

Modal type theory based on the intuitionistic epistemic logic

Abstract

Modal intuitionistic epistemic logic IEL^- was proposed by S.Artemov and T. Protopopescu as the formal foundation for the intuitionistic theory of knowledge. We construct a modal simply typed lambda-calculus which is Curry-Howard isomorphic to IEL^- as formal theory of calculations with applicative functors in functional programming languages like Haskell or Idris. We prove that this typed lambda-calculus has the strong normalization and Church-Rosser properties.

1 Introduction

Modal intuitionistic epistemic logic IEL was proposed by S. Artemov and T. Protopopescu [1]. IEL provides the epistimology and the theory of knowledge as based on BHK-semantics of intuitionistic logic. IEL^- is a variant of IEL , that corresponds to intuitionistic belief. Informally, $\mathbf{K}A$ denotes that A is verified intuitionistically.

Intuitionistic epistemic logic IEL^- is defined with by following axioms and derivation rules:

Definition 1. *Intuitionistic epistemic logic IEL :*

- 1) *IPC axioms;*
 - 2) $\mathbf{K}(A \rightarrow B) \rightarrow (\mathbf{K}A \rightarrow \mathbf{K}B)$ (*normality*);
 - 3) $A \rightarrow \mathbf{K}A$ (*co-reflection*);
- Rule: MP.*

We have the deduction theorem and necessitation rule which is derivable.

V. Krupski and A. Yatmanov provided the sequential calculus for IEL and proved that this calculus is PSPACE-complete [2].

It's not difficult to see that modal axioms in IEL^- and types of the methods of Applicative class in Haskell-like languages (which is described below) are syntactically similar and we are going to show that this coincidence has a non-trivial computational meaning.

Functional programming languages such as Haskell [3], Idris [4], Purescript [5] or Elm [6] have special type classes¹ for calculations with container types like `Functor` and `Applicative`²:

¹Type class in Haskell is a general interface for special group of datatypes.

²Reader may read more about container types in the Haskell standard library documentation[7] or in the next one textbook [8]

```

class Functor f where
  fmap :: (a -> b) -> f a -> f b

class Functor f => Applicative f where
  pure :: a -> f a
  (<*>) :: f (a -> b) -> f a -> f b

```

By *container* (or *computational context*) type we mean some type-operator f , where f is a “function” from $*$ to $*$: type operator takes a simple type (which has kind $*$) and returns another simple type with kind $*$. For more detailed description of the type system with kinds used in Haskell see [12].

The main goal of our research is a relationship between intuitionistic epistemic logic IEL^- and functional programming with effects. We show that relationship by building the type system (which is called λ_K) which is Curry-Howard isomorphic to IEL^- . So we will consider K -modality as an arbitrary applicative functor.

λK consists of the rules for simply typed lambda-calculus and special typing rules for lifting types into the applicative functor K . We assume that our type system will axiomatize the simplest case of computation with effects with one container. We provide proof-theoretical view on this kind of computations in functional programming and prove strong normalization and confluence.

2 Typed lambda-calculus based on IEL^-

At first we define the natural deduction for IEL^- with K -modality and binary connectives \rightarrow and \wedge (we call that calculus $NIEL_{\wedge, \rightarrow}^-$):

Definition 2. *Natural deduction $NIEL_{\wedge, \rightarrow}^-$ for IEL^- with \rightarrow and \wedge :*

$$\begin{array}{c}
\frac{}{\Gamma, \alpha \vdash A} \text{ax} \\
\\
\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \rightarrow_i \qquad \frac{\Gamma \vdash A \rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B} \rightarrow_i \\
\\
\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge_i \qquad \frac{\Gamma \vdash A_1 \wedge A_2}{\Gamma \vdash A_i} \wedge_e, i \in \{1, 2\} \\
\\
\frac{\Gamma, \Delta \vdash A}{K(\Gamma), \Delta \vdash KA} \vdash K_I \qquad \frac{K(\Gamma), \Delta \vdash KA \quad K(\Gamma), KA, \Delta \vdash B}{K(\Gamma), \Delta \vdash B}
\end{array}$$

Lemma 1. $\Gamma \vdash_{NIEL_{\wedge, \rightarrow}^-} A \Rightarrow IEL^- \vdash \bigwedge \Gamma \rightarrow A$.

Proof. Induction on the derivation.

Let us consider cases with modality.

- 1) If $\Gamma, \Delta \vdash_{NIEL_{\wedge, \rightarrow}^-} A$, then $IEL^- \vdash \bigwedge K\Gamma \wedge \bigwedge \Delta \rightarrow KA$.
See [2].

- 2) If $\mathbf{K}\Gamma, \Delta \vdash_{NIEL_{\wedge, \rightarrow}^-} \mathbf{K}A$ and $\mathbf{K}\Gamma, \mathbf{K}A, \Delta \vdash B$, then $IEL^- \vdash \bigwedge \mathbf{K}\Gamma \wedge \bigwedge \Delta \rightarrow B$.
- | | |
|--|------------------------|
| (1) $\bigwedge \mathbf{K}\Gamma \wedge \bigwedge \Delta \rightarrow \mathbf{K}A$ | assumption |
| (2) $\bigwedge \mathbf{K}\Gamma \wedge \mathbf{K}A \wedge \bigwedge \Delta \rightarrow B$ | assumption |
| (3) $\mathbf{K}A \wedge (\bigwedge \mathbf{K}\Gamma \wedge \bigwedge \Delta) \rightarrow B$ | conjunction properties |
| (4) $(\mathbf{K}A \wedge (\bigwedge \mathbf{K}\Gamma \wedge \bigwedge \Delta) \rightarrow B) \rightarrow (\mathbf{K}A \rightarrow ((\bigwedge \mathbf{K}\Gamma \wedge \bigwedge \Delta) \rightarrow B))$ | IPC theorem |
| (5) $\mathbf{K}A \rightarrow ((\bigwedge \mathbf{K}\Gamma \wedge \bigwedge \Delta) \rightarrow B)$ | MP |
| (6) $\bigwedge \mathbf{K}\Gamma \wedge \bigwedge \Delta \rightarrow ((\bigwedge \mathbf{K}\Gamma \wedge \bigwedge \Delta) \rightarrow B)$ | (1), (5), transitivity |
| (7) $(\bigwedge \mathbf{K}\Gamma \wedge \bigwedge \Delta) \rightarrow B$ | (6), contraction |
-

Lemma 2.

If $IEL^- \vdash A$, then $NIEL^- \vdash A$.

Proof. Straightforward derivation of modal axioms in $NIEL$. □

At the next step we build the typed lambda-calculus based on $NIEL_{\wedge, \rightarrow}^-$ by proof-assigning in rules.

At first, we define lambda-terms and types for this lambda-calculus.

Definition 3. *The set of terms:*

Let \mathbb{V} be the set of variables. The set Λ_K of terms is defined by the grammar:

$$\Lambda_K ::= \mathbb{V} \mid (\lambda \Lambda. \Lambda_K) \mid (\Lambda_K \Lambda_K) \mid (\Lambda_K, \Lambda_K) \mid (\pi_1 \Lambda_K) \mid (\pi_2 \Lambda_K) \mid (\text{pure } \Lambda_K) \mid (\text{let pure } \Lambda_K = \Lambda_K \text{ in } \Lambda_K)$$

Definition 4. *The set of types:*

Let \mathbb{T} be the set of atomic types. The set \mathbb{T}_K of types with applicative functor K is generated by the grammar:

$$\mathbb{T}_K ::= \mathbb{T} \mid (\mathbb{T}_K \rightarrow \mathbb{T}_K) \mid (\mathbb{T}_K \times \mathbb{T}_K) \mid (K\mathbb{T}_K) \quad (1)$$

Context, domain of context and range of context are defined standardly [11][12].

We divide into two parts, the first one zone is a modal part (environment with applicative types), the second one is a intuitionistic part (environment with pure types). From a logical point of view, $\Gamma \mid \Delta \vdash M : A$ denotes that M is a proof of $\mathbf{K}\Gamma, \Delta \vdash A$. This idea similar to [19].

Our type system is based on the Curry-style typing rules:

Definition 5. *Modal typed lambda calculus λK based on $NIEL_{\wedge, \rightarrow}^-$:*

$$\frac{}{\Gamma, x : A \vdash x : A} \text{ax}$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x. M : A \rightarrow B} \rightarrow_i$$

$$\frac{\Gamma \vdash x : A \quad \Gamma \vdash y : B}{\Gamma \vdash \langle x, y \rangle : A \times B} \times_i$$

$$\frac{\Gamma \vdash x : A}{\Gamma \vdash \text{pure } x : KA} K_I$$

$$\frac{\Gamma \vdash f : A \rightarrow B \quad \Gamma \vdash x : A}{\Gamma \vdash fx : B} \rightarrow_e$$

$$\frac{\Gamma \vdash M : A_1 \times A_2}{\Gamma \vdash \pi_i M : A_i} \times_e, i \in \{1, 2\}$$

$$\frac{\Gamma \vdash \vec{M} : \mathbf{K}\vec{A} \quad \vec{x} : \vec{A} \vdash N : B}{\Gamma \vdash \text{let pure } \vec{N} = \vec{M} \text{ in } N : \mathbf{K}B} \text{let}_{\mathbf{K}}$$

\mathbf{K}_I -typing rule is the same as \bigcirc -introduction in lax logic (also known as monadic metalanguage [17]) and in typed lambda-calculus which is derived by proof-assignment for lax-logic proofs. \mathbf{K}_I allows to inject an object of type α into the functor. \mathbf{K}_I reflects the Haskell method **pure** for Applicative class. It plays the same role as the **return** method in Monad class.

Here are some examples of derivation trees.

$$\frac{\frac{x : A \vdash x : A}{x : A \vdash \text{pure } x : \mathbf{K}A} \mathbf{K}_I}{\vdash (\lambda x. \text{pure } x) : A \rightarrow \mathbf{K}A} \rightarrow_i$$

$$\frac{\frac{f : \mathbf{K}(A \rightarrow B) \vdash f : \mathbf{K}(A \rightarrow B) \quad x : \mathbf{K}A \vdash x : \mathbf{K}A \quad \frac{g : A \rightarrow B \quad y : A}{g : A \rightarrow B, y : A \vdash gy : B}}{f : \mathbf{K}(A \rightarrow B), x : \mathbf{K}A \vdash \text{let pure } \langle g, y \rangle = \langle f, x \rangle \text{ in } gy : \mathbf{K}B}}{f : \mathbf{K}(A \rightarrow B) \vdash \lambda x. \text{let pure } \langle g, y \rangle = \langle f, x \rangle \text{ in } gy : \mathbf{K}A \rightarrow \mathbf{K}B} \rightarrow_i$$

$$\frac{\frac{f : A \rightarrow B \vdash f : A \rightarrow B}{f : A \rightarrow B \vdash \text{pure } f : \mathbf{K}(A \rightarrow B)} \quad x : \mathbf{K}A \vdash x : \mathbf{K}A \quad \frac{g : A \rightarrow B \quad y : A}{g : A \rightarrow B, y : A \vdash gy : B}}{f : A \rightarrow B, x : \mathbf{K}A \vdash \text{let pure } \langle g, y \rangle = \langle \text{pure } f, x \rangle \text{ in } gy : \mathbf{K}B} \rightarrow_i$$

$$\frac{f : A \rightarrow B \vdash \lambda x. \text{let pure } \langle g, y \rangle = \langle \text{pure } f, x \rangle \text{ in } gy : \mathbf{K}A \rightarrow \mathbf{K}B}{\lambda f. \lambda x. \text{let pure } \langle g, y \rangle = \langle \text{pure } f, x \rangle \text{ in } gy : (A \rightarrow B) \rightarrow \mathbf{K}A \rightarrow \mathbf{K}B} \rightarrow_i$$

Now we define free variables and substitutions. β -reduction, multi-step β -reduction and β -equality are defined standardly:

Definition 6. Set $FV(M)$ of free variables for arbitrary term M :

- 1) $FV(x) = \{x\}$;
- 2) $FV(\lambda x. M) = FV(M) \setminus \{x\}$;
- 3) $FV(MN) = FV(M) \cup FV(N)$;
- 4) $FV((M, N)) = FV(M) \cup FV(N)$;
- 5) $FV(\pi_i p) \subseteq FV(p)$, $i \in \{1, 2\}$;
- 6) $FV(\text{pure } M) = FV(M)$;
- 7) $FV(\text{let pure } \vec{x} = \vec{M} \text{ in } N) = \bigcup_{i=1}^n FV(M_i)$, where $n = |\vec{M}|$.

Definition 7. Substitution:

- 1) $x[x := N] = N$, $x[y := N] = x$;

- 2) $(MN)[x := P] = M[x := P]N[x := P];$
- 3) $(\lambda x.M)[x := N] = \lambda x.M[x := N];$
- 4) $(M, N)[x := P] = (M[x := P], N[x := P]);$
- 5) $(\pi_i M)[x := P] = \pi_i(M[x := P]), i \in \{1, 2\};$
- 6) $(\mathbf{pure} M)[x := P] = \mathbf{pure}(M[x := P]);$
- 7) $(\mathbf{let} \mathbf{pure} \vec{N} = \vec{M} \mathbf{in} M)[x := P] = \mathbf{let} \mathbf{pure} \vec{N} = (\vec{M}[x := P]) \mathbf{in} M.$

In $\lambda\mathbf{K}$ we have the following computational rules:

Definition 8. *β -reduction rules for $\lambda\mathbf{K}$.*

- 1) $(\lambda x.M)N \rightarrow_\beta M[x := N]$
- 2) $\pi_1\langle M, N \rangle \rightarrow_\beta M$
- 3) $\pi_2\langle M, N \rangle \rightarrow_\beta N$

3 Basic lemmas

Now we will prove standard lemmas for contexts in type systems³:

Definition 9. *The domain of a context Γ :*

Let $\Gamma = \{x_1 : A_1, \dots, x_n : A_n\}$. Then the domain of Γ , or $\text{dom}(\Gamma)$, is a set $\{x_1, \dots, x_n\}$.

Lemma 3. *If $\Gamma \vdash M : A$, then $FV(M) \subseteq \text{dom}(\Gamma)$*

Proof. Induction on the derivation of $\Gamma \vdash M : A$. □

Lemma 4. *Generation for $\lambda\mathbf{K}$.*

- 1) $\Gamma \vdash \mathbf{pure} M : \mathbf{K}A$ implies that $\Gamma \vdash M : A$;
- 2) $\Gamma \vdash \mathbf{let} \mathbf{pure} \vec{N} = \vec{M} \mathbf{in} M : \mathbf{K}B$ implies that $\Gamma \vdash \vec{M} : \mathbf{K}\vec{A}$ and $\vec{N} : \vec{A} \vdash M : B$.

Proof.

Induction on the derivation of $\Gamma \vdash \mathbf{pure} M : \mathbf{K}A$ and $\Gamma \vdash \mathbf{let} \mathbf{pure} \vec{N} = \vec{M} \mathbf{in} M : \mathbf{K}B$ respectively. □

The next one lemma allows that weakening structural rule is admissable.

Lemma 5. *Weakening for $\lambda\mathbf{K}$.*

Let $\Gamma \vdash M : A$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash M : A$.

Proof.

Induction on derivation of $\Gamma \vdash M : \alpha$. Let us assume $\Gamma \subseteq \Delta$.

- 1) Let $\Gamma \vdash x : A$, such that $\Gamma = \Delta, x : A$ and $\Theta \subseteq \Gamma$. Let $\Sigma = \Theta \setminus \Gamma$, or, which is the same, $\Sigma = \Theta \setminus \Delta, x : A$, then $\Sigma, \Delta, x : A \vdash x : \alpha$, or, $\Theta \vdash x : A$.
- 2) Let $\Gamma \vdash \mathbf{pure} M : \mathbf{K}A$ and $\Gamma \subseteq \Theta$.

³We will not prove cases with \rightarrow -constructor, they are proved standardly in the same lemmas for simply typed lambda calculus, for example see [11][12][14]. We will consider only modal cases

If $\Gamma \vdash \mathbf{pure} M : \mathbf{K}A$, then $\Gamma \vdash M : A$ by generation and, by hypothesis, $\Theta \vdash M : A$, so $\Theta \vdash \mathbf{pure} M : \mathbf{K}A$ by applying \mathbf{K}_I -rule.

3) Let $\Gamma \vdash \mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} N : \mathbf{K}B$ and $\Gamma \subseteq \Delta$.

By generation $\Gamma \vdash \vec{M} : \mathbf{K}\vec{A}$ and $\vec{x} : \vec{A} \vdash N : \mathbf{K}B$.

By hypothesis we have $\Delta \vdash \vec{M} : \mathbf{K}\vec{A}$. So $\Delta \vdash \mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} N : \mathbf{K}B$ by $\mathbf{let_K}$. □

Lemma 6. *Considering for $\lambda\mathbf{K}$.*

If $\Gamma \vdash M : A$, then $\Gamma \upharpoonright FV(M) \vdash M : A$, where $\Gamma \upharpoonright FV(M)$ is a subcontext of Γ , such that $dom(\Gamma \upharpoonright FV(M)) = dom(\Gamma) \cap FV(M)$.

Proof. Induction by derivation. We consider the base of induction and the case with $\mathbf{let_K}$. The rest cases are proven by the same way.

1) Let $\Gamma \vdash x : A$, where $\Gamma = \Delta, x : A$, $x \in \mathbb{V}$.

$FV(x) = \{x\}$, then $dom(\Gamma) \cap \{x\} = \{x\}$. So $(\Delta, x : A) \upharpoonright FV(x) = \{x : A\}$, then $x : A \vdash x : A$ by axiom.

2) Let $\Gamma \vdash \mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} N : \mathbf{K}B$.

By generation $\Gamma \vdash \vec{M} : \mathbf{K}\vec{A}$ and $\vec{x} : \vec{A} \vdash N : \mathbf{K}B$.

By hypothesis $\Gamma \upharpoonright FV(\vec{M}) \vdash \vec{M} : \mathbf{K}\vec{A}$.

So $\Gamma \upharpoonright FV(\vec{M}) \vdash \mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} N : \mathbf{K}B$ by $\mathbf{let_K}$. □

Lemma 7. *If $\Gamma, x : A \vdash M : B$ and $\Gamma \vdash N : A$, then $\Gamma \vdash (M[x := N]) : B$*

Proof.

1) Let $\Gamma, x : A \vdash \mathbf{pure} M : \mathbf{K}B$ and $\Gamma \vdash N : A$.

If $\Gamma, x : \alpha \vdash \mathbf{pure} M : \mathbf{K}\beta$.

By generation, $\Gamma, x : A \vdash M : B$.

So, by induction hypothesis, $\Gamma \vdash (M[x := N]) : B$. Then $\Gamma \vdash \mathbf{pure} (M[x := N]) : \mathbf{K}B$ by \mathbf{K}_I , but $\mathbf{pure} (M[x := N]) = (\mathbf{pure} M[x := N])$ by substitution definition.

So $\Gamma \vdash (\mathbf{pure} M[x := N]) : \mathbf{K}B$

2) Let $\Gamma, y : C \vdash \mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} N : \mathbf{K}B$ and $\Gamma \vdash P : C$.

By generation $\Gamma, y : C \vdash \vec{M} : \mathbf{K}\vec{A}$ and $\vec{x} : \vec{A} \vdash N : \mathbf{K}B$.

By assumption $\Gamma \vdash \overrightarrow{M[y := P]} : \mathbf{K}\vec{A}$.

So $\Gamma \vdash \mathbf{let pure} \vec{x} = \overrightarrow{M[y := P]} \mathbf{in} N : \mathbf{K}B$ by $\mathbf{let_K}$. □

Definition 10. *Type substitution*

The substitution of type C for type variable B in some type A is defined by:

1) $A[A := B] = B$;

2) $A'[A := C] = A'$;

3) $(A_1 \alpha A_2)[B := C] = A_1[B := C] \alpha A_2[A := B]$, where $\alpha \in \{\times, \rightarrow\}$;

4) $(\mathbf{K}A)[B := C] = \mathbf{K}(A[B := C])$.

Lemma 8. *Substitution for types.*

Let $\Gamma \vdash M : A$, then $\Gamma[B := C] \vdash M : A[B := C]$, where $\Gamma[B := C]$ denotes $\{x_1 : A_1[B := C], \dots, x_n : A_n[B := C]\}$.

1) Let $\Gamma, x : A \vdash x : A$, so $\Gamma[B := C], x : A[B := C] \vdash x : A[B := C]$.

2) Let $\Gamma \vdash \mathbf{pure} M : \mathbf{K}A$. By generation $\Gamma \vdash M : A$.

By assumption $\Gamma[B := C] \vdash M : A[B := C]$.

By \mathbf{K}_I , $\Gamma[B := C] \vdash \mathbf{pure} M : \mathbf{K}(A[B := C])$.

But, by substitution definition, $\mathbf{K}(A[B := C]) = (\mathbf{K}A)[B := C]$.

Then $\Gamma[B := C] \vdash \mathbf{pure} M : (\mathbf{K}A)[B := C]$.

3) $\Gamma \vdash \mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} M : \mathbf{K}B$.

By generation $\Gamma \vdash \vec{M} : \mathbf{K}\vec{A}$ and $\vec{x} : \vec{A} \vdash M : \mathbf{K}B$.

$\Gamma[D := C] \vdash \vec{M} : \mathbf{K}\vec{A}[D := C]$ and $\vec{x} : \vec{A}[D := C] \vdash M : B[D := C]$ by assumption.

Then $\Gamma[D := C] \vdash \mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} M : (B[D := C])$.

Theorem 1. *Subject reduction*

Let $\Gamma \vdash M : \alpha$ and $M \rightarrow_\beta N$, then $\Gamma \vdash N : \alpha$

Proof.

□

4 Strong normalization

We modify and apply Tait's technique of logical relation for modalities. Strong normalization proof with Tait's method for simply typed lambda calculus is described here [13].

Theorem 2. *Let $M \in \Lambda_{\mathbf{K}}$, then any sequence of reduction $M \rightarrow_\beta M_1 \dots$ terminates.*

Proof. We build the smallest of subset of strongly normalizing terms of modal types and show that an arbitrary term belongs to this subset.

Definition 11. *The set of strongly computable terms of type $\phi \in \mathbb{T}_{\mathbf{K}}$, SC_ϕ :*

- Let $\phi = \mathbf{K}\alpha$ and $\alpha \in \mathbb{T}$, then:

$$SC_{\mathbf{K}\alpha} = \{M : \mathbf{K}\alpha \mid M \text{ is strongly normalizing}\} \quad (2)$$

- Let $\phi = \mathbf{K}(\tau \rightarrow \psi)$ and $\tau, \psi \in \mathbb{T}_{\mathbf{K}}$, then:

$$SC_{\mathbf{K}(\tau \rightarrow \psi)} = \{M : \mathbf{K}(\tau \rightarrow \psi) \mid \forall N \in SC_{\mathbf{K}\tau}, M \star N \in SC_{\mathbf{K}\psi}\} \quad (3)$$

- Let $\phi = \mathbf{K}(\tau_1 \times \tau_2)$ and $\tau_1, \tau_2 \in \mathbb{T}_{\mathbf{K}}$, then:

$$SC_{\mathbf{K}(\tau_1 \times \tau_2)} = \{P : \mathbf{K}(\tau_1 \times \tau_2) \mid \mathbf{pure} (\lambda x. \pi_i x) \star P \in SC_{\mathbf{K}\tau_i}, i \in \{1, 2\}\} \quad (4)$$

Lemma 9.

If $M \in SC_\alpha$, then M is strongly normalizing.

Proof.

- 1) If $M \in SC_{\mathbf{K}\alpha}$ and $\alpha \in \mathbb{T}$, then M is strongly normalizing by the definition of $SC_{\mathbf{K}\alpha}$.
- 2) Let $M \in SC_{\mathbf{K}(\tau \rightarrow \psi)}$, so by every $N \in SC_{\mathbf{K}\tau}$, $M \star N \in SC_{\mathbf{K}\psi}$, which is strongly normalizing by hypothesis. So M is strongly normalizing.
- 3) Let $M \in SC_{\mathbf{K}(\tau_1 \times \tau_2)}$, so $\mathbf{pure}(\lambda x. \pi_i x) \star M \in SC_{\mathbf{K}\tau_i}$, $i \in \{1, 2\}$, which are strongly normalizing. So M is strongly normalizing. \square

Lemma 10.

Let $M \rightarrow_\beta M'$ and $M \in SC_\alpha$, then $M' \in SC_\alpha$.

Proof.

- 1) Let $M \rightarrow_\beta M'$ and $M \in SC_{\mathbf{K}\alpha}$, where $\alpha \in \mathbb{T}$.
 M has the longest reduction path (which we denote as $p(M)$). So $p(M') < p(M)$, then $M' \in SC_{\mathbf{K}\alpha}$.
- 2) Let $M \in SC_{\mathbf{K}(\alpha \rightarrow \beta)}$ and $M \rightarrow_\beta M'$. Let $N \in SC_{\mathbf{K}\alpha}$. So $M \star N \in SC_{\mathbf{K}\beta}$.
If $M \rightarrow_\beta M'$, then $M \star N \rightarrow_\beta M' \star N$ by reduction rule, so $M' \star N \in SC_{\mathbf{K}\beta}$ and $M' \in SC_{\mathbf{K}(\alpha \rightarrow \beta)}$ by hypothesis.
- 3) Let $M \in SC_{\mathbf{K}(\tau_1 \times \tau_2)}$ and $M \rightarrow_\beta M'$.
So $\mathbf{pure}(\lambda x. \pi_i x) \star M \rightarrow_\beta \mathbf{pure}(\lambda x. \pi_i x) \star M'$, $i \in \{1, 2\}$ by reduction rule.
So $\mathbf{pure}(\lambda x. \pi_i x) \star M' \in SC_{\mathbf{K}\tau_i}$ and $M' \in SC_{\mathbf{K}(\tau_1 \times \tau_2)}$. \square

Definition 12. *Neutral term:*

We define a term M to be neutral if it has of the next forms:

- 1) $M = x$, where $x \in \mathbb{V}$;
- 2) $M = (PQ)$;
- 3) $M = \pi_i M$, $i \in \{1, 2\}$;
- 4) $M = P \star Q$;
- 5) If M is a neutral, then $\mathbf{pure} M$ is a neutral.

Lemma 11. Let $M \rightarrow_\beta M'$ and $M' \in SC_\alpha$ for every one-step reduction. So if M' is a neutral, then $M \in SC_\alpha$.

Proof.

Simple induction on the structure of M' . \square

Lemma 12.

Let $x_1 : \phi_1, \dots, x_n : \phi_n \vdash M : \phi$ and for all $i \in \{1, \dots, n\}$, $N_i \in SC_{\phi_i}$, then $(M[x_1 := N_1, \dots, x_n := N_n]) \in SC_\phi$.

Proof.

- 1) If ϕ is an atomic and M is a variable, then this condition holds straightforwardly.
- 2) Let $\Gamma = \{x_1 : \phi_1, \dots, x_n : \phi_n\}$, $\Gamma \vdash \mathbf{pure} M : \mathbf{K}\alpha$ and for all $i \in \{1, \dots, n\}$, $N_i \in SC_{\phi_i}$.
Then by $\Gamma \vdash M : \alpha$ by generation and $(M[x_1 := N_1, \dots, x_n := N_n]) \in SC_\alpha$ by induction hypothesis.
Hence, $\Gamma \vdash \mathbf{pure} M : \mathbf{K}\alpha$ and $(\mathbf{pure} M([x_1 := N_1, \dots, x_n := N_n])) \in SC_{\mathbf{K}\alpha}$ by definition of $SC_{\mathbf{K}\alpha}$.

3) Let $\Gamma = \{x_1 : \phi_1, \dots, x_n : \phi_n\}$, $\Gamma : \phi_n \vdash M \star P : \mathbf{K}\beta$ and for all $i \in \{1, \dots, n\}$, $N_i \in SC_{\phi_i}$.

Then $\Gamma \vdash M : \mathbf{K}(\alpha \rightarrow \beta)$, $\Gamma \vdash P : \mathbf{K}\alpha$ by generation.

But by induction hypothesis $M[x_1 := N_1, \dots, x_n := N_n] \in SC_{\mathbf{K}(\alpha \rightarrow \beta)}$ and $P[x_1 := N_1, \dots, x_n := N_n] \in SC_{\mathbf{K}\alpha}$.

Then, by definition of $SC_{\mathbf{K}\beta}$, $((M[x_1 := N_1, \dots, x_n := N_n]) \star (P[x_1 := N_1, \dots, x_n := N_n])) \in SC_{\mathbf{K}\beta}$, i.e. $(M \star N([x_1 := N_1, \dots, x_n := N_n])) \in SC_{\mathbf{K}\beta}$. \square

Corollary 1.

If $\vdash M : \alpha$, then M is strongly normalizing.

Proof. $M \in SC_\alpha$ by Lemma 10, so M is strongly normalizing. \square

\square

\square

5 Confluence

In the confluence proof (below) we treat the cases with **pure** and \star similar to [15] [18].

Definition 13. *Alphabet for the labelled terms:*

variables: $x, y, z, x_1, y_1, z_1, \dots$;

lambdas: $\lambda, \lambda_0, \lambda_1, \lambda_2, \dots$;

constructors for an applicative functor: **pure**, \star ;

parentheses $(,)$.

Definition 14. *The set of labelled terms Λ'_K inductively defined as a set of words on the alphabet described above:*

- 1) $x \in \Lambda'_K$;
- 2) If $M \in \Lambda'_K$, then $(\lambda x.M) \in \Lambda'_K$;
- 3) If $M, N \in \Lambda'_K$, then $(MN) \in \Lambda'_K$;
- 4) If $M \in \Lambda'_K$, then **pure** $M \in \Lambda'_K$;
- 5) If $M, N \in \Lambda'_K$, then $M \star N \in \Lambda'_K$;
- 6) If $M, N \in \Lambda'_K$, then for all $i \in \mathbb{N}$, $((\lambda_i x.M)N) \in \Lambda'_K$.

Definition 15. *Erasing map*

Erasing map is a map $|\cdot| : \Lambda'_K \rightarrow \Lambda_K$, such that:

- 1) $|x| = x$;
- 2) $|(\lambda x.M)| = \lambda x. |M|$;
- 3) $|(MN)| = |M| |N|$;
- 4) $|(\mathbf{pure} M)| = \mathbf{pure} |M|$;
- 5) $|M \star N| = |M| \star |N|$;
- 6) $|((\lambda_i x.M)N)| = (\lambda x. |M|) |N|$

Example 1.

$$|\mathbf{pure} ((\lambda_i x.M)N) \star P| = \mathbf{pure} (\lambda x. |M|) |N| \star |P|$$

Definition 16. Substitution for Λ'_K :

- 1) $x[x := N] = N, x[y := N] = x$;
- 2) $(MN)[x := N] = M[x := N]N[x := N]$;
- 3) $(\lambda x.M)[x := N] = \lambda x.M[x := N]$;
- 4) $(\mathbf{pure} M)[x := P] = \mathbf{pure} (M[x := P])$;
- 5) $(M \star N)[x := P] = (M[x := P]) \star (N[x := P])$;
- 6) $(\lambda_i x.M)N[y := P] = (\lambda_i x.M[y := P])(N[y := P])$.

Definition 17. One-step reduction $\rightarrow_{\beta'}$ for Λ'_K :

- 1) $(\lambda x.M)N \rightarrow_{\beta'} M[x := N]$;
- 2) $\mathbf{pure} (\lambda x.x) \star M \rightarrow_{\beta'} M$;
- 3) $\mathbf{pure} (\lambda f g x.f(gx)) \star M \star N \star P \rightarrow_{\beta'} M \star (N \star P)$;
- 4) $(\mathbf{pure} M) \star (\mathbf{pure} N) \rightarrow_{\beta'} \mathbf{pure} (MN)$;
- 5) $M \star (\mathbf{pure} N) \rightarrow_{\beta'} \mathbf{pure} (\lambda f.fN) \star M$;
- 6) $(\lambda_i x.M)N \rightarrow_{\beta'} M[x := N]$.

Multi-step reduction $\rightarrow_{\beta'}$ is a reflexive-transitive closure of $\rightarrow_{\beta'}$.

Definition 18. Let us define a map $\phi : \Lambda'_K \rightarrow \Lambda_K$ inductively as follows:

- 1) $\phi(x) = x$;
- 2) $\phi(MN) = \phi(M)\phi(N)$;
- 3) $\phi(\lambda x.M) = \lambda x.\phi(M)$;
- 4) $\phi(\mathbf{pure} M) = \mathbf{pure} (\phi(M))$;
- 5) $\phi(M \star N) = \phi(M) \star \phi(N)$;
- 6) $\phi((\lambda_i x.M)N) = \phi(M)[x := \phi(N)]$.

Example 2.

$$\phi(\mathbf{pure} ((\lambda_i x.M)N) \star P) = \mathbf{pure} (\phi(M)[x := \phi(N)]) \star \phi(P)$$

Lemma 13.

- 1) Let $M, N \in \Lambda'_K$ and $|M| \rightarrow_{\beta} |N|$, then $M \rightarrow_{\beta'} N$.
- 2) Let $M, N \in \Lambda'_K$ and $M \rightarrow_{\beta'} N$, then $|M| \rightarrow_{\beta} |N|$.

Proof.

Induction on the generation of \rightarrow_{β} ($\rightarrow_{\beta'}$).

1) Let us consider homomorphism rule. The rest applicative reduction rules are considered similary.

Let $(\mathbf{pure} M') \star (\mathbf{pure} N'), \mathbf{pure} (M'N') \in \Lambda'_K$.

So $|(\mathbf{pure} M') \star (\mathbf{pure} N')| = (\mathbf{pure} |M'|) \star (\mathbf{pure} |N'|)$ and $|\mathbf{pure} (M'N')| = \mathbf{pure} (|M'| |N'|)$.

By reduction rule, $(\mathbf{pure} |M'|) \star (\mathbf{pure} |N'|) \rightarrow_{\beta} \mathbf{pure} (|M'| |N'|)$.

But $(\mathbf{pure} M') \star (\mathbf{pure} N') \rightarrow_{\beta'} \mathbf{pure} (M'N')$ by reduction rule for $\rightarrow_{\beta'}$.

2) Let us consider interchange rule.

Let $M \star (\mathbf{pure} N), \mathbf{pure} (\lambda f.fN) \star M \in \Lambda'_K$ and $M \star (\mathbf{pure} N) \rightarrow_{\beta'} \mathbf{pure} (\lambda f.fN) \star M$.

But $|M \star (\mathbf{pure} N)| = |M| \star (\mathbf{pure} |N|)$ and $|\mathbf{pure} (\lambda f.fN) \star M| = \mathbf{pure} (\lambda f.f |N|) \star |M|$.

So $|M| \star (\mathbf{pure} |N|) \rightarrow_{\beta} \mathbf{pure} (\lambda f.f |N|) \star |M|$ by β -reduction rule.

It is easy to see, that the statement for $\rightarrow_{\beta'}$ and \rightarrow_{β} immedeatly follows from transitivity of multi-step rediction for labelled terms and for usual terms respectively.

□

Lemma 14.

$$\phi(M[x := N]) = \phi(M)[x := \phi(N)].$$

Proof. Induction on M .

1) Let $M = x$. Then $\phi(x[x := N]) = \phi(N)$.

On the other hand, $\phi(x)[x := \phi(N)] = x[x := \phi(N)] = \phi(N)$.

So $\phi(x[x := N]) = \phi(x)[x := \phi(N)]$.

2) Let $M = y$ and $y \neq x$. Then $\phi(y[x := N]) = \phi(y) = y$.

But $\phi(y)[x := \phi(N)] = y[x := \phi(N)] = y$.

Therefore $\phi(y[x := N]) = \phi(y)[x := \phi(N)]$.

3) Let $M = \mathbf{pure} M'$. Then $\phi(\mathbf{pure} M'[x := N]) = \mathbf{pure} \phi(M'[x := N])$.

By hypothesis, $\mathbf{pure} (\phi(M'[x := N])) = \mathbf{pure} (\phi(M')[x := \phi(N)])$, which is $(\mathbf{pure} \phi(M'))[x := \phi(N)]$ by substitution definition.

4) Let $M = M' \star N'$. So $\phi((M' \star N')[x := N]) = \phi(M'[x := N] \star N'[x := N])$.

By definition of ϕ ,

$$\phi(M'[x := N] \star N'[x := N]) = \phi(M'[x := N]) \star \phi(N'[x := N]).$$

But by induction hypothesis,

$$\phi(M'[x := N]) = \phi(M')[x := \phi(N)] \text{ and }$$

$$\phi(N'[x := N]) = \phi(N')[x := \phi(N)].$$

Hence,

$$\phi(M'[x := N]) \star \phi(N'[x := N]) = \phi(M')[x := \phi(N)] \star \phi(N')[x := \phi(N)].$$

So,

$$\phi(M')[x := \phi(N)] \star \phi(N')[x := \phi(N)] = (\phi(M') \star \phi(N'))[x := \phi(N)].$$

And by definition of ϕ , $(\phi(M') \star \phi(N'))[x := \phi(N)] = \phi(M' \star N')[x := \phi(N)]$.

□

Lemma 15.

Let $M, N \in \Lambda'_K$ and $M \rightarrow_{\beta'} N$, then $\phi(M) \rightarrow_{\beta} \phi(N)$.

Proof.

1) Let $\mathbf{pure} (\lambda x.x) \star M, M \in \Lambda'_K$ and $\mathbf{pure} (\lambda x.x) \star M \rightarrow_{\beta'} M$.

But $\phi(\mathbf{pure} (\lambda x.x) \star M) = \mathbf{pure} (\lambda x.x) \star \phi(M)$.

So $\mathbf{pure} (\lambda x.x) \star \phi(M) \rightarrow_{\beta} \phi(M)$ by β -reduction rule.

2) Let $\mathbf{pure} (\lambda f g x.f(gx)) \star M \star N \star P, M \star (N \star P) \in \Lambda'_K$ and $\mathbf{pure} (\lambda f g x.f(gx)) \star M \star N \star P \rightarrow_{\beta'} M \star (N \star P)$.

By the definition of ϕ :

$$\phi(\mathbf{pure} (\lambda f g x.f(gx)) \star M \star N \star P) = \mathbf{pure} (\lambda f g x.f(gx)) \star \phi(M) \star \phi(N) \star \phi(P);$$

$$M \star (N \star P) = \phi(M) \star (\phi(N) \star \phi(P)).$$

Hence, $\mathbf{pure} (\lambda f g x.f(gx)) \star \phi(M) \star \phi(N) \star \phi(P) \rightarrow_{\beta} \phi(M) \star (\phi(N) \star \phi(P))$ by β -reduction rule.

3) Let $(\mathbf{pure} M) \star (\mathbf{pure} N), \mathbf{pure} (MN) \in \Lambda'_K$ and $(\mathbf{pure} M) \star (\mathbf{pure} N) \rightarrow_{\beta} \mathbf{pure} (MN)$.

By the definition of ϕ :

$$\phi((\mathbf{pure} M) \star (\mathbf{pure} N)) = (\mathbf{pure} \phi(M)) \star (\mathbf{pure} \phi(N));$$

$\phi(\mathbf{pure}(MN)) = \mathbf{pure}(\phi(M)\phi(N))$.
 So, by reduction rule, $(\mathbf{pure}\phi(M)) \star (\mathbf{pure}\phi(N)) \rightarrow_{\beta} \mathbf{pure}(\phi(M)\phi(N))$.

4) Let $M \star (\mathbf{pure})$, $\mathbf{pure}(\lambda f.fN) \star M$ and $M \star (\mathbf{pure}N) \rightarrow_{\beta'} (\lambda f.fN) \star M$.
 $\phi(M \star (\mathbf{pure}N)) = \phi(M) \star (\mathbf{pure}\phi(N))$
 $\phi((\lambda f.fN) \star M) = (\lambda f.f\phi(N)) \star \phi(M)$.
 So, $\phi(M) \star (\mathbf{pure}\phi(N)) \rightarrow_{\beta} \mathbf{pure}(\lambda f.f\phi(N)) \star \phi(M)$. \square

Lemma 16.

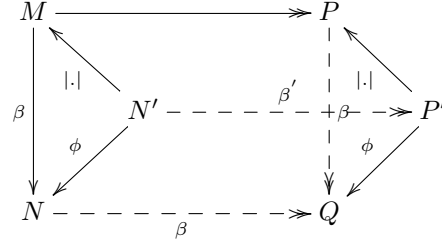
Let $M \in \Lambda'_K$. Then $|M| \rightarrow_{\beta} \phi(M)$.

Proof. Induction on the structure of M . \square

Lemma 17. *Strip lemma.*

If $M \rightarrow_{\beta} N$ and $M \rightarrow_{\beta} P$. Then there exists some term Q , such that $N \rightarrow_{\beta} Q$ and $P \rightarrow_{\beta} Q$.

Proof. Proof is similar to [15] [18]. We build the following diagram



\square

which is commutes by lemmas 11 – 14.

Theorem 3. *Confluence.*

If $M \rightarrow_{\beta} N$ and $M \rightarrow_{\beta} P$. Then there exists some term Q , such that $N \rightarrow_{\beta} Q$ and $P \rightarrow_{\beta} Q$.

Proof.

By unfolding $M \rightarrow_{\beta} N$ as the sequence of one-step reductions $M \rightarrow_{\beta} M_1 \rightarrow_{\beta} \dots \rightarrow_{\beta} M_n \rightarrow_{\beta} N$ and applying strip lemma on every step. \square

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