# On $R$-models

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**Definition 1.**

$$\frac{}{A \to A}\ ax$$

$$\frac{\Gamma \to A \qquad \Delta, B, \Theta \to C}{\Delta, \Gamma, A\backslash B, \Theta \to C}\ \backslash \to$$

$$\frac{\Gamma \to A \qquad \Delta, B, \Theta \to C}{\Delta, B/A, \Gamma, \Theta \to C}\ / \to$$

$$\frac{\Gamma, A, B, \Delta \to C}{\Gamma, A \bullet B, \Delta \to C}\ \bullet \to$$

$$\frac{A, \Pi \to B}{\Pi \to A\backslash B}\ \to \backslash$$

$$\frac{\Pi, A \to B}{\Pi \to B/A}\ \to /$$

$$\frac{\Gamma \to A \qquad \Delta \to B}{\Gamma, \Delta \to A \bullet B}\ \to \bullet$$

**Definition 2.**

*$R$-model is a pair $\mathcal{M} = \langle W, R, v \rangle$, where $R$ is a transitive relation on $W$ and $v : Tp \to 2^R$ is a valuation, such that:*

1. $\mathcal{M}, w \Vdash p_i \Leftrightarrow w \in v(p_i)$;
2. $\mathcal{M}, \langle a, b \rangle \Vdash A \bullet B \Leftrightarrow$ *there exists* $c \in W$, $\mathcal{M}, a \Vdash A$ *and* $\mathcal{M}, b \Vdash B$;
3. $\mathcal{M}, \langle a, b \rangle \Vdash A\backslash B \Leftrightarrow$ *for all* $c \in R^{-1}(a)$, $\mathcal{M}, \langle c, a \rangle \Vdash A$ *implies* $\mathcal{M}, \langle c, b \rangle \Vdash B$;
4. $\mathcal{M}, \langle a, b \rangle \Vdash B/A \Leftrightarrow$ *for all* $c \in R(a)$, $\mathcal{M}, \langle a, c \rangle \Vdash A$ *implies* $\mathcal{M}, \langle b, c \rangle \Vdash B$;
5. $\mathcal{M}, \langle a, b \rangle \Vdash \Gamma \to A \Leftrightarrow \mathcal{M}, \langle a, b \rangle \Vdash \Gamma$ *implies* $\mathcal{M}, \langle a, b \rangle \Vdash A$

where $\mathcal{M}, \langle a, b \rangle \Vdash \Gamma$ denotes $\mathcal{M}, \langle a, b \rangle \Vdash A_1 \bullet \cdots \bullet A_n$, or, equivalently, there exist $c_1, \dots, c_{n-1}$, such that $\mathcal{M}, \langle a, c_1 \rangle \Vdash A_1, \mathcal{M}, \langle c_1, c_2 \rangle \Vdash A_2, \dots, \mathcal{M}, \langle c_{n-1}, b \rangle \Vdash A_n$ implies that $\mathcal{M}, \langle a, b \rangle \Vdash B$.

**Theorem 1.** *Let $\mathbb{F}$ is a $R$-frame, then $\mathbb{F} \models L$.*

**Definition 3.**

*Let $\mathcal{F}_1$, $\mathcal{F}_2$ be R-frames and $\mathcal{M}_1 = \langle \mathcal{F}_1, v_1 \rangle$, $\mathcal{M}_2 = \langle \mathcal{F}_2, v_2 \rangle$ be R-models.*
*A map $f : W_1 \to W_2$ is said to be a R-frame p-morphism if the following conditions hold:*

1. *$f$ is onto;*
2. *$\forall a, b \in W_1 (aR_1 b \Rightarrow f(a) R_2 f(b))$ (monotonicity);*
3. *$\forall d \in W_1, c \in W_2, f(d) R_2 c \Rightarrow \exists c' \in W_1, f(c') = c \,\&\, dR_1 c'$ (left lift property);*
4. *$\forall d \in W_1, c \in W_2, cR_2 f(d) \Rightarrow \exists c' \in W_1, f(c') = c \,\&\, c'R_1 d$ (right lift property);*

*A map $f : \mathcal{F}_1 \to \mathcal{F}_2$ is R-model p-moprhism, iff:*

- $\mathcal{M}_1, \langle a, b \rangle \Vdash p_i \Leftrightarrow \mathcal{M}_2, \langle f(a), f(b) \rangle \Vdash p_i$

**Lemma 1.** *Let $f : \mathcal{M}_1 \twoheadrightarrow \mathcal{M}_2$, then $\mathcal{M}_1, \langle a, b \rangle \Vdash A \Leftrightarrow \mathcal{M}_2, \langle f(a), f(b) \rangle \Vdash A$, for all $a, b \in W_1$ and for all $A \in Fm$*

*Proof.*

1. $\Rightarrow$
   (a) Basic case: follows from the definition.
   (b) Let $A = B \bullet C$ and $\mathcal{M}_1, \langle a, b \rangle \Vdash B \bullet C$, then there exists $c \in W_1$, such that $\mathcal{M}_1, \langle a, c \rangle \Vdash B$ and $\mathcal{M}_1, \langle c, b \rangle \Vdash C$.
   Then, $aR_1 c$ and $cR_1 b$, so $f(a) R_2 f(c)$ and $f(c) R_2 f(b)$.
   Thus, by IH, $\mathcal{M}_2, \langle f(a), f(c) \rangle \Vdash B$ and $\mathcal{M}_2, \langle f(c), f(b) \rangle \Vdash C$, so $\mathcal{M}_2, \langle f(a), f(b) \rangle \Vdash B \bullet C$.
   (c) Let $A = B \backslash C$ and $\mathcal{M}_1, \langle a, b \rangle \Vdash B \backslash C$. Let $c \in W_2$, such that $cR_2 f(a)$.
   Then, by left lift property, there exist $d \in W_1$, such that $f(d) = c$ and $dR_1 a$.
   Thus, $\mathcal{M}_1, \langle d, a \rangle \Vdash A$ implies $\mathcal{M}_1, \langle d, b \rangle \Vdash B$.
   By IH, $\mathcal{M}_2, \langle c, f(a) \rangle \Vdash A$ implies $\mathcal{M}_2, \langle c, f(b) \rangle \Vdash B$, then $\mathcal{M}_2, \langle f(a), f(b) \rangle \Vdash A \backslash B$.
   (d) Similarly to (c), but via right lift property.
   (e) Let $\mathcal{M}_1, \langle a, b \rangle \Vdash A_1, \dots, A_n \to A$.
   Thus, there exist $c_1, \dots, c_{n-1} \in W_1$, such that $\mathcal{M}_1, \langle a, c_1 \rangle \Vdash A_1, \dots, \mathcal{M}_1, \langle c_{n-2}, c_{n-1} \rangle \Vdash A_n$ implies that $\mathcal{M}_1, \langle c_{n-1}, b \rangle \Vdash B$.
   By IH, $\mathcal{M}_2, \langle f(a), f(c_1) \rangle \Vdash A_1, \dots, \mathcal{M}_1, \langle f(c_{n-1}), f(b) \rangle \Vdash A_n$.
   So, $\mathcal{M}_2, \langle f(a), f(b) \rangle \Vdash A_1 \bullet \dots \bullet A_n$.
   Thus, $\mathcal{M}_2, \langle f(a), f(b) \rangle \Vdash B$.
2. $\Leftarrow$
   (a) Basic case: follows from the definition.
   (b) Let $A = B \bullet C$. Let $\mathcal{M}_2, \langle f(a), f(b) \rangle \Vdash B \bullet C$. Then there exists $c \in W_2$, such that $\mathcal{M}_2, \langle f(a), c \rangle \Vdash B$ and $\mathcal{M}_2, \langle c, f(b) \rangle \Vdash C$.
   So far as $f$ is surjection, then there exists $d \in W_1$, such that $c = f(d)$, then $\mathcal{M}_2, \langle f(a), f(d) \rangle \Vdash B$ and $\mathcal{M}_2, \langle f(d), f(b) \rangle \Vdash C$, and, by IH, $\mathcal{M}_1, \langle a, d \rangle \Vdash B$ and $\mathcal{M}_1, \langle d, b \rangle \Vdash C$, then $\mathcal{M}_1, \langle a, b \rangle \Vdash B \bullet C$.

(c) Let $A = B\backslash C$ and $\mathcal{M}_2, \langle f(a), f(b)\rangle \Vdash B\backslash C$.
Let $c \in W_1$ and $cR_1a$, then $f(c)R_1f(a)$ by monotonicity, so $\mathcal{M}_2, \langle f(c), f(a)\rangle \Vdash A$ implies $\mathcal{M}_2, \langle f(c), f(b)\rangle \Vdash B$.
By IH, $\mathcal{M}_1, \langle c, a\rangle \Vdash A$ implies $\mathcal{M}_1, \langle c, b\rangle \Vdash B$. Thus, $\mathcal{M}_1, \langle c, a\rangle \Vdash A\backslash B$.

(d) Similarly to (c).

(e) Let $\mathcal{M}_2, \langle f(a), f(b)\rangle \Vdash \Gamma \to A$, so by case (b) and IH, $\mathcal{M}_2, \langle a, b\rangle\Gamma$, so $\mathcal{M}_\infty, \langle a, b\rangle A$.

□

**Lemma 2.**

1. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be R-models and $\mathcal{M}_1 \twoheadrightarrow \mathcal{M}_2$. Then $\mathcal{M}_1 \models A$ iff $\mathcal{M}_2 \models A$.*
2. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be R-frames and $\mathcal{F}_1 \twoheadrightarrow \mathcal{F}_2$, then $\mathcal{F}_1 \models A$ implies $\mathcal{F}_2 \models A$.*
3. *$\mathcal{F}_1 \cong \mathcal{F}_2$, then $Log(\mathcal{F}_1) = Log(\mathcal{F}_2)$.*

*Proof.* Standardly. □