Models of Lambek calculus with subexponentials

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Abstract

1 Introduction

Categorial grammars were initially introduced by Ajdukiewicz and Bar-Hillel [?] [2]. In the 1950s, Lambek proposed the way of proof-theoretical consideration of such grammars [3]. In this approach, language is logic and parsing is derivation via inference rules. Let us consider a quite simple example. One needs to parse this quote from the poem by Oscar Wilde called "Impression du Matin" [4]:

The Thames nocture of blue and gold
Changed to Harmony in grey

The first is to assign the correpsonding syntactic caterogies to the words as follows:

The	Thames	nocturne	of	blue	and	gold
$(np/n)/np$	np	n	$np\backslash(np/ad)$	ad	$ad/(ad\backslash ad)$	ad

Changed	to	Harmony	in	grey	
$np\backslash(s/p)$	p/np	np	$np\backslash(np/ad)$	ad	$\to s$

Here the basic reduction rules are:

- $A, A\backslash B \to B$
- $B/A, A \to B$

Here we assign the special syntactic categories to the words of this sentence. np denotes "noun phrase", n – noun, ad – adjective, p – phrase, s – sentence. This sequent denotes that this Oscar Wilde's quote is a well-formed sentence. The verb "changed" has type "np under (s over p)". In other words, one needs to apply some noun phrase ("The Thames nocturne of blue and gold") from the left, apply some phrase from the right ("Changed to Harmony in grey") and obtain sentence after that. The other syntactic categories might be considered similarly.

The general case of such derivations in categorial grammars is axiomatised via Lambek calculus [3], non-commutative linear logic.

$$\frac{}{A \to A}\,\mathrm{ax}$$

$$\frac{\Gamma \to A \qquad \Delta, B, \Theta \to C}{\Delta, \Gamma, A\backslash B, \Theta \to C}\,\backslash\to \qquad \frac{A, \Pi \to B}{\Pi \to A\backslash B}\to\backslash, \Pi \text{ is non-empty}$$

$$\frac{\Gamma \to A \qquad \Delta, B, \Theta \to C}{\Delta, B/A, \Gamma, \Theta \to C}\,/\to \qquad \frac{\Pi, A \to B}{\Pi \to B/A}\to /, \Pi \text{ is non-empty}$$

$$\frac{\Gamma, A, B, \Delta \to C}{\Gamma, A \bullet B, \Delta \to C}\bullet\to \qquad \frac{\Gamma \to A \qquad \Delta \to B}{\Gamma, \Delta \to A \bullet B}\to\bullet$$

$$\frac{\Gamma \to A \qquad \Pi, A, \Delta \to B}{\Gamma, \Pi, \Delta \to B}\,\mathbf{cut}$$

Note that, non-commutativity is no coincidence. We cannot assume that a sentence remains well-formed regardless of the word order.

As you could see, non-commutativity yields two implications: the left one (or, left division) and the right one (or, right division). Moreover, the right introduction rules for these divisions have the special restriction that claims non-emptiness of the sequence Π in the antecedent. This restriction denotes that we have no empty word that makes no sense linguistically. On the other hand, we can avoid this restriction. This calculus is called L^*. Note that, the empty word might be introduced explicilty via the constant $\mathbf{1}$. The logic $L_{\mathbf{1}}$ is a conservative extension of L^* with the following inference rules:

$$\frac{\Gamma, \Delta \to A}{\Gamma, \mathbf{1}, \Delta \to A}\,\mathbf{1}\to \qquad \frac{}{\to \mathbf{1}}\to\mathbf{1}$$

Also, we have a multiplicative conjunction or non-commutative tensor. This connective plays the role of concatenation.

One may extend Lambek calculus with so-called additive conjunction and disjunction. The inference rules for these connections are quite similar to inference rules for conjunction and disjunction in intuitionistic sequent calculus. In L-models, additive connections are intersection and union of languages:

$$\frac{\Gamma, A_i, \Delta \to B}{\Gamma, A_1 \wedge A_2, \Delta \to B}\wedge\to, i = 1,2 \qquad \frac{\Gamma \to A \qquad \Gamma \to B}{\Gamma \to A \wedge B}\to\wedge$$

$$\frac{\Gamma, A, \Delta \to C \qquad \Gamma, B, \Delta \to C}{\Gamma, A \vee B, \Delta \to C}\vee\to \qquad \frac{\Gamma \to A_i}{\Gamma \to A_1 \vee A_2}\to\vee, i = 1,2$$

2 The Lambek Calculus with subexponentials

Let us consider the following phrase as an example.

The young lady whom Childe Harold met before his pilgrimage

There is "the young lady" in the middle of this phrase. We are incapable of processing such cases in the core Lambek calculus. That is, there is no ability to extract from the middle. "Met" is a phrasal verb. Whom did Childe Halord meet? He met the young lady.

The young lady whom$_i$ Childe Harold met e_i before his pilgrimage

For this purpose, we introduce the exponential modality with exchange rule. The left one rule is a dereliction, which is similar to the left $\Box$ introduction in modal logic T.

$$\frac{\Gamma, A, \Delta \to B}{\Gamma, !A, \Delta \to B}\ (! \to) \qquad\qquad \frac{\Gamma, !A, \Delta, \Theta \to C}{\Gamma, \Delta, !A, \Theta \to C}\ \mathbf{ex}$$

The another one phrase is related to so-called parasitic extraction:

The letter that Young Werther sent to Charlotte without reading

In addition to medial extraction, we used "the letter" twice in this phrase. Thus, one need to multiply our linguistical resources in a restricted way.

The letter that$_i$ Young Werther sent e_i to Charlotte without reading e_i

Subexponential modality with non-local contraction allows one to do such operations:

$$\frac{\Gamma, !A, \Delta, !A, \Theta \to B}{\Gamma, \Delta, !A, \Theta \to B}\ \mathbf{contr} \qquad\qquad \frac{\Gamma, !A, \Delta, !A, \Theta \to B}{\Gamma, !A, \Delta, \Theta \to B}\ \mathbf{contr}$$

Note that, the usual form of contraction yields the cut inadmissibility in contrast to non-local contraction, which generalises the contraction.

It is useful to have many modalities and distinguish them in accordance with their abilities. In other words, we are going to consider the polymodal case. Let us introduce a subexponential signature, which is a preorder with upwardly closed subsets, where W denotes weakening, etc. Note that, a subexponential signature might have any cardinality you prefer: finite or infinite.

Definition 1. *A subexponential signature is an ordered quintuple:*
$\Sigma = \langle \mathcal{I}, \preceq, \mathcal{W}, \mathcal{C}, \mathcal{E} \rangle$, *where* $\langle \mathcal{I}, \preceq \rangle$ *is a preorder.* $\mathcal{W}, \mathcal{C}, \mathcal{E}$ *are upwardly closed subsets of* I *and* $\mathcal{W} \cap \mathcal{C} \subseteq \mathcal{E}$.

The last condition claims that if there are weakening and contraction, then one may also exchange as follows.

$$\cfrac{\cfrac{\Gamma, !A, \Delta, \Theta \to B}{\Gamma, !A, \Delta, !A, \Theta \to B}\ \mathbf{weak}}{\Gamma, \Delta, !A, \Theta \to B}\ \mathbf{ncontr}$$

Definition 2.
$\mathcal{F}_\Sigma ::= Tp \mid (\mathcal{F}_\Sigma / \mathcal{F}_\Sigma) \mid (\mathcal{F}_\Sigma \backslash \mathcal{F}_\Sigma) \mid (\mathcal{F}_\Sigma \bullet \mathcal{F}_\Sigma) \mid (\mathcal{F}_\Sigma \vee \mathcal{F}_\Sigma) \mid (\mathcal{F}_\Sigma \wedge \mathcal{F}_\Sigma) \mid (!_s \mathcal{F}_\Sigma)_{s \in \Sigma}$

Definition 3. *Noncommutative linear logic with subexponentials (SMALC$_\Sigma$), where* Σ *is a subexponential signature.*

$$\frac{\Gamma, A, \Delta \to C}{\Gamma, !^s A, \Delta \to C}\ ! \to \qquad\qquad \frac{!^{s_1} A_1, \ldots, !^{s_n} A_n \to A}{!^{s_1} A_1, \ldots, !^{s_n} A_n \to !^s A}\ \to !, \forall j, s_j \geq s$$

$$\frac{\Gamma, !^s A, \Delta, !^s A, \Theta \to B}{\Gamma, !^s A, \Delta, \Theta \to B}\ \mathbf{ncontr}_1, s \in C \qquad\qquad \frac{\Gamma, !^s A, \Delta, !^s A, \Theta \to B}{\Gamma, \Delta, !^s A, \Theta \to B}\ \mathbf{ncontr}_2, s \in C$$

$$\frac{\Gamma, \Delta, !^s A, \Theta \to B}{\Gamma, !^s A, \Delta, \Theta \to A}\ \mathbf{ex}_1, s \in E \qquad\qquad \frac{\Gamma, !^s A, \Delta, \Theta \to B}{\Gamma, \Delta, !^s A, \Theta \to A}\ \mathbf{ex}_2, s \in E$$

$$\frac{\Gamma, \Delta \to B}{\Gamma, !^s A, \Delta \to B}\ \mathbf{weak}_!, s \in C$$

After that, we introduce inference polymodal inference rules for non-commutative subexponentials. One can apply substructural rules only if there is the relevant index on the current modality. Also, the right introduction rule is a sort of the generalised right $\Box$-introduction a la $K4$. Modality $!_s$ might be introduced on the right only if its index is less than any other subexponential index from the antecedent. That is, if we have already used stronger modality, then we may apply the weaker one.

Let us consider the current proof-theoretical and algorithmic results on Lambek calculus with additives and subexponentials. The first of all, the cut rule is admissible. Generally, this calculus is undecidable, but the fragment without non-local contraction is PSPACE-complete. These results were obtained by Kanovich, Kuznetsov, Nigam, and Scedrov.

Theorem 1.

1. *Cut-rule is admissable;*
2. *$SMALC_\Sigma$ is undecidable, if $C \neq \varnothing$;*
3. *If C is empty, then the decidability problem of $SMALC_\Sigma$ belongs to PSPACE.*

3 Quantale background

Now we introduce a reader to quantales quite briefly. For more details, one may take a look at these books [?] [?]. The initial idea of considering quantales in linear logic belongs to Yetter [?].

Definition 4. *Quantale*
A quantale is a triple $\mathcal{Q} = \langle A, \bigvee, \cdot \rangle$, where $\langle A, \bigvee \rangle$ is a complete join semilattice and $\langle A, \cdot \rangle$ is a semigroup such that for each indexing set I:

1. $a \cdot \bigvee\limits_{i \in I} b_i = \bigvee\limits_{i \in I} (a \cdot b_i)$;
2. $\bigvee\limits_{i \in I} a_i \cdot b = \bigvee\limits_{i \in I} (a_i \cdot b)$

A quanlate is called unital, if $\langle A, \cdot \rangle$ is a monoid.

- Let A be a semigroup (monoid), then $\langle \mathcal{P}(A), \cdot, \subseteq \rangle$ is a free (unital) quantale.
- Let R be a ring and $Sub(R)$ be a set of additive subgroups of R. We define $A \cdot B$ as an additive subgroup generated by finite sums of products ab and order is defined by inclusion.
- Any locale is a quantale with $\cdot = \wedge$.

It is easy to see, that any (unital) quantale is a residual (monoid) semigroup. We define divisions as follows:

1. $a \backslash b = \bigvee \{c \mid a \cdot c \leqslant b\}$
2. $b / a = \bigvee \{c \mid c \cdot a \leqslant b\}$

Residuality property for those divisions holds straightforwardly:

$$b \leqslant a\backslash c \Leftrightarrow a \cdot b \leqslant c \Leftrightarrow a \leqslant b/c$$

Definition 5. *Let $\mathcal{Q}_1$, $\mathcal{Q}_2$ be quantales. A quantale homomorphism is a map $f : \mathcal{Q}_1 \to \mathcal{Q}_2$, such that:*

1. *for all $a, b \in \mathcal{Q}_1$, $f(a \cdot b) = f(a) \cdot f(b)$;*
2. *for all indexing set I, $f(\bigvee\limits_{i \in I} a_i) = \bigvee\limits_{i \in I} f(a_i)$.*

If $\mathcal{Q}_1$, $\mathcal{Q}_2$ are unital quantales, then a unital homomorphism if a quantale homomorphism such that $f(\varepsilon) = \varepsilon$.

Definition 6.
Let $\mathcal{Q} = \langle A, \bigvee, \cdot \rangle$ be a quantale. $\mathcal{S} \subseteq \mathcal{Q}$ is said to be a subquantale, if $\mathcal{S}$ is closed under multiplication and joins.

There occurs the following simple statement:

Proposition 1.
Let $\mathcal{Q}_1$, $\mathcal{Q}_2$ be quantales and $\mathcal{S} \subseteq \mathcal{Q}_1$ is a subquantale of $\mathcal{Q}_1$.
Then, if $f : \mathcal{Q}_1 \to \mathcal{Q}_2$ is a quantale homomorphism, then $f(\mathcal{S}) \subseteq \mathcal{Q}_2$ is a subquantale of $\mathcal{Q}_2$.

Proof.
Follows from the same statements for semigroup and semilattice homomorphisms. □

Definition 7.
Let $\mathcal{Q} = \langle A, \bigvee, \cdot \rangle$ be a quantale. The centre of a quantale is the subquantale $\mathcal{Z}(\mathcal{Q}) = \{a \in A \mid \forall b \in A, a \cdot b = b \cdot a\}$

We introduce the notion of quantic conucleus, an interior operation on quantale, which is a lax homomorphism with respect to semigroup multiplication. For unital quantales, this conucleus preserves a neutral.

Definition 8.
An open modality (or quantic conucleus) on quantale $\mathcal{Q}$ is a map $I : \mathcal{Q} \to \mathcal{Q}$, such that

1. *$Ix \leqslant x$;*
2. *$Ix = I^2x$;*
3. *$x \leqslant y \Rightarrow Ix \leqslant Iy$;*
4. *$Ix \cdot Iy = I(Ix \cdot Iy)$.*

For unital quantale, we require that $Ie = e$.

Here and below, I^2x denotes $I(Ix)$.
Note that, one may replace the last condition on the one $Ix \cdot Iy \leqslant I(x \cdot y)$. Thus, a quantic conuleus is the special case of lax monoidal monad from a category-theoretic point of view.

Definition 9.
We define a partial order on quantic conuclei on $\mathcal{Q}$ as $I_1 \leqslant I_2 \Leftrightarrow \forall a \in \mathcal{Q}, I_1a \leqslant I_2a$.

Lemma 1. *$I_1a_1 \cdot I_2a_2 \leqslant I(I_1a_1 \cdot I_2a_2)$, where $I_i \leqslant I, i = 1, 2$.*

Proof.
$I_1a_1 \cdot I_2a_2 \leqslant I_1(I_1a_1) \cdot I_2(I_2a_2) \leqslant I(I_1a_1) \cdot I(I_2a_2) \leqslant I(I_1a_1 \cdot I_2a_2)$ □

Let us define the special cases of quantic conuclei that it would be useful for our purpose:

Definition 10.

1. *A quantic conucleus is called central, if for all $a, b \in \mathcal{Q}, \Box a \cdot b = b \cdot \Box a$.*
2. *A quantic conucleus is called strong square-increasing, if for all $a, b \in \mathcal{Q}, \Box a \cdot b \leqslant \Box a \cdot b \cdot \Box a$ and $b \cdot \Box a \leqslant \Box a \cdot b \cdot \Box a$.*
3. *A quantic conucleus is called unital, if $\forall a \in Q, \Box a \leqslant e$.*

Note that centrality and strong square-increasing yield centrality.

Lemma 2.
Let I be an open modality on some unital quantale $\mathcal{Q} = \langle A, \bigvee, \cdot, e\rangle$. Then, if I is unital and weak idempotent, then I is central.

Proof.
$b \cdot Ia \leqslant$
Right strong square-increase
$Ia \cdot b \cdot Ia \leqslant$
Unitality
$Ia \cdot b \cdot e \leqslant$
Identity
$Ia \cdot b \leqslant$
Left strong square-increase
$Ia \cdot b \cdot Ia \leqslant$
Unitality
$e \cdot b \cdot Ia \leqslant$
Identity
$b \cdot Ia$
Hence, $b \cdot Ia = Ia \cdot b$, so for all $a \in \mathcal{Q}, Ia \in \mathcal{Z}(\mathcal{Q})$.
□

Proposition 2.
Let $\mathcal{Q}$ be a quantale and $S \subseteq \mathcal{Q}$ a subquantale, then $I : \mathcal{Q} \to \mathcal{Q}$, such that $\Box a = \bigvee\{s \in S \mid s \leqslant a\}$, is an open modality.

Proof. See Rosenthal. □

Proposition 3.
Let $\mathcal{Q}$ be a quantale and $\mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \mathcal{Q}$.
Then $\Box_{\mathcal{S}_1}(a) \leqslant \Box_{\mathcal{S}_1}(a)$.

Proof. Immediatly. □

Proposition 4.
Let $\mathcal{Q}$ be a quantale and $\mathcal{S} \subseteq \mathcal{Q}$ a subquantale, then the following operations are open modalities:

1. $\Box_z(a) = \bigvee\{s \in S \,|s \leqslant a, s \in \mathcal{Z}(\mathcal{Q})\}$;

2. $\Box_{\mathbb{1}}(a) = \bigvee\{s \in S \,|s \leqslant a, s \leqslant \mathbb{1}\}$;

3. $\Box_{idem}(a) = \bigvee\{s \in S \,|s \leqslant a, \forall b \in Q, b \cdot s \vee s \cdot b \leqslant s \cdot b \cdot s\}$;

4. *Operations with combinations of conditions above;*

Proof. Immediatly. □

Proposition 5.

1. $\forall a \in \mathcal{Q}, \Box_{\mathbb{1},idem}(a) \leqslant \Box_z(a)$.

2. $\forall a \in \mathcal{Q}, \Box_{z,\mathbb{1},idem} = \Box_{\mathbb{1},idem}(a)$

Proof. Follows from Lemma 3. □

We define a subexponential interpretation in a quite sophisticated manner. Suppose we have a contravariant map from signature to the set of subquantales. Here, contravariance denotes that the stronger subexponential index (in sense of preorder) maps to the weaker subquantale. The second one function maps a subquantale to its quantic conucleus according to the previous proposition. We match the index pursuant to its subset. In other words, if $s \in W$, then the result of subexponential interpretation is an unital quantic conucleus, etc. More formally:

Definition 11. *Interpretation of subexponential signature*

Let $\mathcal{Q}$ be a quantale and $\Sigma = \langle \mathcal{I}, \preceq, \mathcal{W}, \mathcal{C}, \mathcal{E}\rangle$ be a subexponential signature. Suppose, we have $S : \Sigma \to Sub(\mathcal{Q})$, a contravariant map from this subexponential signature to the set of subquantales of $\mathcal{Q}$, and map $M : \mathcal{P}(\mathcal{Q}) \to \Box_{\mathcal{Q}}$, where $\Box_{\mathcal{Q}}$ is the set of quantic conuclei on $\mathcal{Q}$. Thus, a subexponential interpretation is the map $\sigma = M \circ S$ such that:

$$\sigma(s_i) = \begin{cases} I_i : \mathcal{Q} \to \mathcal{Q} \text{ s.t. } \forall a \in Q, \Box_i(a) = \{s \in S_i \mid s \leqslant a\}, \\ \quad \text{if } s_i \notin W \cap C \cap E \\ I_i : \mathcal{Q} \to \mathcal{Q} \text{ s.t. } \forall a \in Q, \Box_i(a) = \{s \in S_i \mid s \leqslant a, s \leqslant \mathbb{1}\}, \\ \quad \text{if } s_i \in W \\ I_i : \mathcal{Q} \to \mathcal{Q} \text{ s.t. } \forall a \in Q, \Box_i(a) = \{s \in S_i \mid s \leqslant a, s \in \mathcal{Z}(\mathcal{Q})\}, \\ \quad \text{if } s_i \in E \\ I_i : \mathcal{Q} \to \mathcal{Q} \text{ s.t. } \forall a \in Q, \Box_i(a) = \{s \in S_i \mid s \leqslant a, \forall b, b \cdot s \vee s \cdot b \leqslant s \cdot b \cdot s\}, \\ \quad \text{if } s_i \in E \\ \text{otherwise, if } s_i \text{ belongs to some intersection of subsets, then we combine the relevant conditions} \end{cases}$$

Definition 12. *Let $\mathcal{Q}$ be an unital quantale, $f : Tp \to \mathcal{Q}$ a valuation and $\sigma : I \to \Box_{\mathcal{Q}}$ a subexponential interpretation, then interpretation is defined inductively:*

$$\begin{aligned} &[\![p_i]\!] = f(p_i) \\ &[\![\mathbb{1}]\!] = e \\ &[\![A \bullet B]\!] = [\![A]\!] \cdot [\![B]\!] \\ &[\![A \backslash B]\!] = [\![A]\!] \backslash [\![B]\!] \\ &[\![A / B]\!] = [\![A]\!] / [\![B]\!] \\ &[\![A \& B]\!] = [\![A]\!] \wedge [\![B]\!] \\ &[\![A \vee B]\!] = [\![A]\!] \vee [\![B]\!] \\ &[\![!_{s_i} A]\!] = \sigma(s_i)[\![A]\!] \end{aligned}$$

Definition 13. $\Gamma \models A \Leftrightarrow \forall f, \forall \sigma, [\![\Gamma]\!] \leqslant [\![A]\!]$

An interpretation and an entailment relation are defined standardly via valuation map and subexponential interpretation. There occur soundness and completeness theorems:

Theorem 2. $\Gamma \rightarrow A \Rightarrow [\![\Gamma]\!] \leqslant [\![A]\!]$

Proof.

We consider the promotion case, the rest modal cases are immediatly shown.
Let $!_{s_1} A_1, \ldots, !_{s_n} A_n \rightarrow A$ and $\forall i, s \preceq s_i$. Then $\forall a \in Q, \sigma(s_i)(a) \leqslant \sigma(s)(a)$.
By IH, $\sigma(s_1)[\![A_1]\!] \cdot \cdots \cdot \sigma(s_n)[\![A_n]\!] \leqslant [\![A]\!]$. Thus, $\sigma(s)(\sigma(s_1)[\![A_1]\!] \cdot \cdots \cdot \sigma(s_n)[\![A_n]\!]) \leqslant \sigma(s)([\![A]\!])$.
By Lemma, $\sigma(s_1)[\![A_1]\!] \cdot \cdots \cdot \sigma(s_n)[\![A_n]\!] \leqslant \sigma(s)(\sigma(s_1)[\![A_1]\!] \cdot \cdots \cdot \sigma(s_n)[\![A_n]\!])$.
So, $\sigma(s_1)[\![A_1]\!] \cdot \cdots \cdot \sigma(s_n)[\![A_n]\!] \leqslant \sigma(s)([\![A]\!])$. □

Now we prove completeness a la MacNeille completetion. As a matter of fact, we generalised the technique provided by Brown and Gurr for the polymodal case.

Definition 14.
Let $\mathcal{F} \subseteq Fm$, an ideal is a subset $\mathcal{I} \subseteq \mathcal{F}$, such that:

- *If $B \in \mathcal{I}$ and $A \rightarrow B$, then $A \in \mathcal{I}$;*
- *If $A, B \in \mathcal{I}$, then $A \vee B \in \mathcal{I}$.*

Definition 15.
Let $S \subseteq \mathcal{F} \subseteq Fm$, then $\bigvee S = \bigcap \{\mathcal{I} \subseteq \mathcal{F} \mid S \subseteq \mathcal{I}\}$

The following conditions hold similarly to [**?**]:

Lemma 3.

1. *$\bigvee S$ is an ideal;*
2. *$A \subseteq Fm$, then $\{B \mid B \rightarrow A\} = \bigvee \{A\}$;*
3. *$\bigvee \{A\} \subseteq \bigvee \{B\}$ iff $A \rightarrow B$;*
4. *Let $\mathcal{Q} = \{\bigvee S \mid S \subseteq Fm\}$ and $\bigvee \mathcal{A} \cdot \bigvee \mathcal{B} = \bigvee \{A \bullet B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$. Then $\langle \mathcal{Q}, \subseteq, \cdot, \bigvee \mathbf{1} \rangle$ is a unital quantale.*

We extend this construction for polymodal case as follows:

Lemma 4.

1. *Let $!_s \in I$ and $A\mathcal{F}_\Sigma$, then $\Box_s(\bigvee \{A\}) = \bigvee \{!_s B \mid !_s B \rightarrow A\}$ is a quantic conucleus*
2. *Let $A \in \mathcal{F}_\Sigma$, then $\Box_s \bigvee \{A\} = \bigvee \{!_s A\}$, for each $s \in \mathcal{I}$*
3. *Let $i, j \in I$ and $i \preceq j$, then for all $A \in \mathcal{F}_\Sigma$, $\Box_j(\bigvee \{A\}) \subseteq \Box_i(\bigvee \{A\})$.*

Proof.

1. Similarly to [Yetter].
2. Let $A \in Fm$ and $s \in \mathcal{I}$.

   Let $!_s B \in \Box_s \bigvee \{A\}$, then $!_s B \rightarrow A$, then $!_s B \rightarrow !_s A$ by promotion. So, $!_s B \in \bigvee \{!_s A\}$.

   Let $C \in \bigvee \{!_s A\}$, then $C \rightarrow !_s A$, so $!_s C \rightarrow !_s A$ by dereliction, but $!_s A \rightarrow A$, hence $!_s C \rightarrow A$ by cut. So, $!_s C \in \Box_s \bigvee \{A\}$.

3. Let $i, j \in I$ and $i \preceq j$, then forall $A \in \mathcal{F}_\Sigma$, $!_j A \rightarrow !_i A$ by promotion. Then $\bigvee\{!_j A\} \subseteq \bigvee\{!_i A\}$, so $\Box_j(\bigvee\{A\}) \subseteq \Box_i(\bigvee\{A\})$.

$\Box$

Lemma 5.

For all $A \in \mathcal{F}_\Sigma$,

1. *Let $s \in W$, then $\Box_s\{A\} \subseteq \{\mathbf{1}\}$;*
2. *Let $s \in E$, then $\Box_s(\bigvee\{A\}) \cdot \bigvee\{B\} = \bigvee\{B\} \cdot \Box_s(\bigvee\{A\})$.*
3. *Let $s \in C$, then $(\Box_s \bigvee A \cdot \bigvee B) \cup (\bigvee B \cdot \Box_s \bigvee A) \subseteq \Box_s \bigvee A \cdot \bigvee B \cdot \Box_s \bigvee A$, for all $B \subseteq Fm$.*

Proof.

1. Follows from $!_s A \rightarrow \mathbf{1}$, so $s \in W$;
2. Follows from $!_s A \bullet B \leftrightarrow B \bullet !_s A$;
3. Follows from $!_s A \bullet B \rightarrow !_s A \bullet B \bullet !_s A$ and similarly for $B \bullet !_s A$.

$\Box$

Definition 16.

Let $\mathcal{Q}$ be a syntactic quantale defined above and $\mathcal{I} = \langle \mathcal{I}, \preceq, \mathcal{W}, \mathcal{C}, \mathcal{E} \rangle$ be a subexponential signature.

We define a map $\Box : \mathcal{I} \rightarrow Mod_{\mathcal{Q}}$ as follows:

$\Box(i)(\bigvee\{A\}) = \{!_i B \mid !_i B \rightarrow A\}$.

Lemma 6. *$\Box$ is a subexponential interpretation.*

Proof. Follows from lemmas above. $\Box$

Lemma 7.

Let Q be a quantale constructed above and $(\Box_{s_i})_{s_i \in \Sigma}$ be a family of quantic conuclei on Q. Then there exist a model $\langle Q, [\![.]\!] \rangle$, such that $[\![A]\!] = \bigvee\{A\}$, $A \in Fm$.

Proof.

We define an interpreation as follows:

1. $[\![p_i]\!] = \bigvee\{p_i\}$
2. $[\![\mathbf{1}]\!] = \bigvee\{\mathbf{1}\}$
3. $[\![A \bullet B]\!] = \bigvee\{A \bullet B\}$
4. $[\![A/B]\!] = \bigvee\{A/B\}$
5. $[\![B \backslash A]\!] = \bigvee\{B \backslash A\}$
6. $[\![A \& B]\!] = \bigvee\{A \& B\}$
7. $[\![A \vee B]\!] = \bigvee\{A \vee B\}$
8. $[\![!_s A]\!] = \Box(s)(\bigvee\{A\}) = \bigvee\{!_s A\}$.

$\Box$

Theorem 3. $\Gamma \models A \Rightarrow \Gamma \rightarrow A$.

Proof. Follows from lemmas above. $\Box$

4 Relational semantics

Definition 17.
Let A be a set. Then relational quantale on A is a triple $\mathcal{Q} = \langle \mathcal{A}, \bigvee, \mathcal{I} \rangle$ where $\mathcal{A} \subseteq 2^{A \times A}$:

- *$\langle \mathcal{A}, \bigvee, \subseteq \rangle$ is a complete semi-lattice;*
- *Multiplication is defined as $R \circ S = \{\langle a, c \rangle \mid \exists b \in A, \langle a, b \rangle \in R$ and $\langle b, c \rangle \in S\}$*
- *$\langle \mathcal{A}, \circ, \mathcal{I} \rangle$ is a monoid;*
- *For each indexing set J, $R \circ \bigvee_{i \in I} S_i = \bigvee_{i \in I} (R \circ S_i)$ and $\bigvee_{i \in I} R_i \circ S = \bigvee_{i \in I} (R_i \circ S)$.*

Lemma 8.
Let $\mathcal{Q} = \langle A, \leqslant, \cdot, \bigvee \rangle$ be a unital quantale and $\mathcal{S}$ is a subquantale of $\mathcal{Q}$.
Then $\langle \mathcal{Q}, I_{\mathcal{S}} \rangle$ is isomorphic to some relational quantale of A with some quantic conucleus $\hat{I}$, where $I_{\mathcal{S}}$ is a quantic conclues on $\mathcal{S}$.

Proof.
Consider a relational quantale proposed by Brown and Gurr.
This quantale is 4-tuple $\theta(\mathcal{Q}) = \langle \mathcal{R}, \subseteq, \circ, \bigvee \rangle$ defined as follows:

1. $\theta(a) = \{\langle b, c \rangle \mid b \leqslant a \cdot c\}$;
2. $\theta(a \cdot b) = \theta(a) \circ \theta(b)$;
3. $\theta(\bigvee\limits_{i \in I} a_i) = \bigvee\limits_{i \in I} \theta(a_i)$;
4. $\theta(\varepsilon) = \{\langle b, c \rangle \mid b \cdot \varepsilon \leqslant c\} = \{\langle b, c \rangle \mid b \leqslant c\}$

Let $\mathcal{S} \subseteq \mathcal{Q}$, so $\Box_{\mathcal{S}} a := \bigvee \{s \mid s \in S, s \leqslant a\}$ is quantic conucleus.
So, $\theta(S) \subseteq \theta(\mathcal{Q})$ is a subquantale of $\theta(\mathcal{Q})$.
Let us define $\hat{\Box}\theta(a) := \bigvee \{\theta(s) \mid \theta(s) \in \theta(S), \theta(s) \subseteq \theta(a)\}$, so
$\theta(\Box_{\mathcal{S}} a) = \{\langle p, q \rangle \mid p \leqslant \Box_{\mathcal{S}} a \cdot q\} =$
$\{\langle p, q \rangle \mid p \leqslant \bigvee \{s \mid s \in \mathcal{S}, s \leqslant a\} \cdot q\} =$
Homomorphism
$\theta(\bigvee_{s \in S, s \leqslant a} s) =$
Homomorphism preserves sups
$\bigvee_{s \in S, s \leqslant a} \theta(s) =$
Unfolding
$\bigvee \{\theta(s) \mid s \in S, s \leqslant a\} =$
Unfolding
$\bigvee \{\theta(s) \mid \theta(s) \in \theta(S), \theta(s) \subseteq \theta(a)\} = \hat{\Box}\theta(a)$
Thus, $\hat{\Box}\theta(a) = \theta(\Box_{\mathcal{S}} a)$.

□

Theorem 4. *Let $\mathfrak{Q} = \langle \mathcal{Q}, I \rangle$ be a quantale with quantic conucleus. Then $\mathfrak{Q}$ is isomorphic to relational quantale on Q with some conucleus.*

Proof. Let I be a quantic conucleus. Then $\mathcal{Q}_I = \{a \in \mathcal{Q} \mid I(a) = a\}$ is a subquantale of $\mathcal{Q}$.
Let us define $I'(a) = \bigvee \{s \in \mathcal{Q}_I \mid s \leqslant a\}$.
It is clear that I' is a quantic conucleus. Let us show that for all $a \in \mathcal{Q}$ $I(a) = I'(a)$.

$I'(a) = \bigvee\{s \in \mathcal{Q}_I \mid s \leqslant a\} =$
$\bigvee\{s \in \mathcal{Q}_I \mid I(s) \leqslant a\} =$
$\bigvee\{s \in \mathcal{Q}_I \mid I(s) \leqslant I(a)\} \leqslant I(a)$

On the other hand, $I(a) \leqslant a$. So $I(a) \leqslant \bigvee\{s \in \mathcal{Q}_I \mid s \leqslant a\} = I'(a)$.

Thus, we showed that $I(a) = I'(a)$. One need to apply the previous statement with conucleus I'. □

5 Syntactic concept lattices

Definition 18. *Let $\mathcal{L}$ be a finite alphabet and $L \subseteq \mathcal{L}^*$ be a language.*

We define maps $[.]^{\rhd} : \mathcal{P}(\mathcal{L}^*) \to \mathcal{P}(\mathcal{L}^* \times \mathcal{L}^*)$ and $[.]^{\lhd} : \mathcal{P}(\mathcal{L}^* \times \mathcal{L}^*) \to \mathcal{P}(\mathcal{L}^*)$ as follows:

1. *$M \subseteq \mathcal{L}^*$, $M^{\rhd} = \{(x, y) \mid \forall w \in M, xwy \in L\}$*
2. *$C \subseteq \mathcal{L}^* \times \mathcal{L}^*$, $C^{\lhd} = \{w \mid \forall (x, y) \in C, xwy \in L\}$*

Note that compositions $[.]^{\lhd\rhd}$ and $[.]^{\rhd\lhd}$ form closure operators, by the way $[.]^{\lhd}$ and $[.]^{\rhd}$ are connected via contravariant Galois connection.

Definition 19. *A syntactic concept is a pair $\langle S, C\rangle$, where $S \subseteq \mathcal{L}^*$ and $C \subseteq \mathcal{L}^* \times \mathcal{L}^*$, such that $S^{\rhd} = C$ and $C^{\lhd} = S$.*

Following to Wurm, by the concept we mean a closed set of strings, that is, A is a concept iff $A \rhd \lhd = A$. Moreover, $\langle \mathcal{B}_{\mathcal{L}}, \bigvee, \bigwedge\rangle$, where $\mathcal{B}_{\mathcal{L}}$ is the set of $\rhd\lhd$-closed subsets of $\mathcal{L}^*$.

We define a product of concepts as $A \circ B = (A \cdot B)^{\rhd\lhd} = \{ab \mid a \in A, b \in B\}^{\rhd\lhd}$.

Residuals are defined explicitly as follows:

$$A \backslash B = \{(aB, b) \mid (a, b) \in A^{\rhd}\}^{\lhd}$$
$$B / A = \{(a, Bb) \mid (a, b) \in A^{\rhd}\}^{\lhd}$$

It is easy to see that the following condition hold for that residuals:

$$A \backslash B = \bigvee\{C \mid A \circ C \leqslant B\}$$
$$B / A = \bigvee\{C \mid C \circ A \leqslant B\}$$

Definition 20. *The syntax concept lattice of a language L is a structure $\langle \mathcal{B}_L, \circ, \bigvee, \bigwedge, /, \backslash\rangle$, where $\mathcal{B}_L$ is the set of $[.]^{\rhd\lhd}$-closed subsets of $\mathcal{L}^*$.*

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