

# Modal type theory based on the intuitionistic epistemic logic

## Abstract

Modal intuitionistic epistemic logic  $IEL^-$  was proposed by S.Artemov and T. Protopopescu as the formal foundation for the intuitionistic theory of knowledge. We construct a modal simply typed lambda-calculus which is Curry-Howard isomorphic to  $IEL^-$  as formal theory of calculations with applicative functors in functional programming languages like Haskell or Idris.

## 1 Introduction

Modal intuitionistic epistemic logic  $IEL$  was proposed by S. Artemov and T. Protopopescu [1].  $IEL$  provides the epistimology and the theory of knowledge as based on BHK-semantics of intuitionistic logic.  $IEL^-$  is a variant of  $IEL$ , that corresponds to intuitionistic belief. Informally,  $\mathbf{K}A$  denotes that  $A$  is verified intuitionistically.

Intuitionistic epistemic logic  $IEL^-$  is defined with by following axioms and derivation rules:

**Definition 1.** *Intuitionistic epistemic logic  $IEL$ :*

- 1) *IPC axioms;*
  - 2)  $\mathbf{K}(A \rightarrow B) \rightarrow (\mathbf{K}A \rightarrow \mathbf{K}B)$  (*normality*);
  - 3)  $A \rightarrow \mathbf{K}A$  (*co-reflection*);
- Rule: MP.*

We have the deduction theorem and necessitation rule which is derivable.

V. Krupski and A. Yatmanov provided the sequential calculus for  $IEL$  and proved that this calculus is PSPACE-complete [2].

Functional programming languages such as Haskell [3], Idris [4], Purescript [5] Elm [6] or Scala [?] have special type classes<sup>1</sup> for calculations with container types like `Functor` and `Applicative`<sup>2</sup>:

```
class Functor f where
  fmap :: (a -> b) -> f a -> f b

class Functor f => Applicative f where
  pure :: a -> f a
  (<*>) :: f (a -> b) -> f a -> f b
```

---

<sup>1</sup>Type class in Haskell is a general interface for special group of datatypes.

<sup>2</sup>Reader may read more about container types in the Haskell standard library documentation[7] or in the next one textbook [8]

By *container* (or *computational context*) type we mean some type-operator  $f$ , where  $f$  is a “function” from  $*$  to  $*$ : type operator takes a simple type (which has kind  $*$ ) and returns another simple type with kind  $*$ . For more detailed description of the type system with kinds used in Haskell see [12].

The motivation for using an applicative functor is quite natural. Applicative functor allows to generalize the action of a functor for functions with arbitrary number of arguments, for instance:

```
liftA2 :: Applicative f => (a -> b -> c) -> f a -> f b -> f c
liftA2 f x y = pure f <*> x <*> y
```

It’s not difficult to see that modal axioms in  $IEL^-$  and types of the methods of Applicative class in Haskell-like languages (which is described below) are syntactically similar and we are going to show that this coincidence has a non-trivial computational meaning.

The main goal of our research is a relationship between intuitionistic epistemic logic  $IEL^-$  and functional programming with effects. We show that relationship by building the type system (which is called  $\lambda_K$ ) which is Curry-Howard isomorphic to  $IEL^-$ . So we will consider  $K$ -modality as an arbitrary applicative functor.

$\lambda_K$  consists of the rules for simply typed lambda-calculus and special typing rules for lifting types into the applicative functor  $K$ . We assume that our type system will axiomatize the simplest case of computation with effects with one container. We provide proof-theoretical view on this kind of computations in functional programming and prove strong normalization and confluence.

## 2 Typed lambda-calculus based on $IEL^-$

At first we define the natural deduction for  $IEL^-$  :

**Definition 2.** *Natural deduction  $NIEL$  for  $IEL^-$  is an extension of intuitionistic natural deduction with additional derivation rules for modality:*

$$\frac{\Gamma \vdash A}{\Gamma \vdash KA} K_I \qquad \frac{\Gamma \vdash K\vec{A} \quad \vec{A} \vdash B}{\Gamma \vdash KB}$$

Where  $\Gamma \vdash K\vec{A}$  is a syntax sugar for  $\Gamma \vdash KA_1, \dots, \Gamma \vdash KA_n$ .

**Lemma 1.**  $\Gamma \vdash_{NIEL_{\wedge, \rightarrow}^-} A \Rightarrow IEL^- \vdash \bigwedge \Gamma \rightarrow A$ .

*Proof.* Induction on the derivation.

Let us consider cases with modality.

- 1) If  $\Gamma \vdash_{NIEL_{\wedge, \rightarrow}^-} A$ , then  $IEL^- \vdash \bigwedge \Gamma \rightarrow KA$ .
  - (1)  $\bigwedge \Gamma \rightarrow A$  assumption
  - (2)  $A \rightarrow KA$  co-reflection
  - (3)  $(\bigwedge \Gamma \rightarrow A) \rightarrow ((A \rightarrow KA) \rightarrow (\bigwedge \Gamma \rightarrow KA))$  IPC theorem
  - (4)  $(A \rightarrow KA) \rightarrow (\bigwedge \Gamma \rightarrow KA)$  from (1), (3) and MP
  - (5)  $\bigwedge \Gamma \rightarrow KA$  from (2), (4) and MP

- 2) If  $\Gamma \vdash_{NIEL_{\wedge, \rightarrow}^-} \mathbf{K}\vec{A}$  and  $\vec{A} \vdash B$ , then  $IEL^- \vdash \bigwedge \Gamma \rightarrow \mathbf{K}B$ .
- (1)  $\bigwedge \Gamma \rightarrow \bigwedge_{i=1}^n \mathbf{K}A_i$  assumption
  - (2)  $\bigwedge_{i=1}^n \mathbf{K}A_i \rightarrow \mathbf{K} \bigwedge_{i=1}^n A_i$  IEL theorem
  - (3)  $\bigwedge \Gamma \rightarrow \mathbf{K} \bigwedge_{i=1}^n A_i$  from (1), (2) and transitivity
  - (4)  $\bigwedge_{i=1}^n A_i \rightarrow B$  assumption
  - (5)  $(\bigwedge_{i=1}^n A_i \rightarrow B) \rightarrow \mathbf{K}(\bigwedge_{i=1}^n A_i \rightarrow B)$  co-reflection
  - (6)  $\mathbf{K}(\bigwedge_{i=1}^n A_i \rightarrow B)$  from (2), (3) and MP
  - (7)  $\mathbf{K} \bigwedge_{i=1}^n A_i \rightarrow \mathbf{K}B$  from (6) and normality
  - (8)  $\bigwedge \Gamma \rightarrow \mathbf{K}B$  from (3), (7) and transitivity

□

**Lemma 2.** *If  $IEL^- \vdash A$ , then  $NIEL^- \vdash A$ .*

*Proof.* Straightforward derivation of modal axioms in  $NIEL^-$ . We consider this derivation below using terms. □

At the next step we build the typed lambda-calculus based on  $NIEL_{\wedge, \rightarrow}^-$  by proof-assignment in rules.

At first, we define lambda-terms and types for this lambda-calculus.

**Definition 3.** *The set of terms:*

*Let  $\mathbb{V}$  be the set of variables. The set  $\Lambda_{\mathbf{K}}$  of terms is defined by the grammar:*

$$\Lambda_{\mathbf{K}} ::= \mathbb{V} \mid (\lambda \Lambda. \Lambda_{\mathbf{K}}) \mid (\Lambda_{\mathbf{K}} \Lambda_{\mathbf{K}}) \mid (\Lambda_{\mathbf{K}}, \Lambda_{\mathbf{K}}) \mid (\pi_1 \Lambda_{\mathbf{K}}) \mid (\pi_2 \Lambda_{\mathbf{K}}) \mid (\text{pure } \Lambda_{\mathbf{K}}) \mid (\text{let pure } \Lambda_{\mathbf{K}} = \Lambda_{\mathbf{K}} \text{ in } \Lambda_{\mathbf{K}})$$

**Definition 4.** *The set of types:*

*Let  $\mathbb{T}$  be the set of atomic types. The set  $\mathbb{T}_{\mathbf{K}}$  of types with applicative functor  $\mathbf{K}$  is generated by the grammar:*

$$\mathbb{T}_{\mathbf{K}} ::= \mathbb{T} \mid (\mathbb{T}_{\mathbf{K}} \rightarrow \mathbb{T}_{\mathbf{K}}) \mid (\mathbb{T}_{\mathbf{K}} \times \mathbb{T}_{\mathbf{K}}) \mid (\mathbf{K}\mathbb{T}_{\mathbf{K}}) \quad (1)$$

Context, domain of context and range of context are defined standardly [11][12].

Our type system is based on the Curry-style typing rules:

**Definition 5.** *Modal typed lambda calculus  $\lambda_{\mathbf{K}}$  based on  $NIEL_{\wedge, \rightarrow}^-$ :*

$$\frac{}{\Gamma, x : A \vdash x : A} \text{ax}$$

$$\begin{array}{c}
\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x. M : A \rightarrow B} \rightarrow_i \qquad \frac{\Gamma \vdash M : A \rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B} \rightarrow_e \\
\\
\frac{\Gamma \vdash M : A \quad \Gamma \vdash N : B}{\Gamma \vdash \langle M, N \rangle : A \times B} \times_i \qquad \frac{\Gamma \vdash M : A_1 \times A_2}{\Gamma \vdash \pi_i M : A_i} \times_e, i \in \{1, 2\} \\
\\
\frac{\Gamma \vdash M : A}{\Gamma \vdash \mathbf{pure} M : \mathbf{K}A} \mathbf{K}_I \qquad \frac{\Gamma \vdash \vec{M} : \mathbf{K}\vec{A} \quad \vec{x} : \vec{A} \vdash N : B}{\Gamma \vdash \mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} N : \mathbf{K}B} \mathbf{let_K}
\end{array}$$

$\mathbf{K}_I$ -typing rule is the same as  $\bigcirc$ -introduction in lax logic (also known as monadic metalanguage [17]) and in typed lambda-calculus which is derived by proof-assignment for lax-logic proofs.  $\mathbf{K}_I$  allows to inject an object of type  $\alpha$  into the functor.  $\mathbf{K}_I$  reflects the Haskell method **pure** for Applicative class. It plays the same role as the **return** method in Monad class.

$\mathbf{let_K}$  is similar to the  $\square$ -rule in typed lambda calculus for intuitionistic normal modal logic  $\mathbf{IK}$ , which is described in [19].

In fact, our calculus is the extension of typed lambda calculus for  $\mathbf{IK}$  with typing rule appropriate to co-reflection.

Here are some examples of closed terms:

- $(\lambda x. \mathbf{pure} x) : A \rightarrow \mathbf{K}A$ ;
- $\lambda f. \lambda x. \mathbf{let pure} g, y = f, x \mathbf{in} gy : \mathbf{K}(A \rightarrow B) \rightarrow \mathbf{K}A \rightarrow \mathbf{K}B$
- $\lambda f. \lambda x. \mathbf{let pure} g, y = \mathbf{pure} f, x \mathbf{in} gy : (A \rightarrow B) \rightarrow \mathbf{K}A \rightarrow \mathbf{K}B$

Now we define free variables and substitutions.  $\beta$ -reduction, multi-step  $\beta$ -reduction and  $\beta$ -equality are defined standardly:

**Definition 6.** Set  $FV(M)$  of free variables for arbitrary term  $M$ :

- 1)  $FV(x) = \{x\}$ ;
- 2)  $FV(\lambda x. M) = FV(M) \setminus \{x\}$ ;
- 3)  $FV(MN) = FV(M) \cup FV(N)$ ;
- 4)  $FV(\langle M, N \rangle) = FV(M) \cup FV(N)$ ;
- 5)  $FV(\pi_i M) \subseteq FV(M)$ ,  $i \in \{1, 2\}$ ;
- 6)  $FV(\mathbf{pure} M) = FV(M)$ ;
- 7)  $FV(\mathbf{let pure} \vec{N} = \vec{M} \mathbf{in} M) = \bigcup_{i=1}^n FV(M)$ , where  $n = |\vec{M}|$ .

**Definition 7.** Substitution:

- 1)  $x[x := N] = N$ ,  $x[y := N] = x$ ;
- 2)  $(MN)[x := N] = M[x := N]N[x := N]$ ;
- 3)  $(\lambda x. M)[x := N] = \lambda x. M[x := N]$ ;
- 4)  $(M, N)[x := P] = (M[x := P], N[x := P])$ ;
- 5)  $(\pi_i M)[x := P] = \pi_i(M[x := P])$ ,  $i \in \{1, 2\}$ ;
- 6)  $(\mathbf{pure} M)[x := P] = \mathbf{pure}(M[x := P])$ ;
- 7)  $(\mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} N)[y := P] = \mathbf{let pure} \vec{x} = (\vec{M}[y := P]) \mathbf{in} N$ .

**Definition 8.** *Type substitution*

The substitution of type  $C$  for type variable  $B$  in type  $A$  inductively defined as follows:

- 1)  $B[B := C] = B$  and  $D[B := C] = D$ , if  $B \neq D$ ;
- 2)  $(A_1 \alpha A_2)[B := C] = (A_1[B := C])\alpha(A_2[B := C])$ , where  $\alpha \in \{\rightarrow, \times\}$ ;
- 3)  $(\mathbf{K}A)[B := C] = \mathbf{K}(A[B := C])$ .
- 4) Let  $\Gamma$  be the context, then  $\Gamma[B := C] = \{x : (A[B := C]) \mid x : A \in \Gamma\}$

**Definition 9.**  *$\beta$ -reduction and  $\eta$ -reduction rules for  $\lambda\mathbf{K}$ .*

- 1)  $(\lambda x.M)N \rightarrow_\beta M[x := N]$ ;
- 2)  $\pi_1\langle M, N \rangle \rightarrow_\beta M$ ;
- 3)  $\pi_2\langle M, N \rangle \rightarrow_\beta N$ ;
- 4)  $\text{let pure } \langle \vec{x}, y, \vec{z} \rangle = \langle \vec{M}, \text{let pure } \vec{w} = \vec{N} \text{ in } Q, \vec{P} \rangle \text{ in } R \rightarrow_\beta$   
 $\text{let pure } \langle \vec{x}, \vec{w}, \vec{z} \rangle = \langle \vec{M}, \vec{N}, \vec{P} \rangle \text{ in } R[y := Q]$
- 5)  $\text{let pure } \vec{x} = \text{pure } \vec{M} \text{ in } N \rightarrow_\beta \text{pure } N[\vec{x} := \vec{M}]$
- 6)  $\text{let pure } \_ = \_ \text{ in } N \rightarrow_\beta \text{pure } N$ ;
- 7)  $\lambda x.f x \rightarrow_\eta f$ ;
- 8)  $\langle \pi_1 P, \pi_2 P \rangle \rightarrow_\eta P$ ;
- 9)  $\text{let pure } x = M \text{ in } x \rightarrow_\eta M$ ;
- 10)  $M \rightarrow_{\beta\eta} N \Rightarrow \text{pure } M \rightarrow_{\beta\eta} \text{pure } N$

### 3 Basic lemmas

Now we will prove standard lemmas for contexts in type systems<sup>3</sup>:

**Lemma 3.** *Basic lemmas for  $\mathbf{K}_I$ .*

- i) Let  $\Gamma \vdash \text{pure } M : \mathbf{K}A$ , then  $\Gamma \vdash M : A$ ;
- ii) Let  $\Gamma \vdash M : A$  and  $\Gamma \subseteq \Delta$ , then  $\Delta \vdash M : A$ ;
- iii) Let  $\Gamma, x : A \vdash \text{pure } M : \mathbf{K}B$  and  $\Gamma \vdash N : A$ , then  $\Gamma \vdash \text{pure } M[x := N] : \mathbf{K}B$ .
- iv) Let  $\Gamma \vdash \text{pure } M : \mathbf{K}A$ , then  $\Gamma[B := C] \vdash \text{pure } M : \mathbf{K}(A[B := C])$ .

*Proof.*

- i) Induction on  $\Gamma \vdash \text{pure } M : \mathbf{K}A$  and  $\Gamma \vdash \text{let pure } \vec{x} = \vec{N} \text{ in } N : \mathbf{K}B$  correspondently.
- ii) Let  $\Gamma \vdash \text{pure } M : \mathbf{K}A$ . Then  $\Gamma \vdash M : A$  by generation and  $\Delta \vdash M : A$  by assumption. So  $\Delta \vdash \text{pure } M : \mathbf{K}A$  by  $\mathbf{K}_I$ .
- iii) Let  $\Gamma, x : A \vdash \text{pure } M : \mathbf{K}B$  and  $\Gamma \vdash N : A$ .  
By generation  $\Gamma, x : A \vdash M : B$  and by assumption  $\Gamma \vdash M[x := N] : B$ .  
By  $K_I$ ,  $\Gamma \vdash \text{pure } (M[x := N]) : \mathbf{K}B$ .
- iv) Let  $\Gamma \vdash \text{pure } M : \mathbf{K}A$ . By generation  $\Gamma \vdash M : A$  and by assumption  $\Gamma[B := C] \vdash M : A[B := C]$ .  
By  $K_I$   $\Gamma \vdash \text{pure } M : \mathbf{K}(A[B := C])$ . □

**Theorem 1.** *Subject reduction*

Let  $\Gamma \vdash M : A$  and  $M \rightarrow_{\beta\eta} N$ , then  $\Gamma \vdash N : A$

<sup>3</sup>We will not prove cases with  $\rightarrow$ -constructor, they are proved standardly in the same lemmas for simply typed lambda calculus, for example see [11] [12] [14]. We will consider only modal cases

*Proof.* For cases with application, abstraction and pairs see [12] [13].

- 1) Let  $\Gamma \vdash \mathbf{let\ pure} \langle \vec{x}, y, \vec{z} \rangle = \langle \vec{M}, \mathbf{let\ pure} \vec{w} = \vec{N} \text{ in } Q, \vec{P} \rangle \text{ in } R : \mathbf{KB}$ ,  
then  $\Gamma \vdash \mathbf{let\ pure} \langle \vec{x}, \vec{w}, \vec{z} \rangle = \langle \vec{M}, \vec{N}, \vec{P} \rangle \text{ in } R[y := Q] : \mathbf{KB}$
  - 2) Let  $\Gamma \vdash \mathbf{let\ pure} x = M \text{ in } x : \mathbf{KA}$ , then  $\Gamma \vdash M : \mathbf{KA}$ .
- See [19]. □

**Theorem 2.**

- i)  $\rightarrow_\beta$  is strongly normalizing;
- ii)  $\rightarrow_\beta$  is confluent.

*Proof.*

- i) Strong normalization for **IK** was proved by Kakutani for call-by-value and for call-by name [19] [20].

It is easy to extend Kakutani's result for **pure**-rules.

- ii) Confluence can be proved by extending Barendregt's technique with term underlying for untyped or simply typed lambda calculus [15].

Hence, it is sufficient to prove strip lemma. □

**Theorem 3.**

*Normal form in call-by-name  $\lambda_{\mathbf{K}}$  has the subformula property.*

*Proof.* By induction on the structure of term. Case with **let pure**  $\vec{x} = \vec{M} \text{ in } N$  was considered by Kakutani [19] [20]. Similary, if **pure**  $M$  is a normal form, so  $M$  is a normal form too by hypothesis. □

## 4 Categorical semantics

**Definition 10.** *Monoidal functor*

Let  $\langle \mathcal{C}, \otimes_1, \mathbb{1} \rangle$  and  $\langle \mathcal{D}, \otimes_2, \mathbb{1}' \rangle$  are monoidal categories.

A monoidal functor  $\mathcal{F} : \langle \mathcal{C}, \otimes_1, \mathbb{1} \rangle \rightarrow \langle \mathcal{D}, \otimes_2, \mathbb{1}' \rangle$  is a functor  $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$  with additional natural transformations, which satisfy the well-known conditions described in [23]:

- 1)  $u : \mathbb{1}' \rightarrow \mathcal{F}\mathbb{1}$ ;
- 2)  $*_{A,B} : \mathcal{F}A \otimes_2 \mathcal{F}B \rightarrow \mathcal{F}(A \otimes_1 B)$ .

**Definition 11.** *Applicative functor*

An applicative functor is a triple  $\langle \mathcal{C}, \mathcal{K}, \eta \rangle$ , where  $\mathcal{C}$  is a symmetric monoidal category,  $\mathcal{K}$  is a monoidal and  $\eta : Id_{\mathcal{C}} \Rightarrow \mathcal{K}$  is a natural transformation (similar to unit in monad), such that:

- 1)  $u = \eta_{\mathbb{1}}$ ;
- 2)  $*_{A,B} \circ (\eta_A \otimes \eta_B) = \eta_{A \otimes B}$ ;
- 3) Weak commutativity condition:

$$\begin{array}{ccccc}
 A \otimes \mathcal{K}B & \xrightarrow{\eta_A \otimes id_{\mathcal{K}B}} & \mathcal{K}A \otimes \mathcal{K}B & \xrightarrow{*_{A,B}} & \mathcal{K}(A \otimes B) \\
 \sigma_{A,\mathcal{K}B} \downarrow & & & & \downarrow \mathcal{K}(\sigma_{A,B}) \\
 \mathcal{K}B \otimes A & \xrightarrow{id_{\mathcal{K}B} \otimes \eta_A} & \mathcal{K}B \otimes \mathcal{K}A & \xrightarrow{*_{B,A}} & \mathcal{K}(B \otimes A)
 \end{array}$$

## 4.1 Soundness and completeness

### Theorem 4. Soundness

Let  $\Gamma \vdash M : A$  and  $M =_{\beta\eta} N$ , then  $\llbracket \Gamma \vdash M : A \rrbracket = \llbracket \Gamma \vdash N : A \rrbracket$

*Proof.*

**Definition 12.** Semantical translation from  $\lambda_K$  to CCC with applicative functor  $\mathcal{K}$ :

1) Interpretation for types:

$$\llbracket A \rrbracket := \hat{A}, A \in \mathbb{T};$$

$$\llbracket A \rightarrow B \rrbracket := \llbracket A \rrbracket \rightarrow \llbracket B \rrbracket;$$

$$\llbracket A \times B \rrbracket := \llbracket A \rrbracket \times \llbracket B \rrbracket.$$

2) Interpretation for modal types:  $\llbracket KA \rrbracket = \mathcal{K}\llbracket A \rrbracket$ ;

3) Interpretation for contexts:

$$\llbracket \Gamma = \{x_1 : A_1, \dots, x_n : A_n\} \rrbracket := \llbracket \Gamma \rrbracket = \llbracket A_1 \rrbracket \times \dots \times \llbracket A_n \rrbracket;$$

4) Interpretation for typing assignment:  $\llbracket \Gamma \vdash M : A \rrbracket := \llbracket M \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket$ .

5) Interpretation for typing rules:

$$\begin{array}{c} \frac{}{\llbracket \Gamma, x : A \vdash x : A \rrbracket = \pi_2 : \llbracket \Gamma \rrbracket \times \llbracket A \rrbracket \rightarrow \llbracket A \rrbracket} \\ \frac{\llbracket \Gamma, x : A \vdash M : B \rrbracket = f : \llbracket \Gamma \rrbracket \times \llbracket A \rrbracket \rightarrow \llbracket B \rrbracket}{\llbracket \Gamma \vdash (\lambda x.M) : A \rightarrow B \rrbracket = \Lambda(f) : \llbracket \Gamma \rrbracket \rightarrow \llbracket B \rrbracket^{[A]}} \\ \frac{\llbracket \Gamma \vdash M : A \rightarrow B \rrbracket = \llbracket M \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket B \rrbracket^{[A]} \quad \llbracket \Gamma \vdash N : A \rrbracket = \llbracket N \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket}{\llbracket \Gamma \vdash (MN) : B \rrbracket = \llbracket \Gamma \rrbracket \xrightarrow{\langle \llbracket M \rrbracket, \llbracket N \rrbracket \rangle} \llbracket B \rrbracket^{[A]} \times \llbracket A \rrbracket \xrightarrow{\epsilon} \llbracket B \rrbracket} \\ \frac{\llbracket \Gamma \vdash M : A \rrbracket = f : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket \quad \llbracket \Gamma \vdash N : B \rrbracket = g : \llbracket \Gamma \rrbracket \rightarrow \llbracket B \rrbracket}{\llbracket \Gamma \vdash (M, N) : A \times B \rrbracket = \langle f, g \rangle : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket \times \llbracket B \rrbracket} \\ \frac{\llbracket \Gamma \vdash p : A_1 \times A_2 \rrbracket = f : \llbracket \Gamma \rrbracket \rightarrow \llbracket A_1 \rrbracket \times \llbracket A_2 \rrbracket}{\llbracket \Gamma \vdash \pi_i p : A_i \rrbracket = \llbracket \Gamma \rrbracket \xrightarrow{f} \llbracket A_1 \rrbracket \times \llbracket A_2 \rrbracket \xrightarrow{\pi_i} \llbracket A_i \rrbracket} \quad i \in \{1, 2\} \\ \frac{\llbracket \Gamma \vdash M : A \rrbracket = \llbracket M \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket}{\llbracket \Gamma \vdash \mathbf{pure} M : KA \rrbracket := \llbracket \Gamma \rrbracket \xrightarrow{\llbracket M \rrbracket} \llbracket A \rrbracket \xrightarrow{\eta_{[A]}} \mathcal{K}\llbracket A \rrbracket} \\ \frac{\llbracket \Gamma \vdash \vec{M} : K\vec{A} \rrbracket = \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle : \llbracket \Gamma \rrbracket \rightarrow \prod_{i=1}^n \mathcal{K}\llbracket A_i \rrbracket \quad \llbracket \vec{x} : \vec{A} \vdash N : B \rrbracket = \llbracket N \rrbracket : \prod_{i=1}^n \llbracket A_i \rrbracket \rightarrow \llbracket B \rrbracket}{\llbracket \Gamma \vdash \mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} M : KB \rrbracket = \mathcal{K}(\llbracket N \rrbracket) \circ *_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle : \llbracket \Gamma \rrbracket \rightarrow \mathcal{K}\llbracket B \rrbracket} \end{array}$$

### Definition 13. Simultaneous substitution

Let  $\Gamma = \{x_1 : A_1, \dots, x_n : A_n\}$ ,  $\Gamma \vdash M : A$  and for all  $i \in \{1, \dots, n\}$ ,  $\Gamma \vdash M_i : A_i$ .

We define simultaneous substitution  $M[\vec{x} := \vec{M}]$  recursively by:

1)  $x_i[\vec{x} := \vec{M}] = M_i$ ;

2)  $(\lambda x.M)[\vec{x} := \vec{M}] = \lambda x.(M[\vec{x} := \vec{M}])$ ;

3)  $(MN)[\vec{x} := \vec{M}] = (M[\vec{x} := \vec{M}])(N[\vec{x} := \vec{M}])$ ;

4)  $\langle M, N \rangle = \langle (M[\vec{x} := \vec{M}]), (N[\vec{x} := \vec{M}]) \rangle$ ;

- 5)  $(\pi_i P)[\vec{x} := \vec{M}] = \pi_i(P[\vec{x} = \vec{M}]);$   
 6)  $(\mathbf{pure} M)[\vec{x} := \vec{M}] = \mathbf{pure} (M[\vec{x} = \vec{M}]);$   
 7)  $(\mathbf{let} \mathbf{pure} \vec{x} = \vec{M} \mathbf{in} N)[\vec{y} := \vec{P}] = \mathbf{let} \mathbf{pure} \vec{x} = (\vec{M}[\vec{y} := \vec{P}]) \mathbf{in} N$

**Lemma 4.**

$$\llbracket M[x_1 := M_1, \dots, x_n := M_n] \rrbracket = \llbracket M \rrbracket \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle.$$

*Proof.*

$$1) \llbracket \Gamma \vdash (\mathbf{pure} M)[\vec{x} := \vec{M}] : \mathbf{KA} \rrbracket = \llbracket \Gamma \vdash \mathbf{pure} M : \mathbf{KA} \rrbracket \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle.$$

$$\begin{aligned} \llbracket \Gamma \vdash (\mathbf{pure} M)[\vec{x} := \vec{M}] : \mathbf{KA} \rrbracket &= \llbracket \Gamma \vdash \mathbf{pure} (M[\vec{x} := \vec{M}]) : \mathbf{KA} \rrbracket && \text{Substitution definition} \\ &= \eta_{\llbracket A \rrbracket} \circ \llbracket (M[\vec{x} := \vec{M}]) \rrbracket && \text{Translation for pure} \\ &= \eta_{\llbracket A \rrbracket} \circ (\llbracket M \rrbracket \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle) && \text{Induction hypothesis} \\ &= (\eta_{\llbracket A \rrbracket} \circ \llbracket M \rrbracket) \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle && \text{Associativity of composition} \\ &= \llbracket \Gamma \vdash \mathbf{pure} M : \mathbf{KA} \rrbracket \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle && \text{Translation for pure} \end{aligned}$$

$$2) \llbracket \Gamma \vdash (\mathbf{let} \mathbf{pure} \vec{x} = \vec{M} \mathbf{in} N)[\vec{y} := \vec{P}] : \mathbf{KB} \rrbracket = \llbracket \Gamma \vdash \mathbf{let} \mathbf{pure} \vec{x} = \vec{M} \mathbf{in} N : \mathbf{KB} \rrbracket \circ \langle \llbracket P_1 \rrbracket, \dots, \llbracket P_n \rrbracket \rangle$$

$$\begin{aligned} &\llbracket \Gamma \vdash (\mathbf{let} \mathbf{pure} \vec{x} = \vec{M} \mathbf{in} N)[\vec{y} := \vec{P}] : \mathbf{KB} \rrbracket = \\ &\text{Substitution definition} \\ &\llbracket \Gamma \vdash \mathbf{let} \mathbf{pure} \vec{x} = (\vec{M}[\vec{y} := \vec{P}]) \mathbf{in} N : \mathbf{KB} \rrbracket = \\ &\text{Interpretation for } \mathbf{let}_{\mathbf{K}} \\ &\mathcal{K}(\llbracket N \rrbracket) \circ *_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ \llbracket \Gamma \vdash (\vec{M}[\vec{y} := \vec{P}]) : \mathbf{KA} \rrbracket = \\ &\text{Induction hypothesis} \\ &\mathcal{K}(\llbracket N \rrbracket) \circ *_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ (\llbracket \vec{M} \rrbracket \circ \langle \llbracket P_1 \rrbracket, \dots, \llbracket P_n \rrbracket \rangle) = \\ &\text{Associativity of composition} \\ &(\mathcal{K}(\llbracket N \rrbracket) \circ *_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ \llbracket \vec{M} \rrbracket) \circ \langle \llbracket P_1 \rrbracket, \dots, \llbracket P_n \rrbracket \rangle = \\ &\text{By interpretation} \\ &\llbracket \Gamma \vdash (\mathbf{let} \mathbf{pure} \vec{x} = \vec{M} \mathbf{in} N) \rrbracket \circ \langle \llbracket P_1 \rrbracket, \dots, \llbracket P_n \rrbracket \rangle \end{aligned}$$

□

**Lemma 5.**

Let  $\Gamma \vdash M : A$  and  $M \rightarrow_{\beta\eta} N$ , then  $\llbracket \Gamma \vdash M : A \rrbracket = \llbracket \Gamma \vdash N : A \rrbracket$ ;

*Proof.*

Cases with  $\beta$ -reductions for  $\mathbf{let}_{\mathbf{K}}$  are shown in [20]. Let us consider cases with  $\mathbf{pure}$ .

$$1) \llbracket \Gamma \vdash \mathbf{let} \mathbf{pure} \vec{x} = \vec{M} \mathbf{in} N : \mathbf{KB} \rrbracket = \llbracket \Gamma \vdash \mathbf{pure} N[\vec{x} := \vec{M}] : \mathbf{KB} \rrbracket$$



$$\begin{aligned}
& \llbracket \Gamma \vdash \text{let pure } \vec{x} = \text{pure } \vec{M} \text{ in } N : \mathbf{KB} \rrbracket = \\
& \quad \text{By interpretation} \\
& \mathcal{K}(\llbracket N \rrbracket) \circ *_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ \langle \eta_{\llbracket A_1 \rrbracket} \circ \llbracket M_1 \rrbracket, \dots, \eta_{\llbracket A_n \rrbracket} \circ \llbracket M_n \rrbracket \rangle = \\
& \quad \text{By the property of a pair of morphisms} \\
& \mathcal{K}(\llbracket N \rrbracket) \circ *_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ (\eta_{\llbracket A_1 \rrbracket} \times \dots \times \eta_{\llbracket A_n \rrbracket}) \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle = \\
& \quad \text{Associativity of composition} \\
& \mathcal{K}(\llbracket N \rrbracket) \circ (*_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ (\eta_{\llbracket A_1 \rrbracket} \times \dots \times \eta_{\llbracket A_n \rrbracket})) \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle = \\
& \quad \text{By the definition of an applicative functor} \\
& \mathcal{K}(\llbracket N \rrbracket) \circ \eta_{\llbracket A_1 \rrbracket \times \dots \times \llbracket A_n \rrbracket} \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle = \\
& \quad \text{Naturality of } \eta \\
& \eta_{\llbracket B \rrbracket} \circ \llbracket N \rrbracket \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle = \\
& \quad \text{Associativity of composition} \\
& \eta_{\llbracket B \rrbracket} \circ (\llbracket N \rrbracket \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle) = \\
& \quad \text{Simultaneous substitution lemma} \\
& \eta_{\llbracket B \rrbracket} \circ \llbracket N[\vec{x} := \vec{M}] \rrbracket \\
& \quad \text{By interpretation} \\
& \llbracket \Gamma \vdash \text{pure } (N[\vec{x} := \vec{M}]) : \mathbf{KB} \rrbracket \\
2) \llbracket \vdash \text{let pure } \_ = \_ \text{ in } N : \mathbf{KA} \rrbracket = \llbracket \vdash \text{pure } N : \mathbf{KA} \rrbracket.
\end{aligned}$$

$$\begin{aligned}
\llbracket \vdash \text{let pure } \_ = \_ \text{ in } N : \mathbf{KA} \rrbracket &= \quad \text{By interpretation} \\
\mathcal{K}(\llbracket N \rrbracket) \circ \eta_{\mathbf{1}} &= \quad \text{Naturality for } \eta \\
\eta_{\llbracket A \rrbracket} \circ \llbracket N \rrbracket &= \quad \text{By interpretation} \\
\llbracket \vdash \text{pure } N : \mathbf{KA} \rrbracket
\end{aligned}$$

3)  
If  $\Gamma \vdash M : A$  and  $M \rightarrow_{\beta\eta} N$ , then  $\llbracket \Gamma \vdash \text{pure } M : \mathbf{KA} \rrbracket = \llbracket \Gamma \vdash \text{pure } N : \mathbf{KA} \rrbracket$ .

If  $\Gamma \vdash M : A$  and  $M \rightarrow_{\beta\eta} N$ , then  $\Gamma \vdash N : A$  by subject reduction.  
By assumption  $\llbracket \Gamma \vdash M : A \rrbracket = \llbracket \Gamma \vdash N : A \rrbracket$ .  
So  $\eta_{\llbracket A \rrbracket} \circ \llbracket \Gamma \vdash M : A \rrbracket = \eta_{\llbracket A \rrbracket} \circ \llbracket \Gamma \vdash N : A \rrbracket$ .  
Hence  $\llbracket \Gamma \vdash \text{pure } M : \mathbf{KA} \rrbracket = \llbracket \Gamma \vdash \text{pure } N : \mathbf{KA} \rrbracket$ .

□

□

**Theorem 5. Completeness**

Let  $\llbracket \Gamma \vdash M : A \rrbracket = \llbracket \Gamma \vdash N : A \rrbracket$ , then  $M =_{\beta\eta} N$ .

*Proof.*

We will consider term model for simply typed lambda calculus  $\times$  and  $\rightarrow$  standardly described in [22].

**Definition 14.** Let us define an endofunctor  $\mathcal{K} : \mathcal{C}(\lambda) \rightarrow \mathcal{C}(\lambda)$ , such that for all  $[x, M] \in \text{Hom}_{\mathcal{C}(\lambda)}(A, B)$ ,  $\mathbf{K}([x, M]) = [y, \text{let pure } x = y \text{ in } M] \in \text{Hom}_{\mathcal{C}(\lambda)}(\mathbf{KA}, \mathbf{KB})$  (denotation:  $fmap$   $f$  for an arbitrary arrow  $f$ ).

**Lemma 6. Functoriality**

- i)  $fmap (g \circ f) = fmap (g) \circ fmap (f)$ ;
- ii)  $fmap (id_A) = id_{\mathbf{KA}}$ .

*Proof.* Easy checking using reduction rules.  $\square$

**Definition 15.** Let us define natural transformations:

- 1)  $\eta : Id \Rightarrow \mathcal{K}$ , s. t.  $\forall A \in Ob_{\mathcal{C}(\lambda)}$ ,  $\eta_A = [x, \mathbf{pure} x] \in Hom_{\mathcal{C}(\lambda)}(A, \mathbf{K}A)$ ;
- 2)  $*_{A,B} : \mathbf{K}A \times \mathbf{K}B \rightarrow \mathbf{K}(A \times B)$ , s. t.  $\forall A, B \in Ob_{\mathcal{C}(\lambda)}$ ,  $*_{A,B} = [p, \mathbf{let pure} x, y = \pi_1 p, \pi_2 p \mathbf{ in } \langle x, y \rangle] \in Hom_{\mathcal{C}(\lambda)}(\mathbf{K}A \times \mathbf{K}B, \mathbf{K}(A \times B))$ .

Implementation for  $*$  in our term model is a modification of  $\mathbf{let_K}$ -rule:

$$\frac{\frac{p : \mathbf{K}A \times \mathbf{K}B \vdash p : \mathbf{K}A \times \mathbf{K}B}{p : \mathbf{K}A \times \mathbf{K}B \vdash \pi_1 p : \mathbf{K}A} \quad \frac{p : \mathbf{K}A \times \mathbf{K}B \vdash p : \mathbf{K}A \times \mathbf{K}B}{p : \mathbf{K}A \times \mathbf{K}B \vdash \pi_2 p : \mathbf{K}B} \quad \frac{x : A \vdash x : A \quad y : B \vdash y : B}{x : A, y : B \vdash \langle x, y \rangle : A \times B}}{p : \mathbf{K}A \times \mathbf{K}B \vdash \mathbf{let pure} \langle x, y \rangle = \langle \pi_1 p, \pi_2 p \rangle \mathbf{ in } \langle x, y \rangle : \mathbf{K}(A \times B)}$$

**Lemma 7.** *Naturality for  $\eta$  and for  $*$*

- i)  $fmap f \circ \eta_A = \eta_B \circ f$ ;
- ii)  $fmap (f \times g) \circ *_{A,B} = *_{C,D} \circ (fmap f) \times (fmap g)$ .
- iii)  $*_{A,B} \circ (\eta_A \times \eta_B) = \eta_{A \times B}$ ;

*Proof.*

- i)  $fmap f \circ \eta_A = \eta_B \circ f$

$$\begin{array}{ll} \eta_B \circ f = & \text{By the definition} \\ [y, \mathbf{pure} y] \circ [x, M] = & \text{By the definition of composition} \\ [x, \mathbf{pure} y[y := M]] = & \text{By substitution} \\ [x, \mathbf{pure} M] & \end{array}$$

On the other hand:

$$\begin{array}{ll} fmap f \circ \eta_A = & \text{By the definition} \\ [z, \mathbf{let pure} x = z \mathbf{ in } M] \circ [x, \mathbf{pure} x] = & \text{By the definition of composition} \\ [x, \mathbf{let pure} x = z \mathbf{ in } M[z := \mathbf{pure} x]] = & \text{By substitution} \\ [x, \mathbf{let pure} x = \mathbf{pure} x \mathbf{ in } M] = & \beta\text{-reduction rule} \\ [x, \mathbf{pure} M[x := x]] = & \text{By substitution} \\ [x, \mathbf{pure} M] & \end{array}$$

- ii)  $fmap (f \times g) \circ *_{A,B} = *_{C,D} \circ (fmap f) \times (fmap g)$

See [19].

- iii)  $*_{A,B} \circ (\eta_A \times \eta_B) = \eta_{A \times B}$

$$\begin{aligned}
& *_{A,B} \circ (\eta_A \times \eta_B) = \\
& \text{By unfolding} \\
& [q, \text{let } \mathbf{pure} \ x, y = \pi_1 q, \pi_2 q \text{ in } \langle x, y \rangle] \circ [p, \langle \mathbf{pure} (\pi_1 p), \mathbf{pure} (\pi_2 p) \rangle] = \\
& \text{Composition} \\
& [p, \text{let } \mathbf{pure} \ x, y = \pi_1 q, \pi_2 q \text{ in } \langle x, y \rangle [q := \langle \mathbf{pure} (\pi_1 p), \mathbf{pure} (\pi_2 p) \rangle]] = \\
& \text{By substitution} \\
& [p, \text{let } \mathbf{pure} \ x, y = \pi_1 (\langle \mathbf{pure} (\pi_1 p), \mathbf{pure} (\pi_2 p) \rangle), \pi_2 (\langle \mathbf{pure} (\pi_1 p), \mathbf{pure} (\pi_2 p) \rangle) \text{ in } \langle x, y \rangle] = \\
& \text{Reduction rules} \\
& [p, \text{let } \mathbf{pure} \ x, y = \mathbf{pure} (\pi_1 p), \mathbf{pure} (\pi_2 p) \text{ in } \langle x, y \rangle] = \\
& \text{Reduction rule} \\
& [p, \mathbf{pure} (\langle x, y \rangle [x := \pi_1 p, y := \pi_2 p])] = \\
& \text{Substitution} \\
& [p, \mathbf{pure} \langle \pi_1 p, \pi_2 p \rangle] = \\
& \eta\text{-reduction} \\
& [p, \mathbf{pure} p] = \\
& \text{By definition} \\
& \eta_{A \times B}
\end{aligned}$$

□

Tensorial strength is defined as follows:

**Definition 16.** *Tensorial strength*

Let  $[p, \langle \mathbf{pure} (\pi_1 p), \pi_2 p \rangle] \in \text{Hom}_{\mathcal{C}(\lambda)}(A \times \mathbf{KB}, \mathbf{K}(A \times B))$ .

So tensorial strength is defined as  $\tau_{A,B} = *_{A,B} \circ [p, \langle \mathbf{pure} (\pi_1 p), \pi_2 p \rangle]$ .

It is clearly that tensorial strength defined above can be simplified as follows:

$$\begin{aligned}
& *_{A,B} \circ [p, \langle \mathbf{pure} (\pi_1 p), \pi_2 p \rangle] = & \text{By definition} \\
& [p', \text{let } \mathbf{pure} \ x, y = \pi_1 p', \pi_2 p' \text{ in } \langle x, y \rangle] \circ [p, \langle \mathbf{pure} (\pi_1 p), \pi_2 p \rangle] = & \text{By composition} \\
& [p, \text{let } \mathbf{pure} \ x, y = \pi_1 p', \pi_2 p' \text{ in } \langle x, y \rangle [p' := \langle \mathbf{pure} (\pi_1 p), \pi_2 p \rangle]] = & \text{By substitution} \\
& [p, \text{let } \mathbf{pure} \ x, y = \pi_1 (\langle \mathbf{pure} (\pi_1 p), \pi_2 p \rangle), \pi_2 (\langle \pi_1 p, \mathbf{pure} (\pi_2 p) \rangle) \text{ in } \langle x, y \rangle] = & \text{By } \beta\text{-reduction rules} \\
& [p, \text{let } \mathbf{pure} \ x, y = \mathbf{pure} (\pi_1 p), \pi_2 p \text{ in } \langle x, y \rangle]
\end{aligned}$$

**Lemma 8.** *Weak commutativity.*

$$\begin{aligned}
& \text{fmap } ([p, \langle \pi_2 p, \pi_1 p \rangle]) \circ \tau_{A,B} = \\
& *_{B,A} \circ [q, \langle \pi_1 q, \mathbf{pure} (\pi_2 q) \rangle] \circ [p, \langle \pi_2 p, \pi_1 p \rangle]
\end{aligned}$$

*Proof.*

$\text{fmap } ([r, \langle \pi_2 r, \pi_1 r \rangle]) \circ \tau_{A,B} =$   
 By the definition of  $\tau$   
 $\text{fmap } ([r, \langle \pi_2 r, \pi_1 r \rangle]) \circ [p, \text{let pure } x, y = \text{pure } (\pi_1 p), \pi_2 p \text{ in } \langle x, y \rangle] =$   
 By the definition of  $\text{fmap}$   
 $[q, \text{let pure } r = q \text{ in } \langle \pi_2 r, \pi_1 r \rangle] \circ [p, \text{let pure } x, y = \text{pure } (\pi_1 p), \pi_2 p \text{ in } \langle x, y \rangle] =$   
 Composition  
 $[p, \text{let pure } r = q \text{ in } \langle \pi_2 r, \pi_1 r \rangle [q := \text{let pure } x, y = \text{pure } (\pi_1 p), \pi_2 p \text{ in } \langle x, y \rangle]] =$   
 By  $\beta$ -reduction rules  
 $[p, \text{let pure } r = (\text{let pure } x, y = \text{pure } (\pi_1 p), \pi_2 p \text{ in } \langle x, y \rangle) \text{ in } \langle \pi_2 r, \pi_1 r \rangle] =$   
 By  $\beta$ -reduction rules  
 $[p, \text{let pure } x, y = \text{pure } (\pi_1 p), \pi_2 p \text{ in } \langle \pi_2 r, \pi_1 r \rangle [r := \langle x, y \rangle]] =$   
 By substitution  
 $[p, \text{let pure } x, y = \text{pure } (\pi_1 p), \pi_2 p \text{ in } \langle \pi_2 \langle x, y \rangle, \pi_1 \langle x, y \rangle \rangle] =$   
 By  $\beta$ -reduction rules  
 $[p, \text{let pure } x, y = \text{pure } (\pi_1 p), \pi_2 p \text{ in } \langle y, x \rangle] =$

On the other hand  
 $*_{B,A} \circ [q, \langle \pi_1 q, \text{pure } (\pi_2 q) \rangle] \circ [p, \langle \pi_2 p, \pi_1 p \rangle] =$   
 By the definition of  $*$   
 $[r, \text{let pure } y, x = \pi_1 r, \pi_2 r \text{ in } \langle y, x \rangle] \circ [q, \langle \pi_1 q, \text{pure } (\pi_2 q) \rangle] \circ [p, \langle \pi_2 p, \pi_1 p \rangle] =$   
 Composition  
 $[r, \text{let pure } y, x = \pi_1 r, \pi_2 r \text{ in } \langle y, x \rangle] \circ [p, \langle \pi_1 q, \text{pure } (\pi_2 q) \rangle [q := \langle \pi_2 p, \pi_1 p \rangle]] =$   
 By substitution and by  $\beta$ -reduction rules  
 $[r, \text{let pure } y, x = \pi_1 r, \pi_2 r \text{ in } \langle y, x \rangle] \circ [p, \langle \pi_2 p, \text{pure } (\pi_1 p) \rangle] =$   
 Composition  
 $[p, \text{let pure } y, x = \pi_1 r, \pi_2 r \text{ in } \langle y, x \rangle [r := \langle \pi_2 p, \text{pure } (\pi_1 p) \rangle]] =$   
 By substitution and by  $\beta$ -reduction rules  
 $[p, \text{let pure } y, x = \pi_2 p, \text{pure } (\pi_1 p) \text{ in } \langle y, x \rangle] =$   
 By symmetricity of assingment  
 $[p, \text{let pure } x, y = \text{pure } (\pi_1 p), \pi_2 p \text{ in } \langle y, x \rangle]$

□

**Lemma 9.** *K is an applicative functor*

*Proof.* Immediately follows from previous lemmas in the section.

□

□

## References

- [1] Artemov S. and Protopopescu T., “Intuitionistic Epistemic Logic”, *The Review of Symbolic Logic*, 2016, vol. 9, no 2. pp. 266-298.
- [2] Krupski V. N. and Yatmanov A., “Sequent Calculus for Intuitionistic Epistemic Logic IEL”, *Logical Foundations of Computer Science: International Symposium, LFCS 2016, Deerfield Beach, FL, USA, January 4-7, 2016. Proceedings*, 2016, pp. 187-201.
- [3] Haskell Language. // URL: <https://www.haskell.org>. (Date: 1.08.2017)

- [4] Idris. A Language with Dependent Types.// URL:<https://www.idris-lang.org>. (Date: 1.08.2017)
- [5] Purescript. A strongly-typed functional programming language that compiles to JavaScript. URL: <http://www.purescript.org>. (Date: 1.08.2017)
- [6] Elm. A delightful language for reliable webapps. // URL: <http://elm-lang.org>. (Date: 1.08.2017)
- [7] Hackage, “The base package” // URL: <https://hackage.haskell.org/package/base-4.10.0.0> (Date: 1.08.2017)
- [8] Lipovaca M, “Learn you a Haskell for Great Good!”. //URL: <http://learnyouahaskell.com/chapters> (Date: 1.08.2017)
- [9] McBride C. and Paterson R., “Applicative programming with effects”, *Journal of Functional Programming*, 2008, vol. 18, no 01. pp 1-13.
- [10] McBride C. and Paterson R, “Functional Pearl. Idioms: applicative programming with effects”, *Journal of Functional Programming*, 2005. vol. 18, no 01. pp 1-20.
- [11] R. Nederpelt and H. Geuvers, “Type Theory and Formal Proof: An Introduction”. *Cambridge University Press*, New York, NY, USA, 2014. pp. 436.
- [12] Sorensen M. H. and Urzyczyn P, “Lectures on the Curry-Howard isomorphism”, *Studies in Logic and the Foundations of Mathematics*, vol. 149, *Elsevier Science*, 1998. pp 261.
- [13] Pierce B. C., “Types and Programming Languages”. *Cambridge, Mass: The MIT Press*, 2002. pp. 605.
- [14] Girard J.-Y., Taylor P. and Lafont Y, “Proofs and Types”, *Cambridge University Press*, New York, NY, USA, 1989. pp. 175.
- [15] Barendregt. H. P., “Lambda calculi with types” // Abramsky S., Gabbay Dov M., and S. E. Maibaum, “Handbook of logic in computer science (vol. 2), Osborne Handbooks Of Logic In Computer Science”, Vol. 2. *Oxford University Press, Inc.*, New York, NY, USA, 1993. pp 117-309.
- [16] Hindley J. Roger, “Basic Simple Type Theory”. *Cambridge University Press*, New York, NY, USA, 1997. pp. 185.
- [17] Pfenning F. and Davies R., “A judgmental reconstruction of modal logic”, *Mathematical Structures in Computer Science*, vol. 11, no 4, 2001, pp. 511-540.
- [18] H.P. Barendregt. The Lambda Calculus — Its Syntax and Semantics. *Studies in Logic and the Foundations of Mathematics*, vol. 103. Amsterdam: North-Holland, 1985.
- [19] Yoshihiko KAKUTANI, A Curry-Howard Correspondence for Intuitionistic Normal Modal Logic, Computer Software, Released February 29, 2008, Online ISSN , Print ISSN 0289-6540.

- [20] Kakutani Y. (2007) Call-by-Name and Call-by-Value in Normal Modal Logic. In: Shao Z. (eds) Programming Languages and Systems. APLAS 2007. Lecture Notes in Computer Science, vol 4807. Springer, Berlin, Heidelberg
- [21] T. Abe. Completeness of modal proofs in first-order predicate logic. Computer Software, JSSST Journal, 24:165 – 177, 2007.
- [22] Lambek, J. and Scott P.J. (1986) Introduction to Higher Order Categorical Logic, Cambridge Studies in Advanced Mathematics 7, Cambridge: Cambridge University Press.
- [23] Samuel Eilenberg and Max Kelly, Closed categories. Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965).