Quantale model of noncommutative linear logic with subexponentials

1 Calculus

Definition 1. *A subexponential signature is an ordered quintuple:*
$\Sigma = \langle I, \preceq, W, C, E \rangle$,

where $I = \{s_1, \dots, s_n\}$, $\langle I, \preceq \rangle$ is a preorder. W, C, E are subsets of I and $W \cup C \subseteq E$.

Definition 2. *Noncommutative linear logic with subexponentials ($SMALC_\Sigma$), where Σ is a subexponential signature.*

$$\frac{}{A \Rightarrow A}\ ax$$

$$\frac{\Gamma \Rightarrow A \qquad \Delta, B, \Theta \Rightarrow C}{\Delta, \Gamma, A\backslash B, \Theta \Rightarrow C}\ \backslash \to \qquad \frac{A, \Pi \Rightarrow B}{\Pi \Rightarrow A\backslash B}\ \to \backslash$$

$$\frac{\Gamma \Rightarrow A \qquad \Delta, B, \Theta \Rightarrow C}{\Delta, B/A, \Gamma, \Theta \Rightarrow C}\ / \to \qquad \frac{\Pi, A \Rightarrow B}{\Pi \Rightarrow B/A}\ \to /$$

$$\frac{\Gamma, A, B, \Delta \Rightarrow C}{\Gamma, A \bullet B, \Delta \Rightarrow C}\ \bullet \to \qquad \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \bullet B}\ \to \bullet$$

$$\frac{\Gamma, A_i, \Delta \Rightarrow B}{\Gamma, A_1 \& A_2, \Delta \Rightarrow B}\ \&, i = 1,2 \to \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \& B}\ \to \&$$

$$\frac{\Gamma, A, \Delta \Rightarrow C \qquad \Gamma, B, \Delta \Rightarrow C}{\Gamma, A \vee B, \Delta \Rightarrow C}\ \vee \to \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}\ \to \vee, i = 1,2$$

$$\frac{\Gamma, \Delta \Rightarrow A}{\Gamma, \mathbf{1}, \Delta \Rightarrow A}\ \mathbf{1} \to \qquad \frac{}{\Rightarrow \mathbf{1}}\ \to \mathbf{1}$$

$$\frac{\Gamma, A, \Delta \Rightarrow C}{\Gamma, !^s A, \Delta \Rightarrow C}\ ! \to \qquad \frac{!^{s_1} A_1, \dots, !^{s_n} A_n \Rightarrow A}{!^{s_1} A_1, \dots, !^{s_n} A_n \Rightarrow !^s A}\ \to !, \forall j, s_j \succeq s$$

$$\frac{\Gamma, \Delta \Rightarrow B}{\Gamma, !^s A, \Delta \Rightarrow B}\ \mathbf{weak}_!, s \in C$$

$$\frac{\Gamma, !^s A, \Delta, !^s A, \Theta \Rightarrow B}{\Gamma, !^s A, \Delta, \Theta \Rightarrow B}\ \mathbf{ncontr}_1, s \in C$$

$$\frac{\Gamma, !^s A, \Delta, !^s A, \Theta \Rightarrow B}{\Gamma, \Delta, !^s A, \Theta \Rightarrow B}\ \mathbf{ncontr}_2, s \in C$$

$$\frac{\Gamma, \Delta, !^s A, \Theta \Rightarrow B}{\Gamma, !^s A, \Delta, \Theta \Rightarrow A}\ \mathbf{ex}_1, s \in E$$

$$\frac{\Gamma, !^s A, \Delta, \Theta \Rightarrow B}{\Gamma, \Delta, !^s A, \Theta \Rightarrow A}\ \mathbf{ex}_1, s \in E$$

Lemma 1. *Let $A \Leftrightarrow B$, then $C[p_i := A] \Leftrightarrow C[p_i := B]$*

Proof. By induction on C. □

Lemma 2.

- $!_{s_i}\Gamma \to A$ *iff* $!_{s_i}\Gamma \to !_{s_i}A$.
- $!_{s_i}A \leftrightarrow !_{s_i}(!_{s_i}A)$

Proof.

1. $!_{s_i}\Gamma \to A$ iff $!_{s_i}\Gamma \to !_{s_i}A$;

$$\frac{!_{s_i}\Gamma \to A}{!_{s_i}\Gamma \to !_{s_i}A}\ \to !_{s_i}$$

$$\dfrac{!_{s_i}\Gamma \to !_{s_i}A \qquad \dfrac{A \to A}{!_{s_i}A \to A}\ !_{s_i}\to}{!_{s_i}\Gamma \to A}\ \text{cut}$$

2. $!_{s_i}A \leftrightarrow !_{s_i}!_{s_i}A$

$$\dfrac{\dfrac{\dfrac{A \to A}{!_{s_i}A \to A}}{!_{s_i}A \to !_{s_i}A}}{!_{s_i}!_{s_i}A \to !_{s_i}A}$$

□

2 Semantics

Definition 3. *Quantale*

A quantale is a triple $\langle A, \bigvee, \cdot\rangle$, such that $\langle A, \bigvee\rangle$ is a complete lattice and $\langle A, \cdot\rangle$ is a semigroup. A quanlate is called unital, if $\langle A, \cdot\rangle$ is a monoid.

It is easy to see, that any (unital) quantale is a residual (monoid) semigroup. We define divisions as follows:

1. $a\backslash b = \bigvee\{c \mid a \cdot c \leqslant b\}$
2. $b/a = \bigvee\{c \mid c \cdot a \leqslant b\}$

Definition 4.
Let $\langle A, \bigvee, \cdot \rangle$ be a quantale. The center of a quantale is the set $Z(Q) = \{a \in Q \mid \forall b \in Q, a \cdot b = b \cdot a\}$

Definition 5. *An open modality (or quantic conucleus) on quantale Q is a map $I : Q \to Q$, such that*

1. $I(x) \leqslant x$;
2. $I(x) = I(I(x))$;
3. $x \leqslant y \Rightarrow I(x) \leqslant I(y)$;
4. $I(x) \cdot I(y) = I(I(x) \cdot I(y))$.

Lemma 3.
Let $\langle A, \bigvee, \cdot \rangle$ be a quantale and $I : Q \to Q$ is an open modality on Q, then $I(x) \cdot I(y) \leqslant I(x \cdot y)$.

Proof.
$I(x) \cdot I(y) \leqslant x \cdot y$, then $I(I(x) \cdot I(y)) \leqslant I(x \cdot y)$, but $I(x) \cdot I(y) \leqslant I(I(x) \cdot I(y))$. Thus, $I(x) \cdot I(y) \leqslant I(x \cdot y)$. □

Definition 6. *An open modality is called central, if $\forall a, b \in Q, I(a) \cdot b = b \cdot I(a)$.*

Definition 7. *An open modality is called weak idempotent, if $\forall a, b \in Q, I(a) \cdot b \leqslant I(a) \cdot b \cdot I(a)$ and $b \cdot I(a) \leqslant I(a) \cdot b \cdot I(a)$.*

Definition 8. *An open modality is called unital, if $\forall a \in Q, I(a) \leqslant e$.*

Lemma 4. *Let I be an interior on some unital quantale $\langle Q, \bigvee, \cdot, e \rangle$. Then, if I is unital and weak idempotent, then I is central.*

Proof.
$b \cdot I(a) \leqslant$
Right weak idempotence
$I(a) \cdot b \cdot I(a) \leqslant$
Unitality
$I(a) \cdot b \cdot I(e) \leqslant$
Identity
$I(a) \cdot b \leqslant$
Left weak idempotence
$I(a) \cdot b \cdot I(a) \leqslant$
Unitality
$e \cdot b \cdot I(a) \leqslant$
Identity
$b \cdot I(a)$
Hence, $b \cdot I(a) = I(a) \cdot b$

□

Proposition 1.
Let Q be a quantale and $S \subseteq Q$ a subquantale, then $I : Q \to Q$, such that $I(a) = \bigvee\{s \in S \mid x \leqslant a\}$, is an open modality. Moreover, $\{x \in Q \mid I(x) = x\} = S$.

Proof. See □

Proposition 2.

Let Q be a quantale and $S_1, S_2 \subseteq Q$, such that $S_1 \subseteq S_2$.
Then $I_1(a) \leqslant I_2(a)$.

Proof.
Let $a \in Q$, so $\{s \in S_1 \mid s \leqslant a\} \subseteq \{s \in S_2 \mid s \leqslant a\}$, so $\bigvee\{s \in S_1 \mid s \leqslant a\} \subseteq \bigvee\{s \in S_2 \mid s \leqslant a\}$. Thus, $I_1(a) \leqslant I_2(a)$. ☐

Proposition 3.

Let Q be a quantale and $S \subseteq Q$ a subquantale, then the following operations are open modalities:

1. $I_z(a) = \bigvee\{s \in S \,|s \leqslant a, s \in Z(Q)\}$;
2. $I_{\mathbb{1}}(a) = \bigvee\{s \in S \,|s \leqslant a, s \leqslant \mathbb{1}\}$;
3. $I_{idem}(a) = \bigvee\{s \in S \,|s \leqslant a, \forall b \in Q, b \cdot s \vee s \cdot b \leqslant s \cdot b \cdot s\}$;
4. $I_{z,\mathbb{1}}, I_{z,idem}, I_{\mathbb{1},idem}, I_{z,\mathbb{1},idem}$.

Proof. Immediatly. ☐

Proposition 4.

1. $\forall a \in Q, I_{\mathbb{1},idem}(a) \leqslant I_z(a)$.
2. $\forall a \in Q, I_{z,\mathbb{1},idem} = I_{\mathbb{1},idem}(a)$

Proof. Follows from Lemma 3. ☐

Proposition 5.

1. $I_z(a) \vee I_{\mathbb{1}}(a) \vee I_{idem}(a) \leqslant I(a)$
2. $I_{z,\mathbb{1},idem} \leqslant I_{z,\mathbb{1}}(a) \wedge I_{z,idem}(a)$

Lemma 5. *$\forall a \in Q, I_1(a) \leqslant I_2(I_1(a))$, if $I_1(a) \leqslant I_2(a)$.*

Proof. $I_1(a) \leqslant I_1(I_1(a)) \leqslant I_2(I_1(a))$ ☐

Lemma 6. *$I_1(a_1) \cdot I_2(a_2) \leqslant I^{'}(I_1(a_1) \cdot I_2(a_2))$, where $I_i \leqslant I^{'}, i = 1, 2$.*

Proof.
$I_1(a_1) \cdot I_2(a_2) \leqslant$
$I_1(I_1(a_1)) \cdot I_2(I_2(a_2)) \leqslant$
$I^{'}(I_1(a_1)) \cdot I^{'}(I_2(a_2)) \leqslant$
$I^{'}(I_1(a_1) \cdot I_2(a_2))$ ☐

Definition 9. *Interpretation of subexponential signature*

Let $\Sigma = \langle I, \preceq, W, C, E\rangle$ be a subexponential signature, where $|I| = n$ and $\mathcal{S} = \{\Box_1, \ldots, \Box_n\}$ be a set of open modalities on quantale Q. Subexponential interpretation is a contravariant map $\sigma : I \to \mathcal{S}$ defined as follows:

$$\sigma(s_i) = \begin{cases} \Box_i : Q \to Q,\ s.t. \forall a \in Q, \Box_i(a) = \{s \in S_i \mid s \leqslant a\}, \\ \quad \textit{if } s_i \notin W \cap C \cap E \\ \Box_i : Q \to Q,\ s.t. \forall a \in Q, \Box_i(a) = \{s \in S_i \mid s \leqslant a, \leqslant \mathbb{1}\}, \\ \quad \textit{if } s_i \in W \\ \Box_i : Q \to Q,\ s.t. \forall a \in Q, \Box_i(a) = \{s \in S_i \mid s \leqslant a, \in Z(Q)\}, \\ \quad \textit{if } s_i \in E \\ \Box_i : Q \to Q,\ s.t. \forall a \in Q, \Box_i(a) = \{s \in S_i \mid s \leqslant a, \forall b, b \cdot s \vee s \cdot b \leqslant s \cdot b \cdot s\}, \\ \quad \textit{if } s_i \in E \\ \textit{otherwise, if } s_i \textit{ belongs to some intersection of subsets, then we combine the relevant conditions} \end{cases}$$

Definition 10. *Let Q be a quantale, $f : Tp \to Q$ a valuation and $\sigma : I \to \mathcal{S}$ a subexponential interpretation, then interpretation is defined inductively:*

$$\begin{aligned} &[\![p_i]\!] = f(p_i) \\ &[\![\mathbb{1}]\!] = e \\ &[\![A \bullet B]\!] = [\![A]\!] \cdot [\![B]\!] \\ &[\![A \backslash B]\!] = [\![A]\!] \backslash [\![B]\!] \\ &[\![A / B]\!] = [\![A]\!] / [\![B]\!] \\ &[\![A \& B]\!] = [\![A]\!] \wedge [\![B]\!] \\ &[\![A \vee B]\!] = [\![A]\!] \vee [\![B]\!] \\ &[\![!_{s_i} A]\!] = \sigma(s_i)[\![A]\!] \end{aligned}$$

Theorem 1. $\Gamma \to A \Rightarrow [\![\Gamma]\!] \leqslant [\![A]\!]$

Proof. We consider the case with polymodal promotion rule.

1. Let $!_{s_1} A_1, \ldots, !_{s_n} A_n \to A$ and $\forall i, s \preceq s_i$.
   Then $\forall a \in Q, \sigma(s_i)(a) \leqslant \sigma(s)(a)$.
   By IH, $\sigma(s_1)[\![A_1]\!] \cdot \cdots \cdot \sigma(s_n)[\![A_n]\!] \leqslant [\![A]\!]$.
   Thus, $\sigma(s)(\sigma(s_1)[\![A_1]\!] \cdot \cdots \cdot \sigma(s_n)[\![A_n]\!]) \leqslant \sigma(s)([\![A]\!])$.
   By Lemma 5, $\sigma(s_1)[\![A_1]\!] \cdot \cdots \cdot \sigma(s_n)[\![A_n]\!] \leqslant \sigma(s)(\sigma(s_1)[\![A_1]\!] \cdot \cdots \cdot \sigma(s_n)[\![A_n]\!])$.
   So, $\sigma(s_1)[\![A_1]\!] \cdot \cdots \cdot \sigma(s_n)[\![A_n]\!] \leqslant \sigma(s)([\![A]\!])$.

2. Let $\Gamma, A, \Delta \to B$.
   By IH, $[\![\Gamma]\!] \cdot [\![A]\!] \cdot [\![\Delta]\!] \leqslant [\![B]\!]$.
   By the definition, $\sigma(s_i)([\![A]\!]) \leqslant [\![A]\!]$.
   So, $[\![\Gamma]\!] \cdot \sigma(s_i)([\![A]\!]) \cdot [\![\Delta]\!] \leqslant [\![B]\!]$

3. Let $\Gamma, \Delta \to B$, $A \in Fm$, and $s_i \in W$.
   So, $[\![\Gamma]\!] \cdot [\![\Delta]\!] \leqslant [\![B]\!]$, then $[\![\Gamma]\!] \cdot e \cdot [\![\Delta]\!] \leqslant [\![B]\!]$, where $e \in Q$ is unit.
   By the definition of unital open modality, $\sigma(s_i)([\![A]\!]) \leqslant e$.
   Thus, $[\![\Gamma]\!] \cdot \sigma(s_i)([\![A]\!]) \cdot [\![\Delta]\!] \leqslant [\![B]\!]$.

4. Let $\Gamma, !_{s_i} A, \Delta, !_{s_i} A, \Pi \to B$ and $s_i \in C$.
   By IH, $[\![\Gamma]\!] \cdot \sigma(s_i)([\![A]\!]) \cdot [\![\Delta]\!] \cdot \sigma(s_i)([\![A]\!]) \cdot [\![\Pi]\!] \leqslant [\![B]\!]$.
   By the definition, $\sigma(s_i)([\![A]\!]) \cdot [\![\Delta]\!] \leqslant \sigma(s_i)([\![A]\!]) \cdot [\![\Delta]\!] \cdot \sigma(s_i)([\![A]\!])$.
   Then $[\![\Gamma]\!] \cdot \sigma(s_i)([\![A]\!]) \cdot [\![\Delta]\!] \cdot [\![\Pi]\!] \leqslant [\![B]\!]$

5. Let $\Gamma, !_{s_i} A, \Delta, \Pi \to B$ and $s_i \in E$, so $\sigma(s_i)(a) \in Z(Q)$ for all $a \in Q$ by the definition.
   By IH, $[\![\Gamma]\!] \cdot \sigma(s_i)([\![A]\!]) \cdot [\![\Delta]\!] \cdot [\![\Pi]\!] \leqslant [\![B]\!]$.
   Hence, $[\![\Gamma]\!] \cdot [\![\Delta]\!] \cdot \sigma(s_i)([\![A]\!]) \cdot [\![\Pi]\!] \leqslant [\![B]\!]$

$\square$

3 Quantale completeness

Definition 11.
Let $\mathcal{F} \subseteq Fm$, an ideal is a subset $\mathcal{I} \subseteq \mathcal{F}$, such that:

- *If $B \in \mathcal{I}$ and $A \to B$, then $A \in \mathcal{I}$;*
- *If $A, B \in \mathcal{I}$, then $A \vee B \in \mathcal{I}$.*

Definition 12.
Let $S \subseteq \mathcal{F} \subseteq Fm$, then $\bigvee S = \bigcap\{\mathcal{I} \subseteq \mathcal{F} \mid S \subseteq \mathcal{I}\}$

Proposition 6. *$\bigvee S$ is an ideal.*

Lemma 7. *$A \in \mathcal{F}$, then $\{B \mid B \to A\} = \bigvee\{A\}$.*

Proof.
Let $A \in \mathcal{F}$. Then $\{B \mid B \to A\} \subseteq \bigvee\{A\}$.
On the other hand, $\{B \mid B \to A\}$ is an ideal, hence, $\{A\} \subseteq \{B \mid B \to A\}$. $\square$

Lemma 8. *$\bigvee\{A\} \subseteq \bigvee\{B\}$ iff $A \to B$.*

Proof. Let $\bigvee\{A\} \subseteq \bigvee\{B\}$, then $\{C|C \to A\} \subseteq \{C \mid C \to B\}$.
$A \in \{C|C \to A\}$, so far as $A \to A$, but $\{C \mid C \to B\}$, hence $A \to B$.
On the other hand, let $A \to B$ and $C \in \bigvee\{A\}$. Then $C \in \bigvee\{C' \mid C' \to A\}$, hence, $C \to A$, thus $C \to B$ by cut. So, $C \in \bigvee\{A\}$.

$\square$

Lemma 9. *Let $\mathcal{Q} = \{\bigvee S | S \subseteq Fm\}$ and $\bigvee \mathcal{A} \cdot \bigvee \mathcal{B} = \{A \bullet B | A \in \mathcal{A}, B \in \mathcal{B}\}$. Then $\langle \mathcal{Q}, \subseteq, \cdot, \bigvee \mathbf{1}\rangle$ is a quantale.*

Proof. See $\square$

Lemma 10. *Interior lemma.*
Let $Q_1 \subseteq \mathcal{Q}$, define a map $\square : Q \to Q$, such that $\square(A) = \{Q \in Q_1 \mid Q \subseteq A\}$. Then $\square$ is a quantic conucleus.

Lemma 11. *Let $!_s \in I$, $I \notin W \cap E \cap C$ and $Q \subseteq \mathcal{Q}$. Then there exist a subset $Q \subseteq \mathcal{Q}$ and a quantic conucleus $\square_s(\bigvee\{A\}) = \{\bigvee Q \in \mathcal{Q} \mid \}$*

Proof. $\square$

Proof. See $\square$

Lemma 12. *Let $Q \subseteq \mathcal{Q}$, then the following operators are quantic conuclei:*

1. *$\square_z(A) = \bigvee\{\bigvee\{W\} \in Q \mid \bigvee\{W\} \subseteq \bigvee\{A\}, \bigvee\{W\} \in Z(Q)\}$;*

2. $\Box_{\mathbf{1}}(A) = \bigvee\{\bigvee\{W\} \in Q \mid \bigvee\{W\} \subseteq \bigvee\{A\}, \bigvee\{W\} \subseteq \bigvee\{\mathbf{1}\}\}$;

3. $\Box_{idem}(A) = \bigvee\{\bigvee\{W\} \in Q \mid \bigvee\{W\} \subseteq \bigvee\{A\}, \forall B \in Fm, (\bigvee\{B\} \cdot \bigvee\{W\}) \cup (\bigvee\{W\} \cdot \bigvee\{B\}) \subseteq \bigvee\{W\} \cdot \bigvee\{A\} \cdot \bigvee\{W\}\}$;

4. $\Box_{z,\mathbb{1}}, \Box_{z,idem}, \Box_{\mathbb{1},idem}, \Box_{z,\mathbb{1},idem}$.

Proof. Follow from one of lemmas above. $\Box$

Lemma 13. *Let $!_s \in I$, $I \notin W \cap E \cap C$, then $\Box_s(\bigvee A) = \bigvee\{\bigvee(!_s W) \mid \bigvee !_s W \subseteq \bigvee A\}$ is a quantic conucleus.*

Proof.

1. $\Box_s(\bigvee A) \subseteq \bigvee A$;

   $\bigvee(!_s W) \in \Box_s(\bigvee A)$, then $\bigvee(!_s W) \in \bigvee\{\bigvee(!_s W) \mid \bigvee !_s W \subseteq \bigvee A\}$. Hence, $\bigvee(!_s F) \subseteq \bigvee\{A\}$, so, $!_s F \to A$. Therefore, $!_s F \in \bigvee\{A\}$.

2. $\Box_s(\Box_s(\bigvee A)) = \bigvee \Box_s(\bigvee A)$;

   $\Box_s(\Box_s(\bigvee A)) = \{\bigvee(!_s !_s F) \mid \bigvee(!_s !_s F) \subseteq \bigvee A\}$.

   Let $\bigvee(!_s !_s F) \in \Box_s(\Box_s(\bigvee A))$, then $!_s !_s F \to A$, hence $!_s F \to A$ by equivalence, so $\bigvee(!_s !_s F) \in \Box_s(\bigvee A)$

3. $\bigvee A \subseteq \bigvee B \Rightarrow \Box_s(\bigvee A) \subseteq \Box_s(\bigvee B)$;

4. $\Box_s \bigvee A \cdot \Box_s \bigvee A = \Box_s(\Box_s \bigvee A \cdot \Box_s \bigvee A)$.

$\Box$

Lemma 14. *Let Q be a quantale constructed above and $\Box_1, \ldots, \Box_n$ be a family of quantic conuclei on Q. Then there exist a model $\langle Q, [\![.]\!]\rangle$, such that $[\![A]\!] = \bigvee\{A\}$, $A \in Fm$.*

Proof. $\Box$

Theorem 2. $\Gamma \models A \Rightarrow \Gamma \to A$.