Models of Lambek calculus enriched with subexponentials

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Definition 1. *Let W be a set. A Kripke frame is a 3-tuple $\mathcal{W} = \langle W, R, I \rangle$, where R is a ternary relation on W and I is an unary relation on W with additional requirements:*

1. $R^2 a(bc)d \Leftrightarrow R^2(ab)cd$;
2. $I(b) \Leftrightarrow Raba \& Rbaa$;
3. $a \leqslant b \Leftrightarrow I(c) \& Rcab$;
4.

Definition 2. *A Kripke model is a triple $\mathcal{M} = \langle \mathcal{W}, v \rangle$, where $\mathcal{W}$ is a Kripke frame and $v : Tp \to 2^W$.*

A forcing relation is defined as follows:

1. $\mathcal{M}, w \Vdash p_i \Leftrightarrow w \in v(p_i)$;
2. $\mathcal{M}, w \Vdash \mathbf{1} \Leftrightarrow I(w)$;
3. $\mathcal{M}, w \Vdash A \bullet B \Leftrightarrow \exists u, v \in W, Ruvw$ *and* $\mathcal{W}, u \Vdash A$ *and* $\mathcal{W}, v \Vdash B$;
4. $\mathcal{M}, w \Vdash A \backslash B \Leftrightarrow \forall u, v \in W, Ruwv$ *and* $\mathcal{W}, u \Vdash A$ *implies* $\mathcal{W}, v \Vdash B$;
5. $\mathcal{M}, w \Vdash B / A \Leftrightarrow \forall u, v \in W, Rwuv$ *and* $\mathcal{W}, u \Vdash A$ *implies* $\mathcal{W}, v \Vdash B$;
6. $\mathcal{M}, w \Vdash A \vee B \Leftrightarrow \exists u, v \in W, Swuv$ *and* $\mathcal{W}, u \Vdash A$ *and* $\mathcal{W}, v \Vdash B$;
7. $\mathcal{M}, w \Vdash A \wedge B \Leftrightarrow \mathcal{W}, w \Vdash A$ *and* $\mathcal{W}, w \Vdash A$;
8. $\mathcal{M}, w \Vdash \Gamma \to A \Leftrightarrow \mathcal{W}, w \Vdash \bullet\Gamma$ *implies* $\mathcal{W}, w \Vdash A$.

Theorem 1. *Soundness*

Let $\mathbb{F}$ be a class of Kripke frames, then $Log(\mathbb{F}) = L_{\mathbf{1}}^{*} \wedge \vee$.

Proof.

1. Let $\Gamma \to A$. By IH, for each $\mathcal{W}, w \Vdash \Gamma \to A$.
2. Let $\Gamma, A, \Delta \to C$ and $\Gamma, B, \Delta \to C$,
3. Let $\Gamma \to A$ and $\Gamma \to B$

4. Let $\Gamma \to A_1 \wedge A_2$, so $\mathcal{W}, w \Vdash \Gamma \to A_i$, $i = 1, 2$.

$\square$

Theorem 2. *Strong completeness*
$\Gamma \models A$ *implies that* $L_{\mathbf{1}}^* \wedge \vee \vdash \Gamma \to A$

Proof. $\square$

Definition 3. *Mininal normal modal Lambek calculus (L**K**) is* $L_{\mathbf{1}}^* \wedge \vee$ *with additional rule:*

$$\frac{\Gamma \to A}{!\Gamma \to !A}$$

Definition 4. *A modal Kripke frame is a 4-tuple* $\mathcal{W}_! = \langle W, R, Q, I \rangle$*, where* Q *is a preorder on* W*, a Kripke model is tuple* $\mathcal{M}\langle \mathcal{W}_!, v \rangle$*. A forcing relation for* $!$ *is defined as follows:*

$\mathcal{M}, w \Vdash !A \Leftrightarrow \forall v, Q(w, v) \Rightarrow \mathcal{W}, v \Vdash A$;

Theorem 3. *Soundness*
Let $\mathbb{F}$ *be a class of modal Kripke frames, then* $Log(\mathbb{F}) = L_{\mathbf{1}}^* \wedge \vee_{\mathbf{S4}}$.

Proof.
$\Gamma \models A$, then $L_{\mathbf{1}}^* \wedge \vee_{\mathbf{S4}} \vdash \Gamma \to A$ $\square$