_sA$ ;

• if  $s \in E$ , then for all  $A \in Ob(\mathcal{C})$ , there is an isomorpism,  $e_A : F_sA \otimes B \cong B \otimes F_sA$ ;

• if  $s_1 \in W$ ,  $s_2 \in I$  and  $s_1 \leq s_2$ , then there is a morphism  $w_{As_2} : F_{s_2}A \to \mathbb{1}$  for all  $A \in Ob(\mathcal{C})$  and ditto for E and C;

• Let  $\bigotimes_{s\in J,i=0}^n F_s A$ , where  $J\subset I$ , and  $s'\in I$ , s.t.  $s\geq s'$  for all  $s\in I'$ ; Then there exists morphism a morphism  $\theta_{\bigotimes_{s\in J,i=1}^n F_{sj}A_i}:\bigotimes_{s\in J,i=0}^n F_s A\to F_{s'}(\bigotimes_{s\in J,i=0}^n F_s A)$ , such that  $\bigotimes_{s\in J,i=1}^n F_{sj}A_i,\theta_{\bigotimes_{s\in J,i=1}^n F_{sj}A_i}$  is a coalgebra on  $F_s$ .

**Theorem 1.** The following statements are equivalent:

- $SMLC_{\Sigma} + (cut) \vdash \Gamma \Rightarrow A$
- $SMLC_{\Sigma} \vdash \Gamma \Rightarrow A$
- $\exists f, f : \llbracket \Gamma \rrbracket \to \llbracket A \rrbracket$

Proof.

 $(1) \Rightarrow (2)$ : cut elimination.

•  $(2) \Rightarrow (3)$ : Soundness:

$$id_A:A\to A$$

$$\frac{f:\Gamma\to A \qquad g:\Delta\otimes B\otimes\Theta\to C}{g\circ (id_\Delta\otimes (ev_{A,B_l}\circ (f\otimes id_{A\multimap B}))\otimes id_\Theta):\Delta\otimes (\Gamma\otimes A\multimap B)\otimes\Theta\to C}$$

$$\frac{f:A\otimes\Pi\to B}{\Lambda_l(f):\Pi\to A\multimap B}$$

$$\begin{split} f: \Gamma \to A & g: \Delta \otimes B \otimes \Theta \to C \\ \hline g \circ (id_{\Delta} \otimes (ev_{A,B_{l}} \circ (id_{B \circ -A} \otimes f)) \otimes id_{\Theta}): \Delta \otimes (B \circ -A \otimes \Gamma) \otimes \Theta \to C \\ \hline & \frac{f: \Pi \otimes A \to B}{\Lambda_{r}(f): \Pi \to B \circ -A} \\ \hline & \frac{f: \Gamma \otimes A \otimes B \otimes \Delta \to C}{f \circ (\alpha_{\Gamma,A,B} \otimes id_{\Delta}): \Gamma \otimes (A \otimes B) \otimes \Delta \to C} \\ \hline & \frac{f: \Gamma \to A \quad g: \Delta \to B}{f \otimes g: \Gamma \otimes \Delta \to A \otimes B} \\ \hline & \frac{f: \Gamma \otimes A_{i} \otimes \Delta \to B}{f \circ (id_{\Gamma} \otimes \pi_{i} id_{\Delta}): \Gamma \otimes (A_{1} \times A_{2}) \otimes \Delta \to B} \\ \hline & \frac{f: \Gamma \to A \quad g: \Gamma \to B}{\langle f, g \rangle: \Gamma \to A \times B} \\ \hline & \frac{f: \Gamma \to A \quad g: \Gamma \to B}{\langle f, g \rangle: \Gamma \to A \times B} \\ \hline & \frac{f: \Gamma \otimes A \otimes \Delta \to C \quad g: \Gamma \otimes B \otimes \Delta \to C}{id_{\Gamma} \otimes [f, g] \otimes id_{\Delta}: \Gamma \otimes (A + B) \otimes \Delta \to C} \\ \hline & \frac{id_{1}: \mathbb{1} \to \mathbb{1}}{} \\ \hline & \frac{f: \Gamma \otimes A \otimes \Delta \to B}{f \circ (id_{\Gamma} \otimes \delta_{s}^{A} \otimes id_{\Delta}): \Gamma \otimes F_{s} A \otimes \Delta \to B} \\ \hline & \frac{f: \Gamma \otimes A \otimes \Delta \to B}{f \circ (id_{\Gamma} \otimes \delta_{s}^{A} \otimes id_{\Delta}): \Gamma \otimes F_{s} A \otimes \Delta \to B} \\ \hline & \frac{f: \Gamma \otimes A \otimes \Delta \to B}{f \circ (id_{\Gamma} \otimes \delta_{s}^{A} \otimes id_{\Delta}): \Gamma \otimes F_{s} A \otimes \Delta \to B} \\ \hline & \frac{f: \Gamma \otimes A \otimes \Delta \to B}{f \circ (id_{\Gamma} \otimes \delta_{s}^{A} \otimes id_{\Delta}): \Gamma \otimes F_{s} A \otimes \Delta \to B} \\ \hline & \frac{f: \Gamma \otimes A \otimes \Delta \to B}{F_{s}(f): F_{s}(F_{s}, A_{1} \otimes \cdots \otimes F_{s_{n}} A_{n} \to F_{s} B)} \\ \hline & F_{s}(f): \theta_{\otimes_{s=J,i=1}}^{\circ}, F_{s}, F_{s}, F_{s}, F_{s}, F_{s}, A_{n} \to F_{s} B} \\ \hline & F_{s}(f): \theta_{\otimes_{s=J,i=1}}^{\circ}, F_{s}, F_{$$

• Completeness:

## 1 Concrete model

**Definition 7.** Quantale