

Categorical model of noncommutative linear logic with subexponentials

Definition 1. A subexponential signature is an ordered quintuple:

$$\Sigma = \langle I, \leq, W, C, E \rangle,$$

where $I = \{s_1, \dots, s_n\}$, $\langle I, \leq \rangle$ is a preorder. W, C, E are subsets of I and $W \cup C \subseteq E$.

Definition 2. Noncommutative linear logic with subexponentials ($SMALC_\Sigma$), where Σ is a subexponential signature.

$$\begin{array}{c}
 \overline{A \Rightarrow A} \text{ }^{ax} \\
 \\
 \frac{\Gamma \Rightarrow A \quad \Delta, B, \Theta \Rightarrow C}{\Delta, \Gamma, A \backslash B, \Theta \Rightarrow C} \backslash \rightarrow \qquad \frac{A, \Pi \Rightarrow B}{\Pi \Rightarrow A \backslash B} \rightarrow \backslash \\
 \\
 \frac{\Gamma \Rightarrow A \quad \Delta, B, \Theta \Rightarrow C}{\Delta, B / A, \Gamma, \Theta \Rightarrow C} / \rightarrow \qquad \frac{\Pi, A \Rightarrow B}{\Pi \Rightarrow B / A} \rightarrow / \\
 \\
 \frac{\Gamma, A, B, \Delta \Rightarrow C}{\Gamma, A \bullet B, \Delta \Rightarrow C} \bullet \rightarrow \qquad \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \bullet B} \rightarrow \bullet \\
 \\
 \frac{\Gamma, A_i, \Delta \Rightarrow B}{\Gamma, A_1 \& A_2, \Delta \Rightarrow B} \&, i = 1, 2 \rightarrow \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \& B} \rightarrow \& \\
 \\
 \frac{\Gamma, A, \Delta \Rightarrow C \quad \Gamma, B, \Delta \Rightarrow C}{\Gamma, A \vee B, \Delta \Rightarrow C} \vee \rightarrow \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2} \rightarrow \vee, i = 1, 2 \\
 \\
 \frac{\Gamma, \Delta \Rightarrow A}{\Gamma, \mathbf{1}, \Delta \Rightarrow A} \mathbf{1} \rightarrow \qquad \frac{}{\Rightarrow \mathbf{1}} \rightarrow \mathbf{1} \\
 \\
 \frac{\Gamma, A, \Delta \Rightarrow C}{\Gamma, !^s A, \Delta \Rightarrow C} ! \rightarrow \qquad \frac{!^{s_1} A_1, \dots, !^{s_n} A_n \Rightarrow A}{!^{s_1} A_1, \dots, !^{s_n} A_n \Rightarrow !^s A} \rightarrow !, \forall j, s_j \geq s \\
 \\
 \frac{\Gamma, \Delta \Rightarrow B}{\Gamma, !^s A, \Delta \Rightarrow B} \text{weak}_!, s \in C \\
 \\
 \frac{\Gamma, !^s A, \Delta, !^s A, \Theta \Rightarrow B}{\Gamma, !^s A, \Delta, \Theta \Rightarrow B} \text{ncontr}_1, s \in C \\
 \\
 \frac{\Gamma, !^s A, \Delta, !^s A, \Theta \Rightarrow B}{\Gamma, \Delta, !^s A, \Theta \Rightarrow B} \text{ncontr}_2, s \in C
 \end{array}$$

$$\frac{\Gamma, \Delta, !^s A, \Theta \Rightarrow B}{\Gamma, !^s A, \Delta, \Theta \Rightarrow A} \mathbf{ex}_1, s \in E$$

$$\frac{\Gamma, !^s A, \Delta, \Theta \Rightarrow B}{\Gamma, \Delta, !^s A, \Theta \Rightarrow A} \mathbf{ex}_1, s \in E$$

Lemma 1. Let $A \Leftrightarrow B$, then $C[p_i := A] \Leftrightarrow C[p_i := B]$

Proof. By induction on C . □

Definition 3. Monoidal comonad

A monoidal comonad on some monoidal category \mathcal{C} is a triple $\langle \mathcal{F}, \epsilon, \delta \rangle$, where \mathcal{F} is a monoidal endofunctor and $\epsilon : \mathcal{F} \Rightarrow Id_{\mathcal{C}}$ (counit) and $\epsilon : \mathcal{F} \Rightarrow \mathcal{F}^2$ (comultiplication), such that the following diagrams commute:

$$\begin{array}{ccc} \mathcal{F}A \otimes \mathcal{F}B & \xrightarrow{\phi_{A,B}} & \mathcal{F}(A \otimes B) \\ \delta_A \otimes \delta_B \downarrow & & \searrow \delta_{A \otimes B} \\ \mathcal{F}\mathcal{F}A \otimes \mathcal{F}\mathcal{F}B & \xrightarrow{\phi_{\mathcal{F}A, \mathcal{F}B}} & \mathcal{F}(\mathcal{F}A \otimes \mathcal{F}B) \\ & \nearrow \mathcal{F}(\phi_{A,B}) & \end{array} \quad \begin{array}{ccc} \mathcal{F}A \otimes \mathcal{F}B & \xrightarrow{\phi_{A,B}} & \mathcal{F}(A \otimes B) \\ \epsilon_A \otimes \epsilon_B \searrow & & \swarrow \epsilon_{A \otimes B} \\ & A \otimes B & \end{array}$$

$$\begin{array}{ccc} \mathbb{1} & \xrightarrow{\phi} & \mathcal{F}\mathbb{1} \\ \phi \downarrow & & \downarrow \delta_{\mathbb{1}} \\ \mathcal{F}\mathbb{1} & \xrightarrow{\mathcal{F}(\phi)} & \mathcal{F}\mathcal{F}\mathbb{1} \end{array}$$

$$\begin{array}{ccc} \mathbb{1} & \xrightarrow{id_{\mathbb{1}}} & \mathbb{1} \\ \phi \searrow & & \nearrow \epsilon_{\mathbb{1}} \\ & \mathcal{F}\mathbb{1} & \end{array}$$

Definition 4. Biclosed monoidal category

Let \mathcal{C} be a monoidal category. Biclosed monoidal category is a monoidal category with the following additional data:

1. Bifunctors $_ \multimap _, _ \multimap _ : \mathcal{C}^{op} \times \mathcal{C} \rightarrow \mathcal{C}$;
2. Natural isomorphism $\mathbf{curry}_{A,B,C} : Hom(A \otimes B, C) \cong (B, A \multimap C)$;
3. Natural isomorphism $\mathbf{curry}'_{A,B,C} : Hom(A \otimes B, C) \cong (A, C \multimap B)$;
4. For each $A, B \in Ob_{\mathcal{C}}$, there are exist arrows $ev_{A,B} : A \otimes (A \Rightarrow B) \rightarrow B$ and $ev'_{A,B} : (B \Leftarrow A) \otimes A \rightarrow B$, such that for all $f : A \otimes C \rightarrow B$:
 - (a) $\Lambda_l \circ (id_A \otimes \mathbf{curry}(f)) = f$;
 - (b) $\Lambda_r \circ (\mathbf{curry}'(f) \otimes id_A) = f$

Definition 5. Let F be endofunctor and $A \in Ob_{\mathcal{C}}$, then a coalgebra of F is a tuple $\langle A, \theta \rangle$, where $\theta : A \rightarrow FA$.

Given coalgebras $\langle A, \theta \rangle$ and $\langle A, \psi \rangle$, a homomorphism is a morphism $f : A \rightarrow B$, s.t. the diagram below commutes:

$$\begin{array}{ccc} A & \xrightarrow{\theta} & FA \\ f \downarrow & & \downarrow Ff \\ B & \xrightarrow{\psi} & FB \end{array}$$

that is, $Ff \circ \theta = \psi \circ f$

Definition 6. *Subexponential model structure*

Let $\Sigma = \langle I, \leq, W, C, E \rangle$ be a subexponential signature and \mathcal{C} be a biclosed monoidal category, then a subexponential model structure is $\langle \mathcal{C}, \{\mathcal{F}_s\}_{s \in I} \rangle$ with the following additional data:

- for all $s \in I$, \mathcal{F}_s is a monoidal comonad;
- if $s \in W$, then for all $A \in \text{Ob}(\mathcal{C})$, there exists a morphism $w_{As} : F_s A \rightarrow \mathbb{1}$;
- if $s \in C$, then for all $A \in \text{Ob}(\mathcal{C})$, there exists morphisms $w_{Al} : F_s A \otimes A \otimes F_s A \rightarrow F_s A \otimes B$ and $w_{Ar} : F_s A \otimes A \otimes F_s A \rightarrow B \otimes F_s A$;
- if $s \in E$, then for all $A \in \text{Ob}(\mathcal{C})$, there is an isomorphism, $e_A : F_s A \otimes B \cong B \otimes F_s A$;
- if $s_1 \in W$, $s_2 \in I$ and $s_1 \leq s_2$, then there is a morphism $w_{As_2} : F_{s_2} A \rightarrow \mathbb{1}$ for all $A \in \text{Ob}(\mathcal{C})$ and ditto for E and C ;
- Let $\bigotimes_{s \in J, i=0}^n F_s A$, where $J \subset I$, and $s' \in I$, s.t. $s \geq s'$ for all $s \in J$; Then there exists morphism a morphism $\theta_{\bigotimes_{s \in J, i=1}^n F_{s_j} A_i} : \bigotimes_{s \in J, i=0}^n F_s A \rightarrow F_{s'}(\bigotimes_{s \in J, i=0}^n F_s A)$, such that $\langle \bigotimes_{s \in J, i=1}^n F_{s_j} A_i, \theta_{\bigotimes_{s \in J, i=1}^n F_{s_j} A_i} \rangle$ is a coalgebra on $F_{s'}$.

Definition 7. Let $\langle \mathcal{C}, \{\mathcal{F}_s\}_{s \in I} \rangle$ be a subexponential model structure for subexponential signature $\Sigma = \langle I, \leq, W, C, E \rangle$. Let $v : Tp \rightarrow \text{Ob}(\mathcal{C})$ be a valuation map. Then the interpretation function $\llbracket \cdot \rrbracket$ is defined as follows:

- (1) $\llbracket \mathbb{1} \rrbracket = \mathbb{1}$
- (2) $\llbracket A \setminus B \rrbracket = \llbracket A \rrbracket \multimap \llbracket B \rrbracket$
- (3) $\llbracket A / B \rrbracket = \llbracket A \rrbracket \multimap \llbracket B \rrbracket$
- (4) $\llbracket A \bullet B \rrbracket = \llbracket A \rrbracket \otimes \llbracket B \rrbracket$
- (5) $\llbracket !_s A \rrbracket = F_s \llbracket A \rrbracket$

Theorem 1. *The following statements are equivalent:*

- $SMLC_\Sigma + (\text{cut}) \vdash \Gamma \Rightarrow A$
- $SMLC_\Sigma \vdash \Gamma \Rightarrow A$
- $\exists f, f : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket$

Proof.

(1) \Rightarrow (2): cut elimination.

- (2) \Rightarrow (3): Soundness:

$$\overline{id_A : A \rightarrow A}$$

$$\frac{f : \Gamma \rightarrow A \quad g : \Delta \otimes B \otimes \Theta \rightarrow C}{g \circ (id_\Delta \otimes (ev_{A,B_l} \circ (f \otimes id_{A \multimap B})) \otimes id_\Theta) : \Delta \otimes (\Gamma \otimes A \multimap B) \otimes \Theta \rightarrow C}$$

$$\frac{f : A \otimes \Pi \rightarrow B}{\Lambda_l(f) : \Pi \rightarrow A \multimap B}$$

$$\frac{f : \Gamma \rightarrow A \quad g : \Delta \otimes B \otimes \Theta \rightarrow C}{g \circ (id_\Delta \otimes (ev_{A,B_l} \circ (id_{B \multimap A} \otimes f)) \otimes id_\Theta) : \Delta \otimes (B \multimap A \otimes \Gamma) \otimes \Theta \rightarrow C}$$

$$\frac{f : \Pi \otimes A \rightarrow B}{\Lambda_r(f) : \Pi \rightarrow B \multimap A}$$

$$\frac{f : \Gamma \otimes A \otimes B \otimes \Delta \rightarrow C}{f \circ (\alpha_{\Gamma,A,B} \otimes id_\Delta) : \Gamma \otimes (A \otimes B) \otimes \Delta \rightarrow C}$$

$$\frac{f : \Gamma \rightarrow A \quad g : \Delta \rightarrow B}{f \otimes g : \Gamma \otimes \Delta \rightarrow A \otimes B}$$

$$\frac{f : \Gamma \otimes A_i \otimes \Delta \rightarrow B}{f \circ (id_\Gamma \otimes \pi_i id_\Delta) : \Gamma \otimes (A_1 \times A_2) \otimes \Delta \rightarrow B}$$

$$\frac{f : \Gamma \rightarrow A \quad g : \Gamma \rightarrow B}{\langle f, g \rangle : \Gamma \rightarrow A \times B}$$

$$\frac{f : \Gamma \otimes A \otimes \Delta \rightarrow C \quad g : \Gamma \otimes B \otimes \Delta \rightarrow C}{id_\Gamma \otimes [f, g] \otimes id_\Delta : \Gamma \otimes (A + B) \otimes \Delta \rightarrow C}$$

$$\overline{id_{\mathbb{1}} : \mathbb{1} \rightarrow \mathbb{1}}$$

$$\frac{f : \Gamma \otimes \Delta \rightarrow A}{f \circ (\rho_\Gamma \otimes id_\Delta) : (\Gamma \otimes \mathbb{1}) \otimes \Delta \rightarrow A}$$

$$\frac{f : \Gamma \otimes A \otimes \Delta \rightarrow B}{f \circ (id_\Gamma \otimes \delta_s^A \otimes id_\Delta) : \Gamma \otimes F_s A \otimes \Delta \rightarrow B}$$

$$\frac{f : F_{s_1} A_1 \otimes \cdots \otimes F_{s_n} A_n \rightarrow B}{F_s(f) : F_s(F_{s_1} A_1 \otimes \cdots \otimes F_{s_n} A_n) \rightarrow F_s B}$$

$$\frac{F_s(f) \circ \theta_{\otimes_{s \in J, i=1}^n F_{s_j} A_i} : F_{s_1} A_1 \otimes \cdots \otimes F_{s_n} A_n \rightarrow F_s B}{F_s(f) \circ \theta_{\otimes_{s \in J, i=1}^n F_{s_j} A_i} : F_{s_1} A_1 \otimes \cdots \otimes F_{s_n} A_n \rightarrow F_s B}$$

$$\frac{f : \Gamma \otimes \Delta \rightarrow A}{f \circ (\rho_\Gamma \otimes id_\Delta) : (\Gamma \otimes \mathbb{1}) \otimes \Delta \rightarrow A}$$

$$\frac{f \circ (\rho_\Gamma \otimes id_\Delta) \circ (id_\Gamma \otimes w_{A_s}) \otimes id_\Delta : (\Gamma \otimes F_s A) \otimes \Delta \rightarrow A}{f \circ (\rho_\Gamma \otimes id_\Delta) \circ (id_\Gamma \otimes w_{A_s}) \otimes id_\Delta : (\Gamma \otimes F_s A) \otimes \Delta \rightarrow A}$$

$$\begin{array}{c}
\frac{f : \Gamma \otimes (F_s A \otimes B \otimes F_s A) \otimes \Delta \rightarrow C}{f \circ (id_\Gamma \otimes c_{A_s}^l \otimes id_\Delta) : \Gamma \otimes (F_s A \otimes B) \otimes \Delta \rightarrow C} \\
\\
\frac{f : \Gamma \otimes (F_s A \otimes B \otimes F_s A) \otimes \Delta \rightarrow C}{(id_\Gamma \otimes c_{A_s}^r \otimes id_\Delta) \circ f : \Gamma \otimes (B \otimes F_s A) \otimes \Delta \rightarrow C} \\
\\
\frac{f : \Gamma \otimes (\Delta \otimes F_s A) \otimes \Theta \rightarrow B}{(id_\Gamma \otimes (id_\Delta \otimes e_{A_s}) \otimes id_\Theta) \circ f : \Gamma \otimes (F_s A \otimes \Delta) \otimes \Theta \rightarrow B} \\
\\
\frac{f : \Gamma \otimes (F_s A \otimes \Delta) \otimes \Theta \rightarrow B}{(id_\Gamma \otimes (id_\Delta \otimes e_{A_s}^{-1}) \otimes id_\Theta) \circ f : \Gamma \otimes (\Delta \otimes F_s A) \otimes \Theta \rightarrow B}
\end{array}$$

- Completeness:

Definition 8.

□

1 Concrete model

Definition 9. *Quantale A quantale is a triple $\langle A, \bigvee, \cdot \rangle$, such that $\langle A, \bigvee \rangle$ is a complete lattice and $\langle A, \cdot \rangle$ is a semigroup. A quantale is called unital, if $\langle A, \cdot \rangle$ is a monoid.*

It is easy to see, that any (unital) quantale is a residual (monoid) semigroup. We define divisions as follows:

1. $a \backslash b = \bigvee \{c \mid a \cdot c \leq b\}$
2. $b / a = \bigvee \{c \mid c \cdot a \leq b\}$

Definition 10. *Let $\langle A, \bigvee, \cdot \rangle$ be a quantale. The center of a quantale is the set $Z(Q) = \{a \in Q \mid \forall b \in Q, a \cdot b = b \cdot a\}$*

Definition 11. *An open modality on quantale Q is a map $I : Q \rightarrow Q$, such that*

1. $I(x) \leq x$;
2. $I(x) = I(I(x))$;
3. $x \leq y \Rightarrow I(x) \leq I(y)$;
4. $I(x) \cdot I(y) = I(I(x) \cdot I(y))$.

Lemma 2.

Let $\langle A, \bigvee, \cdot \rangle$ be a quantale and $I : Q \rightarrow Q$ is an open modality on Q , then $I(x) \cdot I(y) \leq I(x \cdot y)$.

Proof.

$I(x) \cdot I(y) \leq x \cdot y$, then $I(I(x) \cdot I(y)) \leq I(x \cdot y)$, but $I(x) \cdot I(y) \leq I(I(x) \cdot I(y))$. Thus, $I(x) \cdot I(y) \leq I(x \cdot y)$. □

Definition 12. *An open modality is called central, if $\forall a, b \in Q, I(a) \cdot b = b \cdot I(a)$.*

Definition 13. An open modality is called *weak idempotent*, if $\forall a, b \in Q, I(a) \cdot b \leq I(a) \cdot b \cdot I(a)$ and $b \cdot I(a) \leq I(a) \cdot b \cdot I(a)$.

Definition 14. An open modality is called *unital*, if $\forall a \in Q, I(a) \leq e$.

Lemma 3. Let I be an interior on unital quantale $\langle Q, \vee, \cdot, e \rangle$. Then, if I is unital and weak idempotent, then I is a central.

Proof.

$$\begin{aligned}
& b \cdot I(a) \leq \\
& \quad \text{Right weak idempotence} \\
& I(a) \cdot b \cdot I(a) \leq \\
& \quad \text{Unitality} \\
& I(a) \cdot b \cdot I(e) \leq \\
& \quad \text{Identity} \\
& I(a) \cdot b \leq \\
& \quad \text{Left weak idempotence} \\
& I(a) \cdot b \cdot I(a) \leq \\
& \quad \text{Unitality} \\
& e \cdot b \cdot I(a) \leq \\
& \quad \text{Identity} \\
& b \cdot I(a)
\end{aligned}$$

□