

# Quantale model of Lambek calculus with subexponentials

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## 1 Calculus

**Definition 1.** A subexponential signature is an ordered quintuple:

$$\Sigma = \langle I, \leq, W, C, E \rangle,$$

where  $I = \{s_1, \dots, s_n\}$ ,  $\langle I, \leq \rangle$  is a preorder.  $W, C, E$  are subsets of  $I$  and  $W \cup C \subseteq E$ .

**Definition 2.** Noncommutative linear logic with subexponentials ( $SMALC_\Sigma$ ), where  $\Sigma$  is a subexponential signature.

$$\begin{array}{c}
 \overline{A \rightarrow A} \text{ }^{ax} \\
 \\
 \frac{\Gamma \rightarrow A \quad \Delta, B, \Theta \rightarrow C}{\Delta, \Gamma, A \backslash B, \Theta \rightarrow C} \backslash \rightarrow \qquad \frac{A, \Pi \rightarrow B}{\Pi \rightarrow A \backslash B} \rightarrow \backslash \\
 \\
 \frac{\Gamma \rightarrow A \quad \Delta, B, \Theta \rightarrow C}{\Delta, B / A, \Gamma, \Theta \rightarrow C} / \rightarrow \qquad \frac{\Pi, A \rightarrow B}{\Pi \rightarrow B / A} \rightarrow / \\
 \\
 \frac{\Gamma, A, B, \Delta \rightarrow C}{\Gamma, A \bullet B, \Delta \rightarrow C} \bullet \rightarrow \qquad \frac{\Gamma \rightarrow A \quad \Delta \rightarrow B}{\Gamma, \Delta \rightarrow A \bullet B} \rightarrow \bullet \\
 \\
 \frac{\Gamma, A_i, \Delta \rightarrow B}{\Gamma, A_1 \& A_2, \Delta \rightarrow B} \&, i = 1, 2 \rightarrow \qquad \frac{\Gamma \rightarrow A \quad \Gamma \rightarrow B}{\Gamma \rightarrow A \& B} \rightarrow \& \\
 \\
 \frac{\Gamma, A, \Delta \rightarrow C \quad \Gamma, B, \Delta \rightarrow C}{\Gamma, A \vee B, \Delta \rightarrow C} \vee \rightarrow \qquad \frac{\Gamma \rightarrow A_i}{\Gamma \rightarrow A_1 \vee A_2} \rightarrow \vee, i = 1, 2 \\
 \\
 \frac{\Gamma, \Delta \rightarrow A}{\Gamma, \mathbf{1}, \Delta \rightarrow A} \mathbf{1} \rightarrow \qquad \overline{\rightarrow \mathbf{1}} \rightarrow \mathbf{1} \\
 \\
 \frac{\Gamma, A, \Delta \rightarrow C}{\Gamma, !^s A, \Delta \rightarrow C} ! \rightarrow \qquad \frac{!^{s_1} A_1, \dots, !^{s_n} A_n \rightarrow A}{!^{s_1} A_1, \dots, !^{s_n} A_n \rightarrow !^s A} \rightarrow !, \forall j, s_j \geq s \\
 \\
 \frac{\Gamma, \Delta \rightarrow B}{\Gamma, !^s A, \Delta \rightarrow B} \text{weak}_!, s \in C
 \end{array}$$

$$\frac{\Gamma, !^s A, \Delta, !^s A, \Theta \rightarrow B}{\Gamma, !^s A, \Delta, \Theta \rightarrow B} \text{ncontr}_1, s \in C$$

$$\frac{\Gamma, !^s A, \Delta, !^s A, \Theta \rightarrow B}{\Gamma, \Delta, !^s A, \Theta \rightarrow B} \text{ncontr}_2, s \in C$$

$$\frac{\Gamma, \Delta, !^s A, \Theta \rightarrow B}{\Gamma, !^s A, \Delta, \Theta \rightarrow A} \text{ex}_1, s \in E$$

$$\frac{\Gamma, !^s A, \Delta, \Theta \rightarrow B}{\Gamma, \Delta, !^s A, \Theta \rightarrow A} \text{ex}_1, s \in E$$

**Proposition 1.**  $!_{s_i} A \leftrightarrow !_{s_i} (!_{s_i} A)$

*Proof.*

$$\frac{\frac{\frac{A \rightarrow A}{!_{s_i} A \rightarrow A}}{!_{s_i} A \rightarrow !_{s_i} A}}{!_{s_i} !_{s_i} A \rightarrow !_{s_i} A}$$

□

## 2 Semantics

**Definition 3.** *Quantale*

A quantale is a triple  $\langle A, \vee, \cdot \rangle$ , such that  $\langle A, \vee \rangle$  is a complete lattice and  $\langle A, \cdot \rangle$  is a semi-group. A quantale is called unital, if  $\langle A, \cdot \rangle$  is a monoid.

It is easy to see, that any (unital) quantale is a residual (monoid) semigroup. We define divisions as follows:

1.  $a \backslash b = \bigvee \{c \mid a \cdot c \leq b\}$
2.  $b / a = \bigvee \{c \mid c \cdot a \leq b\}$

**Definition 4.**

Let  $\mathcal{Q} = \langle A, \vee, \cdot \rangle$  be a quantale. The center of a quantale is the set  $\mathcal{Z}(\mathcal{Q}) = \{a \in A \mid \forall b \in A, a \cdot b = b \cdot a\}$

**Definition 5.** An open modality (or quantic conucleus) on quantale  $\mathcal{Q}$  is a map  $\Box : \mathcal{Q} \rightarrow \mathcal{Q}$ , such that

1.  $\Box(x) \leq x$ ;
2.  $\Box(x) = \Box(\Box(x))$ ;
3.  $x \leq y \Rightarrow \Box(x) \leq \Box(y)$ ;
4.  $\Box(x) \cdot \Box(y) = \Box(\Box(x) \cdot \Box(y))$ .

**Definition 6.** We define a partial order on open modalities on  $\mathcal{Q}$  as  $\Box_1 \leq \Box_2 \Leftrightarrow \forall a \in \mathcal{Q}, \Box_1(a) \leq \Box_2(a)$ .

**Lemma 1.** Let  $\mathcal{Q}$  be a quantale and  $\square_{\mathcal{Q}}$  be a set of all open modalities on  $\mathcal{Q}$ . Then  $\square_{\mathcal{Q}}$  is a locally small category.

*Proof.*  $\langle \square_{\mathcal{Q}}, \leq \rangle$  form a partial order, so  $\langle \square_{\mathcal{Q}}, \leq \rangle$  is a locally small category.  $\square$

**Lemma 2.**

Let  $\mathcal{Q} = \langle A, \bigvee, \cdot \rangle$  be a quantale and  $\square : \mathcal{Q} \rightarrow \mathcal{Q}$  is an open modality on  $\mathcal{Q}$ , then  $\square(x) \cdot \square(y) \leq \square(x \cdot y)$ .

*Proof.*

$\square(x) \cdot \square(y) \leq x \cdot y$ , then  $\square(\square(x) \cdot \square(y)) \leq \square(x \cdot y)$ , but  $\square(x) \cdot \square(y) \leq \square(\square(x) \cdot \square(y))$ . Thus,  $\square(x) \cdot \square(y) \leq \square(x \cdot y)$ .  $\square$

**Definition 7.** An open modality is called central, if  $\forall a, b \in \mathcal{Q}, \square(a) \cdot b = b \cdot \square(a)$ .

**Definition 8.** An open modality is called weak idempotent, if  $\forall a, b \in \mathcal{Q}, \square(a) \cdot b \leq \square(a) \cdot b \cdot \square(a)$  and  $b \cdot \square(a) \leq \square(a) \cdot b \cdot \square(a)$ .

**Definition 9.** An open modality is called unital, if  $\forall a \in \mathcal{Q}, \square(a) \leq e$ .

**Lemma 3.** Let  $\square$  be an open modality on some unital quantale  $\mathcal{Q} = \langle A, \bigvee, \cdot, e \rangle$ . Then, if  $\square$  is unital and weak idempotent, then  $\square$  is central.

*Proof.*

$$\begin{aligned}
& b \cdot \square(a) \leq \\
& \quad \text{Right weak idempotence} \\
& \square(a) \cdot b \cdot \square(a) \leq \\
& \quad \text{Unitality} \\
& \square(a) \cdot b \cdot e \leq \\
& \quad \text{Identity} \\
& \square(a) \cdot b \leq \\
& \quad \text{Left weak idempotence} \\
& \square(a) \cdot b \cdot \square(a) \leq \\
& \quad \text{Unitality} \\
& e \cdot b \cdot \square(a) \leq \\
& \quad \text{Identity} \\
& b \cdot \square(a)
\end{aligned}$$

Hence,  $b \cdot \square(a) = \square(a) \cdot b$ , so  $\forall a \in A, \square(a) \in \mathcal{Z}(\mathcal{Q})$ .  $\square$

**Proposition 2.**

Let  $\mathcal{Q}$  be a quantale and  $S \subseteq \mathcal{Q}$  a subquantale, then  $\square : \mathcal{Q} \rightarrow \mathcal{Q}$ , such that  $\square(a) = \bigvee \{s \in S \mid s \leq a\}$ , is an open modality.

*Proof.* See  $\square$

**Proposition 3.**

Let  $\mathcal{Q}$  be a quantale and  $S_1 \subseteq S_2 \subseteq \mathcal{Q}$ .

Then  $\square_1(a) \leq \square_2(a)$ .

*Proof.*

Let  $a \in \mathcal{Q}$ , so  $\{s \in S_1 \mid s \leq a\} \subseteq \{s \in S_2 \mid s \leq a\}$ , so  $\bigvee \{s \in S_1 \mid s \leq a\} \leq \bigvee \{s \in S_2 \mid s \leq a\}$ . Thus,  $\square_1(a) \leq \square_2(a)$ .  $\square$

**Proposition 4.**

Let  $\mathcal{Q}$  be a quantale and  $S \subseteq \mathcal{Q}$  a subquantale, then the following operations are open modalities:

1.  $\Box_z(a) = \bigvee \{s \in S \mid s \leq a, s \in \mathcal{Z}(\mathcal{Q})\};$
2.  $\Box_1(a) = \bigvee \{s \in S \mid s \leq a, s \leq 1\};$
3.  $\Box_{idem}(a) = \bigvee \{s \in S \mid s \leq a, \forall b \in \mathcal{Q}, b \cdot s \vee s \cdot b \leq s \cdot b \cdot s\};$
4.  $\Box_{z,1}, I_{z,idem}, I_{1,idem}, I_{z,1,idem}.$

*Proof.* Immediately. □

**Proposition 5.**

1.  $\forall a \in \mathcal{Q}, \Box_{1,idem}(a) \leq \Box_z(a).$
2.  $\forall a \in \mathcal{Q}, \Box_{z,1,idem} = \Box_{1,idem}(a)$

*Proof.* Follows from Lemma 3. □

**Proposition 6.**

1.  $\Box_z(a) \vee \Box_1(a) \vee \Box_{idem}(a) \leq \Box(a)$
2.  $\Box_{z,1,idem} \leq \Box_{z,1}(a) \wedge \Box_{z,idem}(a)$

**Lemma 4.**  $\forall a \in \mathcal{Q}, \Box_1(a) \leq \Box_2(\Box_1(a)),$  if  $\Box_1(a) \leq \Box_2(a).$

*Proof.*  $\Box_1(a) \leq \Box_1(\Box_1(a)) \leq \Box_2(\Box_1(a))$  □

**Lemma 5.**  $\Box_1(a_1) \cdot \Box_2(a_2) \leq \Box'(\Box_1(a_1) \cdot \Box_2(a_2)),$  where  $\Box_i \leq \Box', i = 1, 2.$

*Proof.*

$$\begin{aligned} & \Box_1(a_1) \cdot \Box_2(a_2) \leq \\ & \Box_1(\Box_1(a_1)) \cdot \Box_2(\Box_2(a_2)) \leq \\ & \Box'(\Box_1(a_1)) \cdot \Box'(\Box_2(a_2)) \leq \\ & \Box'(\Box_1(a_1) \cdot \Box_2(a_2)) \end{aligned}$$
□

**Definition 10.** Interpretation of subexponential signature

Let  $\Sigma = \langle I, \leq, W, C, E \rangle$  be a subexponential signature, where  $|I| = n$  and  $\Box_{\mathcal{Q}}$  is a category of open modalities on a quantale  $\mathcal{Q}$ . Subexponential interpretation is a contravariant functor  $\sigma : I \rightarrow \Box_{\mathcal{Q}}$  defined as follows:

$$\sigma(s_i) = \begin{cases} \Box_i : \mathcal{Q} \rightarrow \mathcal{Q}, \text{ s.t. } \forall a \in \mathcal{Q}, \Box_i(a) = \{s \in S_i \mid s \leq a\}, \\ \quad \text{if } s_i \notin W \cap C \cap E \\ \Box_i : \mathcal{Q} \rightarrow \mathcal{Q}, \text{ s.t. } \forall a \in \mathcal{Q}, \Box_i(a) = \{s \in S_i \mid s \leq a, s \leq 1\}, \\ \quad \text{if } s_i \in W \\ \Box_i : \mathcal{Q} \rightarrow \mathcal{Q}, \text{ s.t. } \forall a \in \mathcal{Q}, \Box_i(a) = \{s \in S_i \mid s \leq a, s \in \mathcal{Z}(\mathcal{Q})\}, \\ \quad \text{if } s_i \in E \\ \Box_i : \mathcal{Q} \rightarrow \mathcal{Q}, \text{ s.t. } \forall a \in \mathcal{Q}, \Box_i(a) = \{s \in S_i \mid s \leq a, \forall b, b \cdot s \vee s \cdot b \leq s \cdot b \cdot s\}, \\ \quad \text{if } s_i \in E \\ \text{otherwise, if } s_i \text{ belongs to some intersection of subsets, then we combine the relevant conditions} \end{cases}$$

**Definition 11.** Let  $\mathcal{Q}$  be a quantale,  $f : Tp \rightarrow \mathcal{Q}$  a valuation and  $\sigma : I \rightarrow \square_{\mathcal{Q}}$  a subexponential interpretation, then interpretation is defined inductively:

$$\begin{aligned} \llbracket p_i \rrbracket &= f(p_i) \\ \llbracket 1 \rrbracket &= e \\ \llbracket A \bullet B \rrbracket &= \llbracket A \rrbracket \cdot \llbracket B \rrbracket \\ \llbracket A \setminus B \rrbracket &= \llbracket A \rrbracket \setminus \llbracket B \rrbracket \\ \llbracket A/B \rrbracket &= \llbracket A \rrbracket / \llbracket B \rrbracket \\ \llbracket A \& B \rrbracket &= \llbracket A \rrbracket \wedge \llbracket B \rrbracket \\ \llbracket A \vee B \rrbracket &= \llbracket A \rrbracket \vee \llbracket B \rrbracket \\ \llbracket !_{s_i} A \rrbracket &= \sigma(s_i) \llbracket A \rrbracket \end{aligned}$$

**Definition 12.**  $\Gamma \models A \Leftrightarrow \forall f, \forall \sigma, \llbracket \Gamma \rrbracket \leq \llbracket A \rrbracket$

**Theorem 1.**  $\Gamma \rightarrow A \Rightarrow \llbracket \Gamma \rrbracket \leq \llbracket A \rrbracket$

*Proof.* We consider cases with modal rules.

1. Let  $!_{s_1} A_1, \dots, !_{s_n} A_n \rightarrow A$  and  $\forall i, s \leq s_i$ .  
Then  $\forall a \in Q, \sigma(s_i)(a) \leq \sigma(s)(a)$ .  
By IH,  $\sigma(s_1) \llbracket A_1 \rrbracket \cdot \dots \cdot \sigma(s_n) \llbracket A_n \rrbracket \leq \llbracket A \rrbracket$ .  
Thus,  $\sigma(s)(\sigma(s_1) \llbracket A_1 \rrbracket \cdot \dots \cdot \sigma(s_n) \llbracket A_n \rrbracket) \leq \sigma(s)(\llbracket A \rrbracket)$ .  
By Lemma 5,  $\sigma(s_1) \llbracket A_1 \rrbracket \cdot \dots \cdot \sigma(s_n) \llbracket A_n \rrbracket \leq \sigma(s)(\sigma(s_1) \llbracket A_1 \rrbracket \cdot \dots \cdot \sigma(s_n) \llbracket A_n \rrbracket)$ .  
So,  $\sigma(s_1) \llbracket A_1 \rrbracket \cdot \dots \cdot \sigma(s_n) \llbracket A_n \rrbracket \leq \sigma(s)(\llbracket A \rrbracket)$ .
2. Let  $\Gamma, A, \Delta \rightarrow B$ .  
By IH,  $\llbracket \Gamma \rrbracket \cdot \llbracket A \rrbracket \cdot \llbracket \Delta \rrbracket \leq \llbracket B \rrbracket$ .  
By the definition,  $\sigma(s_i)(\llbracket A \rrbracket) \leq \llbracket A \rrbracket$ .  
So,  $\llbracket \Gamma \rrbracket \cdot \sigma(s_i)(\llbracket A \rrbracket) \cdot \llbracket \Delta \rrbracket \leq \llbracket B \rrbracket$ .
3. Let  $\Gamma, \Delta \rightarrow B$ ,  $A \in Fm$ , and  $s_i \in W$ .  
So,  $\llbracket \Gamma \rrbracket \cdot \llbracket \Delta \rrbracket \leq \llbracket B \rrbracket$ , then  $\llbracket \Gamma \rrbracket \cdot e \cdot \llbracket \Delta \rrbracket \leq \llbracket B \rrbracket$ , where  $e \in Q$  is unit.  
By the definition of unital open modality,  $\sigma(s_i)(\llbracket A \rrbracket) \leq e$ .  
Thus,  $\llbracket \Gamma \rrbracket \cdot \sigma(s_i)(\llbracket A \rrbracket) \cdot \llbracket \Delta \rrbracket \leq \llbracket B \rrbracket$ .
4. Let  $\Gamma, !_{s_i} A, \Delta, !_{s_i} A, \Pi \rightarrow B$  and  $s_i \in C$ .  
By IH,  $\llbracket \Gamma \rrbracket \cdot \sigma(s_i)(\llbracket A \rrbracket) \cdot \llbracket \Delta \rrbracket \cdot \sigma(s_i)(\llbracket A \rrbracket) \cdot \llbracket \Pi \rrbracket \leq \llbracket B \rrbracket$ .  
By the definition,  $\sigma(s_i)(\llbracket A \rrbracket) \cdot \llbracket \Delta \rrbracket \leq \sigma(s_i)(\llbracket A \rrbracket) \cdot \llbracket \Delta \rrbracket \cdot \sigma(s_i)(\llbracket A \rrbracket)$ .  
Then  $\llbracket \Gamma \rrbracket \cdot \sigma(s_i)(\llbracket A \rrbracket) \cdot \llbracket \Delta \rrbracket \cdot \llbracket \Pi \rrbracket \leq \llbracket B \rrbracket$ .
5. Let  $\Gamma, !_{s_i} A, \Delta, \Pi \rightarrow B$  and  $s_i \in E$ , so  $\sigma(s_i)(a) \in \mathcal{Z}(\mathcal{Q})$  for all  $a \in Q$  by the definition.  
By IH,  $\llbracket \Gamma \rrbracket \cdot \sigma(s_i)(\llbracket A \rrbracket) \cdot \llbracket \Delta \rrbracket \cdot \llbracket \Pi \rrbracket \leq \llbracket B \rrbracket$   
Hence,  $\llbracket \Gamma \rrbracket \cdot \llbracket \Delta \rrbracket \cdot \sigma(s_i)(\llbracket A \rrbracket) \cdot \llbracket \Pi \rrbracket \leq \llbracket B \rrbracket$ .

□

### 3 Quantale completeness

**Definition 13.**

Let  $\mathcal{F} \subseteq Fm$ , an ideal is a subset  $\mathcal{I} \subseteq \mathcal{F}$ , such that:

- If  $B \in \mathcal{I}$  and  $A \rightarrow B$ , then  $A \in \mathcal{I}$ ;
- If  $A, B \in \mathcal{I}$ , then  $A \vee B \in \mathcal{I}$ .

**Definition 14.**

Let  $S \subseteq \mathcal{F} \subseteq Fm$ , then  $\bigvee S = \bigcap \{\mathcal{I} \subseteq \mathcal{F} \mid S \subseteq \mathcal{I}\}$

**Proposition 7.**  $\bigvee S$  is an ideal.

**Lemma 6.**  $A \subseteq Fm$ , then  $\{B \mid B \rightarrow A'\} = \bigvee A$ .

*Proof.*

Let  $A \subseteq Fm$ . Then  $\{B \mid B \rightarrow A', A' \in A\} \subseteq \bigvee A$ , so far as  $A' \rightarrow A'$  by axiom.

On the other hand,  $\{B \mid B \rightarrow A', A' \in A\}$  is an ideal, hence,  $A \subseteq \{B \mid B \rightarrow A', A' \in A\}$ .  $\square$

**Lemma 7.**  $\bigvee A \subseteq \bigvee B$  iff  $\forall A' \in A, \forall B' \in B, A' \rightarrow B'$ .

*Proof.* Let  $\bigvee A \subseteq \bigvee B$ , then  $\{C \mid C \rightarrow A', A' \in A\} \subseteq \{D \mid D \rightarrow B', B' \in B\}$ .

Thus, for all  $A' \in A$ ,  $A' \in \{C \mid C \rightarrow A', A' \in A\}$ , then  $A' \in \{D \mid D \rightarrow B', B' \in B\}$ , hence  $A' \rightarrow B'$ , for all  $B' \in B$ .

On the other hand, let  $A' \rightarrow B'$  for all  $A' \in A$ ,  $B' \in B$  and  $C \in \bigvee A$ .

Thus,  $C \rightarrow A'$ , then  $C \rightarrow B'$  by cut, so  $C \in B'$ .  $\square$

**Lemma 8.** Let  $\mathcal{Q} = \{\bigvee S \mid S \subseteq Fm\}$  and  $\bigvee \mathcal{A} \cdot \bigvee \mathcal{B} = \bigvee \{A \bullet B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$ . Then  $\langle \mathcal{Q}, \subseteq, \cdot, \bigvee 1 \rangle$  is a quantale.

*Proof.* See  $\square$

**Lemma 9.** Let  $!_s \in I$ ,  $I \notin W \cap E \cap C$ , then  $\Box_s(\bigvee A) = \bigvee \{B \mid B \rightarrow !_s A', A' \in A\}$  is a quantic conucleus.

*Proof.*

$$1. \Box_s(\bigvee A) \subseteq \bigvee A;$$

Let  $B \in \Box_s(\bigvee A)$ , then for all  $A' \in A$ ,  $B \rightarrow !_s A'$ , but  $!_s A' \rightarrow A'$ , then  $B \rightarrow A'$ , so  $B \in \bigvee A$ .

$$2. \Box_s(\Box_s(\bigvee A)) = \bigvee \Box_s(\bigvee A);$$

$$\begin{aligned} \Box_s(\Box_s(\bigvee A)) &= \\ \{B \mid B \rightarrow !_s !_s A', A' \in A\} &= \quad, \text{ that follows from equivalence } !_s !_s B \leftrightarrow !_s B. \\ \{B \mid B \rightarrow !_s A', A' \in A\} \end{aligned}$$

$$3. \bigvee A \subseteq \bigvee B \Rightarrow \Box_s(\bigvee A) \subseteq \Box_s(\bigvee B);$$

Follows from admissibility of K-rule for all  $s \in I$ .

$$4. \Box_s \bigvee A \cdot \Box_s \bigvee B = \Box_s(\Box_s \bigvee A \cdot \Box_s \bigvee B).$$

$$\begin{aligned} \Box_s \bigvee A \cdot \Box_s \bigvee B &= \\ \bigvee \{C \bullet D \mid C \bullet D \rightarrow !_s A' \bullet !_s B'\} &= \\ \bigvee \{C \bullet D \mid C \bullet D \rightarrow !_s (!_s A' \bullet !_s B')\} &= \\ \Box_s(\Box_s \bigvee A \cdot \Box_s \bigvee B) \end{aligned}$$

□

**Lemma 10.**

1. Let  $s \in W$ , then for all  $A \subseteq Fm$ ,  $\mathbf{1} \in \Box_s(\bigvee A)$ ;
2. Let  $s \in E$ , then  $\Box_s(\bigvee A) \cdot \bigvee B = \bigvee B \cdot \Box_s(\bigvee A)$ .
3. Let  $s \in C$ , then  $(\Box_s \bigvee A \cdot \bigvee B) \cup (\bigvee B \cdot \Box_s \bigvee A) \subseteq \Box_s \bigvee A \cdot \bigvee B \cdot \Box_s \bigvee A$ , for all  $B \subseteq Fm$ .

*Proof.* 1. Let  $s \in W$ , then for all  $A \subseteq Fm$ ,  $\Box_s(\bigvee A) = \{!_s B \mid !_s B \rightarrow A', A' \in A\}$ . But,  $!_s B \rightarrow \mathbf{1}$ , hence,  $\mathbf{1} \in \Box_s(\bigvee A)$ , so far as  $\Box_s(\bigvee A)$  is an ideal.

2.

$$\begin{aligned} \Box_s(\bigvee A) \cdot \bigvee B &= \\ \bigvee \{!_s C \bullet D \mid !_s C \bullet D \rightarrow A' \bullet B', A' \in A, B' \in B\} &= \\ \bigvee \{D \bullet !_s C \mid D \bullet !_s C \rightarrow A' \bullet B', A' \in A, B' \in B\} &= \\ \bigvee B \cdot \Box_s(\bigvee A) \end{aligned}$$

3.

$$\Box_s \bigvee A \cdot \bigvee B = \bigvee \{!_s C \bullet D \mid !_s C \bullet D \rightarrow A' \bullet B'\}. \quad !_s C \bullet D \rightarrow !_s C \bullet D \bullet !_s C, \text{ hence } \Box_s \bigvee A \cdot \bigvee B \subseteq \Box_s \bigvee A \cdot \bigvee B \cdot \Box_s \bigvee A.$$

Similarly with  $\bigvee B \cdot \Box_s \bigvee A$ .

□

**Lemma 11.**

Let  $i, j \in I$  and  $i \leq j$ , then for all  $A \subseteq Fm$ ,  $\Box_j(\bigvee A) \subseteq \Box_i(\bigvee A)$ .

*Proof.* Let  $i, j \in I$  and  $i \leq j$ . Let  $B \in \Box_j(\bigvee A)$ , then  $\forall A', B \rightarrow !_j A'$ .

But  $!_j A \rightarrow !_i A$ . Then  $B \rightarrow !_i A$  by hence. So,  $B \in \Box_i(\bigvee A)$ .

□

**Definition 15.** Let  $Q$  be a syntactic quantale as proposed above and  $\mathcal{I} = \langle I, \leq, W, C, E \rangle$  be a subexponential signature.

We define a map  $\Box : \mathcal{I} \rightarrow Mod_Q$  as follows:

$$\Box(i)(\bigvee A) = \{B \mid B \rightarrow !_i A\}.$$

**Lemma 12.**  $\Box$  is a subexponential interpretation.

*Proof.* Follows from lemmas 10 and 11.

□

**Lemma 13.**

Let  $Q$  be a quantale constructed above and  $\Box_1, \dots, \Box_n$  be a family of quantic conuclei on  $Q$ . Then there exist a model  $\langle Q, \llbracket \cdot \rrbracket \rangle$ , such that  $\llbracket A \rrbracket = \bigvee \{A\}$ ,  $A \in Fm$ .

*Proof.*

We define an interpretation as follows:

1.  $\llbracket p_i \rrbracket = \bigvee \{p_i\}$
2.  $\llbracket \mathbf{1} \rrbracket = \bigvee \{\mathbf{1}\}$
3.  $\llbracket A \bullet B \rrbracket = \bigvee \{A \bullet B\}$

4.  $\llbracket A/B \rrbracket = \bigvee \{A/B\}$
5.  $\llbracket B \setminus A \rrbracket = \bigvee \{B \setminus A\}$
6.  $\llbracket A \& B \rrbracket = \bigvee \{A \& B\}$
7.  $\llbracket A \vee B \rrbracket = \bigvee \{A \vee B\}$
8.  $\llbracket !_s A \rrbracket = \Box(s)(\bigvee A) = \{B \mid B \rightarrow !_s A\} = \bigvee \{!_s A\}$

□

**Theorem 2.**  $\Gamma \models A \Rightarrow \Gamma \rightarrow A$ .

*Proof.* Follows from lemmas 6, 12, 13.

□