On R-models

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Definition 1.

$$\frac{}{A \to A}\, ax$$

$$\frac{\Gamma \to A \qquad \Delta, B, \Theta \to C}{\Delta, \Gamma, A\backslash B, \Theta \to C}\, \backslash \to \qquad \frac{A, \Pi \to B}{\Pi \to A\backslash B} \to \backslash$$

$$\frac{\Gamma \to A \qquad \Delta, B, \Theta \to C}{\Delta, B/A, \Gamma, \Theta \to C}\, / \to \qquad \frac{\Pi, A \to B}{\Pi \to B/A} \to /$$

$$\frac{\Gamma, A, B, \Delta \to C}{\Gamma, A \bullet B, \Delta \to C} \bullet \to \qquad \frac{\Gamma \to A \qquad \Delta \to B}{\Gamma, \Delta \to A \bullet B} \to \bullet$$

Definition 2.
R-model is a pair $\mathcal{M} = \langle W, R, v \rangle$, where R is a transitive relation on W and $v : Tp \to 2^R$ is a valuation, such that:

1. $\mathcal{M}, w \models p_i \Leftrightarrow w \in v(p_i)$;
2. $\mathcal{M}, \langle a, b \rangle \models A \bullet B \Leftrightarrow$ *there exists* $c \in W$, $\mathcal{M}, a \models A$ *and* $\mathcal{M}, b \models B$;
3. $\mathcal{M}, \langle a, b \rangle \models A\backslash B \Leftrightarrow$ *for all* $c \in R^{-1}(a)$, $\mathcal{M}, \langle c, a \rangle \models A$ *implies* $\mathcal{M}, \langle c, b \rangle \models B$;
4. $\mathcal{M}, \langle a, b \rangle \models B/A \Leftrightarrow$ *for all* $c \in R(a)$, $\mathcal{M}, \langle a, c \rangle \models A$ *implies* $\mathcal{M}, \langle b, c \rangle \models B$;
5. $\mathcal{M}, \langle a, b \rangle \models \Gamma \to A \Leftrightarrow \mathcal{M}, \langle a, b \rangle \models \Gamma$ *implies* $\mathcal{M}, \langle a, b \rangle \models A$

where $\mathcal{M}, \langle a, b \rangle \models \Gamma$ denotes $\mathcal{M}, \langle a, b \rangle \models A_1 \bullet \cdots \bullet A_n$, or, equivalently, there exist $c_1, \dots, c_{n-1}$, such that $\mathcal{M}, \langle a, c_1 \rangle \models A_1, \mathcal{M}, \langle c_1, c_2 \rangle \models A_2, \dots, \mathcal{M}, \langle c_{n-1}, b \rangle \models A_n$ implies that $\mathcal{M}, \langle a, b \rangle \models B$.

Theorem 1. *Let $\mathbb{F}$ be a R-frame, then $\mathbb{F} \models L$.*

Definition 3.
Let $\mathcal{F}_1$, $\mathcal{F}_2$ be R-frames and $\mathcal{M}_1 = \langle \mathcal{F}_1, v_1 \rangle$, $\mathcal{M}_2 = \langle \mathcal{F}_2, v_2 \rangle$ be R-models.
A map $f : W_1 \to W_2$ is said to be a R-frame p-morphism if the following conditions hold:

1. *f is onto;*
2. $\forall a, b \in W_1 (a R_1 b \Rightarrow f(a) R_2 f(b))$ *(monotonicity);*

3. $\forall d \in W_1, c \in W_2, f(d)R_2c \Rightarrow \exists c' \in W_1, f(c') = c \,\&\, dR_1c'$ *(left lift property);*

4. $\forall d \in W_1, c \in W_2, cR_2f(d) \Rightarrow \exists c' \in W_1, f(c') = c \,\&\, c'R_1d$ *(right lift property).*

A map $f : \mathcal{F}_1 \to \mathcal{F}_2$ *is R-model p-moprhism, iff:*

$$\mathcal{M}_1, \langle a, b\rangle \models p_i \Leftrightarrow \mathcal{M}_2, \langle f(a), f(b)\rangle \models p_i$$

Lemma 1. *Let* $f : \mathcal{M}_1 \twoheadrightarrow \mathcal{M}_2$*, then* $\mathcal{M}_1, \langle a, b\rangle \models A \Leftrightarrow \mathcal{M}_2, \langle f(a), f(b)\rangle \models A$*, for all* $a, b \in W_1$ *and for all* $A \in Fm$

Proof.

1. $\Rightarrow$
   (a) Basic case: follows from the definition.
   (b) Let $A = B \bullet C$ and $\mathcal{M}_1, \langle a, b\rangle \models B \bullet C$, then there exists $c \in W_1$, such that $\mathcal{M}_1, \langle a, c\rangle \models B$ and $\mathcal{M}_1, \langle c, b\rangle \models C$.
   Then, aR_1c and cR_1b, so $f(a)R_2f(c)$ and $f(c)R_2f(b)$.
   Thus, by IH, $\mathcal{M}_2, \langle f(a), f(c)\rangle \models B$ and $\mathcal{M}_2, \langle f(c), f(b)\rangle \models C$, so $\mathcal{M}_2, \langle f(a), f(b)\rangle \models B \bullet C$.
   (c) Let $A = B\backslash C$ and $\mathcal{M}_1, \langle a, b\rangle \models B\backslash C$. Let $c \in W_2$, such that $cR_2f(a)$.
   Then, by left lift property, there exist $d \in W_1$, such that $f(d) = c$ and dR_1a.
   Thus, $\mathcal{M}_1, \langle d, a\rangle \models A$ implies $\mathcal{M}_1, \langle d, b\rangle \models B$.
   By IH, $\mathcal{M}_2, \langle c, f(a)\rangle \models A$ implies $\mathcal{M}_2, \langle c, f(b)\rangle \models B$, then $\mathcal{M}_2, \langle f(a), f(b)\rangle \models A\backslash B$.
   (d) Similarly to (c), but via right lift property.

2. $\Leftarrow$
   (a) Basic case: follows from the definition.
   (b) Let $A = B \bullet C$. Let $\mathcal{M}_2, \langle f(a), f(b)\rangle \models B \bullet C$. Then there exists $c \in W_2$, such that $\mathcal{M}_2, \langle f(a), c\rangle \models B$ and $\mathcal{M}_2, \langle c, f(b)\rangle \models C$.
   So far as f is onto, then there exists $d \in W_1$, such that $c = f(d)$, then $\mathcal{M}_2, \langle f(a), f(d)\rangle \models B$ and $\mathcal{M}_2, \langle f(d), f(b)\rangle \models C$, and, by IH, $\mathcal{M}_1, \langle a, d\rangle \models B$ and $\mathcal{M}_1, \langle d, b\rangle \models C$, then $\mathcal{M}_1, \langle a, b\rangle \models B \bullet C$.
   (c) Let $A = B\backslash C$ and $\mathcal{M}_2, \langle f(a), f(b)\rangle \models B\backslash C$.
   Let $c \in W_1$ and cR_1a, then $f(c)R_1f(a)$ by monotonicity, so $\mathcal{M}_2, \langle f(c), f(a)\rangle \models A$ implies $\mathcal{M}_2, \langle f(c), f(b)\rangle \models B$.
   By IH, $\mathcal{M}_1, \langle c, a\rangle \models A$ implies $\mathcal{M}_1, \langle c, b\rangle \models B$. Thus, $\mathcal{M}_1, \langle c, a\rangle \models A\backslash B$.
   (d) Similarly to (c).

$\square$

Lemma 2.

1. Let $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *be R-models and* $f : \mathcal{M}_1 \twoheadrightarrow \mathcal{M}_2$*. Then* $\mathcal{M}_1 \models A$ *iff* $\mathcal{M}_2 \models A$*.*

2. Let $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *be R-frames and* $f : \mathcal{F}_1 \twoheadrightarrow \mathcal{F}_2$*, then* $\mathcal{F}_1 \models A$ *implies* $\mathcal{F}_2 \models A$*.*

Proof.

1. • Only if:
   Let $\mathcal{M}_1 \models A$. Let $c, d \in W_2$. So far as f is onto, then there exists $a, b \in W_1$, such that $f(a) = c$ and $f(b) = d$.
   Then $\mathcal{M}_1, \langle a, b\rangle \models A$, thus $\mathcal{M}_2\langle f(a), f(b)\rangle \models A$. That is, $\mathcal{M}_2\langle c, d\rangle \models A$
   • If:
   Follows from the previous lemma.

2. Let $f : \mathcal{F}_1 \twoheadrightarrow \mathcal{F}_2$ and $\mathcal{F}_1 \models A$.
   Let $\mathcal{M}_1 = \langle \mathcal{F}_1, v_1\rangle$ and $\mathcal{M}_2 = \langle \mathcal{F}_2, v_2\rangle$, such that for all $p \in Tp$, $\mathcal{M}_1, \langle a, b\rangle \models p \Leftrightarrow_{def} \mathcal{M}_1, \langle f(a), f(b)\rangle \models p$. Thus, $\mathcal{M}_1 \twoheadrightarrow \mathcal{M}_2$ and $\mathcal{M}_1 \models A$. Thus, $\mathcal{M}_2 \models A$.

□

Definition 4.

1. *Let $\mathcal{F} = \langle W, R\rangle$ be a transitive frame and $V \subseteq W$, such that $V \neq \varnothing$ and V is downward and upward closed with respect to R. Then a frame $\mathcal{F}\restriction V = \langle V, R \cap V \times V\rangle$ is a generated subframe.*

2. *If, $\mathcal{M} = \langle W, R, \theta$ and $V \subseteq W$, such that $\mathcal{F}\restriction V$ is a genereted subframe. Then $\mathcal{M}\restriction V = \langle V, R \cap V \times V, \theta'\rangle$ is a generated submodel, where $\theta'(p) = \theta(p) \cap V$.*

As usual, if $V = \{x\}$ is a singleton, then $\mathcal{F}\restriction x = \langle W\restriction x, R \cap x\rangle$ is a rooted frame (model).

Lemma 3. *Let $\mathcal{M} = \langle W, R, \theta$ and V is R-closed subset of W, then:*

1. *If $a, b \in V$, then $\mathcal{M}, \langle a, b\rangle \models A$ iff $\mathcal{M}\restriction V, \langle a, b\rangle \models A$;*

2. *$Log(\mathcal{F}) \subseteq Log(\mathcal{F}\restriction V)$*

Proof.

1. (a) Let $a, b \in V$ and $\langle a, b\rangle \in \theta(p)$, then $\mathcal{M}\restriction V, \langle a, b\rangle \models p$, for $p \in Tp$.
   (b) Let $A \rightleftharpoons B \bullet C$ and $\mathcal{M}, \langle a, b\rangle \models B \bullet C$, then there exists $c \in W$, such that aRC and $\mathcal{M}, \langle a, c\rangle \models B$, cRb and $\mathcal{M}, \langle c, b\rangle \models C$. So far as V is upwardly and downwardly closed, then $c \in V$. Then, by IH, $\mathcal{M}\restriction V, \langle a, c\rangle \models B$, cRb and $\mathcal{M}\restriction V, \langle c, b\rangle \models C$. So $\mathcal{M}\restriction V\langle a, b\rangle \models B \bullet C$.
   (c) Let $A \rightleftharpoons B\backslash C$ and $\mathcal{M}, \langle a, b\rangle \models B\backslash C$. Then for all $c \in R^{-1}(a)$, $\mathcal{M}, \langle c, a\rangle \models B$ implies $\mathcal{M}, \langle c, b\rangle \models C$. Let $\mathcal{M}, \langle c, a\rangle \models B$. By IH, $\mathcal{M}\restriction V, \langle c, a\rangle \models B$.
   $c \in V$, because V is R-closed upwardly and downloadly. On the other hand, cRb, so $b \in V$. Hence, $\langle c, b\rangle \in \mathcal{F}\restriction V$. By IH, $\mathcal{M}\restriction V, \langle c, b\rangle \models C$. Hence, $\mathcal{M}\restriction V, \langle a, b\rangle \models A\backslash B$.

2. Let $\mathcal{F}\restriction V \not\models \Gamma \to A$. Then there exist a valuation v and $\langle a, b\rangle \in R \cap V \times V$, such that $\mathcal{M}\restriction V, \langle a, b\rangle \models \Gamma$ and $\mathcal{M}\restriction V, \langle a, b\rangle \not\models A$.
   Let us define a valuation v' for the frame $\mathcal{F}$, such that $v(p) = v'(p)$. Then, by the first part, $\mathcal{M}, \langle a, b\rangle \models \Gamma$ and $\mathcal{M}, \langle a, b\rangle \not\models A$. So, $\mathcal{F} \not\models \Gamma \to A$.

□

Definition 5.
By $L^*_{\mathbf{S4}}$ ($L_{\mathbf{S4}}$) we mean the calculus L^* (L) extended with the following inference rules:

$$\frac{\Gamma, A, \Delta \to B}{\Gamma, !A, \Delta \to B}\ ! \to \qquad\qquad \frac{!\Gamma \to A}{!\Gamma \to !A} \to !$$

Proposition 1.

$L_{\mathbf{S4}}$ ($L^*_{\mathbf{S4}}$) might be equivalently defined as $L'_{\mathbf{S4}}(L^*_{\mathbf{S4}}) = L(L^*) \oplus !A \to A \oplus !A \to !!A$ closed under monotonicity !-rule

Proof.

It is clear that $L_{\mathbf{S4}} \subseteq L'_{\mathbf{S4}}$. The converse inclusion is proved straightforwardly as follows:

$$\frac{!A \to A \qquad \Gamma, A, \Delta \to A}{\Gamma, !A, \Delta \to B}\,\mathbf{cut} \qquad\qquad \frac{\{!A_i \to !!A_i\}_{i=1,\ldots,n} \qquad \dfrac{!A_1, !A_2, \ldots, !A_n \to B}{!!A_1, !!A_2, \ldots, !!A_n \to !B}\,\mathbf{mon}}{!A_1, !A_2, \ldots, !A_n \to !B}\,\mathbf{cut}$$

□

Definition 6.

A modal R-model $\mathfrak{M}$ is a R-model with additional truth condition:
$\mathfrak{M}, \langle a, b\rangle \models !A \Leftrightarrow \exists R_{1\langle a,b\rangle} \subseteq R$ such that $\forall w \in R_1\ \mathfrak{M}, w \models A$

Theorem 2. *Let $R \subseteq W \times W$ be a (reflexive) transitive relation and $\mathcal{F} = \langle W, R\rangle$ a modal R-model, then $L_{\mathbf{S4}} \subseteq Log(F)$.*

We prove this theorem modulo the previous proposition for simplicity.

Proof.

1. Let $\mathcal{F} \models !A$, then for all $\langle a, b\rangle \in R$ and for each valuation v, $\langle \mathcal{F}, v\rangle, \langle a, b\rangle \models !A$. Thus, $\langle \mathcal{F}, v\rangle, \langle a, b\rangle \models A$.
2. Let $\mathcal{F} \models !A$, then for all $\langle a, b\rangle \in R$, $\langle \mathcal{F}, v\rangle, \langle a, b\rangle \models !A$, where v is a valuation. It means that there exists $U_{\langle a,b\rangle} \subseteq R$ such that for each $u \in U$, $\langle \mathcal{F}, v\rangle, u \models A$.

□

Lemma 4. *Let $\mathcal{F}_1 = \langle W_1, R_1\rangle$ and $\mathcal{F}_2 = \langle W_2, R_2\rangle$ be R-frames and $f : \mathcal{F}_1 \twoheadrightarrow \mathcal{F}_2$ a p-morphism. Then p-morphism properties also hold*

Proof.

Let us show that $\langle \mathcal{F}_1, v_1\rangle, \langle a, b\rangle \models !A \Leftrightarrow \langle \mathcal{F}_2, v_2\rangle, \langle f(a), f(b)\rangle \models !A$

($\Rightarrow$) Let $\langle \mathcal{F}_1, v_1\rangle, \langle a, b\rangle \models !A$. Then there exists a subrelation R_1 such that $\langle a, b\rangle$ and for each $p \in R_1, \langle \mathcal{F}_1, v_1\rangle, p \models A$. Let us define $R'_1 = \{\langle f(x), f(y)\rangle \mid \langle x, y\rangle \in R_1\} \subseteq R_2$. By IH, for each $p' \in R'_1$, $\langle \mathcal{F}_1, v_1\rangle, p \models A$. Thus, there exists a subrelation that contains $\langle f(a), f(b)\rangle$ in each pair of which A is true. So, $\langle \mathcal{F}_2, v_2\rangle, \langle f(a), f(b)\rangle \models !A$.

($\Leftarrow$) Let $\langle \mathcal{F}_2, v_2\rangle, \langle f(a), f(b)\rangle \models !A$. Then there exists $R'_2 \subseteq R_2$ such that $\langle f(a), f(b)\rangle \in R'_2$ and for each $\langle c, d\rangle$, $\langle \mathcal{F}_2, v_2\rangle, \langle c, d\rangle \models A$. By lifting property, □

Let $\mathbb{Z}^*$ be the set of all finite sequences of integers. Let us define the following relation R.