

# Modal type theory based on the intuitionistic epistemic logic

## Abstract

Modal intuitionistic epistemic logic  $IEL^-$  was proposed by S.Artemov and T. Protopopescu as the formal foundation for the intuitionistic theory of knowledge. We construct a modal simply typed lambda-calculus which is Curry-Howard isomorphic to  $IEL^-$  as formal theory of calculations with applicative functors in functional programming languages like Haskell or Idris.

## 1 Introduction

Modal intuitionistic epistemic logic  $IEL$  was proposed by S. Artemov and T. Protopopescu [1].  $IEL$  provides the epistimology and the theory of knowledge as based on BHK-semantics of intuitionistic logic.  $IEL^-$  is a variant of  $IEL$ , that corresponds to intuitionistic belief. Informally,  $\mathbf{K}A$  denotes that  $A$  is verified intuitionistically.

Intuitionistic epistemic logic  $IEL^-$  is defined with by following axioms and derivation rules:

**Definition 1.** *Intuitionistic epistemic logic  $IEL$ :*

- 1) *IPC axioms;*
  - 2)  $\mathbf{K}(A \rightarrow B) \rightarrow (\mathbf{K}A \rightarrow \mathbf{K}B)$  (*normality*);
  - 3)  $A \rightarrow \mathbf{K}A$  (*co-reflection*);
- Rule: MP.*

We have the deduction theorem and necessitation rule which is derivable.

V. Krupski and A. Yatmanov provided the sequential calculus for  $IEL$  and proved that this calculus is PSPACE-complete [2].

Functional programming languages such as Haskell [3], Idris [4], Purescript [5] Elm [6] or Scala [?] have special type classes<sup>1</sup> for calculations with container types like `Functor` and `Applicative`<sup>2</sup>:

```
class Functor f where
  fmap :: (a -> b) -> f a -> f b

class Functor f => Applicative f where
  pure :: a -> f a
  (<*>) :: f (a -> b) -> f a -> f b
```

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<sup>1</sup>Type class in Haskell is a general interface for special group of datatypes.

<sup>2</sup>Reader may read more about container types in the Haskell standard library documentation[7] or in the next one textbook [8]

By *container* (or *computational context*) type we mean some type-operator  $f$ , where  $f$  is a “function” from  $*$  to  $*$ : type operator takes a simple type (which has kind  $*$ ) and returns another simple type with kind  $*$ . For more detailed description of the type system with kinds used in Haskell see [12].

The motivation for using an applicative functor is quite natural. Applicative functor allows to generalize the action of a functor for functions with arbitrary number of arguments, for instance:

```
liftA2 :: Applicative f => (a -> b -> c) -> f a -> f b -> f c
liftA2 f x y = pure f <*> x <*> y
```

It’s not difficult to see that modal axioms in  $IEL^-$  and types of the methods of Applicative class in Haskell-like languages (which is described below) are syntactically similar and we are going to show that this coincidence has a non-trivial computational meaning.

The main goal of our research is a relationship between intuitionistic epistemic logic  $IEL^-$  and functional programming with effects. We show that relationship by building the type system (which is called  $\lambda_{\mathbf{K}}$ ) which is Curry-Howard isomorphic to  $IEL^-$ . So we will consider  $\mathbf{K}$ -modality as an arbitrary applicative functor.

$\lambda_{\mathbf{K}}$  consists of the rules for simply typed lambda-calculus and special typing rules for lifting types into the applicative functor  $\mathbf{K}$ . We assume that our type system will axiomatize the simplest case of computation with effects with one container. We provide proof-theoretical view on this kind of computations in functional programming and prove strong normalization and confluence.

## 2 Typed lambda-calculus based on $IEL^-$

At first we define the natural deduction for  $IEL^-$  :

**Definition 2.** *Natural deduction  $NIEL$  for  $IEL^-$  is an extension of intuitionistic natural deduction with additional derivation rules for modality:*

$$\frac{\Gamma \vdash A}{\Gamma \vdash \mathbf{K}A} K_I \qquad \frac{\Gamma \vdash \mathbf{K}\vec{A} \quad \vec{A} \vdash B}{\Gamma \vdash \mathbf{K}B}$$

Where  $\Gamma \vdash \mathbf{K}\vec{A}$  is a syntax sugar for  $\Gamma \vdash \mathbf{K}A_1, \dots, \Gamma \vdash \mathbf{K}A_n$ .

**Lemma 1.**  $\Gamma \vdash_{NIEL_{\wedge, \rightarrow}^-} A \Rightarrow IEL^- \vdash \bigwedge \Gamma \rightarrow A$ .

*Proof.* Induction on the derivation.

Let us consider cases with modality.

- 1) If  $\Gamma \vdash_{NIEL_{\wedge, \rightarrow}^-} A$ , then  $IEL^- \vdash \bigwedge \Gamma \rightarrow \mathbf{K}A$ .
  - (1)  $\bigwedge \Gamma \rightarrow A$  assumption
  - (2)  $A \rightarrow \mathbf{K}A$  co-reflection
  - (3)  $(\bigwedge \Gamma \rightarrow A) \rightarrow ((A \rightarrow \mathbf{K}A) \rightarrow (\bigwedge \Gamma \rightarrow \mathbf{K}A))$  IPC theorem
  - (4)  $(A \rightarrow \mathbf{K}A) \rightarrow (\bigwedge \Gamma \rightarrow \mathbf{K}A)$  from (1), (3) and MP
  - (5)  $\bigwedge \Gamma \rightarrow \mathbf{K}A$  from (2), (4) and MP

- 2) If  $\Gamma \vdash_{NIEL_{\wedge, \rightarrow}^-} \mathbf{K}\vec{A}$  and  $\vec{A} \vdash B$ , then  $IEL^- \vdash \bigwedge \Gamma \rightarrow \mathbf{K}B$ .
- (1)  $\bigwedge \Gamma \rightarrow \bigwedge_{i=1}^n \mathbf{K}A_i$  assumption
  - (2)  $\bigwedge_{i=1}^n \mathbf{K}A_i \rightarrow \mathbf{K} \bigwedge_{i=1}^n A_i$  IEL theorem
  - (3)  $\bigwedge \Gamma \rightarrow \mathbf{K} \bigwedge_{i=1}^n A_i$  from (1), (2) and transitivity
  - (4)  $\bigwedge_{i=1}^n A_i \rightarrow B$  assumption
  - (5)  $(\bigwedge_{i=1}^n A_i \rightarrow B) \rightarrow \mathbf{K}(\bigwedge_{i=1}^n A_i \rightarrow B)$  co-reflection
  - (6)  $\mathbf{K}(\bigwedge_{i=1}^n A_i \rightarrow B)$  from (2), (3) and MP
  - (7)  $\mathbf{K} \bigwedge_{i=1}^n A_i \rightarrow \mathbf{K}B$  from (6) and normality
  - (8)  $\bigwedge \Gamma \rightarrow \mathbf{K}B$  from (3), (7) and transitivity

□

**Lemma 2.** *If  $IEL^- \vdash A$ , then  $NIEL^- \vdash A$ .*

*Proof.* Straightforward derivation of modal axioms in  $NIEL^-$ . We consider this derivation below using terms. □

At the next step we build the typed lambda-calculus based on  $NIEL_{\wedge, \rightarrow}^-$  by proof-assigning in rules.

At first, we define lambda-terms and types for this lambda-calculus.

**Definition 3.** *The set of terms:*

*Let  $\mathbb{V}$  be the set of variables. The set  $\Lambda_{\mathbf{K}}$  of terms is defined by the grammar:*

$$\Lambda_{\mathbf{K}} ::= \mathbb{V} \mid (\lambda \Lambda. \Lambda_{\mathbf{K}}) \mid (\Lambda_{\mathbf{K}} \Lambda_{\mathbf{K}}) \mid (\Lambda_{\mathbf{K}}, \Lambda_{\mathbf{K}}) \mid (\pi_1 \Lambda_{\mathbf{K}}) \mid (\pi_2 \Lambda_{\mathbf{K}}) \mid (\text{pure } \Lambda_{\mathbf{K}}) \mid (\text{let pure } \Lambda_{\mathbf{K}} = \Lambda_{\mathbf{K}} \text{ in } \Lambda_{\mathbf{K}})$$

**Definition 4.** *The set of types:*

*Let  $\mathbb{T}$  be the set of atomic types. The set  $\mathbb{T}_{\mathbf{K}}$  of types with applicative functor  $\mathbf{K}$  is generated by the grammar:*

$$\mathbb{T}_{\mathbf{K}} ::= \mathbb{T} \mid (\mathbb{T}_{\mathbf{K}} \rightarrow \mathbb{T}_{\mathbf{K}}) \mid (\mathbb{T}_{\mathbf{K}} \times \mathbb{T}_{\mathbf{K}}) \mid (\mathbf{K}\mathbb{T}_{\mathbf{K}}) \quad (1)$$

Context, domain of context and range of context are defined standardly [11][12].

Our type system is based on the Curry-style typing rules:

**Definition 5.** *Modal typed lambda calculus  $\lambda_{\mathbf{K}}$  based on  $NIEL_{\wedge, \rightarrow}^-$ :*

$$\frac{}{\Gamma, x : A \vdash x : A} \text{ax}$$

$$\begin{array}{c}
\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x. M : A \rightarrow B} \rightarrow_i \qquad \frac{\Gamma \vdash M : A \rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B} \rightarrow_e \\
\\
\frac{\Gamma \vdash M : A \quad \Gamma \vdash N : B}{\Gamma \vdash \langle M, N \rangle : A \times B} \times_i \qquad \frac{\Gamma \vdash M : A_1 \times A_2}{\Gamma \vdash \pi_i M : A_i} \times_e, i \in \{1, 2\} \\
\\
\frac{\Gamma \vdash M : A}{\Gamma \vdash \mathbf{pure} M : \mathbf{K}A} \mathbf{K}_I \qquad \frac{\Gamma \vdash \vec{M} : \mathbf{K}\vec{A} \quad \vec{x} : \vec{A} \vdash N : B}{\Gamma \vdash \mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} N : \mathbf{K}B} \mathbf{let_K}
\end{array}$$

$\mathbf{K}_I$ -typing rule is the same as  $\bigcirc$ -introduction in lax logic (also known as monadic metalanguage [17]) and in typed lambda-calculus which is derived by proof-assignment for lax-logic proofs.  $\mathbf{K}_I$  allows to inject an object of type  $\alpha$  into the functor.  $\mathbf{K}_I$  reflects the Haskell method **pure** for Applicative class. It plays the same role as the **return** method in Monad class.

$\mathbf{let_K}$  is similar to the  $\square$ -rule in typed lambda calculus for intuitionistic normal modal logic  $\mathbf{IK}$ , which is described in [19].

In fact, our calculus is the extension of typed lambda calculus for  $\mathbf{IK}$  with typing rule appropriate to co-reflection.

Here are some examples of closed terms:

- $(\lambda x. \mathbf{pure} x) : A \rightarrow \mathbf{K}A$ ;
- $\lambda f. \lambda x. \mathbf{let pure} g, y = f, x \mathbf{in} gy : \mathbf{K}(A \rightarrow B) \rightarrow \mathbf{K}A \rightarrow \mathbf{K}B$
- $\lambda f. \lambda x. \mathbf{let pure} g, y = \mathbf{pure} f, x \mathbf{in} gy : (A \rightarrow B) \rightarrow \mathbf{K}A \rightarrow \mathbf{K}B$

Now we define free variables and substitutions.  $\beta$ -reduction, multi-step  $\beta$ -reduction and  $\beta$ -equality are defined standardly:

**Definition 6.** Set  $FV(M)$  of free variables for arbitrary term  $M$ :

- 1)  $FV(x) = \{x\}$ ;
- 2)  $FV(\lambda x. M) = FV(M) \setminus \{x\}$ ;
- 3)  $FV(MN) = FV(M) \cup FV(N)$ ;
- 4)  $FV(\langle M, N \rangle) = FV(M) \cup FV(N)$ ;
- 5)  $FV(\pi_i M) \subseteq FV(M)$ ,  $i \in \{1, 2\}$ ;
- 6)  $FV(\mathbf{pure} M) = FV(M)$ ;
- 7)  $FV(\mathbf{let pure} \vec{N} = \vec{M} \mathbf{in} M) = \bigcup_{i=1}^n FV(M)$ , where  $n = |\vec{M}|$ .

**Definition 7.** Substitution:

- 1)  $x[x := N] = N$ ,  $x[y := N] = x$ ;
- 2)  $(MN)[x := N] = M[x := N]N[x := N]$ ;
- 3)  $(\lambda x. M)[x := N] = \lambda x. M[x := N]$ ;
- 4)  $(M, N)[x := P] = (M[x := P], N[x := P])$ ;
- 5)  $(\pi_i M)[x := P] = \pi_i(M[x := P])$ ,  $i \in \{1, 2\}$ ;
- 6)  $(\mathbf{pure} M)[x := P] = \mathbf{pure}(M[x := P])$ ;
- 7)  $(\mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} N)[y := P] = \mathbf{let pure} \vec{x} = (\vec{M}[y := P]) \mathbf{in} N$ .

**Definition 8.** *Type substitution*

The substitution of type  $C$  for type variable  $B$  in type  $A$  inductively defined as follows:

- 1)  $B[B := C] = B$  and  $D[B := C] = D$ , if  $B \neq D$ ;
- 2)  $(A_1 \alpha A_2)[B := C] = (A_1[B := C])\alpha(A_2[B := C])$ , where  $\alpha \in \{\rightarrow, \times\}$ ;
- 3)  $(\mathbf{K}A)[B := C] = \mathbf{K}(A[B := C])$ .
- 4) Let  $\Gamma$  be the context, then  $\Gamma[B := C] = \{x : (A[B := C]) \mid x : A \in \Gamma\}$

**Definition 9.**  *$\beta$ -reduction and  $\eta$ -reduction rules for  $\lambda\mathbf{K}$ .*

- 1)  $(\lambda x.M)N \rightarrow_\beta M[x := N]$ ;
- 2)  $\pi_1 \langle M, N \rangle \rightarrow_\beta M$ ;
- 3)  $\pi_2 \langle M, N \rangle \rightarrow_\beta N$ ;
- 4)  $\text{let pure } \langle \vec{x}, y, \vec{z} \rangle = \langle \vec{M}, \text{let pure } \vec{w} = \vec{N} \text{ in } Q, \vec{P} \rangle \text{ in } R \rightarrow_\beta \text{let pure } \langle \vec{x}, \vec{w}, \vec{z} \rangle = \langle \vec{M}, \vec{N}, \vec{P} \rangle \text{ in } R[y := Q]$
- 5)  $\text{let pure } \vec{x} = \text{pure } \vec{M} \text{ in } N \rightarrow_\beta \text{pure } N[\vec{x} := \vec{M}]$
- 6)  $\lambda x.f x \rightarrow_\eta f$ ;
- 7)  $\langle \pi_1 P, \pi_2 P \rangle \rightarrow_\eta P$ ;
- 8)  $\text{let pure } x = M \text{ in } x \rightarrow_\eta M$ ;

By default we use call-by-name evaluation strategy.

### 3 Basic lemmas

Now we will prove standard lemmas for contexts in type systems<sup>3</sup>:

**Lemma 3.** *Generation for  $\mathbf{K}_I$ .*

Let  $\Gamma \vdash \text{pure } M : \mathbf{K}A$ , then  $\Gamma \vdash M : A$ ;

*Proof.* Induction on the structure of  $\text{pure } M$ . □

**Lemma 4.** *Basic lemmas .*

- i) Let  $\Gamma \vdash M : A$  and  $\Gamma \subseteq \Delta$ , then  $\Delta \vdash M : A$ ;
- ii) Let  $\Gamma, x : A \vdash M : B$  and  $\Gamma \vdash N : A$ , then  $\Gamma \vdash M[x := N] : B$ .
- iii) Let  $\Gamma \vdash M : A$ , then  $\Gamma[B := C] \vdash M : (A[B := C])$ .

*Proof.*

- i-ii-iii) Induction on  $\Gamma \vdash M : A$ . □

**Theorem 1.** *Subject reduction*

Let  $\Gamma \vdash M : A$  and  $M \rightarrow_{\beta\eta} N$ , then  $\Gamma \vdash N : A$

*Proof.* For cases with application, abstraction and pairs see [12] [13].

- 1) Let  $\Gamma \vdash \text{let pure } \vec{x}, y, \vec{z} = \langle \vec{M}, \text{let pure } \vec{w} = \vec{N} \text{ in } Q, \vec{P} \rangle \text{ in } R : \mathbf{K}B$ , then  $\Gamma \vdash \text{let pure } \vec{x}, \vec{w}, \vec{z} = \langle \vec{M}, \vec{N}, \vec{P} \rangle \text{ in } R[y := Q] : \mathbf{K}B$
- 2) Let  $\Gamma \vdash \text{let pure } x = M \text{ in } x : \mathbf{K}A$ , then  $\Gamma \vdash M : \mathbf{K}A$ .  
See [19].
- 3) If the derivation ends in

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<sup>3</sup>We will not prove cases with  $\rightarrow$ -constructor, they are proved standardly in the same lemmas for simply typed lambda calculus, for example see [11] [12] [14]. We will consider only modal cases

$$\frac{\Gamma \vdash \mathbf{pure} \vec{M} : \mathbf{K}\vec{A} \quad \vec{x} : \vec{A} \vdash N : B}{\Gamma \vdash \mathbf{let pure} \vec{x} = \mathbf{pure} \vec{M} \mathbf{in} N : \mathbf{K}B}$$

So  $\Gamma \vdash \vec{M} : \vec{A}$  by generation and  $\Gamma \vdash N[\vec{x} := \vec{M}] : B$  by weakening and substitution.

Then we can transform this into the next derivation:

$$\frac{\Gamma \vdash N[\vec{x} := \vec{M}] : B}{\Gamma \vdash \mathbf{pure} N[\vec{x} := \vec{M}] : \mathbf{K}B} \mathbf{K}_I$$

□

**Theorem 2.**

$\rightarrow_\beta$  is strongly normalizing;

*Proof.*

We modify and apply Tait's technique of logical relation for modalities. For strong normalization proof with Tait's method for simply typed lambda calculus see [13].

**Definition 10.** The set of strongly computable terms:

- $SC_A = \{M : A \mid M \text{ is strongly normalizing}\}$  for  $A \in \mathbb{T}$ ;
- $SC_{A \rightarrow B} = \{M : A \rightarrow B \mid \forall N \in SC_A, MN \in SC_B\}$ , for  $A, B \in \mathbb{T}_{\mathbf{K}}$
- $SC_{\mathbf{K}A} = \{M : \mathbf{K}A \mid M \text{ is strongly normalizing}\}$  for  $A \in \mathbb{T}$ ;
- $SC_{\mathbf{K}(A \rightarrow B)} = \{M : \mathbf{K}(A \rightarrow B) \mid \forall f \in SC_{A \rightarrow B}, \forall x \in SC_A, \forall N \in SC_{\mathbf{K}A}, \mathbf{let pure} f, x = M, N \mathbf{in} fx \in SC_{\mathbf{K}B}\}$  for  $A, B \in \mathbb{T}_{\mathbf{K}}$ .

**Lemma 5.**

- If  $M \in SC_A$ , then  $M$  is strongly normalizing;
- Let  $M \in SC_A$  and  $M \rightarrow_\beta N$ , then  $N \in SC_A$ ;
- Let  $N$  is non-introduced,  $N \in SC_A$ . Then, if  $M \rightarrow_\beta N$ , then  $M \in SC_A$ ;

*Proof.*

By induction on the structure of  $A$ .

1)  $A \equiv \mathbf{K}A$ , where  $A \in \mathbb{T}$ .

i) Follows from the definition;

ii) Immediately;

iii) Let  $N$  is non-introduced and  $N \in SC_A$ , such that  $M \rightarrow_\beta N$ . Any reduction path  $M \rightarrow_\beta \dots$  passes through  $M \rightarrow_\beta N$ .

$N$  is strongly normalizing, so  $M$  too.

2)  $A \equiv \mathbf{K}(B \rightarrow C)$

i) Suppose  $M \in SC_{\mathbf{K}(B \rightarrow C)}$ . Let  $N \in SC_{\mathbf{K}B}$ . So  $\mathbf{let pure} f, x = M, N \mathbf{in} fx \in SC_{\mathbf{K}C}$ .

So  $M$  is strongly normalizing, since  $\mathbf{let pure} f, x = M, N \mathbf{in} fx$  is strongly normalizing.

ii) Let  $M_1 \in SC_{\mathbf{K}(B \rightarrow C)}$  and  $M_1 \rightarrow_\beta M_2$ . Fix  $N \in SC_{\mathbf{K}B}$ .

Then **let pure**  $f, x = M_1, N$  **in**  $fx \in SC_{\mathbf{K}C}$ .  
Hence, **let pure**  $f, x = M_1, N$  **in**  $fx \in SC_{\mathbf{K}C} \rightarrow_\beta$  **let pure**  $f, x = M_2, N$  **in**  $fx$ .  
So **let pure**  $f, x = M_2, N$  **in**  $fx \in SC_{\mathbf{K}C}$ . Then  $M_2 \in SC_{\mathbf{K}(B \rightarrow C)}$ .  
iii) Let  $M_2$  be non-introduced,  $M_2 \in SC_{\mathbf{K}(B \rightarrow C)}$  and  $M_1 \rightarrow_\beta M_2$ .  
Let  $N \in SC_{\mathbf{K}B}$ . So **let pure**  $f, x = M_2, N$  **in**  $fx \in SC_{\mathbf{K}C}$ .  
So **let pure**  $f, x = M_1, N$  **in**  $fx \rightarrow_\beta$  **let pure**  $f, x = M_2, N$  **in**  $fx \in SC_{\mathbf{K}C}$ .  
Thus **let pure**  $f, x = M_1, N$  **in**  $fx \in SC_{\mathbf{K}C}$  by IH, so  $M_1 \in SC_{\mathbf{K}(B \rightarrow C)}$ .  $\square$

**Lemma 6.**

If  $M \in SC_A$ , then **pure**  $M \in SC_{\mathbf{K}A}$

*Proof.* Induction on the structure of  $M$ .  $\square$

**Lemma 7.**

Let  $x_1 : A_1, \dots, x_n : A_n \vdash M : A$ , then for all  $i, M_i \in SC_{A_i}$ . Then  $M[x_1 := M_1, \dots, x_n := M_n] \in SC_A$ .

*Proof.*

1) Let the derivation ends in:

$$\frac{x_1 : A_1, \dots, x_n : A_n \vdash M : A}{x_1 : A_1, \dots, x_n : A_n \vdash \mathbf{pure} M : \mathbf{K}A}$$

By assumption  $M[x_1 := M_1, \dots, x_n := M_n] \in SC_A$ , so **pure**  $M[x_1 := M_1, \dots, x_n := M_n] \in SC_{\mathbf{K}A}$ .

2) Let the derivation ends in:

$$\frac{x_1 : A_1, \dots, x_n : A_n \vdash \vec{M}' : \mathbf{K}\vec{A} \quad \vec{x} : \vec{A} \vdash N : B}{x_1 : A_1, \dots, x_n : A_n \vdash \mathbf{let pure} \vec{x} = \vec{M}' \mathbf{in} N : \mathbf{K}B}$$

By IH for all  $i \in \{1, \dots, \text{length}(\vec{M}')\}$ ,  $M'_i[x_1 := M_1, \dots, x_n := M_n] \in SC_{A_i}$ .

So **let pure**  $\vec{x} = \vec{M}'[x_1 := M_1, \dots, x_n := M_n]$  **in**  $N \in SC_{\mathbf{K}B}$ , otherwise we can build infinite reduction path in  $\vec{M}'[x_1 := M_1, \dots, x_n := M_n]$ .  $\square$

**Corollary 1.** All terms are strongly computable, therefore are strongly normalizing.  $\square$

**Theorem 3.**

$\rightarrow_\beta$  is confluent.

*Proof.* We modify and apply Barendregt's technique with term underlying. We will consider the fragment of the grammar for terms without constructors for pairs for simplicity.

**Definition 11.** The set of underlined terms.

- $x \in \mathbb{V} \Rightarrow x \in \underline{\Lambda}$ ;
- $M \in \underline{\Lambda} \Rightarrow (\lambda x.M) \in \underline{\Lambda}$ ;

- $M, N \in \underline{\Lambda} \Rightarrow (MN) \in \underline{\Lambda};$
- $M \in \underline{\Lambda} \Rightarrow (\mathbf{pure} M) \in \underline{\Lambda};$
- $\vec{x} \in \mathbb{V}, \vec{M}, N \in \underline{\Lambda} \Rightarrow \mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} N \in \underline{\Lambda};$
- $M, N \in \underline{\Lambda} \Rightarrow (\lambda_i x. M)N \in \underline{\Lambda}, \text{ for all } i \in \mathbb{N}.$

**Definition 12.** *Substitution for term with labelled lambda:*

$$((\lambda_i x. M)N)[y := Z] = (\lambda_i x. M[y := Z])(N[y := Z])$$

**Definition 13.** *Index erasing*

Let us define map  $|\cdot| : \underline{\Lambda} \rightarrow \Lambda$  as follows:

- $|x| = x;$
- $|\lambda x. M| = \lambda x. |M|;$
- $|MN| = |M||N|;$
- $|\mathbf{pure} M| = \mathbf{pure} |M|;$
- $|\mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} N| = \mathbf{let pure} \vec{x} = |\vec{M}| \mathbf{in} |N|;$
- $|(\lambda_i x. M)N| = (\lambda x. M)N$

**Definition 14.** *Reduction rules:*

- $(\lambda x. M)N \rightarrow_{\underline{\beta}} M[x := N];$
- $\mathbf{let pure} \langle \vec{x}, y, \vec{z} \rangle = \langle \vec{M}, \mathbf{let pure} \vec{w} = \vec{N} \mathbf{in} Q, \vec{P} \rangle \text{ in } R \rightarrow_{\underline{\beta}} \mathbf{let pure} \langle \vec{x}, \vec{w}, \vec{z} \rangle = \langle \vec{M}, \vec{N}, \vec{P} \rangle \text{ in } R[y := Q];$
- $\mathbf{let pure} \vec{x} = \mathbf{pure} \vec{M} \mathbf{in} N \rightarrow_{\underline{\beta}} \mathbf{pure} N[\vec{x} := \vec{M}];$
- $(\lambda x_i. M)N \rightarrow_{\underline{\beta}} M[x := N]$

$\rightarrow_{\underline{\beta}}$  is a reflexive-transitive closure of  $\rightarrow_{\underline{\beta}}$ .

**Definition 15.** *Indexed redex erasing:*

Let us define the next map  $\phi : \underline{\Lambda} \rightarrow \Lambda$ :

- $\phi(x) = x;$
- $\phi(\lambda x. M) = \lambda x. \phi(M);$
- $\phi(MN) = \phi(M)\phi(N);$
- $\phi(\mathbf{pure} M) = \mathbf{pure} \phi(M);$
- $\phi(\mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} N) = \mathbf{let pure} \vec{x} = \phi(\vec{M}) \mathbf{in} \phi(N);$
- $\phi((\lambda_i x. M)N) = M[x := N]$

**Lemma 8.**  $\forall \underline{M}, \underline{N} \in \underline{\Lambda} \forall M, N \in \Lambda, \text{ if } |\underline{M}| = M, |\underline{N}| = N, \text{ then}$

- $\text{If } M \twoheadrightarrow_{\underline{\beta}} N, \text{ then } \underline{M} \rightarrow_{\underline{\beta}} \underline{N}$



- *Vice versa*

*Proof.* Induction on the generation  $\rightarrow_\beta$  and  $\rightarrow_{\underline{\beta}}$  correspondently. The general statement follows from transitivity of multi-step reductions of both types.  $\square$

**Lemma 9.**  $\phi(M[x := N]) = \phi(M)[x := \phi(N)]$ .

*Proof.* We treat only cases with **pure** and with **let**. For the rest cases see [15].

- 1)
 

$\phi(\mathbf{pure}(M[x := N])) =$	By the definition of $\phi$
$\mathbf{pure}(\phi(M[x := N])) =$	Induction hypothesis
$\mathbf{pure}(\phi(M)[x := \phi(N)]) =$	Substitution definition
$(\mathbf{pure} \phi(M))[x := \phi(N)]$	
- 2)
 

$\phi((\mathbf{let} \mathbf{pure} \vec{x} = \vec{M} \mathbf{in} N)[y := P]) =$	Substitution definition
$\phi(\mathbf{let} \mathbf{pure} \vec{x} = (\vec{M}[y := P]) \mathbf{in} N) =$	By the definition of $\phi$
$\mathbf{let} \mathbf{pure} \vec{x} = \phi(\vec{M}[y := P]) \mathbf{in} \phi(N) =$	Induction hypothesis
$\mathbf{let} \mathbf{pure} \vec{x} = (\phi(\vec{M})[y := \phi(P)]) \mathbf{in} \phi(N) =$	Substitution definition
$(\mathbf{let} \mathbf{pure} \vec{x} = \phi(\vec{M}) \mathbf{in} \phi(N))[y := \phi(P)]$	

$\square$

**Lemma 10.**

- If  $M \rightarrow_{\underline{\beta}} N$ , then  $\phi(M) \rightarrow_\beta \phi(N)$
- If  $|M| = N$  and  $\phi(M) = P$ , then  $N \rightarrow_\beta P$ .

*Proof.*

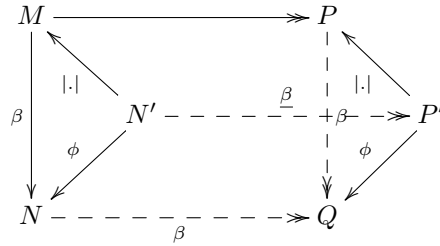
- i) Induction on the generation of  $\rightarrow_{\underline{\beta}}$  using previous lemma.
- ii) Induction on the structure of  $M$ .

$\square$

**Lemma 11.** *Strip lemma.*

If  $M \rightarrow_\beta N$  and  $M \rightarrow_\beta P$ . Then there exists some term  $Q$ , such that  $N \rightarrow_\beta Q$  and  $P \rightarrow_\beta Q$ .

*Proof.* Proof is similar to [15] [18]. We build the following diagram, which commutes by lemmas 8 and 10.



$\square$

**Corollary 2.** If  $M \rightarrow_\beta N$  and  $M \rightarrow_\beta P$ . Then there exists some term  $Q$ , such that  $N \rightarrow_\beta Q$  and  $P \rightarrow_\beta Q$ .

*Proof.* Unfold  $M \rightarrow_\beta N$  as the sequence of one-step reductions and apply strip lemma on the every step.  $\square$

$\square$

**Theorem 4.**

*Normal form in  $\lambda_K$  has the subformula property.*

*Proof.* By induction on the structure of term. Case with **let pure**  $\vec{x} = \vec{M}$  **in**  $N$  was considered by Kakutani [19] [20]. Similary, if **pure**  $M$  is a normal form, so  $M$  is a normal form too by hypothesis.  $\square$

## 4 Categorical semantics

**Definition 16.** *Monoidal functor*

Let  $\langle \mathcal{C}, \otimes_1, \mathbb{1} \rangle$  and  $\langle \mathcal{D}, \otimes_2, \mathbb{1}' \rangle$  are monoidal categories.

A monoidal functor  $\mathcal{F} : \langle \mathcal{C}, \otimes_1, \mathbb{1} \rangle \rightarrow \langle \mathcal{D}, \otimes_2, \mathbb{1}' \rangle$  is a functor  $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$  with additional natural transformations, which satisfy the well-known conditions described in [23]:

- 1)  $u : \mathbb{1}' \rightarrow \mathcal{F}\mathbb{1}$ ;
- 2)  $*_{A,B} : \mathcal{F}A \otimes_2 \mathcal{F}B \rightarrow \mathcal{F}(A \otimes_1 B)$ .

**Definition 17.** *Applicative functor*

An applicative functor is a triple  $\langle \mathcal{C}, \mathcal{K}, \eta \rangle$ , where  $\mathcal{C}$  is a symmetric monoidal category,  $\mathcal{K}$  is a monoidal endofunctor and  $\eta : Id_{\mathcal{C}} \Rightarrow \mathcal{K}$  is a natural transformation (similar to unit in monad), such that:

- 1)  $u = \eta_{\mathbb{1}}$ ;
- 2)  $*_{A,B} \circ (\eta_A \otimes \eta_B) = \eta_{A \otimes B}$ ;
- 3) Weak commutativity condition:

$$\begin{array}{ccccc}
 A \otimes \mathcal{K}B & \xrightarrow{\eta_A \otimes id_{\mathcal{K}B}} & \mathcal{K}A \otimes \mathcal{K}B & \xrightarrow{*_{A,B}} & \mathcal{K}(A \otimes B) \\
 \sigma_{A,\mathcal{K}B} \downarrow & & & & \downarrow \mathcal{K}(\sigma_{A,B}) \\
 \mathcal{K}B \otimes A & \xrightarrow{id_{\mathcal{K}B} \otimes \eta_A} & \mathcal{K}B \otimes \mathcal{K}A & \xrightarrow{*_{B,A}} & \mathcal{K}(B \otimes A)
 \end{array}$$

### 4.1 Soundness and completeness

**Theorem 5.** *Soundness*

Let  $\Gamma \vdash M : A$  and  $M =_{\beta_\eta} N$ , then  $\llbracket \Gamma \vdash M : A \rrbracket = \llbracket \Gamma \vdash N : A \rrbracket$

*Proof.*

**Definition 18.** *Semantical translation from  $\lambda_K$  to some cartesian closed category  $\mathcal{C}$  with applicative functor  $\mathcal{K}$ :*

- 1) *Interpretation for types:*

$\llbracket A \rrbracket := \hat{A}, A \in \mathbb{T}$ , where  $\hat{A}$  is an object of  $\mathcal{C}$  obtained by some given assignment;

$\llbracket A \rightarrow B \rrbracket := \llbracket A \rrbracket \rightarrow \llbracket B \rrbracket$ ;

$\llbracket A \times B \rrbracket := \llbracket A \rrbracket \times \llbracket B \rrbracket$ .

- 2) *Interpretation for modal types:*  $\llbracket KA \rrbracket = \mathcal{K}\llbracket A \rrbracket$ ;

- 3) *Interpretation for contexts:*

- $\llbracket \Gamma = \{x_1 : A_1, \dots, x_n : A_n\} \rrbracket := \llbracket \Gamma \rrbracket = \llbracket A_1 \rrbracket \times \dots \times \llbracket A_n \rrbracket$ ;  
 4) *Interpretation for typing assignment:*  $\llbracket \Gamma \vdash M : A \rrbracket := \llbracket M \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket$ .  
 5) *Interpretation for typing rules:*

$$\begin{array}{c}
\frac{}{\llbracket \Gamma, x : A \vdash x : A \rrbracket = \pi_2 : \llbracket \Gamma \rrbracket \times \llbracket A \rrbracket \rightarrow \llbracket A \rrbracket} \\
\frac{\llbracket \Gamma, x : A \vdash M : B \rrbracket = f : \llbracket \Gamma \rrbracket \times \llbracket A \rrbracket \rightarrow \llbracket B \rrbracket}{\llbracket \Gamma \vdash (\lambda x.M) : A \rightarrow B \rrbracket = \Lambda(f) : \llbracket \Gamma \rrbracket \rightarrow \llbracket B \rrbracket^{[A]}} \\
\frac{\llbracket \Gamma \vdash M : A \rightarrow B \rrbracket = \llbracket M \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket B \rrbracket^{[A]} \quad \llbracket \Gamma \vdash N : A \rrbracket = \llbracket N \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket}{\llbracket \Gamma \vdash (MN) : B \rrbracket = \llbracket \Gamma \rrbracket \xrightarrow{\langle \llbracket M \rrbracket, \llbracket N \rrbracket \rangle} \llbracket B \rrbracket^{[A]} \times \llbracket A \rrbracket \xrightarrow{\epsilon} \llbracket B \rrbracket} \\
\frac{\llbracket \Gamma \vdash M : A \rrbracket = f : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket \quad \llbracket \Gamma \vdash N : B \rrbracket = g : \llbracket \Gamma \rrbracket \rightarrow \llbracket B \rrbracket}{\llbracket \Gamma \vdash (M, N) : A \times B \rrbracket = \langle f, g \rangle : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket \times \llbracket B \rrbracket} \\
\frac{\llbracket \Gamma \vdash p : A_1 \times A_2 \rrbracket = f : \llbracket \Gamma \rrbracket \rightarrow \llbracket A_1 \rrbracket \times \llbracket A_2 \rrbracket}{\llbracket \Gamma \vdash \pi_i p : A_i \rrbracket = \llbracket \Gamma \rrbracket \xrightarrow{f} \llbracket A_1 \rrbracket \times \llbracket A_2 \rrbracket \xrightarrow{\pi_i} \llbracket A_i \rrbracket} \quad i \in \{1, 2\} \\
\frac{\llbracket \Gamma \vdash M : A \rrbracket = \llbracket M \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket}{\llbracket \Gamma \vdash \mathbf{pure} M : \mathbf{KA} \rrbracket := \llbracket \Gamma \rrbracket \xrightarrow{\llbracket M \rrbracket} \llbracket A \rrbracket \xrightarrow{\eta_{[A]}} \mathcal{K}[\llbracket A \rrbracket]} \\
\frac{\llbracket \Gamma \vdash \vec{M} : \mathbf{KA} \rrbracket = \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle : \llbracket \Gamma \rrbracket \rightarrow \prod_{i=1}^n \mathcal{K}[\llbracket A_i \rrbracket] \quad \llbracket \vec{x} : \vec{A} \vdash N : B \rrbracket = \llbracket N \rrbracket : \prod_{i=1}^n \llbracket A_i \rrbracket \rightarrow \llbracket B \rrbracket}{\llbracket \Gamma \vdash \mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} M : \mathbf{KB} \rrbracket = \mathcal{K}(\llbracket N \rrbracket) \circ *_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle : \llbracket \Gamma \rrbracket \rightarrow \mathcal{K}[\llbracket B \rrbracket]}
\end{array}$$

**Definition 19.** *Simultaneous substitution*

Let  $\Gamma = \{x_1 : A_1, \dots, x_n : A_n\}$ ,  $\Gamma \vdash M : A$  and for all  $i \in \{1, \dots, n\}$ ,  $\Gamma \vdash M_i : A_i$ .

We define simultaneous substitution  $M[\vec{x} := \vec{M}]$  recursively by:

- 1)  $x_i[\vec{x} := \vec{M}] = M_i$ ;
- 2)  $(\lambda x.M)[\vec{x} := \vec{M}] = \lambda x.(M[\vec{x} := \vec{M}])$ ;
- 3)  $(MN)[\vec{x} := \vec{M}] = (M[\vec{x} := \vec{M}])(N[\vec{x} := \vec{M}])$ ;
- 4)  $\langle M, N \rangle = \langle (M[\vec{x} := \vec{M}]), (N[\vec{x} := \vec{M}]) \rangle$ ;
- 5)  $(\pi_i P)[\vec{x} := \vec{M}] = \pi_i(P[\vec{x} := \vec{M}])$ ;
- 6)  $(\mathbf{pure} M)[\vec{x} := \vec{M}] = \mathbf{pure} (M[\vec{x} := \vec{M}])$ ;
- 7)  $(\mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} N)[\vec{y} := \vec{P}] = \mathbf{let pure} \vec{x} = (\vec{M}[\vec{y} := \vec{P}]) \mathbf{in} N$

**Lemma 12.**

$$\llbracket M[x_1 := M_1, \dots, x_n := M_n] \rrbracket = \llbracket M \rrbracket \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle.$$

*Proof.*

$$1) \llbracket \Gamma \vdash (\mathbf{pure} M)[\vec{x} := \vec{M}] : \mathbf{KA} \rrbracket = \llbracket \Gamma \vdash \mathbf{pure} M : \mathbf{KA} \rrbracket \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle.$$

$$\begin{array}{ll}
\llbracket \Gamma \vdash (\mathbf{pure} M)[\vec{x} := \vec{M}] : \mathbf{KA} \rrbracket = \llbracket \Gamma \vdash \mathbf{pure} (M[\vec{x} := \vec{M}]) : \mathbf{KA} \rrbracket & \text{Substitution definition} \\
= \eta_{[A]} \circ \llbracket (M[\vec{x} := \vec{M}]) \rrbracket & \text{Translation for pure} \\
= \eta_{[A]} \circ (\llbracket M \rrbracket \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle) & \text{Induction hypothesis} \\
= (\eta_{[A]} \circ \llbracket M \rrbracket) \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle & \text{Associativity of composition} \\
= \llbracket \Gamma \vdash \mathbf{pure} M : \mathbf{KA} \rrbracket \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle & \text{Translation for pure}
\end{array}$$

$$2) \quad \llbracket \Gamma \vdash (\text{let pure } \vec{x} = \vec{M} \text{ in } N)[\vec{y} := \vec{P}] : \mathbf{KB} \rrbracket = \llbracket \Gamma \vdash \text{let pure } \vec{x} = \vec{M} \text{ in } N : \mathbf{KB} \rrbracket \circ \langle \llbracket P_1 \rrbracket, \dots, \llbracket P_n \rrbracket \rangle$$

$$\begin{aligned} & \llbracket \Gamma \vdash (\text{let pure } \vec{x} = \vec{M} \text{ in } N)[\vec{y} := \vec{P}] : \mathbf{KB} \rrbracket = \\ & \text{Substitution definition} \\ & \llbracket \Gamma \vdash \text{let pure } \vec{x} = (\vec{M}[\vec{y} := \vec{P}]) \text{ in } N : \mathbf{KB} \rrbracket = \\ & \text{Interpretation for } \text{let}_{\mathbf{K}} \\ & \mathcal{K}(\llbracket N \rrbracket) \circ *_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ \llbracket \Gamma \vdash (\vec{M}[\vec{y} := \vec{P}]) : \mathbf{KA} \rrbracket = \\ & \text{Induction hypothesis} \\ & \mathcal{K}(\llbracket N \rrbracket) \circ *_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ (\llbracket \vec{M} \rrbracket \circ \langle \llbracket P_1 \rrbracket, \dots, \llbracket P_n \rrbracket \rangle) = \\ & \text{Associativity of composition} \\ & (\mathcal{K}(\llbracket N \rrbracket) \circ *_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ \llbracket \vec{M} \rrbracket) \circ \langle \llbracket P_1 \rrbracket, \dots, \llbracket P_n \rrbracket \rangle = \\ & \text{By interpretation} \\ & \llbracket \Gamma \vdash (\text{let pure } \vec{x} = \vec{M} \text{ in } N) \circ \langle \llbracket P_1 \rrbracket, \dots, \llbracket P_n \rrbracket \rangle \end{aligned}$$

□

**Lemma 13.**

Let  $\Gamma \vdash M : A$  and  $M \rightarrow_{\beta\eta} N$ , then  $\llbracket \Gamma \vdash M : A \rrbracket = \llbracket \Gamma \vdash N : A \rrbracket$ ;

*Proof.*

Cases with  $\beta$ -reductions for  $\text{let}_{\mathbf{K}}$  are shown in [20]. Let us consider cases with **pure**.

$$1) \quad \llbracket \Gamma \vdash \text{let pure } \vec{x} = \text{pure } \vec{M} \text{ in } N : \mathbf{KB} \rrbracket = \llbracket \Gamma \vdash \text{pure } N[\vec{x} := \vec{M}] : \mathbf{KB} \rrbracket$$

$$\begin{aligned} & \llbracket \Gamma \vdash \text{let pure } \vec{x} = \text{pure } \vec{M} \text{ in } N : \mathbf{KB} \rrbracket = \\ & \text{By interpretation} \\ & \mathcal{K}(\llbracket N \rrbracket) \circ *_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ \langle \eta_{\llbracket A_1 \rrbracket} \circ \llbracket M_1 \rrbracket, \dots, \eta_{\llbracket A_n \rrbracket} \circ \llbracket M_n \rrbracket \rangle = \\ & \text{By the property of a pair of morphisms} \\ & \mathcal{K}(\llbracket N \rrbracket) \circ *_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ (\eta_{\llbracket A_1 \rrbracket} \times \dots \times \eta_{\llbracket A_n \rrbracket}) \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle = \\ & \text{Associativity of composition} \\ & \mathcal{K}(\llbracket N \rrbracket) \circ (*_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ (\eta_{\llbracket A_1 \rrbracket} \times \dots \times \eta_{\llbracket A_n \rrbracket})) \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle = \\ & \text{By the definition of an applicative functor} \\ & \mathcal{K}(\llbracket N \rrbracket) \circ \eta_{\llbracket A_1 \rrbracket \times \dots \times \llbracket A_n \rrbracket} \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle = \\ & \text{Naturality of } \eta \\ & \eta_{\llbracket B \rrbracket} \circ \llbracket N \rrbracket \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle = \\ & \text{Associativity of composition} \\ & \eta_{\llbracket B \rrbracket} \circ (\llbracket N \rrbracket \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle) = \\ & \text{Simultaneous substitution lemma} \\ & \eta_{\llbracket B \rrbracket} \circ \llbracket N[\vec{x} := \vec{M}] \rrbracket \\ & \text{By interpretation} \\ & \llbracket \Gamma \vdash \text{pure } (N[\vec{x} := \vec{M}]) : \mathbf{KB} \rrbracket \end{aligned}$$

□

□

**Theorem 6. Completeness**

Let  $\llbracket \Gamma \vdash M : A \rrbracket = \llbracket \Gamma \vdash N : A \rrbracket$ , then  $M =_{\beta\eta} N$ .

*Proof.*

We will consider term model for simply typed lambda calculus  $\times$  and  $\rightarrow$  standardly described in [22]:

**Definition 20.** *Equivalence on term pairs:*

Let us define relation  $\sim_{A,B} \subseteq \mathbb{V} \times \Lambda_{\mathbf{K}}$ , such that:

$$(x, M) \sim_{A,B} (y, N) \Leftrightarrow x : A \vdash M : B \ \& \ y : A \vdash N : A \ \& \ M =_{\beta\eta} N[y := x];$$

We will denote equivalence class as  $[x, M]_{A,B} = \{(y, N) \mid (x, M) \sim_{A,B} (y, N)\}$  (we will drop indeces below).

**Definition 21.** *Category  $\mathcal{C}(\lambda)$ :*

- $Ob_{\mathcal{C}} = \{\hat{A} \mid A \in \mathbb{T}\} \cup \{\mathbb{1}\};$
- $Hom_{\mathcal{C}(\lambda)}(\hat{A}, \hat{B}) = (\mathbb{V} \times \Lambda_{\mathbf{K}}) / \sim_{A,B};$
- Let  $[x, M] \in Hom_{\mathcal{C}(\lambda)}(\hat{A}, \hat{B})$  and  $[y, N] \in Hom_{\mathcal{C}(\lambda)}(\hat{B}, \hat{C})$ , then  $[y, M] \circ [x, M] = [x, N[y := M]];$
- Identity morphism  $id_{\hat{A}} = [x, x] \in Hom_{\mathcal{C}(\lambda)}(\hat{A}, \hat{A});$
- $\mathbb{1}$  is a terminal object;
- $\widehat{A \times B} = \hat{A} \times \hat{B};$
- Canonical projection is defined as  $[x, \pi_i x] \in Hom_{\mathcal{C}(\lambda)}(\hat{A}_1 \times \hat{A}_2, \hat{A}_i)$  for  $i \in \{1, 2\};$
- $\widehat{A \rightarrow B} = \hat{B}^{\hat{A}};$
- Evaluation arrow  $\epsilon = [x, (\pi_2 x)(\pi_1 x)] \in Hom_{\mathcal{C}(\lambda)}(\hat{B}^{\hat{A}} \times \hat{A}, \hat{B}).$

It is sufficient to show  $\mathbf{K}$  is an applicative functor on  $\mathcal{C}(\lambda)$ .

**Definition 22.** Let us define an endofunctor  $\mathcal{K} : \mathcal{C}(\lambda) \rightarrow \mathcal{C}(\lambda)$ , such that for all  $[x, M] \in Hom_{\mathcal{C}(\lambda)}(\hat{A}, \hat{B})$ ,  $\mathbf{K}([x, M]) = [y, \text{let pure } x = y \text{ in } M] \in Hom_{\mathcal{C}(\lambda)}(\mathbf{K}\hat{A}, \mathbf{K}\hat{B})$  (denotation:  $fmap f$  for an arbitrary arrow  $f$ ).

**Lemma 14.** *Functoriality*

- $fmap (g \circ f) = fmap (g) \circ fmap (f);$
- $fmap (id_{\hat{A}}) = id_{\mathbf{K}\hat{A}}.$

*Proof.* Easy checking using reduction rules. □

**Definition 23.** Let us define natural transformations:

- 1)  $\eta : Id \Rightarrow \mathcal{K}$ , s. t.  $\forall \hat{A} \in Ob_{\mathcal{C}(\lambda)}$ ,  $\eta_{\hat{A}} = [x, \text{pure } x] \in Hom_{\mathcal{C}(\lambda)}(\hat{A}, \mathbf{K}\hat{A});$
- 2)  $*_{A,B} : \mathbf{K}\hat{A} \times \mathbf{K}\hat{B} \rightarrow \mathbf{K}(\hat{A} \times \hat{B})$ , s. t.  $\forall \hat{A}, \hat{B} \in Ob_{\mathcal{C}(\lambda)}$ ,  $*_{\hat{A}, \hat{B}} = [p, \text{let pure } x, y = \pi_1 p, \pi_2 p \text{ in } \langle x, y \rangle] \in Hom_{\mathcal{C}(\lambda)}(\mathbf{K}\hat{A} \times \mathbf{K}\hat{B}, \mathbf{K}(\hat{A} \times \hat{B})).$

Implementation for  $*$  in our term model is a modification of  $\text{let}_{\mathbf{K}}$ -rule:

$$\frac{\frac{p : \mathbf{K}A \times \mathbf{K}B \vdash p : \mathbf{K}A \times \mathbf{K}B}{p : \mathbf{K}A \times \mathbf{K}B \vdash \pi_1 p : \mathbf{K}A} \quad \frac{p : \mathbf{K}A \times \mathbf{K}B \vdash p : \mathbf{K}A \times \mathbf{K}B}{p : \mathbf{K}A \times \mathbf{K}B \vdash \pi_2 p : \mathbf{K}B} \quad \frac{x : A \vdash x : A \quad y : B \vdash y : B}{x : A, y : B \vdash \langle x, y \rangle : A \times B}}{p : \mathbf{K}A \times \mathbf{K}B \vdash \text{let pure } \langle x, y \rangle = \langle \pi_1 p, \pi_2 p \rangle \text{ in } \langle x, y \rangle : \mathbf{K}(A \times B)}$$

**Lemma 15.**

$\mathbf{K}$  is a monoidal endofunctor

*Proof.*

See [19]

□

**Lemma 16.** *Naturality for  $\eta$  and for  $*$*

- $\text{fmap } f \circ \eta_A = \eta_B \circ f;$
- $*_{\hat{A}, \hat{B}} \circ (\eta_A \times \eta_B) = \eta_{\hat{A} \times \hat{B}};$

*Proof.*

i)  $\text{fmap } f \circ \eta_{\hat{A}} = \eta_{\hat{B}} \circ f$

$$\begin{aligned} \eta_{\hat{B}} \circ f &= && \text{By the definition} \\ [y, \mathbf{pure } y] \circ [x, M] &= && \text{By the definition of composition} \\ [x, \mathbf{pure } y[y := M]] &= && \text{By substitution} \\ [x, \mathbf{pure } M] \end{aligned}$$

On the other hand:

$$\begin{aligned} \text{fmap } f \circ \eta_{\hat{A}} &= && \text{By the definition} \\ [z, \mathbf{let } \mathbf{pure } x = z \mathbf{ in } M] \circ [x, \mathbf{pure } x] &= && \text{By the definition of composition} \\ [x, \mathbf{let } \mathbf{pure } x = z \mathbf{ in } M[z := \mathbf{pure } x]] &= && \text{By substitution} \\ [x, \mathbf{let } \mathbf{pure } x = \mathbf{pure } x \mathbf{ in } M] &= && \beta\text{-reduction rule} \\ [x, \mathbf{pure } M[x := x]] &= && \text{By substitution} \\ [x, \mathbf{pure } M] \end{aligned}$$

ii)  $*_{\hat{A}, \hat{B}} \circ (\eta_{\hat{A}} \times \eta_{\hat{B}}) = \eta_{\hat{A} \times \hat{B}}$

$$\begin{aligned} *_{\hat{A}, \hat{B}} \circ (\eta_{\hat{A}} \times \eta_{\hat{B}}) &= \\ \text{By unfolding} & \\ [q, \mathbf{let } \mathbf{pure } x, y = \pi_1 q, \pi_2 q \mathbf{ in } \langle x, y \rangle] \circ [p, \langle \mathbf{pure } (\pi_1 p), \mathbf{pure } (\pi_2 p) \rangle] &= \\ \text{Composition} & \\ [p, \mathbf{let } \mathbf{pure } x, y = \pi_1 q, \pi_2 q \mathbf{ in } \langle x, y \rangle [q := \langle \mathbf{pure } (\pi_1 p), \mathbf{pure } (\pi_2 p) \rangle]] &= \\ \text{By substitution} & \\ [p, \mathbf{let } \mathbf{pure } x, y = \pi_1 (\langle \mathbf{pure } (\pi_1 p), \mathbf{pure } (\pi_2 p) \rangle), \pi_2 (\langle \mathbf{pure } (\pi_1 p), \mathbf{pure } (\pi_2 p) \rangle) \mathbf{ in } \langle x, y \rangle] &= \\ \text{Reduction rules} & \\ [p, \mathbf{let } \mathbf{pure } x, y = \mathbf{pure } (\pi_1 p), \mathbf{pure } (\pi_2 p) \mathbf{ in } \langle x, y \rangle] &= \\ \text{Reduction rule} & \\ [p, \mathbf{pure } (\langle x, y \rangle [x := \pi_1 p, y := \pi_2 p])] &= \\ \text{Substitution} & \\ [p, \mathbf{pure } \langle \pi_1 p, \pi_2 p \rangle] &= \\ \eta\text{-reduction} & \\ [p, \mathbf{pure } p] &= \\ \text{By definition} & \\ \eta_{\hat{A} \times \hat{B}} \end{aligned}$$

□

Tensorial strength is defined as follows:

**Definition 24.** *Tensorial strength*

Let  $[p, \langle \mathbf{pure}(\pi_1 p), \pi_2 p \rangle] \in Hom_{\mathcal{C}(\lambda)}(\hat{A} \times \mathbf{K}\hat{B}, \mathbf{K}\hat{A} \times \mathbf{K}\hat{B})$ .

So tensorial strength is defined as  $\tau_{\hat{A}, \hat{B}} = *_{\hat{A}, \hat{B}} \circ [p, \langle \mathbf{pure}(\pi_1 p), \pi_2 p \rangle]$ .

It is clearly that tensorial strength defined above can be simplified as follows:

$$\begin{aligned}
& *_{\hat{A}, \hat{B}} \circ [p, \langle \mathbf{pure}(\pi_1 p), \pi_2 p \rangle] = && \text{By definition} \\
& [p', \mathbf{let pure } x, y = \pi_1 p', \pi_2 p' \mathbf{ in } \langle x, y \rangle] \circ [p, \langle \mathbf{pure}(\pi_1 p), \pi_2 p \rangle] = && \text{By composition} \\
& [p, \mathbf{let pure } x, y = \pi_1 p', \pi_2 p' \mathbf{ in } \langle x, y \rangle] [p' := \langle \mathbf{pure}(\pi_1 p), \pi_2 p \rangle] = && \text{By substitution} \\
& [p, \mathbf{let pure } x, y = \pi_1(\langle \mathbf{pure}(\pi_1 p), \pi_2 p \rangle), \pi_2(\langle \pi_1 p, \mathbf{pure}(\pi_2 p) \rangle) \mathbf{ in } \langle x, y \rangle] = && \text{By } \beta\text{-reduction rules} \\
& [p, \mathbf{let pure } x, y = \mathbf{pure}(\pi_1 p), \pi_2 p \mathbf{ in } \langle x, y \rangle]
\end{aligned}$$

**Lemma 17.** *Weak commutativity.*

$$\begin{aligned}
& fmap([p, \langle \pi_2 p, \pi_1 p \rangle]) \circ \tau_{\hat{A}, \hat{B}} = \\
& *_{\hat{B}, \hat{A}} \circ [q, \langle \pi_1 q, \mathbf{pure}(\pi_2 q) \rangle] \circ [p, \langle \pi_2 p, \pi_1 p \rangle]
\end{aligned}$$

*Proof.*

$$\begin{aligned}
& fmap([p, \langle \pi_2 r, \pi_1 r \rangle]) \circ \tau_{\hat{A}, \hat{B}} = \\
& \text{By the definition of } \tau \\
& fmap([p, \langle \pi_2 r, \pi_1 r \rangle]) \circ [p, \mathbf{let pure } x, y = \mathbf{pure}(\pi_1 p), \pi_2 p \mathbf{ in } \langle x, y \rangle] = \\
& \text{By the definition of } fmap \\
& [q, \mathbf{let pure } r = q \mathbf{ in } \langle \pi_2 r, \pi_1 r \rangle] \circ [p, \mathbf{let pure } x, y = \mathbf{pure}(\pi_1 p), \pi_2 p \mathbf{ in } \langle x, y \rangle] = \\
& \text{Composition} \\
& [p, \mathbf{let pure } r = q \mathbf{ in } \langle \pi_2 r, \pi_1 r \rangle] [q := \mathbf{let pure } x, y = \mathbf{pure}(\pi_1 p), \pi_2 p \mathbf{ in } \langle x, y \rangle] = \\
& \text{By } \beta\text{-reduction rules} \\
& [p, \mathbf{let pure } r = (\mathbf{let pure } x, y = \mathbf{pure}(\pi_1 p), \pi_2 p \mathbf{ in } \langle x, y \rangle) \mathbf{ in } \langle \pi_2 r, \pi_1 r \rangle] = \\
& \text{By } \beta\text{-reduction rules} \\
& [p, \mathbf{let pure } x, y = \mathbf{pure}(\pi_1 p), \pi_2 p \mathbf{ in } \langle \pi_2 r, \pi_1 r \rangle] [r := \langle x, y \rangle] = \\
& \text{By substitution} \\
& [p, \mathbf{let pure } x, y = \mathbf{pure}(\pi_1 p), \pi_2 p \mathbf{ in } \langle \pi_2 \langle x, y \rangle, \pi_1 \langle x, y \rangle \rangle] = \\
& \text{By } \beta\text{-reduction rules} \\
& [p, \mathbf{let pure } x, y = \mathbf{pure}(\pi_1 p), \pi_2 p \mathbf{ in } \langle y, x \rangle] =
\end{aligned}$$

On the other hand

$$\begin{aligned}
& *_{\hat{B}, \hat{A}} \circ [q, \langle \pi_1 q, \mathbf{pure}(\pi_2 q) \rangle] \circ [p, \langle \pi_2 p, \pi_1 p \rangle] = \\
& \text{By the definition of } * \\
& [r, \mathbf{let pure } y, x = \pi_1 r, \pi_2 r \mathbf{ in } \langle y, x \rangle] \circ [q, \langle \pi_1 q, \mathbf{pure}(\pi_2 q) \rangle] \circ [p, \langle \pi_2 p, \pi_1 p \rangle] = \\
& \text{Composition} \\
& [r, \mathbf{let pure } y, x = \pi_1 r, \pi_2 r \mathbf{ in } \langle y, x \rangle] \circ [p, \langle \pi_1 q, \mathbf{pure}(\pi_2 q) \rangle] [q := \langle \pi_2 p, \pi_1 p \rangle] = \\
& \text{By substitution and by } \beta\text{-reduction rules} \\
& [r, \mathbf{let pure } y, x = \pi_1 r, \pi_2 r \mathbf{ in } \langle y, x \rangle] \circ [p, \langle \pi_2 p, \mathbf{pure}(\pi_1 p) \rangle] = \\
& \text{Composition} \\
& [p, \mathbf{let pure } y, x = \pi_1 r, \pi_2 r \mathbf{ in } \langle y, x \rangle] [r := \langle \pi_2 p, \mathbf{pure}(\pi_1 p) \rangle] = \\
& \text{By substitution and by } \beta\text{-reduction rules} \\
& [p, \mathbf{let pure } y, x = \pi_2 p, \mathbf{pure}(\pi_1 p) \mathbf{ in } \langle y, x \rangle] = \\
& \text{By symmetricity of assingment} \\
& [p, \mathbf{let pure } x, y = \mathbf{pure}(\pi_1 p), \pi_2 p \mathbf{ in } \langle y, x \rangle]
\end{aligned}$$

□

**Lemma 18.** *K is an applicative functor*

*Proof.* Immediately follows from previous lemmas in the section. □

Similar to [24], we apply the translation from  $\lambda_K$  to some cartesian closed category with an arbitrary applicative functor  $\mathcal{K}$ , then we have  $\llbracket \Gamma \vdash M : A \rrbracket = [x, M[x_i := \pi_i x]]$ , so  $M =_{\beta\eta} N \Leftrightarrow \llbracket \Gamma \vdash M : A \rrbracket = \llbracket \Gamma \vdash N : A \rrbracket$ .  $\square$

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