Definition 1. *Monoidal comonad*

A monoidal comonad on some monoidal category $\mathcal{C}$ is a triple $\langle \mathcal{F}, \epsilon, \delta \rangle$, where $\mathcal{F}$ is a monoidal endofunctor and $\epsilon : \mathcal{F} \Rightarrow Id_{\mathcal{C}}$ (counit) and $\epsilon : \mathcal{F} \Rightarrow \mathcal{F}^2$ (comultiplication), such that the following diagrams commute:

$$\begin{tikzcd}
\mathcal{F}A \otimes \mathcal{F}B \arrow[r, "\phi_{A,B}"] \arrow[dd, "\delta_A \otimes \delta_B"'] & \mathcal{F}(A \otimes B) \arrow[dr, "\delta_{A \otimes B}"] & \\
 & & \mathcal{F}\mathcal{F}(A \otimes B) \\
\mathcal{F}\mathcal{F}A \otimes \mathcal{F}\mathcal{F}B \arrow[r, "\phi_{\mathcal{F}A,\mathcal{F}B}"'] & \mathcal{F}(\mathcal{F}A \otimes \mathcal{F}B) \arrow[ur, "\mathcal{F}(\phi_{A,B})"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}A \otimes \mathcal{F}B \arrow[rr, "\phi_{A,B}"] \arrow[dr, "\epsilon_A \otimes \epsilon_B"'] & & \mathcal{F}(A \otimes B) \arrow[dl, "\epsilon_{A \otimes B}"] \\
 & A \otimes B &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "\phi"] \arrow[d, "\phi"'] & \mathcal{F}\mathbb{1} \arrow[d, "\delta_{\mathbb{1}}"] \\
\mathcal{F}\mathbb{1} \arrow[r, "\mathcal{F}(\phi)"'] & \mathcal{F}\mathcal{F}\mathbb{1}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{1} \arrow[rr, "id_{\mathbb{1}}"] \arrow[dr, "\phi"'] & & \mathbb{1} \\
 & \mathcal{F}\mathbb{1} \arrow[ur, "\epsilon_{\mathbb{1}}"'] &
\end{tikzcd}$$

Definition 2. *Biclosed monoidal category*

Let $\mathcal{C}$ be a monoidal category. Biclosed monoidal category is a monoidal category with the following additional data:

1. *Bifunctors $\_\_ \multimapinv \_\_, \_\_ \multimap \_\_ : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$;*
2. *Natural isomophism $\mathbf{curry}_{A,B,C} : Hom(A \otimes B, C) \cong (B, A \multimap C)$;*
3. *Natural isomophism $\mathbf{curry}'_{A,B,C} : Hom(A \otimes B, C) \cong (A, C \multimap B)$;*
4. *For each $A, B \in Ob_{\mathcal{C}}$, there are exist arrows $ev_{A,B} : A \otimes (A \Rightarrow B) \to B$ and $ev'_{A,B} : (B \Leftarrow A) \otimes A \to B$, such that for all $f : A \otimes C \to B$:*
   - *(a) $\Lambda_l \circ (id_A \otimes \mathbf{curry}(f)) = f$;*
   - *(b) $\Lambda_r \circ (\mathbf{curry}'(f) \otimes id_A) = f$*

Definition 3. *Let F be endofunctor and $A \in Ob\mathcal{C}$, then a coalgebra of F is a tuple $\langle A, \theta \rangle$, where $\theta : A \to FA$.*

Given coalgebras $\langle A, \theta \rangle$ and $\langle A, \psi \rangle$, a homomorphism is a morphism $f : A \to B$, s.t. the diagram below commutes:

$$\begin{tikzcd}
A \arrow[r, "\theta"] \arrow[d, "f"'] & FA \arrow[d, "Ff"] \\
B \arrow[r, "\psi"'] & FB
\end{tikzcd}$$

that is, $Ff \circ \theta = \psi \circ f$

Definition 4. *Subexponential model structure*

Let $\Sigma = \langle I, \preceq, W, C, E\rangle$ be a subexponential signature and $\mathcal{C}$ be a biclosed monoidal category, then a subexponential model structure is $\langle \mathcal{C}, \{\mathcal{F}_s\}_{s\in I}\rangle$ with the following additional data:

- *for all $s \in I$, $\mathcal{F}_s$ is a monoidal comonad;*
- *if $s \in W$, then for all $A \in Ob(\mathcal{C})$, there exists a morphism $w_{A_s} : F_s A \to \mathbb{1}$;*
- *if $s \in C$, then for all $A \in Ob(\mathcal{C})$, there exists morphisms $w_{A_l} : F_s A \otimes A \otimes F_s A \to F_s A \otimes B$ and $w_{A_r} : F_s A \otimes A \otimes F_s A \to B \otimes F_s A$;*
- *if $s \in E$, then for all $A \in Ob(\mathcal{C})$, there is an isomorpism, $e_A : F_s A \otimes B \cong B \otimes F_s A$;*
- *if $s_1 \in W$, $s_2 \in I$ and $s_1 \preceq s_2$, then there is a morphism $w_{A_{s_2}} : F_{s_2} A \to \mathbb{1}$ for all $A \in Ob(\mathcal{C})$ and ditto for E and C;*
- *Let $\bigotimes_{s\in J, i=0}^{n} F_s A$, where $J \subset I$, and $s' \in I$, s.t. $s \geq s'$ for all $s \in I'$; Then there exists morphism a morphism $\theta_{\bigotimes_{s\in J,i=1}^{n} F_{s\,j} A_i} : \bigotimes_{s\in J, i=0}^{n} F_s A \to F_{s'}(\bigotimes_{s\in J, i=0}^{n} F_s A)$, such that $\langle \bigotimes_{s\in J,i=1}^{n} F_{s\,j} A_i, \theta_{\bigotimes_{s\in J,i=1}^{n} F_{s\,j} A_i}\rangle$ is a coalgebra on F_s.*

Definition 5. *Let $\langle \mathcal{C}, \{\mathcal{F}_s\}_{s\in I}\rangle$ be a subexponential model structure for subexponential signature $\Sigma = \langle I, \preceq, W, C, E\rangle$. Let $v : Tp \to Ob(\mathcal{C})$ be a valuation map. Then the interpretation function $[\![.]\!]$ is defined as follows:*

(1) $[\![\mathbf{1}]\!] = \mathbb{1}$
(2) $[\![A \backslash B]\!] = [\![A]\!] \multimap [\![B]\!]$
(3) $[\![A / B]\!] = [\![A]\!] \multimapinv [\![B]\!]$
(4) $[\![A \bullet B]\!] = [\![A]\!] \otimes [\![B]\!]$
(5) $[\![!_s A]\!] = F_s[\![A]\!]$

Theorem 1. *The following statements are equivalent:*

- $SMLC_\Sigma + (cut) \vdash \Gamma \Rightarrow A$
- $SMLC_\Sigma \vdash \Gamma \Rightarrow A$
- $\exists f, f : [\![\Gamma]\!] \to [\![A]\!]$

  Proof.
  (1) $\Rightarrow$ (2): cut elimination.

- (2) $\Rightarrow$ (3): Soundness:

$$\frac{}{id_A : A \to A}$$

$$\frac{f : \Gamma \to A \qquad g : \Delta \otimes B \otimes \Theta \to C}{g \circ (id_\Delta \otimes (ev_{A,B_l} \circ (f \otimes id_{A \multimap B})) \otimes id_\Theta) : \Delta \otimes (\Gamma \otimes A \multimap B) \otimes \Theta \to C}$$

$$\frac{f : A \otimes \Pi \to B}{\Lambda_l(f) : \Pi \to A \multimap B}$$

$$\frac{f : \Gamma \to A \qquad g : \Delta \otimes B \otimes \Theta \to C}{g \circ (id_\Delta \otimes (ev_{A,B_l} \circ (id_{B \multimapinv A} \otimes f)) \otimes id_\Theta) : \Delta \otimes (B \multimapinv A \otimes \Gamma) \otimes \Theta \to C}$$

$$\frac{f : \Pi \otimes A \to B}{\Lambda_r(f) : \Pi \to B \multimapinv A}$$

$$\frac{f : \Gamma \otimes A \otimes B \otimes \Delta \to C}{f \circ (\alpha_{\Gamma,A,B} \otimes id_\Delta) : \Gamma \otimes (A \otimes B) \otimes \Delta \to C}$$

$$\frac{f : \Gamma \to A \qquad g : \Delta \to B}{f \otimes g : \Gamma \otimes \Delta \to A \otimes B}$$

$$\frac{f : \Gamma \otimes A_i \otimes \Delta \to B}{f \circ (id_\Gamma \otimes \pi_i id_\Delta) : \Gamma \otimes (A_1 \times A_2) \otimes \Delta \to B}$$

$$\frac{f : \Gamma \to A \qquad g : \Gamma \to B}{\langle f, g \rangle : \Gamma \to A \times B}$$

$$\frac{f : \Gamma \otimes A \otimes \Delta \to C \qquad g : \Gamma \otimes B \otimes \Delta \to C}{id_\Gamma \otimes [f, g] \otimes id_\Delta : \Gamma \otimes (A + B) \otimes \Delta \to C}$$

$$\frac{}{id_{\mathbb{1}} : \mathbb{1} \to \mathbb{1}}$$

$$\frac{f : \Gamma \otimes \Delta \to A}{f \circ (\rho_\Gamma \otimes id_\Delta) : (\Gamma \otimes \mathbb{1}) \otimes \Delta \to A}$$

$$\frac{f : \Gamma \otimes A \otimes \Delta \to B}{f \circ (id_\Gamma \otimes \delta_s^A \otimes id_\Delta) : \Gamma \otimes F_s A \otimes \Delta \to B}$$

$$\frac{\dfrac{f : F_{s_1} A_1 \otimes \cdots \otimes F_{s_n} A_n \to B}{F_s(f) : F_s(F_{s_1} A_1 \otimes \cdots \otimes F_{s_n} A_n) \to F_s B}}{F_s(f) \circ \theta_{\bigotimes_{s \in J, i=1}^n F_{s\,j} A_i} : F_{s_1} A_1 \otimes \cdots \otimes F_{s_n} A_n \to F_s B}$$

$$\frac{\dfrac{f : \Gamma \otimes \Delta \to A}{f \circ (\rho_\Gamma \otimes id_\Delta) : (\Gamma \otimes \mathbb{1}) \otimes \Delta \to A}}{f \circ (\rho_\Gamma \otimes id_\Delta) \circ (id_\Gamma \otimes w_{A\,s}) \otimes id_\Delta : (\Gamma \otimes F_s A) \otimes \Delta \to A}$$

$$\frac{f : \Gamma \otimes (F_s A \otimes B \otimes F_s A) \otimes \Delta \to C}{f \circ (id_\Gamma \otimes {c_{A}}_s^l \otimes id_\Delta) : \Gamma \otimes (F_s A \otimes B) \otimes \Delta \to C}$$

$$\frac{f : \Gamma \otimes (F_s A \otimes B \otimes F_s A) \otimes \Delta \to C}{(id_\Gamma \otimes {c_{A}}_s^r \otimes id_\Delta) \circ f : \Gamma \otimes (B \otimes F_s A) \otimes \Delta \to C}$$

$$\frac{f : \Gamma \otimes (\Delta \otimes F_s A) \otimes \Theta \to B}{(id_\Gamma \otimes (id_\Delta \otimes e_{A\,s}) \otimes id_\Theta) \circ f : \Gamma \otimes (F_s A \otimes \Delta) \otimes \Theta \to B}$$

$$\frac{f : \Gamma \otimes (F_s A \otimes \Delta) \otimes \Theta \to B}{(id_\Gamma \otimes (id_\Delta \otimes {e_{A}}_s^{-1}) \otimes id_\Theta) \circ f : \Gamma \otimes (\Delta \otimes F_s A) \otimes \Theta \to B}$$

- Completeness:

  Definition 6.

$\square$

1 Concrete model