Models of Lambek calculus enriched with subexponentials

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Abstract

1 The Lambek Calculus with subexponentials

Definition 1. *A subexponential signature is an ordered quintuple:*
$\Sigma = \langle I, \preceq, W, C, E \rangle$,

where $I = \{s_1, \dots, s_n\}$, $\langle I, \preceq \rangle$ is a preorder. W, C, E are upwardly closed subsets of I and $W \cap C \subseteq E$.

Definition 2.
$\mathcal{F}_\Sigma ::= Tp \mid (\mathcal{F}_\Sigma / \mathcal{F}_\Sigma) \mid (\mathcal{F}_\Sigma \backslash \mathcal{F}_\Sigma) \mid (\mathcal{F}_\Sigma \bullet \mathcal{F}_\Sigma) \mid (\mathcal{F}_\Sigma \vee \mathcal{F}_\Sigma) \mid (\mathcal{F}_\Sigma \wedge \mathcal{F}_\Sigma) \mid !_s \mathcal{F}_\Sigma$

Definition 3. *Noncommutative linear logic with subexponentials ($SMALC_\Sigma$), where Σ is a subexponential signature.*

$$\frac{}{A \to A}\ ax$$

$$\frac{\Gamma \to A \qquad \Delta, B, \Theta \to C}{\Delta, \Gamma, A \backslash B, \Theta \to C}\ \backslash \to \qquad \frac{A, \Pi \to B}{\Pi \to A \backslash B}\ \to \backslash$$

$$\frac{\Gamma \to A \qquad \Delta, B, \Theta \to C}{\Delta, B / A, \Gamma, \Theta \to C}\ / \to \qquad \frac{\Pi, A \to B}{\Pi \to B / A}\ \to /$$

$$\frac{\Gamma, A, B, \Delta \to C}{\Gamma, A \bullet B, \Delta \to C}\ \bullet \to \qquad \frac{\Gamma \to A \qquad \Delta \to B}{\Gamma, \Delta \to A \bullet B}\ \to \bullet$$

$$\frac{\Gamma, A_i, \Delta \to B}{\Gamma, A_1 \& A_2, \Delta \to B}\ \&, i = 1,2 \to \qquad \frac{\Gamma \to A \qquad \Gamma \to B}{\Gamma \to A \& B}\ \to \&$$

$$\frac{\Gamma, A, \Delta \to C \qquad \Gamma, B, \Delta \to C}{\Gamma, A \vee B, \Delta \to C}\ \vee \to \qquad \frac{\Gamma \to A_i}{\Gamma \to A_1 \vee A_2}\ \to \vee, i = 1,2$$

$$\frac{\Gamma, \Delta \to A}{\Gamma, \mathbf{1}, \Delta \to A}\ \mathbf{1} \to \qquad \frac{}{\to \mathbf{1}}\ \to \mathbf{1}$$

$$\frac{\Gamma, A, \Delta \to C}{\Gamma, !^s A, \Delta \to C}\ ! \to \qquad \frac{!^{s_1} A_1, \dots, !^{s_n} A_n \to A}{!^{s_1} A_1, \dots, !^{s_n} A_n \to !^s A}\ \to !, \forall j, s_j \geq s$$

Structural rules:

$$\frac{\Gamma, !^s A, \Delta, !^s A, \Theta \to B}{\Gamma, !^s A, \Delta, \Theta \to B}\ \mathbf{ncontr}_1, s \in C \qquad \frac{\Gamma, !^s A, \Delta, !^s A, \Theta \to B}{\Gamma, \Delta, !^s A, \Theta \to B}\ \mathbf{ncontr}_2, s \in C$$

$$\frac{\Gamma, \Delta, !^s A, \Theta \to B}{\Gamma, !^s A, \Delta, \Theta \to A}\ \mathbf{ex}_1, s \in E \qquad \frac{\Gamma, !^s A, \Delta, \Theta \to B}{\Gamma, \Delta, !^s A, \Theta \to A}\ \mathbf{ex}_2, s \in E$$

$$\frac{\Gamma, \Delta \to B}{\Gamma, !^s A, \Delta \to B}\ \mathbf{weak}_!, s \in C \qquad \frac{\Gamma \to A \qquad \Pi, A, \Delta \to B}{\Gamma, \Pi, \Delta \to B}\ \mathbf{cut}$$

Theorem 1.

1. *Cut-rule is admissable;*
2. *$SMALC_\Sigma$ is undecidable, if $C \neq \varnothing$;*
3. *If C is empty, then the decidability problem of $SMALC_\Sigma$ belongs to PSPACE.*

2 Semantics

Definition 4. *Quantale*

A quantale is a triple $\langle A, \bigvee, \cdot\rangle$, such that $\langle A, \bigvee\rangle$ is a complete lattice and $\langle A, \cdot\rangle$ is a semigroup. A quanlate is called unital, if $\langle A, \cdot\rangle$ is a monoid.

Some example of quantales:

- Let A be a semigroup (monoid), then $\langle \mathcal{P}(A), \cdot, \subseteq\rangle$ is a free (unital) quantale.
- Let R be a ring and $Sub(R)$ be a set of additive subgroups of R. We define $A \cdot B$ as an additive subgroup generated by finite sums of products ab and order is defined by inclusion.
- Any locale is a quantale with $\cdot = \wedge$.

It is easy to see, that any (unital) quantale is a residual (monoid) semigroup. We define divisions as follows:

1. $a \backslash b = \bigvee\{c \mid a \cdot c \leqslant b\}$
2. $b / a = \bigvee\{c \mid c \cdot a \leqslant b\}$

Definition 5.

Let $\mathcal{Q} = \langle A, \bigvee, \cdot\rangle$ be a quantale. The center of a quantale is the set $\mathcal{Z}(\mathcal{Q}) = \{a \in A \mid \forall b \in A, a \cdot b = b \cdot a\}$

Definition 6. *An open modality (or quantic conucleus) on quantale $\mathcal{Q}$ is a map $\Box : \mathcal{Q} \to \mathcal{Q}$, such that*

1. *$\Box x \leqslant x$;*
2. *$\Box x = \Box\Box x$;*
3. *$x \leqslant y \Rightarrow \Box x \leqslant \Box y$;*
4. *$\Box x \cdot \Box y = \Box(\Box x \cdot \Box y)$.*

For unital quantale, we require that $\Box e = e$.

Note that, we may replace the last condition on equivalent condition $\Box(x) \cdot \Box(y) \leqslant \Box(x \cdot y)$.

Definition 7. *We define a partial order on open modalities on $\mathcal{Q}$ as $\Box_1 \leqslant \Box_2 \Leftrightarrow \forall a \in Q, \Box_1(a) \leqslant \Box_2(a)$.*

Lemma 1. *Let $\mathcal{Q}$ be a quantale and $\Box_{\mathcal{Q}}$ be a set of all open modalities on $\mathcal{Q}$. Then $\Box_{\mathcal{Q}}$ is a small category.*

Proof. $\langle \Box_{\mathcal{Q}}, \leqslant \rangle$ form a partial order, so $\langle \Box_{\mathcal{Q}}, \leqslant \rangle$ is a small category. □

Definition 8.

1. *An open modality is called central, if $\forall a, b \in Q, \Box(a) \cdot b = b \cdot \Box(a)$.*
2. *An open modality is called pseudo-idempotent, if $\forall a, b \in Q, \Box(a) \cdot b \leqslant \Box(a) \cdot b \cdot \Box(a)$ and $b \cdot \Box(a) \leqslant \Box(a) \cdot b \cdot \Box(a)$.*
3. *An open modality is called unital, if $\forall a \in Q, \Box(a) \leqslant e$.*

Lemma 2. *Let $\Box$ be an open modality on some unital quantale $\mathcal{Q} = \langle A, \bigvee, \cdot, e \rangle$. Then, if $\Box$ is unital and weak idempotent, then $\Box$ is central.*

Proof.

$b \cdot \Box(a) \leqslant$

Right weak idempotence

$\Box(a) \cdot b \cdot \Box(a) \leqslant$

Unitality

$\Box(a) \cdot b \cdot e \leqslant$

Identity

$\Box(a) \cdot b \leqslant$

Left weak idempotence

$\Box(a) \cdot b \cdot \Box(a) \leqslant$

Unitality

$e \cdot b \cdot \Box(a) \leqslant$

Identity

$b \cdot \Box(a)$

Hence, $b \cdot \Box(a) = \Box(a) \cdot b$, so $\forall a \in A, \Box(a) \in \mathcal{Z}(\mathcal{Q})$.

□

Proposition 1.

Let $\mathcal{Q}$ be a quantale and $S \subseteq \mathcal{Q}$ a subquantale, then $\Box : \mathcal{Q} \to \mathcal{Q}$, such that $\Box(a) = \bigvee\{s \in S \mid s \leqslant a\}$, is an open modality.

Proof. See □

Proposition 2.

Let $\mathcal{Q}$ be a quantale and $S_1 \subseteq S_2 \subseteq \mathcal{Q}$.

Then $\Box_1(a) \leqslant \Box_2(a)$.

Proof. Immediatly. □

Proposition 3.

Let $\mathcal{Q}$ be a quantale and $S \subseteq \mathcal{Q}$ a subquantale, then the following operations are open modalities:

1. $\Box_z(a) = \bigvee\{s \in S \,|s \leqslant a, s \in \mathcal{Z}(\mathcal{Q})\}$;

2. $\Box_{\mathbb{1}}(a) = \bigvee\{s \in S \,|s \leqslant a, s \leqslant \mathbb{1}\}$;

3. $\Box_{idem}(a) = \bigvee\{s \in S \,|s \leqslant a, \forall b \in Q, b \cdot s \vee s \cdot b \leqslant s \cdot b \cdot s\}$;

4. $\Box_{z,\mathbb{1}}, I_{z,idem}, I_{\mathbb{1},idem}, I_{z,\mathbb{1},idem}$.

Proof. Immediatly. $\Box$

Proposition 4.

1. $\forall a \in \mathcal{Q}, \Box_{\mathbb{1},idem}(a) \leqslant \Box_z(a)$.

2. $\forall a \in \mathcal{Q}, \Box_{z,\mathbb{1},idem} = \Box_{\mathbb{1},idem}(a)$

Proof. Follows from Lemma 3. $\Box$

Lemma 3. $\Box_1(a_1) \cdot \Box_2(a_2) \leqslant \Box^{'}(\Box_1(a_1) \cdot \Box_2(a_2))$, *where* $\Box_i \leqslant \Box^{'}, i = 1, 2$.

Proof.

$$\begin{aligned}
&\Box_1(a_1) \cdot \Box_2(a_2) \leqslant \\
&\Box_1(\Box_1(a_1)) \cdot \Box_2(\Box_2(a_2)) \leqslant \\
&\Box^{'}(\Box_1(a_1)) \cdot \Box^{'}(\Box_2(a_2)) \leqslant \\
&\Box^{'}(\Box_1(a_1) \cdot \Box_2(a_2))
\end{aligned}$$

$\Box$

Definition 9. *Interpretation of subexponential signature*

Let $\Sigma = \langle I, \preceq, W, C, E\rangle$ *be a subexponential signature, where* $|I| = n$ *and* $\Box_{\mathcal{Q}}$ *is a category of open modalities on a quantale* $\mathcal{Q}$. *Subexponential interpretation is a contravariant functor* $\sigma : I \to \Box_{\mathcal{Q}}$ *defined as follows:*

$$\sigma(s_i) = \begin{cases}
\Box_i : \mathcal{Q} \to \mathcal{Q},\ s.t. \forall a \in Q, \Box_i(a) = \{s \in S_i \mid s \leqslant a\}, \\
\quad \text{if } s_i \notin W \cap C \cap E \\
\Box_i : \mathcal{Q} \to \mathcal{Q},\ s.t. \forall a \in Q, \Box_i(a) = \{s \in S_i \mid s \leqslant a, s \leqslant \mathbb{1}\}, \\
\quad \text{if } s_i \in W \\
\Box_i : \mathcal{Q} \to \mathcal{Q},\ s.t. \forall a \in Q, \Box_i(a) = \{s \in S_i \mid s \leqslant a, s \in \mathcal{Z}(\mathcal{Q})\}, \\
\quad \text{if } s_i \in E \\
\Box_i : \mathcal{Q} \to \mathcal{Q},\ s.t. \forall a \in Q, \Box_i(a) = \{s \in S_i \mid s \leqslant a, \forall b, b \cdot s \vee s \cdot b \leqslant s \cdot b \cdot s\}, \\
\quad \text{if } s_i \in E \\
\text{otherwise, if } s_i \text{ belongs to some intersection of subsets, then we combine the relevant conditions}
\end{cases}$$

Definition 10. *Let* $\mathcal{Q}$ *be an unital quantale,* $f : Tp \to \mathcal{Q}$ *a valuation and* $\sigma : I \to \Box_{\mathcal{Q}}$ *a subexponential interpretation, then interpretation is defined inductively:*

$$\begin{aligned}
&[\![p_i]\!] = f(p_i) \\
&[\![\mathbb{1}]\!] = e \\
&[\![A \bullet B]\!] = [\![A]\!] \cdot [\![B]\!] \\
&[\![A \backslash B]\!] = [\![A]\!] \backslash [\![B]\!] \\
&[\![A / B]\!] = [\![A]\!] / [\![B]\!] \\
&[\![A \& B]\!] = [\![A]\!] \wedge [\![B]\!] \\
&[\![A \vee B]\!] = [\![A]\!] \vee [\![B]\!] \\
&[\![!_{s_i} A]\!] = \sigma(s_i)[\![A]\!]
\end{aligned}$$

Definition 11. $\Gamma \models A \Leftrightarrow \forall f, \forall \sigma, [\![\Gamma]\!] \leqslant [\![A]\!]$

Theorem 2. $\Gamma \to A \Rightarrow [\![\Gamma]\!] \leqslant [\![A]\!]$

Proof. We consider the promotion case, the rest modal cases are immediatly shown.
Let $!_{s_1} A_1, \dots, !_{s_n} A_n \to A$ and $\forall i, s \preceq s_i$.
Then $\forall a \in Q, \sigma(s_i)(a) \leqslant \sigma(s)(a)$.
By IH, $\sigma(s_1)[\![A_1]\!] \cdot \dots \cdot \sigma(s_n)[\![A_n]\!] \leqslant [\![A]\!]$.
Thus, $\sigma(s)(\sigma(s_1)[\![A_1]\!] \cdot \dots \cdot \sigma(s_n)[\![A_n]\!]) \leqslant \sigma(s)([\![A]\!])$.
By Lemma 5, $\sigma(s_1)[\![A_1]\!] \cdot \dots \cdot \sigma(s_n)[\![A_n]\!] \leqslant \sigma(s)(\sigma(s_1)[\![A_1]\!] \cdot \dots \cdot \sigma(s_n)[\![A_n]\!])$.
So, $\sigma(s_1)[\![A_1]\!] \cdot \dots \cdot \sigma(s_n)[\![A_n]\!] \leqslant \sigma(s)([\![A]\!])$. □

3 Quantale completeness

Definition 12.
Let $\mathcal{F} \subseteq Fm$, an ideal is a subset $\mathcal{I} \subseteq \mathcal{F}$, such that:

- *If $B \in \mathcal{I}$ and $A \to B$, then $A \in \mathcal{I}$;*
- *If $A, B \in \mathcal{I}$, then $A \vee B \in \mathcal{I}$.*

Definition 13.
Let $S \subseteq \mathcal{F} \subseteq Fm$, then $\bigvee S = \bigcap\{\mathcal{I} \subseteq \mathcal{F} \mid S \subseteq \mathcal{I}\}$

Proposition 5. *$\bigvee S$ is an ideal.*

Lemma 4. *$A \subseteq Fm$, then $\{B \mid B \to A\} = \bigvee\{A\}$.*

Proof.
Let $A \in Fm$. Then $\{B \mid B \to A', A' \in A\} \subseteq \bigvee\{A\}$, so far as $\bigvee A$ is an ideal.
On the other hand, $\{B \mid B \to A\}$ is an ideal, it is easy to see that this set is closed under $\vee$.
So, $\bigvee A \subseteq \{B \mid B \to A\}$. □

Lemma 5. *$\bigvee\{A\} \subseteq \bigvee\{B\}$ iff $A \to B$.*

Proof. Let $\bigvee\{A\} \subseteq \bigvee\{B\}$, then $\{C | C \to A\} \subseteq \{D \mid D \to B\}$.
Thus, $A \in \{C | C \to A\}$, then $A \in \{D \mid D \to B\}$, hence $A \to B$.
On the other hand, let $A \to B$ and $C \in \bigvee\{A\}$.
Thus, $C \to A$, then $C \to B$ by cut.
□

Lemma 6. *Let $\mathcal{Q} = \{\bigvee S \mid S \subseteq Fm\}$ and $\bigvee \mathcal{A} \cdot \bigvee \mathcal{B} = \bigvee\{A \bullet B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$. Then $\langle \mathcal{Q}, \subseteq, \cdot, \bigvee \mathbf{1}\rangle$ is a quantale.*

Proof. See □

Lemma 7. *Let $!_s \in I$ and A be an arbitrary formula, then $\Box_s(\bigvee\{A\}) = \bigvee\{B \mid !_s B \to A\}$ is a quantic conucleus.*

Proof.
See Yetter. □

Lemma 8. *Let A be a formula, then $\Box_s \bigvee\{A\} = \bigvee\{!_s A\}$, for each $s \in \mathcal{I}$.*

Proof. Let $A \in Fm$ and $s \in \mathcal{I}$.

Let $!_s B \in \Box_s \bigvee\{A\}$, then $!_s B \to A$, then $!_s B \to !_s A$ by promotion. So, $!_s B \in \bigvee\{!_s A\}$.

Let $C \in \bigvee\{!_s A\}$, then $C \to !_s A$, so $!_s C \to !_s A$ by dereliction, but $!_s A \to A$, hence $!_s C \to A$ by cut. So, $!_s C \in \Box_s \bigvee\{A\}$. $\Box$

Lemma 9.

Let $i, j \in I$ and $i \preceq j$, then for all $A \in Fm$, $\Box_j(\bigvee\{A\}) \subseteq \Box_i(\bigvee\{A\})$.

Proof.

Let $i, j \in I$ and $i \preceq j$. Then forall $A \in Fm$, $!_j A \to !_i A$ by promotion. Then $\bigvee\{!_j A\} \subseteq \bigvee\{!_i A\}$, so $\Box_j(\bigvee\{A\}) \subseteq \Box_i(\bigvee\{A\})$. $\Box$

Lemma 10.

1. *Let $s \in W$, then for all $A \subseteq Fm$, $\Box_s\{A\} \subseteq \{\mathbf{1}\}$;*
2. *Let $s \in E$, then $\Box_s(\bigvee\{A\}) \cdot \bigvee\{B\} = \bigvee\{B\} \cdot \Box_s(\bigvee\{A\})$.*
3. *Let $s \in C$, then $(\Box_s \bigvee A \cdot \bigvee B) \cup (\bigvee B \cdot \Box_s \bigvee A) \subseteq \Box_s \bigvee A \cdot \bigvee B \cdot \Box_s \bigvee A$, for all $B \subseteq Fm$.*

Proof.

Follows from $!_s A \to \mathbf{1}$, so $s \in W$;

Follows from $!_s A \bullet B \leftrightarrow B \bullet !_s A$;

Follows from $!_s A \bullet B \to !_s A \bullet B \bullet !_s A$ and similarly for $B \bullet !_s A$. $\Box$

Definition 14.

Let Q be a syntactic quantale as proposed above and $\mathcal{I} = \langle I, \preceq, W, C, E\rangle$ be a subexponential signature.

We define a map $\Box : \mathcal{I} \to Mod_{\mathcal{Q}}$ as follows:

$\Box(i)(\bigvee\{A\}) = \{!_i B \mid !_i B \to A\}$.

Lemma 11. *$\Box$ is a subexponential interpretation.*

Proof. Follows from lemmas above. $\Box$

Lemma 12.

Let Q be a quantale constructed above and $\Box_1, \ldots, \Box_n$ be a family of quantic conuclei on Q. Then there exist a model $\langle Q, [\![.]\!]\rangle$, such that $[\![A]\!] = \bigvee\{A\}$, $A \in Fm$.

Proof.

We define an interpreation as follows:

1. $[\![p_i]\!] = \bigvee\{p_i\}$
2. $[\![\mathbf{1}]\!] = \bigvee\{\mathbf{1}\}$
3. $[\![A \bullet B]\!] = \bigvee\{A \bullet B\}$
4. $[\![A/B]\!] = \bigvee\{A/B\}$
5. $[\![B\backslash A]\!] = \bigvee\{B\backslash A\}$
6. $[\![A \& B]\!] = \bigvee\{A \& B\}$
7. $[\![A \vee B]\!] = \bigvee\{A \vee B\}$

8. $[\![!_s A]\!] = \Box(s)(\bigvee\{A\}) = \bigvee\{!_s A\}$.

$\square$

Theorem 3. $\Gamma \models A \Rightarrow \Gamma \to A$.

Proof. Follows from lemmas above. $\square$

Definition 15. *Monoidal comonad*

A monoidal comonad on some monoidal category $\mathcal{C}$ is a triple $\langle \mathcal{F}, \epsilon, \delta \rangle$, where $\mathcal{F}$ is a monoidal endofunctor and $\epsilon : \mathcal{F} \Rightarrow Id_{\mathcal{C}}$ (counit) and $\epsilon : \mathcal{F} \Rightarrow \mathcal{F}^2$ (comultiplication), such that the following diagrams commute:

$$\begin{tikzcd}
\mathcal{F}A \otimes \mathcal{F}B \arrow[r, "\phi_{A,B}"] \arrow[d, "\delta_A \otimes \delta_B"'] & \mathcal{F}(A \otimes B) \arrow[dr, "\delta_{A \otimes B}"] & \\
 & & \mathcal{F}\mathcal{F}(A \otimes B) \\
\mathcal{F}\mathcal{F}A \otimes \mathcal{F}\mathcal{F}B \arrow[r, "\phi_{\mathcal{F}A,\mathcal{F}B}"'] & \mathcal{F}(\mathcal{F}A \otimes \mathcal{F}B) \arrow[ur, "\mathcal{F}(\phi_{A,B})"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}A \otimes \mathcal{F}B \arrow[rr, "\phi_{A,B}"] \arrow[dr, "\epsilon_A \otimes \epsilon_B"'] & & \mathcal{F}(A \otimes B) \arrow[dl, "\epsilon_{A \otimes B}"] \\
 & A \otimes B &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "\phi"] \arrow[d, "\phi"'] & \mathcal{F}\mathbb{1} \arrow[d, "\delta_{\mathbb{1}}"] \\
\mathcal{F}\mathbb{1} \arrow[r, "\mathcal{F}(\phi)"'] & \mathcal{F}\mathcal{F}\mathbb{1}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{1} \arrow[rr, "id_{\mathbb{1}}"] \arrow[dr, "\phi"'] & & \mathbb{1} \\
 & \mathcal{F}\mathbb{1} \arrow[ur, "\epsilon_{\mathbb{1}}"'] &
\end{tikzcd}$$

Definition 16. *Biclosed monoidal category*

Let $\mathcal{C}$ be a monoidal category. Biclosed monoidal category is a monoidal category with the following additional data:

1. *Bifunctors* $\_\_ \multimapinv \_\_, \_\_ \multimap \_\_ : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$;
2. *Natural isomophism* $\mathbf{curry}_{A,B,C} : Hom(A \otimes B, C) \cong (B, A \multimap C)$;
3. *Natural isomophism* $\mathbf{curry}'_{A,B,C} : Hom(A \otimes B, C) \cong (A, C \multimap B)$;
4. *For each $A, B \in Ob_{\mathcal{C}}$, there are exist arrows $ev_{A,B} : A \otimes (A \Rightarrow B) \to B$ and $ev'_{A,B} : (B \Leftarrow A) \otimes A \to B$, such that for all $f : A \otimes C \to B$:*
   (a) $\Lambda_l \circ (id_A \otimes \mathbf{curry}(f)) = f$;
   (b) $\Lambda_r \circ (\mathbf{curry}'(f) \otimes id_A) = f$

Definition 17. *Let F be endofunctor and $A \in Ob\mathcal{C}$, then a coalgebra of F is a tuple $\langle A, \theta \rangle$, where $\theta : A \to FA$.*

Given coalgebras $\langle A, \theta\rangle$ and $\langle A, \psi\rangle$, a homomorphism is a morphism $f : A \to B$, s.t. the diagram below commutes:

$$\begin{tikzcd} A \arrow[r, "\theta"] \arrow[d, "f"'] & FA \arrow[d, "Ff"] \\ B \arrow[r, "\psi"'] & FB \end{tikzcd}$$

that is, $Ff \circ \theta = \psi \circ f$

Definition 18. *Subexponential model structure*

Let $\Sigma = \langle I, \preceq, W, C, E\rangle$ be a subexponential signature and $\mathcal{C}$ be a biclosed monoidal category, then a subexponential model structure is $\langle \mathcal{C}, \{\mathcal{F}_s\}_{s\in I}\rangle$ with the following additional data:

- *for all $s \in I$, $\mathcal{F}_s$ is a monoidal comonad;*
- *if $s \in W$, then for all $A \in Ob(\mathcal{C})$, there exists a morphism $w_{A_s} : F_s A \to \mathbb{1}$;*
- *if $s \in C$, then for all $A \in Ob(\mathcal{C})$, there exists morphisms $w_{A_l} : F_s A \otimes A \otimes F_s A \to F_s A \otimes B$ and $w_{A_r} : F_s A \otimes A \otimes F_s A \to B \otimes F_s A$;*
- *if $s \in E$, then for all $A \in Ob(\mathcal{C})$, there is an isomorpism, $e_A : F_s A \otimes B \cong B \otimes F_s A$;*
- *if $s_1 \in W$, $s_2 \in I$ and $s_1 \preceq s_2$, then there is a morphism $w_{A_{s_2}} : F_{s_2} A \to \mathbb{1}$ for all $A \in Ob(\mathcal{C})$ and ditto for E and C;*
- *Let $\bigotimes_{s\in J, i=0}^{n} F_s A$, where $J \subset I$, and $s^{'} \in I$, s.t. $s \geq s^{'}$ for all $s \in I'$; Then there exists morphism a morphism $\theta_{\bigotimes_{s\in J,i=1}^{n} F_{s\,j} A_i} : \bigotimes_{s\in J,i=0}^{n} F_s A \to F_{s'}(\bigotimes_{s\in J,i=0}^{n} F_s A)$, such that $\langle \bigotimes_{s\in J,i=1}^{n} F_{s\,j} A_i, \theta_{\bigotimes_{s\in J,i=1}^{n} F_{s\,j} A_i}\rangle$ is a coalgebra on F_s.*

Definition 19. *Let $\langle \mathcal{C}, \{\mathcal{F}_s\}_{s\in I}\rangle$ be a subexponential model structure for subexponential signature $\Sigma = \langle I, \preceq, W, C, E\rangle$. Let $v : Tp \to Ob(\mathcal{C})$ be a valuation map. Then the interpretation function $[\![.]\!]$ is defined as follows:*

(1) $[\![\mathbf{1}]\!] = \mathbb{1}$
(2) $[\![A \backslash B]\!] = [\![A]\!] \multimap [\![B]\!]$
(3) $[\![A/B]\!] = [\![A]\!] \multimapinv [\![B]\!]$
(4) $[\![A \bullet B]\!] = [\![A]\!] \otimes [\![B]\!]$
(5) $[\![!_s A]\!] = F_s[\![A]\!]$

Theorem 4. *The following statements are equivalent:*

- $SMLC_\Sigma + (cut) \vdash \Gamma \Rightarrow A$
- $SMLC_\Sigma \vdash \Gamma \Rightarrow A$
- $\exists f, f : [\![\Gamma]\!] \to [\![A]\!]$

  Proof.
  (1) $\Rightarrow$ (2): cut elimination.

- (2) $\Rightarrow$ (3): Soundness:

$$\frac{f : \Gamma \otimes A \otimes \Delta \to B}{f \circ (id_\Gamma \otimes \delta_s^A \otimes id_\Delta) : \Gamma \otimes F_s A \otimes \Delta \to B}$$

$$\cfrac{\cfrac{f : F_{s_1} A_1 \otimes \cdots \otimes F_{s_n} A_n \to B}{F_s(f) : F_s(F_{s_1} A_1 \otimes \cdots \otimes F_{s_n} A_n) \to F_s B}}{F_s(f) \circ \theta_{\bigotimes_{s \in J, i=1}^{n} F_{s j} A_i} : F_{s_1} A_1 \otimes \cdots \otimes F_{s_n} A_n \to F_s B}$$

$$\cfrac{\cfrac{f : \Gamma \otimes \Delta \to A}{f \circ (\rho_\Gamma \otimes id_\Delta) : (\Gamma \otimes \mathbb{1}) \otimes \Delta \to A}}{f \circ (\rho_\Gamma \otimes id_\Delta) \circ (id_\Gamma \otimes w_{A s}) \otimes id_\Delta : (\Gamma \otimes F_s A) \otimes \Delta \to A}$$

$$\frac{f : \Gamma \otimes (F_s A \otimes B \otimes F_s A) \otimes \Delta \to C}{f \circ (id_\Gamma \otimes c_{A s}^{\,l} \otimes id_\Delta) : \Gamma \otimes (F_s A \otimes B) \otimes \Delta \to C}$$

$$\frac{f : \Gamma \otimes (F_s A \otimes B \otimes F_s A) \otimes \Delta \to C}{(id_\Gamma \otimes c_{A s}^{\,r} \otimes id_\Delta) \circ f : \Gamma \otimes (B \otimes F_s A) \otimes \Delta \to C}$$

$$\frac{f : \Gamma \otimes (\Delta \otimes F_s A) \otimes \Theta \to B}{(id_\Gamma \otimes (id_\Delta \otimes e_{A s}) \otimes id_\Theta) \circ f : \Gamma \otimes (F_s A \otimes \Delta) \otimes \Theta \to B}$$

$$\frac{f : \Gamma \otimes (F_s A \otimes \Delta) \otimes \Theta \to B}{(id_\Gamma \otimes (id_\Delta \otimes e_{A s}^{-1}) \otimes id_\Theta) \circ f : \Gamma \otimes (\Delta \otimes F_s A) \otimes \Theta \to B}$$

- Completeness:

  Definition 20.

□

4 Relational semantics

Definition 21.
Let A be a set. Then relational quantale on A is a tuple $\mathcal{Q} = \langle \mathcal{A}, \mathcal{I} \rangle$ where $\mathcal{A} \subseteq 2^{A \times A}$:

- *$\langle \mathcal{A}, \subseteq \rangle$ is a complete semi-lattice;*
- *Multiplication is defined as $R \circ S = \{\langle a, c \rangle \mid \exists b \in A, \langle a, b \rangle \in R \text{ and } \langle b, c \rangle \in S\}$*
- *$\langle \mathcal{A}, \circ, \mathcal{I} \rangle$ is a monoid;*
- *For each indexing set J, $R \circ \bigvee_{j \in J} S_j = \bigvee_{j \in J} (R \circ S_j)$ and $\bigvee_{j \in J} R_j \circ S = \bigvee_{j \in J} (R_j \circ S)$.*

Let us define forcing relation $\Vdash$ between elements of $\mathcal{Q}$ and types as follows for arbitrary valuation map $w : Tp \to \mathcal{A}$:

1. Let $p_i \in Tp$, then $\langle a, b \rangle \Vdash p_i \Leftrightarrow \langle a, b \rangle \in w(p_i)$;
2. $\langle a, b \rangle \Vdash \mathbf{1} \Leftrightarrow \langle a, b \rangle \in \mathcal{I}$;

3. $\langle a,b\rangle \Vdash A\backslash B \Leftrightarrow \forall c \in A, \langle c,a\rangle \Vdash A \Rightarrow \langle c,b\rangle \Vdash A$;

4. $\langle a,b\rangle \Vdash B/A \Leftrightarrow \forall c \in A, \langle a,c\rangle \Vdash A \Rightarrow \langle b,c\rangle \Vdash A$;

5. $\langle a,b\rangle \Vdash A \bullet B \Leftrightarrow \exists c, \langle a,c\rangle \Vdash A$ and $\langle c,b\rangle \Vdash B$;

6. $\langle a,b\rangle \Vdash A \vee B \Leftrightarrow \langle a,b\rangle \Vdash A$ or $\langle a,b\rangle \Vdash B$;

7. $\langle a,b\rangle \Vdash A\&B \Leftrightarrow \langle a,b\rangle \Vdash A$ and $\langle a,b\rangle \Vdash B$;

8. $\langle a,b\rangle \Vdash !A \Leftrightarrow \exists S_{\langle a,b\rangle} \forall \langle a',b'\rangle \in S_{\langle a,b\rangle}, \langle a',b'\rangle \Vdash A$

9. $\langle a,b\rangle \Vdash \Gamma \to A \Leftrightarrow \langle a,b\rangle \Vdash A_1 \bullet \dots \bullet A_n \Rightarrow \langle a,b\rangle \vdash A$;

Proposition 6.
Let $\mathcal{R}$ be any R-model:

- $\mathcal{R} \models !A \to !!A$;
- $\mathcal{R} \models !A \to A$.

Proof.

1. Let us show that $\mathcal{Q} \Vdash !A \to A$.

   Let $\mathcal{Q} \Vdash !A$ and v be a valuation, hence $\langle a,b\rangle \Vdash !A$. Hence there exists $S_{\langle a,b\rangle} \subseteq \mathcal{Q}$, such that $\forall \langle a',b'\rangle \in S_{\langle a,b\rangle}, \langle a',b'\rangle \Vdash A$, but $\langle a,b\rangle \in S_{\langle a,b\rangle}$, so $\langle a,b\rangle \Vdash A$.

2. Let us show that $\mathcal{Q} \models !A \to !!A$.

   Let $\langle a,b\rangle \Vdash !A$. Let us show that $\langle a,b\rangle \Vdash !!A$.

   So there exists $S_{\langle a,b\rangle} \subseteq \mathcal{Q}$, such that $\forall \langle a',b'\rangle \in S_{\langle a,b\rangle}, \langle a',b'\rangle \Vdash A$.

   Let $S' \subseteq S_{\langle a,b\rangle}$, then for each $\langle a',b'\rangle \in S', \langle a',b'\rangle \Vdash A$, so $\forall \langle a',b'\rangle \in S_{\langle a,b\rangle}, \langle a',b'\rangle \models !A$, so $\langle a,b\rangle \models !!A$.

$\square$

Theorem 5. *Soundness*
Let $\mathcal{R}$ be a class of R-frames, then $Log(\mathcal{R}) = L_{\mathbf{1}}^{*}A_{\mathbf{S4}}$

Proof.
Modal cases follows from the previous lemma. $\square$

Theorem 6.
Let $W : Fm \to 2^R$, s.t. $W(A) = \{\langle a,b\rangle \mid \mathcal{F}, \langle a,b\rangle\}$, then:

1. *For each $A \in Fm$, $W(A)$ is a subrelation of R;*
2. *(a) $W(A\backslash B) = \bigcup\{\}$*
   (b) $W(A/B) =$
   (c) $W(A \circ B) =$
   (d) $W(\mathbf{1}) =$
   (e) $W(A \vee B) =$
   (f) $W(A \wedge B) =$

Definition 22. *Algebra generated by R-frame.*

Theorem 7.
Any relational quantale is isomorphic to some algebra generated by R-frame.

Theorem 8.

Theorem 9.
If $\Gamma \models_R A$, *then* $L_{\mathbf{1}}^{*} A_{\mathbf{S4}} \vdash \Gamma \to A$.

Proof. $\square$

Theorem 10.