Soundness for modal type theory based on the intuitionistic epistemic logic

1 Modal lambda calculus based on IEL⁻

Definition 1. The set of terms:

Let V is a set of variables. The set Λ_K of terms is defined by the grammar:

$$\Lambda_{\mathbf{K}} ::= \mathbb{V} \mid (\lambda \Lambda. \Lambda_{\mathbf{K}}) \mid (\Lambda_{\mathbf{K}} \Lambda_{\mathbf{K}}) \mid (\Lambda_{\mathbf{K}}, \Lambda_{\mathbf{K}}) \mid (\pi_{i} \Lambda_{\mathbf{K}}) \mid (pure \Lambda_{\mathbf{K}}) \mid (\Lambda_{\mathbf{K}} \star \Lambda_{\mathbf{K}}) \quad (1)$$
where $i \in \{1, 2\}$.

Definition 2. The set of types:

Let \mathbb{T} is a set of atomic types. The set $\mathbb{T}_{\mathbf{K}}$ of types with applicative functor \mathbf{K} is generated by the grammar:

$$\mathbb{T}_{K} ::= \mathbb{T} \mid (\mathbb{T}_{K} \to \mathbb{T}_{K}) \mid (\mathbb{T}_{K} \times \mathbb{T}_{K}) \mid (K\mathbb{T}_{K})$$
(2)

Our type system is based on the Curry-style typing rules:

Definition 3. Modal typed lambda calculus $\lambda \mathbf{K}$ based on $NIEL_{\wedge,\rightarrow}^-$:

$$\frac{\Gamma, x : \alpha \vdash x : \alpha}{\Gamma \vdash \lambda x. M : \alpha \to \beta} \xrightarrow{\lambda_{i}} \xrightarrow{\Gamma} \underbrace{\frac{\Gamma \vdash x : \alpha}{\Gamma \vdash \lambda x. M : \alpha \to \beta}} \xrightarrow{\gamma_{i}} \times_{i}$$

$$\frac{\Gamma \vdash x : \alpha}{\Gamma \vdash (x, y) : \alpha \times \beta} \times_{i}$$

$$\frac{\Gamma, \vdash x : \alpha}{\Gamma \vdash pure \ x : \mathbf{K}\alpha} \mathbf{K}_{I}$$

$$\frac{\Gamma \vdash f : \alpha \to \beta}{\Gamma \vdash fx : \beta} \xrightarrow{\Gamma \vdash x : \alpha} \xrightarrow{\gamma_{e}} \xrightarrow{\Gamma} \underbrace{\Gamma \vdash f : \mathbf{K}(\alpha \to \beta)}_{\Gamma \vdash \pi_{i}p : \alpha_{i}} \times_{e}, i \in \{1, 2\}$$

$$\frac{\Gamma \vdash f : \mathbf{K}(\alpha \to \beta)}{\Gamma \vdash f \star x : \mathbf{K}\beta} \xrightarrow{\Gamma \vdash x : \mathbf{K}\alpha} \mathbf{K}_{app}$$

Definition 4. β -reduction rules:

- 1) $(\lambda x.M)N \rightarrow_{\beta} M[x := N];$
- 2) $\pi_i \langle M_1, M_2 \rangle \rightarrow_{\beta} M_i, i \in \{1, 2\};$
- 3) pure $(\lambda x.x) \star M \to_{\beta} M$;
- 4) pure $(\lambda fgx.f(gx)) \star M \star N \star P \rightarrow_{\beta} M \star (N \star P);$
- 5) $(pure\ M) \star (pure\ N) \rightarrow_{\beta} pure\ (MN);$
- 6) $M \star pure \ N \rightarrow_{\beta} (\lambda f. fN) \star M$;

Definition 5. η -reduction rules for applicative functor:

- 1) pure $(\lambda x. fx) \to_{\eta} pure f$;
- 2) pure $\langle \pi_1 p, \pi_2 p \rangle \rightarrow_{\eta} pure p$;
- 3) $\lambda x.f \star x \to_{\eta} f$.

2 Categorical model.

Let us define monoidal categories and strong lax monoidal functors.

Definition 6. Monoidal category.

A monoidal category C is a category with:

- 1) A bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the tensor product;
- 2) An object $\mathbb{1} \in Ob(\mathcal{C})$ called the unit;
- 3) A natural isomorphism such that for all $A, B, C \in Ob(\mathcal{C})$:

$$\alpha_{A,B,C}: (A \otimes B) \otimes C \xrightarrow{\cong} A \otimes (B \otimes C)$$

where α is called associator.

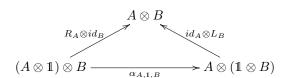
4) A natural isomorphism (left unitor) for all $A \in Ob(C)$:

$$L_A: (\mathbb{1} \otimes A) \xrightarrow{\cong} A$$

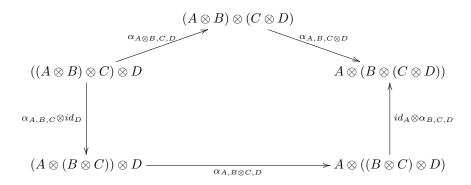
5) A natural isomorphism (right unitor) for all $A \in Ob(\mathcal{C})$:

$$R_A: (A\otimes 1) \xrightarrow{\cong} A$$

6) The next one diagram commutes (the triangle identity):



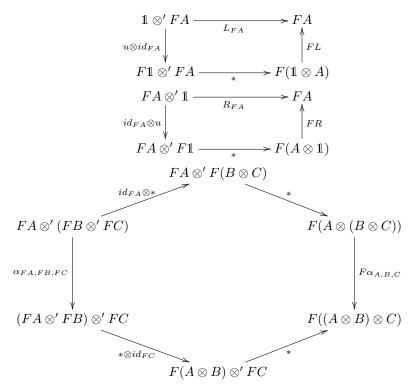
7) The next one diagram commutes too (the pentagon identity):



A monoidal category is symmetrical iff $\forall A, B \in Ob(\mathcal{C}), A \otimes B \cong B \otimes A$.

Definition 7. A lax monoidal functor between monoinal categories $\langle \mathcal{C}, \otimes, \mathbb{1} \rangle$ and $\langle \mathcal{D}, \otimes', \mathbb{1} \rangle$ is a functor $F : \langle \mathcal{C}, \otimes, \mathbb{1} \rangle \to \langle \mathcal{D}, \otimes', \mathbb{1} \rangle$ with the next natural transformations:

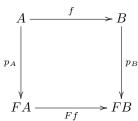
- 1) $u: \mathbb{1} \to F\mathbb{1}$ (unit property);
- 2) $*: FA \otimes' FB \rightarrow F(A \otimes B)$ (application property); and with the next commuting diagrams:



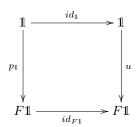
Definition 8. Applicative functor.

Let $\langle \mathcal{C}, \otimes, \mathbb{1} \rangle$ is a symmetrical monoidal category. Applicative functor is an endofunctor $F : \langle \mathcal{C}, \otimes, \mathbb{1} \rangle \to \langle \mathcal{C}, \otimes, \mathbb{1} \rangle$ with a natural transformation $p : Id_{\mathcal{C}} \Rightarrow F$ with the next properties:

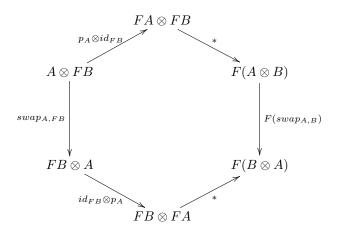
1) A natural transformation p is defined as follows for all $A \in Ob(\mathcal{C})$ the following diagram commutes:



2) $p_1 = u$:



- 3) $p \circ * = * \circ (p \otimes p);$
- 4) The following diagram commutes (weak commutativity condition):



3 Soundness

Definition 9. Semantical translation from $\lambda_{\mathbf{K}}$ to CCC with applicative functor:

- 1) Interpretation for types: $[\![A]\!] := \hat{A}, A \in \mathbb{T}, [\![A \to B]\!] := [\![A]\!] \to [\![B]\!], [\![A \times B]\!] := [\![A]\!] \times [\![B]\!];$
- 2) Interpretation for modal types: $\llbracket \mathbf{K}A \rrbracket = \mathcal{K} \llbracket A \rrbracket$, where \mathcal{K} is an applicative functor;
- 3) Interpretaion for contexts: $\llbracket \Gamma = \{x_1 : A_1, ..., x_n : A_n\} \rrbracket := \llbracket \Gamma \rrbracket = \llbracket A_1 \rrbracket \times ... \times \llbracket A_n \rrbracket;$
- 4) Interpretation for typing assignment: $\llbracket \Gamma \vdash M : A \rrbracket := \llbracket M \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket$, where $\llbracket M \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket \in \mathcal{C}$;
 - 5) Interpretation for typing rules:

2) $[(M \star N)[\vec{x} := \vec{M}] = [M \star N] \circ \langle [M_1], \dots, [M_n] \rangle$.

$$\begin{split} & [\![(M\star N)[\vec{x}:=\vec{M}]\!] = [\![(M[\vec{x}:=\vec{M}]\!])\star(N[\vec{x}:=\vec{M}]\!)]\!] \\ &= p_{\epsilon} \circ * \circ \langle [\![(M[\vec{x}:=\vec{M}]\!])]\!], [\![(N[\vec{x}:=\vec{M}]\!])]\!] \rangle \\ &= p_{\epsilon} \circ * \circ \langle [\![M]\!] \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle, [\![N]\!] \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle \\ &= p_{\epsilon} \circ * \circ \langle [\![M]\!], [\![N]\!] \rangle \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle \\ &= (p_{\epsilon} \circ * \circ \langle [\![M]\!], [\![N]\!]) \rangle \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle \\ &= [\![M\star N]\!] \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle \end{split}$$

Definition of substitution Translation for \star Induction hypothesis Property of morphism product Associativity of composition Translation for \star

Lemma 2.

If
$$M \to_{\beta} N$$
, then $[\![M]\!] = [\![N]\!]$.

1) [pure $(\lambda x.x) \star M$] = [M];

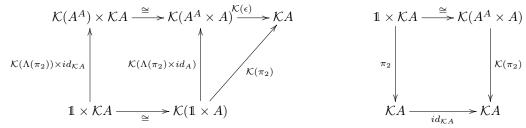
$$\frac{\llbracket x:A\vdash x:A\rrbracket=\pi_2:\mathbb{I}\times\llbracket A\rrbracket\to\llbracket A\rrbracket}{\llbracket\vdash \lambda x.x:A\to A\rrbracket=\Lambda(\pi_2):\mathbb{I}\to\llbracket A\rrbracket^{\llbracket A\rrbracket}}$$

$$\boxed{\llbracket\vdash \text{pure }(\lambda x.x):\mathbf{K}(A\to A)\rrbracket=p_{\llbracket A\rrbracket^{\llbracket A\rrbracket}}\circ\Lambda(\pi_2):\mathbb{I}\to\mathcal{K}(\llbracket A\rrbracket^{\llbracket A\rrbracket})}$$

But by the following diagram:

$$p_{\llbracket A \rrbracket \llbracket A \rrbracket} \circ \Lambda(\pi_2) = id_{\mathbb{I}} \circ \mathcal{K}(\Lambda(\pi_2))$$
$$= \mathcal{K}(\Lambda(\pi_2))$$

Let us consider the next commutative diagram:



Hence:
$$\mathbb{I}M : \mathbf{K}\Delta$$

2) $[\![(\text{pure }\lambda fgx.f(gx))\star M\star N\star P]\!]=[\![M\star (N\star P)]\!]$ The first part of equality:

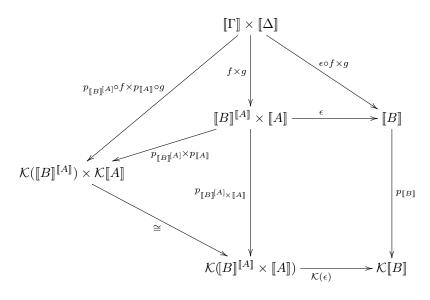
- 3) $\llbracket (\text{pure } M) \star (\text{pure } N) \rrbracket = \llbracket \text{pure } (MN) \rrbracket;$
- 1) The left part of the equality:

$$\begin{split} & \quad \quad \|\Gamma \vdash M : A \to B\| = f : [\![\Gamma]\!] \to [\![B]\!]^{[A]} \\ & \quad \|\Gamma \vdash \text{pure } M : \mathbf{K}(A \to B)\| = p_{[\![B]\!]^{[A]}} \circ f : [\![\Gamma]\!] \to \mathcal{K}([\![B]\!]^{[A]}) \\ & \quad \quad \|\Delta \vdash N : A\| = g : [\![\Delta]\!] \to [\![A]\!] \\ & \quad \quad \|\Delta \vdash \text{pure } N : \mathbf{K}A\| = p_{[\![A]\!]} \circ g : [\![\Delta]\!] \to \mathcal{K}[\![A]\!] \end{split}$$

 $\llbracket \Gamma, \Delta \vdash (\text{pure } M) \star (\text{pure } N) : \mathbf{K}B \rrbracket = \mathcal{K}(\epsilon) \circ (\cong) \circ (p_{\llbracket B \rrbracket^{[A]}} \circ f \times p_{\llbracket A \rrbracket} \circ g) : \Gamma \times \Delta \to \mathcal{K}B$

2) The second part of equality:

$$\frac{\llbracket \Gamma \vdash M : A \to B \rrbracket = f : \llbracket \Gamma \rrbracket \to \llbracket B \rrbracket^{[A]} \qquad \llbracket \Delta \vdash N : A \rrbracket = g : \llbracket \Delta \rrbracket \to \llbracket A \rrbracket}{\llbracket \Gamma, \Delta \vdash MN : B \rrbracket = \epsilon \circ f \times g : \llbracket \Gamma \rrbracket \times \llbracket \Delta \rrbracket \to \llbracket B \rrbracket}$$
$$\frac{\llbracket \Gamma, \Delta \vdash \text{pure } (MN) : \mathbf{K}B \rrbracket = p_{\llbracket B \rrbracket} \circ (\epsilon \circ (f \times g)) : \llbracket \Gamma \rrbracket \times \llbracket \Delta \rrbracket \to \mathcal{K} \llbracket B \rrbracket}$$



$$\begin{split} \llbracket \Gamma, \Delta \vdash (\text{pure } M) \star (\text{pure } N) : \mathbf{K}B \rrbracket &= \mathcal{K}(\epsilon) \circ (\cong) \circ (p_{\llbracket B \rrbracket^{\llbracket A \rrbracket}} \circ f \times p_{\llbracket A \rrbracket} \circ g) \\ &= K(\epsilon) \circ (\cong) \circ p_{\llbracket B \rrbracket^{\llbracket A \rrbracket}} \times p_{\llbracket A \rrbracket} \circ f \times g \\ &= K(\epsilon) \circ p_{\llbracket B \rrbracket} \circ \epsilon \circ f \times g \\ &= \llbracket \Gamma, \Delta \vdash \text{pure } (MN) : \mathcal{K}B \rrbracket \end{split}$$

4)
$$\begin{bmatrix} [N:A,M:\mathbf{K}(A\to B) \vdash M \star \text{pure } N:\mathbf{K}B]] = \\ [N:A,M:\mathbf{K}(A\to B) \vdash \text{pure } (\lambda f.fN) \star M:\mathbf{K}B] \end{bmatrix}$$

It is easy to see that the following diagram commutes:

$$\mathcal{K}(\llbracket B \rrbracket^{(\llbracket B \rrbracket^{\llbracket A \rrbracket)}}) \times \mathcal{K}(\llbracket B \rrbracket^{\llbracket A \rrbracket}) \xrightarrow{\cong} \mathcal{K}(\llbracket B \rrbracket^{\llbracket A \rrbracket)} \times \llbracket B \rrbracket^{\llbracket A \rrbracket}) \xrightarrow{\mathcal{K}(\epsilon)} \mathcal{K}(\epsilon)$$

$$\mathcal{K}(\Lambda(\epsilon \circ (\pi_2, \pi_1))) \times id_{\mathbb{R}^{\llbracket B \rrbracket}[A \rrbracket)} \xrightarrow{\mathcal{K}(\epsilon)} \mathcal{K}(\mathbb{R}^{\llbracket B \rrbracket^{\llbracket A \rrbracket})} \times \mathcal{K}(\mathbb{R}^{\llbracket B \rrbracket^{\llbracket A \rrbracket})} \xrightarrow{\mathcal{K}(\epsilon)} \mathcal{K}(\mathbb{R}^{\llbracket B \rrbracket^{\llbracket A \rrbracket})} \times \mathcal{K}(\mathbb{R}^{\llbracket B \rrbracket^{\llbracket A \rrbracket})} \times \mathcal{K}(\mathbb{R}^{\llbracket B \rrbracket^{\llbracket A \rrbracket})} \xrightarrow{\mathcal{K}(\epsilon)} \mathcal{K}(\mathbb{R}^{\llbracket B \rrbracket^{\llbracket A \rrbracket})} \times \mathcal{K}(\mathbb{R}^{\llbracket A \rrbracket}) \times \mathcal$$

Lemma 3. If $M \to_{\eta} N$, then $\llbracket M \rrbracket = \llbracket N \rrbracket$.

 $\llbracket N:A,M:\mathbf{K}(A\to B)\vdash M\star \text{pure }N:\mathbf{K}B
rbracket$

Proof.

1) $\llbracket \text{pure } (\lambda x. fx) \rrbracket = \llbracket \text{pure } f \rrbracket.$

2) [pure $\langle \pi_1 M, \pi_2 M \rangle$] = [pure M]

3)
$$[M : \mathbf{K}(A \times B) \vdash \text{pure } (\lambda x. \lambda y. \langle x, y \rangle) \star (\text{pure } (\lambda x. \pi_1) \star M) \star (\text{pure } (\lambda x. \pi_2) \star M : \mathbf{K}(A \times B)]] = [M : \mathbf{K}(A \times B) \vdash M : \mathbf{K}(A \times B)]$$

i) The first step

Let us consider interpetation for \vdash pure $(\lambda x.\lambda y.\langle x,y\rangle)$: $\mathbf{K}(A\to B\to A\times B)$:

$$\frac{\pi_2:\mathbb{1}\times(\llbracket A\rrbracket\times\llbracket B\rrbracket)\to\llbracket A\rrbracket\times\llbracket B\rrbracket}{\pi_2\circ\alpha:(\mathbb{1}\times\llbracket A\rrbracket)\times\llbracket B\rrbracket\to\llbracket A\rrbracket\times\llbracket B\rrbracket} \\ \frac{\Lambda(\pi_2\circ\alpha):\mathbb{1}\times\llbracket A\rrbracket\to\llbracket A\rrbracket\times\llbracket B\rrbracket^{\llbracket B\rrbracket}}{\Lambda(\Lambda(\pi_2\circ\alpha)):\mathbb{1}\to\llbracket A\rrbracket\times\llbracket B\rrbracket^{\llbracket B\rrbracket}} \\ \frac{\Gamma(\Lambda(\pi_2\circ\alpha)):\mathbb{1}\to \Gamma(\Lambda)^{\llbracket A\rrbracket\times\llbracket B\rrbracket^{\llbracket A\rrbracket}}}{\Gamma(\Lambda(\pi_2\circ\alpha)):\mathbb{1}\to \Gamma(\Pi)^{\llbracket A\rrbracket\times\llbracket B\rrbracket^{\llbracket A\rrbracket}}}$$

By naturality, $p_{\llbracket A \rrbracket \times \llbracket B \rrbracket \llbracket^B \rrbracket^{\llbracket A \rrbracket}} \circ \Lambda(\Lambda(\pi_2 \circ \alpha)) = \mathcal{K}(\Lambda(\Lambda(\pi_2 \circ \alpha)).$

At first let us show that the following diagram commutes in any CCC:

$$(\llbracket A \times B \rrbracket^{\llbracket B \rrbracket^{\llbracket A \rrbracket}} \times \llbracket A \rrbracket) \times \llbracket B \rrbracket \xrightarrow{\epsilon \circ (\epsilon \times id_{\llbracket B \rrbracket})} \llbracket A \rrbracket \times \llbracket B \rrbracket$$

$$(\Lambda(\Lambda(\pi_2 \circ \alpha)) \times id_{\llbracket A \rrbracket}) \times id_{\llbracket B \rrbracket}$$

$$([1] \times \llbracket A \rrbracket) \times \llbracket B \rrbracket$$

$$\epsilon \circ (\epsilon \times id_{\llbracket B \rrbracket}) \circ (\Lambda(\Lambda(\pi_2 \circ \alpha)) \times id_{\llbracket A \rrbracket}) \times id_{\llbracket B \rrbracket} =$$

by the definition of morphism product

 $\epsilon \circ (\epsilon \times id_{\llbracket B \rrbracket}) \circ \langle \Lambda(\Lambda(\pi_2 \circ \alpha)) \circ \pi_1, id_{\llbracket A \rrbracket} \circ \pi_2 \rangle \times id_{\llbracket B \rrbracket} =$

by the definition of morphism product

$$\epsilon \circ (\epsilon \times id_{\llbracket B \rrbracket}) \circ \langle \langle \Lambda(\Lambda(\pi_2 \circ \alpha)) \circ \pi_1, id_{\llbracket A \rrbracket} \circ \pi_2 \rangle \circ \pi_1, id_{\llbracket B \rrbracket} \circ \pi_2 \rangle$$

by the property of morphism product

$$\epsilon \circ \langle \epsilon \circ \langle \Lambda(\Lambda(\pi_2 \circ \alpha)) \circ \pi_1, id_{\llbracket A \rrbracket} \circ \pi_2 \rangle \circ \pi_1, id_{\llbracket B \rrbracket} \circ id_{\llbracket B \rrbracket} \circ \pi_2 \rangle =$$

by the definition of morphism product and by identity

$$\epsilon \circ \langle \epsilon \circ (\Lambda(\Lambda(\pi_2 \circ \alpha)) \times id_{\llbracket A \rrbracket}) \circ \pi_1, id_{\llbracket B \rrbracket} \circ \pi_2 \rangle =$$

by exponentiation and currying property

 $\epsilon \circ \langle \Lambda(\pi_2 \circ \alpha) \circ \pi_1, id_{\llbracket B \rrbracket} \circ \pi_2 \rangle =$

by the definition of morphism product

 $\epsilon \circ \Lambda(\pi_2 \circ \alpha) \times id_{\llbracket B \rrbracket}$

by exponentiation and currying property

 $\pi_2 \circ \alpha$

 $^{\square}$

Lemma 4.

- $1) \; \llbracket M \rrbracket = \llbracket N \rrbracket, \; \textit{if} \; \llbracket \textit{pure} \; M \rrbracket = \llbracket \textit{pure} \; N \rrbracket;$
- 2) Let $\llbracket M \rrbracket = \llbracket N \rrbracket$, then $\llbracket M \star P \rrbracket = \llbracket N \star P \rrbracket$;
- 3) Let $\llbracket M \rrbracket = \llbracket N \rrbracket$, then $\llbracket P \star M \rrbracket = \llbracket P \star N \rrbracket$.

Proof.

1)

i) "only if"-part.

Let $\llbracket M \rrbracket : \llbracket \Gamma \rrbracket \to \llbracket A \rrbracket$, $\llbracket N \rrbracket : \llbracket \Gamma \rrbracket \to \llbracket A \rrbracket$ and $\llbracket M \rrbracket = \llbracket N \rrbracket$. So $p \circ \llbracket M \rrbracket = p \circ \llbracket N \rrbracket$, hence $\llbracket \text{pure } M \rrbracket = \llbracket \text{pure } N \rrbracket$.

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