

On R -models

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Definition 1.

R -model is a pair $\mathcal{M} = \langle W, R, v \rangle$, where R is a transitive relation on W and $v : Tp \rightarrow 2^R$ is a valuation, such that:

1. $\mathcal{M}, w \Vdash p_i \Leftrightarrow w \in v(p_i)$;
2. $\mathcal{M}, \langle a, b \rangle \Vdash A \bullet B \Leftrightarrow$ there exists $c \in W$, $\mathcal{M}, a \Vdash A$ and $\mathcal{M}, b \Vdash B$;
3. $\mathcal{M}, \langle a, b \rangle \Vdash A \setminus B \Leftrightarrow$ for all $c \in R^{-1}(a)$, $\mathcal{M}, \langle c, a \rangle \Vdash A$ implies $\mathcal{M}, \langle c, b \rangle \Vdash B$;
4. $\mathcal{M}, \langle a, b \rangle \Vdash B/A \Leftrightarrow$ for all $c \in R(a)$, $\mathcal{M}, \langle a, c \rangle \Vdash A$ implies $\mathcal{M}, \langle b, c \rangle \Vdash B$;
5. $\mathcal{M}, \langle a, b \rangle \Vdash \Gamma \rightarrow A \Leftrightarrow \mathcal{M}, \langle a, b \rangle \Vdash \Gamma$ implies $\mathcal{M}, \langle a, b \rangle \Vdash A$

where $\mathcal{M}, \langle a, b \rangle \Vdash \Gamma$ denotes $\mathcal{M}, \langle a, b \rangle \Vdash A_1 \bullet \dots \bullet A_n$, or, equivalently, there exist c_1, \dots, c_{n-1} , such that $\mathcal{M}, \langle a, c_1 \rangle \Vdash A_1, \mathcal{M}, \langle c_1, c_2 \rangle \Vdash A_2, \dots, \mathcal{M}, \langle c_{n-1}, b \rangle \Vdash A_n$ implies that $\mathcal{M}, \langle a, b \rangle \Vdash B$.

Theorem 1. Let \mathbb{F} is a R -frame, then $\mathbb{F} \models L$.

Definition 2.

Let $\mathcal{F}_1, \mathcal{F}_2$ be R -frames and $\mathcal{M}_1 = \langle \mathcal{F}_1, v_1 \rangle, \mathcal{M}_2 = \langle \mathcal{F}_2, v_2 \rangle$ be R -models.

A map $f : W_1 \rightarrow W_2$ is said to be a R -frame p -morphism if the following conditions hold:

1. f is onto;
2. $\forall a, b \in W_1 (aR_1b \Rightarrow f(a)R_2f(b))$ (monotonicity);
3. $\forall d \in W_1, c \in W_2, f(d)R_2c \Rightarrow \exists c' \in W_1, f(c') = c \ \& \ dR_1c'$ (left lift property);
4. $\forall d \in W_1, c \in W_2, cR_2f(d) \Rightarrow \exists c' \in W_1, f(c') = c \ \& \ c'R_1d$ (right lift property);

A map $f : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ is R -model p -morphism, iff:

- $\mathcal{M}_1, \langle a, b \rangle \Vdash p_i \Leftrightarrow \mathcal{M}_2, \langle f(a), f(b) \rangle \Vdash p_i$

Lemma 1. Let $f : \mathcal{M}_1 \Rightarrow \mathcal{M}_2$, then $\mathcal{M}_1, \langle a, b \rangle \Vdash A \Leftrightarrow \mathcal{M}_2, \langle f(a), f(b) \rangle \Vdash A$, for all $a, b \in W_1$ and for all $A \in Fm$

Proof.

1. \Rightarrow

- (a) Basic case: follows from the definition.
- (b) Let $A = B \bullet C$ and $\mathcal{M}_1, \langle a, b \rangle \Vdash B \bullet C$, then there exists $c \in W_1$, such that $\mathcal{M}_1, \langle a, c \rangle \Vdash B$ and $\mathcal{M}_1, \langle c, b \rangle \Vdash C$.
Then, aR_1c and cR_1b , so $f(a)R_2f(c)$ and $f(c)R_2f(b)$.
Thus, by IH, $\mathcal{M}_2, \langle f(a), f(c) \rangle \Vdash B$ and $\mathcal{M}_2, \langle f(c), f(b) \rangle \Vdash C$, so $\mathcal{M}_2, \langle f(a), f(b) \rangle \Vdash B \bullet C$.
- (c) Let $A = B \setminus C$ and $\mathcal{M}_1, \langle a, b \rangle \Vdash B \setminus C$. Let $c \in W_2$, such that $cR_2f(a)$.
Then, by left lift property, there exist $d \in W_1$, such that $f(d) = c$ and dR_1a .
Thus, $\mathcal{M}_1, \langle d, a \rangle \Vdash A$ implies $\mathcal{M}_1, \langle d, b \rangle \Vdash B$.
By IH, $\mathcal{M}_2, \langle c, f(a) \rangle \Vdash A$ implies $\mathcal{M}_2, \langle c, f(b) \rangle \Vdash B$, then $\mathcal{M}_2, \langle f(a), f(b) \rangle \Vdash A \setminus B$.
- (d) Similarly to (c), but via right lift property.
- (e) Let $\mathcal{M}_1, \langle a, b \rangle \Vdash A_1, \dots, A_n \rightarrow A$.
Thus, there exist $c_1, \dots, c_{n-1} \in W_1$, such that $\mathcal{M}_1, \langle a, c_1 \rangle \Vdash A_1, \dots, \mathcal{M}_1, \langle c_{n-2}, c_{n-1} \rangle \Vdash A_n$ implies that $\mathcal{M}_1, \langle c_{n-1}, b \rangle \Vdash B$.
By IH, $\mathcal{M}_2, \langle f(a), f(c_1) \rangle \Vdash A_1, \dots, \mathcal{M}_1, \langle f(c_{n-1}), f(b) \rangle \Vdash A_n$.
So, $\mathcal{M}_2, \langle f(a), f(b) \rangle \Vdash A_1 \bullet \dots \bullet A_n$.
Thus, $\mathcal{M}_2, \langle f(a), f(b) \rangle \Vdash B$.

2. \Leftarrow

- (a) Basic case: follows from the definition.
- (b) Let $A = B \bullet C$. Let $\mathcal{M}_2, \langle f(a), f(b) \rangle \Vdash B \bullet C$. Then there exists $c \in W_2$, such that $\mathcal{M}_2, \langle f(a), c \rangle \Vdash B$ and $\mathcal{M}_2, \langle c, f(b) \rangle \Vdash C$.
So far as f is surjection, then there exists $d \in W_1$, such that $c = f(d)$, then $\mathcal{M}_2, \langle f(a), f(d) \rangle \Vdash B$ and $\mathcal{M}_2, \langle f(d), f(b) \rangle \Vdash C$, and, by IH, $\mathcal{M}_1, \langle a, d \rangle \Vdash B$ and $\mathcal{M}_1, \langle d, b \rangle \Vdash C$, then $\mathcal{M}_1, \langle a, b \rangle \Vdash B \bullet C$.
- (c) Let $A = B \setminus C$ and $\mathcal{M}_2, \langle f(a), f(b) \rangle \Vdash B \setminus C$.
Let $c \in W_1$ and cR_1a , then $f(c)R_1f(a)$ by monotonicity, so $\mathcal{M}_2, \langle f(c), f(a) \rangle \Vdash A$ implies $\mathcal{M}_2, \langle f(c), f(b) \rangle \Vdash B$.
By IH, $\mathcal{M}_1, \langle c, a \rangle \Vdash A$ implies $\mathcal{M}_1, \langle c, b \rangle \Vdash B$. Thus, $\mathcal{M}_1, \langle c, a \rangle \Vdash A \setminus B$.
- (d) Similarly to (c).
- (e) Let $\mathcal{M}_2, \langle f(a), f(b) \rangle \Vdash \Gamma \rightarrow A$, so by case (b) and IH, $\mathcal{M}_2, \langle a, b \rangle \Vdash \Gamma$, so $\mathcal{M}_\infty, \langle a, b \rangle \Vdash A$.

□

Lemma 2.

1. Let \mathcal{M}_1 and \mathcal{M}_2 be R -models and $\mathcal{M}_1 \twoheadrightarrow \mathcal{M}_2$. Then $\mathcal{M}_1 \models A$ iff $\mathcal{M}_2 \models A$.
2. Let \mathcal{F}_1 and \mathcal{F}_2 be R -frames and $\mathcal{F}_1 \twoheadrightarrow \mathcal{F}_2$, then $\mathcal{F}_1 \models A$ implies $\mathcal{F}_2 \models A$.
3. $\mathcal{F}_1 \cong \mathcal{F}_2$, then $\text{Log}(\mathcal{F}_1) = \text{Log}(\mathcal{F}_2)$.

Proof. Standardly.

□