

Modal type theory based on the intuitionistic epistemic logic

Abstract

Modal intuitionistic epistemic logic IEL^- was proposed by S.Artemov and T. Protopopescu as the formal foundation for the intuitionistic theory of knowledge. We construct a modal simply typed lambda-calculus which is Curry-Howard isomorphic to IEL^- as formal theory of calculations with applicative functors in functional programming languages like Haskell or Idris. We prove that this typed lambda-calculus has the strong normalization and Church-Rosser properties.

1 Introduction

Modal intuitionistic epistemic logic IEL was proposed by S. Artemov and T. Protopopescu [1]. IEL provides the epistemology and the theory of knowledge as based on BHK-semantics of intuitionistic logic. IEL^- is a variant of IEL , that corresponds to intuitionistic belief. Informally, $\mathbf{K}A$ denotes that A is verified intuitionistically.

Intuitionistic epistemic logic IEL^- is defined with by following axioms and derivation rules:

Definition 1. *Intuitionistic epistemic logic IEL :*

- 1) *IPC axioms;*
 - 2) $\mathbf{K}(A \rightarrow B) \rightarrow (\mathbf{K}A \rightarrow \mathbf{K}B)$ (*normality*);
 - 3) $A \rightarrow \mathbf{K}A$ (*co-reflection*);
- Rule: MP.*

We have the deduction theorem and necessitation rule which is derivable.

V. Krupski and A. Yatmanov provided the sequential calculus for IEL and proved that this calculus is PSPACE-complete [2].

It's not difficult to see that modal axioms in IEL^- and types of the methods of Applicative class in Haskell-like languages (which is described below) are syntactically similar and we are going to show that this coincidence has a non-trivial computational meaning.

Functional programming languages such as Haskell [3], Idris [4], Purescript [5] or Elm [6] have special type classes¹ for calculations with container types like `Functor` and `Applicative`²:

¹Type class in Haskell is a general interface for special group of datatypes.

²Reader may read more about container types in the Haskell standard library documentation[7] or in the next one textbook [8]

```

class Functor f where
  fmap :: (a -> b) -> f a -> f b

class Functor f => Applicative f where
  pure :: a -> f a
  (<*>) :: f (a -> b) -> f a -> f b

```

By *container* (or *computational context*) type we mean some type-operator f , where f is a “function” from $*$ to $*$: type operator takes a simple type (which has kind $*$) and returns another simple type type with kind $*$. For more detailed description of the type system with kinds used in Haskell see [12].

The main goal of our research is a relationship between intuitionistic epistemic logic IEL^- and functional programming with effects. We show that relationship by building the type system (which is called $\lambda_{\mathbf{K}}$) which is Curry-Howard isomorphic to IEL^- . So we will consider \mathbf{K} -modality as an arbitrary applicative functor.

λK consists of the rules for simply typed lambda-calculus and special typing rules for lifting types into the applicative functor \mathbf{K} . We assume that our type system will axiomatize the simplest case of computation with effects with one container. We provide proof-theoretical view on this kind of computations in functional programming and prove strong normalization and confluence.

2 Typed lambda-calculus based on IEL^-

At first we define the natural deduction for IEL^- with \mathbf{K} -modality and binary connectives \rightarrow and \wedge (we call that calculus $NIEL_{\wedge, \rightarrow}^-$):

Definition 2. *Natural deduction $NIEL_{\wedge, \rightarrow}^-$ for IEL^- with \rightarrow and \wedge :*

$$\begin{array}{c}
\frac{}{\Gamma, \alpha \vdash A} \text{ax} \\
\\
\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \rightarrow_i \qquad \frac{\Gamma \vdash A \rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B} \rightarrow_i \\
\\
\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge_i \qquad \frac{\Gamma \vdash A_1 \wedge A_2}{\Gamma \vdash A_i} \wedge_e, i \in \{1, 2\} \\
\\
\frac{\Gamma \vdash A}{\Gamma \vdash \mathbf{K}A} \mathbf{K}_I \qquad \frac{\Gamma \vdash \mathbf{K}\vec{A} \quad \vec{A} \vdash B}{\Gamma \vdash \mathbf{K}B}
\end{array}$$

Where $\Gamma \vdash \mathbf{K}\vec{A}$ is a syntax sugar for $\Gamma \vdash \mathbf{K}A_1, \dots, \Gamma \vdash \mathbf{K}A_n$.

Lemma 1. $\Gamma \vdash_{NIEL_{\wedge, \rightarrow}^-} A \Rightarrow IEL^- \vdash \bigwedge \Gamma \rightarrow A$.

Proof. Induction on the derivation.

Let us consider cases with modality.

1) If $\Gamma \vdash_{NIEL_{\wedge, \rightarrow}^-} A$, then $IEL^- \vdash \bigwedge \Gamma \rightarrow \mathbf{K}A$.

- (1) $\bigwedge \Gamma \rightarrow A$ assumption
- (2) $A \rightarrow \mathbf{K}A$ co-reflection
- (3) $(\bigwedge \Gamma \rightarrow A) \rightarrow ((A \rightarrow \mathbf{K}A) \rightarrow (\bigwedge \Gamma \rightarrow \mathbf{K}A))$ IPC theorem
- (4) $(A \rightarrow \mathbf{K}A) \rightarrow (\bigwedge \Gamma \rightarrow \mathbf{K}A)$ from (1), (3) and MP
- (5) $\bigwedge \Gamma \rightarrow \mathbf{K}A$ from (2), (4) and MP

2) If $\Gamma \vdash_{NIEL_{\bigwedge, \rightarrow}^-} \mathbf{K}\vec{A}$ and $\vec{A} \vdash B$, then $IEL^- \vdash \bigwedge \Gamma \rightarrow \mathbf{K}B$.

- (1) $\bigwedge \Gamma \rightarrow \bigwedge_{i=1}^n \mathbf{K}A_i$ assumption
- (2) $\bigwedge_{i=1}^n \mathbf{K}A_i \rightarrow \mathbf{K} \bigwedge_{i=1}^n A_i$ IEL theorem
- (3) $\bigwedge \Gamma \rightarrow \mathbf{K} \bigwedge_{i=1}^n A_i$ from (1), (2) and transitivity
- (4) $\bigwedge_{i=1}^n A_i \rightarrow B$ assumption
- (5) $(\bigwedge_{i=1}^n A_i \rightarrow B) \rightarrow \mathbf{K}(\bigwedge_{i=1}^n A_i \rightarrow B)$ co-reflection
- (6) $\mathbf{K}(\bigwedge_{i=1}^n A_i \rightarrow B)$ from (2), (3) and MP
- (7) $\mathbf{K} \bigwedge_{i=1}^n A_i \rightarrow \mathbf{K}B$ from (6) and normality
- (8) $\bigwedge \Gamma \rightarrow \mathbf{K}B$ from (3), (7) and transitivity

□

Lemma 2. *If $IEL^- \vdash A$, then $NIEL^- \vdash A$.*

Proof. Straightforward derivation of modal axioms in $NIEL^-$. We consider this derivation below using terms. □

At the next step we build the typed lambda-calculus based on $NIEL_{\bigwedge, \rightarrow}^-$ by proof-assignment in rules.

At first, we define lambda-terms and types for this lambda-calculus.

Definition 3. *The set of terms:*

Let \mathbb{V} be the set of variables. The set Λ_K of terms is defined by the grammar:

$$\Lambda_K ::= \mathbb{V} \mid (\lambda \Lambda. \Lambda_K) \mid (\Lambda_K \Lambda_K) \mid (\Lambda_K, \Lambda_K) \mid (\pi_1 \Lambda_K) \mid (\pi_2 \Lambda_K) \mid (\text{pure } \Lambda_K) \mid (\text{let pure } \Lambda_K = \Lambda_K \text{ in } \Lambda_K)$$

Definition 4. *The set of types:*

Let \mathbb{T} be the set of atomic types. The set \mathbb{T}_K of types with applicative functor K is generated by the grammar:

$$\mathbb{T}_K ::= \mathbb{T} \mid (\mathbb{T}_K \rightarrow \mathbb{T}_K) \mid (\mathbb{T}_K \times \mathbb{T}_K) \mid (K\mathbb{T}_K) \quad (1)$$

Context, domain of context and range of context are defined standardly [11][12].

Our type system is based on the Curry-style typing rules:

Definition 5. *Modal typed lambda calculus λK based on $NIEL_{\bigwedge, \rightarrow}^-$:*

$$\frac{}{\Gamma, x : A \vdash x : A} \text{ ax}$$

$$\begin{array}{c}
\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \rightarrow B} \rightarrow_i \qquad \frac{\Gamma \vdash f : A \rightarrow B \quad \Gamma \vdash x : A}{\Gamma \vdash fx : B} \rightarrow_e \\
\\
\frac{\Gamma \vdash M : A \quad \Gamma \vdash N : B}{\Gamma \vdash \langle x, y \rangle : A \times B} \times_i \qquad \frac{\Gamma \vdash M : A_1 \times A_2}{\Gamma \vdash \pi_i M : A_i} \times_e, i \in \{1, 2\} \\
\\
\frac{\Gamma \vdash x : A}{\Gamma \vdash \mathbf{pure} \ x : \mathbf{KA}} \mathbf{K}_I \qquad \frac{\Gamma \vdash \vec{M} : \mathbf{KA} \quad \vec{x} : \vec{A} \vdash M : B}{\Gamma \vdash \mathbf{let} \ \mathbf{pure} \ \vec{x} = \vec{M} \ \mathbf{in} \ M : \mathbf{KB}} \mathbf{let}_{\mathbf{K}}
\end{array}$$

\mathbf{K}_I -typing rule is the same as \bigcirc -introduction in lax logic (also known as monadic metalanguage [17]) and in typed lambda-calculus which is derived by proof-assignment for lax-logic proofs. \mathbf{K}_I allows to inject an object of type α into the functor. \mathbf{K}_I reflects the Haskell method **pure** for Applicative class. It plays the same role as the **return** method in Monad class.

$\mathbf{let}_{\mathbf{K}}$ is similar to \square_I -rule in typed lambda calculus for intuitionistic normal modal logic \mathbf{IK} , which is described in [19].

Here are some examples of derivation trees.

$$\begin{array}{c}
\frac{x : A \vdash x : A}{x : A \vdash \mathbf{pure} \ x : \mathbf{KA}} \mathbf{K}_I \\
\hline
\vdash (\lambda x. \mathbf{pure} \ x) : A \rightarrow \mathbf{KA} \rightarrow_i \\
\\
\frac{f : \mathbf{K}(A \rightarrow B) \vdash f : \mathbf{K}(A \rightarrow B) \quad x : \mathbf{KA} \vdash x : \mathbf{KA} \quad \frac{g : A \rightarrow B \quad y : A}{g : A \rightarrow B, y : A \vdash gy : B}}{f : \mathbf{K}(A \rightarrow B), x : \mathbf{KA} \vdash \mathbf{let} \ \mathbf{pure} \ \langle g, y \rangle = \langle f, x \rangle \ \mathbf{in} \ gy : \mathbf{KB}} \\
\hline
\frac{f : \mathbf{K}(A \rightarrow B) \vdash \lambda x. \mathbf{let} \ \mathbf{pure} \ \langle g, y \rangle = \langle f, x \rangle \ \mathbf{in} \ gy : \mathbf{KA} \rightarrow \mathbf{KB}}{\vdash \lambda f. \lambda x. \mathbf{let} \ \mathbf{pure} \ \langle g, y \rangle = \langle f, x \rangle \ \mathbf{in} \ gy : \mathbf{K}(A \rightarrow B) \rightarrow \mathbf{KA} \rightarrow \mathbf{KB}} \\
\\
\frac{f : A \rightarrow B \vdash f : A \rightarrow B \quad \frac{g : A \rightarrow B \quad y : A}{g : A \rightarrow B, y : A \vdash gy : B}}{f : A \rightarrow B \vdash \mathbf{pure} \ f : \mathbf{K}(A \rightarrow B) \quad x : \mathbf{KA} \vdash x : \mathbf{KA}} \\
\hline
\frac{f : A \rightarrow B, x : \mathbf{KA} \vdash \mathbf{let} \ \mathbf{pure} \ \langle g, y \rangle = \langle \mathbf{pure} \ f, x \rangle \ \mathbf{in} \ gy : \mathbf{KB}}{f : A \rightarrow B \vdash \lambda x. \mathbf{let} \ \mathbf{pure} \ \langle g, y \rangle = \langle \mathbf{pure} \ f, x \rangle \ \mathbf{in} \ gy : \mathbf{KA} \rightarrow \mathbf{KB}} \\
\hline
\vdash \lambda f. \lambda x. \mathbf{let} \ \mathbf{pure} \ \langle g, y \rangle = \langle \mathbf{pure} \ f, x \rangle \ \mathbf{in} \ gy : (A \rightarrow B) \rightarrow \mathbf{KA} \rightarrow \mathbf{KB}
\end{array}$$

Now we define free variables and substitutions. β -reduction, multi-step β -reduction and β -equality are defined standardly:

Definition 6. Set $FV(M)$ of free variables for arbitrary term M :

- 1) $FV(x) = \{x\}$;
- 2) $FV(\lambda x.M) = FV(M) \setminus \{x\}$;
- 3) $FV(MN) = FV(M) \cup FV(N)$;
- 4) $FV((M, N)) = FV(M) \cup FV(N)$;
- 5) $FV(\pi_i p) \subseteq FV(p)$, $i \in \{1, 2\}$;
- 6) $FV(\mathbf{pure} \ M) = FV(M)$;
- 7) $FV(\mathbf{let} \ \mathbf{pure} \ \vec{N} = \vec{M} \ \mathbf{in} \ M) = \bigcup_{i=1}^n FV(M)$, where $n = |\vec{M}|$.

Definition 7. *Substitution:*

- 1) $x[x := N] = N$, $x[y := N] = x$;
- 2) $(MN)[x := N] = M[x := N]N[x := N]$;
- 3) $(\lambda x.M)[x := N] = \lambda x.M[x := N]$;
- 4) $(M, N)[x := P] = (M[x := P], N[x := P])$;
- 5) $(\pi_i M)[x := P] = \pi_i(M[x := P])$, $i \in \{1, 2\}$;
- 6) $(\mathbf{pure} M)[x := P] = \mathbf{pure}(M[x := P])$;
- 7) $(\mathbf{let pure} \vec{N} = \vec{M} \mathbf{in} M)[x := P] = \mathbf{let pure} \vec{N} = (\vec{M}[x := P]) \mathbf{in} M$.

Definition 8. β -reduction and η -reduction rules for $\lambda\mathbf{K}$.

- 1) $(\lambda x.M)N \rightarrow_\beta M[x := N]$;
- 2) $\pi_1 \langle M, N \rangle \rightarrow_\beta M$;
- 3) $\pi_2 \langle M, N \rangle \rightarrow_\beta N$;
- 4) $\mathbf{let pure} \langle \vec{x}, y, \vec{z} \rangle = \langle \vec{M}, \mathbf{let pure} \vec{w} = \vec{N} \mathbf{in} Q, \vec{P} \rangle \mathbf{in} R \rightarrow_\beta \mathbf{let pure} \langle \vec{x}, \vec{w}, \vec{z} \rangle = \langle \vec{M}, \vec{N}, \vec{P} \rangle \mathbf{in} R[y := Q]$
- 5) $\mathbf{pure} ((\lambda x.M)N) \rightarrow_\beta \mathbf{pure}(M[x := N])$;
- 6) $\mathbf{pure} (\pi_i \langle M_1, M_2 \rangle) \rightarrow_\beta \mathbf{pure} M_i$, where $i \in \{1, 2\}$.
- 7) $\mathbf{pure} (\mathbf{let pure} \langle \vec{x}, y, \vec{z} \rangle = \langle \vec{M}, \mathbf{let pure} \vec{w} = \vec{N} \mathbf{in} Q, \vec{P} \rangle \mathbf{in} R) \rightarrow_\beta \mathbf{pure} (\mathbf{let pure} \langle \vec{x}, \vec{w}, \vec{z} \rangle = \langle \vec{M}, \vec{N}, \vec{P} \rangle \mathbf{in} R[y := Q])$
- 8) $\lambda x.f x \rightarrow_\eta f$;
- 9) $\langle \pi_1 P, \pi_2 P \rangle \rightarrow_\eta P$;
- 10) $\mathbf{let pure} x = M \mathbf{in} x \rightarrow_\eta M$;
- 11) $\mathbf{pure} (\lambda x.f x) \rightarrow_\eta \mathbf{pure} f$;
- 12) $\mathbf{pure} (\langle \pi_1 P, \pi_2 P \rangle) \rightarrow_\eta \mathbf{pure} P$;
- 13) $\mathbf{pure} (\mathbf{let pure} x = M \mathbf{in} x) \rightarrow_\eta \mathbf{pure} M$.

3 Basic lemmas

Now we will prove standard lemmas for contexts in type systems³:

Definition 9. *The domain of a context Γ :*

Let $\Gamma = \{x_1 : A_1, \dots, x_n : A_n\}$. Then the domain of Γ , or $\text{dom}(\Gamma)$, is a set $\{x_1, \dots, x_n\}$.

Lemma 3. *If $\Gamma \vdash M : A$, then $\text{FV}(M) \subseteq \text{dom}(\Gamma)$*

Proof. Induction on the derivation of $\Gamma \vdash M : A$. □

Lemma 4. *Generation for $\lambda\mathbf{K}$.*

- 1) $\Gamma \vdash \mathbf{pure} M : \mathbf{K}\alpha$ implies that $\Gamma \vdash M : \alpha$;
- 2) $\Gamma \vdash \mathbf{let pure} \vec{N} = \vec{M} \mathbf{in} M : \mathbf{K}B$ implies that $\Gamma \vdash \vec{M} : \mathbf{K}\vec{A}$ and $\vec{N} : \vec{A} \vdash M : B$.

Proof.

Induction on the derivation of $\Gamma \vdash \mathbf{pure} M : \mathbf{K}\alpha$ and $\Gamma \vdash \mathbf{let pure} \vec{N} = \vec{M} \mathbf{in} M : \mathbf{K}B$ respectively. □

³We will not prove cases with \rightarrow -constructor, they are proved standardly in the same lemmas for simply typed lambda calculus, for example see [11][12][14]. We will consider only modal cases

The next one lemma allows that weakening structural rule is admissable.

Lemma 5. *Weakening for $\lambda\mathbf{K}$.*

Let $\Gamma \vdash M : A$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash M : A$.

Proof.

Induction on derivation of $\Gamma \vdash M : A$. Let us assume $\Gamma \subseteq \Delta$.

1) Let $\Gamma \vdash x : A$, such that $\Gamma = \Delta, x : A$ and $\Theta \subseteq \Gamma$. Let $\Sigma = \Theta \setminus \Gamma$, or, which is the same, $\Sigma = \Theta \setminus \Delta, x : A$, then $\Sigma, \Delta, x : A \vdash x : A$, or, $\Theta \vdash x : A$.

2) Let $\Gamma \vdash \mathbf{pure} M : \mathbf{K}A$ and $\Gamma \subseteq \Theta$.

By generation $\Gamma \vdash M : A$

By hypothesis, $\Theta \vdash M : A$, so $\Theta \vdash \mathbf{pure} M : \mathbf{K}A$ by applying \mathbf{K}_I -rule.

3) Let $\Gamma \vdash \mathbf{let pure} \vec{x} = \vec{M} \text{ in } N : \mathbf{K}B$ and $\Gamma \subseteq \Theta$.

By generation $\Gamma \vdash \vec{M} : \mathbf{K}\vec{A}$ and $\vec{x} : \vec{A} \vdash N : B$.

By assumption $\Theta \vdash \vec{M} : \mathbf{K}\vec{A}$.

Hence $\Theta \vdash \mathbf{let pure} \vec{x} = \vec{M} \text{ in } N : \mathbf{K}B$.

□

Lemma 6. *Considering for $\lambda\mathbf{K}$.*

If $\Gamma \vdash M : \alpha$, then $\Gamma \uparrow FV(M) \vdash M : \alpha$, where $\Gamma \uparrow FV(M)$ is a subcontext of Γ , such that $\text{dom}(\Gamma \uparrow FV(M)) = \text{dom}(\Gamma) \cap FV(M)$.

Proof.

1) Let $\Gamma \vdash x : A$, where $\Gamma = \Delta, x : \alpha$, $x \in \mathbb{V}$.

$FV(x) = \{x\}$, then $\text{dom}(\Gamma) \cap \{x\} = \{x\}$. So $(\Delta, x : A) \uparrow FV(x) = \{x : A\}$, then $x : A \vdash x : A$ by axiom.

2) Let $\Gamma \vdash \mathbf{pure} M : \mathbf{K}A$.

By generation, $\Gamma \vdash M : A$ and $\Gamma \uparrow FV(M) \vdash M : A$ by hypothesis.

So $\Gamma \uparrow FV(M) \vdash \mathbf{pure} M : \mathbf{K}A$ by \mathbf{K}_I .

3) Let $\Gamma \vdash \mathbf{let pure} \vec{x} = \vec{M} \text{ in } N : \mathbf{K}B$.

By generation, $\Gamma \vdash \vec{M} : \mathbf{K}\vec{A}$ and $\vec{x} : \vec{A} \vdash N : B$.

By assumption, $\Gamma \uparrow FV(\vec{M}) \vdash \vec{M} : \vec{A}$.

By $\mathbf{let_K}$, $\Gamma \uparrow FV(\vec{M}) \vdash \mathbf{let pure} \vec{x} = \vec{M} \text{ in } N : \mathbf{K}B$

□

Lemma 7. *If $\Gamma, x : A \vdash M : B$ and $\Gamma \vdash N : A$, then $\Gamma \vdash (M[x := N]) : B$*

Proof.

1) Let $\Gamma, x : A \vdash \mathbf{pure} M : \mathbf{K}B$ and $\Gamma \vdash N : A$.

By generation, $\Gamma, x : A \vdash M : B$.

By assumption, $\Gamma \vdash (M[x := N]) : B$

Then $\Gamma \vdash \mathbf{pure} (M[x := N]) : \mathbf{K}B$ by \mathbf{K}_I .

But $\mathbf{pure} (M[x := N]) = (\mathbf{pure} M[x := N])$ by substitution definition, so $\Gamma \vdash (\mathbf{pure} M[x := N]) : \mathbf{K}B$

2) Let $\Gamma, y : A \vdash \mathbf{let pure} \vec{x} = \vec{M} \text{ in } N : \mathbf{K}B$ and $\Gamma \vdash N : A$.

By generation, $\Gamma, y : A \vdash \vec{M} : \mathbf{K}\vec{A}$ and $\vec{x} : \vec{A} \vdash N : B$.
 By hypothesis, $\Gamma \vdash \vec{M}[x := N] : \mathbf{K}\vec{A}$.
 Hence $\Gamma \vdash \mathbf{let\ pure} \vec{x} = \vec{M}[x := N] \mathbf{in} N : \mathbf{K}B$.

□

Theorem 1. *Subject reduction*

Let $\Gamma \vdash M : \alpha$ and $M \rightarrow_\beta N$, then $\Gamma \vdash N : \alpha$

We consider only modal β -reduction rules. The general statement for \rightarrow_β follows from transitivity of multi-step β -reduction.

Proof.

1) Let $\Gamma \vdash \mathbf{let\ pure} \langle \vec{x}, y, \vec{z} \rangle = \langle \vec{M}, \mathbf{let\ pure} \vec{w} = \vec{N} \mathbf{in} Q, \vec{P} \rangle \mathbf{in} R : \mathbf{K}B$

By generation we have $\Gamma \vdash \vec{M} : \mathbf{K}\vec{A}_1$, $\Gamma \vdash \mathbf{let\ pure} \vec{w} = \vec{N} \mathbf{in} Q : \mathbf{K}\vec{A}_2$,
 $\Gamma \vdash \vec{P} : \mathbf{K}\vec{A}_3$ and $\vec{x} : \vec{A}_1, y : A_2, \vec{z} : \vec{A}_3 \vdash R : B$.

If $\Gamma \vdash \mathbf{let\ pure} \vec{w} = \vec{N} \mathbf{in} Q : \mathbf{K}\vec{A}_2$, then

$\Gamma \vdash \vec{N} : \mathbf{K}\vec{A}_4$ and $\vec{w} : \vec{A}_4 \vdash Q : A_2$.

Then $\vec{x} : \vec{A}_1, \vec{w} : \vec{A}_4, \vec{z} : \vec{A}_3 \vdash R[y := Q] : B$ by substitution lemma and weakening.

Hence $\Gamma \vdash \mathbf{let\ pure} \langle \vec{x}, \vec{w}, \vec{z} \rangle = \langle \vec{M}, \vec{N}, \vec{P} \rangle \mathbf{in} R[y := Q] : \mathbf{K}B$ by $\mathbf{let_K}$.

2) Let $\Gamma \vdash \mathbf{pure}((\lambda x.M)N) : \mathbf{K}B$.

By generation $\Gamma \vdash (\lambda x.M)N : B$, but $\Gamma \vdash M[x := N] : B$, then, by \mathbf{K}_I ,
 $\Gamma \vdash \mathbf{pure}(M[x := N]) : \mathbf{K}B$.

3) Let $\Gamma \vdash \mathbf{pure}(\pi_i \langle M_1, M_2 \rangle) : \mathbf{K}A_i$, where $i \in \{1, 2\}$.

By generation $\Gamma \vdash \pi_i \langle M_1, M_2 \rangle : A_i$ and $\Gamma \vdash M_i : A_i$.

Hence $\Gamma \vdash \mathbf{pure} M_i : \mathbf{K}A_i$ by \mathbf{K}_I .

4) Let $\Gamma \vdash \mathbf{let\ pure} (\langle \vec{x}, y, \vec{z} \rangle = \langle \vec{M}, \mathbf{let\ pure} \vec{w} = \vec{N} \mathbf{in} Q, \vec{P} \rangle \mathbf{in} R) : \mathbf{K}^2B$.

By generation $\Gamma \vdash \mathbf{let\ pure} \langle \vec{x}, y, \vec{z} \rangle = \langle \vec{M}, \mathbf{let\ pure} \vec{w} = \vec{N} \mathbf{in} Q, \vec{P} \rangle \mathbf{in} R : \mathbf{K}B$,

hence $\Gamma \vdash \mathbf{let\ pure} \langle \vec{x}, \vec{w}, \vec{z} \rangle = \langle \vec{M}, \vec{N}, \vec{P} \rangle \mathbf{in} R[y := Q] : \mathbf{K}B$ by the first case.

So $\Gamma \vdash \mathbf{pure}(\mathbf{let\ pure} \langle \vec{x}, \vec{w}, \vec{z} \rangle = \langle \vec{M}, \vec{N}, \vec{P} \rangle \mathbf{in} R[y := Q]) : \mathbf{K}^2B$ by \mathbf{K}_I .

□

4 Strong normalization

We modify and apply Tait's technique of logical relation for modalities. Strong normalization proof with Tait's method for simply typed lambda calculus is described here [13].

Theorem 2. *Let $M \in \Lambda_K$, then any sequence of reduction $M \rightarrow_\beta M_1 \dots$ terminates.*

Proof. We build the smallest of subset of strongly normalizing terms of modal types and show that an arbitrary term belongs to this subset.

Definition 10. The set of strongly computable terms of type $\phi \in \mathbb{T}_K$, SC_ϕ :

- Let $\phi = K\alpha$ and $\alpha \in \mathbb{T}$, then:

$$SC_{K\alpha} = \{M : K\alpha \mid M \text{ is strongly normalizing}\} \quad (2)$$

- Let $\phi = K(\tau \rightarrow \psi)$ and $\tau, \psi \in \mathbb{T}_K$, then:

$$SC_{K(\tau \rightarrow \psi)} = \{M : K(\tau \rightarrow \psi) \mid \forall N \in SC_{K\tau}, M \star N \in SC_{K\psi}\} \quad (3)$$

- Let $\phi = K(\tau_1 \times \tau_2)$ and $\tau_1, \tau_2 \in \mathbb{T}_K$, then:

$$SC_{K(\tau_1 \times \tau_2)} = \{P : K(\tau_1 \times \tau_2) \mid \mathbf{pure}(\lambda x. \pi_i x) \star P \in SC_{K\tau_i}, i \in \{1, 2\}\} \quad (4)$$

Lemma 8.

If $M \in SC_\alpha$, then M is strongly normalizing.

Proof.

- 1) If $M \in SC_{K\alpha}$ and $\alpha \in \mathbb{T}$, then M is strongly normalizing by the definition of $SC_{K\alpha}$.
- 2) Let $M \in SC_{K(\tau \rightarrow \psi)}$, so by every $N \in SC_{K\tau}$, $M \star N \in SC_{K\psi}$, which is strongly normalizing by hypothesis. So M is strongly normalizing.
- 3) Let $M \in SC_{K(\tau_1 \times \tau_2)}$, so $\mathbf{pure}(\lambda x. \pi_i x) \star M \in SC_{K\tau_i}$, $i \in \{1, 2\}$, which are strongly normalizing. So M is strongly normalizing. \square

Lemma 9.

Let $M \rightarrow_\beta M'$ and $M \in SC_\alpha$, then $M' \in SC_\alpha$.

Proof.

- 1) Let $M \rightarrow_\beta M'$ and $M \in SC_{K\alpha}$, where $\alpha \in \mathbb{T}$.
 M has the longest reduction path (which we denote as $p(M)$). So $p(M') < p(M)$, then $M' \in SC_{K\alpha}$.
- 2) Let $M \in SC_{K(\alpha \rightarrow \beta)}$ and $M \rightarrow_\beta M'$. Let $N \in SC_{K\alpha}$. So $M \star N \in SC_{K\beta}$.
If $M \rightarrow_\beta M'$, then $M \star N \rightarrow_\beta M' \star N$ by reduction rule, so $M' \star N \in SC_{K\beta}$ and $M' \in SC_{K(\alpha \rightarrow \beta)}$ by hypothesis.
- 3) Let $M \in SC_{K(\tau_1 \times \tau_2)}$ and $M \rightarrow_\beta M'$.
So $\mathbf{pure}(\lambda x. \pi_i x) \star M \rightarrow_\beta \mathbf{pure}(\lambda x. \pi_i x) \star M'$, $i \in \{1, 2\}$ by reduction rule.
So $\mathbf{pure}(\lambda x. \pi_i x) \star M' \in SC_{K\tau_i}$ and $M' \in SC_{K(\tau_1 \times \tau_2)}$. \square

Definition 11. Neutral term:

We define a term M to be neutral if it has of the next forms:

- 1) $M = x$, where $x \in \mathbb{V}$;
- 2) $M = (PQ)$;
- 3) $M = \pi_i M$, $i \in \{1, 2\}$;
- 4) $M = P \star Q$;
- 5) If M is a neutral, then $\mathbf{pure} M$ is a neutral.

Lemma 10. *Let $M \rightarrow_\beta M'$ and $M' \in SC_\alpha$ for every one-step reduction. So if M' is a neutral, then $M \in SC_\alpha$.*

Proof.

Simple induction on the structure of M' . □

Lemma 11.

Let $x_1 : \phi_1, \dots, x_n : \phi_n \vdash M : \phi$ and for all $i \in \{1, \dots, n\}$, $N_i \in SC_{\phi_i}$, then $(M[x_1 := N_1, \dots, x_n := N_n]) \in SC_\phi$.

Proof.

1) If ϕ is an atomic and M is a variable, then this condition holds straightforwardly.

2) Let $\Gamma = \{x_1 : \phi_1, \dots, x_n : \phi_n\}$, $\Gamma \vdash \mathbf{pure} M : \mathbf{K}\alpha$ and for all $i \in \{1, \dots, n\}$, $N_i \in SC_{\phi_i}$.

Then by $\Gamma \vdash M : \alpha$ by generation and $(M[x_1 := N_1, \dots, x_n := N_n]) \in SC_\alpha$ by induction hypothesis.

Hence, $\Gamma \vdash \mathbf{pure} M : \mathbf{K}\alpha$ and $(\mathbf{pure} M([x_1 := N_1, \dots, x_n := N_n])) \in SC_{\mathbf{K}\alpha}$ by definition of $SC_{\mathbf{K}\alpha}$.

3) Let $\Gamma = \{x_1 : \phi_1, \dots, x_n : \phi_n\}$, $\Gamma : \phi_n \vdash M \star P : \mathbf{K}\beta$ and for all $i \in \{1, \dots, n\}$, $N_i \in SC_{\phi_i}$.

Then $\Gamma \vdash M : \mathbf{K}(\alpha \rightarrow \beta)$, $\Gamma \vdash P : \mathbf{K}\alpha$ by generation.

But by induction hypothesis $M[x_1 := N_1, \dots, x_n := N_n] \in SC_{\mathbf{K}(\alpha \rightarrow \beta)}$ and $P[x_1 := N_1, \dots, x_n := N_n] \in SC_{\mathbf{K}\alpha}$.

Then, by definition of $SC_{\mathbf{K}\beta}$, $((M[x_1 := N_1, \dots, x_n := N_n]) \star (P[x_1 := N_1, \dots, x_n := N_n])) \in SC_{\mathbf{K}\beta}$, i.e. $(M \star N([x_1 := N_1, \dots, x_n := N_n])) \in SC_{\mathbf{K}\beta}$. □

Corollary 1.

If $\vdash M : \alpha$, then M is strongly normalizing.

Proof. $M \in SC_\alpha$ by Lemma 10, so M is strongly normalizing. □

5 Confluence

In the confluence proof (below) we treat the cases with **pure** and \star similar to [15] [18].

Definition 12. *Alphabet for the labelled terms:*

variables: $x, y, z, x_1, y_1, z_1, \dots$;

lambdas: $\lambda, \lambda_0, \lambda_1, \lambda_2, \dots$;

constructors for an applicative functor: **pure**, \star ;

parentheses $(,)$.

Definition 13. The set of labelled terms Λ'_K inductively defined as a set of words on the alphabet described above:

- 1) $x \in \Lambda'_K$;
- 2) If $M \in \Lambda'_K$, then $(\lambda x.M) \in \Lambda'_K$;
- 3) If $M, N \in \Lambda'_K$, then $(MN) \in \Lambda'_K$;
- 4) If $M \in \Lambda'_K$, then $\mathbf{pure} M \in \Lambda'_K$;
- 5) If $M, N \in \Lambda'_K$, then $M \star N \in \Lambda'_K$;
- 6) If $M, N \in \Lambda'_K$, then for all $i \in \mathbb{N}$, $((\lambda_i x.M)N) \in \Lambda'_K$.

Definition 14. Erasing map

Erasing map is a map $|\cdot| : \Lambda'_K \rightarrow \Lambda_K$, such that:

- 1) $|x| = x$;
- 2) $|(\lambda x.M)| = \lambda x.|M|$;
- 3) $|(MN)| = |M||N|$;
- 4) $|(\mathbf{pure} M)| = \mathbf{pure} |M|$;
- 5) $|M \star N| = |M| \star |N|$;
- 6) $|((\lambda_i x.M)N)| = (\lambda x.|M|)|N|$

Example 1.

$$|\mathbf{pure} ((\lambda_i x.M)N) \star P| = \mathbf{pure} (\lambda x.|M|)|N| \star |P|$$

Definition 15. Substitution for Λ'_K :

- 1) $x[x := N] = N$, $x[y := N] = x$;
- 2) $(MN)[x := N] = M[x := N]N[x := N]$;
- 3) $(\lambda x.M)[x := N] = \lambda x.M[x := N]$;
- 4) $(\mathbf{pure} M)[x := P] = \mathbf{pure} (M[x := P])$;
- 5) $(M \star N)[x := P] = (M[x := P]) \star (N[x := P])$;
- 6) $(\lambda_i x.M)N[y := P] = (\lambda_i x.M[y := P])(N[y := P])$.

Definition 16. One-step reduction $\rightarrow_{\beta'}$ for Λ'_K :

- 1) $(\lambda x.M)N \rightarrow_{\beta'} M[x := N]$;
- 2) $\mathbf{pure} (\lambda x.x) \star M \rightarrow_{\beta'} M$;
- 3) $\mathbf{pure} (\lambda f g x.f(gx)) \star M \star N \star P \rightarrow_{\beta'} M \star (N \star P)$;
- 4) $(\mathbf{pure} M) \star (\mathbf{pure} N) \rightarrow_{\beta'} \mathbf{pure} (MN)$;
- 5) $M \star (\mathbf{pure} N) \rightarrow_{\beta'} \mathbf{pure} (\lambda f.fN) \star M$;
- 6) $(\lambda_i x.M)N \rightarrow_{\beta'} M[x := N]$.

Multi-step reduction $\twoheadrightarrow_{\beta'}$ is a reflexive-transitive closure of $\rightarrow_{\beta'}$.

Definition 17. Let us define a map $\phi : \Lambda'_K \rightarrow \Lambda_K$ inductively as follows:

- 1) $\phi(x) = x$;
- 2) $\phi(MN) = \phi(M)\phi(N)$;
- 3) $\phi(\lambda x.M) = \lambda x.\phi(M)$;
- 4) $\phi(\mathbf{pure} M) = \mathbf{pure} (\phi(M))$;
- 5) $\phi(M \star N) = \phi(M) \star \phi(N)$;
- 6) $\phi((\lambda_i x.M)N) = \phi(M)[x := \phi(N)]$.

Example 2.

$$\phi(\mathbf{pure} ((\lambda_i x.M)N) \star P) = \mathbf{pure} (\phi(M)[x := \phi(N)]) \star \phi(P)$$

Lemma 12.

- 1) Let $M, N \in \Lambda'_K$ and $|M| \twoheadrightarrow_{\beta} |N|$, then $M \twoheadrightarrow_{\beta'} N$.
- 2) Let $M, N \in \Lambda'_K$ and $M \twoheadrightarrow_{\beta'} N$, then $|M| \twoheadrightarrow_{\beta} |N|$.

Proof.

Induction on the generation of \rightarrow_β ($\rightarrow_{\beta'}$).

1) Let us consider homomorphism rule. The rest applicative reduction rules are considered similary.

Let $(\mathbf{pure} M') \star (\mathbf{pure} N'), \mathbf{pure} (M'N') \in \Lambda'_K$.

So $|(\mathbf{pure} M') \star (\mathbf{pure} N')| = (\mathbf{pure} |M'|) \star (\mathbf{pure} |N'|)$ and $|\mathbf{pure} (M'N')| = \mathbf{pure} (|M'| |N'|)$.

By reduction rule, $(\mathbf{pure} |M'|) \star (\mathbf{pure} |N'|) \rightarrow_\beta \mathbf{pure} (|M'| |N'|)$.

But $(\mathbf{pure} M') \star (\mathbf{pure} N') \rightarrow_{\beta'} \mathbf{pure} (M'N')$ by reduction rule for $\rightarrow_{\beta'}$.

2) Let us consider interchange rule.

Let $M \star (\mathbf{pure} N), \mathbf{pure} (\lambda f.fN) \star M \in \Lambda'_K$ and $M \star (\mathbf{pure} N) \rightarrow_{\beta'} \mathbf{pure} (\lambda f.fN) \star M$.

But $|M \star (\mathbf{pure} N)| = |M| \star (\mathbf{pure} |N|)$ and $|\mathbf{pure} (\lambda f.fN) \star M| = \mathbf{pure} (\lambda f.f |N|) \star |M|$.

So $|M| \star (\mathbf{pure} |N|) \rightarrow_\beta \mathbf{pure} (\lambda f.f |N|) \star |M|$ by β -reduction rule.

It is easy to see, that the statement for $\rightarrow_{\beta'}$ and \rightarrow_β immedeatly follows from transitivity of multi-step rediction for labelled terms and for usual terms respectively.

□

Lemma 13.

$$\phi(M[x := N]) = \phi(M)[x := \phi(N)].$$

Proof. Induction on M .

1) Let $M = x$. Then $\phi(x[x := N]) = \phi(N)$.

On the other hand, $\phi(x)[x := \phi(N)] = x[x := \phi(N)] = \phi(N)$.

So $\phi(x[x := N]) = \phi(x)[x := \phi(N)]$.

2) Let $M = y$ and $y \neq x$. Then $\phi(y[x := N]) = \phi(y) = y$.

But $\phi(y)[x := \phi(N)] = y[x := \phi(N)] = y$.

Therefore $\phi(y[x := N]) = \phi(y)[x := \phi(N)]$.

3) Let $M = \mathbf{pure} M'$. Then $\phi(\mathbf{pure} M'[x := N]) = \mathbf{pure} \phi(M'[x := N])$.

By hypothesis, $\mathbf{pure} (\phi(M'[x := N])) = \mathbf{pure} (\phi(M')[x := \phi(N)])$, which is $(\mathbf{pure} \phi(M'))[x := \phi(N)]$ by substitution definition.

4) Let $M = M' \star N'$. So $\phi((M' \star N')[x := N]) = \phi(M'[x := N] \star N'[x := N])$.

By definition of ϕ ,

$$\phi(M'[x := N] \star N'[x := N]) = \phi(M'[x := N]) \star \phi(N'[x := N]).$$

But by induction hypothesis,

$$\phi(M'[x := N]) = \phi(M')[x := \phi(N)] \text{ and }$$

$$\phi(N'[x := N]) = \phi(N')[x := \phi(N)].$$

Hence,

$$\phi(M'[x := N]) \star \phi(N'[x := N]) = \phi(M')[x := \phi(N)] \star \phi(N')[x := \phi(N)].$$

So,

$$\phi(M')[x := \phi(N)] \star \phi(N')[x := \phi(N)] = (\phi(M') \star \phi(N'))[x := \phi(N)].$$

And by definition of ϕ , $(\phi(M') \star \phi(N'))[x := \phi(N)] = \phi(M' \star N')[x := \phi(N)]$.

□

Lemma 14.

Let $M, N \in \Lambda'_K$ and $M \rightarrow_{\beta'} N$, then $\phi(M) \rightarrow_{\beta} \phi(N)$.

Proof.

1) Let $\mathbf{pure}(\lambda x.x) \star M, M \in \Lambda'_K$ and $\mathbf{pure}(\lambda x.x) \star M \rightarrow_{\beta'} M$.

But $\phi(\mathbf{pure}(\lambda x.x) \star M) = \mathbf{pure}(\lambda x.x) \star \phi(M)$.

So $\mathbf{pure}(\lambda x.x) \star \phi(M) \rightarrow_{\beta} \phi(M)$ by β -reduction rule.

2) Let $\mathbf{pure}(\lambda f g x.f(gx)) \star M \star N \star P, M \star (N \star P) \in \Lambda'_K$ and $\mathbf{pure}(\lambda f g x.f(gx)) \star M \star N \star P \rightarrow_{\beta'} M \star (N \star P)$.

By the definition of ϕ :

$\phi(\mathbf{pure}(\lambda f g x.f(gx)) \star M \star N \star P) = \mathbf{pure}(\lambda f g x.f(gx)) \star \phi(M) \star \phi(N) \star \phi(P);$
 $M \star (N \star P) = \phi(M) \star (\phi(N) \star \phi(P)).$

Hence, $\mathbf{pure}(\lambda f g x.f(gx)) \star \phi(M) \star \phi(N) \star \phi(P) \rightarrow_{\beta} \phi(M) \star (\phi(N) \star \phi(P))$ by β -reduction rule.

3) Let $(\mathbf{pure} M) \star (\mathbf{pure} N), \mathbf{pure}(MN) \in \Lambda'_K$ and $(\mathbf{pure} M) \star (\mathbf{pure} N) \rightarrow_{\beta} \mathbf{pure}(MN)$.

By the definition of ϕ :

$\phi((\mathbf{pure} M) \star (\mathbf{pure} N)) = (\mathbf{pure} \phi(M)) \star (\mathbf{pure} \phi(N));$

$\phi(\mathbf{pure}(MN)) = \mathbf{pure}(\phi(M)\phi(N)).$

So, by reduction rule, $(\mathbf{pure} \phi(M)) \star (\mathbf{pure} \phi(N)) \rightarrow_{\beta} \mathbf{pure}(\phi(M)\phi(N)).$

4) Let $M \star (\mathbf{pure}), \mathbf{pure}(\lambda f.fN) \star M$ and $M \star (\mathbf{pure} N) \rightarrow_{\beta'} (\lambda f.fN) \star M$.

$\phi(M \star (\mathbf{pure} N)) = \phi(M) \star (\mathbf{pure} \phi(N))$

$\phi((\lambda f.fN) \star M) = (\lambda f.f\phi(N)) \star \phi(M).$

So, $\phi(M) \star (\mathbf{pure} \phi(N)) \rightarrow_{\beta} \mathbf{pure}(\lambda f.f\phi(N)) \star \phi(M).$ □

Lemma 15.

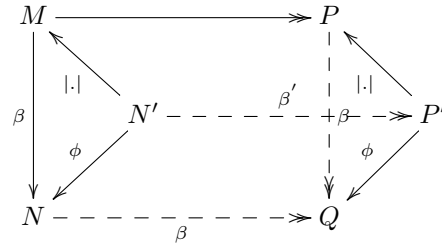
Let $M \in \Lambda'_K$. Then $|M| \rightarrow_{\beta} \phi(M)$.

Proof. Induction on the structure of M . □

Lemma 16. Strip lemma.

If $M \rightarrow_{\beta} N$ and $M \rightarrow_{\beta} P$. Then there exists some term Q , such that $N \rightarrow_{\beta} Q$ and $P \rightarrow_{\beta} Q$.

Proof. Proof is similar to [15] [18]. We build the following diagram



□

which is commutes by lemmas 11 – 14.

Theorem 3. *Confluence.*

If $M \rightarrow_\beta N$ and $M \rightarrow_\beta P$. Then there exists some term Q , such that $N \rightarrow_\beta Q$ and $P \rightarrow_\beta Q$.

Proof.

By unfolding $M \rightarrow_\beta N$ as the sequence of one-step reductions $M \rightarrow_\beta M_1 \rightarrow_\beta \dots \rightarrow_\beta M_n \rightarrow_\beta N$ and applying strip lemma on every step. □

6 Acknowledgement.

Author would like to thank his supervisor V.L.Vasukov, V.N. Krupski for general idea and wise advice, V. de Paiva, V.I. Shalack, A.V. Rodin and M. Taldykin for discussing, critics and consulting.

References

- [1] Artemov S. and Protopopescu T., “Intuitionistic Epistemic Logic”, *The Review of Symbolic Logic*, 2016, vol. 9, no 2. pp. 266-298.
- [2] Krupski V. N. and Yatmanov A., “Sequent Calculus for Intuitionistic Epistemic Logic IEL”, *Logical Foundations of Computer Science: International Symposium, LFCs 2016, Deerfield Beach, FL, USA, January 4-7, 2016. Proceedings*, 2016, pp. 187-201.
- [3] Haskell Language. // URL: <https://www.haskell.org>. (Date: 1.08.2017)
- [4] Idris. A Language with Dependent Types.// URL:<https://www.idris-lang.org>. (Date: 1.08.2017)
- [5] Purescript. A strongly-typed functional programming language that compiles to JavaScript. URL: <http://www.purescript.org>. (Date: 1.08.2017)
- [6] Elm. A delightful language for reliable webapps. // URL: <http://elm-lang.org>. (Date: 1.08.2017)
- [7] Hackage, “The base package” // URL: <https://hackage.haskell.org/package/base-4.10.0.0> (Date: 1.08.2017)
- [8] Lipovaca M, “Learn you a Haskell for Great Good!”. //URL: <http://learnyouahaskell.com/chapters> (Date: 1.08.2017)
- [9] McBride C. and Paterson R., “Applicative programming with effects”, *Journal of Functional Programming*, 2008, vol. 18, no 01. pp 1-13.
- [10] McBride C. and Paterson R, “Functional Pearl. Idioms: applicative programming with effects”, *Journal of Functional Programming*, 2005. vol. 18, no 01. pp 1-20.
- [11] R. Nederpelt and H. Geuvers, “Type Theory and Formal Proof: An Introduction”. *Cambridge University Press*, New York, NY, USA, 2014. pp. 436.

- [12] Sorensen M. H. and Urzyczyn P, “Lectures on the Curry-Howard isomorphism”, *Studies in Logic and the Foundations of Mathematics*, vol. 149, *Elsevier Science*, 1998. pp 261.
- [13] Pierce B. C., “Types and Programming Languages”. *Cambridge, Mass: The MIT Press*, 2002. pp. 605.
- [14] Girard J.-Y., Taylor P. and Lafont Y, “Proofs and Types”, *Cambridge University Press*, New York, NY, USA, 1989. pp. 175.
- [15] Barendregt. H. P., “Lambda calculi with types” // Abramsky S., Gabbay Dov M., and S. E. Maibaum, “Handbook of logic in computer science (vol. 2), Osborne Handbooks Of Logic In Computer Science”, Vol. 2. *Oxford University Press, Inc.*, New York, NY, USA, 1993. pp 117-309.
- [16] Hindley J. Roger, “Basic Simple Type Theory”. *Cambridge University Press*, New York, NY, USA, 1997. pp. 185.
- [17] Pfenning F. and Davies R., “A judgmental reconstruction of modal logic”, *Mathematical Structures in Computer Science*, vol. 11, no 4, 2001, pp. 511-540.
- [18] H.P. Barendregt. The Lambda Calculus — Its Syntax and Semantics. *Studies in Logic and the Foundations of Mathematics*, vol. 103. Amsterdam: North-Holland, 1985.
- [19] Yoshihiko KAKUTANI, A Curry-Howard Correspondence for Intuitionistic Normal Modal Logic, Computer Software, Released February 29, 2008, Online ISSN , Print ISSN 0289-6540.