

Models of Lambek calculus enriched with subexponentials

Daniel Rogozin^{1,2}

¹Lomonosov Moscow State University

²Serokell OÜ

Abstract

1 The Lambek Calculus with subexponentials

Definition 1. A subexponential signature is an ordered quintuple:

$$\Sigma = \langle \mathcal{I}, \leq, \mathcal{W}, \mathcal{C}, \mathcal{E} \rangle,$$

where $I = \{s_1, \dots, s_n\}$, \mathcal{I}, \leq is a preorder. W, C, E are upwardly closed subsets of I and $\mathcal{W} \cap \mathcal{C} \subseteq \mathcal{E}$.

Definition 2.

$$\mathcal{F}_\Sigma ::= Tp \mid (\mathcal{F}_\Sigma / \mathcal{F}_\Sigma) \mid (\mathcal{F}_\Sigma \backslash \mathcal{F}_\Sigma) \mid (\mathcal{F}_\Sigma \bullet \mathcal{F}_\Sigma) \mid (\mathcal{F}_\Sigma \vee \mathcal{F}_\Sigma) \mid (\mathcal{F}_\Sigma \wedge \mathcal{F}_\Sigma) \mid !_s \mathcal{F}_{\Sigma, s \in \mathcal{I}}$$

Definition 3. Noncommutative linear logic with subexponentials ($SMALC_\Sigma$), where Σ is a subexponential signature.

$$\begin{array}{c} \overline{A \rightarrow A} \text{ }^{ax} \\ \\ \frac{\Gamma \rightarrow A \quad \Delta, B, \Theta \rightarrow C}{\Delta, \Gamma, A \backslash B, \Theta \rightarrow C} \backslash \rightarrow \qquad \frac{A, \Pi \rightarrow B}{\Pi \rightarrow A \backslash B} \rightarrow \backslash \\ \\ \frac{\Gamma \rightarrow A \quad \Delta, B, \Theta \rightarrow C}{\Delta, B / A, \Gamma, \Theta \rightarrow C} / \rightarrow \qquad \frac{\Pi, A \rightarrow B}{\Pi \rightarrow B / A} \rightarrow / \\ \\ \frac{\Gamma, A, B, \Delta \rightarrow C}{\Gamma, A \bullet B, \Delta \rightarrow C} \bullet \rightarrow \qquad \frac{\Gamma \rightarrow A \quad \Delta \rightarrow B}{\Gamma, \Delta \rightarrow A \bullet B} \rightarrow \bullet \\ \\ \frac{\Gamma, A_i, \Delta \rightarrow B}{\Gamma, A_1 \& A_2, \Delta \rightarrow B} \&, i = 1, 2 \rightarrow \qquad \frac{\Gamma \rightarrow A \quad \Gamma \rightarrow B}{\Gamma \rightarrow A \& B} \rightarrow \& \\ \\ \frac{\Gamma, A, \Delta \rightarrow C \quad \Gamma, B, \Delta \rightarrow C}{\Gamma, A \vee B, \Delta \rightarrow C} \vee \rightarrow \qquad \frac{\Gamma \rightarrow A_i}{\Gamma \rightarrow A_1 \vee A_2} \rightarrow \vee, i = 1, 2 \\ \\ \frac{\Gamma, \Delta \rightarrow A}{\Gamma, 1, \Delta \rightarrow A} 1 \rightarrow \qquad \overline{\rightarrow 1} \rightarrow 1 \\ \\ \frac{\Gamma, A, \Delta \rightarrow C}{\Gamma, !^s A, \Delta \rightarrow C} ! \rightarrow \qquad \frac{!^{s_1} A_1, \dots, !^{s_n} A_n \rightarrow A}{!^{s_1} A_1, \dots, !^{s_n} A_n \rightarrow !^s A} \rightarrow !, \forall j, s_j \geq s \end{array}$$

Structural rules:

$$\begin{array}{c}
\frac{\Gamma, !^s A, \Delta, !^s A, \Theta \rightarrow B}{\Gamma, !^s A, \Delta, \Theta \rightarrow B} \text{ncontr}_1, s \in C \qquad \frac{\Gamma, !^s A, \Delta, !^s A, \Theta \rightarrow B}{\Gamma, \Delta, !^s A, \Theta \rightarrow B} \text{ncontr}_2, s \in C \\
\\
\frac{\Gamma, \Delta, !^s A, \Theta \rightarrow B}{\Gamma, !^s A, \Delta, \Theta \rightarrow A} \text{ex}_1, s \in E \qquad \frac{\Gamma, !^s A, \Delta, \Theta \rightarrow B}{\Gamma, \Delta, !^s A, \Theta \rightarrow A} \text{ex}_2, s \in E \\
\\
\frac{\Gamma, \Delta \rightarrow B}{\Gamma, !^s A, \Delta \rightarrow B} \text{weak}_1, s \in C \qquad \frac{\Gamma \rightarrow A \quad \Pi, A, \Delta \rightarrow B}{\Gamma, \Pi, \Delta \rightarrow B} \text{cut}
\end{array}$$

Definition 4. By $L_1^* \wedge \vee_{\mathbf{s4}}$ we mean $SMALC_\Sigma$, where $\Sigma = \{s\}$ and $C = W = E = \emptyset$.

Theorem 1.

1. Cut-rule is admissable;
2. $SMALC_\Sigma$ is undecidable, if $C \neq \emptyset$;
3. If C is empty, then the decidability problem of $SMALC_\Sigma$ belongs to PSPACE.

2 Semantics

Definition 5. *Quantale*

A quantale is a triple $\mathcal{Q} = \langle A, \vee, \cdot \rangle$, such that $\langle A, \vee \rangle$ is a complete lattice and $\langle A, \cdot \rangle$ is a semigroup, such that for all indexing set I :

1. $a \cdot \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \cdot b_i)$;
2. $\bigvee_{i \in I} a_i \cdot b = \bigvee_{i \in I} (a_i \cdot b)$

A quantale is called unital, if $\langle A, \cdot \rangle$ is a monoid.

Some example of quantales:

- Let A be a semigroup (monoid), then $\langle \mathcal{P}(A), \cdot, \subseteq \rangle$ is a free (unital) quantale.
- Let R be a ring and $Sub(R)$ be a set of additive subgroups of R . We define $A \cdot B$ as an additive subgroup generated by finite sums of products ab and order is defined by inclusion.
- Any locale is a quantale with $\cdot = \wedge$.

It is easy to see, that any (unital) quantale is a residual (monoid) semigroup. We define divisions as follows:

1. $a \backslash b = \bigvee \{c \mid a \cdot c \leq b\}$
2. $b / a = \bigvee \{c \mid c \cdot a \leq b\}$

Definition 6. Let $\mathcal{Q}_1, \mathcal{Q}_2$ be quantales. A quantale homomorphism is a map $f : \mathcal{Q}_1 \rightarrow \mathcal{Q}_2$, such that:

1. for all $a, b \in \mathcal{Q}_1$, $f(a \cdot b) = f(a) \cdot f(b)$;

2. for all indexing set I , $f(\bigvee_{i \in I} a_i) = \bigvee_{i \in I} f(a_i)$.

If $\mathcal{Q}_1, \mathcal{Q}_2$ are unital quantales, then a unital homomorphism is a quantale homomorphism, such that $f(\varepsilon) = \varepsilon$.

Definition 7.

Let $\mathcal{Q} = \langle A, \bigvee, \cdot \rangle$ be a quantale. $\mathcal{S} \subseteq \mathcal{Q}$ is said to be a subquantale, if \mathcal{S} is closed under multiplication and sups.

There occurs the following simple statement:

Proposition 1.

Let $\mathcal{Q}_1, \mathcal{Q}_2$ be quantales and $\mathcal{S} \subseteq \mathcal{Q}_1$ is a subquantale of \mathcal{Q}_1 .

Then, if $f : \mathcal{Q}_1 \rightarrow \mathcal{Q}_2$ is a quantale homomorphism, then $f(\mathcal{S}) \subseteq \mathcal{Q}_2$ is a subquantale of \mathcal{Q}_2 .

In other words, a homomorphic image of subquantale is a subquantale.

Proof. It is clearly that $f(\mathcal{S}) \subseteq \mathcal{Q}_2$ is a submonoid of \mathcal{Q}_2 .

Let $a_i \in \mathcal{S}$ for each $i \in I$, so $\bigvee_{i \in I} a_i \in \mathcal{S}$, but $f(a_i) \in f(\mathcal{S})$ for any $i \in I$, so $f(\bigvee_{i \in I} a_i) = \bigvee_{i \in I} f(a_i) \in f(\mathcal{S})$, so $f(\mathcal{S})$ is closed under joins, so $f(\mathcal{S})$ is a subquantale of \mathcal{Q}_2 \square

Definition 8.

Let $\mathcal{Q} = \langle A, \bigvee, \cdot \rangle$ be a quantale. The center of a quantale is the subquantale $Z(\mathcal{Q}) = \{a \in A \mid \forall b \in A, a \cdot b = b \cdot a\}$

Definition 9.

An open modality (or quantic conucleus) on quantale \mathcal{Q} is a map $\Box : \mathcal{Q} \rightarrow \mathcal{Q}$, such that

1. $\Box x \leq x$;
2. $\Box x = \Box \Box x$;
3. $x \leq y \Rightarrow \Box x \leq \Box y$;
4. $\Box x \cdot \Box y = \Box(\Box x \cdot \Box y)$.

For unital quantale, we require that $\Box e = e$.

Note that, we may replace the last condition on equivalent condition $\Box(x) \cdot \Box(y) \leq \Box(x \cdot y)$.

Definition 10.

We define a partial order on open modalities on \mathcal{Q} as $\Box_1 \leq \Box_2 \Leftrightarrow \forall a \in \mathcal{Q}, \Box_1(a) \leq \Box_2(a)$.

Lemma 1. $\Box_1 a_1 \cdot \Box_2 a_2 \leq \Box'(\Box_1 a_1 \cdot \Box_2 a_2)$, where $\Box_i \leq \Box, i = 1, 2$.

Proof.

$$\begin{aligned} & \Box_1 a_1 \cdot \Box_2 a_2 \leq \\ & \Box_1(\Box_1 a_1) \cdot \Box_2(\Box_2 a_2) \leq \\ & \Box(\Box_1 a_1) \cdot \Box(\Box_2 a_2) \leq \\ & \Box(\Box_1(a_1) \cdot \Box_2(a_2)) \end{aligned}$$

\square

Definition 11.

1. An open modality is called central, if for all $a, b \in \mathcal{Q}$, $\Box(a) \cdot b = b \cdot \Box(a)$.

2. An open modality is called *weak square-increasing*, if for all $a, b \in \mathcal{Q}$, $\Box a \cdot b \leq \Box a \cdot b \cdot \Box a$ and $b \cdot \Box a \leq \Box a \cdot b \cdot \Box a$.

3. An open modality is called *unital*, if $\forall a \in \mathcal{Q}, \Box a \leq e$.

Lemma 2.

Let \Box be an open modality on some unital quantale $\mathcal{Q} = \langle A, \bigvee, \cdot, e \rangle$. Then, if \Box is unital and weak idempotent, then \Box is central.

Proof.

$$\begin{aligned}
& b \cdot \Box a \leq \\
& \quad \text{Right weak square-increase} \\
& \Box a \cdot b \cdot \Box a \leq \\
& \quad \text{Unitality} \\
& \Box a \cdot b \cdot e \leq \\
& \quad \text{Identity} \\
& \Box a \cdot b \leq \\
& \quad \text{Left weak square-increase} \\
& \Box a \cdot b \cdot \Box a \leq \\
& \quad \text{Unitality} \\
& e \cdot b \cdot \Box a \leq \\
& \quad \text{Identity} \\
& b \cdot \Box(a)
\end{aligned}$$

Hence, $b \cdot \Box a = \Box a \cdot b$, so for all $a \in \mathcal{Q}, \Box a \in \mathcal{Z}(\mathcal{Q})$.

□

Proposition 2.

Let \mathcal{Q} be a quantale and $S \subseteq \mathcal{Q}$ a subquantale, then $\Box : \mathcal{Q} \rightarrow \mathcal{Q}$, such that $\Box(a) = \bigvee \{s \in S \mid s \leq a\}$, is an open modality.

Proof. See

□

Proposition 3.

Let \mathcal{Q} be a quantale and $\mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \mathcal{Q}$.

Then $\Box_{\mathcal{S}_1}(a) \leq \Box_{\mathcal{S}_2}(a)$.

Proof. Immediately.

□

Proposition 4.

Let \mathcal{Q} be a quantale and $\mathcal{S} \subseteq \mathcal{Q}$ a subquantale, then the following operations are open modalities:

1. $\Box_z(a) = \bigvee \{s \in \mathcal{S} \mid s \leq a, s \in \mathcal{Z}(\mathcal{Q})\};$
2. $\Box_1(a) = \bigvee \{s \in \mathcal{S} \mid s \leq a, s \leq 1\};$
3. $\Box_{idem}(a) = \bigvee \{s \in \mathcal{S} \mid s \leq a, \forall b \in \mathcal{Q}, b \cdot s \vee s \cdot b \leq s \cdot b \cdot s\};$
4. $\Box_{z,1}, I_{z,idem}, I_{1,idem}, I_{z,1,idem}.$

Proof. Immediately.

□

Proposition 5.

1. $\forall a \in \mathcal{Q}, \Box_{\mathbf{1}, idem}(a) \leq \Box_z(a)$.
2. $\forall a \in \mathcal{Q}, \Box_{z, \mathbf{1}, idem} = \Box_{\mathbf{1}, idem}(a)$

Proof. Follows from Lemma 3. □

Definition 12. *Interpretation of subexponential signature*

Let $\Sigma = \langle I, \leq, W, C, E \rangle$ be a subexponential signature, where $|I| = n$ and $\Box_{\mathcal{Q}}$ is a category of open modalities on a quantale \mathcal{Q} . Subexponential interpretation is a contravariant functor $\sigma : I \rightarrow \Box_{\mathcal{Q}}$ defined as follows:

$$\sigma(s_i) = \begin{cases} \Box_i : \mathcal{Q} \rightarrow \mathcal{Q}, s.t. \forall a \in \mathcal{Q}, \Box_i(a) = \{s \in S_i \mid s \leq a\}, \\ \quad \text{if } s_i \notin W \cap C \cap E \\ \Box_i : \mathcal{Q} \rightarrow \mathcal{Q}, s.t. \forall a \in \mathcal{Q}, \Box_i(a) = \{s \in S_i \mid s \leq a, s \leq \mathbf{1}\}, \\ \quad \text{if } s_i \in W \\ \Box_i : \mathcal{Q} \rightarrow \mathcal{Q}, s.t. \forall a \in \mathcal{Q}, \Box_i(a) = \{s \in S_i \mid s \leq a, s \in \mathcal{Z}(\mathcal{Q})\}, \\ \quad \text{if } s_i \in E \\ \Box_i : \mathcal{Q} \rightarrow \mathcal{Q}, s.t. \forall a \in \mathcal{Q}, \Box_i(a) = \{s \in S_i \mid s \leq a, \forall b, b \cdot s \vee s \cdot b \leq s \cdot b \cdot s\}, \\ \quad \text{if } s_i \in E \\ \text{otherwise, if } s_i \text{ belongs to some intersection of subsets, then we combine the relevant conditions} \end{cases}$$

Definition 13. Let \mathcal{Q} be an unital quantale, $f : Tp \rightarrow \mathcal{Q}$ a valuation and $\sigma : I \rightarrow \Box_{\mathcal{Q}}$ a subexponential interpretation, then interpretation is defined inductively:

$$\begin{aligned} \llbracket p_i \rrbracket &= f(p_i) \\ \llbracket \mathbf{1} \rrbracket &= e \\ \llbracket A \bullet B \rrbracket &= \llbracket A \rrbracket \cdot \llbracket B \rrbracket \\ \llbracket A \setminus B \rrbracket &= \llbracket A \rrbracket \setminus \llbracket B \rrbracket \\ \llbracket A/B \rrbracket &= \llbracket A \rrbracket / \llbracket B \rrbracket \\ \llbracket A \&B \rrbracket &= \llbracket A \rrbracket \wedge \llbracket B \rrbracket \\ \llbracket A \vee B \rrbracket &= \llbracket A \rrbracket \vee \llbracket B \rrbracket \\ \llbracket !_{s_i} A \rrbracket &= \sigma(s_i) \llbracket A \rrbracket \end{aligned}$$

Definition 14. $\Gamma \models A \Leftrightarrow \forall f, \forall \sigma, \llbracket \Gamma \rrbracket \leq \llbracket A \rrbracket$

Theorem 2. $\Gamma \rightarrow A \Rightarrow \llbracket \Gamma \rrbracket \leq \llbracket A \rrbracket$

Proof. We consider the promotion case, the rest modal cases are immediatly shown.

Let $!_{s_1} A_1, \dots, !_{s_n} A_n \rightarrow A$ and $\forall i, s \leq s_i$.

Then $\forall a \in \mathcal{Q}, \sigma(s_i)(a) \leq \sigma(s)(a)$.

By IH, $\sigma(s_1) \llbracket A_1 \rrbracket \cdot \dots \cdot \sigma(s_n) \llbracket A_n \rrbracket \leq \llbracket A \rrbracket$.

Thus, $\sigma(s)(\sigma(s_1) \llbracket A_1 \rrbracket \cdot \dots \cdot \sigma(s_n) \llbracket A_n \rrbracket) \leq \sigma(s)(\llbracket A \rrbracket)$.

By Lemma 5, $\sigma(s_1) \llbracket A_1 \rrbracket \cdot \dots \cdot \sigma(s_n) \llbracket A_n \rrbracket \leq \sigma(s)(\sigma(s_1) \llbracket A_1 \rrbracket \cdot \dots \cdot \sigma(s_n) \llbracket A_n \rrbracket)$.

So, $\sigma(s_1) \llbracket A_1 \rrbracket \cdot \dots \cdot \sigma(s_n) \llbracket A_n \rrbracket \leq \sigma(s)(\llbracket A \rrbracket)$. □

3 Quantale completeness

Definition 15.

Let $\mathcal{F} \subseteq Fm$, an ideal is a subset $\mathcal{I} \subseteq \mathcal{F}$, such that:

- If $B \in \mathcal{I}$ and $A \rightarrow B$, then $A \in \mathcal{I}$;

- If $A, B \in \mathcal{I}$, then $A \vee B \in \mathcal{I}$.

Definition 16.

Let $S \subseteq \mathcal{F} \subseteq Fm$, then $\bigvee S = \bigcap \{\mathcal{I} \subseteq \mathcal{F} \mid S \subseteq \mathcal{I}\}$

Proposition 6. $\bigvee S$ is an ideal.

Lemma 3. $A \subseteq Fm$, then $\{B \mid B \rightarrow A\} = \bigvee \{A\}$.

Proof.

Let $A \in Fm$. Then $\{B \mid B \rightarrow A', A' \in A\} \subseteq \bigvee \{A\}$, so far as $\bigvee A$ is an ideal.

On the other hand, $\{B \mid B \rightarrow A\}$ is an ideal, it is easy to see that this set is closed under \vee . So, $\bigvee A \subseteq \{B \mid B \rightarrow A\}$. \square

Lemma 4. $\bigvee \{A\} \subseteq \bigvee \{B\}$ iff $A \rightarrow B$.

Proof. Let $\bigvee \{A\} \subseteq \bigvee \{B\}$, then $\{C \mid C \rightarrow A\} \subseteq \{D \mid D \rightarrow B\}$.

Thus, $A \in \{C \mid C \rightarrow A\}$, then $A \in \{D \mid D \rightarrow B\}$, hence $A \rightarrow B$.

On the other hand, let $A \rightarrow B$ and $C \in \bigvee \{A\}$.

Thus, $C \rightarrow A$, then $C \rightarrow B$ by cut. \square

Lemma 5. Let $\mathcal{Q} = \{\bigvee S \mid S \subseteq Fm\}$ and $\bigvee \mathcal{A} \cdot \bigvee \mathcal{B} = \bigvee \{A \bullet B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$. Then $\langle \mathcal{Q}, \subseteq, \cdot, \bigvee \mathbf{1} \rangle$ is a quantale.

Proof. See \square

Lemma 6. Let $!_s \in I$ and $A \in \mathcal{F}_\Sigma$, then $\Box_s(\bigvee \{A\}) = \bigvee \{!_s B \mid !_s B \rightarrow A\}$ is a quantic conucleus.

Proof.

See Yetter. \square

Lemma 7. Let $A \in \mathcal{F}_\Sigma$, then $\Box_s \bigvee \{A\} = \bigvee \{!_s A\}$, for each $s \in \mathcal{I}$.

Proof. Let $A \in Fm$ and $s \in \mathcal{I}$.

Let $!_s B \in \Box_s \bigvee \{A\}$, then $!_s B \rightarrow A$, then $!_s B \rightarrow !_s A$ by promotion. So, $!_s B \in \bigvee \{!_s A\}$.

Let $C \in \bigvee \{!_s A\}$, then $C \rightarrow !_s A$, so $!_s C \rightarrow !_s A$ by dereliction, but $!_s A \rightarrow A$, hence $!_s C \rightarrow A$ by cut. So, $!_s C \in \Box_s \bigvee \{A\}$. \square

Lemma 8.

Let $i, j \in I$ and $i \leq j$, then for all $A \in \mathcal{F}_\Sigma$, $\Box_j(\bigvee \{A\}) \subseteq \Box_i(\bigvee \{A\})$.

Proof.

Let $i, j \in I$ and $i \leq j$, then for all $A \in \mathcal{F}_\Sigma$, $!_j A \rightarrow !_i A$ by promotion. Then $\bigvee \{!_j A\} \subseteq \bigvee \{!_i A\}$, so $\Box_j(\bigvee \{A\}) \subseteq \Box_i(\bigvee \{A\})$. \square

Lemma 9.

For all $A \in \mathcal{F}_\Sigma$,

1. Let $s \in W$, then $\Box_s \{A\} \subseteq \{\mathbf{1}\}$;
2. Let $s \in E$, then $\Box_s(\bigvee \{A\}) \cdot \bigvee \{B\} = \bigvee \{B\} \cdot \Box_s(\bigvee \{A\})$.
3. Let $s \in C$, then $(\Box_s \bigvee A \cdot \bigvee B) \cup (\bigvee B \cdot \Box_s \bigvee A) \subseteq \Box_s \bigvee A \cdot \bigvee B \cdot \Box_s \bigvee A$, for all $B \subseteq Fm$.

Proof.

Follows from $!_s A \rightarrow \mathbf{1}$, so $s \in W$;

Follows from $!_s A \bullet B \leftrightarrow B \bullet !_s A$;

Follows from $!_s A \bullet B \rightarrow !_s A \bullet B \bullet !_s A$ and similarly for $B \bullet !_s A$. \square

Definition 17.

Let Q be a syntactic quantale as proposed above and $\mathcal{I} = \langle I, \leq, W, C, E \rangle$ be a subexponential signature.

We define a map $\square : \mathcal{I} \rightarrow \text{Mod}_{\mathcal{Q}}$ as follows:

$$\square(i)(\bigvee \{A\}) = \{!_i B \mid !_i B \rightarrow A\}.$$

Lemma 10. \square is a subexponential interpretation.

Proof. Follows from lemmas above. \square

Lemma 11.

Let Q be a quantale constructed above and $\square_1, \dots, \square_n$ be a family of quantic conuclei on Q . Then there exist a model $\langle Q, \llbracket \cdot \rrbracket \rangle$, such that $\llbracket A \rrbracket = \bigvee \{A\}$, $A \in \text{Fm}$.

Proof.

We define an interpretation as follows:

1. $\llbracket p_i \rrbracket = \bigvee \{p_i\}$
2. $\llbracket \mathbf{1} \rrbracket = \bigvee \{\mathbf{1}\}$
3. $\llbracket A \bullet B \rrbracket = \bigvee \{A \bullet B\}$
4. $\llbracket A/B \rrbracket = \bigvee \{A/B\}$
5. $\llbracket B \setminus A \rrbracket = \bigvee \{B \setminus A\}$
6. $\llbracket A \& B \rrbracket = \bigvee \{A \& B\}$
7. $\llbracket A \vee B \rrbracket = \bigvee \{A \vee B\}$
8. $\llbracket !_s A \rrbracket = \square(s)(\bigvee \{A\}) = \bigvee \{!_s A\}$.

\square

Theorem 3. $\Gamma \models A \Rightarrow \Gamma \rightarrow A$.

Proof. Follows from lemmas above. \square

4 Relational semantics

Definition 18.

Let A be a set. Then relational quantale on A is a tuple $\mathcal{Q} = \langle \mathcal{A}, \mathcal{I} \rangle$ where $\mathcal{A} \subseteq 2^{A \times A}$:

- $\langle \mathcal{A}, \subseteq \rangle$ is a complete semi-lattice;
- Multiplication is defined as $R \circ S = \{ \langle a, c \rangle \mid \exists b \in A, \langle a, b \rangle \in R \text{ and } \langle b, c \rangle \in S \}$
- $\langle \mathcal{A}, \circ, \mathcal{I} \rangle$ is a monoid;
- For each indexing set J , $R \circ \bigvee_{i \in I} S_i = \bigvee_{i \in I} (R \circ S_i)$ and $\bigvee_{i \in I} R_i \circ S = \bigvee_{i \in I} (R_i \circ S)$.

Theorem 4.

Let $\mathcal{Q} = \langle A, \leq, \cdot, \vee \rangle$ be a unital quantale and \mathcal{S} is a subquantale of \mathcal{Q} .

Then $\langle \mathcal{Q}, \Box_{\mathcal{S}} \rangle$ is isomorphic to some relational quantale of A with some quantic conucleus $\hat{\Box}$.

Proof.

Consider a relational quantale proposed by Brown and Gurr.

This quantale is 4-tuple $\theta(\mathcal{Q}) = \langle \mathcal{R}, \subseteq, \circ, \vee \rangle$ defined as follows:

1. $\theta(a) = \{ \langle b, c \rangle \mid b \leq a \cdot c \};$
2. $\theta(a \cdot b) = \theta(a) \circ \theta(b);$
3. $\theta(\bigvee_{i \in I} a_i) = \bigvee_{i \in I} \theta(a_i);$
4. $\theta(\varepsilon) = \{ \langle b, c \rangle \mid b \cdot \varepsilon \leq c \} = \{ \langle b, c \rangle \mid b \leq c \}$

Let $\mathcal{S} \subseteq \mathcal{Q}$, so $\Box_{\mathcal{S}} a := \bigvee \{ s \mid s \in \mathcal{S}, s \leq a \}$ is quantic conucleus.

So, $\theta(\mathcal{S}) \subseteq \theta(\mathcal{Q})$ is a subquantale of $\theta(\mathcal{Q})$.

Let us define $\hat{\Box}\theta(a) := \bigvee \{ \theta(s) \mid \theta(s) \in \theta(\mathcal{S}), \theta(s) \subseteq \theta(a) \}$, so

$$\begin{aligned} \theta(\Box_{\mathcal{S}} a) &= \{ \langle p, q \rangle \mid p \leq \Box_{\mathcal{S}} a \cdot q \} = \\ &= \{ \langle p, q \rangle \mid p \leq \bigvee \{ s \mid s \in \mathcal{S}, s \leq a \} \cdot q \} = \end{aligned}$$

Homomorphism

$$\theta(\bigvee_{s \in \mathcal{S}, s \leq a} s) =$$

Homomorphism preserves sups

$$\bigvee_{s \in \mathcal{S}, s \leq a} \theta(s) =$$

Unfolding

$$\bigvee \{ \theta(s) \mid s \in \mathcal{S}, s \leq a \} =$$

Unfolding

$$\bigvee \{ \theta(s) \mid \theta(s) \in \theta(\mathcal{S}), \theta(s) \subseteq \theta(a) \} = \hat{\Box}\theta(a)$$

So, $\hat{\Box}\theta(a) = \theta(\Box_{\mathcal{S}} a)$.

□

Theorem 5. $\Gamma \models A$, then $L_1^* \wedge \vee_{\mathbf{S4}} \vdash \Gamma \rightarrow A$