Models of Lambek calculus with subexponentials

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Abstract

1 The Lambek Calculus with subexponentials

Definition 1. *A subexponential signature is an ordered quintuple:*
$$\Sigma = \langle \mathcal{I}, \preceq, \mathcal{W}, \mathcal{C}, \mathcal{E} \rangle,$$

where $I = \{s_1, \ldots, s_n\}$, $\mathcal{I}, \preceq\rangle$ is a preorder. W, C, E are upwardly closed subsets of I and $\mathcal{W} \cap \mathcal{C} \subseteq \mathcal{E}$.

Definition 2.
$$\mathcal{F}_\Sigma ::= Tp \,|\, (\mathcal{F}_\Sigma / \mathcal{F}_\Sigma) \,|\, (\mathcal{F}_\Sigma \backslash \mathcal{F}_\Sigma) \,|\, (\mathcal{F}_\Sigma \bullet \mathcal{F}_\Sigma) \,|\, (\mathcal{F}_\Sigma \vee \mathcal{F}_\Sigma) \,|\, (\mathcal{F}_\Sigma \wedge \mathcal{F}_\Sigma) \,|\, (!_s \mathcal{F}_\Sigma)_{s \in \Sigma}$$

Definition 3. *Noncommutative linear logic with subexponentials ($SMALC_\Sigma$), where Σ is a subexponential signature.*

$$\overline{A \to A} \ ax$$

$$\frac{\Gamma \to A \qquad \Delta, B, \Theta \to C}{\Delta, \Gamma, A\backslash B, \Theta \to C} \backslash \to \qquad\qquad \frac{A, \Pi \to B}{\Pi \to A\backslash B} \to \backslash$$

$$\frac{\Gamma \to A \qquad \Delta, B, \Theta \to C}{\Delta, B/A, \Gamma, \Theta \to C} / \to \qquad\qquad \frac{\Pi, A \to B}{\Pi \to B/A} \to /$$

$$\frac{\Gamma, A, B, \Delta \to C}{\Gamma, A \bullet B, \Delta \to C} \bullet \to \qquad\qquad \frac{\Gamma \to A \qquad \Delta \to B}{\Gamma, \Delta \to A \bullet B} \to \bullet$$

$$\frac{\Gamma, A_i, \Delta \to B}{\Gamma, A_1 \& A_2, \Delta \to B} \&, i = 1, 2 \to \qquad\qquad \frac{\Gamma \to A \qquad \Gamma \to B}{\Gamma \to A \& B} \to \&$$

$$\frac{\Gamma, A, \Delta \to C \qquad \Gamma, B, \Delta \to C}{\Gamma, A \vee B, \Delta \to C} \vee \to \qquad\qquad \frac{\Gamma \to A_i}{\Gamma \to A_1 \vee A_2} \to \vee, i = 1, 2$$

$$\frac{\Gamma, \Delta \to A}{\Gamma, \mathbf{1}, \Delta \to A} \mathbf{1} \to \qquad\qquad \frac{}{\to \mathbf{1}} \to \mathbf{1}$$

$$\frac{\Gamma, A, \Delta \to C}{\Gamma, !^s A, \Delta \to C} ! \to \qquad\qquad \frac{!^{s_1} A_1, \ldots, !^{s_n} A_n \to A}{!^{s_1} A_1, \ldots, !^{s_n} A_n \to !^s A} \to !, \forall j, s_j \succeq s$$

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Structural rules:

$$\frac{\Gamma, !^s A, \Delta, !^s A, \Theta \to B}{\Gamma, !^s A, \Delta, \Theta \to B} \ \mathbf{ncontr}_1, s \in C \qquad \frac{\Gamma, !^s A, \Delta, !^s A, \Theta \to B}{\Gamma, \Delta, !^s A, \Theta \to B} \ \mathbf{ncontr}_2, s \in C$$

$$\frac{\Gamma, \Delta, !^s A, \Theta \to B}{\Gamma, !^s A, \Delta, \Theta \to A} \ \mathbf{ex}_1, s \in E \qquad \frac{\Gamma, !^s A, \Delta, \Theta \to B}{\Gamma, \Delta, !^s A, \Theta \to A} \ \mathbf{ex}_2, s \in E$$

$$\frac{\Gamma, \Delta \to B}{\Gamma, !^s A, \Delta \to B} \ \mathbf{weak}_!, s \in C \qquad \frac{\Gamma \to A \qquad \Pi, A, \Delta \to B}{\Gamma, \Pi, \Delta \to B} \ \mathbf{cut}$$

Theorem 1.

1. *Cut-rule is admissable;*

2. *$SMALC_\Sigma$ is undecidable, if $C \neq \varnothing$;*

3. *If C is empty, then the decidability problem of $SMALC_\Sigma$ belongs to PSPACE.*

2 Semantics

Definition 4. *Quantale*
A quantale is a triple $\mathcal{Q} = \langle A, \bigvee, \cdot \rangle$, such that $\langle A, \bigvee \rangle$ is a complete lattice and $\langle A, \cdot \rangle$ is a semigroup, such that for all indexing set I:

1. $a \cdot \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \cdot b_i)$;

2. $\bigvee_{i \in I} a_i \cdot b = \bigvee_{i \in I} (a_i \cdot b)$

A quanlate is called unital, if $\langle A, \cdot \rangle$ is a monoid.

Some example of quantales:

- Let A be a semigroup (monoid), then $\langle \mathcal{P}(A), \cdot, \subseteq \rangle$ is a free (unital) quantale.

- Let R be a ring and $Sub(R)$ be a set of additive subgroups of R. We define $A \cdot B$ as an additive subgroup generated by finite sums of products ab and order is defined by inclusion.

- Any locale is a quantale with $\cdot = \wedge$.

It is easy to see, that any (unital) quantale is a residual (monoid) semigroup. We define divisions as follows:

1. $a \backslash b = \bigvee \{c \mid a \cdot c \leqslant b\}$

2. $b / a = \bigvee \{c \mid c \cdot a \leqslant b\}$

Definition 5. *Let $\mathcal{Q}_1$, $\mathcal{Q}_2$ be quantales. A quantale homomorphism is a map $f : \mathcal{Q}_1 \to \mathcal{Q}_2$, such that:*

1. for all $a, b \in \mathcal{Q}_1$, $f(a \cdot b) = f(a) \cdot f(b)$;

2. for all indexing set I, $f(\bigvee_{i \in I} a_i) = \bigvee_{i \in I} f(a_i)$.

If $\mathcal{Q}_1$, $\mathcal{Q}_2$ are unital quantales, then a unital homomorphism if a quantale homomorphism, such that $f(\varepsilon) = \varepsilon$.

Definition 6.

Let $\mathcal{Q} = \langle A, \bigvee, \cdot \rangle$ be a quantale. $\mathcal{S} \subseteq \mathcal{Q}$ is said to be a subquantale, if $\mathcal{S}$ is closed under multiplication and sups.

There occurs the following simple statement:

Proposition 1.

Let $\mathcal{Q}_1$, $\mathcal{Q}_2$ be quantales and $\mathcal{S} \subseteq \mathcal{Q}_1$ is a subquantale of $\mathcal{Q}_1$.

Then, if $f : \mathcal{Q}_1 \to \mathcal{Q}_2$ is a quantale homomorphism, then $f(\mathcal{S}) \subseteq \mathcal{Q}_2$ is a subquantale of $\mathcal{Q}_2$. In other words, a homomorphic image of subquantale is a subquantale.

Proof.

It is clearly that $f(\mathcal{S}) \subseteq \mathcal{Q}_2$ is a submonoid of $\mathcal{Q}_2$. Let $a_i \in S$ for each $i \in I$, so $\bigvee_{i \in I} a_i \in S$, but $f(a_i) \in f(\mathcal{S})$ for any $i \in I$, so $f(\bigvee_{i \in I} a_i) = \bigvee_{i \in I}(f(a)) \in f(\mathcal{S})$, so $f(\mathcal{S})$ is closed under joins, so $f(\mathcal{S})$ is a subquantale of $\mathcal{Q}_2$ $\qquad\square$

Definition 7.

Let $\mathcal{Q} = \langle A, \bigvee, \cdot \rangle$ be a quantale. The centre of a quantale is the subquantale $\mathcal{Z}(\mathcal{Q}) = \{a \in A \mid \forall b \in A, a \cdot b = b \cdot a\}$

Definition 8.

An open modality (or quantic conucleus) on quantale $\mathcal{Q}$ is a map $\square : \mathcal{Q} \to \mathcal{Q}$, such that

1. $\square x \leqslant x$;

2. $\square x = \square\square x$;

3. $x \leqslant y \Rightarrow \square x \leqslant \square y$;

4. $\square x \cdot \square y = \square(\square x \cdot \square y)$.

For unital quantale, we require that $\square e = e$.

Note that, we may replace the last condition on equivalent condition $\square(x) \cdot \square(y) \leqslant \square(x \cdot y)$.

Definition 9.

We define a partial order on open modalities on $\mathcal{Q}$ as $\square_1 \leqslant \square_2 \Leftrightarrow \forall a \in Q, \square_1(a) \leqslant \square_2(a)$.

Lemma 1. $\square_1 a_1 \cdot \square_2 a_2 \leqslant \square(\square_1 a_1 \cdot \square_2 a_2)$, where $\square_i \leqslant \square, i = 1, 2$.

Proof.

$\square_1 a_1 \cdot \square_2 a_2 \leqslant$
$\square_1(\square_1 a_1) \cdot \square_2(\square_2 a_2) \leqslant$
$\square(\square_1 a_1) \cdot \square(\square_2 a_2) \leqslant$ $\qquad\square$
$\square(\square_1(a_1) \cdot \square_2(a_2))$

Definition 10.

1. An open modality is called central, if for all $a, b \in \mathcal{Q}, \square a \cdot b = b \cdot \square a$.

2. An open modality is called weak square-increasing, if for all $a, b \in \mathcal{Q}, \square a \cdot b \leqslant \square a \cdot b \cdot \square a$ and $b \cdot \square a \leqslant \square a \cdot b \cdot \square a$.

3. An open modality is called unital, if $\forall a \in Q, \Box a \leqslant e$.

Lemma 2.

Let $\Box$ be an open modality on some unital quantale $\mathcal{Q} = \langle A, \bigvee, \cdot, e \rangle$. Then, if $\Box$ is unital and weak idempotent, then $\Box$ is central.

Proof.

$$b \cdot \Box a \leqslant$$
  Right weak square-increase
$$\Box a \cdot b \cdot \Box a \leqslant$$
  Unitality
$$\Box a \cdot b \cdot e \leqslant$$
  Identity
$$\Box a \cdot b \leqslant$$
  Left weak square-increase
$$\Box a \cdot b \cdot \Box a \leqslant$$
  Unitality
$$e \cdot b \cdot \Box a \leqslant$$
  Identity
$$b \cdot \Box a$$

Hence, $b \cdot \Box a = \Box a \cdot b$, so for all $a \in \mathcal{Q}, \Box a \in \mathcal{Z}(\mathcal{Q})$.

$\square$

Proposition 2.

Let $\mathcal{Q}$ be a quantale and $S \subseteq \mathcal{Q}$ a subquantale, then $\Box : \mathcal{Q} \to \mathcal{Q}$, such that $\Box a = \bigvee \{ s \in S \,|\, s \leqslant a \}$, is an open modality.

Proof. See Rosenthal. $\square$

Proposition 3.

Let $\mathcal{Q}$ be a quantale and $\mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \mathcal{Q}$.
Then $\Box_{\mathcal{S}_1}(a) \leqslant \Box_{\mathcal{S}_1}(a)$.

Proof. Immediatly. $\square$

Proposition 4.

Let $\mathcal{Q}$ be a quantale and $\mathcal{S} \subseteq \mathcal{Q}$ a subquantale, then the following operations are open modalities:

1. $\Box_z(a) = \bigvee \{ s \in S \,|\, s \leqslant a, s \in \mathcal{Z}(\mathcal{Q}) \}$;

2. $\Box_{\mathbb{1}}(a) = \bigvee \{ s \in S \,|\, s \leqslant a, s \leqslant \mathbb{1} \}$;

3. $\Box_{idem}(a) = \bigvee \{ s \in S \,|\, s \leqslant a, \forall b \in Q, b \cdot s \vee s \cdot b \leqslant s \cdot b \cdot s \}$;

4. *Operations with combinations of conditions above;*

Proof. Immediatly. $\square$

Proposition 5.

1. $\forall a \in \mathcal{Q}, \Box_{\mathbb{1}, idem}(a) \leqslant \Box_z(a)$.

2. $\forall a \in \mathcal{Q}, \Box_{z, \mathbb{1}, idem} = \Box_{\mathbb{1}, idem}(a)$

Proof. Follows from Lemma 3. □

Definition 11. *Interpretation of subexponential signature*

Let $\Sigma = \langle I, \preceq, W, C, E \rangle$ be a subexponential signature, where $|I| = n$ and $\Box_{\mathcal{Q}}$ is a category of open modalities on a quantale $\mathcal{Q}$. Subexponential interpretation is a contravariant functor $\sigma : I \to \Box_{\mathcal{Q}}$ defined as follows:

$$\sigma(s_i) = \begin{cases} \Box_i : \mathcal{Q} \to \mathcal{Q}, \ s.t. \forall a \in Q, \Box_i(a) = \{s \in S_i \mid s \leqslant a\}, \\ \quad \text{if } s_i \notin W \cap C \cap E \\ \Box_i : \mathcal{Q} \to \mathcal{Q}, \ s.t. \forall a \in Q, \Box_i(a) = \{s \in S_i \mid s \leqslant a, s \leqslant \mathbb{1}\}, \\ \quad \text{if } s_i \in W \\ \Box_i : \mathcal{Q} \to \mathcal{Q}, \ s.t. \forall a \in Q, \Box_i(a) = \{s \in S_i \mid s \leqslant a, s \in \mathcal{Z}(\mathcal{Q})\}, \\ \quad \text{if } s_i \in E \\ \Box_i : \mathcal{Q} \to \mathcal{Q}, \ s.t. \forall a \in Q, \Box_i(a) = \{s \in S_i \mid s \leqslant a, \forall b, b \cdot s \vee s \cdot b \leqslant s \cdot b \cdot s\}, \\ \quad \text{if } s_i \in E \\ \text{otherwise, if } s_i \text{ belongs to some intersection of subsets, then we combine the relevant conditions} \end{cases}$$

Definition 12. Let $\mathcal{Q}$ be an unital quantale, $f : Tp \to \mathcal{Q}$ a valuation and $\sigma : I \to \Box_{\mathcal{Q}}$ a subexponential interpretation, then interpretation is defined inductively:

$$\begin{aligned} [\![p_i]\!] &= f(p_i) \\ [\![\mathbb{1}]\!] &= e \\ [\![A \bullet B]\!] &= [\![A]\!] \cdot [\![B]\!] \\ [\![A \backslash B]\!] &= [\![A]\!] \backslash [\![B]\!] \\ [\![A / B]\!] &= [\![A]\!] / [\![B]\!] \\ [\![A \& B]\!] &= [\![A]\!] \wedge [\![B]\!] \\ [\![A \vee B]\!] &= [\![A]\!] \vee [\![B]\!] \\ [\![!_{s_i} A]\!] &= \sigma(s_i)[\![A]\!] \end{aligned}$$

Definition 13. $\Gamma \models A \Leftrightarrow \forall f, \forall \sigma, [\![\Gamma]\!] \leqslant [\![A]\!]$

Theorem 2. $\Gamma \to A \Rightarrow [\![\Gamma]\!] \leqslant [\![A]\!]$

Proof. We consider the promotion case, the rest modal cases are immediatly shown.

Let $!_{s_1} A_1, \ldots, !_{s_n} A_n \to A$ and $\forall i, s \preceq s_i$.

Then $\forall a \in Q, \sigma(s_i)(a) \leqslant \sigma(s)(a)$.

By IH, $\sigma(s_1)[\![A_1]\!] \cdot \ldots \cdot \sigma(s_n)[\![A_n]\!] \leqslant [\![A]\!]$.

Thus, $\sigma(s)(\sigma(s_1)[\![A_1]\!] \cdot \ldots \cdot \sigma(s_n)[\![A_n]\!]) \leqslant \sigma(s)([\![A]\!])$.

By Lemma 5, $\sigma(s_1)[\![A_1]\!] \cdot \ldots \cdot \sigma(s_n)[\![A_n]\!] \leqslant \sigma(s)(\sigma(s_1)[\![A_1]\!] \cdot \ldots \cdot \sigma(s_n)[\![A_n]\!])$.

So, $\sigma(s_1)[\![A_1]\!] \cdot \ldots \cdot \sigma(s_n)[\![A_n]\!] \leqslant \sigma(s)([\![A]\!])$. □

3 Quantale completeness

Definition 14.

Let $\mathcal{F} \subseteq Fm$, an ideal is a subset $\mathcal{I} \subseteq \mathcal{F}$, such that:

- If $B \in \mathcal{I}$ and $A \to B$, then $A \in \mathcal{I}$;

- If $A, B \in \mathcal{I}$, then $A \vee B \in \mathcal{I}$.

Definition 15.

Let $S \subseteq \mathcal{F} \subseteq Fm$, then $\bigvee S = \bigcap \{\mathcal{I} \subseteq \mathcal{F} \mid S \subseteq \mathcal{I}\}$

The following conditions hold similarly to [**?**]:

Lemma 3.

1. $\bigvee S$ *is an ideal;*

2. $A \subseteq Fm$, *then* $\{B \mid B \to A\} = \bigvee\{A\};$

3. $\bigvee\{A\} \subseteq \bigvee\{B\}$ *iff* $A \to B;$

4. *Let* $\mathcal{Q} = \{\bigvee S \mid S \subseteq Fm\}$ *and* $\bigvee \mathcal{A} \cdot \bigvee \mathcal{B} = \bigvee\{A \bullet B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$. *Then* $\langle \mathcal{Q}, \subseteq, \cdot, \bigvee \mathbf{1} \rangle$ *is a unital quantale.*

We extend this construction for polymodal case as follows:

Lemma 4. *Let* $!_s \in I$ *and* $A \mathcal{F}_\Sigma$, *then* $\square_s(\bigvee\{A\}) = \bigvee\{!_s B \mid !_s B \to A\}$ *is a quantic conucleus.*

Proof.
See Yetter. $\qquad\square$

Lemma 5. *Let* $A \in \mathcal{F}_\Sigma$, *then* $\square_s \bigvee\{A\} = \bigvee\{!_s A\}$, *for each* $s \in \mathcal{I}$.

Proof. Let $A \in Fm$ and $s \in \mathcal{I}$.

Let $!_s B \in \square_s \bigvee\{A\}$, then $!_s B \to A$, then $!_s B \to !_s A$ by promotion. So, $!_s B \in \bigvee\{!_s A\}$.

Let $C \in \bigvee\{!_s A\}$, then $C \to !_s A$, so $!_s C \to !_s A$ by dereliction, but $!_s A \to A$, hence $!_s C \to A$ by cut. So, $!_s C \in \square_s \bigvee\{A\}$. $\qquad\square$

Lemma 6.

Let $i, j \in I$ *and* $i \leq j$, *then for all* $A \in \mathcal{F}_\Sigma$, $\square_j(\bigvee\{A\}) \subseteq \square_i(\bigvee\{A\})$.

Proof.

Let $i, j \in I$ and $i \leq j$, then forall $A \in \mathcal{F}_\Sigma$, $!_j A \to !_i A$ by promotion. Then $\bigvee\{!_j A\} \subseteq \bigvee\{!_i A\}$, so $\square_j(\bigvee\{A\}) \subseteq \square_i(\bigvee\{A\})$. $\qquad\square$

Lemma 7.

For all $A \in \mathcal{F}_\Sigma$,

1. *Let* $s \in W$, *then* $\square_s\{A\} \subseteq \{\mathbf{1}\};$

2. *Let* $s \in E$, *then* $\square_s(\bigvee\{A\}) \cdot \bigvee\{B\} = \bigvee\{B\} \cdot \square_s(\bigvee\{A\})$.

3. *Let* $s \in C$, *then* $(\square_s \bigvee A \cdot \bigvee B) \cup (\bigvee B \cdot \square_s \bigvee A) \subseteq \square_s \bigvee A \cdot \bigvee B \cdot \square_s \bigvee A$, *for all* $B \subseteq Fm$.

Proof.

1. Follows from $!_s A \to \mathbf{1}$, so $s \in W$;

2. Follows from $!_s A \bullet B \leftrightarrow B \bullet !_s A$;

3. Follows from $!_s A \bullet B \to !_s A \bullet B \bullet !_s A$ and similarly for $B \bullet !_s A$.

$\qquad\square$

Definition 16.

Let $\mathcal{Q}$ be a syntactic quantale as proposed above and $\mathcal{I} = \langle I, \preceq, W, C, E \rangle$ be a subexponential signature.

We define a map $\Box : \mathcal{I} \to Mod_{\mathcal{Q}}$ as follows:

$\Box(i)(\bigvee\{A\}) = \{!_i B \mid !_i B \to A\}.$

Lemma 8. $\Box$ is a subexponential interpretation.

Proof. Follows from lemmas above. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Lemma 9.

Let Q be a quantale constructed above and $(\Box_{s_i})_{s_i \in \Sigma}$ be a family of quantic conuclei on Q. Then there exist a model $\langle Q, [\![.]\!] \rangle$, such that $[\![A]\!] = \bigvee\{A\}$, $A \in Fm$.

Proof.

We define an interpreation as follows:

1. $[\![p_i]\!] = \bigvee\{p_i\}$

2. $[\![\mathbf{1}]\!] = \bigvee\{\mathbf{1}\}$

3. $[\![A \bullet B]\!] = \bigvee\{A \bullet B\}$

4. $[\![A/B]\!] = \bigvee\{A/B\}$

5. $[\![B\backslash A]\!] = \bigvee\{B\backslash A\}$

6. $[\![A \& B]\!] = \bigvee\{A \& B\}$

7. $[\![A \vee B]\!] = \bigvee\{A \vee B\}$

8. $[\![!_s A]\!] = \Box(s)(\bigvee\{A\}) = \bigvee\{!_s A\}.$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Theorem 3. $\Gamma \models A \Rightarrow \Gamma \to A.$

Proof. Follows from lemmas above. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

4 Relational semantics

Definition 17.

Let A be a set. Then relational quantale on A is a triple $\mathcal{Q} = \langle \mathcal{A}, \bigvee, \mathcal{I} \rangle$ where $\mathcal{A} \subseteq 2^{A \times A}$:

- $\langle \mathcal{A}, \bigvee, \subseteq \rangle$ is a complete semi-lattice;

- Multiplication is defined as $R \circ S = \{\langle a, c \rangle \mid \exists b \in A, \langle a, b \rangle \in R \text{ and } \langle b, c \rangle \in S\}$

- $\langle \mathcal{A}, \circ, \mathcal{I} \rangle$ is a monoid;

- For each indexing set J, $R \circ \bigvee_{i \in I} S_i = \bigvee_{i \in I}(R \circ S_i)$ and $\bigvee_{i \in I} R_i \circ S = \bigvee_{i \in I}(R_i \circ S)$.

Theorem 4.

Let $\mathcal{Q} = \langle A, \leqslant, \cdot, \bigvee \rangle$ be a unital quantale and $\mathcal{S}$ is a subquantale of $\mathcal{Q}$.

Then $\langle \mathcal{Q}, \Box_{\mathcal{S}} \rangle$ is isomorphic to some relational quantale of A wit some quantic conucleus $\hat{\Box}$.

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Proof.

Consider a relational quantale proposed by Brown and Gurr.

This quantale is 4-tuple $\theta(\mathcal{Q}) = \langle \mathcal{R}, \subseteq, \circ, \bigvee \rangle$ defined as follows:

1. $\theta(a) = \{\langle b, c \rangle \mid b \leqslant a \cdot c\}$;

2. $\theta(a \cdot b) = \theta(a) \circ \theta(b)$;

3. $\theta(\bigvee\limits_{i \in I} a_i) = \bigvee\limits_{i \in I} \theta(a_i)$;

4. $\theta(\varepsilon) = \{\langle b, c \rangle \mid b \cdot \varepsilon \leqslant c\} = \{\langle b, c \rangle \mid b \leqslant c\}$

Let $\mathcal{S} \subseteq \mathcal{Q}$, so $\square_{\mathcal{S}} a := \bigvee\{s \mid s \in S, s \leqslant a\}$ is quantic conucleus.

So, $\theta(S) \subseteq \theta(\mathcal{Q})$ is a subquantale of $\theta(\mathcal{Q})$.

Let us define $\hat{\square}\theta(a) := \bigvee\{\theta(s) \mid \theta(s) \in \theta(S), \theta(s) \subseteq \theta(a)\}$, so

$\theta(\square_{\mathcal{S}} a) = \{\langle p, q \rangle \mid p \leqslant \square_{\mathcal{S}} a \cdot q\} =$

$\{\langle p, q \rangle \mid p \leqslant \bigvee\{s \mid s \in \mathcal{S}, s \leqslant a\} \cdot q\} =$

Homomorphism

$\theta(\bigvee_{s \in S, s \leqslant a} s) =$

Homomorphism preserves sups

$\bigvee_{s \in S, s \leqslant a} \theta(s) =$

Unfolding

$\bigvee\{\theta(s) \mid s \in S, s \leqslant a\} =$

Unfolding

$\bigvee\{\theta(s) \mid \theta(s) \in \theta(S), \theta(s) \subseteq \theta(a)\} = \hat{\square}\theta(a)$

Thus, $\hat{\square}\theta(a) = \theta(\square_{\mathcal{S}} a)$.

$\square$

Theorem 5. *Let $\mathfrak{Q} = \langle \mathcal{Q}, I \rangle$ be a quantale with quantic conucleus. Then $\mathfrak{Q}$ is isomorphic to relational quantale on Q with some quantic conucleus.*

Proof. Let I be a quantic conucleus. Then $\mathcal{Q}_I = \{a \in \mathcal{Q} \mid I(a) = a\}$ is a subquantale of $\mathcal{Q}$.

Let us define $I'(a) = \bigvee\{s \in \mathcal{Q}_I \mid s \leqslant a\}$.

It is clear that I' is a quantic conucleus. Let us show that for all $a \in \mathcal{Q}$ $I(a) = I'(a)$.

$I'(a) = \bigvee\{s \in \mathcal{Q}_I \mid s \leqslant a\} =$

$\bigvee\{s \in \mathcal{Q}_I \mid I(s) \leqslant a\} =$

$\bigvee\{s \in \mathcal{Q}_I \mid I(s) \leqslant I(a)\} \leqslant I(a)$

It is clear that, $I(a) \leqslant a$. So $I(a) \leqslant \bigvee\{s \in \mathcal{Q}_I \mid s \leqslant a\} = I'(a)$.

Thus, we showed that $I(a) = I'(a)$. One need to apply the previous theorem with I'. $\square$

5 Syntactic concept lattices

Definition 18. *Let $\mathcal{L}$ be a finite alphabet and $L \subseteq \mathcal{L}^*$ be a language.*

We define maps $[.]^{\rhd} : \mathcal{P}(\mathcal{L}^*) \to \mathcal{P}(\mathcal{L}^* \times \mathcal{L}^*)$ and $[.]^{\lhd} : \mathcal{P}(\mathcal{L}^* \times \mathcal{L}^*) \to \mathcal{P}(\mathcal{L}^*)$ as follows:

1. *$M \subseteq \mathcal{L}^*$, $M^{\rhd} = \{(x, y) \mid \forall w \in M, xwy \in L\}$;*

2. *$C \subseteq \mathcal{L}^* \times \mathcal{L}^*$, $C^{\lhd} = \{w \mid \forall (x, y) \in C, xwy \in L\}$*

Note that compositions $[.]^{\lhd\rhd}$ and $[.]^{\rhd\lhd}$ form closure operators, by the way $[.]^{\lhd}$ and $[.]^{\rhd}$ are connected via contravariant Galois connection.

Definition 19. *A syntactic concept is a pair $\langle S, C \rangle$, where $S \subseteq \mathcal{L}^*$ and $C \subseteq \mathcal{L}^* \times \mathcal{L}^*$, such that $S^{\triangleright} = C$ and $C^{\triangleleft} = S$.*

Following to Wurm, by the concept we mean a closed set of strings, that is, A is a concept iff $A \triangleright \triangleleft = A$. Moreover, $\langle \mathcal{B}_{\mathcal{L}}, \bigvee, \bigwedge \rangle$, where $\mathcal{B}_{\mathcal{L}}$ is the set of $\bowtie$-closed subsets of $\mathcal{L}^*$.

We define a product of concepts as $A \circ B = (A \cdot B)^{\bowtie} = \{ab \mid a \in A, b \in B\}^{\bowtie}$.

Residuals are defined explicitly as follows:

$$A \backslash B = \{(aB, b) \mid (a, b) \in A^{\triangleright}\}^{\triangleleft}$$
$$B / A = \{(a, Bb) \mid (a, b) \in A^{\triangleright}\}^{\triangleleft}$$

It is easy to see, that the following condition hold for that residuals:

$$A \backslash B = \bigvee \{C \mid A \circ C \leqslant B\}$$
$$B / A = \bigvee \{C \mid C \circ A \leqslant B\}$$

Definition 20. *The syntax concept lattice of a language L is a structure $\langle \mathcal{B}_L, \circ, \bigvee, \bigwedge, /, \backslash \rangle$, where $\mathcal{B}_L$ is the set of $[.]^{\bowtie}$-closed subsets of $\mathcal{L}^*$.*

6 Thanks