

Modal type theory based on the intuitionistic epistemic logic

Abstract

Modal intuitionistic epistemic logic IEL^- was proposed by S.Artemov and T. Protopopescu as the formal foundation for the intuitionistic theory of knowledge. We construct a modal simply typed lambda-calculus which is Curry-Howard isomorphic to IEL^- as formal theory of calculations with applicative functors in functional programming languages like Haskell or Idris. We prove that this typed lambda-calculus has the strong normalization and Church-Rosser properties.

1 Introduction

Modal intuitionistic epistemic logic IEL was proposed by S. Artemov and T. Protopopescu [1]. IEL provides the epistemology and the theory of knowledge as based on BHK-semantics of intuitionistic logic. IEL^- is a variant of IEL , that corresponds to intuitionistic belief. Informally, $\mathbf{K}A$ denotes that A is verified intuitionistically.

Intuitionistic epistemic logic IEL^- is defined with by following axioms and derivation rules:

Definition 1. *Intuitionistic epistemic logic IEL :*

- 1) *IPC axioms;*
 - 2) $\mathbf{K}(A \rightarrow B) \rightarrow (\mathbf{K}A \rightarrow \mathbf{K}B)$ (*normality*);
 - 3) $A \rightarrow \mathbf{K}A$ (*co-reflection*);
- Rule: MP.*

We have the deduction theorem and necessitation rule which is derivable.

V. Krupski and A. Yatmanov provided the sequential calculus for IEL and proved that this calculus is PSPACE-complete [2].

It's not difficult to see that modal axioms in IEL^- and types of the methods of Applicative class in Haskell-like languages (which is described below) are syntactically similar and we are going to show that this coincidence has a non-trivial computational meaning.

Functional programming languages such as Haskell [3], Idris [4], Purescript [5] or Elm [6] have special type classes¹ for calculations with container types like `Functor` and `Applicative`²:

¹Type class in Haskell is a general interface for special group of datatypes.

²Reader may read more about container types in the Haskell standard library documentation[7] or in the next one textbook [8]

```

class Functor f where
  fmap :: (a -> b) -> f a -> f b

class Functor f => Applicative f where
  pure :: a -> f a
  (<*>) :: f (a -> b) -> f a -> f b

```

By *container* (or *computational context*) type we mean some type-operator f , where f is a “function” from $*$ to $*$: type operator takes a simple type (which has kind $*$) and returns another simple type type with kind $*$. For more detailed description of the type system with kinds used in Haskell see [12].

The main goal of our research is a relationship between intuitionistic epistemic logic IEL^- and functional programming with effects. We show that relationship by building the type system (which is called $\lambda_{\mathbf{K}}$) which is Curry-Howard isomorphic to IEL^- . So we will consider \mathbf{K} -modality as an arbitrary applicative functor.

λK consists of the rules for simply typed lambda-calculus and special typing rules for lifting types into the applicative functor \mathbf{K} . We assume that our type system will axiomatize the simplest case of computation with effects with one container. We provide proof-theoretical view on this kind of computations in functional programming and prove strong normalization and confluence.

2 Typed lambda-calculus based on IEL^-

At first we define the natural deduction for IEL^- with \mathbf{K} -modality and binary connectives \rightarrow and \wedge (we call that calculus $NIEL_{\wedge, \rightarrow}^-$):

Definition 2. *Natural deduction $NIEL_{\wedge, \rightarrow}^-$ for IEL^- with \rightarrow and \wedge :*

$$\begin{array}{c}
\frac{}{\Gamma, A \vdash A} \text{ ax} \\
\\
\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \rightarrow_i \qquad \frac{\Gamma \vdash A \rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B} \rightarrow_i \\
\\
\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge_i \qquad \frac{\Gamma \vdash A_1 \wedge A_2}{\Gamma \vdash A_i} \wedge_e, i \in \{1, 2\} \\
\\
\frac{\Gamma \vdash A}{\Gamma \vdash \mathbf{K}A} \mathbf{K}_I \qquad \frac{\Gamma \vdash \mathbf{K}\vec{A} \quad \vec{A} \vdash B}{\Gamma \vdash \mathbf{K}B}
\end{array}$$

Where $\Gamma \vdash \mathbf{K}\vec{A}$ is a syntax sugar for $\Gamma \vdash \mathbf{K}A_1, \dots, \Gamma \vdash \mathbf{K}A_n$.

Lemma 1. $\Gamma \vdash_{NIEL_{\wedge, \rightarrow}^-} A \Rightarrow IEL^- \vdash \bigwedge \Gamma \rightarrow A$.

Proof. Induction on the derivation.

Let us consider cases with modality.

1) If $\Gamma \vdash_{NIEL_{\wedge, \rightarrow}^-} A$, then $IEL^- \vdash \bigwedge \Gamma \rightarrow \mathbf{K}A$.

- | | | |
|-----|---|----------------------|
| (1) | $\bigwedge \Gamma \rightarrow A$ | assumption |
| (2) | $A \rightarrow \mathbf{K}A$ | co-reflection |
| (3) | $(\bigwedge \Gamma \rightarrow A) \rightarrow ((A \rightarrow \mathbf{K}A) \rightarrow (\bigwedge \Gamma \rightarrow \mathbf{K}A))$ | IPC theorem |
| (4) | $(A \rightarrow \mathbf{K}A) \rightarrow (\bigwedge \Gamma \rightarrow \mathbf{K}A)$ | from (1), (3) and MP |
| (5) | $\bigwedge \Gamma \rightarrow \mathbf{K}A$ | from (2), (4) and MP |

2) If $\Gamma \vdash_{NIEL_{\wedge, \rightarrow}^-} \mathbf{K}\vec{A}$ and $\vec{A} \vdash B$, then $IEL^- \vdash \bigwedge \Gamma \rightarrow \mathbf{K}B$.

- | | | |
|-----|---|--------------------------------|
| (1) | $\bigwedge \Gamma \rightarrow \bigwedge_{i=1}^n \mathbf{K}A_i$ | assumption |
| (2) | $\bigwedge_{i=1}^n \mathbf{K}A_i \rightarrow \mathbf{K} \bigwedge_{i=1}^n A_i$ | IEL theorem |
| (3) | $\bigwedge \Gamma \rightarrow \mathbf{K} \bigwedge_{i=1}^n A_i$ | from (1), (2) and transitivity |
| (4) | $\bigwedge_{i=1}^n A_i \rightarrow B$ | assumption |
| (5) | $(\bigwedge_{i=1}^n A_i \rightarrow B) \rightarrow \mathbf{K}(\bigwedge_{i=1}^n A_i \rightarrow B)$ | co-reflection |
| (6) | $\mathbf{K}(\bigwedge_{i=1}^n A_i \rightarrow B)$ | from (2), (3) and MP |
| (7) | $\mathbf{K} \bigwedge_{i=1}^n A_i \rightarrow \mathbf{K}B$ | from (6) and normality |
| (8) | $\bigwedge \Gamma \rightarrow \mathbf{K}B$ | from (3), (7) and transitivity |

□

Lemma 2. *If $IEL^- \vdash A$, then $NIEL^- \vdash A$.*

Proof. Straightforward derivation of modal axioms in $NIEL^-$. We consider this derivation below using terms. □

At the next step we build the typed lambda-calculus based on $NIEL_{\wedge, \rightarrow}^-$ by proof-assigning in rules.

At first, we define lambda-terms and types for this lambda-calculus.

Definition 3. *The set of terms:*

Let \mathbb{V} be the set of variables. The set Λ_K of terms is defined by the grammar:

$$\Lambda_K ::= \mathbb{V} \mid (\lambda \Lambda. \Lambda_K) \mid (\Lambda_K \Lambda_K) \mid (\Lambda_K, \Lambda_K) \mid (\pi_1 \Lambda_K) \mid (\pi_2 \Lambda_K) \mid (\text{pure } \Lambda_K) \mid (\text{let pure } \Lambda_K = \Lambda_K \text{ in } \Lambda_K)$$

Definition 4. *The set of types:*

Let \mathbb{T} be the set of atomic types. The set \mathbb{T}_K of types with applicative functor K is generated by the grammar:

$$\mathbb{T}_K ::= \mathbb{T} \mid (\mathbb{T}_K \rightarrow \mathbb{T}_K) \mid (\mathbb{T}_K \times \mathbb{T}_K) \mid (K\mathbb{T}_K) \quad (1)$$

Context, domain of context and range of context are defined standardly [11][12].

Our type system is based on the Curry-style typing rules:

Definition 5. *Modal typed lambda calculus λK based on $NIEL_{\wedge, \rightarrow}^-$:*

$$\frac{}{\Gamma, x : A \vdash x : A} \text{ ax}$$

$$\begin{array}{c}
\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x. M : A \rightarrow B} \rightarrow_i \qquad \frac{\Gamma \vdash f : A \rightarrow B \quad \Gamma \vdash x : A}{\Gamma \vdash fx : B} \rightarrow_e \\
\\
\frac{\Gamma \vdash M : A \quad \Gamma \vdash N : B}{\Gamma \vdash \langle x, y \rangle : A \times B} \times_i \qquad \frac{\Gamma \vdash M : A_1 \times A_2}{\Gamma \vdash \pi_i M : A_i} \times_e, i \in \{1, 2\} \\
\\
\frac{\Gamma \vdash x : A}{\Gamma \vdash \mathbf{pure} \ x : \mathbf{KA}} \mathbf{K}_I \qquad \frac{\Gamma \vdash \vec{M} : \mathbf{KA} \quad \vec{x} : \vec{A} \vdash M : B}{\Gamma \vdash \mathbf{let} \ \mathbf{pure} \ \vec{x} = \vec{M} \ \mathbf{in} \ M : \mathbf{KB}} \mathbf{let}_{\mathbf{K}}
\end{array}$$

\mathbf{K}_I -typing rule is the same as \bigcirc -introduction in lax logic (also known as monadic metalanguage [17]) and in typed lambda-calculus which is derived by proof-assignment for lax-logic proofs. \mathbf{K}_I allows to inject an object of type α into the functor. \mathbf{K}_I reflects the Haskell method **pure** for Applicative class. It plays the same role as the **return** method in Monad class.

$\mathbf{let}_{\mathbf{K}}$ is similar to \square_I -rule in typed lambda calculus for intuitionistic normal modal logic \mathbf{IK} , which is described in [19].

Here are some examples of derivation trees.

$$\begin{array}{c}
\frac{x : A \vdash x : A}{x : A \vdash \mathbf{pure} \ x : \mathbf{KA}} \mathbf{K}_I \\
\hline
\vdash (\lambda x. \mathbf{pure} \ x) : A \rightarrow \mathbf{KA} \rightarrow_i \\
\\
\frac{f : \mathbf{K}(A \rightarrow B) \vdash f : \mathbf{K}(A \rightarrow B) \quad x : \mathbf{KA} \vdash x : \mathbf{KA} \quad \frac{g : A \rightarrow B \quad y : A}{g : A \rightarrow B, y : A \vdash gy : B}}{f : \mathbf{K}(A \rightarrow B), x : \mathbf{KA} \vdash \mathbf{let} \ \mathbf{pure} \ \langle g, y \rangle = \langle f, x \rangle \ \mathbf{in} \ gy : \mathbf{KB}} \\
\hline
\frac{f : \mathbf{K}(A \rightarrow B) \vdash \lambda x. \mathbf{let} \ \mathbf{pure} \ \langle g, y \rangle = \langle f, x \rangle \ \mathbf{in} \ gy : \mathbf{KA} \rightarrow \mathbf{KB}}{\vdash \lambda f. \lambda x. \mathbf{let} \ \mathbf{pure} \ \langle g, y \rangle = \langle f, x \rangle \ \mathbf{in} \ gy : \mathbf{K}(A \rightarrow B) \rightarrow \mathbf{KA} \rightarrow \mathbf{KB}} \\
\\
\frac{f : A \rightarrow B \vdash f : A \rightarrow B \quad \frac{g : A \rightarrow B \quad y : A}{g : A \rightarrow B, y : A \vdash gy : B}}{f : A \rightarrow B \vdash \mathbf{pure} \ f : \mathbf{K}(A \rightarrow B) \quad x : \mathbf{KA} \vdash x : \mathbf{KA}} \\
\hline
\frac{f : A \rightarrow B, x : \mathbf{KA} \vdash \mathbf{let} \ \mathbf{pure} \ \langle g, y \rangle = \langle \mathbf{pure} \ f, x \rangle \ \mathbf{in} \ gy : \mathbf{KB}}{f : A \rightarrow B \vdash \lambda x. \mathbf{let} \ \mathbf{pure} \ \langle g, y \rangle = \langle \mathbf{pure} \ f, x \rangle \ \mathbf{in} \ gy : \mathbf{KA} \rightarrow \mathbf{KB}} \\
\hline
\vdash \lambda f. \lambda x. \mathbf{let} \ \mathbf{pure} \ \langle g, y \rangle = \langle \mathbf{pure} \ f, x \rangle \ \mathbf{in} \ gy : (A \rightarrow B) \rightarrow \mathbf{KA} \rightarrow \mathbf{KB}
\end{array}$$

Now we define free variables and substitutions. β -reduction, multi-step β -reduction and β -equality are defined standardly:

Definition 6. Set $FV(M)$ of free variables for arbitrary term M :

- 1) $FV(x) = \{x\}$;
- 2) $FV(\lambda x. M) = FV(M) \setminus \{x\}$;
- 3) $FV(MN) = FV(M) \cup FV(N)$;
- 4) $FV(\langle M, N \rangle) = FV(M) \cup FV(N)$;
- 5) $FV(\pi_i M) \subseteq FV(M)$, $i \in \{1, 2\}$;
- 6) $FV(\mathbf{pure} \ M) = FV(M)$;
- 7) $FV(\mathbf{let} \ \mathbf{pure} \ \vec{N} = \vec{M} \ \mathbf{in} \ M) = \bigcup_{i=1}^n FV(M)$, where $n = |\vec{M}|$.

Definition 7. *Substitution:*

- 1) $x[x := N] = N, x[y := N] = x;$
- 2) $(MN)[x := N] = M[x := N]N[x := N];$
- 3) $(\lambda x.M)[x := N] = \lambda x.M[x := N];$
- 4) $(M, N)[x := P] = (M[x := P], N[x := P]);$
- 5) $(\pi_i M)[x := P] = \pi_i(M[x := P]), i \in \{1, 2\};$
- 6) $(\mathbf{pure} M)[x := P] = \mathbf{pure}(M[x := P]);$
- 7) $(\mathbf{let pure } \vec{x} = \vec{M} \mathbf{ in } M)[y := P] = \mathbf{let pure } \vec{x} = (\vec{M}[y := P]) \mathbf{ in } M.$

Definition 8. β -reduction and η -reduction rules for $\lambda\mathbf{K}$.

- 1) $(\lambda x.M)N \rightarrow_\beta M[x := N];$
- 2) $\pi_1\langle M, N \rangle \rightarrow_\beta M;$
- 3) $\pi_2\langle M, N \rangle \rightarrow_\beta N;$
- 4) $\mathbf{let pure } \langle \vec{x}, y, \vec{z} \rangle = \langle \vec{M}, \mathbf{let pure } \vec{w} = \vec{N} \mathbf{ in } Q, \vec{P} \rangle \mathbf{ in } R \rightarrow_\beta \mathbf{let pure } \langle \vec{x}, \vec{w}, \vec{z} \rangle = \langle \vec{M}, \vec{N}, \vec{P} \rangle \mathbf{ in } R[y := Q]$
- 5) $\mathbf{pure}((\lambda x.M)N) \rightarrow_\beta \mathbf{pure}(M[x := N]);$
- 6) $\mathbf{pure}(\pi_i\langle M_1, M_2 \rangle) \rightarrow_\beta \mathbf{pure} M_i, \text{ where } i \in \{1, 2\}.$
- 7) $\mathbf{pure}(\mathbf{let pure } \langle \vec{x}, y, \vec{z} \rangle = \langle \vec{M}, \mathbf{let pure } \vec{w} = \vec{N} \mathbf{ in } Q, \vec{P} \rangle \mathbf{ in } R) \rightarrow_\beta \mathbf{pure}(\mathbf{let pure } \langle \vec{x}, \vec{w}, \vec{z} \rangle = \langle \vec{M}, \vec{N}, \vec{P} \rangle \mathbf{ in } R[y := Q])$
- 8) $\lambda x.f x \rightarrow_\eta f;$
- 9) $\langle \pi_1 P, \pi_2 P \rangle \rightarrow_\eta P;$
- 10) $\mathbf{let pure } _ = _ \mathbf{ in } N \rightarrow_\eta \mathbf{pure} N;$
- 11) $\mathbf{let pure } x = M \mathbf{ in } x \rightarrow_\eta M;$
- 12) $\mathbf{pure}(\lambda x.f x) \rightarrow_\eta \mathbf{pure} f;$
- 13) $\mathbf{pure}(\langle \pi_1 P, \pi_2 P \rangle) \rightarrow_\eta \mathbf{pure} P;$
- 14) $\mathbf{pure}(\mathbf{let pure } x = M \mathbf{ in } x) \rightarrow_\eta \mathbf{pure} M;$
- 15) $\mathbf{pure}(\mathbf{let pure } _ = _ \mathbf{ in } N) \rightarrow_\eta \mathbf{pure}(\mathbf{pure} N).$

3 Basic lemmas

Now we will prove standard lemmas for contexts in type systems³:

Definition 9. *The domain of a context Γ :*

Let $\Gamma = \{x_1 : A_1, \dots, x_n : A_n\}$. Then the domain of Γ , or $\text{dom}(\Gamma)$, is a set $\{x_1, \dots, x_n\}$.

Lemma 3. *If $\Gamma \vdash M : A$, then $FV(M) \subseteq \text{dom}(\Gamma)$*

Proof. Induction on the derivation of $\Gamma \vdash M : A$. □

Lemma 4. *Generation for $\lambda\mathbf{K}$.*

- 1) $\Gamma \vdash \mathbf{pure} M : \mathbf{K}\alpha$ implies that $\Gamma \vdash M : \alpha;$
- 2) $\Gamma \vdash \mathbf{let pure } \vec{N} = \vec{M} \mathbf{ in } M : \mathbf{K}B$ implies that $\Gamma \vdash \vec{M} : \mathbf{K}\vec{A}$ and $\vec{N} : \vec{A} \vdash M : B.$

³We will not prove cases with \rightarrow -constructor, they are proved standardly in the same lemmas for simply typed lambda calculus, for example see [11][12][14]. We will consider only modal cases

Proof.

Induction on the derivation of $\Gamma \vdash \mathbf{pure} M : \mathbf{K}\alpha$ and $\Gamma \vdash \mathbf{let pure} \vec{N} = \vec{M} \mathbf{in} M : \mathbf{K}B$ respectively. \square

The next one lemma allows that weakening structural rule is admissable.

Lemma 5. *Weakening for $\lambda\mathbf{K}$.*

Let $\Gamma \vdash M : A$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash M : A$.

Proof.

Induction on derivation of $\Gamma \vdash M : A$. Let us assume $\Gamma \subseteq \Delta$.

1) Let $\Gamma \vdash x : A$, such that $\Gamma = \Delta, x : A$ and $\Theta \subseteq \Gamma$. Let $\Sigma = \Theta \setminus \Gamma$, or, which is the same, $\Sigma = \Theta \setminus \Delta, x : A$, then $\Sigma, \Delta, x : A \vdash x : A$, or, $\Theta \vdash x : A$.

2) Let $\Gamma \vdash \mathbf{pure} M : \mathbf{K}A$ and $\Gamma \subseteq \Theta$.

By generation $\Gamma \vdash M : A$

By hypothesis, $\Theta \vdash M : A$, so $\Theta \vdash \mathbf{pure} M : \mathbf{K}A$ by applying \mathbf{K}_I -rule.

3) Let $\Gamma \vdash \mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} N : \mathbf{K}B$ and $\Gamma \subseteq \Theta$.

By generation $\Gamma \vdash \vec{M} : \mathbf{K}\vec{A}$ and $\vec{x} : \vec{A} \vdash N : B$.

By assumption $\Theta \vdash \vec{M} : \mathbf{K}\vec{A}$.

Hence $\Theta \vdash \mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} N : \mathbf{K}B$. \square

Lemma 6. *Considering for $\lambda\mathbf{K}$.*

If $\Gamma \vdash M : \alpha$, then $\Gamma \uparrow FV(M) \vdash M : \alpha$, where $\Gamma \uparrow FV(M)$ is a subcontext of Γ , such that $\text{dom}(\Gamma \uparrow FV(M)) = \text{dom}(\Gamma) \cap FV(M)$.

Proof.

1) Let $\Gamma \vdash x : A$, where $\Gamma = \Delta, x : A$, $x \in \mathbb{V}$.

$FV(x) = \{x\}$, then $\text{dom}(\Gamma) \cap \{x\} = \{x\}$. So $(\Delta, x : A) \uparrow FV(x) = \{x : A\}$, then $x : A \vdash x : A$ by axiom.

2) Let $\Gamma \vdash \mathbf{pure} M : \mathbf{K}A$.

By generation, $\Gamma \vdash M : A$ and $\Gamma \uparrow FV(M) \vdash M : A$ by hypothesis.

So $\Gamma \uparrow FV(M) \vdash \mathbf{pure} M : \mathbf{K}A$ by \mathbf{K}_I .

3) Let $\Gamma \vdash \mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} N : \mathbf{K}B$.

By generation, $\Gamma \vdash \vec{M} : \mathbf{K}\vec{A}$ and $\vec{x} : \vec{A} \vdash N : B$.

By assumption, $\Gamma \uparrow FV(\vec{M}) \vdash \vec{M} : \vec{A}$.

By $\mathbf{let_K}$, $\Gamma \uparrow FV(\vec{M}) \vdash \mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} N : \mathbf{K}B$ \square

Lemma 7. *If $\Gamma, x : A \vdash M : B$ and $\Gamma \vdash N : A$, then $\Gamma \vdash (M[x := N]) : B$*

Proof.

1) Let $\Gamma, x : A \vdash \mathbf{pure} M : \mathbf{K}B$ and $\Gamma \vdash N : A$.

By generation, $\Gamma, x : A \vdash M : B$.

By assumption, $\Gamma \vdash (M[x := N]) : B$

Then $\Gamma \vdash \mathbf{pure} (M[x := N]) : \mathbf{KB}$ by \mathbf{K}_I .
 But $\mathbf{pure} (M[x := N]) = (\mathbf{pure} M[x := N])$ by substitution definition, so
 $\Gamma \vdash (\mathbf{pure} M[x := N]) : \mathbf{KB}$

2) Let $\Gamma, y : A \vdash \mathbf{let pure } \vec{x} = \vec{M} \mathbf{ in } N : \mathbf{KB}$ and $\Gamma \vdash N : A$.
 By generation, $\Gamma, y : A \vdash \vec{M} : \mathbf{K}\vec{A}$ and $\vec{x} : \vec{A} \vdash N : B$.
 By hypothesis, $\Gamma \vdash \vec{M}[x := N] : \mathbf{K}\vec{A}$.
 Hence $\Gamma \vdash \mathbf{let pure } \vec{x} = \vec{M}[x := N] \mathbf{ in } N : \mathbf{KB}$.

□

Theorem 1. *Subject reduction*

- i) Let $\Gamma \vdash M : A$ and $M \rightarrow_\beta N$, then $\Gamma \vdash N : A$
- ii) Let $\Gamma \vdash M : A$ and $M \rightarrow_\eta N$, then $\Gamma \vdash N : A$

We consider only modal β -reduction rules. The general statement for \rightarrow_β follows from transitivity of multi-step β -reduction.

Proof.

- i) For multistep β -reduction:

- 1) Let $\Gamma \vdash \mathbf{let pure } \langle \vec{x}, y, \vec{z} \rangle = \langle \vec{M}, \mathbf{let pure } \vec{w} = \vec{N} \mathbf{ in } Q, \vec{P} \rangle \mathbf{ in } R : \mathbf{KB}$
 By generation we have $\Gamma \vdash \vec{M} : \mathbf{K}\vec{A}_1$, $\Gamma \vdash \mathbf{let pure } \vec{w} = \vec{N} \mathbf{ in } Q : \mathbf{K}\vec{A}_2$,
 $\Gamma \vdash \vec{P} : \mathbf{K}\vec{A}_3$ and $\vec{x} : \vec{A}_1, y : A_2, \vec{z} : \vec{A}_3 \vdash R : B$.
 If $\Gamma \vdash \mathbf{let pure } \vec{w} = \vec{N} \mathbf{ in } Q : \mathbf{K}\vec{A}_2$, then
 $\Gamma \vdash \vec{N} : \mathbf{K}\vec{A}_4$ and $\vec{w} : \vec{A}_4 \vdash Q : A_2$.
 Then $\vec{x} : \vec{A}_1, \vec{w} : \vec{A}_4, \vec{z} : \vec{A}_3 \vdash R[y := Q] : B$ by substitution lemma and weakening.
 Hence $\Gamma \vdash \mathbf{let pure } \langle \vec{x}, \vec{w}, \vec{z} \rangle = \langle \vec{M}, \vec{N}, \vec{P} \rangle \mathbf{ in } R[y := Q] : \mathbf{KB}$ by $\mathbf{let_K}$.
- 2) Let $\Gamma \vdash \mathbf{pure}((\lambda x.M)N) : \mathbf{KB}$.
 By generation $\Gamma \vdash (\lambda x.M)N : B$, but $\Gamma \vdash M[x := N] : B$, then, by \mathbf{K}_I ,
 $\Gamma \vdash \mathbf{pure} (M[x := N]) : \mathbf{KB}$.
- 3) Let $\Gamma \vdash \mathbf{pure}(\pi_i \langle M_1, M_2 \rangle) : \mathbf{K}A_i$, where $i \in \{1, 2\}$.
 By generation $\Gamma \vdash \pi_i \langle M_1, M_2 \rangle : A_i$ and $\Gamma \vdash M_i : A_i$.
 Hence $\Gamma \vdash \mathbf{pure} M_i : \mathbf{K}A_i$ by \mathbf{K}_I .

- 4) Let $\Gamma \vdash \mathbf{let pure } (\langle \vec{x}, y, \vec{z} \rangle = \langle \vec{M}, \mathbf{let pure } \vec{w} = \vec{N} \mathbf{ in } Q, \vec{P} \rangle \mathbf{ in } R) : \mathbf{K}^2 B$.
 By generation $\Gamma \vdash \mathbf{let pure } \langle \vec{x}, y, \vec{z} \rangle = \langle \vec{M}, \mathbf{let pure } \vec{w} = \vec{N} \mathbf{ in } Q, \vec{P} \rangle \mathbf{ in } R : \mathbf{KB}$,
 hence $\Gamma \vdash \mathbf{let pure } \langle \vec{x}, \vec{w}, \vec{z} \rangle = \langle \vec{M}, \vec{N}, \vec{P} \rangle \mathbf{ in } R[y := Q] : \mathbf{KB}$ by the first case.
 So $\Gamma \vdash \mathbf{pure} (\mathbf{let pure } \langle \vec{x}, \vec{w}, \vec{z} \rangle = \langle \vec{M}, \vec{N}, \vec{P} \rangle \mathbf{ in } R[y := Q]) : \mathbf{K}^2 B$ by \mathbf{K}_I .

- ii) For multistep η -reduction:

- 1) Let $\vdash \mathbf{let pure } _ = _ \mathbf{ in } N : \mathbf{KA}$.

Then by generation $\vdash N : A$, so $\vdash \mathbf{pure} N : \mathbf{K}A$ by \mathbf{K}_I .

2) Let $\Gamma \vdash \mathbf{let pure} x = M \mathbf{in} x : \mathbf{K}A$.

By generation $\Gamma \vdash M : \mathbf{K}A$ and $x : A \vdash x : A$, hence $\Gamma \vdash M : \mathbf{K}A$.

3) Let $\Gamma \vdash \mathbf{pure} (\lambda x. fx) : \mathbf{K}(A \rightarrow B)$.

By generation $\Gamma \vdash \lambda x. fx : A \rightarrow B$, so $\Gamma \vdash f : A \rightarrow B$, then $\Gamma \vdash \mathbf{pure} f : \mathbf{K}(A \rightarrow B)$ by \mathbf{K}_I .

4) Let $\Gamma \vdash \mathbf{pure} (\langle \pi_1 P, \pi_2 P \rangle) : \mathbf{K}(A \times B)$.

By generation $\Gamma \vdash \langle \pi_1 P, \pi_2 P \rangle : A \times B$, then $\Gamma \vdash P : A \times B$.

By \mathbf{K}_I , $\Gamma \vdash \mathbf{pure} P : \mathbf{K}(A \times B)$.

5) $\Gamma \vdash \mathbf{pure} (\mathbf{let pure} x = M \mathbf{in} x) : \mathbf{K}^2 A$.

Then $\Gamma \vdash M : \mathbf{K}^2 A$ and $x : \mathbf{K}A \vdash x : \mathbf{K}A$, so $\Gamma \vdash M : \mathbf{K}^2 A$.

6) Let $\vdash \mathbf{pure} (\mathbf{let pure} _ = _ \mathbf{in} N) : \mathbf{K}^2 A$.

By generation $\mathbf{let pure} _ = _ \mathbf{in} N : \mathbf{K}A$, so $\vdash N : A$, then $\vdash \mathbf{pure} N : \mathbf{K}$.

□

4 Strong normalization

We modify and apply Tait's technique of logical relation for modalities. Strong normalization proof with Tait's method for simply typed lambda calculus is described here [13].

Theorem 2. *Let $M \in \Lambda_{\mathbf{K}}$, then any sequence of reduction $M \rightarrow_{\beta} M_1 \dots$ terminates.*

5 Confluence

6 Categorical semantics

Definition 10. *Lax monoidal functor*

Let $\langle \mathcal{C}, \oplus_1, \mathbb{1} \rangle$ and $\langle \mathcal{D}, \oplus_2, \mathbb{1}' \rangle$ are monoidal categories.

A lax monoidal functor $\mathcal{F} : \langle \mathcal{C}, \oplus_1, \mathbb{1} \rangle \rightarrow \langle \mathcal{D}, \oplus_2, \mathbb{1}' \rangle$ is a functor $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ with additional natural transformations:

1) $u : \mathbb{1}' \rightarrow \mathcal{F}\mathbb{1}$;

*2) $*_{A,B} \mathcal{F}A \otimes_2 \mathcal{F}B \rightarrow \mathcal{F}(A \otimes_1 B)$*

Definition 11. *Applicative functor*

An applicative functor is a triple $\langle \mathcal{C}, \mathcal{K}, \eta \rangle$, where \mathcal{C} is a symmetric monoidal category, \mathcal{K} is a lax monoidal endofunctor and η is a natural transformation, such that:

1) $u = \eta_{\mathbb{1}}$;

*2) $*_{A,B} \circ (\eta_A \otimes \eta_B) = \eta_{A \otimes B}$;*

3) Weak commutativity condition holds:

$$A \otimes \mathcal{K}B \quad \mathcal{K}A \otimes \mathcal{K}B \quad \mathcal{K}(A \otimes B)$$

$$\mathcal{K}B \otimes A \quad \mathcal{K}B \otimes \mathcal{K}A \quad \mathcal{K}(B \otimes A)$$

By default we will consider an arbitrary closed functor on some cartesian closed category, which is the special case of an applicative functor.

6.1 Soundness

Definition 12. *Semantical translation from λ_K to CCC with applicative functor \mathcal{K} :*

1) *Interpretation for types:*

$$\llbracket A \rrbracket := \hat{A}, A \in \mathbb{T};$$

$$\llbracket A \rightarrow B \rrbracket := \llbracket A \rrbracket \rightarrow \llbracket B \rrbracket;$$

$$\llbracket A \times B \rrbracket := \llbracket A \rrbracket \times \llbracket B \rrbracket.$$

2) *Interpretation for modal types:* $\llbracket \mathbf{K}A \rrbracket = \mathcal{K}\llbracket A \rrbracket$;

3) *Interpretation for contexts:*

$$\llbracket \Gamma = \{x_1 : A_1, \dots, x_n : A_n\} \rrbracket := \llbracket \Gamma \rrbracket = \llbracket A_1 \rrbracket \times \dots \times \llbracket A_n \rrbracket;$$

4) *Interpretation for typing assignment:* $\llbracket \Gamma \vdash M : A \rrbracket := \llbracket M \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket$.

5) *Interpretation for typing rules:*

$$\begin{array}{c} \frac{}{\llbracket \Gamma, x : A \vdash x : A \rrbracket = \pi_2 : \llbracket \Gamma \rrbracket \times \llbracket A \rrbracket \rightarrow \llbracket A \rrbracket} \\[10pt] \frac{\llbracket \Gamma, x : A \vdash M : B \rrbracket = f : \llbracket \Gamma \rrbracket \times \llbracket A \rrbracket \rightarrow \llbracket B \rrbracket}{\llbracket \Gamma \vdash (\lambda x.M) : A \rightarrow B \rrbracket = \Lambda(f) : \llbracket \Gamma \rrbracket \rightarrow \llbracket B \rrbracket^{[A]}} \\[10pt] \frac{\llbracket \Gamma \vdash M : A \rightarrow B \rrbracket = \llbracket M \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket B \rrbracket^{[A]} \quad \llbracket \Gamma \vdash N : A \rrbracket = \llbracket N \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket}{\llbracket \Gamma \vdash (MN) : B \rrbracket = \llbracket \Gamma \rrbracket \xrightarrow{\langle \llbracket M \rrbracket, \llbracket N \rrbracket \rangle} \llbracket B \rrbracket^{[A]} \times \llbracket A \rrbracket \xrightarrow{\epsilon} \llbracket B \rrbracket} \\[10pt] \frac{\llbracket \Gamma \vdash M : A \rrbracket = f : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket \quad \llbracket \Gamma \vdash N : B \rrbracket = g : \llbracket \Gamma \rrbracket \rightarrow \llbracket B \rrbracket}{\llbracket \Gamma \vdash (M, N) : A \times B \rrbracket = \langle f, g \rangle : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket \times \llbracket B \rrbracket} \\[10pt] \frac{\llbracket \Gamma \vdash p : A_1 \times A_2 \rrbracket = f : \llbracket \Gamma \rrbracket \rightarrow \llbracket A_1 \rrbracket \times \llbracket A_2 \rrbracket}{\llbracket \Gamma \vdash \pi_i p : A_i \rrbracket = \llbracket \Gamma \rrbracket \xrightarrow{f} \llbracket A_1 \rrbracket \times \llbracket A_2 \rrbracket \xrightarrow{\pi_i} \llbracket A_i \rrbracket} \quad i \in \{1, 2\} \\[10pt] \frac{\llbracket \Gamma \vdash M : A \rrbracket = \llbracket M \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket}{\llbracket \Gamma \vdash \mathbf{pure} M : \mathbf{K}A \rrbracket := \llbracket \Gamma \rrbracket \xrightarrow{\llbracket M \rrbracket} \llbracket A \rrbracket \xrightarrow{\eta_{[A]}} \mathcal{K}\llbracket A \rrbracket} \\[10pt] \frac{\llbracket \Gamma \vdash \vec{M} : \mathbf{K}\vec{A} \rrbracket = \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle : \llbracket \Gamma \rrbracket \rightarrow \prod_{i=1}^n \mathcal{K}\llbracket A_i \rrbracket \quad \llbracket \vec{x} : \vec{A} \vdash N : B \rrbracket = \llbracket N \rrbracket : \prod_{i=1}^n \llbracket A_i \rrbracket \rightarrow \llbracket B \rrbracket}{\llbracket \Gamma \vdash \mathbf{let} \mathbf{pure} \vec{x} = \vec{M} \mathbf{in} M : \mathbf{K}B \rrbracket = \mathcal{K}(\llbracket N \rrbracket) \circ *_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle : \llbracket \Gamma \rrbracket \rightarrow \mathcal{K}\llbracket B \rrbracket} \end{array}$$

Definition 13. *Simultaneous substitution*

Let $\Gamma = \{x_1 : A_1, \dots, x_n : A_n\}$, $\Gamma \vdash M : A$ and for all $i \in \{1, \dots, n\}$, $\Gamma \vdash M_i : A_i$.

We define simultaneous substitution $M[\vec{x} := \vec{M}]$ recursively by:

- 1) $x_i[\vec{x} := \vec{M}] = M_i$;
- 2) $(\lambda x. M)[\vec{x} := \vec{M}] = \lambda x. (M[\vec{x} := \vec{M}])$;
- 3) $(MN)[\vec{x} := \vec{M}] = (M[\vec{x} := \vec{M}]) (N[\vec{x} := \vec{M}])$;
- 4) $\langle M, N \rangle = \langle (M[\vec{x} := \vec{M}]), (N[\vec{x} := \vec{M}]) \rangle$;
- 5) $(\pi_i P)[\vec{x} := \vec{M}] = \pi_i (P[\vec{x} := \vec{M}])$;
- 6) $(\mathbf{pure} M)[\vec{x} := \vec{M}] = \mathbf{pure} (M[\vec{x} := \vec{M}])$;
- 7) $(\mathbf{let} \mathbf{pure} \vec{x} = \vec{M} \mathbf{in} N)[\vec{y} := \vec{P}] = \mathbf{let} \mathbf{pure} \vec{x} = (\vec{M}[\vec{y} := \vec{P}]) \mathbf{in} N$

Lemma 8.

$$\llbracket M[x_1 := M_1, \dots, x_n := M_n] \rrbracket = \llbracket M \rrbracket \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle.$$

Proof.

$$1) \llbracket \Gamma \vdash (\mathbf{pure} M)[\vec{x} := \vec{M}] : \mathbf{KA} \rrbracket = \llbracket \Gamma \vdash \mathbf{pure} M : \mathbf{KA} \rrbracket \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle.$$

$$\begin{aligned} \llbracket \Gamma \vdash (\mathbf{pure} M)[\vec{x} := \vec{M}] : \mathbf{KA} \rrbracket &= \llbracket \Gamma \vdash \mathbf{pure} (M[\vec{x} := \vec{M}]) : \mathbf{KA} \rrbracket && \text{Substitution definition} \\ &= \eta_{\llbracket A \rrbracket} \circ \llbracket (M[\vec{x} := \vec{M}]) \rrbracket && \text{Translation for pure} \\ &= \eta_{\llbracket A \rrbracket} \circ (\llbracket M \rrbracket \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle) && \text{Induction hypothesis} \\ &= (\eta_{\llbracket A \rrbracket} \circ \llbracket M \rrbracket) \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle && \text{Associativity of composition} \\ &= \llbracket \Gamma \vdash \mathbf{pure} M : \mathbf{KA} \rrbracket \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle && \text{Translation for pure} \end{aligned}$$

$$2) \llbracket \Gamma \vdash (\mathbf{let} \mathbf{pure} \vec{x} = \vec{M} \mathbf{in} N)[\vec{y} := \vec{P}] : \mathbf{KB} \rrbracket = \llbracket \Gamma \vdash \mathbf{let} \mathbf{pure} \vec{x} = \vec{M} \mathbf{in} N : \mathbf{KB} \rrbracket \circ \langle \llbracket P_1 \rrbracket, \dots, \llbracket P_n \rrbracket \rangle$$

$$\begin{aligned} \llbracket \Gamma \vdash (\mathbf{let} \mathbf{pure} \vec{x} = \vec{M} \mathbf{in} N)[\vec{y} := \vec{P}] : \mathbf{KB} \rrbracket &= \\ \text{Substitution definition} & \\ \llbracket \Gamma \vdash \mathbf{let} \mathbf{pure} \vec{x} = (\vec{M}[\vec{y} := \vec{P}]) \mathbf{in} N : \mathbf{KB} \rrbracket &= \\ \text{Interpretation for } \mathbf{let}_{\mathbf{K}} & \\ \mathcal{K}(\llbracket N \rrbracket) \circ *_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ \llbracket \Gamma \vdash (\vec{M}[\vec{y} := \vec{P}]) : \mathbf{KA} \rrbracket &= \\ \text{Induction hypothesis} & \\ \mathcal{K}(\llbracket N \rrbracket) \circ *_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ (\llbracket \vec{M} \rrbracket \circ \langle \llbracket P_1 \rrbracket, \dots, \llbracket P_n \rrbracket \rangle) &= \\ \text{Associativity of composition} & \\ (\mathcal{K}(\llbracket N \rrbracket) \circ *_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ \llbracket \vec{M} \rrbracket) \circ \langle \llbracket P_1 \rrbracket, \dots, \llbracket P_n \rrbracket \rangle &= \\ \text{By interpretation} & \\ \llbracket \Gamma \vdash (\mathbf{let} \mathbf{pure} \vec{x} = \vec{M} \mathbf{in} N) \rrbracket \circ \langle \llbracket P_1 \rrbracket, \dots, \llbracket P_n \rrbracket \rangle & \end{aligned}$$

□

Lemma 9.

- i) Let $\Gamma \vdash M : A$ and $M \rightarrow_{\beta} N$, then $\llbracket \Gamma \vdash M : A \rrbracket = \llbracket \Gamma \vdash N : A \rrbracket$;
- ii) Let $\Gamma \vdash M : A$ and $M \rightarrow_{\eta} N$, then $\llbracket \Gamma \vdash M : A \rrbracket = \llbracket \Gamma \vdash N : A \rrbracket$;

Proof. i) For β -reduction

Cases with β -reductions for $\mathbf{let}_{\mathbf{K}}$ are shown in [20]. Let us consider cases with \mathbf{pure} .

$$1) \llbracket \Gamma \vdash \mathbf{pure} ((\lambda x. M)N) : \mathbf{KB} \rrbracket = \llbracket \Gamma \vdash \mathbf{pure} (M[x := N]) : \mathbf{KB} \rrbracket$$

$$\begin{aligned}
\llbracket \Gamma \vdash \mathbf{pure} (\lambda x.M) N : \mathbf{KB} \rrbracket &= && \text{By interpretation} \\
\eta_{\llbracket B \rrbracket} \circ (\epsilon \circ \langle \Lambda(\llbracket M \rrbracket), \llbracket N \rrbracket \rangle) &= && \text{Property of } \times \\
\eta_{\llbracket B \rrbracket} \circ (\epsilon \circ (\Lambda(\llbracket M \rrbracket) \times id_{\llbracket A \rrbracket}) \circ \langle id_{\llbracket \Gamma \rrbracket}, \llbracket N \rrbracket \rangle) &= && \text{Associativity of composition} \\
\eta_{\llbracket B \rrbracket} \circ ((\epsilon \circ (\Lambda(\llbracket M \rrbracket) \times id_{\llbracket A \rrbracket})) \circ \langle id_{\llbracket \Gamma \rrbracket}, \llbracket N \rrbracket \rangle) &= && \text{Exponentiation property} \\
\eta_{\llbracket B \rrbracket} \circ (\llbracket M \rrbracket \circ \langle id_{\llbracket \Gamma \rrbracket}, \llbracket N \rrbracket \rangle) &= && \text{Substitution lemma} \\
\eta_{\llbracket B \rrbracket} \circ \llbracket M[\vec{x}, x := \vec{x}, N] \rrbracket &= && \text{By interpretation} \\
\llbracket \Gamma \vdash \mathbf{pure} (M[x := N]) : \mathbf{KB} \rrbracket &= &&
\end{aligned}$$

$$2) \llbracket \Gamma \vdash \mathbf{pure} (\pi_i \langle \llbracket M_1 \rrbracket, \llbracket M_2 \rrbracket \rangle) : \mathbf{KA}_i \rrbracket = \llbracket \Gamma \vdash \mathbf{pure} M_i : \mathbf{KA}_i \rrbracket$$

$$\begin{aligned}
\llbracket \Gamma \vdash \mathbf{pure} (\pi_i \langle M_1, M_2 \rangle) : \mathbf{KA}_i \rrbracket &= && \text{By interpretation} \\
\eta_{\llbracket A_i \rrbracket} \circ \pi_i \circ \langle \llbracket M_1 \rrbracket, \llbracket M_2 \rrbracket \rangle &= && \text{Property of } \times \\
\eta_{\llbracket A_i \rrbracket} \circ \llbracket M_i \rrbracket &= && \text{By interpretation} \\
\llbracket \Gamma \vdash \mathbf{pure} M_i : \mathbf{KA}_i \rrbracket &= &&
\end{aligned}$$

□

Theorem 3. Soundness

Let $\Gamma \vdash M : A$ and $M =_{\beta\eta} N$, then $\llbracket \Gamma \vdash M : A \rrbracket = \llbracket \Gamma \vdash N : A \rrbracket$

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