Quantale model of Lambek calculus with subexponentials

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1 Calculus

Definition 1. *A subexponential signature is an ordered quintuple:*
$\Sigma = \langle I, \preceq, W, C, E \rangle$,

where $I = \{s_1, \ldots, s_n\}$, $\langle I, \preceq \rangle$ is a preorder. W, C, E are subsets of I and $W \cup C \subseteq E$.

Definition 2. *Noncommutative linear logic with subexponentials ($SMALC_\Sigma$), where Σ is a subexponential signature.*

$$\frac{}{A \to A}\ ax$$

$$\frac{\Gamma \to A \qquad \Delta, B, \Theta \to C}{\Delta, \Gamma, A \backslash B, \Theta \to C} \ \backslash \to \qquad \frac{A, \Pi \to B}{\Pi \to A \backslash B} \to \backslash$$

$$\frac{\Gamma \to A \qquad \Delta, B, \Theta \to C}{\Delta, B / A, \Gamma, \Theta \to C} \ / \to \qquad \frac{\Pi, A \to B}{\Pi \to B / A} \to /$$

$$\frac{\Gamma, A, B, \Delta \to C}{\Gamma, A \bullet B, \Delta \to C} \bullet \to \qquad \frac{\Gamma \to A \qquad \Delta \to B}{\Gamma, \Delta \to A \bullet B} \to \bullet$$

$$\frac{\Gamma, A_i, \Delta \to B}{\Gamma, A_1 \& A_2, \Delta \to B} \ \&, i = 1, 2 \to \qquad \frac{\Gamma \to A \qquad \Gamma \to B}{\Gamma \to A \& B} \to \&$$

$$\frac{\Gamma, A, \Delta \to C \qquad \Gamma, B, \Delta \to C}{\Gamma, A \vee B, \Delta \to C} \vee \to \qquad \frac{\Gamma \to A_i}{\Gamma \to A_1 \vee A_2} \to \vee, i = 1, 2$$

$$\frac{\Gamma, \Delta \to A}{\Gamma, \mathbf{1}, \Delta \to A} \ \mathbf{1} \to \qquad \frac{}{\to \mathbf{1}} \to \mathbf{1}$$

$$\frac{\Gamma, A, \Delta \to C}{\Gamma, !^s A, \Delta \to C} \ ! \to \qquad \frac{!^{s_1} A_1, \ldots, !^{s_n} A_n \to A}{!^{s_1} A_1, \ldots, !^{s_n} A_n \to !^s A} \to !, \forall j, s_j \succeq s$$

$$\frac{\Gamma, \Delta \to B}{\Gamma, !^s A, \Delta \to B} \ \mathbf{weak}_!, s \in C$$

$$\frac{\Gamma, !^s A, \Delta, !^s A, \Theta \to B}{\Gamma, !^s A, \Delta, \Theta \to B}\ \mathbf{ncontr}_1, s \in C$$

$$\frac{\Gamma, !^s A, \Delta, !^s A, \Theta \to B}{\Gamma, \Delta, !^s A, \Theta \to B}\ \mathbf{ncontr}_2, s \in C$$

$$\frac{\Gamma, \Delta, !^s A, \Theta \to B}{\Gamma, !^s A, \Delta, \Theta \to A}\ \mathbf{ex}_1, s \in E$$

$$\frac{\Gamma, !^s A, \Delta, \Theta \to B}{\Gamma, \Delta, !^s A, \Theta \to A}\ \mathbf{ex}_1, s \in E$$

Proposition 1. $!_{s_i} A \leftrightarrow !_{s_i}(!_{s_i} A)$

Proof.

$$\cfrac{\cfrac{\cfrac{A \to A}{!_{s_i} A \to A}}{!_{s_i} A \to !_{s_i} A}}{!_{s_i} !_{s_i} A \to !_{s_i} A}$$

□

2 Semantics

Definition 3. *Quantale*

A quantale is a triple $\langle A, \bigvee, \cdot \rangle$, such that $\langle A, \bigvee \rangle$ is a complete lattice and $\langle A, \cdot \rangle$ is a semigroup. A quanlate is called unital, if $\langle A, \cdot \rangle$ is a monoid.

It is easy to see, that any (unital) quantale is a residual (monoid) semigroup. We define divisions as follows:

1. $a \backslash b = \bigvee \{c \mid a \cdot c \leqslant b\}$
2. $b / a = \bigvee \{c \mid c \cdot a \leqslant b\}$

Definition 4.

Let $\mathcal{Q} = \langle A, \bigvee, \cdot \rangle$ be a quantale. The center of a quantale is the set $\mathcal{Z}(\mathcal{Q}) = \{a \in A \mid \forall b \in A, a \cdot b = b \cdot a\}$

Definition 5. *An open modality (or quantic conucleus) on quantale $\mathcal{Q}$ is a map $\square : \mathcal{Q} \to \mathcal{Q}$, such that*

1. $\square(x) \leqslant x$;
2. $\square(x) = \square(\square(x))$;
3. $x \leqslant y \Rightarrow \square(x) \leqslant \square(y)$;
4. $\square(x) \cdot \square(y) = \square(\square(x) \cdot \square(y))$.

Definition 6. *We define a partial order on open modalities on $\mathcal{Q}$ as $\square_1 \leqslant \square_2 \Leftrightarrow \forall a \in Q, \square_1(a) \leqslant \square_2(a)$.*

Lemma 1. *Let $\mathcal{Q}$ be a quantale and $\square_{\mathcal{Q}}$ be a set of all open modalities on $\mathcal{Q}$. Then $\square_{\mathcal{Q}}$ is a locally small category.*

Proof. $\langle \square_{\mathcal{Q}}, \leqslant \rangle$ form a partial order, so $\langle \square_{\mathcal{Q}}, \leqslant \rangle$ is a locally small category. $\square$

Lemma 2.
Let $\mathcal{Q} = \langle A, \bigvee, \cdot \rangle$ be a quantale and $\square : \mathcal{Q} \to \mathcal{Q}$ is an open modality on $\mathcal{Q}$, then $\square(x) \cdot \square(y) \leqslant \square(x \cdot y)$.

Proof.
$\square(x) \cdot \square(y) \leqslant x \cdot y$, then $\square(\square(x) \cdot \square(y)) \leqslant \square(x \cdot y)$, but $\square(x) \cdot \square(y) \leqslant \square(\square(x) \cdot \square(y))$. Thus, $\square(x) \cdot \square(y) \leqslant \square(x \cdot y)$. $\square$

Definition 7. *An open modality is called central, if $\forall a, b \in Q, \square(a) \cdot b = b \cdot \square(a)$.*

Definition 8. *An open modality is called weak idempotent, if $\forall a, b \in Q, \square(a) \cdot b \leqslant \square(a) \cdot b \cdot \square(a)$ and $b \cdot \square(a) \leqslant \square(a) \cdot b \cdot \square(a)$.*

Definition 9. *An open modality is called unital, if $\forall a \in Q, \square(a) \leqslant e$.*

Lemma 3. *Let $\square$ be an open modality on some unital quantale $\mathcal{Q} = \langle A, \bigvee, \cdot, e \rangle$. Then, if $\square$ is unital and weak idempotent, then $\square$ is central.*

Proof.
$b \cdot \square(a) \leqslant$
Right weak idempotence
$\square(a) \cdot b \cdot \square(a) \leqslant$
Unitality
$\square(a) \cdot b \cdot e \leqslant$
Identity
$\square(a) \cdot b \leqslant$
Left weak idempotence
$\square(a) \cdot b \cdot \square(a) \leqslant$
Unitality
$e \cdot b \cdot \square(a) \leqslant$
Identity
$b \cdot \square(a)$
Hence, $b \cdot \square(a) = \square(a) \cdot b$, so $\forall a \in A, \square(a) \in \mathcal{Z}(\mathcal{Q})$.
$\square$

Proposition 2.
Let $\mathcal{Q}$ be a quantale and $S \subseteq \mathcal{Q}$ a subquantale, then $\square : \mathcal{Q} \to \mathcal{Q}$, such that $\square(a) = \bigvee\{s \in S \mid x \leqslant a\}$, is an open modality.

Proof. See $\square$

Proposition 3.
Let $\mathcal{Q}$ be a quantale and $S_1 \subseteq S_2 \subseteq \mathcal{Q}$.
Then $\square_1(a) \leqslant \square_2(a)$.

Proof.
Let $a \in Q$, so $\{s \in S_1 \mid s \leqslant a\} \subseteq \{s \in S_2 \mid s \leqslant a\}$, so $\bigvee\{s \in S_1 \mid s \leqslant a\} \subseteq \bigvee\{s \in S_2 \mid s \leqslant a\}$. Thus, $\square_1(a) \leqslant \square_2(a)$. $\square$

Proposition 4.

Let $\mathcal{Q}$ be a quantale and $S \subseteq \mathcal{Q}$ a subquantale, then the following operations are open modalities:

1. $\square_z(a) = \bigvee\{s \in S \,|\, s \leqslant a, s \in \mathcal{Z}(\mathcal{Q})\}$;
2. $\square_{\mathbb{1}}(a) = \bigvee\{s \in S \,|\, s \leqslant a, s \leqslant \mathbb{1}\}$;
3. $\square_{idem}(a) = \bigvee\{s \in S \,|\, s \leqslant a, \forall b \in Q, b \cdot s \vee s \cdot b \leqslant s \cdot b \cdot s\}$;
4. $\square_{z,\mathbb{1}}, I_{z,idem}, I_{\mathbb{1},idem}, I_{z,\mathbb{1},idem}$.

Proof. Immediatly. $\square$

Proposition 5.

1. $\forall a \in \mathcal{Q}, \square_{\mathbb{1},idem}(a) \leqslant \square_z(a)$.
2. $\forall a \in \mathcal{Q}, \square_{z,\mathbb{1},idem} = \square_{\mathbb{1},idem}(a)$

Proof. Follows from Lemma 3. $\square$

Proposition 6.

1. $\square_z(a) \vee \square_{\mathbb{1}}(a) \vee \square_{idem}(a) \leqslant \square(a)$
2. $\square_{z,\mathbb{1},idem} \leqslant \square_{z,\mathbb{1}}(a) \wedge \square_{z,idem}(a)$

Lemma 4. $\forall a \in Q, \square_1(a) \leqslant \square_2(\square_1(a))$, *if* $\square_1(a) \leqslant \square_2(a)$.

Proof. $\square_1(a) \leqslant \square_1(\square_1(a)) \leqslant \square_2(\square_1(a))$ $\square$

Lemma 5. $\square_1(a_1) \cdot \square_2(a_2) \leqslant \square^{'}(\square_1(a_1) \cdot \square_2(a_2))$, *where* $\square_i \leqslant \square^{'}, i = 1, 2$.

Proof.

$$\begin{aligned}
&\square_1(a_1) \cdot \square_2(a_2) \leqslant \\
&\square_1(\square_1(a_1)) \cdot \square_2(\square_2(a_2)) \leqslant \\
&\square^{'}(\square_1(a_1)) \cdot \square^{'}(\square_2(a_2)) \leqslant \\
&\square^{'}(\square_1(a_1) \cdot \square_2(a_2))
\end{aligned}$$

$\square$

Definition 10. *Interpretation of subexponential signature*

Let $\Sigma = \langle I, \preceq, W, C, E\rangle$ be a subexponential signature, where $|I| = n$ and $\square_{\mathcal{Q}}$ is a category of open modalities on a quantale $\mathcal{Q}$. Subexponential interpretation is a contravariant functor $\sigma : I \to \square_{\mathcal{Q}}$ defined as follows:

$$\sigma(s_i) = \begin{cases}
\square_i : \mathcal{Q} \to \mathcal{Q}, \ s.t. \forall a \in Q, \square_i(a) = \{s \in S_i \mid s \leqslant a\}, \\
\quad \textit{if } s_i \notin W \cap C \cap E \\
\square_i : \mathcal{Q} \to \mathcal{Q}, \ s.t. \forall a \in Q, \square_i(a) = \{s \in S_i \mid s \leqslant a, s \leqslant \mathbb{1}\}, \\
\quad \textit{if } s_i \in W \\
\square_i : \mathcal{Q} \to \mathcal{Q}, \ s.t. \forall a \in Q, \square_i(a) = \{s \in S_i \mid s \leqslant a, s \in \mathcal{Z}(\mathcal{Q})\}, \\
\quad \textit{if } s_i \in E \\
\square_i : \mathcal{Q} \to \mathcal{Q}, \ s.t. \forall a \in Q, \square_i(a) = \{s \in S_i \mid s \leqslant a, \forall b, b \cdot s \vee s \cdot b \leqslant s \cdot b \cdot s\}, \\
\quad \textit{if } s_i \in E \\
\textit{otherwise, if } s_i \textit{ belongs to some intersection of subsets, then we combine the relevant conditions}
\end{cases}$$

Definition 11. *Let $\mathcal{Q}$ be a quantale, $f : Tp \to \mathcal{Q}$ a valuation and $\sigma : I \to \Box_{\mathcal{Q}}$ a subexponential interpretation, then interpretation is defined inductively:*

$$\begin{aligned}
&[\![p_i]\!] = f(p_i)\\
&[\![\mathbb{1}]\!] = e\\
&[\![A \bullet B]\!] = [\![A]\!] \cdot [\![B]\!]\\
&[\![A \backslash B]\!] = [\![A]\!] \backslash [\![B]\!]\\
&[\![A / B]\!] = [\![A]\!] / [\![B]\!]\\
&[\![A \& B]\!] = [\![A]\!] \wedge [\![B]\!]\\
&[\![A \vee B]\!] = [\![A]\!] \vee [\![B]\!]\\
&[\![!_{s_i} A]\!] = \sigma(s_i)[\![A]\!]
\end{aligned}$$

Definition 12. $\Gamma \models A \Leftrightarrow \forall f, \forall \sigma, [\![\Gamma]\!] \leqslant [\![A]\!]$

Theorem 1. $\Gamma \to A \Rightarrow [\![\Gamma]\!] \leqslant [\![A]\!]$

Proof. We consider cases with modal rules.

1. Let $!_{s_1} A_1, \dots, !_{s_n} A_n \to A$ and $\forall i, s \preceq s_i$.
   Then $\forall a \in Q, \sigma(s_i)(a) \leqslant \sigma(s)(a)$.
   By IH, $\sigma(s_1)[\![A_1]\!] \cdot \dots \cdot \sigma(s_n)[\![A_n]\!] \leqslant [\![A]\!]$.
   Thus, $\sigma(s)(\sigma(s_1)[\![A_1]\!] \cdot \dots \cdot \sigma(s_n)[\![A_n]\!]) \leqslant \sigma(s)([\![A]\!])$.
   By Lemma 5, $\sigma(s_1)[\![A_1]\!] \cdot \dots \cdot \sigma(s_n)[\![A_n]\!] \leqslant \sigma(s)(\sigma(s_1)[\![A_1]\!] \cdot \dots \cdot \sigma(s_n)[\![A_n]\!])$.
   So, $\sigma(s_1)[\![A_1]\!] \cdot \dots \cdot \sigma(s_n)[\![A_n]\!] \leqslant \sigma(s)([\![A]\!])$.

2. Let $\Gamma, A, \Delta \to B$.
   By IH, $[\![\Gamma]\!] \cdot [\![A]\!] \cdot [\![\Delta]\!] \leqslant [\![B]\!]$.
   By the definition, $\sigma(s_i)([\![A]\!]) \leqslant [\![A]\!]$.
   So, $[\![\Gamma]\!] \cdot \sigma(s_i)([\![A]\!]) \cdot [\![\Delta]\!] \leqslant [\![B]\!]$

3. Let $\Gamma, \Delta \to B$, $A \in Fm$, and $s_i \in W$.
   So, $[\![\Gamma]\!] \cdot [\![\Delta]\!] \leqslant [\![B]\!]$, then $[\![\Gamma]\!] \cdot e \cdot [\![\Delta]\!] \leqslant [\![B]\!]$, where $e \in Q$ is unit.
   By the definition of unital open modality, $\sigma(s_i)([\![A]\!]) \leqslant e$.
   Thus, $[\![\Gamma]\!] \cdot \sigma(s_i)([\![A]\!]) \cdot [\![\Delta]\!] \leqslant [\![B]\!]$.

4. Let $\Gamma, !_{s_i} A, \Delta, !_{s_i} A, \Pi \to B$ and $s_i \in C$.
   By IH, $[\![\Gamma]\!] \cdot \sigma(s_i)([\![A]\!]) \cdot [\![\Delta]\!] \cdot \sigma(s_i)([\![A]\!]) \cdot [\![\Pi]\!] \leqslant [\![B]\!]$.
   By the definition, $\sigma(s_i)([\![A]\!]) \cdot [\![\Delta]\!] \leqslant \sigma(s_i)([\![A]\!]) \cdot [\![\Delta]\!] \cdot \sigma(s_i)([\![A]\!])$.
   Then $[\![\Gamma]\!] \cdot \sigma(s_i)([\![A]\!]) \cdot [\![\Delta]\!] \cdot [\![\Pi]\!] \leqslant [\![B]\!]$

5. Let $\Gamma, !_{s_i} A, \Delta, \Pi \to B$ and $s_i \in E$, so $\sigma(s_i)(a) \in \mathcal{Z}(\mathcal{Q})$ for all $a \in Q$ by the definition.
   By IH, $[\![\Gamma]\!] \cdot \sigma(s_i)([\![A]\!]) \cdot [\![\Delta]\!] \cdot [\![\Pi]\!] \leqslant [\![B]\!]$
   Hence, $[\![\Gamma]\!] \cdot [\![\Delta]\!] \cdot \sigma(s_i)([\![A]\!]) \cdot [\![\Pi]\!] \leqslant [\![B]\!]$.

$\square$

3 Quantale completeness

Definition 13.
Let $\mathcal{F} \subseteq Fm$, an ideal is a subset $\mathcal{I} \subseteq \mathcal{F}$, such that:

- *If $B \in \mathcal{I}$ and $A \to B$, then $A \in \mathcal{I}$;*
- *If $A, B \in \mathcal{I}$, then $A \vee B \in \mathcal{I}$.*

Definition 14.
Let $S \subseteq \mathcal{F} \subseteq Fm$, then $\bigvee S = \bigcap \{\mathcal{I} \subseteq \mathcal{F} \mid S \subseteq \mathcal{I}\}$

Proposition 7. *$\bigvee S$ is an ideal.*

Lemma 6. *$A \subseteq Fm$, then $\{B \mid B \to A\} = \bigvee\{A\}$.*

Proof.
Let $A \in Fm$. Then $\{B \mid B \to A', A' \in A\} \subseteq \bigvee\{A\}$, so far as $\bigvee A$ is an ideal.
On the other hand, $\{B \mid B \to A\}$ is an ideal, it is easy to see that this set is closed under $\vee$. So, $\bigvee A \subseteq \{B \mid B \to A\}$. $\square$

Lemma 7. *$\bigvee\{A\} \subseteq \bigvee\{B\}$ iff $A \to B$.*

Proof. Let $\bigvee\{A\} \subseteq \bigvee\{B\}$, then $\{C|C \to A\} \subseteq \{D \mid D \to B\}$.
Thus, $A \in \{C|C \to A\}$, then $A \in \{D \mid D \to B\}$, hence $A \to B$.
On the other hand, let $A \to B$ and $C \in \bigvee\{A\}$.
Thus, $C \to A$, then $C \to B$ by cut.
$\square$

Lemma 8. *Let $\mathcal{Q} = \{\bigvee S \mid S \subseteq Fm\}$ and $\bigvee \mathcal{A} \cdot \bigvee \mathcal{B} = \bigvee\{A \bullet B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$. Then $\langle \mathcal{Q}, \subseteq, \cdot, \bigvee \mathbf{1}\rangle$ is a quantale.*

Proof. See $\square$

Lemma 9. *Let $!_s \in I$ and A be an arbitrary formula, then $\Box_s(\bigvee\{A\}) = \bigvee\{B \mid !_s B \to A\}$ is a quantic conucleus.*

Proof.
See Yetter. $\square$

Lemma 10. *Let A be a formula, then $\Box_s \bigvee\{A\} = \bigvee\{!_s A\}$, for each $s \in \mathcal{I}$.*

Proof. Let $A \in Fm$ and $s \in \mathcal{I}$.
Let $!_s B \in \Box_s \bigvee\{A\}$, then $!_s B \to A$, then $!_s B \to !_s A$ by promotion. So, $!_s B \in \bigvee\{!_s A\}$.
Let $C \in \bigvee\{!_s A\}$, then $C \to !_s A$, so $!_s C \to !_s A$ by dereliction, but $!_s A \to A$, hence $!_s C \to A$ by cut. So, $!_s C \in \Box_s \bigvee\{A\}$. $\square$

Lemma 11.
Let $i, j \in I$ and $i \preceq j$, then for all $A \in Fm$, $\Box_j(\bigvee\{A\}) \subseteq \Box_i(\bigvee\{A\})$.

Proof.
Let $i, j \in I$ and $i \preceq j$. Then forall $A \in Fm$, $!_j A \to !_i A$ by promotion. Then $\bigvee\{!_j A\} \subseteq \bigvee\{!_i A\}$, so $\Box_j(\bigvee\{A\}) \subseteq \Box_i(\bigvee\{A\})$. $\square$

Lemma 12.

1. *Let* $s \in W$*, then for all* $A \subseteq Fm$, $\Box_s\{A\} \subseteq \{\mathbf{1}\}$*;*

2. *Let* $s \in E$*, then* $\Box_s(\bigvee\{A\}) \cdot \bigvee\{B\} = \bigvee\{B\} \cdot \Box_s(\bigvee\{A\})$.

3. *Let* $s \in C$*, then* $(\Box_s \bigvee A \cdot \bigvee B) \cup (\bigvee B \cdot \Box_s \bigvee A) \subseteq \Box_s \bigvee A \cdot \bigvee B \cdot \Box_s \bigvee A$*, for all* $B \subseteq Fm$.

Proof.

Follows from $!_s A \to \mathbf{1}$, so $s \in W$;

Follows from $!_s A \bullet B \leftrightarrow B \bullet !_s A$;

Follows from $!_s A \bullet B \to !_s A \bullet B \bullet !_s A$ and similarly for $B \bullet !_s A$. $\Box$

Definition 15.

Let Q *be a syntactic quantale as proposed above and* $\mathcal{I} = \langle I, \preceq, W, C, E \rangle$ *be a subexponential signature.*

We define a map $\Box : \mathcal{I} \to Mod_{\mathcal{Q}}$ *as follows:*

$\Box(i)(\bigvee\{A\}) = \{!_i B \,|!_i B \to A\}$.

Lemma 13. $\Box$ *is a subexponential interpretation.*

Proof. Follows from lemmas above. $\Box$

Lemma 14.

Let Q *be a quantale constructed above and* $\Box_1, \ldots, \Box_n$ *be a family of quantic conuclei on* Q*. Then there exist a model* $\langle Q, [\![.]\!] \rangle$*, such that* $[\![A]\!] = \bigvee\{A\}$, $A \in Fm$.

Proof.

We define an interpreation as follows:

1. $[\![p_i]\!] = \bigvee\{p_i\}$
2. $[\![\mathbf{1}]\!] = \bigvee\{\mathbf{1}\}$
3. $[\![A \bullet B]\!] = \bigvee\{A \bullet B\}$
4. $[\![A/B]\!] = \bigvee\{A/B\}$
5. $[\![B\backslash A]\!] = \bigvee\{B\backslash A\}$
6. $[\![A\&B]\!] = \bigvee\{A\&B\}$
7. $[\![A \vee B]\!] = \bigvee\{A \vee B\}$
8. $[\![!_s A]\!] = \Box(s)(\bigvee\{A\}) = \bigvee\{!_s A\}$.

$\Box$

Theorem 2. $\Gamma \models A \Rightarrow \Gamma \to A$.

Proof. Follows from lemmas above. $\Box$