On R-models

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Definition 1.

$$\frac{}{A \to A}\ ax$$

$$\frac{\Gamma \to A \qquad \Delta, B, \Theta \to C}{\Delta, \Gamma, A\backslash B, \Theta \to C}\ \backslash \to \qquad \frac{A, \Pi \to B}{\Pi \to A\backslash B}\ \to \backslash$$

$$\frac{\Gamma \to A \qquad \Delta, B, \Theta \to C}{\Delta, B/A, \Gamma, \Theta \to C}\ / \to \qquad \frac{\Pi, A \to B}{\Pi \to B/A}\ \to /$$

$$\frac{\Gamma, A, B, \Delta \to C}{\Gamma, A \bullet B, \Delta \to C}\ \bullet \to \qquad \frac{\Gamma \to A \qquad \Delta \to B}{\Gamma, \Delta \to A \bullet B}\ \to \bullet$$

Definition 2.

R-model is a pair $\mathcal{M} = \langle W, R, v \rangle$, where R is a transitive relation on W and $v : Tp \to 2^R$ is a valuation, such that:

1. $\mathcal{M}, w \Vdash p_i \Leftrightarrow w \in v(p_i)$;
2. $\mathcal{M}, \langle a, b \rangle \Vdash A \bullet B \Leftrightarrow$ *there exists* $c \in W$, $\mathcal{M}, a \Vdash A$ *and* $\mathcal{M}, b \Vdash B$;
3. $\mathcal{M}, \langle a, b \rangle \Vdash A\backslash B \Leftrightarrow$ *for all* $c \in R^{-1}(a)$, $\mathcal{M}, \langle c, a \rangle \Vdash A$ *implies* $\mathcal{M}, \langle c, b \rangle \Vdash B$;
4. $\mathcal{M}, \langle a, b \rangle \Vdash B/A \Leftrightarrow$ *for all* $c \in R(a)$, $\mathcal{M}, \langle a, c \rangle \Vdash A$ *implies* $\mathcal{M}, \langle b, c \rangle \Vdash B$;
5. $\mathcal{M}, \langle a, b \rangle \Vdash \Gamma \to A \Leftrightarrow \mathcal{M}, \langle a, b \rangle \Vdash \Gamma$ *implies* $\mathcal{M}, \langle a, b \rangle \Vdash A$

where $\mathcal{M}, \langle a, b \rangle \Vdash \Gamma$ denotes $\mathcal{M}, \langle a, b \rangle \Vdash A_1 \bullet \cdots \bullet A_n$, or, equivalently, there exist $c_1, \ldots, c_{n-1}$, such that $\mathcal{M}, \langle a, c_1 \rangle \Vdash A_1, \mathcal{M}, \langle c_1, c_2 \rangle \Vdash A_2, \ldots, \mathcal{M}, \langle c_{n-1}, b \rangle \Vdash A_n$ implies that $\mathcal{M}, \langle a, b \rangle \Vdash B$.

Theorem 1. *Let $\mathbb{F}$ be a R-frame, then $\mathbb{F} \models L$.*

Definition 3.

Let $\mathcal{F}_1$, $\mathcal{F}_2$ be R-frames and $\mathcal{M}_1 = \langle \mathcal{F}_1, v_1 \rangle$, $\mathcal{M}_2 = \langle \mathcal{F}_2, v_2 \rangle$ be R-models.

A map $f : W_1 \to W_2$ is said to be a R-frame p-morphism if the following conditions hold:

1. *f is onto;*
2. $\forall a, b \in W_1 (a R_1 b \Rightarrow f(a) R_2 f(b))$ *(monotonicity);*

3. $\forall d \in W_1, c \in W_2, f(d)R_2c \Rightarrow \exists c' \in W_1, f(c') = c \,\&\, dR_1c'$ *(left lift property);*

4. $\forall d \in W_1, c \in W_2, cR_2f(d) \Rightarrow \exists c' \in W_1, f(c') = c \,\&\, c'R_1d$ *(right lift property);*

A map $f : \mathcal{F}_1 \to \mathcal{F}_2$ *is R-model p-moprhism, iff:*

$$\mathcal{M}_1, \langle a, b\rangle \Vdash p_i \Leftrightarrow \mathcal{M}_2, \langle f(a), f(b)\rangle \Vdash p_i$$

Lemma 1. *Let* $f : \mathcal{M}_1 \twoheadrightarrow \mathcal{M}_2$*, then* $\mathcal{M}_1, \langle a, b\rangle \Vdash A \Leftrightarrow \mathcal{M}_2, \langle f(a), f(b)\rangle \Vdash A$*, for all* $a, b \in W_1$ *and for all* $A \in Fm$

Proof.

1. $\Rightarrow$
   (a) Basic case: follows from the definition.
   (b) Let $A = B \bullet C$ and $\mathcal{M}_1, \langle a, b\rangle \Vdash B \bullet C$, then there exists $c \in W_1$, such that $\mathcal{M}_1, \langle a, c\rangle \Vdash B$ and $\mathcal{M}_1, \langle c, b\rangle \Vdash C$.
   Then, aR_1c and cR_1b, so $f(a)R_2f(c)$ and $f(c)R_2f(b)$.
   Thus, by IH, $\mathcal{M}_2, \langle f(a), f(c)\rangle \Vdash B$ and $\mathcal{M}_2, \langle f(c), f(b)\rangle \Vdash C$, so $\mathcal{M}_2, \langle f(a), f(b)\rangle \Vdash B \bullet C$.
   (c) Let $A = B\backslash C$ and $\mathcal{M}_1, \langle a, b\rangle \Vdash B\backslash C$. Let $c \in W_2$, such that $cR_2f(a)$.
   Then, by left lift property, there exist $d \in W_1$, such that $f(d) = c$ and dR_1a.
   Thus, $\mathcal{M}_1, \langle d, a\rangle \Vdash A$ implies $\mathcal{M}_1, \langle d, b\rangle \Vdash B$.
   By IH, $\mathcal{M}_2, \langle c, f(a)\rangle \Vdash A$ implies $\mathcal{M}_2, \langle c, f(b)\rangle \Vdash B$, then $\mathcal{M}_2, \langle f(a), f(b)\rangle \Vdash A\backslash B$.
   (d) Similarly to (c), but via right lift property.

2. $\Leftarrow$
   (a) Basic case: follows from the definition.
   (b) Let $A = B \bullet C$. Let $\mathcal{M}_2, \langle f(a), f(b)\rangle \Vdash B \bullet C$. Then there exists $c \in W_2$, such that $\mathcal{M}_2, \langle f(a), c\rangle \Vdash B$ and $\mathcal{M}_2, \langle c, f(b)\rangle \Vdash C$.
   So far as f is surjection, then there exists $d \in W_1$, such that $c = f(d)$, then $\mathcal{M}_2, \langle f(a), f(d)\rangle \Vdash B$ and $\mathcal{M}_2, \langle f(d), f(b)\rangle \Vdash C$, and, by IH, $\mathcal{M}_1, \langle a, d\rangle \Vdash B$ and $\mathcal{M}_1, \langle d, b\rangle \Vdash C$, then $\mathcal{M}_1, \langle a, b\rangle \Vdash B \bullet C$.
   (c) Let $A = B\backslash C$ and $\mathcal{M}_2, \langle f(a), f(b)\rangle \Vdash B\backslash C$.
   Let $c \in W_1$ and cR_1a, then $f(c)R_1f(a)$ by monotonicity, so $\mathcal{M}_2, \langle f(c), f(a)\rangle \Vdash A$ implies $\mathcal{M}_2, \langle f(c), f(b)\rangle \Vdash B$.
   By IH, $\mathcal{M}_1, \langle c, a\rangle \Vdash A$ implies $\mathcal{M}_1, \langle c, b\rangle \Vdash B$. Thus, $\mathcal{M}_1, \langle c, a\rangle \Vdash A\backslash B$.
   (d) Similarly to (c).

$\square$

Lemma 2.

1. Let $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *be R-models and* $f : \mathcal{M}_1 \twoheadrightarrow \mathcal{M}_2$*. Then* $\mathcal{M}_1 \models A$ *iff* $\mathcal{M}_2 \models A$*.*

2. Let $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *be R-frames and* $f : \mathcal{F}_1 \twoheadrightarrow \mathcal{F}_2$*, then* $\mathcal{F}_1 \models A$ *implies* $\mathcal{F}_2 \models A$*.*

Proof.

1.
   - Only if:
     Let $\mathcal{M}_1 \models A$. Let $c, d \in W_2$. So far as f is onto, then there exists $a, b \in W_1$, such that $f(a) = c$ and $f(b) = d$.
     Then $\mathcal{M}_1, \langle a, b\rangle \Vdash A$, thus $\mathcal{M}_2\langle f(a), f(b)\rangle \Vdash A$. That is, $\mathcal{M}_2\langle c, d\rangle \Vdash A$
   - If:
     Follows from the previous lemma.

2. Let $f : \mathcal{F}_1 \twoheadrightarrow \mathcal{F}_2$ and $\mathcal{F}_1 \models A$.

   Let $\mathcal{M}_1 = \langle \mathcal{F}_1, v_1\rangle$ and $\mathcal{M}_2 = \langle \mathcal{F}_2, v_2\rangle$, such that for all $p \in Tp$, $\mathcal{M}_1, \langle a, b\rangle \Vdash p \Leftrightarrow_{def} \mathcal{M}_1, \langle f(a), f(b)\rangle \Vdash p$. Thus, $\mathcal{M}_1 \twoheadrightarrow \mathcal{M}_2$ and $\mathcal{M}_1 \models A$. Thus, $\mathcal{M}_2 \models A$.

$\square$