

Quantale model of noncommutative linear logic with subexponentials

1 Calculus

Definition 1. A subexponential signature is an ordered quintuple:

$$\Sigma = \langle I, \leq, W, C, E \rangle,$$

where $I = \{s_1, \dots, s_n\}$, $\langle I, \leq \rangle$ is a preorder. W, C, E are subsets of I and $W \cup C \subseteq E$.

Definition 2. Noncommutative linear logic with subexponentials ($SMALC_\Sigma$), where Σ is a subexponential signature.

$$\begin{array}{c}
 \overline{A \Rightarrow A} \text{ }^{ax} \\
 \\
 \frac{\Gamma \Rightarrow A \quad \Delta, B, \Theta \Rightarrow C}{\Delta, \Gamma, A \backslash B, \Theta \Rightarrow C} \backslash \rightarrow \qquad \frac{A, \Pi \Rightarrow B}{\Pi \Rightarrow A \backslash B} \rightarrow \backslash \\
 \\
 \frac{\Gamma \Rightarrow A \quad \Delta, B, \Theta \Rightarrow C}{\Delta, B / A, \Gamma, \Theta \Rightarrow C} / \rightarrow \qquad \frac{\Pi, A \Rightarrow B}{\Pi \Rightarrow B / A} \rightarrow / \\
 \\
 \frac{\Gamma, A, B, \Delta \Rightarrow C}{\Gamma, A \bullet B, \Delta \Rightarrow C} \bullet \rightarrow \qquad \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \bullet B} \rightarrow \bullet \\
 \\
 \frac{\Gamma, A_i, \Delta \Rightarrow B}{\Gamma, A_1 \& A_2, \Delta \Rightarrow B} \&, i = 1, 2 \rightarrow \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \& B} \rightarrow \& \\
 \\
 \frac{\Gamma, A, \Delta \Rightarrow C \quad \Gamma, B, \Delta \Rightarrow C}{\Gamma, A \vee B, \Delta \Rightarrow C} \vee \rightarrow \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2} \rightarrow \vee, i = 1, 2 \\
 \\
 \frac{\Gamma, \Delta \Rightarrow A}{\Gamma, 1, \Delta \Rightarrow A} 1 \rightarrow \qquad \frac{}{\Rightarrow 1} \rightarrow 1 \\
 \\
 \frac{\Gamma, A, \Delta \Rightarrow C}{\Gamma, !^s A, \Delta \Rightarrow C} ! \rightarrow \qquad \frac{!^{s_1} A_1, \dots, !^{s_n} A_n \Rightarrow A}{!^{s_1} A_1, \dots, !^{s_n} A_n \Rightarrow !^s A} \rightarrow !, \forall j, s_j \geq s \\
 \\
 \frac{\Gamma, \Delta \Rightarrow B}{\Gamma, !^s A, \Delta \Rightarrow B} \text{weak}_!, s \in C \\
 \\
 \frac{\Gamma, !^s A, \Delta, !^s A, \Theta \Rightarrow B}{\Gamma, !^s A, \Delta, \Theta \Rightarrow B} \text{ncontr}_!, s \in C
 \end{array}$$

$$\frac{\Gamma, !^s A, \Delta, !^s A, \Theta \Rightarrow B}{\Gamma, \Delta, !^s A, \Theta \Rightarrow B} \text{ncontr}_2, s \in C$$

$$\frac{\Gamma, \Delta, !^s A, \Theta \Rightarrow B}{\Gamma, !^s A, \Delta, \Theta \Rightarrow A} \text{ex}_1, s \in E$$

$$\frac{\Gamma, !^s A, \Delta, \Theta \Rightarrow B}{\Gamma, \Delta, !^s A, \Theta \Rightarrow A} \text{ex}_1, s \in E$$

Lemma 1. *Let $A \Leftrightarrow B$, then $C[p_i := A] \Leftrightarrow C[p_i := B]$*

Proof. By induction on C . □

Lemma 2. • $!_{s_i} \Gamma \rightarrow A$ iff $!_{s_i} \Gamma \rightarrow !_{s_i} A$.

• $!_{s_i} A \leftrightarrow !_{s_i} (!_{s_i} A)$

Proof.

1. $!_{s_i} \Gamma \rightarrow A$ iff $!_{s_i} \Gamma \rightarrow !_{s_i} A$;

$$\frac{!_{s_i} \Gamma \rightarrow A}{!_{s_i} \Gamma \rightarrow !_{s_i} A} \rightarrow !_{s_i}$$

$$\frac{!_{s_i} \Gamma \rightarrow !_{s_i} A \quad \frac{A \rightarrow A}{!_{s_i} A \rightarrow A} !_{s_i} \rightarrow}{!_{s_i} \Gamma \rightarrow A} \text{cut}$$

2. $!_{s_i} A \leftrightarrow !_{s_i} !_{s_i} A$

$$\frac{\frac{A \rightarrow A}{!_{s_i} A \rightarrow A}}{!_{s_i} A \rightarrow !_{s_i} A} !_{s_i} !_{s_i} A \rightarrow !_{s_i} A$$

□

2 Semantics

Definition 3. *Quantale A quantale is a triple $\langle A, \bigvee, \cdot \rangle$, such that $\langle A, \bigvee \rangle$ is a complete lattice and $\langle A, \cdot \rangle$ is a semigroup. A quantale is called unital, if $\langle A, \cdot \rangle$ is a monoid.*

It is easy to see, that any (unital) quantale is a residual (monoid) semigroup. We define divisions as follows:

$$1. a \backslash b = \bigvee \{c \mid a \cdot c \leq b\}$$

$$2. b / a = \bigvee \{c \mid c \cdot a \leq b\}$$

Definition 4. *Let $\langle A, \bigvee, \cdot \rangle$ be a quantale. The center of a quantale is the set $Z(Q) = \{a \in Q \mid \forall b \in Q, a \cdot b = b \cdot a\}$*

Definition 5. An open modality on quantale Q is a map $I : Q \rightarrow Q$, such that

1. $I(x) \leq x$;
2. $I(x) = I(I(x))$;
3. $x \leq y \Rightarrow I(x) \leq I(y)$;
4. $I(x) \cdot I(y) = I(I(x) \cdot I(y))$.

Lemma 3.

Let $\langle A, \vee, \cdot \rangle$ be a quantale and $I : Q \rightarrow Q$ is an open modality on Q , then $I(x) \cdot I(y) \leq I(x \cdot y)$.

Proof.

$I(x) \cdot I(y) \leq x \cdot y$, then $I(I(x) \cdot I(y)) \leq I(x \cdot y)$, but $I(x) \cdot I(y) \leq I(I(x) \cdot I(y))$. Thus, $I(x) \cdot I(y) \leq I(x \cdot y)$. \square

Definition 6. An open modality is called central, if $\forall a, b \in Q, I(a) \cdot b = b \cdot I(a)$.

Definition 7. An open modality is called weak idempotent, if $\forall a, b \in Q, I(a) \cdot b \leq I(a) \cdot b \cdot I(a)$ and $b \cdot I(a) \leq I(a) \cdot b \cdot I(a)$.

Definition 8. An open modality is called unital, if $\forall a \in Q, I(a) \leq e$.

Lemma 4. Let I be an interior on some unital quantale $\langle Q, \vee, \cdot, e \rangle$. Then, if I is unital and weak idempotent, then I is central.

Proof.

$$\begin{aligned}
& b \cdot I(a) \leq \\
& \quad \text{Right weak idempotence} \\
& I(a) \cdot b \cdot I(a) \leq \\
& \quad \text{Unitality} \\
& I(a) \cdot b \cdot I(e) \leq \\
& \quad \text{Identity} \\
& I(a) \cdot b \leq \\
& \quad \text{Left weak idempotence} \\
& I(a) \cdot b \cdot I(a) \leq \\
& \quad \text{Unitality} \\
& e \cdot b \cdot I(a) \leq \\
& \quad \text{Identity} \\
& b \cdot I(a)
\end{aligned}$$

Hence, $b \cdot I(a) = I(a) \cdot b$

\square

Proposition 1.

Let Q be a quantale and $S \subseteq Q$ a subquantale, then $I : Q \rightarrow Q$, such that $I(a) = \bigvee \{s \in S \mid x \leq a\}$, is an open modality. Moreover, $\{x \in Q \mid I(x) = x\} = S$.

Proof. See

\square

Proposition 2.

Let Q be a quantale and $S_1, S_2 \subseteq Q$, such that $S_1 \subseteq S_2$.

Then $I_1(a) \leq I_2(a)$.

Proof.

Let $a \in Q$, so $\{s \in S_1 \mid s \leq a\} \subseteq \{s \in S_2 \mid s \leq a\}$, so $\bigvee\{s \in S_1 \mid s \leq a\} \subseteq \bigvee\{s \in S_2 \mid s \leq a\}$. Thus, $I_1(a) \leq I_2(a)$. \square

Proposition 3.

Let Q be a quantale and $S \subseteq Q$ a subquantale, then the following operations are open modalities:

1. $I_z(a) = \bigvee\{s \in S \mid s \leq a, s \in Z(Q)\};$
2. $I_{\mathbb{1}}(a) = \bigvee\{s \in S \mid s \leq a, s \leq \mathbb{1}\};$
3. $I_{idem}(a) = \bigvee\{s \in S \mid s \leq a, \forall b \in Q, b \cdot s \vee s \cdot b \leq s \cdot b \cdot s\};$
4. $I_{z, \mathbb{1}}, I_{z, idem}, I_{\mathbb{1}, idem}, I_{z, \mathbb{1}, idem}.$

Proof. Immediately. \square

Proposition 4.

1. $\forall a \in Q, I_{\mathbb{1}, idem}(a) \leq I_z(a).$
2. $\forall a \in Q, I_{z, \mathbb{1}, idem} = I_{\mathbb{1}, idem}(a)$

Proof. Follows from Lemma 3. \square

Proposition 5.

1. $I_z(a) \vee I_{\mathbb{1}}(a) \vee I_{idem}(a) \leq I(a)$
2. $I_{z, \mathbb{1}, idem} \leq I_{z, \mathbb{1}}(a) \wedge I_{z, idem}(a)$

Lemma 5. $\forall a \in Q, I_1(a) \leq I_2(I_1(a))$, if $I_1(a) \leq I_2(a)$.

Proof. $I_1(a) \leq I_1(I_1(a)) \leq I_2(I_1(a))$ \square

Lemma 6. $I_1(a_1) \cdot I_2(a_2) \leq I'(I_1(a_1) \cdot I_2(a_2))$, where $I_i \leq I', i = 1, 2$.

Proof.

$$\begin{aligned} I_1(a_1) \cdot I_2(a_2) &\leq \\ I_1(I_1(a_1)) \cdot I_2(I_2(a_2)) &\leq \\ I'(I_1(a_1)) \cdot I'(I_2(a_2)) &\leq \\ I'(I_1(a_1) \cdot I_2(a_2)) & \end{aligned} \quad \square$$

Definition 9. Interpretation of subexponential signature

Let $\Sigma = \langle I, \leq, W, C, E \rangle$ be a subexponential signature, where $|I| = n$ and $\mathcal{S} = \{\square_1, \dots, \square_n\}$ be a set of open modalities on quantale Q . Subexponential interpretation is a contravariant map $\sigma : I \rightarrow \mathcal{S}$ defined as follows:

$$\sigma(s_i) = \begin{cases} \square_i : Q \rightarrow Q, \text{ s.t. } \forall a \in Q, \square_i(a) = \{s \in S_i \mid s \leq a\}, \\ \quad \text{if } s_i \notin W \cap C \cap E \\ \square_i : Q \rightarrow Q, \text{ s.t. } \forall a \in Q, \square_i(a) = \{s \in S_i \mid s \leq a, s \leq \mathbb{1}\}, \\ \quad \text{if } s_i \in W \\ \square_i : Q \rightarrow Q, \text{ s.t. } \forall a \in Q, \square_i(a) = \{s \in S_i \mid s \leq a, s \in Z(Q)\}, \\ \quad \text{if } s_i \in E \\ \square_i : Q \rightarrow Q, \text{ s.t. } \forall a \in Q, \square_i(a) = \{s \in S_i \mid s \leq a, \forall b, b \cdot s \vee s \cdot b \leq s \cdot b \cdot s\}, \\ \quad \text{if } s_i \in E \\ \text{otherwise, if } s_i \text{ belongs to some intersection of subsets, then we combine the relevant conditions} \end{cases}$$

Definition 10. Let Q be a quantale, $f : Tp \rightarrow Q$ a valuation and $\sigma : I \rightarrow S$ a subexponential interpretation, then interpretation is defined inductively:

$$\begin{aligned} \llbracket p_i \rrbracket &= f(p_i) \\ \llbracket 1 \rrbracket &= e \\ \llbracket A \bullet B \rrbracket &= \llbracket A \rrbracket \cdot \llbracket B \rrbracket \\ \llbracket A \setminus B \rrbracket &= \llbracket A \rrbracket \setminus \llbracket B \rrbracket \\ \llbracket A/B \rrbracket &= \llbracket A \rrbracket / \llbracket B \rrbracket \\ \llbracket A \& B \rrbracket &= \llbracket A \rrbracket \wedge \llbracket B \rrbracket \\ \llbracket A \vee B \rrbracket &= \llbracket A \rrbracket \vee \llbracket B \rrbracket \\ \llbracket !_{s_i} A \rrbracket &= \sigma(s_i) \llbracket A \rrbracket \end{aligned}$$

Theorem 1. $\Gamma \rightarrow A \Rightarrow \llbracket \Gamma \rrbracket \leq \llbracket A \rrbracket$

Proof. We consider the case with polymodal promotion rule.

Let $!_{s_1} A_1, \dots, !_{s_n} A_n \rightarrow A$ and $\forall i, s \leq s_i$. Then $\forall a \in Q, \sigma(s_i)(a) \leq \sigma(s)(a)$.

By IH, $\sigma(s_1) \llbracket A_1 \rrbracket \cdots \sigma(s_n) \llbracket A_n \rrbracket \leq \llbracket A \rrbracket$.

Thus, $\sigma(s)(\sigma(s_1) \llbracket A_1 \rrbracket \cdots \sigma(s_n) \llbracket A_n \rrbracket) \leq \sigma(s)(\llbracket A \rrbracket)$.

By Lemma 5, $\sigma(s_1) \llbracket A_1 \rrbracket \cdots \sigma(s_n) \llbracket A_n \rrbracket \leq \sigma(s)(\sigma(s_1) \llbracket A_1 \rrbracket \cdots \sigma(s_n) \llbracket A_n \rrbracket)$.

So, $\sigma(s_1) \llbracket A_1 \rrbracket \cdots \sigma(s_n) \llbracket A_n \rrbracket \leq \sigma(s)(\llbracket A \rrbracket)$. □

3 Quantale completeness

Definition 11.

Let $\mathcal{F} \subseteq Fm$, an ideal is a subset $\mathcal{I} \subseteq \mathcal{F}$, such that:

- If $B \in \mathcal{I}$ and $A \rightarrow B$, then $A \in \mathcal{I}$;
- If $A, B \in \mathcal{I}$, then $A \vee B \in \mathcal{I}$.

Definition 12.

Let $S \subseteq \mathcal{F} \subseteq Fm$, then $\bigvee S = \bigcap \{\mathcal{I} \subseteq \mathcal{F} \mid S \subseteq \mathcal{I}\}$

Proposition 6. $\bigvee S$ is an ideal.

Lemma 7. $A \in \mathcal{F}$, then $\{B \mid B \rightarrow A\} = \bigvee \{A\}$.

Proof.

Let $A \in \mathcal{F}$. Then $\{B \mid B \rightarrow A\} \subseteq \bigvee \{A\}$.

On the other hand, $\{B \mid B \rightarrow A\}$ is an ideal, hence, $\{A\} \subseteq \{B \mid B \rightarrow A\}$. □

Lemma 8. $\bigvee \{A\} \subseteq \bigvee \{B\}$ iff $A \rightarrow B$.

Proof. Let $\bigvee \{A\} \subseteq \bigvee \{B\}$, then $\{C \mid C \rightarrow A\} \subseteq \{C \mid C \rightarrow B\}$.

$A \in \{C \mid C \rightarrow A\}$, so far as $A \rightarrow A$, but $\{C \mid C \rightarrow B\}$, hence $A \rightarrow B$.

On the other hand, let $A \rightarrow B$ and $C \in \bigvee \{A\}$. Then $C \in \bigvee \{C' \mid C' \rightarrow A\}$, hence, $C \rightarrow A$, thus $C \rightarrow B$ by cut. So, $C \in \bigvee \{B\}$. □

Lemma 9. Let $\mathcal{Q} = \{\bigvee S \mid S \subseteq Fm\}$ and $\bigvee \mathcal{A} \cdot \bigvee \mathcal{B} = \{A \bullet B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$. Then $\langle \mathcal{Q}, \subseteq, \cdot, \bigvee 1 \rangle$ is a quantale.

Proof. See □

Lemma 10. *Interior lemma.*

Let $Q_1 \subseteq \mathcal{Q}$, define a map $\square : Q \rightarrow Q$, such that $\square(A) = \{Q \in Q_1 \mid Q \subseteq A\}$. Then \square is a quantic conucleus.

Lemma 11. Let $!_s \in I$, $I \notin W \cap E \cap C$ and $Q \subseteq \mathcal{Q}$. Then there exist a subset $Q \subseteq \mathcal{Q}$ and a quantic conucleus $\square_s(\bigvee \{A\}) = \{\bigvee Q \in \mathcal{Q} \mid \}$

Proof.

□

Proof. See

□

Lemma 12. Let $Q \subseteq \mathcal{Q}$, then the following operators are quantic conuclei:

1. $\square_z(A) = \bigvee \{\bigvee \{W\} \in Q \mid \bigvee \{W\} \subseteq \bigvee \{A\}, \bigvee \{W\} \in Z(Q)\};$
2. $\square_1(A) = \bigvee \{\bigvee \{W\} \in Q \mid \bigvee \{W\} \subseteq \bigvee \{A\}, \bigvee \{W\} \subseteq \bigvee \{1\}\};$
3. $\square_{idem}(A) = \bigvee \{\bigvee \{W\} \in Q \mid \bigvee \{W\} \subseteq \bigvee \{A\}, \forall B \in Fm, (\bigvee \{B\} \cdot \bigvee \{W\}) \cup (\bigvee \{W\} \cdot \bigvee \{B\}) \subseteq \bigvee \{W\} \cdot \bigvee \{A\} \cdot \bigvee \{W\}\};$
4. $\square_{z,1}, \square_{z,idem}, \square_{1,idem}, \square_{z,1,idem}.$

Proof. Follow from one of lemmas above.

□

Lemma 13. Let $!_s \in I$, $I \notin W \cap E \cap C$, then $\square_s(\bigvee A) = \bigvee \{\bigvee (!_s W) \mid \bigvee !_s W \subseteq \bigvee A\}$ is a quantic conucleus.

Proof.

1. $\square_s(\bigvee A) \subseteq \bigvee A;$
 $\bigvee (!_s W) \in \square_s(\bigvee A)$, then $\bigvee (!_s W) \in \bigvee \{\bigvee (!_s W) \mid \bigvee !_s W \subseteq \bigvee A\}$. Hence, $\bigvee (!_s F) \subseteq \bigvee \{A\}$, so, $!_s F \rightarrow A$. Therefore, $!_s F \in \bigvee \{A\}$.
2. $\square_s(\square_s(\bigvee A)) = \bigvee \square_s(\bigvee A);$
 $\square_s(\square_s(\bigvee A)) = \bigvee \{\bigvee (!_s !_s F) \mid \bigvee (!_s !_s F) \subseteq \bigvee A\}.$
Let $\bigvee (!_s !_s F) \in \square_s(\square_s(\bigvee A))$, then $!_s !_s F \rightarrow A$, hence $!_s F \rightarrow A$ by equivalence, so $\bigvee (!_s !_s F) \in \square_s(\bigvee A)$
3. $\bigvee A \subseteq \bigvee B \Rightarrow \square_s(\bigvee A) \subseteq \square_s(\bigvee B);$
4. $\square_s \bigvee A \cdot \square_s \bigvee A = \square_s(\square_s \bigvee A \cdot \square_s \bigvee A).$

□

Lemma 14. Let Q be a quantale constructed above and $\square_1, \dots, \square_n$ be a family of quantic conuclei on Q . Then there exist a model $\langle Q, \llbracket \cdot \rrbracket \rangle$, such that $\llbracket A \rrbracket = \bigvee \{A\}$, $A \in Fm$.

Proof.

□

Theorem 2. $\Gamma \models A \Rightarrow \Gamma \rightarrow A$.