Models of Lambek calculus with subexponentials

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Abstract

1 The Lambek Calculus with subexponentials

Definition 1. *A subexponential signature is an ordered quintuple:*
$\Sigma = \langle \mathcal{I}, \preceq, \mathcal{W}, \mathcal{C}, \mathcal{E} \rangle$,

where $I = \{s_1, \dots, s_n\}$, $\mathcal{I}, \preceq\rangle$ is a preorder. W, C, E are upwardly closed subsets of I and $\mathcal{W} \cap \mathcal{C} \subseteq \mathcal{E}$.

Definition 2.
$\mathcal{F}_\Sigma ::= Tp \,|\, (\mathcal{F}_\Sigma / \mathcal{F}_\Sigma) \,|\, (\mathcal{F}_\Sigma \backslash \mathcal{F}_\Sigma) \,|\, (\mathcal{F}_\Sigma \bullet \mathcal{F}_\Sigma) \,|\, (\mathcal{F}_\Sigma \vee \mathcal{F}_\Sigma) \,|(\mathcal{F}_\Sigma \wedge \mathcal{F}_\Sigma) \,|(!_s \mathcal{F}_\Sigma)_{s \in \Sigma}$

Definition 3. *Noncommutative linear logic with subexponentials ($SMALC_\Sigma$), where Σ is a subexponential signature.*

$$\frac{}{A \to A}\, ax$$

$$\frac{\Gamma \to A \qquad \Delta, B, \Theta \to C}{\Delta, \Gamma, A \backslash B, \Theta \to C}\, \backslash \to \qquad\qquad \frac{A, \Pi \to B}{\Pi \to A \backslash B}\, \to \backslash$$

$$\frac{\Gamma \to A \qquad \Delta, B, \Theta \to C}{\Delta, B / A, \Gamma, \Theta \to C}\, / \to \qquad\qquad \frac{\Pi, A \to B}{\Pi \to B / A}\, \to /$$

$$\frac{\Gamma, A, B, \Delta \to C}{\Gamma, A \bullet B, \Delta \to C}\, \bullet \to \qquad\qquad \frac{\Gamma \to A \qquad \Delta \to B}{\Gamma, \Delta \to A \bullet B}\, \to \bullet$$

$$\frac{\Gamma, A_i, \Delta \to B}{\Gamma, A_1 \& A_2, \Delta \to B}\, \&, i = 1, 2 \to \qquad\qquad \frac{\Gamma \to A \qquad \Gamma \to B}{\Gamma \to A \& B}\, \to \&$$

$$\frac{\Gamma, A, \Delta \to C \qquad \Gamma, B, \Delta \to C}{\Gamma, A \vee B, \Delta \to C}\, \vee \to \qquad\qquad \frac{\Gamma \to A_i}{\Gamma \to A_1 \vee A_2}\, \to \vee, i = 1, 2$$

$$\frac{\Gamma, \Delta \to A}{\Gamma, \mathbf{1}, \Delta \to A}\, \mathbf{1} \to \qquad\qquad \frac{}{\to \mathbf{1}}\, \to \mathbf{1}$$

$$\frac{\Gamma, A, \Delta \to C}{\Gamma, !^s A, \Delta \to C}\, ! \to \qquad\qquad \frac{!^{s_1} A_1, \dots, !^{s_n} A_n \to A}{!^{s_1} A_1, \dots, !^{s_n} A_n \to !^s A}\, \to !, \forall j, s_j \geq s$$

Structural rules:

$$\frac{\Gamma, !^s A, \Delta, !^s A, \Theta \to B}{\Gamma, !^s A, \Delta, \Theta \to B}\ \mathbf{ncontr}_1, s \in C \qquad \frac{\Gamma, !^s A, \Delta, !^s A, \Theta \to B}{\Gamma, \Delta, !^s A, \Theta \to B}\ \mathbf{ncontr}_2, s \in C$$

$$\frac{\Gamma, \Delta, !^s A, \Theta \to B}{\Gamma, !^s A, \Delta, \Theta \to A}\ \mathbf{ex}_1, s \in E \qquad \frac{\Gamma, !^s A, \Delta, \Theta \to B}{\Gamma, \Delta, !^s A, \Theta \to A}\ \mathbf{ex}_2, s \in E$$

$$\frac{\Gamma, \Delta \to B}{\Gamma, !^s A, \Delta \to B}\ \mathbf{weak}_!, s \in C \qquad \frac{\Gamma \to A \qquad \Pi, A, \Delta \to B}{\Gamma, \Pi, \Delta \to B}\ \mathbf{cut}$$

Theorem 1.

1. *Cut-rule is admissable;*
2. *$SMALC_\Sigma$ is undecidable, if $C \neq \varnothing$;*
3. *If C is empty, then the decidability problem of $SMALC_\Sigma$ belongs to PSPACE.*

2 Semantics

Definition 4. *Quantale*

A quantale is a triple $\mathcal{Q} = \langle A, \bigvee, \cdot \rangle$, such that $\langle A, \bigvee \rangle$ is a complete lattice and $\langle A, \cdot \rangle$ is a semigroup, such that for all indexing set I:

1. $a \cdot \bigvee\limits_{i \in I} b_i = \bigvee\limits_{i \in I} (a \cdot b_i)$;
2. $\bigvee\limits_{i \in I} a_i \cdot b = \bigvee\limits_{i \in I} (a_i \cdot b)$

A quanlate is called unital, if $\langle A, \cdot \rangle$ is a monoid.

Some example of quantales:

- Let A be a semigroup (monoid), then $\langle \mathcal{P}(A), \cdot, \subseteq \rangle$ is a free (unital) quantale.
- Let R be a ring and $Sub(R)$ be a set of additive subgroups of R. We define $A \cdot B$ as an additive subgroup generated by finite sums of products ab and order is defined by inclusion.
- Any locale is a quantale with $\cdot = \wedge$.

It is easy to see, that any (unital) quantale is a residual (monoid) semigroup. We define divisions as follows:

1. $a \backslash b = \bigvee \{c \mid a \cdot c \leqslant b\}$
2. $b / a = \bigvee \{c \mid c \cdot a \leqslant b\}$

Definition 5. *Let $\mathcal{Q}_1$, $\mathcal{Q}_2$ be quantales. A quantale homomorphism is a map $f : \mathcal{Q}_1 \to \mathcal{Q}_2$, such that:*

1. *for all $a, b \in \mathcal{Q}_1$, $f(a \cdot b) = f(a) \cdot f(b)$;*
2. *for all indexing set I, $f(\bigvee\limits_{i \in I} a_i) = \bigvee\limits_{i \in I} f(a_i)$.*

If $\mathcal{Q}_1$, $\mathcal{Q}_2$ are unital quantales, then a unital homomorphism if a quantale homomorphism, such that $f(\varepsilon) = \varepsilon$.

Definition 6.

Let $\mathcal{Q} = \langle A, \bigvee, \cdot \rangle$ be a quantale. $\mathcal{S} \subseteq \mathcal{Q}$ is said to be a subquantale, if $\mathcal{S}$ is closed under multiplication and sups.

There occurs the following simple statement:

Proposition 1.

Let $\mathcal{Q}_1$, $\mathcal{Q}_2$ be quantales and $\mathcal{S} \subseteq \mathcal{Q}_1$ is a subquantale of $\mathcal{Q}_1$.

Then, if $f : \mathcal{Q}_1 \to \mathcal{Q}_2$ is a quantale homomorphism, then $f(\mathcal{S}) \subseteq \mathcal{Q}_2$ is a subquantale of $\mathcal{Q}_2$.

In other words, a homomorphic image of subquantale is a subquantale.

Proof.

It is clearly that $f(\mathcal{S}) \subseteq \mathcal{Q}_2$ is a submonoid of $\mathcal{Q}_2$. Let $a_i \in S$ for each $i \in I$, so $\bigvee\limits_{i \in I} a_i \in S$, but $f(a_i) \in f(\mathcal{S})$ for any $i \in I$, so $f(\bigvee\limits_{i \in I} a_i) = \bigvee\limits_{i \in I} (f(a)) \in f(\mathcal{S})$, so $f(\mathcal{S})$ is closed under joins, so $f(\mathcal{S})$ is a subquantale of $\mathcal{Q}_2$ $\square$

Definition 7.

Let $\mathcal{Q} = \langle A, \bigvee, \cdot \rangle$ be a quantale. The center of a quantale is the subquantale $\mathcal{Z}(\mathcal{Q}) = \{a \in A \mid \forall b \in A, a \cdot b = b \cdot a\}$

Definition 8.

An open modality (or quantic conucleus) on quantale $\mathcal{Q}$ is a map $\Box : \mathcal{Q} \to \mathcal{Q}$, such that

1. $\Box x \leqslant x$;
2. $\Box x = \Box\Box x$;
3. $x \leqslant y \Rightarrow \Box x \leqslant \Box y$;
4. $\Box x \cdot \Box y = \Box(\Box x \cdot \Box y)$.

For unital quantale, we require that $\Box e = e$.

Note that, we may replace the last condition on equivalent condition $\Box(x) \cdot \Box(y) \leqslant \Box(x \cdot y)$.

Definition 9.

We define a partial order on open modalities on $\mathcal{Q}$ as $\Box_1 \leqslant \Box_2 \Leftrightarrow \forall a \in Q, \Box_1(a) \leqslant \Box_2(a)$.

Lemma 1. $\Box_1 a_1 \cdot \Box_2 a_2 \leqslant \Box(\Box_1 a_1 \cdot \Box_2 a_2)$, *where $\Box_i \leqslant \Box, i = 1, 2$.*

Proof.

$$\begin{aligned}
&\Box_1 a_1 \cdot \Box_2 a_2 \leqslant \\
&\Box_1(\Box_1 a_1) \cdot \Box_2(\Box_2 a_2) \leqslant \\
&\Box(\Box_1 a_1) \cdot \Box(\Box_2 a_2) \leqslant \\
&\Box(\Box_1(a_1) \cdot \Box_2(a_2))
\end{aligned}$$

$\square$

Definition 10.

1. *An open modality is called central, if for all $a, b \in \mathcal{Q}, \Box a \cdot b = b \cdot \Box a$.*
2. *An open modality is called weak square-increasing, if for all $a, b \in \mathcal{Q}, \Box a \cdot b \leqslant \Box a \cdot b \cdot \Box a$ and $b \cdot \Box a \leqslant \Box a \cdot b \cdot \Box a$.*

3. *An open modality is called unital, if* $\forall a \in Q, \Box a \leqslant e$.

Lemma 2.

Let $\Box$ *be an open modality on some unital quantale* $\mathcal{Q} = \langle A, \bigvee, \cdot, e \rangle$. *Then, if* $\Box$ *is unital and weak idempotent, then* $\Box$ *is central.*

Proof.

$b \cdot \Box a \leqslant$

Right weak square-increase

$\Box a \cdot b \cdot \Box a \leqslant$

Unitality

$\Box a \cdot b \cdot e \leqslant$

Identity

$\Box a \cdot b \leqslant$

Left weak square-increase

$\Box a \cdot b \cdot \Box a \leqslant$

Unitality

$e \cdot b \cdot \Box a \leqslant$

Identity

$b \cdot \Box a$

Hence, $b \cdot \Box a = \Box a \cdot b$, so for all $a \in \mathcal{Q}, \Box a \in \mathcal{Z}(\mathcal{Q})$.

$\Box$

Proposition 2.

Let $\mathcal{Q}$ *be a quantale and* $S \subseteq \mathcal{Q}$ *a subquantale, then* $\Box : \mathcal{Q} \to \mathcal{Q}$, *such that* $\Box a = \bigvee \{ s \in S \,|\, s \leqslant a \}$, *is an open modality.*

Proof. See Rosenthal. $\Box$

Proposition 3.

Let $\mathcal{Q}$ *be a quantale and* $\mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \mathcal{Q}$.

Then $\Box_{\mathcal{S}_1}(a) \leqslant \Box_{\mathcal{S}_1}(a)$.

Proof. Immediatly. $\Box$

Proposition 4.

Let $\mathcal{Q}$ *be a quantale and* $\mathcal{S} \subseteq \mathcal{Q}$ *a subquantale, then the following operations are open modalities:*

1. $\Box_z(a) = \bigvee \{ s \in S \,|\, s \leqslant a, s \in \mathcal{Z}(\mathcal{Q}) \}$;
2. $\Box_{\mathbb{1}}(a) = \bigvee \{ s \in S \,|\, s \leqslant a, s \leqslant \mathbb{1} \}$;
3. $\Box_{idem}(a) = \bigvee \{ s \in S \,|\, s \leqslant a, \forall b \in Q, b \cdot s \vee s \cdot b \leqslant s \cdot b \cdot s \}$;
4. *Operations with combinations of conditions above;*

Proof. Immediatly. $\Box$

Proposition 5.

1. $\forall a \in \mathcal{Q}, \Box_{\mathbb{1},idem}(a) \leqslant \Box_z(a)$.
2. $\forall a \in \mathcal{Q}, \Box_{z,\mathbb{1},idem} = \Box_{\mathbb{1},idem}(a)$

Proof. Follows from Lemma 3. $\square$

Definition 11. *Interpretation of subexponential signature*

Let $\Sigma = \langle I, \preceq, W, C, E\rangle$ be a subexponential signature, where $|I| = n$ and $\square_{\mathcal{Q}}$ is a category of open modalities on a quantale $\mathcal{Q}$. Subexponential interpretation is a contravariant functor $\sigma : I \to \square_{\mathcal{Q}}$ defined as follows:

$$\sigma(s_i) = \begin{cases} \square_i : \mathcal{Q} \to \mathcal{Q},\ s.t. \forall a \in Q, \square_i(a) = \{s \in S_i \mid s \leqslant a\}, \\ \quad \text{if } s_i \notin W \cap C \cap E \\ \square_i : \mathcal{Q} \to \mathcal{Q},\ s.t. \forall a \in Q, \square_i(a) = \{s \in S_i \mid s \leqslant a, s \leqslant \mathbb{1}\}, \\ \quad \text{if } s_i \in W \\ \square_i : \mathcal{Q} \to \mathcal{Q},\ s.t. \forall a \in Q, \square_i(a) = \{s \in S_i \mid s \leqslant a, s \in \mathcal{Z}(\mathcal{Q})\}, \\ \quad \text{if } s_i \in E \\ \square_i : \mathcal{Q} \to \mathcal{Q},\ s.t. \forall a \in Q, \square_i(a) = \{s \in S_i \mid s \leqslant a, \forall b, b \cdot s \vee s \cdot b \leqslant s \cdot b \cdot s\}, \\ \quad \text{if } s_i \in E \\ \text{otherwise, if } s_i \text{ belongs to some intersection of subsets, then we combine the relevant conditions} \end{cases}$$

Definition 12. *Let $\mathcal{Q}$ be an unital quantale, $f : Tp \to \mathcal{Q}$ a valuation and $\sigma : I \to \square_{\mathcal{Q}}$ a subexponential interpretation, then interpretation is defined inductively:*

$$\begin{aligned} &[\![p_i]\!] = f(p_i) \\ &[\![\mathbb{1}]\!] = e \\ &[\![A \bullet B]\!] = [\![A]\!] \cdot [\![B]\!] \\ &[\![A \backslash B]\!] = [\![A]\!] \backslash [\![B]\!] \\ &[\![A / B]\!] = [\![A]\!] / [\![B]\!] \\ &[\![A \& B]\!] = [\![A]\!] \wedge [\![B]\!] \\ &[\![A \vee B]\!] = [\![A]\!] \vee [\![B]\!] \\ &[\![!_{s_i} A]\!] = \sigma(s_i)[\![A]\!] \end{aligned}$$

Definition 13. $\Gamma \models A \Leftrightarrow \forall f, \forall \sigma, [\![\Gamma]\!] \leqslant [\![A]\!]$

Theorem 2. $\Gamma \to A \Rightarrow [\![\Gamma]\!] \leqslant [\![A]\!]$

Proof. We consider the promotion case, the rest modal cases are immediatly shown.

Let $!_{s_1} A_1, \ldots, !_{s_n} A_n \to A$ and $\forall i, s \preceq s_i$.

Then $\forall a \in Q, \sigma(s_i)(a) \leqslant \sigma(s)(a)$.

By IH, $\sigma(s_1)[\![A_1]\!] \cdot \cdots \cdot \sigma(s_n)[\![A_n]\!] \leqslant [\![A]\!]$.

Thus, $\sigma(s)(\sigma(s_1)[\![A_1]\!] \cdot \cdots \cdot \sigma(s_n)[\![A_n]\!]) \leqslant \sigma(s)([\![A]\!])$.

By Lemma 5, $\sigma(s_1)[\![A_1]\!] \cdot \cdots \cdot \sigma(s_n)[\![A_n]\!] \leqslant \sigma(s)(\sigma(s_1)[\![A_1]\!] \cdot \cdots \cdot \sigma(s_n)[\![A_n]\!])$.

So, $\sigma(s_1)[\![A_1]\!] \cdot \cdots \cdot \sigma(s_n)[\![A_n]\!] \leqslant \sigma(s)([\![A]\!])$. $\square$

3 Quantale completeness

Definition 14.

Let $\mathcal{F} \subseteq Fm$, an ideal is a subset $\mathcal{I} \subseteq \mathcal{F}$, such that:

- *If $B \in \mathcal{I}$ and $A \to B$, then $A \in \mathcal{I}$;*
- *If $A, B \in \mathcal{I}$, then $A \vee B \in \mathcal{I}$.*

Definition 15.

Let $S \subseteq \mathcal{F} \subseteq Fm$, then $\bigvee S = \bigcap\{\mathcal{I} \subseteq \mathcal{F} \mid S \subseteq \mathcal{I}\}$

The following conditions hold similarly to [**?**]:

Lemma 3.

1. *$\bigvee S$ is an ideal;*
2. *$A \subseteq Fm$, then $\{B \mid B \to A\} = \bigvee\{A\}$;*
3. *$\bigvee\{A\} \subseteq \bigvee\{B\}$ iff $A \to B$;*
4. *Let $\mathcal{Q} = \{\bigvee S \mid S \subseteq Fm\}$ and $\bigvee \mathcal{A} \cdot \bigvee \mathcal{B} = \bigvee\{A \bullet B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$. Then $\langle \mathcal{Q}, \subseteq, \cdot, \bigvee \mathbf{1}\rangle$ is a unital quantale.*

We extend this construction for polymodal case as follows:

Lemma 4. *Let $!_s \in I$ and $A\mathcal{F}_\Sigma$, then $\Box_s(\bigvee\{A\}) = \bigvee\{!_sB \mid !_sB \to A\}$ is a quantic conucleus.*

Proof.
See Yetter. $\Box$

Lemma 5. *Let $A \in \mathcal{F}_\Sigma$, then $\Box_s \bigvee\{A\} = \bigvee\{!_sA\}$, for each $s \in \mathcal{I}$.*

Proof. Let $A \in Fm$ and $s \in \mathcal{I}$.
Let $!_sB \in \Box_s \bigvee\{A\}$, then $!_sB \to A$, then $!_sB \to !_sA$ by promotion. So, $!_sB \in \bigvee\{!_sA\}$.
Let $C \in \bigvee\{!_sA\}$, then $C \to !_sA$, so $!_sC \to !_sA$ by dereliction, but $!_sA \to A$, hence $!_sC \to A$ by cut. So, $!_sC \in \Box_s \bigvee\{A\}$. $\Box$

Lemma 6.

Let $i, j \in I$ and $i \preceq j$, then for all $A \in \mathcal{F}_\Sigma$, $\Box_j(\bigvee\{A\}) \subseteq \Box_i(\bigvee\{A\})$.

Proof.
Let $i, j \in I$ and $i \preceq j$, then forall $A \in \mathcal{F}_\Sigma$, $!_jA \to !_iA$ by promotion. Then $\bigvee\{!_jA\} \subseteq \bigvee\{!_iA\}$, so $\Box_j(\bigvee\{A\}) \subseteq \Box_i(\bigvee\{A\})$. $\Box$

Lemma 7.

For all $A \in \mathcal{F}_\Sigma$,

1. *Let $s \in W$, then $\Box_s\{A\} \subseteq \{\mathbf{1}\}$;*
2. *Let $s \in E$, then $\Box_s(\bigvee\{A\}) \cdot \bigvee\{B\} = \bigvee\{B\} \cdot \Box_s(\bigvee\{A\})$.*
3. *Let $s \in C$, then $(\Box_s \bigvee A \cdot \bigvee B) \cup (\bigvee B \cdot \Box_s \bigvee A) \subseteq \Box_s \bigvee A \cdot \bigvee B \cdot \Box_s \bigvee A$, for all $B \subseteq Fm$.*

Proof.

1. Follows from $!_sA \to \mathbf{1}$, so $s \in W$;
2. Follows from $!_sA \bullet B \leftrightarrow B \bullet !_sA$;
3. Follows from $!_sA \bullet B \to !_sA \bullet B \bullet !_sA$ and similarly for $B \bullet !_sA$.

$\Box$

Definition 16.

Let $\mathcal{Q}$ be a syntactic quantale as proposed above and $\mathcal{I} = \langle I, \preceq, W, C, E\rangle$ be a subexponential signature.

We define a map $\Box : \mathcal{I} \to Mod_{\mathcal{Q}}$ as follows:
$\Box(i)(\bigvee\{A\}) = \{!_i B \,|!_i B \to A\}$.

Lemma 8. *$\Box$ is a subexponential interpretation.*

Proof. Follows from lemmas above. $\square$

Lemma 9.

Let Q be a quantale constructed above and $(\Box_{s_i})_{s_i \in \Sigma}$ be a family of quantic conuclei on Q. Then there exist a model $\langle Q, [\![.]\!]\rangle$, such that $[\![A]\!] = \bigvee\{A\}$, $A \in Fm$.

Proof.

We define an interpreation as follows:

1. $[\![p_i]\!] = \bigvee\{p_i\}$
2. $[\![\mathbf{1}]\!] = \bigvee\{\mathbf{1}\}$
3. $[\![A \bullet B]\!] = \bigvee\{A \bullet B\}$
4. $[\![A/B]\!] = \bigvee\{A/B\}$
5. $[\![B\backslash A]\!] = \bigvee\{B\backslash A\}$
6. $[\![A \& B]\!] = \bigvee\{A \& B\}$
7. $[\![A \vee B]\!] = \bigvee\{A \vee B\}$
8. $[\![!_s A]\!] = \Box(s)(\bigvee\{A\}) = \bigvee\{!_s A\}$.

$\square$

Theorem 3. $\Gamma \models A \Rightarrow \Gamma \to A$.

Proof. Follows from lemmas above. $\square$

4 Relational semantics

Definition 17.

Let A be a set. Then relational quantale on A is a triple $\mathcal{Q} = \langle \mathcal{A}, \bigvee, \mathcal{I}\rangle$ where $\mathcal{A} \subseteq 2^{A \times A}$:

- *$\langle \mathcal{A}, \bigvee, \subseteq\rangle$ is a complete semi-lattice;*
- *Multiplication is defined as $R \circ S = \{\langle a, c\rangle \mid \exists b \in A, \langle a, b\rangle \in R$ and $\langle b, c\rangle \in S\}$*
- *$\langle \mathcal{A}, \circ, \mathcal{I}\rangle$ is a monoid;*
- *For each indexing set J, $R \circ \bigvee_{i \in I} S_i = \bigvee_{i \in I}(R \circ S_i)$ and $\bigvee_{i \in I} R_i \circ S = \bigvee_{i \in I}(R_i \circ S)$.*

Theorem 4.

Let $\mathcal{Q} = \langle A, \leqslant, \cdot, \bigvee\rangle$ be a unital quantale and $\mathcal{S}$ is a subquantale of $\mathcal{Q}$.

Then $\langle \mathcal{Q}, \Box_{\mathcal{S}}\rangle$ is isomorphic to some relational quantale of A wit some quantic conucleus $\hat{\Box}$.

Proof.

Consider a relational quantale proposed by Brown and Gurr.

This quantale is 4-tuple $\theta(\mathcal{Q}) = \langle \mathcal{R}, \subseteq, \circ, \bigvee \rangle$ defined as follows:

1. $\theta(a) = \{\langle b, c\rangle \mid b \leqslant a \cdot c\}$;
2. $\theta(a \cdot b) = \theta(a) \circ \theta(b)$;
3. $\theta(\bigvee\limits_{i \in I} a_i) = \bigvee\limits_{i \in I} \theta(a_i)$;
4. $\theta(\varepsilon) = \{\langle b, c\rangle \mid b \cdot \varepsilon \leqslant c\} = \{\langle b, c\rangle \mid b \leqslant c\}$

Let $\mathcal{S} \subseteq \mathcal{Q}$, so $\Box_{\mathcal{S}} a := \bigvee\{s \mid s \in S, s \leqslant a\}$ is quantic conucleus.
So, $\theta(S) \subseteq \theta(\mathcal{Q})$ is a subquantale of $\theta(\mathcal{Q})$.
Let us define $\hat{\Box}\theta(a) := \bigvee\{\theta(s) \mid \theta(s) \in \theta(S), \theta(s) \subseteq \theta(a)\}$, so

$\theta(\Box_{\mathcal{S}} a) = \{\langle p, q\rangle \mid p \leqslant \Box_{\mathcal{S}} a \cdot q\} =$
$\{\langle p, q\rangle \mid p \leqslant \bigvee\{s \mid s \in \mathcal{S}, s \leqslant a\} \cdot q\} =$
Homomorphism
$\theta(\bigvee_{s \in S, s \leqslant a} s) =$
Homomorphism preserves sups
$\bigvee_{s \in S, s \leqslant a} \theta(s) =$
Unfolding
$\bigvee\{\theta(s) \mid s \in S, s \leqslant a\} =$
Unfolding
$\bigvee\{\theta(s) \mid \theta(s) \in \theta(S), \theta(s) \subseteq \theta(a)\} = \hat{\Box}\theta(a)$

So, $\hat{\Box}\theta(a) = \theta(\Box_{\mathcal{S}} a)$.

$\Box$

5 Syntactic concept lattices

Definition 18. *Let $\mathcal{L}$ be a finite alphabet and $L \subseteq \mathcal{L}^*$ be a language.*
We define maps $[.]^{\rhd} : \mathcal{P}(\mathcal{L}^*) \to \mathcal{P}(\mathcal{L}^* \times \mathcal{L}^*)$ and $[.]^{\lhd} : \mathcal{P}(\mathcal{L}^* \times \mathcal{L}^*) \to \mathcal{P}(\mathcal{L}^*)$ as follows:

1. *$M \subseteq \mathcal{L}^*$, $M^{\rhd} = \{(x, y) \mid \forall w \in M, xwy \in L\}$;*
2. *$C \subseteq \mathcal{L}^* \times \mathcal{L}^*$, $C^{\lhd} = \{w \mid \forall (x, y) \in C, xwy \in L\}$*

Note that compositions $[.]^{\lhd\rhd}$ and $[.]^{\rhd\lhd}$ form closure operators, by the way $[.]^{\lhd}$ and $[.]^{\rhd}$ are connected via contravariant Galois connection.

Definition 19. *A syntactic concept is a pair $\langle S, C\rangle$, where $S \subseteq \mathcal{L}^*$ and $C \subseteq \mathcal{L}^* \times \mathcal{L}^*$, such that $S^{\rhd} = C$ and $C^{\lhd} = S$.*