Categorical model of noncommutative linear logic with subexponentials

 $\textbf{Definition 1.} \ \textit{A subexponential signature is an ordered quintuple:}$

$$\Sigma = \langle I, \leq, W, C, E \rangle,$$

where $I = \{s_1, \ldots, s_n\}, \langle I, \leq \rangle$ is a preorder. W, C, E are subsets of I and $W \cup C \subseteq E$.

Definition 2. Noncommutative linear logic with subexponentials $(SMALC_{\Sigma})$, where Σ is a subexponential signature.

$$\frac{\Gamma, \Delta, !^{s} A, \Theta \Rightarrow B}{\Gamma, !^{s} A, \Delta, \Theta \Rightarrow A} \mathbf{ex}_{1}, s \in E$$

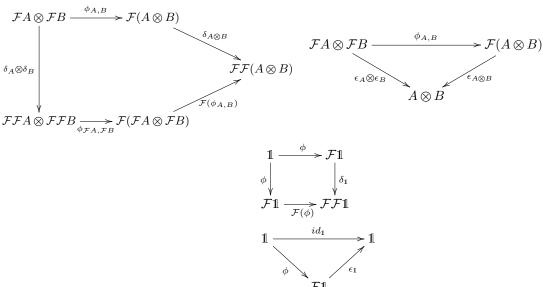
$$\frac{\Gamma, !^s A, \Delta, \Theta \Rightarrow B}{\Gamma, \Delta, !^s A, \Theta \Rightarrow A} \mathbf{ex}_1, s \in E$$

Lemma 1. Let $A \Leftrightarrow B$, then $C[p_i := A] \Leftrightarrow C[p_i := B]$

Proof. By induction on C.

Definition 3. Monoidal comonad

A monoidal comonad on some monoidal category C is a triple $\langle \mathcal{F}, \epsilon, \delta \rangle$, where \mathcal{F} is a monoidal endofunctor and $\epsilon : \mathcal{F} \Rightarrow Id_{\mathcal{C}}$ (counit) and $\epsilon : \mathcal{F} \Rightarrow \mathcal{F}^2$ (comultiplication), such that the following diagrams commute:



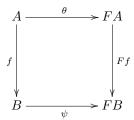
Definition 4. Biclosed monoidal category

Let C be a monoidal category. Biclosed monoidal category is a monoidal category with the following additional data:

- 1. Bifunctors $_$ \circ — $, <math>_$ \rightarrow $_$: $\mathcal{C}^{op} \times \mathcal{C} \rightarrow \mathcal{C}$;
- 2. Natural isomorphism $\mathbf{curry}_{A,B,C} : Hom(A \otimes B, C) \cong (B, A \multimap C);$
- 3. Natural isomorphism $\mathbf{curry}'_{A,B,C} : Hom(A \otimes B, C) \cong (A, C \multimap B);$
- 4. For each $A, B \in Ob_{\mathcal{C}}$, there are exist arrows $ev_{A,B} : A \otimes (A \Rightarrow B) \to B$ and $ev_{A,B}' : (B \Leftarrow A) \otimes A \to B$, such that for all $f : A \otimes C \to B$:
 - (a) $\Lambda_l \circ (id_A \otimes \mathbf{curry}(f)) = f$;
 - (b) $\Lambda_r \circ (\mathbf{curry}'(f) \otimes id_A) = f$

Definition 5. Let F be endofunctor and $A \in Ob\mathcal{C}$, then a coalgebra of F is a tuple $\langle A, \theta \rangle$, where $\theta : A \to FA$.

Given coalgebras $\langle A, \theta \rangle$ and $\langle A, \psi \rangle$, a homomorphism is a morphism $f: A \to B$, s.t. the diagram below commutes:



that is, $Ff \circ \theta = \psi \circ f$

Definition 6. Subexponential model structure

Let $\Sigma = \langle I, \leq, W, C, E \rangle$ be a subexponential signature and \mathcal{C} be a biclosed monoidal category, then a subexponential model structure is $\langle \mathcal{C}, \{\mathcal{F}_s\}_{s \in I} \rangle$ with the following additional data:

- for all $s \in I$, \mathcal{F}_s is a monoidal comonad;
- if $s \in W$, then for all $A \in Ob(\mathcal{C})$, there exists a morphism $w_{A_s} : F_s A \to 1$;
- if $s \in C$, then for all $A \in Ob(C)$, there exists morphisms $w_{Al} : F_sA \otimes A \otimes F_sA \to F_sA \otimes B$ and $w_{Ar} : F_sA \otimes A \otimes F_sA \to B \otimes F_sA$;
- if $s \in E$, then for all $A \in Ob(\mathcal{C})$, there is an isomorpism, $e_A : F_sA \otimes B \cong B \otimes F_sA$;
- if $s_1 \in W$, $s_2 \in I$ and $s_1 \leq s_2$, then there is a morphism $w_{As_2} : F_{s_2}A \to \mathbb{1}$ for all $A \in Ob(\mathcal{C})$ and ditto for E and C;
- Let $\bigotimes_{s\in J,i=0}^n F_s A$, where $J\subset I$, and $s'\in I$, s.t. $s\geq s'$ for all $s\in I'$; Then there exists morphism a morphism $\theta\bigotimes_{s\in J,i=1}^n F_{sj}A_i:\bigotimes_{s\in J,i=0}^n F_s A\to F_{s'}(\bigotimes_{s\in J,i=0}^n F_s A)$, such that $\bigotimes_{s\in J,i=1}^n F_{sj}A_i,\theta\bigotimes_{s\in J,i=1}^n F_{sj}A_i$ is a coalgebra on F_s .

Definition 7. Let $\langle \mathcal{C}, \{\mathcal{F}_s\}_{s\in I} \rangle$ be a subexponential model structure for subexponential signature $\Sigma = \langle I, \leq, W, C, E \rangle$. Let $v: Tp \to Ob(\mathcal{C})$ be a valuation map. Then the interpretation function $[\![.]\!]$ is defined as follows:

- (1) [1] = 1
- $(2) \quad \llbracket A \backslash B \rrbracket = \llbracket A \rrbracket \multimap \llbracket B \rrbracket$
- $(3) \quad \overline{\llbracket A/B \rrbracket} = \overline{\llbracket A \rrbracket} \smile \overline{\llbracket B \rrbracket}$
- $(4) \quad \llbracket A \bullet B \rrbracket = \llbracket A \rrbracket \otimes \llbracket B \rrbracket$
- $(5) \quad \llbracket !_s A \rrbracket = F_s \llbracket A \rrbracket$

Theorem 1. The following statements are equivalent:

- $SMLC_{\Sigma} + (cut) \vdash \Gamma \Rightarrow A$
- $SMLC_{\Sigma} \vdash \Gamma \Rightarrow A$
- $\exists f, f : \llbracket \Gamma \rrbracket \to \llbracket A \rrbracket$

Proof.

- $(1) \Rightarrow (2)$: cut elimination.
- $(2) \Rightarrow (3)$: Soundness:

$$\frac{f: \Gamma \otimes (F_s A \otimes B \otimes F_s A) \otimes \Delta \to C}{f \circ (id_{\Gamma} \otimes c_A{}^l_s \otimes id_{\Delta}): \Gamma \otimes (F_s A \otimes B) \otimes \Delta \to C}$$

$$\frac{f: \Gamma \otimes (F_s A \otimes B \otimes F_s A) \otimes \Delta \to C}{(id_{\Gamma} \otimes c_A{}^r_s \otimes id_{\Delta}) \circ f: \Gamma \otimes (B \otimes F_s A) \otimes \Delta \to C}$$

$$\frac{f: \Gamma \otimes (\Delta \otimes F_s A) \otimes \Theta \to B}{(id_{\Gamma} \otimes (id_{\Delta} \otimes e_{A_s}) \otimes id_{\Theta}) \circ f: \Gamma \otimes (F_s A \otimes \Delta) \otimes \Theta \to B}$$

$$\frac{f: \Gamma \otimes (F_s A \otimes \Delta) \otimes \Theta \to B}{(id_{\Gamma} \otimes (id_{\Delta} \otimes e_{A_s}) \otimes id_{\Theta}) \circ f: \Gamma \otimes (\Delta \otimes F_s A) \otimes \Theta \to B}$$

$$\frac{f: \Gamma \otimes (F_s A \otimes \Delta) \otimes \Theta \to B}{(id_{\Gamma} \otimes (id_{\Delta} \otimes e_{A_s}) \otimes id_{\Theta}) \circ f: \Gamma \otimes (\Delta \otimes F_s A) \otimes \Theta \to B}$$

• Completeness:

Definition 8.

1 Concrete model

Definition 9. Quantale A quantale is a triple $\langle A, \bigvee, \cdot \rangle$, such that $\langle A, \bigvee \rangle$ is a complete lattice and $\langle A, \cdot \rangle$ is a semigroup. A quantate is called unital, if $\langle A, \cdot \rangle$ is a monoid.

It is easy to see, that any (unital) quantale is a residual (monoid) semigroup. We define divisions as follows:

1.
$$a \setminus b = \bigvee \{c \mid a \cdot c \leqslant b\}$$

2.
$$b/a = \bigvee \{c \mid c \cdot a \leq b\}$$

Definition 10. Let $\langle A, \bigvee, \cdot \rangle$ be a quantale. The center of a quantale is the set $Z(Q) = \{a \in Q \mid \forall b \in Q, a \cdot b = b \cdot a\}$

Definition 11. An open modality on quantale Q is a map $I: Q \to Q$, such that

1.
$$I(x) \le x$$
;

2.
$$I(x) = I(I(x));$$

3.
$$x \leq y \Rightarrow I(x) \leq I(y)$$
;

4.
$$I(x) \cdot I(y) = I(I(x) \cdot I(y))$$
.

Lemma 2.

Let $\langle A, \bigvee, \cdot \rangle$ be a quantale and $I: Q \to Q$ is an open modality on Q, then $I(x) \cdot I(y) \leq I(x \cdot y)$.

Proof.

$$I(x) \cdot I(y) \leqslant x \cdot y$$
, then $I(I(x) \cdot I(y)) \leqslant I(x \cdot y)$, but $I(x) \cdot I(y) \leqslant I(I(x) \cdot I(y))$. Thus, $I(x) \cdot I(y) \leqslant I(x \cdot y)$.

Definition 12. An open modality is called central, if $\forall a, b \in Q, I(a) \cdot b = b \cdot I(a)$.

Definition 13. An open modality is called weak idempotent, if $\forall a, b \in Q, I(a) \cdot b \leq I(a) \cdot b \cdot I(a)$ and $b \cdot I(a) \leq I(a) \cdot b \cdot I(a)$.

Definition 14. An open modality is called unital, if $\forall a \in Q, I(a) \leq e$.

Lemma 3. Let I be an interior on unital quantale $\langle Q, \bigvee, \cdot, e \rangle$. Then, if I is unital and weak idempotent, then I is a central.

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Proof. \\ b \cdot I(a) \leqslant \\ \text{Right weak idempotence} \\ I(a) \cdot b \cdot I(a) \leqslant \\ \text{Unitality} \\ I(a) \cdot b \cdot I(e) \leqslant \\ \text{Identity} \\ I(a) \cdot b \leqslant \\ \text{Left weak idempotence} \\ I(a) \cdot b \cdot I(a) \leqslant \\ \text{Unitality} \\ e \cdot b \cdot I(a) \leqslant \\ \text{Identity} \\ b \cdot I(a)
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