Quantale model of Lambek calculus with subexponentials

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1 Calculus

Definition 1. *A subexponential signature is an ordered quintuple:*
$\Sigma = \langle I, \preceq, W, C, E \rangle$,

where $I = \{s_1, \ldots, s_n\}$, $\langle I, \preceq \rangle$ is a preorder. W, C, E are subsets of I and $W \cup C \subseteq E$.

Definition 2. *Noncommutative linear logic with subexponentials ($SMALC_\Sigma$), where Σ is a subexponential signature.*

$$\frac{}{A \Rightarrow A}\, ax$$

$$\frac{\Gamma \Rightarrow A \qquad \Delta, B, \Theta \Rightarrow C}{\Delta, \Gamma, A \backslash B, \Theta \Rightarrow C} \backslash \rightarrow \qquad \frac{A, \Pi \Rightarrow B}{\Pi \Rightarrow A \backslash B} \rightarrow \backslash$$

$$\frac{\Gamma \Rightarrow A \qquad \Delta, B, \Theta \Rightarrow C}{\Delta, B / A, \Gamma, \Theta \Rightarrow C} / \rightarrow \qquad \frac{\Pi, A \Rightarrow B}{\Pi \Rightarrow B / A} \rightarrow /$$

$$\frac{\Gamma, A, B, \Delta \Rightarrow C}{\Gamma, A \bullet B, \Delta \Rightarrow C} \bullet \rightarrow \qquad \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \bullet B} \rightarrow \bullet$$

$$\frac{\Gamma, A_i, \Delta \Rightarrow B}{\Gamma, A_1 \& A_2, \Delta \Rightarrow B} \&, i = 1, 2 \rightarrow \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \& B} \rightarrow \&$$

$$\frac{\Gamma, A, \Delta \Rightarrow C \qquad \Gamma, B, \Delta \Rightarrow C}{\Gamma, A \vee B, \Delta \Rightarrow C} \vee \rightarrow \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2} \rightarrow \vee, i = 1, 2$$

$$\frac{\Gamma, \Delta \Rightarrow A}{\Gamma, \mathbf{1}, \Delta \Rightarrow A} \mathbf{1} \rightarrow \qquad \frac{}{\Rightarrow \mathbf{1}} \rightarrow \mathbf{1}$$

$$\frac{\Gamma, A, \Delta \Rightarrow C}{\Gamma, !^s A, \Delta \Rightarrow C} ! \rightarrow \qquad \frac{!^{s_1} A_1, \ldots, !^{s_n} A_n \Rightarrow A}{!^{s_1} A_1, \ldots, !^{s_n} A_n \Rightarrow !^s A} \rightarrow !, \forall j, s_j \geq s$$

$$\frac{\Gamma, \Delta \Rightarrow B}{\Gamma, !^s A, \Delta \Rightarrow B} \mathbf{weak}_!, s \in C$$

$$\frac{\Gamma, !^s A, \Delta, !^s A, \Theta \Rightarrow B}{\Gamma, !^s A, \Delta, \Theta \Rightarrow B}\ \mathbf{ncontr}_1, s \in C$$

$$\frac{\Gamma, !^s A, \Delta, !^s A, \Theta \Rightarrow B}{\Gamma, \Delta, !^s A, \Theta \Rightarrow B}\ \mathbf{ncontr}_2, s \in C$$

$$\frac{\Gamma, \Delta, !^s A, \Theta \Rightarrow B}{\Gamma, !^s A, \Delta, \Theta \Rightarrow A}\ \mathbf{ex}_1, s \in E$$

$$\frac{\Gamma, !^s A, \Delta, \Theta \Rightarrow B}{\Gamma, \Delta, !^s A, \Theta \Rightarrow A}\ \mathbf{ex}_1, s \in E$$

Lemma 1. *Let* $A \Leftrightarrow B$*, then* $C[p_i := A] \Leftrightarrow C[p_i := B]$

Proof. By induction on C. □

Lemma 2.
- $!_{s_i}\Gamma \to A$ *iff* $!_{s_i}\Gamma \to !_{s_i} A$.
- $!_{s_i} A \leftrightarrow !_{s_i}(!_{s_i} A)$

Proof.

1. $!_{s_i}\Gamma \to A$ iff $!_{s_i}\Gamma \to !_{s_i} A$;

$$\frac{!_{s_i}\Gamma \to A}{!_{s_i}\Gamma \to !_{s_i} A} \to !_{s_i}$$

$$\dfrac{!_{s_i}\Gamma \to !_{s_i} A \qquad \dfrac{A \to A}{!_{s_i} A \to A}\ !_{s_i} \to}{!_{s_i}\Gamma \to A}\ \text{cut}$$

2. $!_{s_i} A \leftrightarrow !_{s_i} !_{s_i} A$

$$\dfrac{\dfrac{\dfrac{A \to A}{!_{s_i} A \to A}}{!_{s_i} A \to !_{s_i} A}}{!_{s_i} !_{s_i} A \to !_{s_i} A}$$

□

2 Semantics

Definition 3. *Quantale*

A quantale is a triple $\langle A, \bigvee, \cdot \rangle$*, such that* $\langle A, \bigvee \rangle$ *is a complete lattice and* $\langle A, \cdot \rangle$ *is a semi-group. A quanlate is called unital, if* $\langle A, \cdot \rangle$ *is a monoid.*

It is easy to see, that any (unital) quantale is a residual (monoid) semigroup. We define divisions as follows:

1. $a \backslash b = \bigvee \{c \mid a \cdot c \leqslant b\}$

2. $b/a = \bigvee\{c \mid c \cdot a \leqslant b\}$

Definition 4.
Let $\langle A, \bigvee, \cdot\rangle$ be a quantale. The center of a quantale is the set $Z(Q) = \{a \in Q \mid \forall b \in Q, a \cdot b = b \cdot a\}$

Definition 5. *An open modality (or quantic conucleus) on quantale Q is a map $I : Q \to Q$, such that*

1. $I(x) \leqslant x$;
2. $I(x) = I(I(x))$;
3. $x \leqslant y \Rightarrow I(x) \leqslant I(y)$;
4. $I(x) \cdot I(y) = I(I(x) \cdot I(y))$.

Definition 6. *We define a partial order on open modalities on Q as $I_1 \leqslant I_2 \Leftrightarrow \forall a \in Q, I_1(a) \leqslant I_2(a)$.*

Lemma 3. *Let Q be a quantale and $\mathcal{I}$ be a set of all open modalities on Q. Then $\mathcal{I}$ for a locally small category.*

Proof. $\langle \mathcal{I}, \leqslant\rangle$ form a partial order, so $\langle \mathcal{I}, \leqslant\rangle$ is a locally small category. □

Lemma 4.
Let $\langle A, \bigvee, \cdot\rangle$ be a quantale and $I : Q \to Q$ is an open modality on Q, then $I(x)\cdot I(y) \leqslant I(x\cdot y)$.

Proof.
$I(x) \cdot I(y) \leqslant x \cdot y$, then $I(I(x) \cdot I(y)) \leqslant I(x \cdot y)$, but $I(x) \cdot I(y) \leqslant I(I(x) \cdot I(y))$. Thus, $I(x) \cdot I(y) \leqslant I(x \cdot y)$. □

Definition 7. *An open modality is called central, if $\forall a, b \in Q, I(a) \cdot b = b \cdot I(a)$.*

Definition 8. *An open modality is called weak idempotent, if $\forall a, b \in Q, I(a) \cdot b \leqslant I(a) \cdot b \cdot I(a)$ and $b \cdot I(a) \leqslant I(a) \cdot b \cdot I(a)$.*

Definition 9. *An open modality is called unital, if $\forall a \in Q, I(a) \leqslant e$.*

Lemma 5. *Let I be an interior on some unital quantale $\langle Q, \bigvee, \cdot, e\rangle$. Then, if I is unital and weak idempotent, then I is central.*

Proof.

$b \cdot I(a) \leqslant$
Right weak idempotence
$I(a) \cdot b \cdot I(a) \leqslant$
Unitality
$I(a) \cdot b \cdot I(e) \leqslant$
Identity
$I(a) \cdot b \leqslant$
Left weak idempotence
$I(a) \cdot b \cdot I(a) \leqslant$
Unitality
$e \cdot b \cdot I(a) \leqslant$
Identity
$b \cdot I(a)$

Hence, $b \cdot I(a) = I(a) \cdot b$

□

Proposition 1.

Let Q be a quantale and $S \subseteq Q$ a subquantale, then $I : Q \to Q$, such that $I(a) = \bigvee\{s \in S \mid x \leqslant a\}$, is an open modality. Moreover, $\{x \in Q \mid I(x) = x\} = S$.

Proof. See □

Proposition 2.

Let Q be a quantale and $S_1, S_2 \subseteq Q$, such that $S_1 \subseteq S_2$.
Then $I_1(a) \leqslant I_2(a)$.

Proof.

Let $a \in Q$, so $\{s \in S_1 \mid s \leqslant a\} \subseteq \{s \in S_2 \mid s \leqslant a\}$, so $\bigvee\{s \in S_1 \mid s \leqslant a\} \subseteq \bigvee\{s \in S_2 \mid s \leqslant a\}$. Thus, $I_1(a) \leqslant I_2(a)$. □

Proposition 3.

Let Q be a quantale and $S \subseteq Q$ a subquantale, then the following operations are open modalities:

1. $I_z(a) = \bigvee\{s \in S \,|s \leqslant a, s \in Z(Q)\}$;
2. $I_{\mathbb{1}}(a) = \bigvee\{s \in S \,|s \leqslant a, s \leqslant \mathbb{1}\}$;
3. $I_{idem}(a) = \bigvee\{s \in S \,|s \leqslant a, \forall b \in Q, b \cdot s \vee s \cdot b \leqslant s \cdot b \cdot s\}$;
4. $I_{z,\mathbb{1}}, I_{z,idem}, I_{\mathbb{1},idem}, I_{z,\mathbb{1},idem}$.

Proof. Immediatly. □

Proposition 4.

1. $\forall a \in Q, I_{\mathbb{1},idem}(a) \leqslant I_z(a)$.
2. $\forall a \in Q, I_{z,\mathbb{1},idem} = I_{\mathbb{1},idem}(a)$

Proof. Follows from Lemma 3. □

Proposition 5.

1. $I_z(a) \vee I_{\mathbb{1}}(a) \vee I_{idem}(a) \leqslant I(a)$
2. $I_{z,\mathbb{1},idem} \leqslant I_{z,\mathbb{1}}(a) \wedge I_{z,idem}(a)$

Lemma 6. $\forall a \in Q, I_1(a) \leqslant I_2(I_1(a))$, *if* $I_1(a) \leqslant I_2(a)$.

Proof. $I_1(a) \leqslant I_1(I_1(a)) \leqslant I_2(I_1(a))$ □

Lemma 7. $I_1(a_1) \cdot I_2(a_2) \leqslant I^{'}(I_1(a_1) \cdot I_2(a_2))$, *where* $I_i \leqslant I^{'}, i = 1, 2$.

Proof.

$$\begin{aligned} &I_1(a_1) \cdot I_2(a_2) \leqslant \\ &I_1(I_1(a_1)) \cdot I_2(I_2(a_2)) \leqslant \\ &I^{'}(I_1(a_1)) \cdot I^{'}(I_2(a_2)) \leqslant \\ &I^{'}(I_1(a_1) \cdot I_2(a_2)) \end{aligned}$$

□

Definition 10. *Interpretation of subexponential signature*

Let $\Sigma = \langle I, \preceq, W, C, E\rangle$ be a subexponential signature, where $|I| = n$ and $\mathcal{Q}$ is a category of open modalities on Q. Subexponential interpretation is a contravariant functor $\sigma : I \to \mathcal{Q}$ defined as follows:

$$\sigma(s_i) = \begin{cases} \Box_i : Q \to Q,\ s.t. \forall a \in Q, \Box_i(a) = \{s \in S_i \mid s \leqslant a\}, \\ \quad \text{if } s_i \notin W \cap C \cap E \\ \Box_i : Q \to Q,\ s.t. \forall a \in Q, \Box_i(a) = \{s \in S_i \mid s \leqslant a, \leqslant \mathbb{1}\}, \\ \quad \text{if } s_i \in W \\ \Box_i : Q \to Q,\ s.t. \forall a \in Q, \Box_i(a) = \{s \in S_i \mid s \leqslant a, \in Z(Q)\}, \\ \quad \text{if } s_i \in E \\ \Box_i : Q \to Q,\ s.t. \forall a \in Q, \Box_i(a) = \{s \in S_i \mid s \leqslant a, \forall b, b \cdot s \vee s \cdot b \leqslant s \cdot b \cdot s\}, \\ \quad \text{if } s_i \in E \\ \text{otherwise, if } s_i \text{ belongs to some intersection of subsets, then we combine the relevant conditions} \end{cases}$$

Definition 11. *Let Q be a quantale, $f : Tp \to Q$ a valuation and $\sigma : I \to \mathcal{S}$ a subexponential interpretation, then interpretation is defined inductively:*

$$\begin{aligned} &[\![p_i]\!] = f(p_i) \\ &[\![\mathbb{1}]\!] = e \\ &[\![A \bullet B]\!] = [\![A]\!] \cdot [\![B]\!] \\ &[\![A \backslash B]\!] = [\![A]\!] \backslash [\![B]\!] \\ &[\![A / B]\!] = [\![A]\!] / [\![B]\!] \\ &[\![A \& B]\!] = [\![A]\!] \wedge [\![B]\!] \\ &[\![A \vee B]\!] = [\![A]\!] \vee [\![B]\!] \\ &[\![!_{s_i} A]\!] = \sigma(s_i)[\![A]\!] \end{aligned}$$

Definition 12. $\Gamma \models A \Leftrightarrow \forall f \forall \sigma, [\![\Gamma]\!] \leqslant [\![A]\!]$

Theorem 1. $\Gamma \to A \Rightarrow [\![\Gamma]\!] \leqslant [\![A]\!]$

Proof. We consider cases with modal rules.

1. Let $!_{s_1} A_1, \ldots, !_{s_n} A_n \to A$ and $\forall i, s \preceq s_i$.
   Then $\forall a \in Q, \sigma(s_i)(a) \leqslant \sigma(s)(a)$.
   By IH, $\sigma(s_1)[\![A_1]\!] \cdot \cdots \cdot \sigma(s_n)[\![A_n]\!] \leqslant [\![A]\!]$.
   Thus, $\sigma(s)(\sigma(s_1)[\![A_1]\!] \cdot \cdots \cdot \sigma(s_n)[\![A_n]\!]) \leqslant \sigma(s)([\![A]\!])$.
   By Lemma 5, $\sigma(s_1)[\![A_1]\!] \cdot \cdots \cdot \sigma(s_n)[\![A_n]\!] \leqslant \sigma(s)(\sigma(s_1)[\![A_1]\!] \cdot \cdots \cdot \sigma(s_n)[\![A_n]\!])$.
   So, $\sigma(s_1)[\![A_1]\!] \cdot \cdots \cdot \sigma(s_n)[\![A_n]\!] \leqslant \sigma(s)([\![A]\!])$.

2. Let $\Gamma, A, \Delta \to B$.
   By IH, $[\![\Gamma]\!] \cdot [\![A]\!] \cdot [\![\Delta]\!] \leqslant [\![B]\!]$.
   By the definition, $\sigma(s_i)([\![A]\!]) \leqslant [\![A]\!]$.
   So, $[\![\Gamma]\!] \cdot \sigma(s_i)([\![A]\!]) \cdot [\![\Delta]\!] \leqslant [\![B]\!]$

3. Let $\Gamma, \Delta \to B$, $A \in Fm$, and $s_i \in W$.
   So, $[\![\Gamma]\!] \cdot [\![\Delta]\!] \leqslant [\![B]\!]$, then $[\![\Gamma]\!] \cdot e \cdot [\![\Delta]\!] \leqslant [\![B]\!]$, where $e \in Q$ is unit.
   By the definition of unital open modality, $\sigma(s_i)([\![A]\!]) \leqslant e$.
   Thus, $[\![\Gamma]\!] \cdot \sigma(s_i)([\![A]\!]) \cdot [\![\Delta]\!] \leqslant [\![B]\!]$.

4. Let $\Gamma, !_{s_i} A, \Delta, !_{s_i} A, \Pi \to B$ and $s_i \in C$.
   By IH, $[\![\Gamma]\!] \cdot \sigma(s_i)([\![A]\!]) \cdot [\![\Delta]\!] \cdot \sigma(s_i)([\![A]\!]) \cdot [\![\Pi]\!] \leqslant [\![B]\!]$.
   By the definition, $\sigma(s_i)([\![A]\!]) \cdot [\![\Delta]\!] \leqslant \sigma(s_i)([\![A]\!]) \cdot [\![\Delta]\!] \cdot \sigma(s_i)([\![A]\!])$.
   Then $[\![\Gamma]\!] \cdot \sigma(s_i)([\![A]\!]) \cdot [\![\Delta]\!] \cdot [\![\Pi]\!] \leqslant [\![B]\!]$

5. Let $\Gamma, !_{s_i} A, \Delta, \Pi \to B$ and $s_i \in E$, so $\sigma(s_i)(a) \in Z(Q)$ for all $a \in Q$ by the definition.
   By IH, $[\![\Gamma]\!] \cdot \sigma(s_i)([\![A]\!]) \cdot [\![\Delta]\!] \cdot [\![\Pi]\!] \leqslant [\![B]\!]$
   Hence, $[\![\Gamma]\!] \cdot [\![\Delta]\!] \cdot \sigma(s_i)([\![A]\!]) \cdot [\![\Pi]\!] \leqslant [\![B]\!]$.

□

3 Quantale completeness

Definition 13.
Let $\mathcal{F} \subseteq Fm$, an ideal is a subset $\mathcal{I} \subseteq \mathcal{F}$, such that:

- *If $B \in \mathcal{I}$ and $A \to B$, then $A \in \mathcal{I}$;*
- *If $A, B \in \mathcal{I}$, then $A \vee B \in \mathcal{I}$.*

Definition 14.
Let $S \subseteq \mathcal{F} \subseteq Fm$, then $\bigvee S = \bigcap \{\mathcal{I} \subseteq \mathcal{F} \mid S \subseteq \mathcal{I}\}$

Proposition 6. *$\bigvee S$ is an ideal.*

Lemma 8. *$A \subseteq Fm$, then $\{B \mid B \to A'\} = \bigvee A$.*

Proof.
Let $A \subseteq Fm$. Then $\{B \mid B \to A', A' \in A\} \subseteq \bigvee A$, so far as $A' \to A'$ by axiom.
On the other hand, $\{B \mid B \to A', A' \in A\}$ is an ideal, hence, $A \subseteq \{B \mid B \to A', A' \in A\}$. □

Lemma 9. *$\bigvee A \subseteq \bigvee B$ iff $\forall A' \in A, \forall B' \in B, A' \to B'$.*

Proof. Let $\bigvee A \subseteq \bigvee B$, then $\{C | C \to A', A' \in A\} \subseteq \{D \mid D \to B', B' \in B\}$.
Thus, for all $A' \in A$, $A' \in \{C | C \to A', A' \in A\}$, then $A' \in \{D \mid D \to B', B' \in B\}$, hence $A' \to B'$, for all $B' \in B$.
On the other hand, let $A' \to B'$ for all $A' \in A$, $B' \in B$ and $C \in \bigvee A$.
Thus, $C \to A'$, then $C \to B'$ by cut, so $C \in B'$.

□

Lemma 10. *Let $\mathcal{Q} = \{\bigvee S \mid S \subseteq Fm\}$ and $\bigvee \mathcal{A} \cdot \bigvee \mathcal{B} = \{A \bullet B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$. Then $\langle \mathcal{Q}, \subseteq, \cdot, \bigvee \mathbf{1} \rangle$ is a quantale.*

Proof. See □

Lemma 11. *Let $!_s \in I$, $I \notin W \cap E \cap C$, then $\Box_s(\bigvee A) = \bigvee \{B \,|\, B \to \bigvee !_s A^{'}, A^{'} \in A\}$ is a quantic conucleus.*

Proof.

1. $\Box_s(\bigvee A) \subseteq \bigvee A$;
   Let $B \in \Box_s(\bigvee A)$, then for all $A^{'} \in A$, $B \to !_s A^{'}$, but $!_s A^{'} \to A^{'}$, then $B \to A^{'}$, so $B \in \bigvee A$.

2. $\Box_s(\Box_s(\bigvee A)) = \bigvee \Box_s(\bigvee A)$;

$\Box_s(\Box_s(\bigvee A)) =$
$\{B \mid B \to \bigvee !_s!_sA', A' \in A\} =$, that follows from equivalence $!_s!_sB \leftrightarrow !_sB$.
$\{B \mid B \to \bigvee !_sA', A' \in A\}$

3. $\bigvee A \subseteq \bigvee B \Rightarrow \Box_s(\bigvee A) \subseteq \Box_s(\bigvee B)$;

Follows from admissiability of K-rule for all $s \in I$.

4. $\Box_s \bigvee A \cdot \Box_s \bigvee B = \Box_s(\Box_s \bigvee A \cdot \Box_s \bigvee B)$.

$\Box_s \bigvee A \cdot \Box_s \bigvee B =$
$\bigvee\{C \bullet D \mid C \bullet D \to !_sA' \bullet !_sB'\} =$
$\bigvee\{C \bullet D \mid C \bullet D \to !_s(!_sA' \bullet !_sB')\} =$
$\Box_s(\Box_s \bigvee A \cdot \Box_s \bigvee B)$

□

Lemma 12.

1. *Let $s \in W$, then for all $A \subseteq Fm$, $\mathbf{1} \in \Box_s(\bigvee A)$;*
2. *Let $s \in E$, then $\Box_s(\bigvee A) \cdot \bigvee B = \bigvee B \cdot \Box_s(\bigvee A)$.*
3. *Let $s \in C$, then $(\Box_s \bigvee A \cdot \bigvee B) \cup (\bigvee B \cdot \Box_s \bigvee A) \subseteq \Box_s \bigvee A \cdot \bigvee B \cdot \Box_s \bigvee A$, for all $B \subseteq Fm$.*

Proof. 1. Let $s \in W$, then for all $A \subseteq Fm$, $\Box_s(\bigvee A) = \{!_sB \mid !_sB \to A', A' \in A\}$. But, $!_sB \to \mathbf{1}$, hence, $1 \in \Box_s(\bigvee A)$, so far as $\Box_s(\bigvee A)$ is an ideal.

2.

$\Box_s(\bigvee A) \cdot \bigvee B =$
$\bigvee\{!_sC \bullet D \mid !_sC \bullet D \to A' \bullet B', A' \in A, B' \in B\} =$
$\bigvee\{D \bullet !_sC \mid D \bullet !_sC \to A' \bullet B', A' \in A, B' \in B\} =$
$\bigvee B \cdot \Box_s(\bigvee A)$

3.

$\Box_s \bigvee A \cdot \bigvee B = \bigvee\{!_sC \bullet D \mid !_sC \bullet D \to A' \bullet B'\}$. $!_sC \bullet D \to !_sC \bullet D \bullet !_sC$, hence $\Box_s \bigvee A \cdot \bigvee B \subseteq \Box_s \bigvee A \cdot \bigvee B \cdot \Box_s \bigvee A$.

Similarly with $\bigvee B \cdot \Box_s \bigvee A$.

□

Lemma 13.
Let $i, j \in I$ and $i \preceq j$, then for all $A \subseteq Fm$, $\Box_j(\bigvee A) \subseteq \Box_i(\bigvee A)$.

Proof. Let $i, j \in I$ and $i \preceq j$. Let $B \in \Box_j(\bigvee A)$, then $\forall A'$, $B \to !_jA'$.

But $!_jA \to !_iA$. Then $B \to !_iA$ by hence. So, $B \in \Box_j(\bigvee A)$.

□

Definition 15. *Let Q be a syntactic quantale as proposed above and $\mathcal{I} = \langle I, \preceq, W, C, E\rangle$ be a subexponential signature.*

We define a map $\Box : \mathcal{I} \to Mod_Q$ as follows:
$\Box(i)(\bigvee A) = \{B \mid B \to !_iA\}$.

Lemma 14. $\Box$ *is a subexponential interpretation.*

Proof. Follows from lemmas 11 and 12. $\Box$

Lemma 15.

Let Q be a quantale constructed above and $\Box_1, \ldots, \Box_n$ be a family of quantic conuclei on Q. Then there exist a model $\langle Q, [\![.]\!]\rangle$, such that $[\![A]\!] = \bigvee\{A\}$, $A \in Fm$.

Proof.

We define an interpreation as follows:

1. $[\![p_i]\!] = \bigvee\{p_i\}$
2. $[\![\mathbf{1}]\!] = \bigvee\{\mathbf{1}\}$
3. $[\![A \bullet B]\!] = \bigvee\{A \bullet B\}$
4. $[\![A/B]\!] = \bigvee\{A/B\}$
5. $[\![B\backslash A]\!] = \bigvee\{B\backslash A\}$
6. $[\![A\&B]\!] = \bigvee\{A\&B\}$
7. $[\![A \vee B]\!] = \bigvee\{A \vee B\}$
8. $[\![!_s A]\!] = \Box(s)(\bigvee A) = \{B \mid B \to !_s A\} = \bigvee\{!_s A\}$

$\Box$

Theorem 2. $\Gamma \models A \Rightarrow \Gamma \to A$.

Proof. Follows from lemmas 9, 12, 13, 14. $\Box$