

Quantale model of noncommutative linear logic with subexponentials

Definition 1. A subexponential signature is an ordered quintuple:

$$\Sigma = \langle I, \leq, W, C, E \rangle,$$

where $I = \{s_1, \dots, s_n\}$, $\langle I, \leq \rangle$ is a preorder. W, C, E are subsets of I and $W \cup C \subseteq E$.

Definition 2. Noncommutative linear logic with subexponentials ($SMALC_\Sigma$), where Σ is a subexponential signature.

$$\begin{array}{c}
 \overline{A \Rightarrow A} \text{ }^{ax} \\
 \\
 \frac{\Gamma \Rightarrow A \quad \Delta, B, \Theta \Rightarrow C}{\Delta, \Gamma, A \backslash B, \Theta \Rightarrow C} \backslash \rightarrow \qquad \frac{A, \Pi \Rightarrow B}{\Pi \Rightarrow A \backslash B} \rightarrow \backslash \\
 \\
 \frac{\Gamma \Rightarrow A \quad \Delta, B, \Theta \Rightarrow C}{\Delta, B / A, \Gamma, \Theta \Rightarrow C} / \rightarrow \qquad \frac{\Pi, A \Rightarrow B}{\Pi \Rightarrow B / A} \rightarrow / \\
 \\
 \frac{\Gamma, A, B, \Delta \Rightarrow C}{\Gamma, A \bullet B, \Delta \Rightarrow C} \bullet \rightarrow \qquad \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \bullet B} \rightarrow \bullet \\
 \\
 \frac{\Gamma, A_i, \Delta \Rightarrow B}{\Gamma, A_1 \& A_2, \Delta \Rightarrow B} \&, i = 1, 2 \rightarrow \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \& B} \rightarrow \& \\
 \\
 \frac{\Gamma, A, \Delta \Rightarrow C \quad \Gamma, B, \Delta \Rightarrow C}{\Gamma, A \vee B, \Delta \Rightarrow C} \vee \rightarrow \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2} \rightarrow \vee, i = 1, 2 \\
 \\
 \frac{\Gamma, \Delta \Rightarrow A}{\Gamma, \mathbf{1}, \Delta \Rightarrow A} \mathbf{1} \rightarrow \qquad \frac{}{\Rightarrow \mathbf{1}} \rightarrow \mathbf{1} \\
 \\
 \frac{\Gamma, A, \Delta \Rightarrow C}{\Gamma, !^s A, \Delta \Rightarrow C} ! \rightarrow \qquad \frac{!^{s_1} A_1, \dots, !^{s_n} A_n \Rightarrow A}{!^{s_1} A_1, \dots, !^{s_n} A_n \Rightarrow !^s A} \rightarrow !, \forall j, s_j \geq s \\
 \\
 \frac{\Gamma, \Delta \Rightarrow B}{\Gamma, !^s A, \Delta \Rightarrow B} \text{weak}_!, s \in C \\
 \\
 \frac{\Gamma, !^s A, \Delta, !^s A, \Theta \Rightarrow B}{\Gamma, !^s A, \Delta, \Theta \Rightarrow B} \text{ncontr}_1, s \in C \\
 \\
 \frac{\Gamma, !^s A, \Delta, !^s A, \Theta \Rightarrow B}{\Gamma, \Delta, !^s A, \Theta \Rightarrow B} \text{ncontr}_2, s \in C
 \end{array}$$

$$\frac{\Gamma, \Delta, !^s A, \Theta \Rightarrow B}{\Gamma, !^s A, \Delta, \Theta \Rightarrow A} \mathbf{ex}_1, s \in E$$

$$\frac{\Gamma, !^s A, \Delta, \Theta \Rightarrow B}{\Gamma, \Delta, !^s A, \Theta \Rightarrow A} \mathbf{ex}_1, s \in E$$

Lemma 1. Let $A \Leftrightarrow B$, then $C[p_i := A] \Leftrightarrow C[p_i := B]$

Proof. By induction on C . □

Definition 3. Monoidal comonad

A monoidal comonad on some monoidal category \mathcal{C} is a triple $\langle \mathcal{F}, \epsilon, \delta \rangle$, where \mathcal{F} is a monoidal endofunctor and $\epsilon : \mathcal{F} \Rightarrow Id_{\mathcal{C}}$ (counit) and $\epsilon : \mathcal{F} \Rightarrow \mathcal{F}^2$ (comultiplication), such that the following diagrams commute:

$$\begin{array}{ccc} \mathcal{F}A \otimes \mathcal{F}B & \xrightarrow{\phi_{A,B}} & \mathcal{F}(A \otimes B) \\ \delta_A \otimes \delta_B \downarrow & & \searrow \delta_{A \otimes B} \\ \mathcal{F}\mathcal{F}A \otimes \mathcal{F}\mathcal{F}B & \xrightarrow{\phi_{\mathcal{F}A, \mathcal{F}B}} & \mathcal{F}(\mathcal{F}A \otimes \mathcal{F}B) \\ & \nearrow \mathcal{F}(\phi_{A,B}) & \end{array} \quad \begin{array}{ccc} \mathcal{F}A \otimes \mathcal{F}B & \xrightarrow{\phi_{A,B}} & \mathcal{F}(A \otimes B) \\ \epsilon_A \otimes \epsilon_B \searrow & & \swarrow \epsilon_{A \otimes B} \\ & A \otimes B & \end{array}$$

$$\begin{array}{ccc} \mathbb{1} & \xrightarrow{\phi} & \mathcal{F}\mathbb{1} \\ \phi \downarrow & & \downarrow \delta_{\mathbb{1}} \\ \mathcal{F}\mathbb{1} & \xrightarrow{\mathcal{F}(\phi)} & \mathcal{F}\mathcal{F}\mathbb{1} \end{array}$$

$$\begin{array}{ccc} \mathbb{1} & \xrightarrow{id_{\mathbb{1}}} & \mathbb{1} \\ \phi \searrow & & \nearrow \epsilon_{\mathbb{1}} \\ & \mathcal{F}\mathbb{1} & \end{array}$$

Definition 4. Biclosed monoidal category

Let \mathcal{C} be a monoidal category. Biclosed monoidal category is a monoidal category with the following additional data:

1. Bifunctors $_ \multimap _, _ \multimap _ : \mathcal{C}^{op} \times \mathcal{C} \rightarrow \mathcal{C}$;
2. Natural isomorphism $\mathbf{curry}_{A,B,C} : Hom(A \otimes B, C) \cong (B, A \multimap C)$;
3. Natural isomorphism $\mathbf{curry}'_{A,B,C} : Hom(A \otimes B, C) \cong (A, C \multimap B)$;
4. For each $A, B \in Ob_{\mathcal{C}}$, there are exist arrows $ev_{A,B} : A \otimes (A \Rightarrow B) \rightarrow B$ and $ev'_{A,B} : (B \Leftarrow A) \otimes A \rightarrow B$, such that for all $f : A \otimes C \rightarrow B$:
 - (a) $\Lambda_l \circ (id_A \otimes \mathbf{curry}(f)) = f$;
 - (b) $\Lambda_r \circ (\mathbf{curry}'(f) \otimes id_A) = f$

Definition 5. Let F be endofunctor and $A \in Ob_{\mathcal{C}}$, then a coalgebra of F is a tuple $\langle A, \theta \rangle$, where $\theta : A \rightarrow FA$.

Given coalgebras $\langle A, \theta \rangle$ and $\langle A, \psi \rangle$, a homomorphism is a morphism $f : A \rightarrow B$, s.t. the diagram below commutes:

$$\begin{array}{ccc} A & \xrightarrow{\theta} & FA \\ f \downarrow & & \downarrow Ff \\ B & \xrightarrow{\psi} & FB \end{array}$$

that is, $Ff \circ \theta = \psi \circ f$

Definition 6. *Subexponential model structure*

Let $\Sigma = \langle I, \leq, W, C, E \rangle$ be a subexponential signature and \mathcal{C} be a biclosed monoidal category, then a subexponential model structure is $\langle \mathcal{C}, \{\mathcal{F}_s\}_{s \in I} \rangle$ with the following additional data:

- for all $s \in I$, \mathcal{F}_s is a monoidal comonad;
- if $s \in W$, then for all $A \in \text{Ob}(\mathcal{C})$, there exists a morphism $w_{As} : F_s A \rightarrow \mathbb{1}$;
- if $s \in C$, then for all $A \in \text{Ob}(\mathcal{C})$, there exists morphisms $w_{Al} : F_s A \otimes A \otimes F_s A \rightarrow F_s A \otimes B$ and $w_{Ar} : F_s A \otimes A \otimes F_s A \rightarrow B \otimes F_s A$;
- if $s \in E$, then for all $A \in \text{Ob}(\mathcal{C})$, there is an isomorphism, $e_A : F_s A \otimes B \cong B \otimes F_s A$;
- if $s_1 \in W$, $s_2 \in I$ and $s_1 \leq s_2$, then there is a morphism $w_{As_2} : F_{s_2} A \rightarrow \mathbb{1}$ for all $A \in \text{Ob}(\mathcal{C})$ and ditto for E and C ;
- Let $\bigotimes_{s \in J, i=0}^n F_s A$, where $J \subset I$, and $s' \in I$, s.t. $s \geq s'$ for all $s \in J$; Then there exists morphism a morphism $\theta_{\bigotimes_{s \in J, i=1}^n F_{s_j} A_i} : \bigotimes_{s \in J, i=0}^n F_s A \rightarrow F_{s'}(\bigotimes_{s \in J, i=0}^n F_s A)$, such that $\langle \bigotimes_{s \in J, i=1}^n F_{s_j} A_i, \theta_{\bigotimes_{s \in J, i=1}^n F_{s_j} A_i} \rangle$ is a coalgebra on $F_{s'}$.

Definition 7. Let $\langle \mathcal{C}, \{\mathcal{F}_s\}_{s \in I} \rangle$ be a subexponential model structure for subexponential signature $\Sigma = \langle I, \leq, W, C, E \rangle$. Let $v : Tp \rightarrow \text{Ob}(\mathcal{C})$ be a valuation map. Then the interpretation function $\llbracket \cdot \rrbracket$ is defined as follows:

- (1) $\llbracket \mathbb{1} \rrbracket = \mathbb{1}$
- (2) $\llbracket A \setminus B \rrbracket = \llbracket A \rrbracket \multimap \llbracket B \rrbracket$
- (3) $\llbracket A / B \rrbracket = \llbracket A \rrbracket \multimap \llbracket B \rrbracket$
- (4) $\llbracket A \bullet B \rrbracket = \llbracket A \rrbracket \otimes \llbracket B \rrbracket$
- (5) $\llbracket !_s A \rrbracket = F_s \llbracket A \rrbracket$

Theorem 1. *The following statements are equivalent:*

- $SMLC_\Sigma + (\text{cut}) \vdash \Gamma \Rightarrow A$
- $SMLC_\Sigma \vdash \Gamma \Rightarrow A$
- $\exists f, f : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket$

Proof.

(1) \Rightarrow (2): cut elimination.

- (2) \Rightarrow (3): Soundness:

$$\overline{id_A : A \rightarrow A}$$

$$\frac{f : \Gamma \rightarrow A \quad g : \Delta \otimes B \otimes \Theta \rightarrow C}{g \circ (id_\Delta \otimes (ev_{A,B_l} \circ (f \otimes id_{A \multimap B})) \otimes id_\Theta) : \Delta \otimes (\Gamma \otimes A \multimap B) \otimes \Theta \rightarrow C}$$

$$\frac{f : A \otimes \Pi \rightarrow B}{\Lambda_l(f) : \Pi \rightarrow A \multimap B}$$

$$\frac{f : \Gamma \rightarrow A \quad g : \Delta \otimes B \otimes \Theta \rightarrow C}{g \circ (id_\Delta \otimes (ev_{A,B_l} \circ (id_{B \multimap A} \otimes f)) \otimes id_\Theta) : \Delta \otimes (B \multimap A \otimes \Gamma) \otimes \Theta \rightarrow C}$$

$$\frac{f : \Pi \otimes A \rightarrow B}{\Lambda_r(f) : \Pi \rightarrow B \multimap A}$$

$$\frac{f : \Gamma \otimes A \otimes B \otimes \Delta \rightarrow C}{f \circ (\alpha_{\Gamma,A,B} \otimes id_\Delta) : \Gamma \otimes (A \otimes B) \otimes \Delta \rightarrow C}$$

$$\frac{f : \Gamma \rightarrow A \quad g : \Delta \rightarrow B}{f \otimes g : \Gamma \otimes \Delta \rightarrow A \otimes B}$$

$$\frac{f : \Gamma \otimes A_i \otimes \Delta \rightarrow B}{f \circ (id_\Gamma \otimes \pi_i id_\Delta) : \Gamma \otimes (A_1 \times A_2) \otimes \Delta \rightarrow B}$$

$$\frac{f : \Gamma \rightarrow A \quad g : \Gamma \rightarrow B}{\langle f, g \rangle : \Gamma \rightarrow A \times B}$$

$$\frac{f : \Gamma \otimes A \otimes \Delta \rightarrow C \quad g : \Gamma \otimes B \otimes \Delta \rightarrow C}{id_\Gamma \otimes [f, g] \otimes id_\Delta : \Gamma \otimes (A + B) \otimes \Delta \rightarrow C}$$

$$\overline{id_{\mathbb{1}} : \mathbb{1} \rightarrow \mathbb{1}}$$

$$\frac{f : \Gamma \otimes \Delta \rightarrow A}{f \circ (\rho_\Gamma \otimes id_\Delta) : (\Gamma \otimes \mathbb{1}) \otimes \Delta \rightarrow A}$$

$$\frac{f : \Gamma \otimes A \otimes \Delta \rightarrow B}{f \circ (id_\Gamma \otimes \delta_s^A \otimes id_\Delta) : \Gamma \otimes F_s A \otimes \Delta \rightarrow B}$$

$$\frac{f : F_{s_1} A_1 \otimes \cdots \otimes F_{s_n} A_n \rightarrow B}{F_s(f) : F_s(F_{s_1} A_1 \otimes \cdots \otimes F_{s_n} A_n) \rightarrow F_s B}$$

$$\frac{F_s(f) \circ \theta_{\otimes_{s \in J, i=1}^n F_{s_j} A_i} : F_{s_1} A_1 \otimes \cdots \otimes F_{s_n} A_n \rightarrow F_s B}{F_s(f) \circ \theta_{\otimes_{s \in J, i=1}^n F_{s_j} A_i} : F_{s_1} A_1 \otimes \cdots \otimes F_{s_n} A_n \rightarrow F_s B}$$

$$\frac{f : \Gamma \otimes \Delta \rightarrow A}{f \circ (\rho_\Gamma \otimes id_\Delta) : (\Gamma \otimes \mathbb{1}) \otimes \Delta \rightarrow A}$$

$$\frac{f \circ (\rho_\Gamma \otimes id_\Delta) \circ (id_\Gamma \otimes w_{A_s}) \otimes id_\Delta : (\Gamma \otimes F_s A) \otimes \Delta \rightarrow A}{f \circ (\rho_\Gamma \otimes id_\Delta) \circ (id_\Gamma \otimes w_{A_s}) \otimes id_\Delta : (\Gamma \otimes F_s A) \otimes \Delta \rightarrow A}$$

$$\begin{array}{c}
\frac{f : \Gamma \otimes (F_s A \otimes B \otimes F_s A) \otimes \Delta \rightarrow C}{f \circ (id_\Gamma \otimes c_{A_s}^l \otimes id_\Delta) : \Gamma \otimes (F_s A \otimes B) \otimes \Delta \rightarrow C} \\
\\
\frac{f : \Gamma \otimes (F_s A \otimes B \otimes F_s A) \otimes \Delta \rightarrow C}{(id_\Gamma \otimes c_{A_s}^r \otimes id_\Delta) \circ f : \Gamma \otimes (B \otimes F_s A) \otimes \Delta \rightarrow C} \\
\\
\frac{f : \Gamma \otimes (\Delta \otimes F_s A) \otimes \Theta \rightarrow B}{(id_\Gamma \otimes (id_\Delta \otimes e_{A_s}) \otimes id_\Theta) \circ f : \Gamma \otimes (F_s A \otimes \Delta) \otimes \Theta \rightarrow B} \\
\\
\frac{f : \Gamma \otimes (F_s A \otimes \Delta) \otimes \Theta \rightarrow B}{(id_\Gamma \otimes (id_\Delta \otimes e_{A_s}^{-1}) \otimes id_\Theta) \circ f : \Gamma \otimes (\Delta \otimes F_s A) \otimes \Theta \rightarrow B}
\end{array}$$

- Completeness:

Definition 8.

□

1 Concrete model

Definition 9. *Quantale A quantale is a triple $\langle A, \bigvee, \cdot \rangle$, such that $\langle A, \bigvee \rangle$ is a complete lattice and $\langle A, \cdot \rangle$ is a semigroup. A quantale is called unital, if $\langle A, \cdot \rangle$ is a monoid.*

It is easy to see, that any (unital) quantale is a residual (monoid) semigroup. We define divisions as follows:

1. $a \backslash b = \bigvee \{c \mid a \cdot c \leq b\}$
2. $b / a = \bigvee \{c \mid c \cdot a \leq b\}$

Definition 10. *Let $\langle A, \bigvee, \cdot \rangle$ be a quantale. The center of a quantale is the set $Z(Q) = \{a \in Q \mid \forall b \in Q, a \cdot b = b \cdot a\}$*

Definition 11. *An open modality on quantale Q is a map $I : Q \rightarrow Q$, such that*

1. $I(x) \leq x$;
2. $I(x) = I(I(x))$;
3. $x \leq y \Rightarrow I(x) \leq I(y)$;
4. $I(x) \cdot I(y) = I(I(x) \cdot I(y))$.

Lemma 2.

Let $\langle A, \bigvee, \cdot \rangle$ be a quantale and $I : Q \rightarrow Q$ is an open modality on Q , then $I(x) \cdot I(y) \leq I(x \cdot y)$.

Proof.

$I(x) \cdot I(y) \leq x \cdot y$, then $I(I(x) \cdot I(y)) \leq I(x \cdot y)$, but $I(x) \cdot I(y) \leq I(I(x) \cdot I(y))$. Thus, $I(x) \cdot I(y) \leq I(x \cdot y)$. □

Definition 12. *An open modality is called central, if $\forall a, b \in Q, I(a) \cdot b = b \cdot I(a)$.*

Definition 13. An open modality is called *weak idempotent*, if $\forall a, b \in Q, I(a) \cdot b \leq I(a) \cdot b \cdot I(a)$ and $b \cdot I(a) \leq I(a) \cdot b \cdot I(a)$.

Definition 14. An open modality is called *unital*, if $\forall a \in Q, I(a) \leq e$.

Lemma 3. Let I be an interior on some unital quantale $\langle Q, \vee, \cdot, e \rangle$. Then, if I is unital and weak idempotent, then I is central.

Proof.

$$\begin{aligned}
& b \cdot I(a) \leq \\
& \quad \text{Right weak idempotence} \\
& I(a) \cdot b \cdot I(a) \leq \\
& \quad \text{Unitality} \\
& I(a) \cdot b \cdot I(e) \leq \\
& \quad \text{Identity} \\
& I(a) \cdot b \leq \\
& \quad \text{Left weak idempotence} \\
& I(a) \cdot b \cdot I(a) \leq \\
& \quad \text{Unitality} \\
& e \cdot b \cdot I(a) \leq \\
& \quad \text{Identity} \\
& b \cdot I(a) \\
& \text{Hence, } b \cdot I(a) = I(a) \cdot b
\end{aligned}$$

□

Proposition 1.

Let Q be a quantale and $S \subseteq Q$ a subquantale, then $I : Q \rightarrow Q$, such that $I(a) = \bigvee \{s \in S \mid x \leq a\}$, is an open modality.

Proof. See

□

Proposition 2.

Let Q be a quantale and $S_1, S_2 \subseteq Q$, such that $S_1 \subseteq S_2$.
Then $I_1(a) \leq I_2(a)$.

Proof.

Let $a \in Q$, so $\{s \in S_1 \mid s \leq a\} \subseteq \{s \in S_2 \mid s \leq a\}$, so $\bigvee \{s \in S_1 \mid s \leq a\} \leq \bigvee \{s \in S_2 \mid s \leq a\}$.
Thus, $I_1(a) \leq I_2(a)$. □

Proposition 3.

Let Q be a quantale and $S \subseteq Q$ a subquantale, then the following operations are open modalities:

1. $I_z(a) = \bigvee \{s \in S \mid s \leq a, s \in Z(Q)\};$
2. $I_{\mathbb{1}}(a) = \bigvee \{s \in S \mid s \leq a, s \leq \mathbb{1}\};$
3. $I_{idem}(a) = \bigvee \{s \in S \mid s \leq a, \forall b \in Q, b \cdot s \vee s \cdot b \leq s \cdot b \cdot s\};$
4. $I_{z, \mathbb{1}}, I_{z, idem}, I_{\mathbb{1}, idem}, I_{z, \mathbb{1}, idem}.$

Proof. Immediately.

□

Proposition 4.

1. $\forall a \in Q, I_{\mathbb{1},idem}(a) \leq I_z(a).$
2. $\forall a \in Q, I_{z,\mathbb{1},idem} = I_{\mathbb{1},idem}(a)$

Proof. Follows from Lemma 3. □

Proposition 5.

1. $I_z(a) \vee I_{\mathbb{1}}(a) \vee I_{idem}(a) \leq I(a)$
2. $I_{z,\mathbb{1},idem} \leq I_{z,\mathbb{1}}(a) \wedge I_{idem}(a)$

Lemma 4. $\forall a \in Q, I_1(a) \leq I_2(I_1(a)),$ if $I_1(a) \leq I_2(a).$

Proof. $I_1(a) \leq I_1(I_1(a)) \leq I_2(I_1(a))$ □

Lemma 5. $I_1(a_1) \cdot I_2(a_2) \leq I'(I_1(a_1) \cdot I_2(a_2)),$ where $I_i \leq I', i = 1, 2.$

Proof.

$$\begin{aligned} I_1(a_1) \cdot I_2(a_2) &\leq \\ I_1(I_1(a_1)) \cdot I_2(I_2(a_2)) &\leq \\ I'(I_1(a_1)) \cdot I'(I_2(a_2)) &\leq \\ I'(I_1(a_1) \cdot I_2(a_2)) & \end{aligned}$$
□

Definition 15. *Interpretation of subexponential signature*

Let $\Sigma = \langle I, \leq, W, C, E \rangle$ be a subexponential signature, where $|I| = n$ and $\mathcal{S} = \{\Box_1, \dots, \Box_n\}$ be a set of open modalities on quantale Q . Subexponential interpretation is a contravariant map $\sigma : I \rightarrow \mathcal{S}$ defined as follows:

$$\sigma(s_i) = \begin{cases} \Box_i : Q \rightarrow Q, \text{ s.t. } \forall a \in Q, \Box_i(a) = \{s \in S_i \mid s \leq a\}, \\ \quad \text{if } s_i \notin W \cap C \cap E \\ \Box_i : Q \rightarrow Q, \text{ s.t. } \forall a \in Q, \Box_i(a) = \{s \in S_i \mid s \leq a, \leq \mathbb{1}\}, \\ \quad \text{if } s_i \in W \\ \Box_i : Q \rightarrow Q, \text{ s.t. } \forall a \in Q, \Box_i(a) = \{s \in S_i \mid s \leq a, \in Z(Q)\}, \\ \quad \text{if } s_i \in E \\ \Box_i : Q \rightarrow Q, \text{ s.t. } \forall a \in Q, \Box_i(a) = \{s \in S_i \mid s \leq a, \forall b, b \cdot s \vee s \cdot b \leq s \cdot b \cdot s\}, \\ \quad \text{if } s_i \in E \\ \text{otherwise, if } s_i \text{ belongs to some intersection of subsets, then we combine the relevant conditions} \end{cases}$$

Definition 16. Let Q be a quantale, $f : Tp \rightarrow Q$ a valuation and $\sigma : I \rightarrow \mathcal{S}$ a subexponential interpretation, then interpretation is defined inductively:

$$\begin{aligned} \llbracket p_i \rrbracket &= f(p_i) \\ \llbracket \mathbb{1} \rrbracket &= e \\ \llbracket A \bullet B \rrbracket &= \llbracket A \rrbracket \cdot \llbracket B \rrbracket \\ \llbracket A \setminus B \rrbracket &= \llbracket A \rrbracket \setminus \llbracket B \rrbracket \\ \llbracket A / B \rrbracket &= \llbracket A \rrbracket / \llbracket B \rrbracket \\ \llbracket A \& B \rrbracket &= \llbracket A \rrbracket \wedge \llbracket B \rrbracket \\ \llbracket A \vee B \rrbracket &= \llbracket A \rrbracket \vee \llbracket B \rrbracket \\ \llbracket !_{s_i} A \rrbracket &= \sigma(s_i) \llbracket A \rrbracket \end{aligned}$$

Theorem 2. $\Gamma \rightarrow A \Rightarrow \llbracket \Gamma \rrbracket \leq \llbracket A \rrbracket$

Proof. We consider the case with polymodal promotion rule.

Let $!_{s_1}A_1, \dots, !_{s_n}A_n \rightarrow A$ and $\forall i, s \leq s_i$. Then $\forall a \in Q, \sigma(s_i)(a) \leq \sigma(s)(a)$.

By IH, $\sigma(s_1)\llbracket A_1 \rrbracket \cdots \sigma(s_n)\llbracket A_n \rrbracket \leq \llbracket A \rrbracket$.

Thus, $\sigma(s)(\sigma(s_1)\llbracket A_1 \rrbracket \cdots \sigma(s_n)\llbracket A_n \rrbracket) \leq \sigma(s)(\llbracket A \rrbracket)$.

By Lemma 5, $\sigma(s_1)\llbracket A_1 \rrbracket \cdots \sigma(s_n)\llbracket A_n \rrbracket \leq \sigma(s)(\sigma(s_1)\llbracket A_1 \rrbracket \cdots \sigma(s_n)\llbracket A_n \rrbracket)$.

So, $\sigma(s_1)\llbracket A_1 \rrbracket \cdots \sigma(s_n)\llbracket A_n \rrbracket \leq \sigma(s)(\llbracket A \rrbracket)$. □

2 Quantale completeness

Definition 17.

Let $\mathcal{F} \subseteq Fm$, an ideal is a subset $\mathcal{I} \subseteq \mathcal{F}$, such that:

- If $B \in \mathcal{I}$ and $A \rightarrow B$, then $A \in \mathcal{I}$;
- If $A, B \in \mathcal{I}$, then $A \vee B \in \mathcal{I}$.

Definition 18.

Let $S \subseteq \mathcal{F} \subseteq Fm$, then $\bigvee S = \bigcap \{\mathcal{I} \subseteq \mathcal{F} \mid S \subseteq \mathcal{I}\}$

Proposition 6. $\bigvee S$ is an ideal.

Lemma 6. $A \in \mathcal{F}$, then $\{B \mid B \rightarrow A\} = \bigvee \{A\}$.

Proof.

Let $A \in \mathcal{F}$. Then $\{B \mid B \rightarrow A\} \subseteq \bigvee \{A\}$.

On the other hand, $\{B \mid B \rightarrow A\}$ is an ideal, hence, $\{A\} \subseteq \{B \mid B \rightarrow A\}$. □

Lemma 7. $\bigvee \{A\} \subseteq \bigvee \{B\}$ iff $A \rightarrow B$.

Proof. Let $\bigvee \{A\} \subseteq \bigvee \{B\}$, then $\{C \mid C \rightarrow A\} \subseteq \{C \mid C \rightarrow B\}$.

$A \in \{C \mid C \rightarrow A\}$, so far as $A \rightarrow A$, but $\{C \mid C \rightarrow B\}$, hence $A \rightarrow B$.

On the other hand, let $A \rightarrow B$ and $C \in \bigvee \{A\}$. Then $C \in \bigvee \{C' \mid C' \rightarrow A\}$, hence, $C \rightarrow A$, thus $C \rightarrow B$ by cut. So, $C \in \bigvee \{B\}$. □

Lemma 8. Let $\mathcal{Q} = \{\bigvee S \mid S \subseteq Fm\}$ and $\bigvee \mathcal{A} \cdot \bigvee \mathcal{B} = \{A \bullet B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$. Then $\langle \mathcal{Q}, \subseteq, \cdot, \bigvee \mathbf{1} \rangle$ is a quantale.

Proof. See □

Lemma 9. Interior lemma.

Let $Q_1 \subseteq \mathcal{Q}$, define a map $\square : Q \rightarrow Q$, such that $\square(A) = \{Q \in Q_1 \mid Q \subseteq A\}$. Then \square is a quantic conucleus.

Proof. See □

Lemma 10. Let $Q \subseteq \mathcal{Q}$, then the following operators are quantic conuclei:

1. $\square_z(A) = \bigvee \{\bigvee \{W\} \in Q \mid \bigvee \{W\} \subseteq \bigvee \{A\}, \bigvee \{W\} \in Z(Q)\};$
2. $\square_1(A) = \bigvee \{\bigvee \{W\} \in Q \mid \bigvee \{W\} \subseteq \bigvee \{A\}, \bigvee \{W\} \subseteq \bigvee \{\mathbf{1}\}\};$
3. $\square_{idem}(A) = \bigvee \{\bigvee \{W\} \in Q \mid \bigvee \{W\} \subseteq \bigvee \{A\}, \forall B \in \mathcal{Q}, (\bigvee \{B\} \cdot \bigvee \{W\}) \cup (\bigvee \{W\} \cdot \bigvee \{B\}) \subseteq \bigvee \{W\} \cdot \bigvee \{A\} \cdot \bigvee \{W\}\};$
4. $\square_{z,1}, \square_{z,idem}, \square_{1,idem}, \square_{z,1,idem}.$

Proof. Follow from one of lemmas above. □