Completeness for modal type theory based on the intuitionistic epistemic logic

Definition 1. *Family of relations* $\sim_{\alpha,\beta} \subseteq \Lambda_{\boldsymbol{K}} \times \Lambda_{\boldsymbol{K}}$, $\alpha, \beta \in \mathbb{T}_{\boldsymbol{K}}$ *is defined by:*

$$(x_1, y_1) \sim_{\alpha,\beta} (x_2, y_2) \Leftrightarrow x_i : \alpha \vdash y_i : \beta, i \in \{1,2\} \;\&\; y_2 =_{\beta\eta} y_1[x_1 := x_2] \quad (1)$$

It is clearly that forall $\alpha, \beta \in \mathbb{T}_{\mathbf{K}}$ is an equivalence relation.

Equivalence class for some $\phi, \psi \in\in \mathbb{T}_{\mathbf{K}}$ and for some (M, N) such that $M : \phi \vdash N : \psi$ is defined as:

$$[(M,N)]_{\phi,\psi} = \{(K,L) | (K,L) \sim_{\phi,\psi} (M,N)\} \quad (2)$$

Let us consider the category $\mathcal{C}_\lambda$:

Definition 2.

- $Ob_{\mathcal{C}_\lambda} = \{\mathbb{1}\} \cup \{\hat{\phi} \mid \phi \in \mathbb{T}_{\boldsymbol{K}}\}$
- $Hom_{\mathcal{C}_\lambda}(\hat{\phi}, \hat{\psi}) = \{[(x_1, y_1)]_{\phi,\psi} \mid x_1 : \phi \vdash y_1 : \psi\}$.
- $[(x_2, y_2)]_{\psi,\tau} \circ [(x_1, y_1)]_{\phi,\psi} = [(x_1, y_2[y_1 := x_2])]_{\phi,\tau}$;
- $id_{\hat{\phi}} = [(x, x)]_{\phi,\phi}$;
- $\widehat{\phi \times \psi} = \hat{\phi} \times \hat{\psi}$;
- $\pi_i = [(x, \pi_i x)]_{\phi_1 \times \phi_2, \phi_i}, i \in \{1,2\}$;
- $\epsilon = [(x, (\pi_1 x)(\pi_2 x))]_{\phi \to \psi \times \phi, \psi}$;
- $\Lambda([(x, M)]_{\phi \times \psi, \tau}) = [(x_1, \lambda x_2.M[x := \langle x_1, x_2 \rangle])]_{\phi, \psi \to \tau}$.

Definition 3. *Let us functor map* $\boldsymbol{K} : \mathcal{C}_\lambda \to \mathcal{C}_\lambda$, *such that:*

- $\boldsymbol{K} : \hat{\phi} \mapsto \hat{\boldsymbol{K}}\phi$;
- $\boldsymbol{K}(\mathbb{1}) = \mathbb{1}$
- $\boldsymbol{K} : [(x_2, y_2)]_{\psi,\tau} \mapsto \boldsymbol{K}([(x_2, y_2)]_{\psi,\tau}) = K([(x_2, y_2)])_{\boldsymbol{K}\psi, \boldsymbol{K}\tau}$;
- $\boldsymbol{K}([(x_2, y_2)]_{\psi,\tau} \circ [(x_1, y_1)]_{\phi,\psi}) = \boldsymbol{K}([(x_1, y_2[y_1 := x_2])])_{\boldsymbol{K}\phi, \boldsymbol{K}\tau}$
- $\boldsymbol{K} : \hat{\phi} \times \hat{\psi} \mapsto \hat{\boldsymbol{K}}\phi \times \hat{\boldsymbol{K}}\psi$;
- $\boldsymbol{K} : \hat{\phi} \to \hat{\psi} \mapsto \hat{\boldsymbol{K}}\phi \to \hat{\boldsymbol{K}}\psi$.

Lemma 1. $\boldsymbol{K}(\hat{\phi} \times \hat{\psi}) \cong \boldsymbol{K}(\hat{\phi}) \times \boldsymbol{K}(\hat{\psi})$.

Proof.

$\langle \mathbf{K}(\pi_1), \mathbf{K}(\pi_2) \rangle = \langle \mathbf{K}([(x, \pi_1 x)])_{\mathbf{K}(\phi \times \psi), \mathbf{K}\phi}, \mathbf{K}([(x, \pi_2 x)])_{\mathbf{K}(\phi \times \psi), \mathbf{K}\psi} \rangle : \mathbf{K}(\hat{\phi} \times \hat{\psi}) \to \mathbf{K}(\hat{\phi}) \times \mathbf{K}(\hat{\psi})$. □