Quantale model of Lambek calculus with subexponentials

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1 Calculus

Definition 1. *A subexponential signature is an ordered quintuple:*
  $\Sigma = \langle I, \preceq, W, C, E \rangle$,

  where $I = \{s_1, \dots, s_n\}$, $\langle I, \preceq \rangle$ is a preorder. W, C, E are upwardly closed subsets of I and $W \cup C \subseteq E$.

Definition 2. *Noncommutative linear logic with subexponentials ($SMALC_\Sigma$), where Σ is a subexponential signature.*

$$\overline{A \to A} \; ax$$

$$\frac{\Gamma \to A \qquad \Delta, B, \Theta \to C}{\Delta, \Gamma, A \backslash B, \Theta \to C} \, \backslash \to \qquad\qquad \frac{A, \Pi \to B}{\Pi \to A \backslash B} \to \backslash$$

$$\frac{\Gamma \to A \qquad \Delta, B, \Theta \to C}{\Delta, B/A, \Gamma, \Theta \to C} \, / \to \qquad\qquad \frac{\Pi, A \to B}{\Pi \to B/A} \to /$$

$$\frac{\Gamma, A, B, \Delta \to C}{\Gamma, A \bullet B, \Delta \to C} \, \bullet \to \qquad\qquad \frac{\Gamma \to A \qquad \Delta \to B}{\Gamma, \Delta \to A \bullet B} \to \bullet$$

$$\frac{\Gamma, A_i, \Delta \to B}{\Gamma, A_1 \& A_2, \Delta \to B} \, \&, i = 1, 2 \to \qquad\qquad \frac{\Gamma \to A \qquad \Gamma \to B}{\Gamma \to A \& B} \to \&$$

$$\frac{\Gamma, A, \Delta \to C \qquad \Gamma, B, \Delta \to C}{\Gamma, A \vee B, \Delta \to C} \, \vee \to \qquad\qquad \frac{\Gamma \to A_i}{\Gamma \to A_1 \vee A_2} \to \vee, i = 1, 2$$

$$\frac{\Gamma, \Delta \to A}{\Gamma, \mathbf{1}, \Delta \to A} \, \mathbf{1} \to \qquad\qquad \overline{\to \mathbf{1}} \to \mathbf{1}$$

$$\frac{\Gamma, A, \Delta \to C}{\Gamma, !^s A, \Delta \to C} \, ! \to \qquad\qquad \frac{!^{s_1} A_1, \dots, !^{s_n} A_n \to A}{!^{s_1} A_1, \dots, !^{s_n} A_n \to !^s A} \to !, \forall j, s_j \succeq s$$

$$\frac{\Gamma, \Delta \to B}{\Gamma, !^s A, \Delta \to B} \, \mathbf{weak}_!, s \in C$$

$$\frac{\Gamma, !^s A, \Delta, !^s A, \Theta \to B}{\Gamma, !^s A, \Delta, \Theta \to B} \ \mathbf{ncontr}_1, s \in C$$

$$\frac{\Gamma, !^s A, \Delta, !^s A, \Theta \to B}{\Gamma, \Delta, !^s A, \Theta \to B} \ \mathbf{ncontr}_2, s \in C$$

$$\frac{\Gamma, \Delta, !^s A, \Theta \to B}{\Gamma, !^s A, \Delta, \Theta \to A} \ \mathbf{ex}_1, s \in E$$

$$\frac{\Gamma, !^s A, \Delta, \Theta \to B}{\Gamma, \Delta, !^s A, \Theta \to A} \ \mathbf{ex}_2, s \in E$$

Proposition 1. $!_{s_i} A \leftrightarrow !_{s_i}(!_{s_i} A)$

Proof.

$$\frac{\dfrac{\dfrac{\dfrac{A \to A}{!_{s_i} A \to A}}{!_{s_i} A \to !_{s_i} A}}{!_{s_i}!_{s_i} A \to !_{s_i} A}}{}$$

$\square$

2 Semantics

Definition 3. *Quantale*
A quantale is a triple $\langle A, \bigvee, \cdot \rangle$, such that $\langle A, \bigvee \rangle$ is a complete lattice and $\langle A, \cdot \rangle$ is a semigroup. A quanlate is called unital, if $\langle A, \cdot \rangle$ is a monoid.

It is easy to see, that any (unital) quantale is a residual (monoid) semigroup. We define divisions as follows:

1. $a \backslash b = \bigvee \{ c \mid a \cdot c \leqslant b \}$

2. $b / a = \bigvee \{ c \mid c \cdot a \leqslant b \}$

Definition 4.
Let $\mathcal{Q} = \langle A, \bigvee, \cdot \rangle$ be a quantale. The center of a quantale is the set $\mathcal{Z}(\mathcal{Q}) = \{ a \in A \mid \forall b \in A, a \cdot b = b \cdot a \}$

Definition 5. *An open modality (or quantic conucleus) on quantale $\mathcal{Q}$ is a map $\square : \mathcal{Q} \to \mathcal{Q}$, such that*

1. *$\square(x) \leqslant x$;*

2. *$\square(x) = \square(\square(x))$;*

3. *$x \leqslant y \Rightarrow \square(x) \leqslant \square(y)$;*

4. *$\square(x) \cdot \square(y) = \square(\square(x) \cdot \square(y))$.*

Definition 6. *We define a partial order on open modalities on $\mathcal{Q}$ as $\square_1 \leqslant \square_2 \Leftrightarrow \forall a \in Q, \square_1(a) \leqslant \square_2(a)$.*

Lemma 1. *Let $\mathcal{Q}$ be a quantale and $\square_{\mathcal{Q}}$ be a set of all open modalities on $\mathcal{Q}$. Then $\square_{\mathcal{Q}}$ is a locally small category.*

Proof. $\langle \square_{\mathcal{Q}}, \leqslant \rangle$ form a partial order, so $\langle \square_{\mathcal{Q}}, \leqslant \rangle$ is a locally small category. $\qquad\square$

Lemma 2.
    Let $\mathcal{Q} = \langle A, \bigvee, \cdot \rangle$ be a quantale and $\square : \mathcal{Q} \to \mathcal{Q}$ is an open modality on $\mathcal{Q}$, then $\square(x) \cdot \square(y) \leqslant \square(x \cdot y)$.

Proof.
    $\square(x) \cdot \square(y) \leqslant x \cdot y$, then $\square(\square(x) \cdot \square(y)) \leqslant \square(x \cdot y)$, but $\square(x) \cdot \square(y) \leqslant \square(\square(x) \cdot \square(y))$. Thus, $\square(x) \cdot \square(y) \leqslant \square(x \cdot y)$. $\qquad\square$

Definition 7. *An open modality is called central, if $\forall a, b \in Q, \square(a) \cdot b = b \cdot \square(a)$.*

Definition 8. *An open modality is called weak idempotent, if $\forall a, b \in Q, \square(a) \cdot b \leqslant \square(a) \cdot b \cdot \square(a)$ and $b \cdot \square(a) \leqslant \square(a) \cdot b \cdot \square(a)$.*

Definition 9. *An open modality is called unital, if $\forall a \in Q, \square(a) \leqslant e$.*

Lemma 3. *Let $\square$ be an open modality on some unital quantale $\mathcal{Q} = \langle A, \bigvee, \cdot, e \rangle$. Then, if $\square$ is unital and weak idempotent, then $\square$ is central.*

Proof.
    $b \cdot \square(a) \leqslant$
      Right weak idempotence
    $\square(a) \cdot b \cdot \square(a) \leqslant$
      Unitality
    $\square(a) \cdot b \cdot e \leqslant$
      Identity
    $\square(a) \cdot b \leqslant$
      Left weak idempotence
    $\square(a) \cdot b \cdot \square(a) \leqslant$
      Unitality
    $e \cdot b \cdot \square(a) \leqslant$
      Identity
    $b \cdot \square(a)$
  Hence, $b \cdot \square(a) = \square(a) \cdot b$, so $\forall a \in A, \square(a) \in \mathcal{Z}(\mathcal{Q})$.
$\qquad\square$

Proposition 2.
    Let $\mathcal{Q}$ be a quantale and $S \subseteq \mathcal{Q}$ a subquantale, then $\square : \mathcal{Q} \to \mathcal{Q}$, such that $\square(a) = \bigvee \{s \in S \mid x \leqslant a\}$, is an open modality.

Proof. See $\qquad\square$

Proposition 3.
    Let $\mathcal{Q}$ be a quantale and $S_1 \subseteq S_2 \subseteq \mathcal{Q}$.
    Then $\square_1(a) \leqslant \square_2(a)$.

Proof.
    Let $a \in Q$, so $\{s \in S_1 \mid s \leqslant a\} \subseteq \{s \in S_2 \mid s \leqslant a\}$, so $\bigvee \{s \in S_1 \mid s \leqslant a\} \subseteq \bigvee \{s \in S_2 \mid s \leqslant a\}$. Thus, $\square_1(a) \leqslant \square_2(a)$. $\qquad\square$

Proposition 4.

Let $\mathcal{Q}$ be a quantale and $S \subseteq \mathcal{Q}$ a subquantale, then the following operations are open modalities:

1. $\square_z(a) = \bigvee\{s \in S \,|\, s \leqslant a, s \in \mathcal{Z}(\mathcal{Q})\}$;

2. $\square_{\mathbb{1}}(a) = \bigvee\{s \in S \,|\, s \leqslant a, s \leqslant \mathbb{1}\}$;

3. $\square_{idem}(a) = \bigvee\{s \in S \,|\, s \leqslant a, \forall b \in Q, b \cdot s \vee s \cdot b \leqslant s \cdot b \cdot s\}$;

4. $\square_{z,\mathbb{1}}, I_{z,idem}, I_{\mathbb{1},idem}, I_{z,\mathbb{1},idem}$.

Proof. Immediatly. $\qquad\square$

Proposition 5.

1. $\forall a \in \mathcal{Q}, \square_{\mathbb{1},idem}(a) \leqslant \square_z(a)$.

2. $\forall a \in \mathcal{Q}, \square_{z,\mathbb{1},idem} = \square_{\mathbb{1},idem}(a)$

Proof. Follows from Lemma 3. $\qquad\square$

Proposition 6.

1. $\square_z(a) \vee \square_{\mathbb{1}}(a) \vee \square_{idem}(a) \leqslant \square(a)$

2. $\square_{z,\mathbb{1},idem} \leqslant \square_{z,\mathbb{1}}(a) \wedge \square_{z,idem}(a)$

Lemma 4. $\forall a \in Q, \square_1(a) \leqslant \square_2(\square_1(a))$, *if* $\square_1(a) \leqslant \square_2(a)$.

Proof. $\square_1(a) \leqslant \square_1(\square_1(a)) \leqslant \square_2(\square_1(a))$ $\qquad\square$

Lemma 5. $\square_1(a_1) \cdot \square_2(a_2) \leqslant \square'(\square_1(a_1) \cdot \square_2(a_2))$, *where* $\square_i \leqslant \square', i = 1, 2$.

Proof.
$$\square_1(a_1) \cdot \square_2(a_2) \leqslant$$
$$\square_1(\square_1(a_1)) \cdot \square_2(\square_2(a_2)) \leqslant$$
$$\square'(\square_1(a_1)) \cdot \square'(\square_2(a_2)) \leqslant$$
$$\square'(\square_1(a_1) \cdot \square_2(a_2))$$
$\qquad\square$

Definition 10. *Interpretation of subexponential signature*

Let $\Sigma = \langle I, \leq, W, C, E \rangle$ be a subexponential signature, where $|I| = n$ and $\square_{\mathcal{Q}}$ is a category of open modalities on a quantale $\mathcal{Q}$. Subexponential interpretation is a contravariant functor $\sigma : I \to \square_{\mathcal{Q}}$ defined as follows:

$$\sigma(s_i) = \begin{cases} \square_i : \mathcal{Q} \to \mathcal{Q}, \ s.t. \forall a \in Q, \square_i(a) = \{s \in S_i \,|\, s \leqslant a\}, \\ \quad \text{if } s_i \notin W \cap C \cap E \\ \square_i : \mathcal{Q} \to \mathcal{Q}, \ s.t. \forall a \in Q, \square_i(a) = \{s \in S_i \,|\, s \leqslant a, s \leqslant \mathbb{1}\}, \\ \quad \text{if } s_i \in W \\ \square_i : \mathcal{Q} \to \mathcal{Q}, \ s.t. \forall a \in Q, \square_i(a) = \{s \in S_i \,|\, s \leqslant a, s \in \mathcal{Z}(\mathcal{Q})\}, \\ \quad \text{if } s_i \in E \\ \square_i : \mathcal{Q} \to \mathcal{Q}, \ s.t. \forall a \in Q, \square_i(a) = \{s \in S_i \,|\, s \leqslant a, \forall b, b \cdot s \vee s \cdot b \leqslant s \cdot b \cdot s\}, \\ \quad \text{if } s_i \in E \\ \text{otherwise, if } s_i \text{ belongs to some intersection of subsets, then we combine the relevant conditions} \end{cases}$$

Definition 11. *Let $\mathcal{Q}$ be a quantale, $f : Tp \to \mathcal{Q}$ a valuation and $\sigma : I \to \Box_{\mathcal{Q}}$ a subexponential interpretation, then interpretation is defined inductively:*

$$
\begin{aligned}
[\![p_i]\!] &= f(p_i) \\
[\![\mathbb{1}]\!] &= e \\
[\![A \bullet B]\!] &= [\![A]\!] \cdot [\![B]\!] \\
[\![A \backslash B]\!] &= [\![A]\!] \backslash [\![B]\!] \\
[\![A / B]\!] &= [\![A]\!] / [\![B]\!] \\
[\![A \& B]\!] &= [\![A]\!] \wedge [\![B]\!] \\
[\![A \vee B]\!] &= [\![A]\!] \vee [\![B]\!] \\
[\![!_{s_i} A]\!] &= \sigma(s_i)[\![A]\!]
\end{aligned}
$$

Definition 12. $\Gamma \models A \Leftrightarrow \forall f, \forall \sigma, [\![\Gamma]\!] \leqslant [\![A]\!]$

Theorem 1. $\Gamma \to A \Rightarrow [\![\Gamma]\!] \leqslant [\![A]\!]$

Proof. We consider cases with modal rules.

1. Let $!_{s_1} A_1, \ldots, !_{s_n} A_n \to A$ and $\forall i, s \leq s_i$.

   Then $\forall a \in Q, \sigma(s_i)(a) \leqslant \sigma(s)(a)$.

   By IH, $\sigma(s_1)[\![A_1]\!] \cdot \cdots \cdot \sigma(s_n)[\![A_n]\!] \leqslant [\![A]\!]$.

   Thus, $\sigma(s)(\sigma(s_1)[\![A_1]\!] \cdot \cdots \cdot \sigma(s_n)[\![A_n]\!]) \leqslant \sigma(s)([\![A]\!])$.

   By Lemma 5, $\sigma(s_1)[\![A_1]\!] \cdot \cdots \cdot \sigma(s_n)[\![A_n]\!] \leqslant \sigma(s)(\sigma(s_1)[\![A_1]\!] \cdot \cdots \cdot \sigma(s_n)[\![A_n]\!])$.

   So, $\sigma(s_1)[\![A_1]\!] \cdot \cdots \cdot \sigma(s_n)[\![A_n]\!] \leqslant \sigma(s)([\![A]\!])$.

2. Let $\Gamma, A, \Delta \to B$.

   By IH, $[\![\Gamma]\!] \cdot [\![A]\!] \cdot [\![\Delta]\!] \leqslant [\![B]\!]$.

   By the definition, $\sigma(s_i)([\![A]\!]) \leqslant [\![A]\!]$.

   So, $[\![\Gamma]\!] \cdot \sigma(s_i)([\![A]\!]) \cdot [\![\Delta]\!] \leqslant [\![B]\!]$

3. Let $\Gamma, \Delta \to B$, $A \in Fm$, and $s_i \in W$.

   So, $[\![\Gamma]\!] \cdot [\![\Delta]\!] \leqslant [\![B]\!]$, then $[\![\Gamma]\!] \cdot e \cdot [\![\Delta]\!] \leqslant [\![B]\!]$, where $e \in Q$ is unit.

   By the definition of unital open modality, $\sigma(s_i)([\![A]\!]) \leqslant e$.

   Thus, $[\![\Gamma]\!] \cdot \sigma(s_i)([\![A]\!]) \cdot [\![\Delta]\!] \leqslant [\![B]\!]$.

4. Let $\Gamma, !_{s_i} A, \Delta, !_{s_i} A, \Pi \to B$ and $s_i \in C$.

   By IH, $[\![\Gamma]\!] \cdot \sigma(s_i)([\![A]\!]) \cdot [\![\Delta]\!] \cdot \sigma(s_i)([\![A]\!]) \cdot [\![\Pi]\!] \leqslant [\![B]\!]$.

   By the definition, $\sigma(s_i)([\![A]\!]) \cdot [\![\Delta]\!] \leqslant \sigma(s_i)([\![A]\!]) \cdot [\![\Delta]\!] \cdot \sigma(s_i)([\![A]\!])$.

   Then $[\![\Gamma]\!] \cdot \sigma(s_i)([\![A]\!]) \cdot [\![\Delta]\!] \cdot [\![\Pi]\!] \leqslant [\![B]\!]$

5. Let $\Gamma, !_{s_i} A, \Delta, \Pi \to B$ and $s_i \in E$, so $\sigma(s_i)(a) \in \mathcal{Z}(\mathcal{Q})$ for all $a \in Q$ by the definition.

   By IH, $[\![\Gamma]\!] \cdot \sigma(s_i)([\![A]\!]) \cdot [\![\Delta]\!] \cdot [\![\Pi]\!] \leqslant [\![B]\!]$

   Hence, $[\![\Gamma]\!] \cdot [\![\Delta]\!] \cdot \sigma(s_i)([\![A]\!]) \cdot [\![\Pi]\!] \leqslant [\![B]\!]$.

$\square$

3 Quantale completeness

Definition 13.

Let $\mathcal{F} \subseteq Fm$, an ideal is a subset $\mathcal{I} \subseteq \mathcal{F}$, such that:

- If $B \in \mathcal{I}$ and $A \to B$, then $A \in \mathcal{I}$;

- If $A, B \in \mathcal{I}$, then $A \vee B \in \mathcal{I}$.

Definition 14.

Let $S \subseteq \mathcal{F} \subseteq Fm$, then $\bigvee S = \bigcap \{\mathcal{I} \subseteq \mathcal{F} \mid S \subseteq \mathcal{I}\}$

Proposition 7. $\bigvee S$ *is an ideal.*

Lemma 6. $A \subseteq Fm$, *then* $\{B \mid B \to A\} = \bigvee\{A\}$.

Proof.

Let $A \in Fm$. Then $\{B \mid B \to A', A' \in A\} \subseteq \bigvee\{A\}$, so far as $\bigvee A$ is an ideal.

On the other hand, $\{B \mid B \to A\}$ is an ideal, it is easy to see that this set is closed under $\vee$. So, $\bigvee A \subseteq \{B \mid B \to A\}$. $\qquad\square$

Lemma 7. $\bigvee\{A\} \subseteq \bigvee\{B\}$ *iff* $A \to B$.

Proof. Let $\bigvee\{A\} \subseteq \bigvee\{B\}$, then $\{C \mid C \to A\} \subseteq \{D \mid D \to B\}$.

Thus, $A \in \{C \mid C \to A\}$, then $A \in \{D \mid D \to B\}$, hence $A \to B$.

On the other hand, let $A \to B$ and $C \in \bigvee\{A\}$.

Thus, $C \to A$, then $C \to B$ by cut.

$\qquad\square$

Lemma 8. *Let* $\mathcal{Q} = \{\bigvee S \mid S \subseteq Fm\}$ *and* $\bigvee \mathcal{A} \cdot \bigvee \mathcal{B} = \bigvee\{A \bullet B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$. *Then* $\langle \mathcal{Q}, \subseteq, \cdot, \bigvee \mathbf{1}\rangle$ *is a quantale.*

Proof. See $\qquad\square$

Lemma 9. *Let* $!_s \in I$ *and* A *be an arbitrary formula, then* $\square_s(\bigvee\{A\}) = \bigvee\{B \mid !_s B \to A\}$ *is a quantic conucleus.*

Proof.

See Yetter. $\qquad\square$

Lemma 10. *Let* A *be a formula, then* $\square_s \bigvee\{A\} = \bigvee\{!_s A\}$, *for each* $s \in \mathcal{I}$.

Proof. Let $A \in Fm$ and $s \in \mathcal{I}$.

Let $!_s B \in \square_s \bigvee\{A\}$, then $!_s B \to A$, then $!_s B \to !_s A$ by promotion. So, $!_s B \in \bigvee\{!_s A\}$.

Let $C \in \bigvee\{!_s A\}$, then $C \to !_s A$, so $!_s C \to !_s A$ by dereliction, but $!_s A \to A$, hence $!_s C \to A$ by cut. So, $!_s C \in \square_s \bigvee\{A\}$. $\qquad\square$

Lemma 11.

Let $i, j \in I$ and $i \leq j$, then for all $A \in Fm$, $\square_j(\bigvee\{A\}) \subseteq \square_i(\bigvee\{A\})$.

Proof.

Let $i, j \in I$ and $i \leq j$. Then forall $A \in Fm$, $!_j A \to !_i A$ by promotion. Then $\bigvee\{!_j A\} \subseteq \bigvee\{!_i A\}$, so $\square_j(\bigvee\{A\}) \subseteq \square_i(\bigvee\{A\})$. $\qquad\square$

Lemma 12.

1. *Let $s \in W$, then for all $A \subseteq Fm$, $\square_s\{A\} \subseteq \{\mathbf{1}\}$;*

2. *Let $s \in E$, then $\square_s(\bigvee\{A\}) \cdot \bigvee\{B\} = \bigvee\{B\} \cdot \square_s(\bigvee\{A\})$.*

3. *Let $s \in C$, then $(\square_s \bigvee A \cdot \bigvee B) \cup (\bigvee B \cdot \square_s \bigvee A) \subseteq \square_s \bigvee A \cdot \bigvee B \cdot \square_s \bigvee A$, for all $B \subseteq Fm$.*

Proof.

Follows from $!_s A \rightarrow \mathbf{1}$, so $s \in W$;

Follows from $!_s A \bullet B \leftrightarrow B \bullet !_s A$;

Follows from $!_s A \bullet B \rightarrow !_s A \bullet B \bullet !_s A$ and similarly for $B \bullet !_s A$. $\qquad \square$

Definition 15.
   Let Q *be a syntactic quantale as proposed above and* $\mathcal{I} = \langle I, \leq, W, C, E \rangle$ *be a subexponential signature.*
   We define a map $\square : \mathcal{I} \rightarrow Mod_Q$ *as follows:*
   $\square(i)(\bigvee\{A\}) = \{!_i B \mid !_i B \rightarrow A\}$.

Lemma 13. $\square$ *is a subexponential interpretation.*

Proof. Follows from lemmas above. $\qquad \square$

Lemma 14.
   Let Q *be a quantale constructed above and* $\square_1, \ldots, \square_n$ *be a family of quantic conuclei on* Q. *Then there exist a model* $\langle Q, \llbracket . \rrbracket \rangle$, *such that* $\llbracket A \rrbracket = \bigvee\{A\}$, $A \in Fm$.

Proof.
   We define an interpreation as follows:

1. $\llbracket p_i \rrbracket = \bigvee\{p_i\}$

2. $\llbracket \mathbf{1} \rrbracket = \bigvee\{\mathbf{1}\}$

3. $\llbracket A \bullet B \rrbracket = \bigvee\{A \bullet B\}$

4. $\llbracket A/B \rrbracket = \bigvee\{A/B\}$

5. $\llbracket B \backslash A \rrbracket = \bigvee\{B \backslash A\}$

6. $\llbracket A \& B \rrbracket = \bigvee\{A \& B\}$

7. $\llbracket A \vee B \rrbracket = \bigvee\{A \vee B\}$

8. $\llbracket !_s A \rrbracket = \square(s)(\bigvee\{A\}) = \bigvee\{!_s A\}$.

$\qquad \square$

Theorem 2. $\Gamma \models A \Rightarrow \Gamma \rightarrow A$.

Proof. Follows from lemmas above. $\qquad \square$