# Soundness for modal type theory based on the intuitionistic epistemic logic

### 1 Modal lambda calculus based on IEL<sup>-</sup>

**Definition 1.** The set of terms:

Let V is a set of variables. The set  $\Lambda_K$  of terms is defined by the grammar:

$$\Lambda_{\mathbf{K}} ::= \mathbb{V} \mid (\lambda \Lambda. \Lambda_{\mathbf{K}}) \mid (\Lambda_{\mathbf{K}} \Lambda_{\mathbf{K}}) \mid (\Lambda_{\mathbf{K}}, \Lambda_{\mathbf{K}}) \mid (\pi_{i} \Lambda_{\mathbf{K}}) \mid (pure \Lambda_{\mathbf{K}}) \mid (\Lambda_{\mathbf{K}} \star \Lambda_{\mathbf{K}}) \quad (1)$$
where  $i \in \{1, 2\}$ .

**Definition 2.** The set of types:

Let  $\mathbb{T}$  is a set of atomic types. The set  $\mathbb{T}_{\mathbf{K}}$  of types with applicative functor  $\mathbf{K}$  is generated by the grammar:

$$\mathbb{T}_{K} ::= \mathbb{T} \mid (\mathbb{T}_{K} \to \mathbb{T}_{K}) \mid (\mathbb{T}_{K} \times \mathbb{T}_{K}) \mid (K\mathbb{T}_{K})$$
(2)

Our type system is based on the Curry-style typing rules:

**Definition 3.** Modal typed lambda calculus  $\lambda \mathbf{K}$  based on  $NIEL_{\wedge,\rightarrow}^-$ :

$$\frac{\Gamma, x : \alpha \vdash x : \alpha}{\Gamma \vdash \lambda x. M : \alpha \to \beta} \xrightarrow{\lambda_{i}} \xrightarrow{\Gamma} \underbrace{\frac{\Gamma \vdash x : \alpha}{\Gamma \vdash \lambda x. M : \alpha \to \beta}} \xrightarrow{\gamma_{i}} \times_{i}$$

$$\frac{\Gamma \vdash x : \alpha}{\Gamma \vdash (x, y) : \alpha \times \beta} \times_{i}$$

$$\frac{\Gamma, \vdash x : \alpha}{\Gamma \vdash pure \ x : \mathbf{K}\alpha} \mathbf{K}_{I}$$

$$\frac{\Gamma \vdash f : \alpha \to \beta}{\Gamma \vdash fx : \beta} \xrightarrow{\Gamma \vdash x : \alpha} \xrightarrow{\gamma_{e}} \xrightarrow{\Gamma} \underbrace{\Gamma \vdash f : \mathbf{K}(\alpha \to \beta)}_{\Gamma \vdash \pi_{i}p : \alpha_{i}} \times_{e}, i \in \{1, 2\}$$

$$\frac{\Gamma \vdash f : \mathbf{K}(\alpha \to \beta)}{\Gamma \vdash f \star x : \mathbf{K}\beta} \xrightarrow{\Gamma \vdash x : \mathbf{K}\alpha} \mathbf{K}_{app}$$

**Definition 4.**  $\beta$ -reduction rules:

- 1)  $(\lambda x.M)N \rightarrow_{\beta} M[x := N];$
- 2)  $\pi_i \langle M_1, M_2 \rangle \rightarrow_{\beta} M_i, i \in \{1, 2\};$
- 3) pure  $(\lambda x.x) \star M \to_{\beta} M$ ;
- 4) pure  $(\lambda fgx.f(gx)) \star M \star N \star P \rightarrow_{\beta} M \star (N \star P);$
- 5)  $(pure\ M) \star (pure\ N) \rightarrow_{\beta} pure\ (MN);$
- 6)  $M \star pure \ N \rightarrow_{\beta} (\lambda f. fN) \star M$ ;

**Definition 5.**  $\eta$ -reduction rules for applicative functor:

- 1) pure  $(\lambda x. fx) \to_{\eta} pure f$ ;
- 2) pure  $\langle \pi_1 p, \pi_2 p \rangle \rightarrow_{\eta} pure p$ ;
- 3)  $\lambda x.f \star x \to_{\eta} f$ .

## 2 Categorical model.

Let us define monoidal categories and strong lax monoidal functors.

**Definition 6.** Monoidal category.

A monoidal category C is a category with:

- 1) A bifunctor  $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$  called the tensor product;
- 2) An object  $\mathbb{1} \in Ob(\mathcal{C})$  called the unit;
- 3) A natural isomorphism such that for all  $A, B, C \in Ob(\mathcal{C})$ :

$$\alpha_{A,B,C}: (A \otimes B) \otimes C \xrightarrow{\cong} A \otimes (B \otimes C)$$

where  $\alpha$  is called associator.

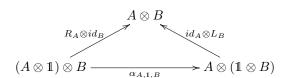
4) A natural isomorphism (left unitor) for all  $A \in Ob(C)$ :

$$L_A: (\mathbb{1} \otimes A) \xrightarrow{\cong} A$$

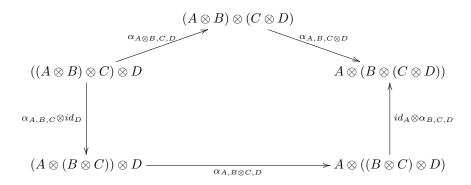
5) A natural isomorphism (right unitor) for all  $A \in Ob(\mathcal{C})$ :

$$R_A: (A\otimes 1) \xrightarrow{\cong} A$$

6) The next one diagram commutes (the triangle identity):



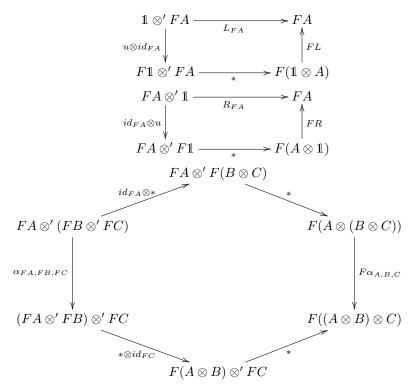
7) The next one diagram commutes too (the pentagon identity):



A monoidal category is symmetrical iff  $\forall A, B \in Ob(\mathcal{C}), A \otimes B \cong B \otimes A$ .

**Definition 7.** A lax monoidal functor between monoinal categories  $\langle \mathcal{C}, \otimes, \mathbb{1} \rangle$  and  $\langle \mathcal{D}, \otimes', \mathbb{1} \rangle$  is a functor  $F : \langle \mathcal{C}, \otimes, \mathbb{1} \rangle \to \langle \mathcal{D}, \otimes', \mathbb{1} \rangle$  with the next natural transformations:

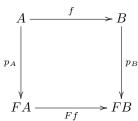
- 1)  $u: \mathbb{1} \to F\mathbb{1}$  (unit property);
- 2)  $*: FA \otimes' FB \rightarrow F(A \otimes B)$  (application property); and with the next commuting diagrams:



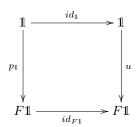
**Definition 8.** Applicative functor.

Let  $\langle \mathcal{C}, \otimes, \mathbb{1} \rangle$  is a symmetrical monoidal category. Applicative functor is an endofunctor  $F : \langle \mathcal{C}, \otimes, \mathbb{1} \rangle \to \langle \mathcal{C}, \otimes, \mathbb{1} \rangle$  with a natural transformation  $p : Id_{\mathcal{C}} \Rightarrow F$  with the next properties:

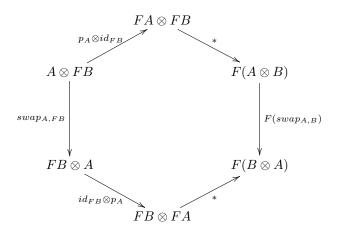
1) A natural transformation p is defined as follows for all  $A \in Ob(\mathcal{C})$  the following diagram commutes:



2)  $p_1 = u$ :



- 3)  $p \circ * = * \circ (p \otimes p);$
- 4) The following diagram commutes (weak commutativity condition):



## 3 Soundness

**Definition 9.** Semantical translation from  $\lambda_{\mathbf{K}}$  to CCC with applicative functor:

- 1) Interpretation for types:  $[\![A]\!] := \hat{A}, A \in \mathbb{T}, [\![A \to B]\!] := [\![A]\!] \to [\![B]\!], [\![A \times B]\!] := [\![A]\!] \times [\![B]\!];$
- 2) Interpretation for modal types:  $\llbracket \mathbf{K}A \rrbracket = \mathcal{K} \llbracket A \rrbracket$ , where  $\mathcal{K}$  is an applicative functor;
- 3) Interpretaion for contexts:  $\llbracket \Gamma = \{x_1 : A_1, ..., x_n : A_n\} \rrbracket := \llbracket \Gamma \rrbracket = \llbracket A_1 \rrbracket \times ... \times \llbracket A_n \rrbracket;$
- 4) Interpretation for typing assignment:  $\llbracket \Gamma \vdash M : A \rrbracket := \llbracket M \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket$ , where  $\llbracket M \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket \in \mathcal{C}$ ;
  - 5) Interpretation for typing rules:

2)  $[(M \star N)[\vec{x} := \vec{M}] = [M \star N] \circ \langle [M_1], \dots, [M_n] \rangle$ .

$$\begin{split} & [\![(M\star N)[\vec{x}:=\vec{M}]\!] = [\![(M[\vec{x}:=\vec{M}]\!])\star(N[\vec{x}:=\vec{M}]\!)]\!] \\ &= p_{\epsilon} \circ * \circ \langle [\![(M[\vec{x}:=\vec{M}]\!])]\!], [\![(N[\vec{x}:=\vec{M}]\!])]\!] \rangle \\ &= p_{\epsilon} \circ * \circ \langle [\![M]\!] \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle, [\![N]\!] \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle \\ &= p_{\epsilon} \circ * \circ \langle [\![M]\!], [\![N]\!] \rangle \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle \\ &= (p_{\epsilon} \circ * \circ \langle [\![M]\!], [\![N]\!]) \rangle \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle \\ &= [\![M\star N]\!] \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle \end{split}$$

Definition of substitution Translation for  $\star$ Induction hypothesis Property of morphism product Associativity of composition Translation for  $\star$ 

#### Lemma 2.

If 
$$M \to_{\beta} N$$
, then  $[\![M]\!] = [\![N]\!]$ .

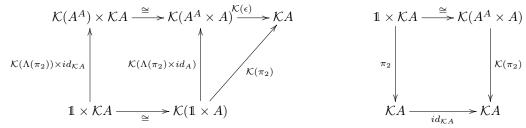
1) [pure  $(\lambda x.x) \star M$ ] = [M];

$$\frac{\llbracket x:A\vdash x:A\rrbracket=\pi_2:\mathbb{I}\times\llbracket A\rrbracket\to\llbracket A\rrbracket}{\llbracket\vdash \lambda x.x:A\to A\rrbracket=\Lambda(\pi_2):\mathbb{I}\to\llbracket A\rrbracket^{\llbracket A\rrbracket}}$$
 
$$\boxed{\llbracket\vdash \text{pure }(\lambda x.x):\mathbf{K}(A\to A)\rrbracket=p_{\llbracket A\rrbracket^{\llbracket A\rrbracket}}\circ\Lambda(\pi_2):\mathbb{I}\to\mathcal{K}(\llbracket A\rrbracket^{\llbracket A\rrbracket})}$$

But by the following diagram:

$$p_{\llbracket A \rrbracket \llbracket A \rrbracket} \circ \Lambda(\pi_2) = id_{\mathbb{I}} \circ \mathcal{K}(\Lambda(\pi_2))$$
$$= \mathcal{K}(\Lambda(\pi_2))$$

Let us consider the next commutative diagram:



Hence: 
$$\mathbb{I}M : \mathbf{K}\Delta$$

2)  $[\![(\text{pure }\lambda fgx.f(gx))\star M\star N\star P]\!]=[\![M\star (N\star P)]\!]$  The first part of equality:

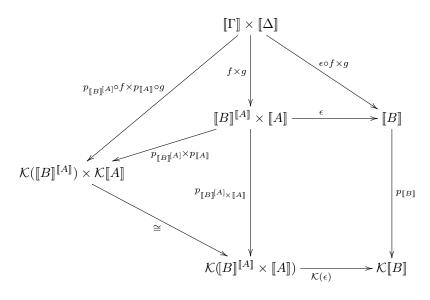
- 3)  $\llbracket (\text{pure } M) \star (\text{pure } N) \rrbracket = \llbracket \text{pure } (MN) \rrbracket;$
- 1) The left part of the equality:

$$\begin{split} & \quad \quad \|\Gamma \vdash M : A \to B\| = f : [\![\Gamma]\!] \to [\![B]\!]^{[A]} \\ & \quad \|\Gamma \vdash \text{pure } M : \mathbf{K}(A \to B)\| = p_{[\![B]\!]^{[A]}} \circ f : [\![\Gamma]\!] \to \mathcal{K}([\![B]\!]^{[A]}) \\ & \quad \quad \|\Delta \vdash N : A\| = g : [\![\Delta]\!] \to [\![A]\!] \\ & \quad \quad \|\Delta \vdash \text{pure } N : \mathbf{K}A\| = p_{[\![A]\!]} \circ g : [\![\Delta]\!] \to \mathcal{K}[\![A]\!] \end{split}$$

 $\llbracket \Gamma, \Delta \vdash (\text{pure } M) \star (\text{pure } N) : \mathbf{K}B \rrbracket = \mathcal{K}(\epsilon) \circ (\cong) \circ (p_{\llbracket B \rrbracket^{[A]}} \circ f \times p_{\llbracket A \rrbracket} \circ g) : \Gamma \times \Delta \to \mathcal{K}B$ 

2) The second part of equality:

$$\frac{\llbracket \Gamma \vdash M : A \to B \rrbracket = f : \llbracket \Gamma \rrbracket \to \llbracket B \rrbracket^{[A]} \qquad \llbracket \Delta \vdash N : A \rrbracket = g : \llbracket \Delta \rrbracket \to \llbracket A \rrbracket}{\llbracket \Gamma, \Delta \vdash MN : B \rrbracket = \epsilon \circ f \times g : \llbracket \Gamma \rrbracket \times \llbracket \Delta \rrbracket \to \llbracket B \rrbracket}$$
$$\frac{\llbracket \Gamma, \Delta \vdash \text{pure } (MN) : \mathbf{K}B \rrbracket = p_{\llbracket B \rrbracket} \circ (\epsilon \circ (f \times g)) : \llbracket \Gamma \rrbracket \times \llbracket \Delta \rrbracket \to \mathcal{K} \llbracket B \rrbracket}$$



$$\begin{split} \llbracket \Gamma, \Delta \vdash (\text{pure } M) \star (\text{pure } N) : \mathbf{K}B \rrbracket &= \mathcal{K}(\epsilon) \circ (\cong) \circ (p_{\llbracket B \rrbracket^{\llbracket A \rrbracket}} \circ f \times p_{\llbracket A \rrbracket} \circ g) \\ &= K(\epsilon) \circ (\cong) \circ p_{\llbracket B \rrbracket^{\llbracket A \rrbracket}} \times p_{\llbracket A \rrbracket} \circ f \times g \\ &= K(\epsilon) \circ p_{\llbracket B \rrbracket} \circ \epsilon \circ f \times g \\ &= \llbracket \Gamma, \Delta \vdash \text{pure } (MN) : \mathcal{K}B \rrbracket \end{split}$$

4) 
$$\begin{bmatrix} [N:A,M:\mathbf{K}(A\to B) \vdash M \star \text{pure } N:\mathbf{K}B] ] = \\ [N:A,M:\mathbf{K}(A\to B) \vdash \text{pure } (\lambda f.fN) \star M:\mathbf{K}B] \end{bmatrix}$$

It is easy to see that the following diagram commutes:

$$\mathcal{K}([\![B]\!]^{([\![B]\!]^{[\![A]\!]}})) \times \mathcal{K}([\![B]\!]^{[\![A]\!]}) \overset{\cong}{\longrightarrow} \mathcal{K}([\![B]\!]^{([\![B]\!]^{[\![A]\!]}}) \times [\![B]\!]^{[\![A]\!]}) \overset{\mathcal{K}(\epsilon)}{\longrightarrow} \mathcal{K}[\![B]\!]$$

$$\mathcal{K}(\Lambda(\epsilon \circ (\pi_2, \pi_1))) \times id_{\mathbb{K}[\![B]\!]^{[\![A]\!]}}) \overset{\wedge}{\longrightarrow} \mathcal{K}([\![B]\!]^{[\![A]\!]}) \overset{\wedge}{\longrightarrow} \mathcal{K}([\![A]\!] \times [\![B]\!]^{[\![A]\!]}) \overset{\wedge}{\longrightarrow} \mathcal{K}([\![B]\!]^{[\![A]\!]}) \overset{$$

**Lemma 3.** If  $M \to_{\eta} N$ , then  $\llbracket M \rrbracket = \llbracket N \rrbracket$ .

Proof.

1) [pure  $(\lambda x. fx)$ ] = [pure f].

$$\begin{split} \llbracket \text{pure } (\lambda x. fx) \rrbracket &= p \circ \llbracket \lambda x. fx \rrbracket & \text{Translation for pure} \\ &= p \circ \llbracket f \rrbracket & \eta\text{-reduction rule for application} \\ &= \llbracket \text{pure } f \rrbracket & \text{Translation for pure} \end{split}$$

2) [pure  $\langle \pi_1 M, \pi_2 M \rangle$ ] = [pure M]

3) [pure  $(\lambda x.\lambda y.\langle x,y\rangle) \star (\text{pure } (\lambda x.\pi_1) \star M) \star (\text{pure } (\lambda x.\pi_2) \star M)] = [\![M]\!]$ 

$$\mathcal{K}(((A \times B)^B)^A) \times (\mathcal{K}(A^{A \times B}) \times \mathcal{K}(A \times B)) \times (\mathcal{K}(B^{A \times B}) \times \mathcal{K}(A \times B))$$

$$\mathcal{K}(((A \times B)^B)^A) \times \mathcal{K}A \times \mathcal{K}B$$

$$\mathcal{K}((A \times B)^B) \times \mathcal{K}B$$

$$\mathcal{K}(A \times B)$$

Lemma 4.

1) [M] = [N], if [pure M] = [pure N];

2) Let  $\llbracket M \rrbracket = \llbracket N \rrbracket$ , then  $\llbracket M \star P \rrbracket = \llbracket N \star P \rrbracket$ ;

3) Let  $\llbracket M \rrbracket = \llbracket N \rrbracket$ , then  $\llbracket P \star M \rrbracket = \llbracket P \star N \rrbracket$ .

Proof.

1)

i) "only if"-part.

Let  $\llbracket M \rrbracket : \llbracket \Gamma \rrbracket \to \llbracket A \rrbracket, \llbracket N \rrbracket : \llbracket \Gamma \rrbracket \to \llbracket A \rrbracket$  and  $\llbracket M \rrbracket = \llbracket N \rrbracket.$  So  $p \circ \llbracket M \rrbracket = p \circ \llbracket N \rrbracket$ , hence  $\llbracket \text{pure } M \rrbracket = \llbracket \text{pure } N \rrbracket.$