Models of Lambek calculus enriched with subexponentials

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Abstract

1 The Lambek Calculus with subexponentials

Definition 1. *A subexponential signature is an ordered quintuple:*
$\Sigma = \langle I, \preceq, W, C, E \rangle$,

where $I = \{s_1, \dots, s_n\}$, $\langle I, \preceq \rangle$ is a preorder. W, C, E are upwardly closed subsets of I and $W \cap C \subseteq E$.

Definition 2.
$\mathcal{F}_\Sigma ::= Tp \mid (\mathcal{F}_\Sigma / \mathcal{F}_\Sigma) \mid (\mathcal{F}_\Sigma \backslash \mathcal{F}_\Sigma) \mid (\mathcal{F}_\Sigma \bullet \mathcal{F}_\Sigma) \mid (\mathcal{F}_\Sigma \vee \mathcal{F}_\Sigma) \mid (\mathcal{F}_\Sigma \wedge \mathcal{F}_\Sigma) \mid !_s \mathcal{F}_\Sigma$

Definition 3. *Noncommutative linear logic with subexponentials ($SMALC_\Sigma$), where Σ is a subexponential signature.*

$$\frac{}{A \to A}\ ax$$

$$\frac{\Gamma \to A \qquad \Delta, B, \Theta \to C}{\Delta, \Gamma, A \backslash B, \Theta \to C}\ \backslash \to \qquad\qquad \frac{A, \Pi \to B}{\Pi \to A \backslash B}\ \to \backslash$$

$$\frac{\Gamma \to A \qquad \Delta, B, \Theta \to C}{\Delta, B / A, \Gamma, \Theta \to C}\ / \to \qquad\qquad \frac{\Pi, A \to B}{\Pi \to B / A}\ \to /$$

$$\frac{\Gamma, A, B, \Delta \to C}{\Gamma, A \bullet B, \Delta \to C}\ \bullet \to \qquad\qquad \frac{\Gamma \to A \qquad \Delta \to B}{\Gamma, \Delta \to A \bullet B}\ \to \bullet$$

$$\frac{\Gamma, A_i, \Delta \to B}{\Gamma, A_1 \& A_2, \Delta \to B}\ \&, i = 1, 2 \to \qquad\qquad \frac{\Gamma \to A \qquad \Gamma \to B}{\Gamma \to A \& B}\ \to \&$$

$$\frac{\Gamma, A, \Delta \to C \qquad \Gamma, B, \Delta \to C}{\Gamma, A \vee B, \Delta \to C}\ \vee \to \qquad\qquad \frac{\Gamma \to A_i}{\Gamma \to A_1 \vee A_2}\ \to \vee, i = 1, 2$$

$$\frac{\Gamma, \Delta \to A}{\Gamma, \mathbf{1}, \Delta \to A}\ \mathbf{1} \to \qquad\qquad \frac{}{\to \mathbf{1}}\ \to \mathbf{1}$$

$$\frac{\Gamma, A, \Delta \to C}{\Gamma, !^s A, \Delta \to C}\ ! \to \qquad\qquad \frac{!^{s_1} A_1, \dots, !^{s_n} A_n \to A}{!^{s_1} A_1, \dots, !^{s_n} A_n \to !^s A}\ \to !, \forall j, s_j \geq s$$

Structural rules:

$$\frac{\Gamma, !^s A, \Delta, !^s A, \Theta \to B}{\Gamma, !^s A, \Delta, \Theta \to B}\ \mathbf{ncontr}_1, s \in C \qquad \frac{\Gamma, !^s A, \Delta, !^s A, \Theta \to B}{\Gamma, \Delta, !^s A, \Theta \to B}\ \mathbf{ncontr}_2, s \in C$$

$$\frac{\Gamma, \Delta, !^s A, \Theta \to B}{\Gamma, !^s A, \Delta, \Theta \to A}\ \mathbf{ex}_1, s \in E \qquad \frac{\Gamma, !^s A, \Delta, \Theta \to B}{\Gamma, \Delta, !^s A, \Theta \to A}\ \mathbf{ex}_2, s \in E$$

$$\frac{\Gamma, \Delta \to B}{\Gamma, !^s A, \Delta \to B}\ \mathbf{weak}_!, s \in C \qquad \frac{\Gamma \to A \qquad \Pi, A, \Delta \to B}{\Gamma, \Pi, \Delta \to B}\ \mathbf{cut}$$

Definition 4. *By* $L_{\mathbf{1}}^{*} \wedge \vee_{\mathbf{S4}}$ *we mean* $SMALC_{\Sigma}$*, where* $\Sigma = \{s\}$ *and* $C = W = E = \varnothing$.

Theorem 1.

1. *Cut-rule is admissable;*
2. $SMALC_{\Sigma}$ *is undecidable, if* $C \neq \varnothing$*;*
3. *If* C *is empty, then the decidability problem of* $SMALC_{\Sigma}$ *belongs to PSPACE.*

2 Semantics

Definition 5. *Quantale*

A quantale is a triple $\langle A, \bigvee, \cdot \rangle$*, such that* $\langle A, \bigvee \rangle$ *is a complete lattice and* $\langle A, \cdot \rangle$ *is a semigroup. A quanlate is called unital, if* $\langle A, \cdot \rangle$ *is a monoid.*

Some example of quantales:

- Let A be a semigroup (monoid), then $\langle \mathcal{P}(A), \cdot, \subseteq \rangle$ is a free (unital) quantale.
- Let R be a ring and $Sub(R)$ be a set of additive subgroups of R. We define $A \cdot B$ as an additive subgroup generated by finite sums of products ab and order is defined by inclusion.
- Any locale is a quantale with $\cdot = \wedge$.

It is easy to see, that any (unital) quantale is a residual (monoid) semigroup. We define divisions as follows:

1. $a \backslash b = \bigvee \{c \mid a \cdot c \leqslant b\}$
2. $b / a = \bigvee \{c \mid c \cdot a \leqslant b\}$

Definition 6.

Let $\mathcal{Q} = \langle A, \bigvee, \cdot \rangle$ *be a quantale. The center of a quantale is the set* $\mathcal{Z}(\mathcal{Q}) = \{a \in A \mid \forall b \in A, a \cdot b = b \cdot a\}$

Definition 7. *An open modality (or quantic conucleus) on quantale* $\mathcal{Q}$ *is a map* $\Box : \mathcal{Q} \to \mathcal{Q}$*, such that*

1. $\Box x \leqslant x$*;*
2. $\Box x = \Box\Box x$*;*
3. $x \leqslant y \Rightarrow \Box x \leqslant \Box y$*;*

4. $\Box x \cdot \Box y = \Box(\Box x \cdot \Box y)$.

For unital quantale, we require that $\Box e = e$.
Note that, we may replace the last condition on equivalent condition $\Box(x) \cdot \Box(y) \leqslant \Box(x \cdot y)$.

Definition 8. *We define a partial order on open modalities on $\mathcal{Q}$ as $\Box_1 \leqslant \Box_2 \Leftrightarrow \forall a \in Q, \Box_1(a) \leqslant \Box_2(a)$.*

Lemma 1. *Let $\mathcal{Q}$ be a quantale and $\Box_{\mathcal{Q}}$ be a set of all open modalities on $\mathcal{Q}$. Then $\Box_{\mathcal{Q}}$ is a small category.*

Proof. $\langle \Box_{\mathcal{Q}}, \leqslant \rangle$ form a partial order, so $\langle \Box_{\mathcal{Q}}, \leqslant \rangle$ is a small category. $\square$

Definition 9.

1. *An open modality is called central, if $\forall a, b \in Q, \Box(a) \cdot b = b \cdot \Box(a)$.*
2. *An open modality is called pseudo-idempotent, if $\forall a, b \in Q, \Box(a) \cdot b \leqslant \Box(a) \cdot b \cdot \Box(a)$ and $b \cdot \Box(a) \leqslant \Box(a) \cdot b \cdot \Box(a)$.*
3. *An open modality is called unital, if $\forall a \in Q, \Box(a) \leqslant e$.*

Lemma 2. *Let $\Box$ be an open modality on some unital quantale $\mathcal{Q} = \langle A, \bigvee, \cdot, e \rangle$. Then, if $\Box$ is unital and weak idempotent, then $\Box$ is central.*

Proof.
$b \cdot \Box(a) \leqslant$
Right weak idempotence
$\Box(a) \cdot b \cdot \Box(a) \leqslant$
Unitality
$\Box(a) \cdot b \cdot e \leqslant$
Identity
$\Box(a) \cdot b \leqslant$
Left weak idempotence
$\Box(a) \cdot b \cdot \Box(a) \leqslant$
Unitality
$e \cdot b \cdot \Box(a) \leqslant$
Identity
$b \cdot \Box(a)$
Hence, $b \cdot \Box(a) = \Box(a) \cdot b$, so $\forall a \in A, \Box(a) \in \mathcal{Z}(\mathcal{Q})$.

$\square$

Proposition 1.
Let $\mathcal{Q}$ be a quantale and $S \subseteq \mathcal{Q}$ a subquantale, then $\Box : \mathcal{Q} \to \mathcal{Q}$, such that $\Box(a) = \bigvee\{s \in S \mid s \leqslant a\}$, is an open modality.

Proof. See $\square$

Proposition 2.
Let $\mathcal{Q}$ be a quantale and $S_1 \subseteq S_2 \subseteq \mathcal{Q}$.
Then $\Box_1(a) \leqslant \Box_2(a)$.

Proof. Immediatly. $\square$

Proposition 3.

Let $\mathcal{Q}$ be a quantale and $S \subseteq \mathcal{Q}$ a subquantale, then the following operations are open modalities:

1. $\Box_z(a) = \bigvee\{s \in S \,|s \leqslant a, s \in \mathcal{Z}(\mathcal{Q})\}$;
2. $\Box_{\mathbb{1}}(a) = \bigvee\{s \in S \,|s \leqslant a, s \leqslant \mathbb{1}\}$;
3. $\Box_{idem}(a) = \bigvee\{s \in S \,|s \leqslant a, \forall b \in Q, b \cdot s \vee s \cdot b \leqslant s \cdot b \cdot s\}$;
4. $\Box_{z,\mathbb{1}}, I_{z,idem}, I_{\mathbb{1},idem}, I_{z,\mathbb{1},idem}$.

Proof. Immediatly. $\square$

Proposition 4.

1. $\forall a \in \mathcal{Q}, \Box_{\mathbb{1},idem}(a) \leqslant \Box_z(a)$.
2. $\forall a \in \mathcal{Q}, \Box_{z,\mathbb{1},idem} = \Box_{\mathbb{1},idem}(a)$

Proof. Follows from Lemma 3. $\square$

Lemma 3. $\Box_1(a_1) \cdot \Box_2(a_2) \leqslant \Box^{'}(\Box_1(a_1) \cdot \Box_2(a_2))$, *where* $\Box_i \leqslant \Box^{'}, i = 1, 2$.

Proof.

$$\begin{aligned}&\Box_1(a_1) \cdot \Box_2(a_2) \leqslant \\ &\Box_1(\Box_1(a_1)) \cdot \Box_2(\Box_2(a_2)) \leqslant \\ &\Box^{'}(\Box_1(a_1)) \cdot \Box^{'}(\Box_2(a_2)) \leqslant \\ &\Box^{'}(\Box_1(a_1) \cdot \Box_2(a_2))\end{aligned}$$

$\square$

Definition 10. *Interpretation of subexponential signature*

Let $\Sigma = \langle I, \preceq, W, C, E\rangle$ be a subexponential signature, where $|I| = n$ and $\Box_{\mathcal{Q}}$ is a category of open modalities on a quantale $\mathcal{Q}$. Subexponential interpretation is a contravariant functor $\sigma : I \to \Box_{\mathcal{Q}}$ defined as follows:

$$\sigma(s_i) = \begin{cases} \Box_i : \mathcal{Q} \to \mathcal{Q},\ s.t. \forall a \in Q, \Box_i(a) = \{s \in S_i \mid s \leqslant a\}, \\ \quad \text{if } s_i \notin W \cap C \cap E \\ \Box_i : \mathcal{Q} \to \mathcal{Q},\ s.t. \forall a \in Q, \Box_i(a) = \{s \in S_i \mid s \leqslant a, s \leqslant \mathbb{1}\}, \\ \quad \text{if } s_i \in W \\ \Box_i : \mathcal{Q} \to \mathcal{Q},\ s.t. \forall a \in Q, \Box_i(a) = \{s \in S_i \mid s \leqslant a, s \in \mathcal{Z}(\mathcal{Q})\}, \\ \quad \text{if } s_i \in E \\ \Box_i : \mathcal{Q} \to \mathcal{Q},\ s.t. \forall a \in Q, \Box_i(a) = \{s \in S_i \mid s \leqslant a, \forall b, b \cdot s \vee s \cdot b \leqslant s \cdot b \cdot s\}, \\ \quad \text{if } s_i \in E \\ \text{otherwise, if } s_i \text{ belongs to some intersection of subsets, then we combine the relevant conditions} \end{cases}$$

Definition 11. *Let $\mathcal{Q}$ be an unital quantale, $f : Tp \to \mathcal{Q}$ a valuation and $\sigma : I \to \Box_{\mathcal{Q}}$ a subexponential interpretation, then interpretation is defined inductively:*

$$\begin{aligned}&[\![p_i]\!] = f(p_i) \\ &[\![\mathbb{1}]\!] = e \\ &[\![A \bullet B]\!] = [\![A]\!] \cdot [\![B]\!] \\ &[\![A \backslash B]\!] = [\![A]\!] \backslash [\![B]\!] \\ &[\![A / B]\!] = [\![A]\!] / [\![B]\!] \\ &[\![A \& B]\!] = [\![A]\!] \wedge [\![B]\!] \\ &[\![A \vee B]\!] = [\![A]\!] \vee [\![B]\!] \\ &[\![!_{s_i} A]\!] = \sigma(s_i)[\![A]\!]\end{aligned}$$

Definition 12. $\Gamma \models A \Leftrightarrow \forall f, \forall \sigma, [\![\Gamma]\!] \leqslant [\![A]\!]$

Theorem 2. $\Gamma \to A \Rightarrow [\![\Gamma]\!] \leqslant [\![A]\!]$

Proof. We consider the promotion case, the rest modal cases are immediatly shown.

Let $!_{s_1} A_1, \dots, !_{s_n} A_n \to A$ and $\forall i, s \preceq s_i$.

Then $\forall a \in Q, \sigma(s_i)(a) \leqslant \sigma(s)(a)$.

By IH, $\sigma(s_1)[\![A_1]\!] \cdot \dots \cdot \sigma(s_n)[\![A_n]\!] \leqslant [\![A]\!]$.

Thus, $\sigma(s)(\sigma(s_1)[\![A_1]\!] \cdot \dots \cdot \sigma(s_n)[\![A_n]\!]) \leqslant \sigma(s)([\![A]\!])$.

By Lemma 5, $\sigma(s_1)[\![A_1]\!] \cdot \dots \cdot \sigma(s_n)[\![A_n]\!] \leqslant \sigma(s)(\sigma(s_1)[\![A_1]\!] \cdot \dots \cdot \sigma(s_n)[\![A_n]\!])$.

So, $\sigma(s_1)[\![A_1]\!] \cdot \dots \cdot \sigma(s_n)[\![A_n]\!] \leqslant \sigma(s)([\![A]\!])$. ∎

3 Quantale completeness

Definition 13.
Let $\mathcal{F} \subseteq Fm$, an ideal is a subset $\mathcal{I} \subseteq \mathcal{F}$, such that:

- *If $B \in \mathcal{I}$ and $A \to B$, then $A \in \mathcal{I}$;*
- *If $A, B \in \mathcal{I}$, then $A \vee B \in \mathcal{I}$.*

Definition 14.
Let $S \subseteq \mathcal{F} \subseteq Fm$, then $\bigvee S = \bigcap \{\mathcal{I} \subseteq \mathcal{F} \mid S \subseteq \mathcal{I}\}$

Proposition 5. *$\bigvee S$ is an ideal.*

Lemma 4. *$A \subseteq Fm$, then $\{B \mid B \to A\} = \bigvee \{A\}$.*

Proof.
Let $A \in Fm$. Then $\{B \mid B \to A', A' \in A\} \subseteq \bigvee \{A\}$, so far as $\bigvee A$ is an ideal.

On the other hand, $\{B \mid B \to A\}$ is an ideal, it is easy to see that this set is closed under $\vee$. So, $\bigvee A \subseteq \{B \mid B \to A\}$. ∎

Lemma 5. *$\bigvee \{A\} \subseteq \bigvee \{B\}$ iff $A \to B$.*

Proof. Let $\bigvee \{A\} \subseteq \bigvee \{B\}$, then $\{C | C \to A\} \subseteq \{D \mid D \to B\}$.

Thus, $A \in \{C | C \to A\}$, then $A \in \{D \mid D \to B\}$, hence $A \to B$.

On the other hand, let $A \to B$ and $C \in \bigvee \{A\}$.

Thus, $C \to A$, then $C \to B$ by cut.

∎

Lemma 6. *Let $\mathcal{Q} = \{\bigvee S \mid S \subseteq Fm\}$ and $\bigvee \mathcal{A} \cdot \bigvee \mathcal{B} = \bigvee \{A \bullet B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$. Then $\langle \mathcal{Q}, \subseteq, \cdot, \bigvee \mathbf{1} \rangle$ is a quantale.*

Proof. See ∎

Lemma 7. *Let $!_s \in I$ and A be an arbitrary formula, then $\Box_s(\bigvee \{A\}) = \bigvee \{B \mid !_s B \to A\}$ is a quantic conucleus.*

Proof.
See Yetter. ∎

Lemma 8. *Let A be a formula, then $\Box_s \bigvee \{A\} = \bigvee \{!_s A\}$, for each $s \in \mathcal{I}$.*

Proof. Let $A \in Fm$ and $s \in \mathcal{I}$.

Let $!_s B \in \Box_s \bigvee\{A\}$, then $!_s B \to A$, then $!_s B \to !_s A$ by promotion. So, $!_s B \in \bigvee\{!_s A\}$.

Let $C \in \bigvee\{!_s A\}$, then $C \to !_s A$, so $!_s C \to !_s A$ by dereliction, but $!_s A \to A$, hence $!_s C \to A$ by cut. So, $!_s C \in \Box_s \bigvee\{A\}$. □

Lemma 9.

Let $i, j \in I$ and $i \preceq j$, then for all $A \in Fm$, $\Box_j(\bigvee\{A\}) \subseteq \Box_i(\bigvee\{A\})$.

Proof.

Let $i, j \in I$ and $i \preceq j$. Then forall $A \in Fm$, $!_j A \to !_i A$ by promotion. Then $\bigvee\{!_j A\} \subseteq \bigvee\{!_i A\}$, so $\Box_j(\bigvee\{A\}) \subseteq \Box_i(\bigvee\{A\})$. □

Lemma 10.

1. *Let $s \in W$, then for all $A \subseteq Fm$, $\Box_s\{A\} \subseteq \{\mathbf{1}\}$;*
2. *Let $s \in E$, then $\Box_s(\bigvee\{A\}) \cdot \bigvee\{B\} = \bigvee\{B\} \cdot \Box_s(\bigvee\{A\})$.*
3. *Let $s \in C$, then $(\Box_s \bigvee A \cdot \bigvee B) \cup (\bigvee B \cdot \Box_s \bigvee A) \subseteq \Box_s \bigvee A \cdot \bigvee B \cdot \Box_s \bigvee A$, for all $B \subseteq Fm$.*

Proof.

Follows from $!_s A \to \mathbf{1}$, so $s \in W$;

Follows from $!_s A \bullet B \leftrightarrow B \bullet !_s A$;

Follows from $!_s A \bullet B \to !_s A \bullet B \bullet !_s A$ and similarly for $B \bullet !_s A$. □

Definition 15.

Let Q be a syntactic quantale as proposed above and $\mathcal{I} = \langle I, \preceq, W, C, E\rangle$ be a subexponential signature.

We define a map $\Box : \mathcal{I} \to Mod_{\mathcal{Q}}$ as follows:

$\Box(i)(\bigvee\{A\}) = \{!_i B \mid !_i B \to A\}$.

Lemma 11. *$\Box$ is a subexponential interpretation.*

Proof. Follows from lemmas above. □

Lemma 12.

Let Q be a quantale constructed above and $\Box_1, \ldots, \Box_n$ be a family of quantic conuclei on Q. Then there exist a model $\langle Q, [\![.]\!]\rangle$, such that $[\![A]\!] = \bigvee\{A\}$, $A \in Fm$.

Proof.

We define an interpreation as follows:

1. $[\![p_i]\!] = \bigvee\{p_i\}$
2. $[\![\mathbf{1}]\!] = \bigvee\{\mathbf{1}\}$
3. $[\![A \bullet B]\!] = \bigvee\{A \bullet B\}$
4. $[\![A/B]\!] = \bigvee\{A/B\}$
5. $[\![B\backslash A]\!] = \bigvee\{B\backslash A\}$
6. $[\![A\&B]\!] = \bigvee\{A\&B\}$
7. $[\![A \vee B]\!] = \bigvee\{A \vee B\}$

8. $[\![!_s A]\!] = \Box(s)(\bigvee\{A\}) = \bigvee\{!_s A\}$.

$\Box$

Theorem 3. $\Gamma \models A \Rightarrow \Gamma \rightarrow A$.

Proof. Follows from lemmas above. $\Box$

4 Relational semantics

Definition 16.
Let A be a set. Then relational quantale on A is a tuple $\mathcal{Q} = \langle \mathcal{A}, \mathcal{I} \rangle$ where $\mathcal{A} \subseteq 2^{A \times A}$:

- *$\langle \mathcal{A}, \subseteq \rangle$ is a complete semi-lattice;*
- *Multiplication is defined as $R \circ S = \{\langle a, c\rangle \mid \exists b \in A, \langle a, b\rangle \in R$ and $\langle b, c\rangle \in S\}$*
- *$\langle \mathcal{A}, \circ, \mathcal{I} \rangle$ is a monoid;*
- *For each indexing set J, $R \circ \bigvee_{j \in J} S_j = \bigvee_{j \in J}(R \circ S_j)$ and $\bigvee_{j \in J} R_j \circ S = \bigvee_{j \in J}(R_j \circ S)$.*

Proposition 6.
Let $\mathcal{Q}_1$, $\mathcal{Q}_2$ be quantales and $\mathcal{S} \subseteq \mathcal{Q}_1$ is a subquantale of $\mathcal{Q}_2$.
Then, if $f : \mathcal{Q}_1 \rightarrow \mathcal{Q}_2$ is a quantale homomorphism, then $f(\mathcal{S}) \subseteq \mathcal{Q}_2$ is a subquantale of $\mathcal{Q}_2$.

Proof. It is clearly that $f(\mathcal{S}) \subseteq \mathcal{Q}_2$ is a submonoid of $\mathcal{Q}_2$.
Let $a \in S$, so $\bigvee a \in S$, but $f(a) \in f(\mathcal{S})$, so $f(\bigvee a) = \bigvee(f(a)) \in f(\mathcal{S})$, so $f(\mathcal{S})$ is closed under joins, so $f(\mathcal{S})$ is a subquantale of $\mathcal{Q}_2$ $\Box$

Theorem 4.
Let $\mathcal{Q} = \langle A, \leqslant, \cdot, \bigvee \rangle$ be a unital quantale and $\mathcal{S}$ is a subquantale of $\mathcal{Q}$.
Then $\langle \mathcal{Q}, \Box_{\mathcal{S}} \rangle$ is isomorphic to some relational quantale of A wit some quantic conucleus $\hat{\Box}$.

Proof. Consider a relational quantale proposed by Brown and Gurr. This quantale is 4-tuple $\theta(\mathcal{Q}) = \langle \mathcal{R}, \subseteq, \circ, \bigvee \rangle$ defined as follows:

1. $\theta(a) = \{\langle b, c\rangle \mid b \leqslant a \cdot c\}$;
2. $\theta(a \cdot b) = \theta(a) \circ \theta(b)$;
3. $\theta(\bigvee a) = \bigvee \theta(a)$;
4. $\theta(\varepsilon) = \{\langle b, c\rangle \mid b \cdot \varepsilon \leqslant c\} = \{\langle b, c\rangle \mid b \leqslant c\}$

Let $\mathcal{S} \subseteq \mathcal{Q}$, we define $\Box a = \{s \mid s \in S, s \leqslant a\}$ is quantic conucleus.
So, $\theta(S) \subseteq \theta(\mathcal{Q})$ is a subquantale of $\theta(\mathcal{Q})$.
Let us define $\theta(\Box)\theta a := \bigvee\{\theta s \mid \theta s \in \theta S, \theta s \subseteq \theta a\}$, so
$\theta(\Box a) = \{\langle p, q\rangle \mid p \leqslant \Box a \cdot q\} =$
$\{\langle p, q\rangle \mid p \leqslant \bigvee\{s \mid s \in S, s \leqslant a\} \cdot q\} =$
$\theta(\bigvee_{s \in S, s \leqslant a} s) =$
$\bigvee_{s \in S, s \leqslant a} \theta(s) =$
$\bigvee\{\theta(s) \mid s \in S, s \leqslant a\} =$
$\bigvee\{\theta(s) \mid \theta(s) \in \theta(S), \theta(s) \subseteq \theta(a)\} = \theta(\Box)\theta(a)$

$\Box$

Theorem 5. $\Gamma \models A$*, then* $L_{\mathbf{1}}^{*} \wedge \vee_{\mathbf{S4}} \vdash \Gamma \rightarrow A$