Quantale model of Lambek calculus with subexponentials

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1 Calculus

Definition 1. *A subexponential signature is an ordered quintuple:*
$\Sigma = \langle I, \preceq, W, C, E \rangle$,

where $I = \{s_1, \ldots, s_n\}$, $\langle I, \preceq \rangle$ is a preorder. W, C, E are subsets of I and $W \cup C \subseteq E$.

Definition 2. *Noncommutative linear logic with subexponentials ($SMALC_\Sigma$), where Σ is a subexponential signature.*

$$\frac{}{A \to A}\ ax$$

$$\frac{\Gamma \to A \qquad \Delta, B, \Theta \to C}{\Delta, \Gamma, A \backslash B, \Theta \to C}\ \backslash \to \qquad \frac{A, \Pi \to B}{\Pi \to A \backslash B}\ \to \backslash$$

$$\frac{\Gamma \to A \qquad \Delta, B, \Theta \to C}{\Delta, B / A, \Gamma, \Theta \to C}\ / \to \qquad \frac{\Pi, A \to B}{\Pi \to B / A}\ \to /$$

$$\frac{\Gamma, A, B, \Delta \to C}{\Gamma, A \bullet B, \Delta \to C}\ \bullet \to \qquad \frac{\Gamma \to A \qquad \Delta \to B}{\Gamma, \Delta \to A \bullet B}\ \to \bullet$$

$$\frac{\Gamma, A_i, \Delta \to B}{\Gamma, A_1 \& A_2, \Delta \to B}\ \&, i = 1,2 \to \qquad \frac{\Gamma \to A \qquad \Gamma \to B}{\Gamma \to A \& B}\ \to \&$$

$$\frac{\Gamma, A, \Delta \to C \qquad \Gamma, B, \Delta \to C}{\Gamma, A \vee B, \Delta \to C}\ \vee \to \qquad \frac{\Gamma \to A_i}{\Gamma \to A_1 \vee A_2}\ \to \vee, i = 1,2$$

$$\frac{\Gamma, \Delta \to A}{\Gamma, \mathbf{1}, \Delta \to A}\ \mathbf{1} \to \qquad \frac{}{\to \mathbf{1}}\ \to \mathbf{1}$$

$$\frac{\Gamma, A, \Delta \to C}{\Gamma, !^s A, \Delta \to C}\ ! \to \qquad \frac{!^{s_1} A_1, \ldots, !^{s_n} A_n \to A}{!^{s_1} A_1, \ldots, !^{s_n} A_n \to !^s A}\ \to !, \forall j, s_j \succeq s$$

$$\frac{\Gamma, \Delta \to B}{\Gamma, !^s A, \Delta \to B}\ \mathbf{weak}_!, s \in C$$

$$\frac{\Gamma, !^s A, \Delta, !^s A, \Theta \to B}{\Gamma, !^s A, \Delta, \Theta \to B}\ \mathbf{ncontr}_1, s \in C$$

$$\frac{\Gamma, !^s A, \Delta, !^s A, \Theta \to B}{\Gamma, \Delta, !^s A, \Theta \to B}\ \mathbf{ncontr}_2, s \in C$$

$$\frac{\Gamma, \Delta, !^s A, \Theta \to B}{\Gamma, !^s A, \Delta, \Theta \to A}\ \mathbf{ex}_1, s \in E$$

$$\frac{\Gamma, !^s A, \Delta, \Theta \to B}{\Gamma, \Delta, !^s A, \Theta \to A}\ \mathbf{ex}_1, s \in E$$

Proposition 1. $!_{s_i} A \leftrightarrow !_{s_i}(!_{s_i} A)$

Proof.

$$\cfrac{\cfrac{\cfrac{A \to A}{!_{s_i} A \to A}}{!_{s_i} A \to !_{s_i} A}}{!_{s_i} !_{s_i} A \to !_{s_i} A}$$

□

2 Semantics

Definition 3. *Quantale*

A quantale is a triple $\langle A, \bigvee, \cdot \rangle$, such that $\langle A, \bigvee \rangle$ is a complete lattice and $\langle A, \cdot \rangle$ is a semigroup. A quanlate is called unital, if $\langle A, \cdot \rangle$ is a monoid.

It is easy to see, that any (unital) quantale is a residual (monoid) semigroup. We define divisions as follows:

1. $a \backslash b = \bigvee \{c \mid a \cdot c \leqslant b\}$
2. $b / a = \bigvee \{c \mid c \cdot a \leqslant b\}$

Definition 4.

Let $\mathcal{Q} = \langle A, \bigvee, \cdot \rangle$ be a quantale. The center of a quantale is the set $\mathcal{Z}(\mathcal{Q}) = \{a \in A \mid \forall b \in A, a \cdot b = b \cdot a\}$

Definition 5. *An open modality (or quantic conucleus) on quantale $\mathcal{Q}$ is a map $\Box : \mathcal{Q} \to \mathcal{Q}$, such that*

1. $\Box(x) \leqslant x$;
2. $\Box(x) = \Box(\Box(x))$;
3. $x \leqslant y \Rightarrow \Box(x) \leqslant \Box(y)$;
4. $\Box(x) \cdot \Box(y) = \Box(\Box(x) \cdot \Box(y))$.

Definition 6. *We define a partial order on open modalities on $\mathcal{Q}$ as $\Box_1 \leqslant \Box_2 \Leftrightarrow \forall a \in Q, \Box_1(a) \leqslant \Box_2(a)$.*

Lemma 1. *Let $\mathcal{Q}$ be a quantale and $\Box_{\mathcal{Q}}$ be a set of all open modalities on $\mathcal{Q}$. Then $\Box_{\mathcal{Q}}$ is a locally small category.*

Proof. $\langle \Box_{\mathcal{Q}}, \leqslant \rangle$ form a partial order, so $\langle \Box_{\mathcal{Q}}, \leqslant \rangle$ is a locally small category. $\square$

Lemma 2.
Let $\mathcal{Q} = \langle A, \bigvee, \cdot \rangle$ be a quantale and $\Box : \mathcal{Q} \to \mathcal{Q}$ is an open modality on $\mathcal{Q}$, then $\Box(x) \cdot \Box(y) \leqslant \Box(x \cdot y)$.

Proof.
$\Box(x) \cdot \Box(y) \leqslant x \cdot y$, then $\Box(\Box(x) \cdot \Box(y)) \leqslant \Box(x \cdot y)$, but $\Box(x) \cdot \Box(y) \leqslant \Box(\Box(x) \cdot \Box(y))$. Thus, $\Box(x) \cdot \Box(y) \leqslant \Box(x \cdot y)$. $\square$

Definition 7. *An open modality is called central, if $\forall a, b \in Q, \Box(a) \cdot b = b \cdot \Box(a)$.*

Definition 8. *An open modality is called weak idempotent, if $\forall a, b \in Q, \Box(a) \cdot b \leqslant \Box(a) \cdot b \cdot \Box(a)$ and $b \cdot \Box(a) \leqslant \Box(a) \cdot b \cdot \Box(a)$.*

Definition 9. *An open modality is called unital, if $\forall a \in Q, \Box(a) \leqslant e$.*

Lemma 3. *Let $\Box$ be an open modality on some unital quantale $\mathcal{Q} = \langle A, \bigvee, \cdot, e \rangle$. Then, if $\Box$ is unital and weak idempotent, then $\Box$ is central.*

Proof.
$b \cdot \Box(a) \leqslant$
Right weak idempotence
$\Box(a) \cdot b \cdot \Box(a) \leqslant$
Unitality
$\Box(a) \cdot b \cdot e \leqslant$
Identity
$\Box(a) \cdot b \leqslant$
Left weak idempotence
$\Box(a) \cdot b \cdot \Box(a) \leqslant$
Unitality
$e \cdot b \cdot \Box(a) \leqslant$
Identity
$b \cdot \Box(a)$
Hence, $b \cdot \Box(a) = \Box(a) \cdot b$, so $\forall a \in A, \Box(a) \in \mathcal{Z}(\mathcal{Q})$.
$\square$

Proposition 2.
Let $\mathcal{Q}$ be a quantale and $S \subseteq \mathcal{Q}$ a subquantale, then $\Box : \mathcal{Q} \to \mathcal{Q}$, such that $\Box(a) = \bigvee\{s \in S \mid x \leqslant a\}$, is an open modality.

Proof. See $\square$

Proposition 3.
Let $\mathcal{Q}$ be a quantale and $S_1 \subseteq S_2 \subseteq \mathcal{Q}$.
Then $\Box_1(a) \leqslant \Box_2(a)$.

Proof.
Let $a \in Q$, so $\{s \in S_1 \mid s \leqslant a\} \subseteq \{s \in S_2 \mid s \leqslant a\}$, so $\bigvee\{s \in S_1 \mid s \leqslant a\} \subseteq \bigvee\{s \in S_2 \mid s \leqslant a\}$. Thus, $\Box_1(a) \leqslant \Box_2(a)$. $\square$

Proposition 4.

Let $\mathcal{Q}$ be a quantale and $S \subseteq \mathcal{Q}$ a subquantale, then the following operations are open modalities:

1. $\Box_z(a) = \bigvee\{s \in S \,|s \leqslant a, s \in \mathcal{Z}(\mathcal{Q})\}$;
2. $\Box_{\mathbb{1}}(a) = \bigvee\{s \in S \,|s \leqslant a, s \leqslant \mathbb{1}\}$;
3. $\Box_{idem}(a) = \bigvee\{s \in S \,|s \leqslant a, \forall b \in Q, b \cdot s \vee s \cdot b \leqslant s \cdot b \cdot s\}$;
4. $\Box_{z,\mathbb{1}}, I_{z,idem}, I_{\mathbb{1},idem}, I_{z,\mathbb{1},idem}$.

Proof. Immediatly. $\Box$

Proposition 5.

1. $\forall a \in \mathcal{Q}, \Box_{\mathbb{1},idem}(a) \leqslant \Box_z(a)$.
2. $\forall a \in \mathcal{Q}, \Box_{z,\mathbb{1},idem} = \Box_{\mathbb{1},idem}(a)$

Proof. Follows from Lemma 3. $\Box$

Proposition 6.

1. $\Box_z(a) \vee \Box_{\mathbb{1}}(a) \vee \Box_{idem}(a) \leqslant \Box(a)$
2. $\Box_{z,\mathbb{1},idem} \leqslant \Box_{z,\mathbb{1}}(a) \wedge \Box_{z,idem}(a)$

Lemma 4. $\forall a \in Q, \Box_1(a) \leqslant \Box_2(\Box_1(a))$*, if* $\Box_1(a) \leqslant \Box_2(a)$.

Proof. $\Box_1(a) \leqslant \Box_1(\Box_1(a)) \leqslant \Box_2(\Box_1(a))$ $\Box$

Lemma 5. $\Box_1(a_1) \cdot \Box_2(a_2) \leqslant \Box^{'}(\Box_1(a_1) \cdot \Box_2(a_2))$*, where* $\Box_i \leqslant \Box^{'}, i = 1, 2$.

Proof.

$$\begin{aligned}
&\Box_1(a_1) \cdot \Box_2(a_2) \leqslant \\
&\Box_1(\Box_1(a_1)) \cdot \Box_2(\Box_2(a_2)) \leqslant \\
&\Box^{'}(\Box_1(a_1)) \cdot \Box^{'}(\Box_2(a_2)) \leqslant \\
&\Box^{'}(\Box_1(a_1) \cdot \Box_2(a_2))
\end{aligned}$$

$\Box$

Definition 10. *Interpretation of subexponential signature*

Let $\Sigma = \langle I, \preceq, W, C, E\rangle$ be a subexponential signature, where $|I| = n$ and $\Box_{\mathcal{Q}}$ is a category of open modalities on a quantale $\mathcal{Q}$. Subexponential interpretation is a contravariant functor $\sigma : I \to \Box_{\mathcal{Q}}$ defined as follows:

$$\sigma(s_i) = \begin{cases}
\Box_i : \mathcal{Q} \to \mathcal{Q}, \text{ s.t.} \forall a \in Q, \Box_i(a) = \{s \in S_i \mid s \leqslant a\}, \\
\quad \text{if } s_i \notin W \cap C \cap E \\
\Box_i : \mathcal{Q} \to \mathcal{Q}, \text{ s.t.} \forall a \in Q, \Box_i(a) = \{s \in S_i \mid s \leqslant a, s \leqslant \mathbb{1}\}, \\
\quad \text{if } s_i \in W \\
\Box_i : \mathcal{Q} \to \mathcal{Q}, \text{ s.t.} \forall a \in Q, \Box_i(a) = \{s \in S_i \mid s \leqslant a, s \in \mathcal{Z}(\mathcal{Q})\}, \\
\quad \text{if } s_i \in E \\
\Box_i : \mathcal{Q} \to \mathcal{Q}, \text{ s.t.} \forall a \in Q, \Box_i(a) = \{s \in S_i \mid s \leqslant a, \forall b, b \cdot s \vee s \cdot b \leqslant s \cdot b \cdot s\}, \\
\quad \text{if } s_i \in E \\
\text{otherwise, if } s_i \text{ belongs to some intersection of subsets, then we combine the relevant conditions}
\end{cases}$$

Definition 11. *Let $\mathcal{Q}$ be a quantale, $f : Tp \to \mathcal{Q}$ a valuation and $\sigma : I \to \Box_{\mathcal{Q}}$ a subexponential interpretation, then interpretation is defined inductively:*

$$\begin{aligned}
&[\![p_i]\!] = f(p_i)\\
&[\![\mathbb{1}]\!] = e\\
&[\![A \bullet B]\!] = [\![A]\!] \cdot [\![B]\!]\\
&[\![A \backslash B]\!] = [\![A]\!] \backslash [\![B]\!]\\
&[\![A / B]\!] = [\![A]\!] / [\![B]\!]\\
&[\![A \& B]\!] = [\![A]\!] \wedge [\![B]\!]\\
&[\![A \vee B]\!] = [\![A]\!] \vee [\![B]\!]\\
&[\![!_{s_i} A]\!] = \sigma(s_i)[\![A]\!]
\end{aligned}$$

Definition 12. $\Gamma \models A \Leftrightarrow \forall f, \forall \sigma, [\![\Gamma]\!] \leqslant [\![A]\!]$

Theorem 1. $\Gamma \to A \Rightarrow [\![\Gamma]\!] \leqslant [\![A]\!]$

Proof. We consider cases with modal rules.

1. Let $!_{s_1} A_1, \ldots, !_{s_n} A_n \to A$ and $\forall i, s \preceq s_i$.
   Then $\forall a \in Q, \sigma(s_i)(a) \leqslant \sigma(s)(a)$.
   By IH, $\sigma(s_1)[\![A_1]\!] \cdot \cdots \cdot \sigma(s_n)[\![A_n]\!] \leqslant [\![A]\!]$.
   Thus, $\sigma(s)(\sigma(s_1)[\![A_1]\!] \cdot \cdots \cdot \sigma(s_n)[\![A_n]\!]) \leqslant \sigma(s)([\![A]\!])$.
   By Lemma 5, $\sigma(s_1)[\![A_1]\!] \cdot \cdots \cdot \sigma(s_n)[\![A_n]\!] \leqslant \sigma(s)(\sigma(s_1)[\![A_1]\!] \cdot \cdots \cdot \sigma(s_n)[\![A_n]\!])$.
   So, $\sigma(s_1)[\![A_1]\!] \cdot \cdots \cdot \sigma(s_n)[\![A_n]\!] \leqslant \sigma(s)([\![A]\!])$.

2. Let $\Gamma, A, \Delta \to B$.
   By IH, $[\![\Gamma]\!] \cdot [\![A]\!] \cdot [\![\Delta]\!] \leqslant [\![B]\!]$.
   By the definition, $\sigma(s_i)([\![A]\!]) \leqslant [\![A]\!]$.
   So, $[\![\Gamma]\!] \cdot \sigma(s_i)([\![A]\!]) \cdot [\![\Delta]\!] \leqslant [\![B]\!]$

3. Let $\Gamma, \Delta \to B$, $A \in Fm$, and $s_i \in W$.
   So, $[\![\Gamma]\!] \cdot [\![\Delta]\!] \leqslant [\![B]\!]$, then $[\![\Gamma]\!] \cdot e \cdot [\![\Delta]\!] \leqslant [\![B]\!]$, where $e \in Q$ is unit.
   By the definition of unital open modality, $\sigma(s_i)([\![A]\!]) \leqslant e$.
   Thus, $[\![\Gamma]\!] \cdot \sigma(s_i)([\![A]\!]) \cdot [\![\Delta]\!] \leqslant [\![B]\!]$.

4. Let $\Gamma, !_{s_i} A, \Delta, !_{s_i} A, \Pi \to B$ and $s_i \in C$.
   By IH, $[\![\Gamma]\!] \cdot \sigma(s_i)([\![A]\!]) \cdot [\![\Delta]\!] \cdot \sigma(s_i)([\![A]\!]) \cdot [\![\Pi]\!] \leqslant [\![B]\!]$.
   By the definition, $\sigma(s_i)([\![A]\!]) \cdot [\![\Delta]\!] \leqslant \sigma(s_i)([\![A]\!]) \cdot [\![\Delta]\!] \cdot \sigma(s_i)([\![A]\!])$.
   Then $[\![\Gamma]\!] \cdot \sigma(s_i)([\![A]\!]) \cdot [\![\Delta]\!] \cdot [\![\Pi]\!] \leqslant [\![B]\!]$

5. Let $\Gamma, !_{s_i} A, \Delta, \Pi \to B$ and $s_i \in E$, so $\sigma(s_i)(a) \in \mathcal{Z}(\mathcal{Q})$ for all $a \in Q$ by the definition.
   By IH, $[\![\Gamma]\!] \cdot \sigma(s_i)([\![A]\!]) \cdot [\![\Delta]\!] \cdot [\![\Pi]\!] \leqslant [\![B]\!]$
   Hence, $[\![\Gamma]\!] \cdot [\![\Delta]\!] \cdot \sigma(s_i)([\![A]\!]) \cdot [\![\Pi]\!] \leqslant [\![B]\!]$.

□

3 Quantale completeness

Definition 13.
Let $\mathcal{F} \subseteq Fm$, an ideal is a subset $\mathcal{I} \subseteq \mathcal{F}$, such that:

- *If $B \in \mathcal{I}$ and $A \to B$, then $A \in \mathcal{I}$;*
- *If $A, B \in \mathcal{I}$, then $A \vee B \in \mathcal{I}$.*

Definition 14.
Let $S \subseteq \mathcal{F} \subseteq Fm$, then $\bigvee S = \bigcap \{\mathcal{I} \subseteq \mathcal{F} \mid S \subseteq \mathcal{I}\}$

Proposition 7. *$\bigvee S$ is an ideal.*

Lemma 6. *$A \subseteq Fm$, then $\{B \mid B \to A'\} = \bigvee A$.*

Proof.
Let $A \subseteq Fm$. Then $\{B \mid B \to A', A' \in A\} \subseteq \bigvee A$, so far as $\bigvee A$ is an ideal.
On the other hand, $\{B \mid B \to A', A' \in A\}$ is an ideal, it is easy to see that this set is closed under $\vee$. So, $\bigvee A \subseteq \{B \mid B \to A', A' \in A\}$. □

Lemma 7. *$\bigvee A \subseteq \bigvee B$ iff $\forall A' \in A, \forall B' \in B, A' \to B'$.*

Proof. Let $\bigvee A \subseteq \bigvee B$, then $\{C|C \to A', A' \in A\} \subseteq \{D \mid D \to B', B' \in B\}$.
Thus, for all $A' \in A$, $A' \in \{C|C \to A', A' \in A\}$, then $A' \in \{D \mid D \to B', B' \in B\}$, hence $A' \to B'$, for all $B' \in B$.
On the other hand, let $A' \to B'$ for all $A' \in A$, $B' \in B$ and $C \in \bigvee A$.
Thus, $C \to A'$, then $C \to B'$ by cut, so $C \in B'$.
□

Lemma 8. *Let $\mathcal{Q} = \{\bigvee S \mid S \subseteq Fm\}$ and $\bigvee \mathcal{A} \cdot \bigvee \mathcal{B} = \bigvee \{A \bullet B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$. Then $\langle \mathcal{Q}, \subseteq, \cdot, \bigvee \mathbf{1} \rangle$ is a quantale.*

Proof. See □

Lemma 9. *Let $!_s \in I$, then $\Box_s(\bigvee A) = \bigvee \{B \mid B \to !_s A^{'}, A^{'} \in A\}$ is a quantic conucleus.*

Proof.

1. $\Box_s(\bigvee A) \subseteq \bigvee A$;
   Let $B \in \Box_s(\bigvee A)$, then for all $A^{'} \in A$, $B \to !_s A^{'}$, but $!_s A^{'} \to A^{'}$, then $B \to A^{'}$, so $B \in \bigvee A$.
2. $\Box_s(\Box_s(\bigvee A)) = \bigvee \Box_s(\bigvee A)$;
   $\Box_s(\Box_s(\bigvee A)) =$
   $\{B \mid B \to !_s !_s A^{'}, A^{'} \in A\} =$, that follows from equivalence $!_s !_s B \leftrightarrow !_s B$.
   $\{B \mid B \to !_s A^{'}, A^{'} \in A\}$
3. $\bigvee A \subseteq \bigvee B \Rightarrow \Box_s(\bigvee A) \subseteq \Box_s(\bigvee B)$;
   Follows from admissiability of K-rule for all $s \in I$.
4. $\Box_s \bigvee A \cdot \Box_s \bigvee B = \Box_s(\Box_s \bigvee A \cdot \Box_s \bigvee B)$.
   $\Box_s \bigvee A \cdot \Box_s \bigvee B =$
   $\bigvee \{C \bullet D \mid C \bullet D \to !_s A^{'} \bullet !_s B^{'}\} =$
   $\bigvee \{C \bullet D \mid C \bullet D \to !_s(!_s A^{'} \bullet !_s B^{'})\} =$
   $\Box_s(\Box_s \bigvee A \cdot \Box_s \bigvee B)$

$\square$

Lemma 10.

1. *Let $s \in W$, then for all $A \subseteq Fm$, $\mathbf{1} \in \square_s(\bigvee A)$;*
2. *Let $s \in E$, then $\square_s(\bigvee A) \cdot \bigvee B = \bigvee B \cdot \square_s(\bigvee A)$.*
3. *Let $s \in C$, then $(\square_s \bigvee A \cdot \bigvee B) \cup (\bigvee B \cdot \square_s \bigvee A) \subseteq \square_s \bigvee A \cdot \bigvee B \cdot \square_s \bigvee A$, for all $B \subseteq Fm$.*

Proof. 1. Let $s \in W$, then for all $A \subseteq Fm$, $\square_s(\bigvee A) = \{!_s B \mid !_s B \to A', A' \in A\}$. But, $!_s B \to \mathbf{1}$, hence, $1 \in \square_s(\bigvee A)$, so far as $\square_s(\bigvee A)$ is an ideal.

2.
$\square_s(\bigvee A) \cdot \bigvee B =$
$\bigvee\{!_s C \bullet D \mid !_s C \bullet D \to A' \bullet B', A' \in A, B' \in B\} =$
$\bigvee\{D \bullet !_s C \mid D \bullet !_s C \to A' \bullet B', A' \in A, B' \in B\} =$
$\bigvee B \cdot \square_s(\bigvee A)$

3.
$\square_s \bigvee A \cdot \bigvee B = \bigvee\{!_s C \bullet D | !_s C \bullet D \to A' \bullet B'\}$. $!_s C \bullet D \to !_s C \bullet D \bullet !_s C$, hence $\square_s \bigvee A \cdot \bigvee B \subseteq \square_s \bigvee A \cdot \bigvee B \cdot \square_s \bigvee A$.

Similarly with $\bigvee B \cdot \square_s \bigvee A$.

$\square$

Lemma 11.
Let $i, j \in I$ and $i \preceq j$, then for all $A \subseteq Fm$, $\square_j(\bigvee A) \subseteq \square_i(\bigvee A)$.

Proof. Let $i, j \in I$ and $i \preceq j$. Let $B \in \square_j(\bigvee A)$, then $\forall A'$, $B \to !_j A'$.

But $!_j A \to !_i A$. Then $B \to !_i A$ by hence. So, $B \in \square_j(\bigvee A)$.

$\square$

Definition 15. *Let Q be a syntactic quantale as proposed above and $\mathcal{I} = \langle I, \preceq, W, C, E\rangle$ be a subexponential signature.*

We define a map $\square : \mathcal{I} \to Mod_{\mathcal{Q}}$ as follows:
$\square(i)(\bigvee A) = \{B \mid B \to !_i A\}$.

Lemma 12. *$\square$ is a subexponential interpretation.*

Proof. Follows from lemmas 10 and 11. $\square$

Lemma 13.
Let Q be a quantale constructed above and $\square_1, \ldots, \square_n$ be a family of quantic conuclei on Q. Then there exist a model $\langle Q, [\![.]\!]\rangle$, such that $[\![A]\!] = \bigvee\{A\}$, $A \in Fm$.

Proof.
We define an interpreation as follows:

1. $[\![p_i]\!] = \bigvee\{p_i\}$
2. $[\![\mathbf{1}]\!] = \bigvee\{\mathbf{1}\}$
3. $[\![A \bullet B]\!] = \bigvee\{A \bullet B\}$

4. $[\![A/B]\!] = \bigvee\{A/B\}$
5. $[\![B\backslash A]\!] = \bigvee\{B\backslash A\}$
6. $[\![A\&B]\!] = \bigvee\{A\&B\}$
7. $[\![A \vee B]\!] = \bigvee\{A \vee B\}$
8. $[\![!_s A]\!] = \Box(s)(\bigvee A) = \{B \mid B \to !_s A\} = \bigvee\{!_s A\}$

$\square$

Theorem 2. *$\Gamma \models A \Rightarrow \Gamma \to A$.*

Proof. Follows from lemmas 6, 12, 13. $\square$