# Categorical model of noncommutative linear logic with subexponentials

**Definition 1.** *A subexponential signature is an ordered quintuple:*
$\Sigma = \langle I, \preceq, W, C, E \rangle$,

where $I = \{s_1, \ldots, s_n\}$, $\langle I, \preceq \rangle$ is a preorder. $W, C, E$ are subsets of $I$ and $W \cup C \subseteq E$.

**Definition 2.** *Noncommutative linear logic with subexponentials ($SMALC_\Sigma$), where $\Sigma$ is a subexponential signature.*

$$\frac{}{A \Rightarrow A}\; ax$$

$$\frac{\Gamma \Rightarrow A \qquad \Delta, B, \Theta \Rightarrow C}{\Delta, \Gamma, A \backslash B, \Theta \Rightarrow C} \;\backslash \rightarrow \qquad \frac{A, \Pi \Rightarrow B}{\Pi \Rightarrow A \backslash B} \rightarrow \backslash$$

$$\frac{\Gamma \Rightarrow A \qquad \Delta, B, \Theta \Rightarrow C}{\Delta, B / A, \Gamma, \Theta \Rightarrow C} \;/ \rightarrow \qquad \frac{\Pi, A \Rightarrow B}{\Pi \Rightarrow B / A} \rightarrow /$$

$$\frac{\Gamma, A, B, \Delta \Rightarrow C}{\Gamma, A \bullet B, \Delta \Rightarrow C} \bullet \rightarrow \qquad \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \bullet B} \rightarrow \bullet$$

$$\frac{\Gamma, A_i, \Delta \Rightarrow B}{\Gamma, A_1 \& A_2, \Delta \Rightarrow B} \;\&, i = 1, 2 \rightarrow \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \& B} \rightarrow \&$$

$$\frac{\Gamma, A, \Delta \Rightarrow C \qquad \Gamma, B, \Delta \Rightarrow C}{\Gamma, A \vee B, \Delta \Rightarrow C} \vee \rightarrow \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2} \rightarrow \vee, i = 1, 2$$

$$\frac{\Gamma, \Delta \Rightarrow A}{\Gamma, \mathbf{1}, \Delta \Rightarrow A} \;\mathbf{1} \rightarrow \qquad \frac{}{\Rightarrow \mathbf{1}} \rightarrow \mathbf{1}$$

$$\frac{\Gamma, A, \Delta \Rightarrow C}{\Gamma, !^s A, \Delta \Rightarrow C} \;! \rightarrow \qquad \frac{!^{s_1} A_1, \ldots, !^{s_n} A_n \Rightarrow A}{!^{s_1} A_1, \ldots, !^{s_n} A_n \Rightarrow !^s A} \rightarrow !, \forall j, s_j \succeq s$$

$$\frac{\Gamma, \Delta \Rightarrow B}{\Gamma, !^s A, \Delta \Rightarrow B} \;\mathbf{weak}_!, s \in C$$

$$\frac{\Gamma, !^s A, \Delta, !^s A, \Theta \Rightarrow B}{\Gamma, !^s A, \Delta, \Theta \Rightarrow B} \;\mathbf{ncontr}_1, s \in C$$

$$\frac{\Gamma, !^s A, \Delta, !^s A, \Theta \Rightarrow B}{\Gamma, \Delta, !^s A, \Theta \Rightarrow B} \;\mathbf{ncontr}_2, s \in C$$

$$\frac{\Gamma, \Delta, !^s A, \Theta \Rightarrow B}{\Gamma, !^s A, \Delta, \Theta \Rightarrow A} \, \mathbf{ex}_1, s \in E$$

$$\frac{\Gamma, !^s A, \Delta, \Theta \Rightarrow B}{\Gamma, \Delta, !^s A, \Theta \Rightarrow A} \, \mathbf{ex}_1, s \in E$$

**Definition 3.** *Monoidal comonad*

*A monoidal comonad on some monoidal category $\mathcal{C}$ is a triple $\langle \mathcal{F}, \epsilon, \delta \rangle$, where $\mathcal{F}$ is a monoidal endofunctor and $\epsilon : \mathcal{F} \Rightarrow Id_{\mathcal{C}}$ (counit) and $\epsilon : \mathcal{F} \Rightarrow \mathcal{F}^2$ (comultiplication), such that the following diagrams commute:*

$$\begin{tikzcd}
\mathcal{F}A \otimes \mathcal{F}B \arrow[r, "\phi_{A,B}"] \arrow[dd, "\delta_A \otimes \delta_B"'] & \mathcal{F}(A \otimes B) \arrow[dr, "\delta_{A \otimes B}"] & \\
& & \mathcal{F}\mathcal{F}(A \otimes B) \\
\mathcal{F}\mathcal{F}A \otimes \mathcal{F}\mathcal{F}B \arrow[r, "\phi_{\mathcal{F}A, \mathcal{F}B}"'] & \mathcal{F}(\mathcal{F}A \otimes \mathcal{F}B) \arrow[ur, "\mathcal{F}(\phi_{A,B})"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}A \otimes \mathcal{F}B \arrow[rr, "\phi_{A,B}"] \arrow[dr, "\epsilon_A \otimes \epsilon_B"'] & & \mathcal{F}(A \otimes B) \arrow[dl, "\epsilon_{A \otimes B}"] \\
& A \otimes B &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "\phi"] \arrow[d, "\phi"'] & \mathcal{F}\mathbb{1} \arrow[d, "\delta_{\mathbb{1}}"] \\
\mathcal{F}\mathbb{1} \arrow[r, "\mathcal{F}(\phi)"'] & \mathcal{F}\mathcal{F}\mathbb{1}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{1} \arrow[rr, "id_{\mathbb{1}}"] \arrow[dr, "\phi"'] & & \mathbb{1} \\
& \mathcal{F}\mathbb{1} \arrow[ur, "\epsilon_{\mathbb{1}}"'] &
\end{tikzcd}$$

**Definition 4.** *Biclosed monoidal category*

*Let $\mathcal{C}$ be a monoidal category. Biclosed monoidal category is a monoidal category with the following additional data:*

1. *Bifunctors $\_\_ \multimapinv \_\_, \_\_ \multimap \_\_ : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$;*
2. *Natural isomophism $\mathbf{curry}_{A,B,C} : Hom(A \otimes B, C) \cong (B, A \multimap C)$;*
3. *Natural isomophism $\mathbf{curry}'_{A,B,C} : Hom(A \otimes B, C) \cong (A, C \multimap B)$;*
4. *For each $A, B \in Ob_{\mathcal{C}}$, there are exist arrows $ev_{A,B} : A \otimes (A \Rightarrow B) \to B$ and $ev'_{A,B} : (B \Leftarrow A) \otimes A \to B$, such that for all $f : A \otimes C \to B$:*
   (a) *$\Lambda_l \circ (id_A \otimes \mathbf{curry}(f)) = f$;*
   (b) *$\Lambda_r \circ (\mathbf{curry}'(f) \otimes id_A) = f$*

**Definition 5.** *Let $F$ be endofunctor and $A \in Ob\mathcal{C}$, then a coalgebra of $F$ is a tuple $\langle A, \theta \rangle$, where $\theta : A \to FA$.*

Given coalgebras $\langle A, \theta \rangle$ and $\langle A, \psi \rangle$, a homomorphism is a morphism $f : A \to B$, s.t. the diagram below commutes:

$$
\begin{tikzcd}
A \arrow[r, "\theta"] \arrow[d, "f"'] & FA \arrow[d, "Ff"] \\
B \arrow[r, "\psi"'] & FB
\end{tikzcd}
$$

that is, $Ff \circ \theta = \psi \circ f$

**Definition 6.** *Subexponential model structure*

*Let $\Sigma = \langle I, \preceq, W, C, E \rangle$ be a subexponential model structure and $\mathcal{C}$ be a biclosed monoidal category, then a subexponential model structure is $\langle \mathcal{C}, \{\mathcal{F}_s\}_{s \in I} \rangle$ with the following additional data:*

- *for all $s \in I$, $\mathcal{F}_s$ is a monoidal comonad;*
- *if $s \in W$, then for all $A \in Ob(\mathcal{C})$, there exists a morphism $w_{A_s} : F_s A \to \mathbb{1}$;*
- *if $s \in C$, then for all $A \in Ob(\mathcal{C})$, there exists morphisms $w_{A_l} : F_s A \otimes B \to F_s A \otimes A \otimes F_s A$ and $w_{A_r} : B \otimes F_s A \to F_s A \otimes A \otimes F_s A$;*
- *if $s \in E$, then for all $A \in Ob(\mathcal{C})$, there is an isomorpism, $e_A : F_s A \otimes B \cong B \otimes F_s A$;*
- *if $s_1 \in W$, $s_2 \in I$ and $s_1 \preceq s_2$, then there is a morphism $w_{A_{s_2}} : F_{s_2} A \to \mathbb{1}$ for all $A \in Ob(\mathcal{C})$ and ditto for $E$ and $C$;*
- *Let $\bigotimes_{s \in J, i=0}^{n} F_s A$, where $J \subset I$, and $s^{'} \in I$, s.t. $s \geq s^{'}$ for all $s \in I'$; Then there exists morphism a morphism $\theta_{\bigotimes_{s \in J, i=1}^{n} F_{sj} A_i} : \bigotimes_{s \in J, i=0}^{n} F_s A \to F_{s'}(\bigotimes_{s \in J, i=0}^{n} F_s A)$, such that $\langle \bigotimes_{s \in J, i=1}^{n} F_{sj} A_i, \theta_{\bigotimes_{s \in J, i=1}^{n} F_{sj} A_i} \rangle$ is a coalgebra on $F_s$.*

**Theorem 1.** *The following statements are equivalent:*

- $SMLC_{\Sigma} + (cut) \vdash \Gamma \Rightarrow A$
- $SMLC_{\Sigma} \vdash \Gamma \Rightarrow A$
- $\exists f, f : [\![\Gamma]\!] \to [\![A]\!]$

  *Proof.*
  (1) $\Rightarrow$ (2): cut elimination.

- (2) $\Rightarrow$ (3): Soundness:

$$
\frac{}{id_A : A \to A}
$$

$$
\frac{f : \Gamma \to A \qquad g : \Delta \otimes B \otimes \Theta \to C}{g \circ (id_{\Delta} \otimes (ev_{A,B_l} \circ (f \otimes id_{A \multimap B})) \otimes id_{\Theta}) : \Delta \otimes (\Gamma \otimes A \multimap B) \otimes \Theta \to C}
$$

$$
\frac{f : A \otimes \Pi \to B}{\Lambda_l(f) : \Pi \to A \multimap B}
$$

$$\frac{f : \Gamma \to A \qquad g : \Delta \otimes B \otimes \Theta \to C}{g \circ (id_\Delta \otimes (ev_{A,B_l} \circ (id_{B \circ\!- A} \otimes f)) \otimes id_\Theta) : \Delta \otimes (B \circ\!- A \otimes \Gamma) \otimes \Theta \to C}$$

$$\frac{f : \Pi \otimes A \to B}{\Lambda_r(f) : \Pi \to B \circ\!- A}$$

$$\frac{f : \Gamma \otimes A \otimes B \otimes \Delta \to C}{f \circ (\alpha_{\Gamma,A,B} \otimes id_\Delta) : \Gamma \otimes (A \otimes B) \otimes \Delta \to C}$$

$$\frac{f : \Gamma \to A \qquad g : \Delta \to B}{f \otimes g : \Gamma \otimes \Delta \to A \otimes B}$$

$$\frac{f : \Gamma \otimes A_i \otimes \Delta \to B}{f \circ (id_\Gamma \otimes \pi_i id_\Delta) : \Gamma \otimes (A_1 \times A_2) \otimes \Delta \to B}$$

$$\frac{f : \Gamma \to A \qquad g : \Gamma \to B}{\langle f, g \rangle : \Gamma \to A \times B}$$

$$\frac{f : \Gamma \otimes A \otimes \Delta \to C \qquad g : \Gamma \otimes B \otimes \Delta \to C}{id_\Gamma \otimes [f, g] \otimes id_\Delta : \Gamma \otimes (A + B) \otimes \Delta \to C}$$

$$\frac{}{id_{\mathbb{1}} : \mathbb{1} \to \mathbb{1}}$$

$$\frac{f : \Gamma \otimes \Delta \to A}{f \circ (\rho_\Gamma \otimes id_\Delta) : (\Gamma \otimes \mathbb{1}) \otimes \Delta \to A}$$

$$\frac{f : \Gamma \otimes A \otimes \Delta \to B}{f \circ (id_\Gamma \otimes \delta_s^A \otimes id_\Delta) : \Gamma \otimes F_s A \otimes \Delta \to B}$$

$$\dfrac{\dfrac{f : F_{s_1} A_1 \otimes \cdots \otimes F_{s_n} A_n \to B}{F_s(f) : F_s(F_{s_1} A_1 \otimes \cdots \otimes F_{s_n} A_n) \to F_s B}}{F_s(f) \circ \theta_{\bigotimes_{s \in J, i=1}^{n} F_{sj} A_i} : F_{s_1} A_1 \otimes \cdots \otimes F_{s_n} A_n \to F_s B}$$

- Completeness:

□

# 1 Concrete model

**Definition 7.** *Quantale*