Categorical model of noncommutative linear logic with subexponentials

 $\textbf{Definition 1.} \ \textit{A subexponential signature is an ordered quintuple:}$

$$\Sigma = \langle I, \leq, W, C, E \rangle,$$

where $I = \{s_1, \ldots, s_n\}, \langle I, \leq \rangle$ is a preorder. W, C, E are subsets of I and $W \cup C \subseteq E$.

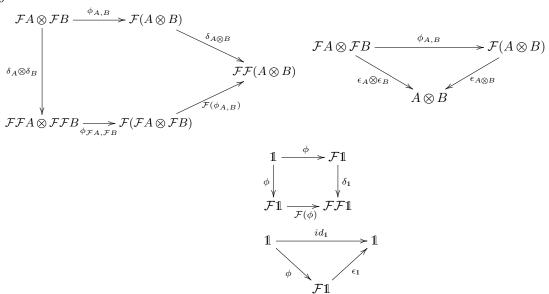
Definition 2. Noncommutative linear logic with subexponentials $(SMALC_{\Sigma})$, where Σ is a subexponential signature.

$$\frac{\Gamma, \Delta, !^{s}A, \Theta \Rightarrow B}{\Gamma, !^{s}A, \Delta, \Theta \Rightarrow A} \mathbf{ex}_{1}, s \in E$$

$$\frac{\Gamma, !^s A, \Delta, \Theta \Rightarrow B}{\Gamma, \Delta, !^s A, \Theta \Rightarrow A} \mathbf{ex}_1, s \in E$$

Definition 3. Monoidal comonad

A monoidal comonad on some monoidal category C is a triple $\langle \mathcal{F}, \epsilon, \delta \rangle$, where \mathcal{F} is a monoidal endofunctor and $\epsilon : \mathcal{F} \Rightarrow Id_{\mathcal{C}}$ (counit) and $\epsilon : \mathcal{F} \Rightarrow \mathcal{F}^2$ (comultiplication), such that the following diagrams commute:



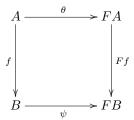
Definition 4. Biclosed monoidal category

Let C be a monoidal category. Biclosed monoidal category is a monoidal category with the following additional data:

- 1. Bifunctors $_ \circ _ , _ \multimap _ : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C};$
- 2. Natural isomophism $\mathbf{curry}_{A,B,C} : Hom(A \otimes B, C) \cong (B, A \multimap C);$
- 3. Natural isomorphism $\mathbf{curry}'_{A,B,C} : Hom(A \otimes B, C) \cong (A, C \multimap B);$
- 4. For each $A, B \in Ob_{\mathcal{C}}$, there are exist arrows $ev_{A,B} : A \otimes (A \Rightarrow B) \to B$ and $ev'_{A,B} : (B \Leftarrow A) \otimes A \to B$, such that for all $f : A \otimes C \to B$:
 - (a) $\Lambda_l \circ (id_A \otimes \mathbf{curry}(f)) = f$;
 - (b) $\Lambda_r \circ (\mathbf{curry}'(f) \otimes id_A) = f$

Definition 5. Let F be endofunctor and $A \in Ob\mathcal{C}$, then a coalgebra of F is a tuple $\langle A, \theta \rangle$, where $\theta : A \to FA$.

Given coalgebras $\langle A, \theta \rangle$ and $\langle A, \psi \rangle$, a homomorphism is a morphism $f: A \to B$, s.t. the diagram below commutes:



that is, $Ff \circ \theta = \psi \circ f$

Definition 6. Subexponential model structure

Let $\Sigma = \langle I, \leq, W, C, E \rangle$ be a subexponential model structure and C be a biclosed monoidal category, then a subexponential model structure is $\langle C, \{\mathcal{F}_s\}_{s \in I} \rangle$ with the following additional data:

- for all $s \in I$, \mathcal{F}_s is a monoidal comonad;
- if $s \in W$, then for all $A \in Ob(\mathcal{C})$, there exists a morphism $w_{As} : F_s A \to 1$;
- if $s \in C$, then for all $A \in Ob(C)$, there exists morphisms $w_{Al} : F_sA \otimes B \to F_sA \otimes A \otimes F_sA$ and $w_{Ar} : B \otimes F_sA \to F_sA \otimes A \otimes F_sA$;
- if $s \in E$, then for all $A \in Ob(\mathcal{C})$, there is an isomorpism, $e_A : F_sA \otimes B \cong B \otimes F_sA$;
- if $s_1 \in W$, $s_2 \in I$ and $s_1 \leq s_2$, then there is a morphism $w_{As_2} : F_{s_2}A \to \mathbb{1}$ for all $A \in Ob(\mathcal{C})$ and ditto for E and C;
- Let $\bigotimes_{s\in J, i=0}^n F_s A$, where $J\subset I$, and $s'\in I$, s.t. $s\geq s'$ for all $s\in I'$; Then there exists morphism a morphism $\theta:\bigotimes_{s\in J, i=0}^n F_s A\to F_{s'}(\bigotimes_{s\in J, i=0}^n F_s A)$, such that $\langle\bigotimes_{s\in J, i=0}^n F_s A, \theta\rangle$ is a coalgebra on F_s .

Theorem 1. The following statements are equivalent:

- $SMLC_{\Sigma} + (cut) \vdash \Gamma \Rightarrow A$
- $SMLC_{\Sigma} \vdash \Gamma \Rightarrow A$
- $\bullet \ \exists f, f: \llbracket \Gamma \rrbracket \to \llbracket A \rrbracket$

Proof. • $(1) \Rightarrow (2)$: cut elimination.

• $(2) \Rightarrow (3)$: Soundness:

$$id_A:A\to A$$

$$\frac{f:\Gamma\to A \qquad g:\Delta\otimes B\otimes\Theta\to C}{g\circ (id_\Delta\otimes (ev_{A,B_l}\circ (f\otimes id_{A\multimap B}))\otimes id_\Theta):\Delta\otimes (\Gamma\otimes A\multimap B)\otimes\Theta\to C}$$

$$\frac{f: A \otimes \Pi \to B}{\Lambda_l(f): \Pi \to A \multimap B}$$

$$\frac{f:\Gamma\to A \qquad g:\Delta\otimes B\otimes\Theta\to C}{g\circ (id_\Delta\otimes (ev_{A,B_l}\circ (id_{B\circ\!-A}\otimes f))\otimes id_\Theta):\Delta\otimes (B\circ\!-A\otimes\Gamma)\otimes\Theta\to C}$$

$$\frac{f: \Pi \otimes A \to B}{\Lambda_r(f): \Pi \to B \multimap A}$$

$$\frac{f: \Gamma \otimes A \otimes B \otimes \Delta \to C}{f \circ (\alpha_{\Gamma,A,B} \otimes id_{\Delta}): \Gamma \otimes (A \otimes B) \otimes \Delta \to C}$$

$$\frac{f: \Gamma \to A \qquad g: \Delta \to B}{f \otimes g: \Gamma \otimes \Delta \to A \otimes B}$$

$$id_1: 1 \to 1$$

$$\frac{f: \Gamma \otimes \Delta \to A}{f \circ (\rho_{\Gamma} \otimes id_{\Delta}): (\Gamma \otimes 1) \otimes \Delta \to A}$$

 \bullet Completeness:

1 Concrete model

 $\textbf{Definition 7.} \ \textit{Quantale}$