Modal type theory based on the intuitionistic epistemic logic

Abstract

Modal intuitionistic epistemic logic IEL$^-$ was proposed by S.Artemov and T. Protopopescu as the formal foundation for the intuitionistic theory of knowledge. We construct a modal simply typed lambda-calculus which is Curry-Howard isomorphic to IEL$^-$ as formal theory of calculations with applicative functors in functional programming languages like Haskell or Idris. We prove that this typed lambda-calculus has the strong normalization and Church-Rosser properties.

1 Introduction

Modal intutionistic epistemic logic IEL was proposed by S. Artemov and T. Proropopescu [1]. IEL provides the epistimology and the theory of knowledge as based on BHK-semantics of intuitionistic logic. IEL$^-$ is a variant of IEL, that corresponds to intuitionistic belief. Informally, $\mathbf{K}A$ denotes that A is verified intuitionistically.

Intuitionistic epistemic logic IEL$^-$ is defined with by following axioms and derivation rules:

Definition 1. *Intuitionistic epistemic logic IEL:*
1) IPC axioms;
2) $\mathbf{K}(A \to B) \to (\mathbf{K}A \to \mathbf{K}B)$ *(normality);*
3) $A \to \mathbf{K}A$ *(co-reflection);*
Rule: MP.

We have the deduction theorem and necessitation rule which is derivable.

V. Krupski and A. Yatmanov provided the sequential calculus for IEL and proved that this calculus is PSPACE-complete [2].

It's not difficult to see that modal axioms in IEL^- and types of the methods of Applicative class in Haskell-like languages (which is described below) are syntactically similar and we are going to show that this coincidence has a non-trivial computational meaning.

Functional programming languages such as Haskell [3], Idris [4], Purescript [5] or Elm [6] have special type classes[1] for calculations with container types like `Functor` and `Applicative` [2]:

[1]Type class in Haskell is a general interface for special group of datatypes.

[2]Reader may read more about container types in the Haskell standard library documentation[7] or in the next one textbook [8]

```
class Functor f where
  fmap :: (a -> b) -> f a -> f b

class Functor f => Applicative f where
  pure :: a -> f a
  (<*>) :: f (a -> b) -> f a -> f b
```

By *container* (or *computational context*) type we mean some type-operator f, where f is a "function" from $*$ to $*$: type operator takes a simple type (which has kind $*$) and returns another simple type type with kind $*$. For more detailed description of the type system with kinds used in Haskell see [12].

The main goal of our research is a relationship between intuitionistic epistemic logic IEL^- and functional programming with effects. We show that relationship by building the type system (which is called $\lambda_{\mathbf{K}}$) which is Curry-Howard isomorphic to IEL^-. So we will consider **K**-modality as an arbitrary applicative functor.

$\lambda_{\mathbf{K}}$ consists of the rules for simply typed lambda-calculus and special typing rules for lifting types into the applicative functor **K**. We assume that our type system will axiomatize the simplest case of computation with effects with one container. We provide proof-theoretical view on this kind of computations in functional programming and prove strong normalization and confluence.

2 Typed lambda-calculus based on IEL$^-$

At first we define the natural deduction for IEL$^-$ with **K**-modality and binary connectives $\to$ and $\wedge$ (we call that calculus NIEL$^-_{\wedge,\to}$):

Definition 2. *Natural deduction $NIEL^-_{\wedge,\to}$ for IEL^- with $\to$ and $\wedge$:*

$$\frac{}{\Gamma, A \vdash A}\ ax$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ \to_i \qquad \frac{\Gamma \vdash A \to B \qquad \Gamma \vdash A}{\Gamma \vdash B}\ \to_i$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge_i \qquad \frac{\Gamma \vdash A_1 \wedge A_2}{\Gamma \vdash A_i}\ \wedge_e, i \in \{1,2\}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \boldsymbol{K}A}\ \boldsymbol{K_I} \qquad \frac{\Gamma \vdash \mathbf{K}\vec{A} \qquad \vec{A} \vdash B}{\Gamma \vdash \mathbf{K}B}$$

Where $\Gamma \vdash \mathbf{K}\vec{A}$ is a syntax sugar for $\Gamma \vdash \mathbf{K}A_1, \dots, \Gamma \vdash \mathbf{K}A_n$.

Lemma 1. $\Gamma \vdash_{NIEL^-_{\wedge,\to}} A \Rightarrow IEL^- \vdash \bigwedge \Gamma \to A$.

Proof. Induction on the derivation.

Let us consider cases with modality.

1) If $\Gamma \vdash_{NIEL^-_{\wedge,\to}} A$, then $IEL^- \vdash \bigwedge \Gamma \to \mathbf{K}A$.

(1)	$\bigwedge \Gamma \to A$	assumption
(2)	$A \to \mathbf{K}A$	co-reflection
(3)	$(\bigwedge \Gamma \to A) \to ((A \to \mathbf{K}A) \to (\bigwedge \Gamma \to \mathbf{K}A))$	IPC theorem
(4)	$(A \to \mathbf{K}A) \to (\bigwedge \Gamma \to \mathbf{K}A)$	from (1), (3) and MP
(5)	$\bigwedge \Gamma \to \mathbf{K}A$	from (2), (4) and MP

2) If $\Gamma \vdash_{NIEL^-_{\wedge,\to}} \mathbf{K}\vec{A}$ and $\vec{A} \vdash B$, then $IEL^- \vdash \bigwedge \Gamma \to \mathbf{K}B$.

(1)	$\bigwedge \Gamma \to \bigwedge_{i=1}^{n} \mathbf{K}A_i$	assumption
(2)	$\bigwedge_{i=1}^{n} \mathbf{K}A_i \to \mathbf{K} \bigwedge_{i=1}^{n} A_i$	IEL theorem
(3)	$\bigwedge \Gamma \to \mathbf{K} \bigwedge_{i=1}^{n} A_i$	from (1), (2) and transitivity
(4)	$\bigwedge_{i=1}^{n} A_i \to B$	assumption
(5)	$(\bigwedge_{i=1}^{n} A_i \to B) \to \mathbf{K}(\bigwedge_{i=1}^{n} A_i \to B)$	co-reflection
(6)	$\mathbf{K}(\bigwedge_{i=1}^{n} A_i \to B)$	from (2), (3) and MP
(7)	$\mathbf{K} \bigwedge_{i=1}^{n} A_i \to \mathbf{K}B$	from (6) and normality
(8)	$\bigwedge \Gamma \to \mathbf{K}B$	from (3), (7) and transitivity

□

Lemma 2. *If $IEL^- \vdash A$, then $NIEL^- \vdash A$.*

Proof. Straightforward derivation of modal axioms in NIEL$^-$. We consider this derivation below using terms. □

At the next step we build the typed lambda-calculus based on NIEL$^-_{\wedge,\to}$ by proof-assingment in rules.

At first, we define lambda-terms and types for this lambda-calculus.

Definition 3. *The set of terms:*

Let $\mathbb{V}$ be the set of variables. The set $\Lambda_{\mathbf{K}}$ of terms is defined by the grammar:

$$\Lambda_{\mathbf{K}} ::= \mathbb{V} \mid (\lambda\Lambda.\Lambda_{\mathbf{K}}) \mid (\Lambda_{\mathbf{K}}\Lambda_{\mathbf{K}}) \mid (\Lambda_{\mathbf{K}}, \Lambda_{\mathbf{K}}) \mid (\pi_1\Lambda_{\mathbf{K}}) \mid (\pi_2\Lambda_{\mathbf{K}}) \mid (\mathbf{pure}\ \Lambda_{\mathbf{K}}) \mid (\mathbf{let\ pure}\ \Lambda_{\mathbf{K}} = \Lambda_{\mathbf{K}}\ \mathbf{in}\ \Lambda_{\mathbf{K}})$$

Definition 4. *The set of types:*

Let $\mathbb{T}$ be the set of atomic types. The set $\mathbb{T}_{\mathbf{K}}$ of types with applicative functor $\mathbf{K}$ is generated by the grammar:

$$\mathbb{T}_{\mathbf{K}} ::= \mathbb{T} \mid (\mathbb{T}_{\mathbf{K}} \to \mathbb{T}_{\mathbf{K}}) \mid (\mathbb{T}_{\mathbf{K}} \times \mathbb{T}_{\mathbf{K}}) \mid (\mathbf{K}\mathbb{T}_{\mathbf{K}}) \qquad (1)$$

Context, domain of context and range of context are defined standardly [11][12].

Our type system is based on the Curry-style typing rules:

Definition 5. *Modal typed lambda calculus $\lambda_{\mathbf{K}}$ based on $NIEL^-_{\wedge,\to}$:*

$$\frac{}{\Gamma, x : A \vdash x : A}\ ax$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B} \to_i \qquad \frac{\Gamma \vdash f : A \to B \qquad \Gamma \vdash x : A}{\Gamma \vdash fx : B} \to_e$$

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash N : B}{\Gamma \vdash \langle x, y \rangle : A \times B} \times_i \qquad \frac{\Gamma \vdash M : A_1 \times A_2}{\Gamma \vdash \pi_i M : A_i} \times_e,\ i \in \{1, 2\}$$

$$\frac{\Gamma \vdash x : A}{\Gamma \vdash \mathbf{pure}\ x : \mathbf{K}A} \boldsymbol{K_I} \qquad \frac{\Gamma \vdash \vec{M} : \mathbf{K}\vec{A} \qquad \vec{x} : \vec{A} \vdash M : B}{\Gamma \vdash \mathbf{let\ pure}\ \vec{x} = \vec{M}\ \mathbf{in}\ M : \mathbf{K}B} let_{\mathbf{K}}$$

$\mathbf{K}_I$-typing rule is the same as $\bigcirc$-introduction in lax logic (also known as monadic metalanguage [17]) and in typed lambda-calculus which is derived by proof-assignment for lax-logic proofs. $\mathbf{K}_I$ allows to inject an object of type α into the functor. $\mathbf{K}_I$ reflects the Haskell method **pure** for Applicative class. It plays the same role as the **return** method in Monad class.

$\text{let}_{\mathbf{K}}$ is the same as the $\Box$-rule in typed lambda calculus for intuitionistic normal modal logic **IK**, which is described in [19].

In fact, our calculus is the extention of typed lambda calculus for **IK** with typing rule appropriate to co-reflection.

Here are some examples of derivation trees.

$$\dfrac{\dfrac{x : A \vdash x : A}{x : A \vdash \mathbf{pure}\, x : \mathbf{K}A} \mathbf{K}_I}{\vdash (\lambda x.\mathbf{pure}\, x) : A \to \mathbf{K}A} \to_i$$

$$\dfrac{\dfrac{\dfrac{f : \mathbf{K}(A \to B) \vdash f : \mathbf{K}(A \to B) \qquad x : \mathbf{K}A \vdash x : \mathbf{K}A \qquad \dfrac{g : A \to B \qquad y : A}{g : A \to B, y : A \vdash gy : B}}{f : \mathbf{K}(A \to B), x : \mathbf{K}A \vdash \mathbf{let\ pure}\ \langle g, y \rangle = \langle f, x \rangle\ \mathbf{in}\ gy : \mathbf{K}B}}{f : \mathbf{K}(A \to B) \vdash \lambda x.\mathbf{let\ pure}\ \langle g, y \rangle = \langle f, x \rangle\ \mathbf{in}\ gy : \mathbf{K}A \to \mathbf{K}B}}{\vdash \lambda f.\lambda x.\mathbf{let\ pure}\ \langle g, y \rangle = \langle f, x \rangle\ \mathbf{in}\ gy : \mathbf{K}(A \to B) \to \mathbf{K}A \to \mathbf{K}B}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{f : A \to B \vdash f : A \to B}{f : A \to B \vdash \mathbf{pure}\ f : \mathbf{K}(A \to B)} \qquad x : \mathbf{K}A \vdash x : \mathbf{K}A \qquad \dfrac{g : A \to B \qquad y : A}{g : A \to B, y : A \vdash gy : B}}{f : A \to B, x : \mathbf{K}A \vdash \mathbf{let\ pure}\ \langle g, y \rangle = \langle \mathbf{pure}\ f, x \rangle\ \mathbf{in}\ gy : \mathbf{K}B}}{f : A \to B \vdash \lambda x.\mathbf{let\ pure}\ \langle g, y \rangle = \langle \mathbf{pure}\ f, x \rangle\ \mathbf{in}\ gy : \mathbf{K}A \to \mathbf{K}B}}{\lambda f.\lambda x.\mathbf{let\ pure}\ \langle g, y \rangle = \langle \mathbf{pure}\ f, x \rangle\ \mathbf{in}\ gy : (A \to B) \to \mathbf{K}A \to \mathbf{K}B}$$

Now we define free variables and substitutions. β-reduction, multi-step β-reduction and β-equality are defined standardly:

Definition 6. *Set $FV(M)$ of free variables for arbitrary term M:*
1) $FV(x) = \{x\}$;
2) $FV(\lambda x.M) = FV(M) \setminus \{x\}$;
3) $FV(MN) = FV(M) \cup FV(N)$;
4) $FV(\langle M, N \rangle) = FV(M) \cup FV(N)$;

5) $FV(\pi_i M) \subseteq FV(M)$*,* $i \in \{1,2\}$*;*
6) $FV(\text{pure } M) = FV(M)$*;*
7) $FV(\textbf{let pure } \vec{N} = \vec{M} \textbf{ in } M) = \bigcup_{i=1}^{n} FV(M)$, *where* $n = |\vec{M}|$.

Definition 7. *Substitution:*
1) $x[x := N] = N$*,* $x[y := N] = x$*;*
2) $(MN)[x := N] = M[x := N]N[x := N]$*;*
3) $(\lambda x.M)[x := N] = \lambda x.M[x := N]$*;*
4) $(M, N)[x := P] = (M[x := P], N[x := P])$*;*
5) $(\pi_i M)[x := P] = \pi_i(M[x := P])$*,* $i \in \{1,2\}$*;*
6) $(\textbf{pure } M)[x := P] = \textbf{pure } (M[x := P])$*;*
7) $(\textbf{let pure } \vec{x} = \vec{M} \textbf{ in } M)[y := P] = \textbf{let pure } \vec{x} = (\vec{M}[y := P]) \textbf{ in } M$*.*

Definition 8. *β-reduction and η-reduction rules for $\lambda\mathbf{K}$.*
1) $(\lambda x.M)N \to_\beta M[x := N]$*;*
2) $\pi_1\langle M, N\rangle \to_\beta M$*;*
3) $\pi_2\langle M, N\rangle \to_\beta N$*;*
4) $\begin{array}{l} \textbf{let pure } \langle \vec{x}, y, \vec{z}\rangle = \langle \vec{M}, \textbf{let pure } \vec{w} = \vec{N} \textbf{ in } Q, \vec{P}\rangle \textit{ in } R \to_\beta \\ \textbf{let pure } \langle \vec{x}, \vec{w}, \vec{z}\rangle = \langle \vec{M}, \vec{N}, \vec{P}\rangle \textbf{ in } R[y := Q] \end{array}$
5) $M \to_\beta N \Rightarrow \textbf{pure M} \to_\beta \textbf{pure N}$
6) $\lambda x.fx \to_\eta f$*;*
7) $\langle \pi_1 P, \pi_2 P\rangle \to_\eta P$*;*
10) $\textbf{let pure } \_\_ = \_\_ \textbf{ in } N \to_\eta \textbf{pure } N$*;*
11) $\textbf{let pure } x = M \textbf{ in } x \to_\eta M$*;*
12) $M \to_\beta N \Rightarrow \textbf{pure M} \to_\eta \textbf{pure N}$

Let us show the next simple observation, which immeadelty follows from the previous definition.

Lemma 3.
If $M \to_{\beta\eta} N$*, then* $\textbf{pure } M \to_{\beta\eta} \textbf{pure } N$*.*

3 Basic lemmas

Now we will prove standard lemmas for contexts in type systems[3]:

4 Strong normalization

We modify and apply Tait's technique of logical relation for modalities. Strong normalization proof with Tait's method for simply typed lambda calculus is described here [13].

Strong normalization for **IK** is proved in [21] [19]. So we consider simply typed lambda calculus with $\mathbf{K}_I$ rule and show that $\lambda_{\to,\times} + \mathbf{K}_I$ is strongly normalizable.

[3]We will not prove cases with $\to$-constructor, they are proved standardly in the same lemmas for simply typed lambda calculus, for example see [11][12][14]. We will consider only modal cases

Theorem 1. *Let $M \in \Lambda_{\mathbf{K}}$, then any sequence of reduction $M \to_\beta M_1 \ldots$ terminates.*

Proof.
We build the subset of strongly normalazing terms and show that an arbitrary term belongs to this subset.

Definition 9. *The set of strongly computable terms for every type $T \in \mathbb{T}_{\mathbf{K}}$.*

- *Let $A \in \mathbb{T}$, then $SC_A = \{M : A \mid M$ is strongly normalizing$\}$;*
- $SC_{A \to B} = \{M : A \to B \mid \forall A \in SC_A, MN \in SC_B\}$;
- $SC_{A_1 \times A_2} = \{M : A \times B \mid \pi_i M \in SC_{A_i}, i \in \{1, 2\}\}$;
- $SC_{\mathbf{K}A} = \{\mathbf{pure}\, M : \mathbf{K}A \mid M \in SC_A\}$

Strong normalization proof reduces to the proof of the next lemma:

Lemma 4.
i) If $M \in SC_A$, then M is stronly normalizing;
ii) If $M \to_\beta M^{'}$ and $M \in SC_A$, then $M^{'}$;
iii) Let $M \to_\beta M^{'}$ and $M^{'} \in SC_A$, then, if M is a neutral term, then $M \in SC_A$.
iv) Let $x_1 : A_1, \ldots, x_n : A_n \vdash M : B$ and $\forall i \in \{1, \ldots, n\}, N_i \in SC_{A_i}$, then $M[\vec{x} := \vec{N}] \in SC_B$.

Proof.
i)
The base case follows from the definition.
Let us consider case with $SC_{\mathbf{K}A}$. If $\mathbf{pure}\, M \in SC_{\mathbf{K}A}$, then $M \in SC_A$ and M is strongly normalizible. So $\mathbf{pure}\, M$ is strongly normalizible, otherwise there would be an infinite reduction path in $\mathbf{pure}\, M$.

ii)
The base case is trivial.
Let $\mathbf{pure}\, M \to_\beta \mathbf{pure}\, M^{'}$ and $\mathbf{pure}\, M \in SN_{\mathbf{K}A}$. By assumption, $M \in SN_A$ and $M \to_\beta M^{'}$, so $M^{'} \in SN_A$. Hence $\mathbf{pure}\, M^{'} \in SC_{\mathbf{K}A}$ by the first statement of the lemma.

iii)
The base case is trivial.
Let $\mathbf{pure}\, M \to_\beta \mathbf{pure}\, M^{'}$ and $\mathbf{pure}\, M^{'} \in SN_{\mathbf{K}A}$.
$\mathbf{pure}\, M^{'}$ is a neutral by the definition. By assumption M is a strongly normalizing. So $\mathbf{pure}\, M$ is a strongly normalizing by the first part of the current lemma.

iv)
Let $x_1 : A_1, \ldots, x_n : A_n \vdash \mathbf{pure}\, M : \mathbf{K}A$ and $\forall i \in \{1, \ldots, n\}, N_i \in SC_{A_i}$.
By generation $x_1 : A_1, \ldots, x_n : A_n \vdash M : A$ and by assumption $M[\vec{x} := \vec{N}] \in SC_B$.
Hence, by the first part of lemma, $\mathbf{pure}\, (M[\vec{x} := \vec{N}]) \in SC_{\mathbf{K}B}$. □

Corollary 1.
Let $\vdash N : A$, then N is strongly normalizing.

Proof.
If $\vdash N : A$, then $N \in SC_A$, hence N is strongly normalizing. □

□

5 Confluence

6 Categorical semantics

Definition 10. *Lax monoidal functor*
Let $\langle \mathcal{C}, \oplus_1, \mathbb{1} \rangle$ and $\langle \mathcal{D}, \oplus_2, \mathbb{1}' \rangle$ are monoidal categories.
A lax monoidal functor $\mathcal{F} : \langle \mathcal{C}, \oplus_1, \mathbb{1} \rangle \to \langle \mathcal{D}, \oplus_2, \mathbb{1}' \rangle$ is a functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ with additional natural transformations:
1) $u : \mathbb{1}' \to \mathcal{F}\mathbb{1}$;
2) $*_{A,B} : \mathcal{F}A \otimes_2 \mathcal{F}B \to \mathcal{F}(A \otimes_1 B)$

Definition 11. *Applicative functor*
An applicative functor is a triple $\langle \mathcal{C}, \mathcal{K}, \eta \rangle$, where $\mathcal{C}$ is a symmetric monoidal category, $\mathcal{K}$ is a lax monoidal endofunctor and η is a natural transformation, such that:
1) $u = \eta_{\mathbb{1}}$;
2) $*_{A,B} \circ (\eta_A \otimes \eta_B) = \eta_{A \otimes B}$;
3) Weak commutativity condition holds:

$$\begin{array}{ccc} A \otimes \mathcal{K}B & \mathcal{K}A \otimes \mathcal{K}B & \mathcal{K}(A \otimes B) \\ & & \\ \mathcal{K}B \otimes A & \mathcal{K}B \otimes \mathcal{K}A & \mathcal{K}(B \otimes A) \end{array}$$

By default we will consider an arbitrary closed functor on some cartersian closed category, which is the special case of an applicative functor.

We identify terminal objects. So $\mathcal{K}(\mathbb{1}) = \mathbb{1}$ and $\eta_{\mathbb{1}} = id_{\mathbb{1}}$ since $\mathcal{K}$ is an endofunctor.

6.1 Soundness

Definition 12. *Semantical translation from $\lambda_{\boldsymbol{K}}$ to CCC with applicative functor $\mathcal{K}$:*
1) Interpretation for types:
$[\![A]\!] := \hat{A}, A \in \mathbb{T}$;
$[\![A \to B]\!] := [\![A]\!] \to [\![B]\!]$;
$[\![A \times B]\!] := [\![A]\!] \times [\![B]\!]$.
2) Interpretation for modal types: $[\![\boldsymbol{K}A]\!] = \mathcal{K}[\![A]\!]$;
3) Interpretaion for contexts:
$[\![\Gamma = \{x_1 : A_1, ..., x_n : A_n\}]\!] := [\![\Gamma]\!] = [\![A_1]\!] \times ... \times [\![A_n]\!]$;
4) Interpretation for typing assignment: $[\![\Gamma \vdash M : A]\!] := [\![M]\!] : [\![\Gamma]\!] \to [\![A]\!]$.
5) Interpretation for typing rules:

$$\frac{}{[\![\Gamma, x : A \vdash x : A]\!] = \pi_2 : [\![\Gamma]\!] \times [\![A]\!] \to [\![A]\!]}$$

$$\frac{[\![\Gamma, x : A \vdash M : B]\!] = f : [\![\Gamma]\!] \times [\![A]\!] \to [\![B]\!]}{[\![\Gamma \vdash (\lambda x.M) : A \to B]\!] = \Lambda(f) : [\![\Gamma]\!] \to [\![B]\!]^{[\![A]\!]}}$$

$$\frac{[\![\Gamma \vdash M : A \to B]\!] = [\![M]\!] : [\![\Gamma]\!] \to [\![B]\!]^{[\![A]\!]} \qquad [\![\Gamma \vdash N : A]\!] = [\![N]\!] : [\![\Gamma]\!] \to [\![A]\!]}{[\![\Gamma \vdash (MN) : B]\!] = [\![\Gamma]\!] \xrightarrow{\langle [\![M]\!], [\![N]\!] \rangle} [\![B]\!]^{[\![A]\!]} \times [\![A]\!] \xrightarrow{\epsilon} [\![B]\!]}$$

$$\frac{[\![\Gamma \vdash M : A]\!] = f : [\![\Gamma]\!] \to [\![A]\!] \qquad [\![\Gamma \vdash N : B]\!] = g : [\![\Gamma]\!] \to [\![B]\!]}{[\![\Gamma \vdash (M, N) : A \times B]\!] = \langle f, g \rangle : [\![\Gamma]\!] \to [\![A]\!] \times [\![B]\!]}$$

$$\frac{[\![\Gamma \vdash p : A_1 \times A_2]\!] = f : [\![\Gamma]\!] \to [\![A_1]\!] \times [\![A_2]\!]}{[\![\Gamma \vdash \pi_i p : A_i]\!] = [\![\Gamma]\!] \xrightarrow{f} [\![A_1]\!] \times [\![A_2]\!] \xrightarrow{\pi_i} [\![A_i]\!]}\ i \in \{1, 2\}$$

$$\frac{[\![\Gamma \vdash M : A]\!] = [\![M]\!] : [\![\Gamma]\!] \to [\![A]\!]}{[\![\Gamma \vdash \mathbf{pure}\, M : \boldsymbol{K}A]\!] := [\![\Gamma]\!] \xrightarrow{[\![M]\!]} [\![A]\!] \xrightarrow{\eta_{[\![A]\!]}} \mathcal{K}[\![A]\!]}$$

$$\frac{[\![\Gamma \vdash \vec{M} : \mathbf{K}\vec{A}]\!] = \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle : [\![\Gamma]\!] \to \prod_{i=1}^{n} \mathcal{K}[\![A_i]\!] \qquad [\![\vec{x} : \vec{A} \vdash N : B]\!] = [\![N]\!] : \prod_{i=1}^{n} [\![A_i]\!] \to [\![B]\!]}{[\![\Gamma \vdash \mathbf{let\ pure}\, \vec{x} = \vec{M}\ \mathbf{in}\ M : \mathbf{K}B]\!] = \mathcal{K}([\![N]\!]) \circ *_{[\![A_1]\!], \dots, [\![A_n]\!]} \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle : [\![\Gamma]\!] \to \mathcal{K}[\![B]\!]}$$

Definition 13. *Simultaneous substitution*

Let $\Gamma = \{x_1 : A_1, ..., x_n : A_n\}$, $\Gamma \vdash M : A$ *and for all* $i \in \{1, ..., n\}$, $\Gamma \vdash M_i : A_i$.

We define simultaneous substitution $M[\vec{x} := \vec{M}]$ *recursively by:*

1) $x_i[\vec{x} := \vec{M}] = M_i$;
2) $(\lambda x.M)[\vec{x} := \vec{M}] = \lambda x.(M[\vec{x} := \vec{M}])$;
3) $(MN)[\vec{x} := \vec{M}] = (M[\vec{x} = \vec{M}])(N[\vec{x} := \vec{M}])$;
4) $\langle M, N \rangle = \langle (M[\vec{x} = \vec{M}]), (N[\vec{x} := \vec{M}]) \rangle$;
5) $(\pi_i P)[\vec{x} := \vec{M}] = \pi_i(P[\vec{x} = \vec{M}])$;
6) $(\mathbf{pure}\, M)[\vec{x} := \vec{M}] = \mathbf{pure}\, (M[\vec{x} = \vec{M}])$;
7) $(\mathbf{let\ pure}\, \vec{x} = \vec{M}\ \mathbf{in}\ N)[\vec{y} := \vec{P}] = \mathbf{let\ pure}\, \vec{x} = (\vec{M}[\vec{y} := \vec{P}])\ \mathbf{in}\ N$

Lemma 5.
$[\![M[x_1 := M_1, \dots, x_n := M_n]]\!] = [\![M]\!] \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle$.

Proof.
1) $[\![\Gamma \vdash (\mathbf{pure}\, M)[\vec{x} := \vec{M}] : \mathbf{K}A]\!] = [\![\Gamma \vdash \mathbf{pure}\, M : \mathbf{K}A]\!] \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle$.

$$\begin{aligned}
[\![\Gamma \vdash (\mathbf{pure}\, M)[\vec{x} := \vec{M}] : \mathbf{K}A]\!] &= [\![\Gamma \vdash \mathbf{pure}\, (M[\vec{x} := \vec{M}]) : \mathbf{K}A]\!] && \text{Substitution definition} \\
&= \eta_{[\![A]\!]} \circ [\![(M[\vec{x} := \vec{M}])]\!] && \text{Translation for pure} \\
&= \eta_{[\![A]\!]} \circ ([\![M]\!] \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle) && \text{Induction hypothesis} \\
&= (\eta_{[\![A]\!]} \circ [\![M]\!]) \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle && \text{Associativity of composition} \\
&= [\![\Gamma \vdash \mathbf{pure}\, M : \mathbf{K}A]\!] \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle && \text{Translation for pure}
\end{aligned}$$

2) $[\![\Gamma \vdash (\mathbf{let\ pure}\, \vec{x} = \vec{M}\ \mathbf{in}\ N)[\vec{y} := \vec{P}] : \mathbf{K}B]\!] = [\![\Gamma \vdash \mathbf{let\ pure}\, \vec{x} = \vec{M}\ \mathbf{in}\ N : \mathbf{K}B]\!] \circ \langle [\![P_1]\!], \dots, [\![P_n]\!] \rangle$

$[\![\Gamma \vdash (\mathbf{let\,pure}\,\vec{x} = \vec{M}\,\mathbf{in}\,N)[\vec{y} := \vec{P}] : \mathbf{K}B]\!] =$
Substitution definition
$[\![\Gamma \vdash \mathbf{let\,pure}\,\vec{x} = (\vec{M}[\vec{y} := \vec{P}])\,\mathbf{in}\,N : \mathbf{K}B]\!] =$
Interpretaion for $let_{\mathbf{K}}$
$\mathcal{K}([\![N]\!]) \circ *_{[\![A_1]\!],\ldots,[\![A_n]\!]} \circ [\![\Gamma \vdash (\vec{M}[\vec{y} := \vec{P}]) \vdash : \mathbf{K}\vec{A}]\!] =$
Induction hypothesis
$\mathcal{K}([\![N]\!]) \circ *_{[\![A_1]\!],\ldots,[\![A_n]\!]} \circ ([\![\vec{M}]\!] \circ \langle [\![P_1]\!], \ldots, [\![P_n]\!]\rangle) =$
Associativity of composition
$(\mathcal{K}([\![N]\!]) \circ *_{[\![A_1]\!],\ldots,[\![A_n]\!]} \circ [\![\vec{M}]\!]) \circ \langle [\![P_1]\!], \ldots, [\![P_n]\!]\rangle =$
By interpretation
$[\![\Gamma \vdash (\mathbf{let\,pure}\,\vec{x} = \vec{M}\,\mathbf{in}\,N]\!] \circ \langle [\![P_1]\!], \ldots, [\![P_n]\!]\rangle$

□

Lemma 6.
i) Let $\Gamma \vdash M : A$ *and* $M \twoheadrightarrow_\beta N$*, then* $[\![\Gamma \vdash M : A]\!] = [\![\Gamma \vdash N : A]\!]$*;*
ii) Let $\Gamma \vdash M : A$ *and* $M \twoheadrightarrow_\eta N$*, then* $[\![\Gamma \vdash M : A]\!] = [\![\Gamma \vdash N : A]\!]$*;*

Proof.
i) For β-reduction

Cases with β-reductions for $let_{\mathbf{K}}$ are shown in [20]. Let us consider cases with **pure**.

1) $[\![\Gamma \vdash \mathbf{pure}\,((\lambda x.M)N) : \mathbf{K}B]\!] = [\![\Gamma \vdash \mathbf{pure}\,(M[x := N]]) : \mathbf{K}B]\!]$

$[\![\Gamma \vdash \mathbf{pure}\,(\lambda x.M)N : \mathbf{K}B]\!] =$	By interpretation
$\eta_{[\![B]\!]} \circ (\epsilon \circ \langle \Lambda([\![M]\!]), [\![N]\!]\rangle) =$	Property of $\times$
$\eta_{[\![B]\!]} \circ (\epsilon \circ (\Lambda([\![M]\!]) \times id_{[\![A]\!]}) \circ \langle id_{[\![\Gamma]\!]}, [\![N]\!]\rangle) =$	Associativity of composition
$\eta_{[\![B]\!]} \circ ((\epsilon \circ (\Lambda([\![M]\!]) \times id_{[\![A]\!]})) \circ \langle id_{[\![\Gamma]\!]}, [\![N]\!]\rangle) =$	Exponentiation property
$\eta_{[\![B]\!]} \circ ([\![M]\!] \circ \langle id_{[\![\Gamma]\!]}, [\![N]\!]\rangle) =$	Substitution lemma
$\eta_{[\![B]\!]} \circ [\![M[\vec{x}, x := \vec{x}, N]]\!] =$	By interpretation
$[\![\Gamma \vdash \mathbf{pure}\,(M[x := N]]) : \mathbf{K}B]\!]$	

2) $[\![\Gamma \vdash \mathbf{pure}\,(\pi_i\langle [\![M_1]\!], [\![M_2]\!]\rangle) : \mathbf{K}A_i]\!] = [\![\Gamma \vdash\ \mathbf{pure} M_i : \mathbf{K}A_i]\!]$

$[\![\Gamma \vdash \mathbf{pure}\,(\pi_i\langle M_1, M_2]\rangle) : \mathbf{K}A_i]\!] =$	By interpretation
$\eta_{[\![A_i]\!]} \circ \pi_i \circ \langle [\![M_1]\!], [\![M_2]\!]\rangle =$	Property of $\times$
$\eta_{[\![A_i]\!]} \circ [\![M_i]\!] =$	By interpetation
$[\![\Gamma \vdash \mathbf{pure}\,M_i : \mathbf{K}A_i]\!]$	

ii) For η-reduction.

1) $[\![\Gamma \vdash \mathbf{pure}\,(\lambda x.Mx) : \mathbf{K}(A \to B)]\!] = [\![\Gamma \vdash \mathbf{pure}\,M : \mathbf{K}(A \to B)]\!]$.

$[\![\Gamma \vdash \mathbf{pure}\,(\lambda x.Mx) : \mathbf{K}(A \to B)]\!] =$	By interpetation
$\eta_{[\![B]\!]^{[\![A]\!]}} \circ \Lambda(\epsilon \circ [\![M]\!] \times id_{[\![A]\!])}$	Exponentiation property
$\eta_{[\![B]\!]^{[\![A]\!]}} \circ [\![M]\!] =$	By interpretation
$[\![\Gamma \vdash \mathbf{pure}\,M : \mathbf{K}(A \to B)]\!]$	

2) $[\![\Gamma \vdash \mathbf{pure}\,\langle \pi_1 M, \pi_2 M\rangle : \mathbf{K}(A \times B)]\!] = [\![\Gamma \vdash \mathbf{pure}\,M : \mathbf{K}(A \times B)]\!]$

$$
\begin{array}{ll}
[\![\Gamma \vdash \mathbf{pure}\ \langle \pi_1 M, \pi_2 M\rangle : \mathbf{K}(A \times B)]\!] = & \text{By interpetation} \\
\eta_{[\![A]\!]\times[\![B]\!]} \circ \langle \pi_1 \circ [\![M]\!], \pi_2 \circ [\![M]\!]\rangle = & \text{By the property of a product of morphisms} \\
\eta_{[\![A]\!]\times[\![B]\!]} \circ [\![M]\!] = & \text{By interpetation} \\
[\![\Gamma \vdash \mathbf{pure}\ M : \mathbf{K}(A \times B)]\!] &
\end{array}
$$

3) $[\![\vdash \mathbf{let\ pure}\ \_\_ = \_\_\ \mathbf{in}\ N : KA]\!] = [\![\vdash \mathbf{pure}\ N : \mathbf{KA}]\!]$.

$$
\begin{array}{ll}
[\![\vdash \mathbf{let\ pure}\ \_\_ = \_\_\ \mathbf{in}\ N : KA]\!] = & \text{By interpetation} \\
\mathcal{K}([\![N]\!]) \circ \eta_{\mathbb{1}} = & \text{Naturality for } \eta \\
\eta_{[\![A]\!]} \circ [\![N]\!] = & \text{By interpretation} \\
[\![\vdash \mathbf{pure}\ N : \mathbf{K}A]\!] &
\end{array}
$$

$\square$

Theorem 2. *Soundness*
Let $\Gamma \vdash M : A$ *and* $M =_{\beta\eta} N$, *then* $[\![\Gamma \vdash M : A]\!] = [\![\Gamma \vdash N : A]\!]$

Proof. Straightforwardly follows from two previous lemmas. $\square$

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