Modal type theory based on the intuitionistic epistemic logic

Abstract

Modal intuitionistic epistemic logic IEL^- was proposed by S.Artemov and T. Protopopescu as the formal foundation for the intuitionistic theory of knowledge. We construct a modal simply typed lambda-calculus which is Curry-Howard isomorphic to IEL^- as formal theory of calculations with applicative functors in functional programming languages like Haskell or Idris. We prove that this typed lambda-calculus has the strong normalization and Church-Rosser properties.

1 Introduction

Modal intutionistic epistemic logic IEL was proposed by S. Artemov and T. Proropopescu [1]. IEL provides the epistimology and the theory of knowledge as based on BHK-semantics of intuitionistic logic. IEL^- is a variant of IEL, that corresponds to intuitionistic belief. Informally, $\mathbf{K}A$ denotes that A is verified intuitionistically.

Intuitionistic epistemic logic IEL^- is defined with by following axioms and derivation rules:

Definition 1. *Intuitionistic epistemic logic IEL:*
1) IPC axioms;
2) $\boldsymbol{K}(A \to B) \to (\boldsymbol{K}A \to \boldsymbol{K}B)$ *(normality);*
3) $A \to \boldsymbol{K}A$ *(co-reflection);*
Rule: MP.

We have the deduction theorem and necessitation rule which is derivable.

V. Krupski and A. Yatmanov provided the sequential calculus for IEL and proved that this calculus is PSPACE-complete [2].

It's not difficult to see that modal axioms in IEL^- and types of the methods of Applicative class in Haskell-like languages (which is described below) are syntactically similar and we are going to show that this coincidence has a non-trivial computational meaning.

Functional programming languages such as Haskell [3], Idris [4], Purescript [5] or Elm [6] have special type classes[1] for calculations with container types like `Functor` and `Applicative` [2]:

[1] Type class in Haskell is a general interface for special group of datatypes.

[2] Reader may read more about container types in the Haskell standard library documentation[7] or in the next one textbook [8]

```haskell
class Functor f where
  fmap :: (a -> b) -> f a -> f b

class Functor f => Applicative f where
  pure :: a -> f a
  (<*>) :: f (a -> b) -> f a -> f b
```

By *container* (or *computational context*) type we mean some type-operator f, where f is a "function" from $*$ to $*$: type operator takes a simple type (which has kind $*$) and returns another simple type type with kind $*$. For more detailed description of the type system with kinds used in Haskell see [12].

The main goal of our research is a relationship between intuitionistic epistemic logic IEL^- and functional programming with effects. We show that relationship by building the type system (which is called $\lambda_{\mathbf{K}}$) which is Curry-Howard isomorphic to IEL^-. So we will consider $\mathbf{K}$-modality as an arbitrary applicative functor.

λK consists of the rules for simply typed lambda-calculus and special typing rules for lifting types into the applicative functor $\mathbf{K}$. We assume that our type system will axiomatize the simplest case of computation with effects with one container. We provide proof-theoretical view on this kind of computations in functional programming and prove strong normalization and confluence.

2 Typed lambda-calculus based on IEL$^-$

Definition 2. *Intuitionistic normal modal logic* **IK**

1) IPC axioms;

2) $\Box(A \to B) \to (\Box A \to \Box B)$*;*

3) Rules: MP and necessiation.

Definition 3. *Translation from* **IK** *into* **IEL**$^-$

1) $\bot^\circ = \bot$*;*

2) $p^\circ = p$*;*

3) $(A \alpha B)^\circ = A^\circ \alpha B^\circ$, $\alpha \in \{\to, \wedge, \vee\}$.

4) $(\Box A)^\circ = \mathbf{K} A^\circ$

Lemma 1. $\mathbf{IK} \vdash A \Rightarrow \mathbf{IEL}^- \vdash A^\circ$

Proof. Straightforward induction on the structure of M. $\Box$

It is clearly that we may prove similar fact for **IEL** by the same way.

At first we define the natural deduction for IEL$^-$ with **K**-modality and binary connectives $\to$ and $\wedge$ (we call that calculus NIEL$^-_{\wedge,\to}$):

Definition 4. *Natural deduction* $NIEL^-_{\wedge,\to}$ *for* IEL^- *with* $\to$ *and* $\wedge$*:*

$$\frac{}{\Gamma, \alpha \vdash A}\ ax$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} \to_i \qquad \frac{\Gamma \vdash A \to B \qquad \Gamma \vdash A}{\Gamma \vdash B} \to_i$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge_i \qquad \frac{\Gamma \vdash A_1 \wedge A_2}{\Gamma \vdash A_i} \wedge_e, i \in \{1, 2\}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \boldsymbol{K}A} \boldsymbol{K}_I \qquad \frac{\Gamma \vdash \mathbf{K}\vec{A} \qquad \vec{A} \vdash B}{\Gamma \vdash \mathbf{K}B}$$

Where $\Gamma \vdash \mathbf{K}\vec{A}$ is a syntax sugar for $\Gamma \vdash \mathbf{K}A_1, \dots, \Gamma \vdash \mathbf{K}A_n$.

Lemma 2. $\Gamma \vdash_{NIEL^-_{\wedge,\to}} A \Rightarrow IEL^- \vdash \bigwedge \Gamma \to A.$

Proof. Induction on the derivation.

Let us consider cases with modality.

1) If $\Gamma \vdash_{NIEL^-_{\wedge,\to}} A$, then $IEL^- \vdash \bigwedge \Gamma \to \mathbf{K}A$.

(1)	$\bigwedge \Gamma \to A$	assumption
(2)	$A \to \mathbf{K}A$	co-reflection
(3)	$(\bigwedge \Gamma \to A) \to ((A \to \mathbf{K}A) \to (\bigwedge \Gamma \to \mathbf{K}A))$	IPC theorem
(4)	$(A \to \mathbf{K}A) \to (\bigwedge \Gamma \to \mathbf{K}A)$	from (1), (3) and MP
(5)	$\bigwedge \Gamma \to \mathbf{K}A$	from (2), (4) and MP

2) If $\Gamma \vdash_{NIEL^-_{\wedge,\to}} \mathbf{K}\vec{A}$ and $\vec{A} \vdash B$, then $IEL^- \vdash \bigwedge \Gamma \to \mathbf{K}B$.

(1)	$\bigwedge \Gamma \to \bigwedge\limits_{i=1}^{n} \mathbf{K}A_i$	assumption
(2)	$\bigwedge\limits_{i=1}^{n} \mathbf{K}A_i \to \mathbf{K} \bigwedge\limits_{i=1}^{n} A_i$	IEL theorem
(3)	$\bigwedge \Gamma \to \mathbf{K} \bigwedge\limits_{i=1}^{n} A_i$	from (1), (2) and transitivity
(4)	$\bigwedge\limits_{i=1}^{n} A_i \to B$	assumption
(5)	$(\bigwedge\limits_{i=1}^{n} A_i \to B) \to \mathbf{K}(\bigwedge\limits_{i=1}^{n} A_i \to B)$	co-reflection
(6)	$\mathbf{K}(\bigwedge\limits_{i=1}^{n} A_i \to B)$	from (2), (3) and MP
(7)	$\mathbf{K} \bigwedge\limits_{i=1}^{n} A_i \to \mathbf{K}B$	from (6) and normality
(8)	$\bigwedge \Gamma \to \mathbf{K}B$	from (3), (7) and transitivity

□

At the next step we build the typed lambda-calculus based on $\text{NIEL}^-_{\wedge,\to}$ by proof-assingment in rules.

At first, we define lambda-terms and types for this lambda-calculus.

Definition 5. *The set of terms:*

Let $\mathbb{V}$ *be the set of variables. The set* $\Lambda_{\boldsymbol{K}}$ *of terms is defined by the grammar:*

$$\Lambda_{\boldsymbol{K}} ::= \mathbb{V} \mid (\lambda\Lambda.\Lambda_{\boldsymbol{K}}) \mid (\Lambda_{\boldsymbol{K}}\Lambda_{\boldsymbol{K}}) \mid (\Lambda_{\boldsymbol{K}}, \Lambda_{\boldsymbol{K}}) \mid (\pi_1\Lambda_{\boldsymbol{K}}) \mid (\pi_2\Lambda_{\boldsymbol{K}}) \mid \\ (\textbf{pure } \Lambda_{\boldsymbol{K}}) \mid (\textbf{let pure } \Lambda_{\boldsymbol{K}} = \Lambda_{\boldsymbol{K}} \textbf{ in } \Lambda_{\boldsymbol{K}})$$

Definition 6. *The set of types:*
Let $\mathbb{T}$ be the set of atomic types. The set $\mathbb{T}_{\boldsymbol{K}}$ of types with applicative functor $\boldsymbol{K}$ is generated by the grammar:

$$\mathbb{T}_{\boldsymbol{K}} ::= \mathbb{T} \mid (\mathbb{T}_{\boldsymbol{K}} \to \mathbb{T}_{\boldsymbol{K}}) \mid (\mathbb{T}_{\boldsymbol{K}} \times \mathbb{T}_{\boldsymbol{K}}) \mid (\boldsymbol{K}\mathbb{T}_{\boldsymbol{K}}) \tag{1}$$

Context, domain of context and range of context are defined standardly [11][12].

Our type system is based on the Curry-style typing rules:

Definition 7. *Modal typed lambda calculus $\lambda\boldsymbol{K}$ based on NIEL$^{-}_{\wedge,\to}$:*

$$\frac{}{\Gamma, x : A \vdash x : A}\, ax$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B} \to_i$$

$$\frac{\Gamma \vdash x : A \qquad \Gamma \vdash y : B}{\Gamma \vdash \langle x, y\rangle : A \times B} \times_i$$

$$\frac{\Gamma \vdash x : A}{\Gamma \vdash \textbf{pure } x : \boldsymbol{K}A}\, \boldsymbol{K}_I$$

$$\frac{\Gamma \vdash f : A \to B \qquad \Gamma \vdash x : A}{\Gamma \vdash fx : B} \to_e$$

$$\frac{\Gamma \vdash M : A_1 \times A_2}{\Gamma \vdash \pi_i M : A_i} \times_e,\ i \in \{1, 2\}$$

$$\frac{\Gamma \vdash \vec{M} : \mathbf{K}\vec{A} \qquad \vec{x} : \vec{A} \vdash N : B}{\Gamma \vdash \textbf{let pure } \vec{N} = \vec{M} \textbf{ in } N : \mathbf{K}B}\, \textbf{let}_{\boldsymbol{K}}$$

$\mathbf{K}_I$-typing rule is the same as $\bigcirc$-introduction in lax logic (also known as monadic metalanguage [17]) and in typed lambda-calculus which is derived by proof-assignment for lax-logic proofs. $\mathbf{K}_I$ allows to inject an object of type α into the functor. $\mathbf{K}_I$ reflects the Haskell method **pure** for Applicative class. It plays the same role as the **return** method in Monad class.

Here are some examples of derivation trees.

$$\cfrac{\cfrac{x : A \vdash x : A}{x : A \vdash \textbf{pure}\, x : \mathbf{K}A}\, \mathbf{K}_I}{\vdash (\lambda x.\textbf{pure}\, x) : A \to \mathbf{K}A} \to_i$$

$$\cfrac{\cfrac{\cfrac{f : \mathbf{K}(A \to B) \vdash f : \mathbf{K}(A \to B) \qquad x : \mathbf{K}A \vdash x : \mathbf{K}A \qquad \cfrac{g : A \to B \qquad y : A}{g : A \to B, y : A \vdash gy : B}}{f : \mathbf{K}(A \to B), x : \mathbf{K}A \vdash \textbf{let pure } \langle g, y\rangle = \langle f, x\rangle \textbf{ in } gy : \mathbf{K}B}}{f : \mathbf{K}(A \to B) \vdash \lambda x.\textbf{let pure } \langle g, y\rangle = \langle f, x\rangle \textbf{ in } gy : \mathbf{K}A \to \mathbf{K}B}}{\vdash \lambda f.\lambda x.\textbf{let pure } \langle g, y\rangle = \langle f, x\rangle \textbf{ in } gy : \mathbf{K}(A \to B) \to \mathbf{K}A \to \mathbf{K}B}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{f : A \to B \vdash f : A \to B}{f : A \to B \vdash \textbf{pure } f : \mathbf{K}(A \to B)} \qquad x : \mathbf{K}A \vdash x : \mathbf{K}A \qquad \dfrac{g : A \to B \qquad y : A}{g : A \to B, y : A \vdash gy : B}}{f : A \to B, x : \mathbf{K}A \vdash \textbf{let pure } \langle g, y\rangle = \langle \textbf{pure } f, x\rangle \textbf{ in } gy : \mathbf{K}B}}{f : A \to B \vdash \lambda x.\textbf{let pure } \langle g, y\rangle = \langle \textbf{pure } f, x\rangle \textbf{ in } gy : \mathbf{K}A \to \mathbf{K}B}}{\lambda f.\lambda x.\textbf{let pure } \langle g, y\rangle = \langle \textbf{pure } f, x\rangle \textbf{ in } gy : (A \to B) \to \mathbf{K}A \to \mathbf{K}B}$$

Now we define free variables and substitutions. β-reduction, multi-step β-reduction and β-equality are defined standardly:

Definition 8. *Set $FV(M)$ of free variables for arbitrary term M:*
1) $FV(x) = \{x\}$;
2) $FV(\lambda x.M) = FV(M) \setminus \{x\}$;
3) $FV(MN) = FV(M) \cup FV(N)$;
4) $FV((M, N)) = FV(M) \cup FV(N)$;
5) $FV(\pi_i p) \subseteq FV(p)$, $i \in \{1, 2\}$;
6) $FV(\textit{pure } M) = FV(M)$;
7) $FV(\textbf{let pure } \vec{x} = \vec{M} \textbf{ in } N) = \bigcup\limits_{i=1}^{n} FV(M_i)$, *where* $n = |\vec{M}|$.

Definition 9. *Substitution:*
1) $x[x := N] = N$, $x[y := N] = x$;
2) $(MN)[x := N] = M[x := N]N[x := N]$;
3) $(\lambda x.M)[x := N] = \lambda x.M[x := N]$;
4) $(M, N)[x := P] = (M[x := P], N[x := P])$;
5) $(\pi_i M)[x := P] = \pi_i(M[x := P])$, $i \in \{1, 2\}$;
6) $(\textbf{pure } M)[x := P] = \textbf{pure } (M[x := P])$;
7) $(\textbf{let pure } \vec{N} = \vec{M} \textbf{ in } M)[x := P] = \textbf{let pure } \vec{N} = (\vec{M}[x := P]) \textbf{ in } M$.

In $\lambda\mathbf{K}$ we have the following computational rules:

Definition 10. *β-reduction rules for $\lambda\mathbf{K}$.*
1) $(\lambda x.M)N \to_\beta M[x := N]$
2) $\pi_1\langle M, N\rangle \to_\beta M$
3) $\pi_2\langle M, N\rangle \to_\beta N$
4) $\textbf{let pure } \vec{y} = \vec{M}_2 \textbf{ in } (\textbf{let pure } \vec{x} = \vec{M}_1 \textbf{ in } N) \to_\beta$
$\textbf{let pure } \vec{x} = (\textbf{let pure } \vec{y} = \vec{M}_2 \textbf{ in } \vec{M}_1) \textbf{ in let pure } \vec{x} = \vec{x} \textbf{ in } N$

3 Basic lemmas

Now we will prove standard lemmas for contexts in type systems[3]:

Definition 11. *The domain of a context Γ:*
Let $\Gamma = \{x_1 : A_1, ..., x_n : A_n\}$. Then the domain of Γ, or $dom(\Gamma)$, is a set $\{x_1, ..., x_n\}$.

Lemma 3. *If $\Gamma \vdash M : A$, then $FV(M) \subseteq dom(\Gamma)$*

[3] We will not prove cases with $\to$-constructor, they are proved standardly in the same lemmas for simply typed lambda calculus, for example see [11][12][14]. We will consider only modal cases

Proof. Induction on the derivation of $\Gamma \vdash M : A$.

$\square$

Lemma 4. *Generation for $\lambda\boldsymbol{K}$.*

*1) $\Gamma \vdash$ **pure** $M : \boldsymbol{K}A$ implies that $\Gamma \vdash M : A$;*

*2) $\Gamma \vdash$ **let pure** $\vec{N} = \vec{M}$ **in** $M : \mathbf{K}B$ implies that $\Gamma \vdash \vec{M} : \mathbf{K}\vec{A}$ and $\vec{N} : \vec{A} \vdash M : B$.*

Proof.

Induction on the derivation of $\Gamma \vdash$ **pure** $M : \mathbf{K}\alpha$ and $\Gamma \vdash$ **let pure** $\vec{N} = \vec{M}$ **in** $M : \mathbf{K}B$ respectively. $\square$

The next one lemma allows that weakening structural rule is admissable.

Lemma 5. *Weakening for $\lambda\boldsymbol{K}$.*

Let $\Gamma \vdash M : A$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash M : A$.

Proof.

Induction on derivation of $\Gamma \vdash M : \alpha$. Let us assume $\Gamma \subseteq \Delta$.

1) Let $\Gamma \vdash x : A$, such that $\Gamma = \Delta, x : A$ and $\Theta \subseteq \Gamma$. Let $\Sigma = \Theta \setminus \Gamma$, or, which is the same, $\Sigma = \Theta \setminus \Delta, x : A$, then $\Sigma, \Delta, x : A \vdash x : \alpha$, or, $\Theta \vdash x : A$.

2) Let $\Gamma \vdash$ **pure** $M : \mathbf{K}A$ and $\Gamma \subseteq \Theta$.

If $\Gamma \vdash$ **pure** $M : \mathbf{K}A$, then $\Gamma \vdash M : A$ by generation and, by hypothesis, $\Theta \vdash M : A$, so $\Theta \vdash$ **pure** $M : \mathbf{K}A$ by applying $\mathbf{K}_I$-rule.

3) Let $\Gamma \vdash$ **let pure** $\vec{x} = \vec{M}$ **in** $N : \mathbf{K}B$ and $\Gamma \subseteq \Delta$.

By generation $\Gamma \vdash \vec{M} : \mathbf{K}\vec{A}$ and $\vec{x} : \vec{A} \vdash N : \mathbf{K}B$.

By hypothesis we have $\Delta \vdash \vec{M} : \mathbf{K}\vec{A}$. So $\Delta \vdash$ **let pure** $\vec{x} = \vec{M}$ **in** $N : \mathbf{K}B$ by $\mathbf{let}_{\mathbf{K}}$.

$\square$

Lemma 6. *Considering for $\lambda\boldsymbol{K}$.*

If $\Gamma \vdash M : A$, then $\Gamma \uparrow FV(M) \vdash M : A$, where $\Gamma \uparrow FV(M)$ is a subcontext of Γ, such that $dom(\Gamma \uparrow FV(M)) = dom(\Gamma) \cap FV(M)$.

Proof. Induction by derivation. We consider the base of induction and the case with $\mathbf{let}_{\mathbf{K}}$. The rest cases are proven by the same way.

1) Let $\Gamma \vdash x : A$, where $\Gamma = \Delta, x : A$, $x \in \mathbb{V}$.

$FV(x) = \{x\}$, then $dom(\Gamma) \cap \{x\} = \{x\}$. So $(\Delta, x : A) \uparrow FV(x) = \{x : A\}$, then $x : A \vdash x : A$ by axiom.

2) Let $\Gamma \vdash$ **let pure** $\vec{x} = \vec{M}$ **in** $N : \mathbf{K}B$.

By generation $\Gamma \vdash \vec{M} : \mathbf{K}\vec{A}$ and $\vec{x} : \vec{A} \vdash N : \mathbf{K}B$.

By hypothesis $\Gamma \uparrow FV(\vec{M}) \vdash \vec{M} : \mathbf{K}\vec{A}$.

So $\Gamma \uparrow FV(\vec{M}) \vdash$ **let pure** $\vec{x} = \vec{M}$ **in** $N : \mathbf{K}B$.

$\square$

Lemma 7. *If $\Gamma, x : A \vdash M : B$ and $\Gamma \vdash N : A$, then $\Gamma \vdash (M[x := N]) : B$*

Proof.

1) Let $\Gamma, x : A \vdash \mathbf{pure}\, M : \mathbf{K}B$ and $\Gamma \vdash N : A$.

If $\Gamma, x : \alpha \vdash \mathbf{pure}\, M : \mathbf{K}\beta$.

By generation, $\Gamma, x : A \vdash M : B$.

So, by induction hypothesis, $\Gamma \vdash (M[x := N]) : B$. Then $\Gamma \vdash \mathbf{pure}\,(M[x := N]) : \mathbf{K}B$ by $\mathbf{K}_I$, but $\mathbf{pure}\,(M[x := N]) = (\mathbf{pure}\, M[x := N])$ by substitution definition.

So $\Gamma \vdash (\mathbf{pure}\, M[x := N]) : \mathbf{K}B$

2) Let $\Gamma, x : \gamma \vdash M \star N : \mathbf{K}\beta$, and $\Gamma \vdash y : \gamma$.

So, by generation, $\Gamma, x : \gamma \vdash M : \mathbf{K}(\alpha \to \beta)$ and $\Gamma, x : \gamma \vdash N : \mathbf{K}\alpha$.

Hence $\Gamma, x : \gamma \vdash (M[x := y]) : \mathbf{K}(\alpha \to \beta)$ and $\Gamma \vdash (N[x := y]) : \mathbf{K}\alpha$ by hypothesis.

So $\Gamma \vdash (M[x := y]) \star (N[x := y]) : \mathbf{K}\beta$, or, $\Gamma \vdash (M \star N)([x := y]) : \mathbf{K}\beta$.

□

Theorem 1. *Subject reduction*

Let $\Gamma \vdash M : \alpha$ *and* $M \twoheadrightarrow_\beta N$, *then* $\Gamma \vdash N : \alpha$

We consider cases with reduction rules which are applicative laws. The general statement for $\twoheadrightarrow_\beta$ follows from transitivity of multi-step β-reduction.

Proof.

1) Let $\Gamma \vdash \mathbf{pure}\,(\lambda x.x) \star M : \mathbf{K}\alpha$. Then $\Gamma \vdash \mathbf{pure}\,(\lambda x.x) : \mathbf{K}(\alpha \to \alpha)$ and $\Gamma \vdash M : \mathbf{K}\alpha$ by generation. Then $\Gamma \vdash M : \mathbf{K}\alpha$ trivially.

2) Let $\Gamma \vdash \mathbf{pure}\,(\lambda fgx.f(gx)) \star M \star N \star P : \mathbf{K}\gamma$.

Then $\Gamma \vdash \mathbf{pure}\,(\lambda fgx.f(gx)) : \mathbf{K}((\beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma)$, $\Gamma \vdash M : \mathbf{K}(\beta \to \gamma)$, $\Gamma \vdash N : \mathbf{K}(\alpha \to \beta)$ and $\Gamma \vdash P : \mathbf{K}\alpha$ by generation.

If $\Gamma \vdash N : \mathbf{K}(\alpha \to \beta)$ and $\Gamma \vdash P : \mathbf{K}\alpha$, then $\Gamma \vdash N \star P : \mathbf{K}\beta$ by $\mathbf{K}_{app}$.

Hence, if $\Gamma \vdash M : \mathbf{K}(\beta \to \gamma)$, then $\Gamma \vdash M \star (N \star P) : \mathbf{K}\gamma$ by $\mathbf{K}_{app}$.

3) Let $\Gamma \vdash (\mathbf{pure}\, M) \star (\mathbf{pure}\, N) : \mathbf{K}\beta$. Then $\Gamma \vdash \mathbf{pure}\, M : \mathbf{K}(\alpha \to \beta)$ and $\Gamma \vdash \mathbf{pure}\, N : \mathbf{K}\alpha$ by generation. Moreover, $\Gamma \vdash M : \alpha \to \beta$ and $\Gamma \vdash N : \alpha$.

Then $\Gamma \vdash MN : \beta$ by application.

Hence, $\Gamma \vdash \mathbf{pure}\,(MN) : \mathbf{K}\beta$ by K_I.

4) Let $\Gamma \vdash M \star (\mathbf{pure}\, N) : \mathbf{K}\beta$.

Then $\Gamma \vdash M : \mathbf{K}(\alpha \to \beta)$ and $\Gamma \vdash \mathbf{pure}\, N : \mathbf{K}\alpha$.

Moreover, $\Gamma \vdash N : \alpha$ by generation.

Let $\Gamma, f : \alpha \to \beta \vdash f : \alpha \to \beta$ and $\Gamma, f : \alpha \to \beta \vdash N : \alpha$ by weakening.

So $\Gamma, f : \alpha \to \beta \vdash fN : \beta$ by application, so $\Gamma \vdash \lambda f.fN : (\alpha \to \beta) \to \beta$ by abstraction.

Then $\Gamma \vdash \mathbf{pure}\,(\lambda f.fN) : \mathbf{K}((\alpha \to \beta) \to \beta)$ by $\mathbf{K}_I$.

Hence, $\Gamma \vdash \mathbf{pure}\,(\lambda f.fN) \star M : \mathbf{K}\beta$.

□

4 Strong normalization

We modify and apply Tait's technique of logical relation for modalities. Strong normalization proof with Tait's method for simply typed lambda calculus is described here [13].

Theorem 2. *Let $M \in \Lambda_{\boldsymbol{K}}$, then any sequence of reduction $M \to_\beta M_1 \dots$ terminates.*

Proof. We build the smallest of subset of strongly normalazing terms of modal types and show that an arbitrary term belongs to this subset.

Definition 12. *The set of strongly computable terms of type $\phi \in \mathbb{T}_{\boldsymbol{K}}$, SC_ϕ:*

- *Let $\phi = \boldsymbol{K}\alpha$ and $\alpha \in \mathbb{T}$, then:*

$$SC_{\boldsymbol{K}\alpha} = \{M : \boldsymbol{K}\alpha \mid M \text{ is strongly normalizing}\} \tag{2}$$

- *Let $\phi = \boldsymbol{K}(\tau \to \psi)$ and $\tau, \psi \in \mathbb{T}_{\boldsymbol{K}}$, then:*

$$SC_{\boldsymbol{K}(\tau\to\psi)} = \{M : \boldsymbol{K}(\tau \to \psi) \mid \forall N \in SC_{\boldsymbol{K}\tau}, M \star N \in SC_{\boldsymbol{K}\psi}\} \tag{3}$$

- *Let $\phi = \boldsymbol{K}(\tau_1 \times \tau_2)$ and $\tau_1, \tau_2 \in \mathbb{T}_{\boldsymbol{K}}$, then:*

$$SC_{\boldsymbol{K}(\tau_1\times\tau_2)} = \{P : \boldsymbol{K}(\tau_1 \times \tau_2) \,|\, \mathbf{pure}\,(\lambda x.\pi_i x) \star P \in SC_{\boldsymbol{K}\tau_i}, i \in \{1,2\}\} \tag{4}$$

Lemma 8.
If $M \in SC_\alpha$, then M is strongly normalizing.

Proof.
1) If $M \in SC_{\mathbf{K}\alpha}$ and $\alpha \in \mathbb{T}$, then M is strongly normalizing by the definition of $SC_{\mathbf{K}\alpha}$.
2) Let $M \in SC_{\mathbf{K}(\tau\to\psi)}$, so by every $N \in SC_{\mathbf{K}\tau}$, $M \star N \in SC_{\mathbf{K}\psi}$, which is strongly normalizing by hypothesis. So M is strongly normalizing.
3) Let $M \in SC_{\mathbf{K}(\tau_1\times\tau_2)}$, so $\mathbf{pure}\,(\lambda x.\pi_i x) \star M \in SC_{\mathbf{K}\tau_i}$, $i \in \{1,2\}$, which are strongly normalizing. So M is strongly normalizing.

□

Lemma 9.
Let $M \to_\beta M'$ and $M \in SC_\alpha$, then $M' \in SC_\alpha$.

Proof.
1) Let $M \to_\beta M'$ and $M \in SC_{\mathbf{K}\alpha}$, where $\alpha \in \mathbb{T}$.
M has the longest reduction path (which we denote as $p(M)$). So $p(M') < p(M)$, then $M' \in SC_{\mathbf{K}\alpha}$.
2) Let $M \in SC_{\mathbf{K}(\alpha\to\beta)}$ and $M \to_\beta M'$. Let $N \in SC_{\mathbf{K}\alpha}$. So $M \star N \in SC_{\mathbf{K}\beta}$.
If $M \to_\beta M'$, then $M \star N \to_\beta M' \star N$ by reduction rule, so $M' \star N \in SC_{\mathbf{K}\beta}$ and $M' \in SC_{\mathbf{K}(\alpha\to\beta)}$ by hypothesis.
3) Let $M \in SC_{\mathbf{K}(\tau_1\times\tau_2)}$ and $M \to_\beta M'$.
So $\mathbf{pure}\,(\lambda x.\pi_i x) \star M \to_\beta \mathbf{pure}\,(\lambda x.\pi_i x) \star M'$, $i \in \{1,2\}$ by reduction rule. So $\mathbf{pure}\,(\lambda x.\pi_i x) \star M' \in SC_{\mathbf{K}\tau_i}$ and $M' \in SC_{\mathbf{K}(\tau_1\times\tau_2)}$.

□

Definition 13. *Neutral term:*

We define a term M to be neutral if it has of the next forms:
1) $M = x$, where $x \in \mathbb{V}$;
2) $M = (PQ)$;
3) $M = \pi_i M$, $i \in \{1, 2\}$;
4) $M = P \star Q$;
*5) If M is a neutral, then **pure** M is a neutral.*

Lemma 10. *Let $M \to_\beta M'$ and $M' \in SC_\alpha$ for every one-step reduction. So if M' is a neutral, then $M \in SC_\alpha$.*

Proof.
Simple induction on the structure of M'.

□

Lemma 11.
Let $x_1 : \phi_1, \ldots, x_n : \phi_n \vdash M : \phi$ and for all $i \in \{1, \ldots, n\}$, $N_i \in SC_{\phi_i}$, then $(M[x_1 := N_1, \ldots, x_n := N_n]) \in SC_\phi$.

Proof.
1) If ϕ is an atomic and M is a variable, then this condition holds straightforwardly.

2) Let $\Gamma = \{x_1 : \phi_1, \ldots, x_n : \phi_n\}$, $\Gamma \vdash \textbf{pure}\, M : \mathbf{K}\alpha$ and for all $i \in \{1, \ldots, n\}, N_i \in SC_{\phi_i}$.

Then by $\Gamma \vdash M : \alpha$ by generation and $(M[x_1 := N_1, \ldots, x_n := N_n]) \in SC_\alpha$ by induction hypothesis.

Hence, $\Gamma \vdash \textbf{pure}\, M : \mathbf{K}\alpha$ and $(\textbf{pure}\, M([x_1 := N_1, \ldots, x_n := N_n])) \in SC_{\mathbf{K}\alpha}$ by definition of $SC_{\mathbf{K}\alpha}$.

3) Let $\Gamma = \{x_1 : \phi_1, \ldots, x_n : \phi_n\}$, $\Gamma : \phi_n \vdash M \star P : \mathbf{K}\beta$ and forall $i \in \{1, \ldots, n\}, N_i \in SC_{\phi_i}$.

Then $\Gamma \vdash M : \mathbf{K}(\alpha \to \beta)$, $\Gamma \vdash P : \mathbf{K}\alpha$ by generation.

But by induction hypothesis $M[x_1 := N_1, \ldots, x_n := N_n] \in SC_{\mathbf{K}(\alpha\to\beta)}$ and $P[x_1 := N_1, \ldots, x_n := N_n] \in SC_{\mathbf{K}\alpha}$.

Then, by definition of $SC_{\mathbf{K}\beta}$, $((M[x_1 := N_1, \ldots, x_n := N_n]) \star (P[x_1 := N_1, \ldots, x_n := N_n])) \in SC_{\mathbf{K}\beta}$, i.e. $(M \star N([x_1 := N_1, \ldots, x_n := N_n])) \in SC_{\mathbf{K}\beta}$.
□

Corollary 1.
If $\vdash M : \alpha$, then M is strongly normalizing.

Proof. $M \in SC_\alpha$ by Lemma 10, so M is strongly normalizing.

□

□

5 Confluence

In the confluence proof (below) we treat the cases with **pure** and $\star$ similar to [15] [18].

Definition 14. *Alphabet for the labelled terms:*
variables: $x, y, z, x_1, y_1, z_1, ...$;
lambdas: $\lambda, \lambda_0, \lambda_1, \lambda_2, ...$;
constructors for an applicative functor: **pure** $, \star$;
parentheses (,).

Definition 15. *The set of labelled terms $\Lambda^{'}_{\boldsymbol{K}}$ inductively defined as a set of words on the alphabet described above:*
1) $x \in \Lambda^{'}$;
2) If $M \in \Lambda^{'}_{\boldsymbol{K}}$, *then* $(\lambda x.M) \in \Lambda^{'}_{\boldsymbol{K}}$;
3) If $M, N \in \Lambda^{'}_{\boldsymbol{K}}$, *then* $(MN) \in \Lambda^{'}_{\boldsymbol{K}}$;
4) If $M \in \Lambda^{'}_{\boldsymbol{K}}$, *then* $\mathbf{pure}\, M \in \Lambda^{'}_{\boldsymbol{K}}$;
5) If $M, N \in \Lambda^{'}_{\boldsymbol{K}}$, *then* $M \star N \in \Lambda^{'}_{\boldsymbol{K}}$;
6) If $M, N \in \Lambda^{'}_{\boldsymbol{K}}$, *then for all* $i \in \mathbb{N}$, $((\lambda_i x.M)N) \in \Lambda^{'}_{\boldsymbol{K}}$.

Definition 16. *Erasing map*
Erasing map is a map $|.| : \Lambda^{'}_{\boldsymbol{K}} \to \Lambda_{\boldsymbol{K}}$, *such that:*
1) $|x| = x$;
2) $|(\lambda x.M)| = \lambda x.|M|$;
3) $|(MN)| = |M||N|$;
4) $|(\mathbf{pure}\, M)| = \mathbf{pure}\, |M|$;
5) $|M \star N| = |M| \star |N|$;
6) $|((\lambda_i x.M)N)| = (\lambda x.|M|)|N|$

Example 1.
$|\mathbf{pure}\, ((\lambda_i x.M)N) \star P| = \mathbf{pure}\, (\lambda x.|M|)|N|) \star |P|$

Definition 17. *Substitution for* $\Lambda^{'}_{\boldsymbol{K}}$:
1) $x[x := N] = N$, $x[y := N] = x$;
2) $(MN)[x := N] = M[x := N]N[x := N]$;
3) $(\lambda x.M)[x := N] = \lambda x.M[x := N]$;
4) $(\mathbf{pure}\, M)[x := P] = \mathbf{pure}\ (M[x := P])$;
5) $(M \star N)[x := P] = (M[x := P]) \star (N[x := P])$;
6) $(\lambda_i x.M)N[y := P] = (\lambda_i x.M[y := P])(N[y := P])$.

Definition 18. *One-step reduction* $\to_{\beta'}$ *for* $\Lambda^{'}_{\boldsymbol{K}}$:
1) $(\lambda x.M)N \to_{\beta'} M[x := N]$;
2) $\mathbf{pure}\, (\lambda x.x) \star M \to_{\beta'} M$;
3) $\mathbf{pure}\, (\lambda fgx.f(gx)) \star M \star N \star P \to_{\beta'} M \star (N \star P)$;
4) $(\mathbf{pure}\, M) \star (\mathbf{pure}\, N) \to_{\beta'} \mathbf{pure}\, (MN)$;
5) $M \star (\mathbf{pure}\, N) \to_{\beta'} \mathbf{pure}\, (\lambda f.fN) \star M$;
6) $(\lambda_i x.M)N \to_{\beta'} M[x := N]$.
Multi-step reduction $\twoheadrightarrow_{\beta'}$ *is a reflexive-transitive closure of* $\to_{\beta'}$.

Definition 19. *Let us define a map* $\phi : \Lambda^{'}_{\boldsymbol{K}} \to \Lambda_{\boldsymbol{K}}$ *inductively as follows:*
1) $\phi(x) = x$;
2) $\phi(MN) = \phi(M)\phi(N)$;

3) $\phi(\lambda x.M) = \lambda x.\phi(M)$*;*
4) $\phi(\mathbf{pure}\, M) = \mathbf{pure}\,(\phi(M))$*;*
5) $\phi(M \star N) = \phi(M) \star \phi(N)$*;*
6) $\phi((\lambda_i x.M)N) = \phi(M)[x := \phi(N)]$.

Example 2.
$\phi(\mathbf{pure}\,((\lambda_i x.M)N) \star P) = \mathbf{pure}\,(\phi(M)[x := \phi(N)]) \star \phi(P)$

Lemma 12.
1) Let $M, N \in \Lambda^{'}_{\boldsymbol{K}}$ *and* $|M| \twoheadrightarrow_\beta |N|$*, then* $M \twoheadrightarrow_{\beta'} N$*.*
2) Let $M, N \in \Lambda^{'}_{\boldsymbol{K}}$ *and* $M \twoheadrightarrow_{\beta'} N$*, then* $|M| \twoheadrightarrow_\beta |N|$*.*

Proof.
Induction on the generation of $\twoheadrightarrow_\beta$ ($\twoheadrightarrow_{\beta'}$).
1) Let us consider homomorphism rule. The rest applicative reduction rules are considered similary.
Let $(\mathbf{pure}\, M') \star (\mathbf{pure}\, N'), \mathbf{pure}\,(M'N') \in \Lambda^{'}_{\mathbf{K}}$.
So $|(\mathbf{pure}\, M') \star (\mathbf{pure}\, N')| = (\mathbf{pure}\, |M'|) \star (\mathbf{pure}\, |N'|)$ and $|\mathbf{pure}\,(M'N')| = \mathbf{pure}\,(|M'||N'|)$.
By reduction rule, $(\mathbf{pure}\, |M'|) \star (\mathbf{pure}\, |N'|) \to_\beta \mathbf{pure}\,(|M'||N'|)$.
But $(\mathbf{pure}\, M') \star (\mathbf{pure}\, N') \to_{\beta'} \mathbf{pure}\,(M'N')$ by reduction rule for $\to_{\beta'}$.

2) Let us consider interchange rule.
Let $M \star (\mathbf{pure}\, N), \mathbf{pure}\,(\lambda f.fN) \star M \in \Lambda^{'}_{\mathbf{K}}$ and $M \star (\mathbf{pure}\, N) \to_{\beta'} \mathbf{pure}\,(\lambda f.fN) \star M$.
But $|M \star (\mathbf{pure}\, N)| = |M| \star (\mathbf{pure}\, |N|)$ and $|\mathbf{pure}\,(\lambda f.fN) \star M| = \mathbf{pure}\,(\lambda f.f|N|) \star |M|$.
So $|M| \star (\mathbf{pure}\, |N|) \to_\beta \mathbf{pure}\,(\lambda f.f|N|) \star |M|$ by β-reduction rule.

It is easy to see, that the statement for $\twoheadrightarrow_{\beta'}$ and $\twoheadrightarrow_\beta$ immedeatly follows from transitivity of multi-step rediction for labelled terms and for usual terms respectively.

□

Lemma 13.
$\phi(M[x := N]) = \phi(M)[x := \phi(N)]$.

Proof. Induction on M.
1) Let $M = x$. Then $\phi(x[x := N]) = \phi(N)$.
On the other hand, $\phi(x)[x := \phi(N)] = x[x := \phi(N)] = \phi(N)$.
So $\phi(x[x := N]) = \phi(x)[x := \phi(N)]$.

2) Let $M = y$ and $y \neq x$. Then $\phi(y[x := N]) = \phi(y) = y$.
But $\phi(y)[x := \phi(N)] = y[x := \phi(N)] = y$.
Therefore $\phi(y[x := N]) = \phi(y)[x := \phi(N)]$.

3) Let $M = \mathbf{pure}\, M'$. Then $\phi(\mathbf{pure}\, M'[x := N]) = \mathbf{pure}\, \phi(M'[x := N])$.
By hypothesis, $\mathbf{pure}\,(\phi(M'[x := N])) = \mathbf{pure}\,(\phi(M')[x := \phi(N)])$, which is $(\mathbf{pure}\, \phi(M'))[x := \phi(N)]$ by substitution definition.

4) Let $M = M' \star N'$. So $\phi((M' \star N')[x := N])) = \phi(M'[x := N] \star N'[x := N])$.

By definition of ϕ,
$\phi(M'[x := N] \star N'[x := N]) = \phi(M'[x := N]) \star \phi(N'[x := N])$.
But by induction hypothesis,
$\phi(M'[x := N]) = \phi(M')[x := \phi(N)]$ and
$\phi(N'[x := N]) = \phi(N')[x := \phi(N)]$.
Hence,
$\phi(M'[x := N]) \star \phi(N'[x := N]) = \phi(M')[x := \phi(N)] \star \phi(N')[x := \phi(N)]$.
So,
$\phi(M')[x := \phi(N)] \star \phi(N')[x := \phi(N)] = (\phi(M') \star \phi(N'))[x := \phi(N)]$.
And by definition of ϕ, $(\phi(M')\star\phi(N'))[x := \phi(N)] = \phi(M'\star N')[x := \phi(N)]$.
□

Lemma 14.
Let $M, N \in \Lambda_{\mathbf{K}}^{'}$ and $M \twoheadrightarrow_{\beta'} N$, then $\phi(M) \twoheadrightarrow_{\beta} \phi(N)$.

Proof.
1) Let $\mathbf{pure}\,(\lambda x.x) \star M, M \in \Lambda_{\mathbf{K}}^{'}$ and $\mathbf{pure}\,(\lambda x.x) \star M \rightarrow_{\beta'} M$.
But $\phi(\mathbf{pure}\,(\lambda x.x) \star M) = \mathbf{pure}\,(\lambda x.x) \star \phi(M)$.
So $\mathbf{pure}\,(\lambda x.x) \star \phi(M) \rightarrow_{\beta} \phi(M)$ by β-reduction rule.

2) Let $\mathbf{pure}\,(\lambda fgx.f(gx))\star M\star N\star P, M\star(N\star P) \in \Lambda_{\mathbf{K}}^{'}$ and $\mathbf{pure}\,(\lambda fgx.f(gx))\star M \star N \star P \rightarrow_{\beta'} M \star (N \star P)$.
By the definition of ϕ:
$\phi(\mathbf{pure}\,(\lambda fgx.f(gx))\star M\star N\star P) = \mathbf{pure}\,(\lambda fgx.f(gx))\star\phi(M)\star\phi(N)\star\phi(P)$;
$M \star (N \star P) = \phi(M) \star (\phi(N) \star \phi(P))$.
Hence, $\mathbf{pure}\,(\lambda fgx.f(gx)) \star \phi(M) \star \phi(N) \star \phi(P) \rightarrow_{\beta} \phi(M) \star (\phi(N) \star \phi(P))$ by β-reduction rule.

3) Let $(\mathbf{pure}\,M)\star(\mathbf{pure}\,N), \mathbf{pure}\,(MN) \in \Lambda_{\mathbf{K}}^{'}$ and $(\mathbf{pure}\,M)\star(\mathbf{pure}\,N) \rightarrow_{\beta} \mathbf{pure}\,(MN)$.
By the definition of ϕ:
$\phi((\mathbf{pure}\,M) \star (\mathbf{pure}\,N)) = (\mathbf{pure}\,\phi(M)) \star (\mathbf{pure}\,\phi(N))$;
$\phi(\mathbf{pure}\,(MN)) = \mathbf{pure}\,(\phi(M)\phi(N))$.
So, by reduction rule, $(\mathbf{pure}\,\phi(M)) \star (\mathbf{pure}\,\phi(N)) \rightarrow_{\beta} \mathbf{pure}\,(\phi(M)\phi(N))$.

4) Let $M \star (\mathbf{pure}\,), \mathbf{pure}\,(\lambda f.fN) \star M$ and $M \star (\mathbf{pure}\,N) \rightarrow_{\beta'} (\lambda f.fN) \star M$.
$\phi(M \star (\mathbf{pure}\,N)) = \phi(M) \star (\mathbf{pure}\,\phi(N))$
$\phi((\lambda f.fN) \star M) = (\lambda f.f\phi(N)) \star \phi(M)$.
So, $\phi(M) \star (\mathbf{pure}\,\phi(N)) \rightarrow_{\beta} \mathbf{pure}\,(\lambda f.f\phi(N)) \star \phi(M)$. □

Lemma 15.
Let $M \in \Lambda_{\boldsymbol{K}}^{'}$. Then $|M| \twoheadrightarrow_{\beta} \phi(M)$.

Proof. Induction on the structure of M.
□

Lemma 16. *Strip lemma.*

If $M \rightarrow_{\beta} N$ and $M \twoheadrightarrow_{\beta} P$. Then there exists some term Q, such that $N \twoheadrightarrow_{\beta} Q$ and $P \twoheadrightarrow_{\beta} Q$.

Proof. Proof is similar to [15] [18]. We build the following diagram

$$\begin{tikzcd}
M \arrow[rr, two heads] \arrow[dd, "\beta"'] & & P \arrow[dd, dashed, "\beta"] & \\
& N' \arrow[ul, "|.|"'] \arrow[dl, "\phi"] \arrow[rr, dashed, two heads, "\beta'"] & & P' \arrow[ul, "|.|"'] \arrow[dl, "\phi"] \\
N \arrow[rr, dashed, two heads, "\beta"'] & & Q &
\end{tikzcd}$$

□

which is commutes by lemmas 11 – 14.

Theorem 3. *Confluence.*

If $M \twoheadrightarrow_\beta N$ and $M \twoheadrightarrow_\beta P$. Then there exists some term Q, such that $N \twoheadrightarrow_\beta Q$ and $P \twoheadrightarrow_\beta Q$.

Proof.
By unfolding $M \twoheadrightarrow_\beta N$ as the sequence of one-step reductions $M \rightarrow_\beta M_1 \rightarrow_\beta ... \rightarrow_\beta M_n \rightarrow_\beta N$ and applying strip lemma on every step.

□

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