Modal type theory based on the intuitionistic epistemic logic

Abstract

Modal intuitionistic epistemic logic IEL^- was proposed by S.Artemov and T. Protopopescu as the formal foundation for the intuitionistic theory of knowledge. We construct a modal simply typed lambda-calculus which is Curry-Howard isomorphic to IEL^- as formal theory of calculations with applicative functors in functional programming languages like Haskell or Idris.

1 Introduction

Modal intutionistic epistemic logic IEL was proposed by S. Artemov and T. Proropopescu [1]. IEL provides the epistimology and the theory of knowledge as based on BHK-semantics of intuitionistic logic. IEL^- is a variant of IEL, that corresponds to intuitionistic belief. Informally, $\mathbf{K}A$ denotes that A is verified intuitionistically.

Intuitionistic epistemic logic IEL^- is defined with by following axioms and derivation rules:

Definition 1. *Intuitionistic epistemic logic IEL:*
1) IPC axioms;
2) $\mathbf{K}(A \to B) \to (\mathbf{K}A \to \mathbf{K}B)$ *(normality);*
3) $A \to \mathbf{K}A$ *(co-reflection);*
Rule: MP.

We have the deduction theorem and necessitation rule which is derivable.

V. Krupski and A. Yatmanov provided the sequential calculus for IEL and proved that this calculus is PSPACE-complete [2].

Functional programming languages such as Haskell [3], Idris [4], Purescript [5] Elm [6] or Scala [**?**] have special type classes[1] for calculations with container types like `Functor` and `Applicative` [2]:

```
class Functor f where
  fmap :: (a -> b) -> f a -> f b

class Functor f => Applicative f where
  pure :: a -> f a
  (<*>) :: f (a -> b) -> f a -> f b
```

[1] Type class in Haskell is a general interface for special group of datatypes.

[2] Reader may read more about container types in the Haskell standard library documentation[7] or in the next one textbook [8]

By *container* (or *computational context*) type we mean some type-operator f, where f is a "function" from $*$ to $*$: type operator takes a simple type (which has kind $*$) and returns another simple type type with kind $*$. For more detailed description of the type system with kinds used in Haskell see [12].

The motivation for using an applicative functor is quite natural. Applicative functor allows to generalize the action of a functor for functions with arbitrary number of arguments, for instance:

```
liftA2 :: Applicative f => (a -> b -> c) -> f a -> f b -> f c
liftA2 f x y = pure f <*> x <*> y
```

It's not difficult to see that modal axioms in IEL^- and types of the methods of Applicative class in Haskell-like languages (which is described below) are syntactically similar and we are going to show that this coincidence has a non-trivial computational meaning.

The main goal of our research is a relationship between intuitionistic epistemic logic IEL^- and functional programming with effects. We show that relationship by building the type system (which is called $\lambda_{\mathbf{K}}$) which is Curry-Howard isomorphic to IEL^-. So we will consider $\mathbf{K}$-modality as an arbitrary applicative functor.

$\lambda_{\mathbf{K}}$ consists of the rules for simply typed lambda-calculus and special typing rules for lifting types into the applicative functor $\mathbf{K}$. We assume that our type system will axiomatize the simplest case of computation with effects with one container. We provide proof-theoretical view on this kind of computations in functional programming and prove strong normalization and confluence.

2 Typed lambda-calculus based on IEL$^-$

At first we define the natural deduction for IEL$^-$:

Definition 2. *Natural deduction NIEL for IEL$^-$ is an extenstion of intuitionistic natural deduction with additional derivation rules for modality:*

$$\frac{\Gamma \vdash A}{\Gamma \vdash \boldsymbol{K}A}\ \boldsymbol{K}_I \qquad\qquad \frac{\Gamma \vdash \mathbf{K}\vec{A} \qquad \vec{A} \vdash B}{\Gamma \vdash \mathbf{K}B}$$

Where $\Gamma \vdash \mathbf{K}\vec{A}$ is a syntax sugar for $\Gamma \vdash \mathbf{K}A_1, \dots, \Gamma \vdash \mathbf{K}A_n$.

Lemma 1. $\Gamma \vdash_{NIEL^-_{\wedge,\to}} A \Rightarrow IEL^- \vdash \bigwedge\Gamma \to A.$

Proof. Induction on the derivation.

Let us consider cases with modality.

1) If $\Gamma \vdash_{NIEL^-_{\wedge,\to}} A$, then $IEL^- \vdash \bigwedge\Gamma \to \mathbf{K}A$.

(1)	$\bigwedge\Gamma \to A$	assumption
(2)	$A \to \mathbf{K}A$	co-reflection
(3)	$(\bigwedge\Gamma \to A) \to ((A \to \mathbf{K}A) \to (\bigwedge\Gamma \to \mathbf{K}A))$	IPC theorem
(4)	$(A \to \mathbf{K}A) \to (\bigwedge\Gamma \to \mathbf{K}A)$	from (1), (3) and MP
(5)	$\bigwedge\Gamma \to \mathbf{K}A$	from (2), (4) and MP

2) If $\Gamma \vdash_{NIEL^{-}_{\wedge,\rightarrow}} \mathbf{K}\vec{A}$ and $\vec{A} \vdash B$, then $IEL^{-} \vdash \bigwedge \Gamma \rightarrow \mathbf{K}B$.

(1)	$\bigwedge \Gamma \rightarrow \bigwedge_{i=1}^{n} \mathbf{K}A_i$	assumption
(2)	$\bigwedge_{i=1}^{n} \mathbf{K}A_i \rightarrow \mathbf{K} \bigwedge_{i=1}^{n} A_i$	IEL theorem
(3)	$\bigwedge \Gamma \rightarrow \mathbf{K} \bigwedge_{i=1}^{n} A_i$	from (1), (2) and transitivity
(4)	$\bigwedge_{i=1}^{n} A_i \rightarrow B$	assumption
(5)	$(\bigwedge_{i=1}^{n} A_i \rightarrow B) \rightarrow \mathbf{K}(\bigwedge_{i=1}^{n} A_i \rightarrow B)$	co-reflection
(6)	$\mathbf{K}(\bigwedge_{i=1}^{n} A_i \rightarrow B)$	from (2), (3) and MP
(7)	$\mathbf{K} \bigwedge_{i=1}^{n} A_i \rightarrow \mathbf{K}B$	from (6) and normality
(8)	$\bigwedge \Gamma \rightarrow \mathbf{K}B$	from (3), (7) and transitivity

□

Lemma 2. *If $IEL^{-} \vdash A$, then $NIEL^{-} \vdash A$.*

Proof. Straightforward derivation of modal axioms in NIEL^{-}. We consider this derivation below using terms. □

At the next step we build the typed lambda-calculus based on NIEL$^{-}_{\wedge,\rightarrow}$ by proof-assingment in rules.

At first, we define lambda-terms and types for this lambda-calculus.

Definition 3. *The set of terms:*

Let $\mathbb{V}$ be the set of variables. The set $\Lambda_{\mathbf{K}}$ of terms is defined by the grammar:

$$\Lambda_{\mathbf{K}} ::= \mathbb{V} \mid (\lambda\Lambda.\Lambda_{\mathbf{K}}) \mid (\Lambda_{\mathbf{K}}\Lambda_{\mathbf{K}}) \mid (\Lambda_{\mathbf{K}}, \Lambda_{\mathbf{K}}) \mid (\pi_1\Lambda_{\mathbf{K}}) \mid (\pi_2\Lambda_{\mathbf{K}}) \mid (\mathbf{pure}\ \Lambda_{\mathbf{K}}) \mid (\mathbf{let\ pure}\ \Lambda_{\mathbf{K}} = \Lambda_{\mathbf{K}}\ \mathbf{in}\ \Lambda_{\mathbf{K}})$$

Definition 4. *The set of types:*

Let $\mathbb{T}$ be the set of atomic types. The set $\mathbb{T}_{\mathbf{K}}$ of types with applicative functor $\mathbf{K}$ is generated by the grammar:

$$\mathbb{T}_{\mathbf{K}} ::= \mathbb{T} \mid (\mathbb{T}_{\mathbf{K}} \rightarrow \mathbb{T}_{\mathbf{K}}) \mid (\mathbb{T}_{\mathbf{K}} \times \mathbb{T}_{\mathbf{K}}) \mid (\mathbf{K}\mathbb{T}_{\mathbf{K}}) \tag{1}$$

Context, domain of context and range of context are defined standardly [11][12].

Our type system is based on the Curry-style typing rules:

Definition 5. *Modal typed lambda calculus $\lambda_{\mathbf{K}}$ based on NIEL$^{-}_{\wedge,\rightarrow}$:*

$$\frac{}{\Gamma, x : A \vdash x : A}\ ax$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B} \to_i \qquad \frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash MN : B} \to_e$$

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash N : B}{\Gamma \vdash \langle M, N \rangle : A \times B} \times_i \qquad \frac{\Gamma \vdash M : A_1 \times A_2}{\Gamma \vdash \pi_i M : A_i} \times_e,\ i \in \{1, 2\}$$

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash \textbf{pure}\ M : \mathbf{K}A} \boldsymbol{K_I} \qquad \frac{\Gamma \vdash \vec{M} : \mathbf{K}\vec{A} \qquad \vec{x} : \vec{A} \vdash N : B}{\Gamma \vdash \textbf{let pure}\ \vec{x} = \vec{M}\ \textbf{in}\ N : \mathbf{K}B} \mathit{let}_{\mathbf{K}}$$

$\mathbf{K}_I$-typing rule is the same as $\bigcirc$-introduction in lax logic (also known as monadic metalanguage [17]) and in typed lambda-calculus which is derived by proof-assignment for lax-logic proofs. $\mathbf{K}_I$ allows to inject an object of type α into the functor. $\mathbf{K}_I$ reflects the Haskell method **pure** for Applicative class. It plays the same role as the **return** method in Monad class.

$\text{let}_{\mathbf{K}}$ is similar to the $\Box$-rule in typed lambda calculus for intuitionistic normal modal logic **IK**, which is described in [19].

In fact, our calculus is the extention of typed lambda calculus for **IK** with typing rule appropriate to co-reflection.

Here are some examples of closed terms:

- $(\lambda x.\textbf{pure}\, x) : A \to \mathbf{K}A$;
- $\lambda f.\lambda x.\textbf{let pure}\, g, y = f, x\, \textbf{in}\, gy : \mathbf{K}(A \to B) \to \mathbf{K}A \to \mathbf{K}B$
- $\lambda f.\lambda x.\textbf{let pure}\, g, y = \textbf{pure}\, f, x\, \textbf{in}\, gy : (A \to B) \to \mathbf{K}A \to \mathbf{K}B$

Now we define free variables and substitutions. β-reduction, multi-step β-reduction and β-equality are defined standardly:

Definition 6. *Set $FV(M)$ of free variables for arbitrary term M:*
1) $FV(x) = \{x\}$;
2) $FV(\lambda x.M) = FV(M) \setminus \{x\}$;
3) $FV(MN) = FV(M) \cup FV(N)$;
4) $FV(\langle M, N \rangle) = FV(M) \cup FV(N)$;
5) $FV(\pi_i M) \subseteq FV(M)$, $i \in \{1, 2\}$;
6) $FV(\mathit{pure}\ M) = FV(M)$;
7) $FV(\textbf{let pure}\ \vec{N} = \vec{M}\ \textbf{in}\ M) = \bigcup_{i=1}^{n} FV(M)$, where $n = |\vec{M}|$.

Definition 7. *Substitution:*
1) $x[x := N] = N$, $x[y := N] = x$;
2) $(MN)[x := N] = M[x := N]N[x := N]$;
3) $(\lambda x.M)[x := N] = \lambda x.M[x := N]$;
4) $(M, N)[x := P] = (M[x := P], N[x := P])$;
5) $(\pi_i M)[x := P] = \pi_i(M[x := P])$, $i \in \{1, 2\}$;
6) $(\textbf{pure}\, M)[x := P] = \textbf{pure}\, (M[x := P])$;
7) $(\textbf{let pure}\, \vec{x} = \vec{M}\, \textbf{in}\, N)[y := P] = \textbf{let pure}\, \vec{x} = (\vec{M}[y := P])\, \textbf{in}\, M$.

Definition 8. *Type substituition*

The substituition of type C for type variable B in type A inductively defined as follows:

1) $B[B := C] = B$ *and* $D[B := C] = D$*, if* $B \neq D$*;*

2) $(A_1 \alpha A_2)[B := C] = (A_1[B := C])\alpha(A_2[B := C])$*, where* $\alpha \in \{\rightarrow, \times\}$*;*

3) $(\mathbf{K}A)[B := C] = \mathbf{K}(A[B := C])$*.*

4) Let Γ *be the context, then* $\Gamma[B := C] = \{x : (A[B := C]) \mid x : A \in \Gamma\}$

Definition 9. *β-reduction and η-reduction rules for $\lambda\mathbf{K}$.*

1) $(\lambda x.M)N \rightarrow_\beta M[x := N]$*;*

2) $\pi_1\langle M, N\rangle \rightarrow_\beta M$*;*

3) $\pi_2\langle M, N\rangle \rightarrow_\beta N$*;*

4) $\begin{array}{l}\mathbf{let\,pure}\, \langle \vec{x}, y, \vec{z}\rangle = \langle \vec{M}, \mathbf{let\,pure}\, \vec{w} = \vec{N}\, \mathbf{in}\, Q, \vec{P}\rangle \; in \; R \rightarrow_\beta \\ \mathbf{let\,pure}\, \langle \vec{x}, \vec{w}, \vec{z}\rangle = \langle \vec{M}, \vec{N}, \vec{P}\rangle\, \mathbf{in}\, R[y := Q]\end{array}$

5) $\mathbf{let\,pure}\, \vec{x} = \mathbf{pure}\, \vec{M}\, \mathbf{in}\, N \rightarrow_\beta \mathbf{pure}\, N[\vec{x} := \vec{M}]$

6) $\mathbf{let\,pure}\, \_\_ = \_\_\, \mathbf{in}\, N \rightarrow_\beta \mathbf{pure}\, N$*;*

7) $\lambda x.fx \rightarrow_\eta f$*;*

8) $\langle \pi_1 P, \pi_2 P\rangle \rightarrow_\eta P$*;*

9) $\mathbf{let\,pure}\, x = M\, \mathbf{in}\, x \rightarrow_\eta M$*;*

10) $M \rightarrow_{\beta\eta} N \Rightarrow \mathbf{pure\,M} \rightarrow_{\beta\eta} \mathbf{pure\,N}$

3 Basic lemmas

Now we will prove standard lemmas for contexts in type systems[3]:

Lemma 3. *Basic lemmas for $\mathbf{K}_I$.*

i) Let $\Gamma \vdash \mathbf{pure}\, M : \mathbf{K}A$*, then* $\Gamma \vdash M : A$*;*

ii) Let $\Gamma \vdash M : A$ *and* $\Gamma \subseteq \Delta$*, then* $\Delta \vdash M : A$*;*

iii) Let $\Gamma, x : A \vdash \mathbf{pure}\, M : \mathbf{K}B$ *and* $\Gamma \vdash N : A$*, then* $\Gamma \vdash \mathbf{pure}\, M[x := N] : \mathbf{K}B$*.*

iv) Let $\Gamma \vdash \mathbf{pure}\, M : \mathbf{K}A$*, then* $\Gamma[B := C] \vdash \mathbf{pure}\, M : \mathbf{K}(A[B := C])$*.*

Proof.

i) Induction on $\Gamma \vdash \mathbf{pure}\, M : \mathbf{K}A$ and $\Gamma \vdash \mathbf{let\,pure}\, \vec{x} = \vec{N}\, \mathbf{in}\, N : \mathbf{K}B$ correspondently.

ii) Let $\Gamma \vdash \mathbf{pure}\, M : \mathbf{K}A$. Then $\Gamma \vdash M : A$ by generation and $\Delta \vdash M : A$ by assumption. So $\Delta \vdash \mathbf{pure}\, M : \mathbf{K}A$ by $\mathbf{K}_I$.

iii) Let $\Gamma, x : A \vdash \mathbf{pure}\, M : \mathbf{K}B$ and $\Gamma \vdash N : A$.

By generation $\Gamma, x : A \vdash M : B$ and by assumption $\Gamma \vdash M[x := N] : B$.

By K_I, $\Gamma \vdash \mathbf{pure}\, (M[x := N]) : \mathbf{K}B$.

iv) Let $\Gamma \vdash \mathbf{pure}\, M : \mathbf{K}A$. By generation $\Gamma \vdash M : A$ and by assumption $\Gamma[B := C] \vdash M : A[B := C]$.

By K_I $\Gamma \vdash \mathbf{pure}\, M : \mathbf{K}(A[B := C])$. □

Theorem 1. *Subject reduction*

Let $\Gamma \vdash M : A$ *and* $M \twoheadrightarrow_{\beta\eta} N$*, then* $\Gamma \vdash N : A$

[3] We will not prove cases with $\rightarrow$-constructor, they are proved standardly in the same lemmas for simply typed lambda calculus, for example see [11] [12] [14]. We will consider only modal cases

Proof. For cases with application, abstraction and pairs see [12] [13].

1) Let $\Gamma \vdash$ **let pure** $\langle \vec{x}, y, \vec{z} \rangle = \langle \vec{M}, \mathbf{let\,pure}\, \vec{w} = \vec{N}\, \mathbf{in}\, Q, \vec{P} \rangle\ in\ R : \mathbf{K}B$, then Γ**let pure** $\langle \vec{x}, \vec{w}, \vec{z} \rangle = \langle \vec{M}, \vec{N}, \vec{P} \rangle$ **in** $R[y := Q] : \mathbf{K}B$

2) Let $\Gamma \vdash$ **let pure** $x = M$ **in** $x : \mathbf{K}A$, then $\Gamma \vdash M : \mathbf{K}A$.

See [19].

$\square$

Theorem 2.

i) $\twoheadrightarrow_\beta$ *is strongly normalizing;*

ii) $\twoheadrightarrow_\beta$ *is confluent.*

Proof.

i) Strong normalization for **IK** was proved by Kakutani for call-by-value and for call-by name [19] [20].

It is easy to extend Kakutani's result for **pure**-rules.

ii) Confluence can be proved by extending Barendregt's technique with term underlying for untyped or simply typed lambda calculus [15].

Hence, it is sufficient to prove strip lemma. $\square$

Theorem 3.

Normal form in call-by-name $\lambda_{\mathbf{K}}$ *has the subformula property.*

Proof. By induction on the structure of term. Case with **let pure** $\vec{x} = \vec{M}$ **in** N was considered by Kakutani [19] [20]. Similary, if **pure** M is a normal form, so M is a normal form too by hypothesis. $\square$

4 Categorical semantics

Definition 10. *Monoidal functor*

Let $\langle \mathcal{C}, \otimes_1, \mathbb{1} \rangle$ *and* $\langle \mathcal{D}, \otimes_2, \mathbb{1}' \rangle$ *are monoidal categories.*

A monoidal functor $\mathcal{F} : \langle \mathcal{C}, \otimes_1, \mathbb{1} \rangle \to \langle \mathcal{D}, \otimes_2, \mathbb{1}' \rangle$ *is a functor* $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ *with additional natural transformations, which satisfy the well-known conditions described in [23]:*

1) $u : \mathbb{1}' \to \mathcal{F}\mathbb{1}$*;*

2) $*_{A,B} : \mathcal{F}A \otimes_2 \mathcal{F}B \to \mathcal{F}(A \otimes_1 B)$.

Definition 11. *Applicative functor*

An applicative functor is a triple $\langle \mathcal{C}, \mathcal{K}, \eta \rangle$*, where* $\mathcal{C}$ *is a symmetric monoidal category,* $\mathcal{K}$ *is a monoidal and* $\eta : Id_{\mathcal{C}} \Rightarrow \mathcal{K}$ *is a natural transformation (similar to unit in monad), such that:*

1) $u = \eta_{\mathbb{1}}$*;*

2) $*_{A,B} \circ (\eta_A \otimes \eta_B) = \eta_{A \otimes B}$*;*

3) Weak commutativity condition:

$$\begin{tikzcd}
A \otimes \mathcal{K}B \arrow[r, "\eta_A \otimes id_{\mathcal{K}B}"] \arrow[d, "\sigma_{A,\mathcal{K}B}"'] & \mathcal{K}A \otimes \mathcal{K}B \arrow[r, "*_{A,B}"] & \mathcal{K}(A \otimes B) \arrow[d, "\mathcal{K}(\sigma_{A,B})"] \\
\mathcal{K}B \otimes A \arrow[r, "id_{\mathcal{K}B} \otimes \eta_A"'] & \mathcal{K}B \otimes \mathcal{K}A \arrow[r, "*_{B,A}"'] & \mathcal{K}(B \otimes A)
\end{tikzcd}$$

4.1 Soundness and completeness

Theorem 4. *Soundness*

Let $\Gamma \vdash M : A$ and $M =_{\beta\eta} N$, then $[\![\Gamma \vdash M : A]\!] = [\![\Gamma \vdash N : A]\!]$

Proof.

Definition 12. *Semantical translation from $\lambda_{\boldsymbol{K}}$ to CCC with applicative functor $\mathcal{K}$:*

1) Interpretation for types:
$[\![A]\!] := \hat{A}, A \in \mathbb{T}$;
$[\![A \to B]\!] := [\![A]\!] \to [\![B]\!]$;
$[\![A \times B]\!] := [\![A]\!] \times [\![B]\!]$.
2) Interpretation for modal types: $[\![\boldsymbol{K}A]\!] = \mathcal{K}[\![A]\!]$;
3) Interpretaion for contexts:
$[\![\Gamma = \{x_1 : A_1, ..., x_n : A_n\}]\!] := [\![\Gamma]\!] = [\![A_1]\!] \times ... \times [\![A_n]\!]$;
4) Interpretation for typing assignment: $[\![\Gamma \vdash M : A]\!] := [\![M]\!] : [\![\Gamma]\!] \to [\![A]\!]$.
5) Interpretation for typing rules:

$$\frac{}{[\![\Gamma, x : A \vdash x : A]\!] = \pi_2 : [\![\Gamma]\!] \times [\![A]\!] \to [\![A]\!]}$$

$$\frac{[\![\Gamma, x : A \vdash M : B]\!] = f : [\![\Gamma]\!] \times [\![A]\!] \to [\![B]\!]}{[\![\Gamma \vdash (\lambda x.M) : A \to B]\!] = \Lambda(f) : [\![\Gamma]\!] \to [\![B]\!]^{[\![A]\!]}}$$

$$\frac{[\![\Gamma \vdash M : A \to B]\!] = [\![M]\!] : [\![\Gamma]\!] \to [\![B]\!]^{[\![A]\!]} \qquad [\![\Gamma \vdash N : A]\!] = [\![N]\!] : [\![\Gamma]\!] \to [\![A]\!]}{[\![\Gamma \vdash (MN) : B]\!] = [\![\Gamma]\!] \xrightarrow{\langle [\![M]\!], [\![N]\!] \rangle} [\![B]\!]^{[\![A]\!]} \times [\![A]\!] \xrightarrow{\epsilon} [\![B]\!]}$$

$$\frac{[\![\Gamma \vdash M : A]\!] = f : [\![\Gamma]\!] \to [\![A]\!] \qquad [\![\Gamma \vdash N : B]\!] = g : [\![\Gamma]\!] \to [\![B]\!]}{[\![\Gamma \vdash (M, N) : A \times B]\!] = \langle f, g \rangle : [\![\Gamma]\!] \to [\![A]\!] \times [\![B]\!]}$$

$$\frac{[\![\Gamma \vdash p : A_1 \times A_2]\!] = f : [\![\Gamma]\!] \to [\![A_1]\!] \times [\![A_2]\!]}{[\![\Gamma \vdash \pi_i p : A_i]\!] = [\![\Gamma]\!] \xrightarrow{f} [\![A_1]\!] \times [\![A_2]\!] \xrightarrow{\pi_i} [\![A_i]\!]}\ i \in \{1, 2\}$$

$$\frac{[\![\Gamma \vdash M : A]\!] = [\![M]\!] : [\![\Gamma]\!] \to [\![A]\!]}{[\![\Gamma \vdash \textbf{pure}\, M : \boldsymbol{K}A]\!] := [\![\Gamma]\!] \xrightarrow{[\![M]\!]} [\![A]\!] \xrightarrow{\eta_{[\![A]\!]}} \mathcal{K}[\![A]\!]}$$

$$\frac{[\![\Gamma \vdash \vec{M} : \mathbf{K}\vec{A}]\!] = \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle : [\![\Gamma]\!] \to \prod_{i=1}^{n} \mathcal{K}[\![A_i]\!] \qquad [\![\vec{x} : \vec{A} \vdash N : B]\!] = [\![N]\!] : \prod_{i=1}^{n} [\![A_i]\!] \to [\![B]\!]}{[\![\Gamma \vdash \textbf{let pure}\, \vec{x} = \vec{M}\ \textbf{in}\ M : \mathbf{K}B]\!] = \mathcal{K}([\![N]\!]) \circ *_{[\![A_1]\!], \dots, [\![A_n]\!]} \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle : [\![\Gamma]\!] \to \mathcal{K}[\![B]\!]}$$

Definition 13. *Simultaneous substitution*

Let $\Gamma = \{x_1 : A_1, ..., x_n : A_n\}$, $\Gamma \vdash M : A$ and for all $i \in \{1, ..., n\}$, $\Gamma \vdash M_i : A_i$.

We define simultaneous substitution $M[\vec{x} := \vec{M}]$ recursively by:
1) $x_i[\vec{x} := \vec{M}] = M_i$;
2) $(\lambda x.M)[\vec{x} := \vec{M}] = \lambda x.(M[\vec{x} := \vec{M}])$;
3) $(MN)[\vec{x} := \vec{M}] = (M[\vec{x} = \vec{M}])(N[\vec{x} := \vec{M}])$;
4) $\langle M, N \rangle = \langle (M[\vec{x} = \vec{M}]), (N[\vec{x} := \vec{M}]) \rangle$;

5) $(\pi_i P)[\vec{x} := \vec{M}] = \pi_i(P[\vec{x} = \vec{M}])$*;*
6) $(\mathbf{pure}\, M)[\vec{x} := \vec{M}] = \mathbf{pure}\,(M[\vec{x} = \vec{M}])$*;*
7) $(\mathbf{let\ pure}\, \vec{x} = \vec{M}\ \mathbf{in}\ N)[\vec{y} := \vec{P}] = \mathbf{let\ pure}\, \vec{x} = (\vec{M}[\vec{y} := \vec{P}])\ \mathbf{in}\ N$

Lemma 4.
$[\![M[x_1 := M_1, \dots, x_n := M_n]]\!] = [\![M]\!] \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle$.

Proof.
1) $[\![\Gamma \vdash (\mathbf{pure}\, M)[\vec{x} := \vec{M}] : \mathbf{K}A]\!] = [\![\Gamma \vdash \mathbf{pure}\, M : \mathbf{K}A]\!] \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle$.

$$
\begin{array}{lll}
[\![\Gamma \vdash (\mathbf{pure}\, M)[\vec{x} := \vec{M}] : \mathbf{K}A]\!] & = [\![\Gamma \vdash \mathbf{pure}\,(M[\vec{x} := \vec{M}]) : \mathbf{K}A]\!] & \text{Substitution definition} \\
& = \eta_{[\![A]\!]} \circ [\![(M[\vec{x} := \vec{M}])]\!] & \text{Translation for pure} \\
& = \eta_{[\![A]\!]} \circ ([\![M]\!] \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle) & \text{Induction hypothesis} \\
& = (\eta_{[\![A]\!]} \circ [\![M]\!]) \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle & \text{Associativity of composition} \\
& = [\![\Gamma \vdash \mathbf{pure}\, M : \mathbf{K}A]\!] \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle & \text{Translation for pure}
\end{array}
$$

2) $$[\![\Gamma \vdash (\mathbf{let\ pure}\, \vec{x} = \vec{M}\ \mathbf{in}\ N)[\vec{y} := \vec{P}] : \mathbf{K}B]\!] = [\![\Gamma \vdash \mathbf{let\ pure}\, \vec{x} = \vec{M}\ \mathbf{in}\ N : \mathbf{K}B]\!] \circ \langle [\![P_1]\!], \dots, [\![P_n]\!] \rangle$$

$[\![\Gamma \vdash (\mathbf{let\ pure}\, \vec{x} = \vec{M}\ \mathbf{in}\ N)[\vec{y} := \vec{P}] : \mathbf{K}B]\!] =$
Substitution definition
$[\![\Gamma \vdash \mathbf{let\ pure}\, \vec{x} = (\vec{M}[\vec{y} := \vec{P}])\ \mathbf{in}\ N : \mathbf{K}B]\!] =$
Interpretaion for $let_{\mathbf{K}}$
$\mathcal{K}([\![N]\!]) \circ *_{[\![A_1]\!], \dots, [\![A_n]\!]} \circ [\![\Gamma \vdash (\vec{M}[\vec{y} := \vec{P}]) \vdash : \mathbf{K}\vec{A}]\!] =$
Induction hypothesis
$\mathcal{K}([\![N]\!]) \circ *_{[\![A_1]\!], \dots, [\![A_n]\!]} \circ ([\![\vec{M}]\!] \circ \langle [\![P_1]\!], \dots, [\![P_n]\!] \rangle) =$
Associativity of composition
$(\mathcal{K}([\![N]\!]) \circ *_{[\![A_1]\!], \dots, [\![A_n]\!]} \circ [\![\vec{M}]\!]) \circ \langle [\![P_1]\!], \dots, [\![P_n]\!] \rangle =$
By interpretation
$[\![\Gamma \vdash (\mathbf{let\ pure}\, \vec{x} = \vec{M}\ \mathbf{in}\ N]\!] \circ \langle [\![P_1]\!], \dots, [\![P_n]\!] \rangle$

□

Lemma 5.
Let $\Gamma \vdash M : A$ *and* $M \twoheadrightarrow_{\beta\eta} N$*, then* $[\![\Gamma \vdash M : A]\!] = [\![\Gamma \vdash N : A]\!]$*;*

Proof.
Cases with β-reductions for $let_{\mathbf{K}}$ are shown in [20]. Let us consider cases with **pure**.

1) $[\![\Gamma \vdash \mathbf{let\ pure}\, \vec{x} = \mathbf{pure}\, \vec{M}\ \mathbf{in}\ N : \mathbf{K}B]\!] = [\![\Gamma \vdash \mathbf{pure}\, N[\vec{x} := \vec{M}] : \mathbf{K}B]\!]$

$[\![\Gamma \vdash \mathbf{let\ pure}\ \vec{x} = \mathbf{pure}\ \vec{M}\ \mathbf{in}\ N : \mathbf{K}B]\!] =$
By interpretation
$\mathcal{K}([\![N]\!]) \circ *_{[\![A_1]\!],\dots,[\![A_n]\!]} \circ \langle \eta_{[\![A_1]\!]} \circ [\![M_1]\!], \dots, \eta_{[\![A_n]\!]} \circ [\![M_n]\!] \rangle =$
By the property of a pair of morphisms
$\mathcal{K}([\![N]\!]) \circ *_{[\![A_1]\!],\dots,[\![A_n]\!]} \circ (\eta_{[\![A_1]\!]} \times \dots \times \eta_{[\![A_n]\!]}) \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle =$
Associativity of composition
$\mathcal{K}([\![N]\!]) \circ (*_{[\![A_1]\!],\dots,[\![A_n]\!]} \circ (\eta_{[\![A_1]\!]} \times \dots \eta_{[\![A_n]\!]})) \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle =$
By the definition of an applicative functor
$\mathcal{K}([\![N]\!]) \circ \eta_{[\![A_1]\!] \times \dots \times [\![A_n]\!]} \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle =$
Naturality of η
$\eta_{[\![B]\!]} \circ [\![N]\!] \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle =$
Associativity of composition
$\eta_{[\![B]\!]} \circ ([\![N]\!] \circ \langle [\![M_1]\!], \dots, [\![M_n]\!]) \rangle =$
Simultaneous substitution lemma
$\eta_{[\![B]\!]} \circ [\![N[\vec{x} := \vec{M}]]\!]$
By interpetation
$[\![\Gamma \vdash \mathbf{pure}\ (N[\vec{x} := \vec{M}]) : \mathbf{K}B]\!]$

2) $[\![\vdash \mathbf{let\ pure}\ \_\_ = \_\_\ \mathbf{in}\ N : KA]\!] = [\![\vdash \mathbf{pure}\ N : \mathbf{KA}]\!]$.

$[\![\vdash \mathbf{let\ pure}\ \_\_ = \_\_\ \mathbf{in}\ N : KA]\!] =$	By interpetation
$\mathcal{K}([\![N]\!]) \circ \eta_{\mathbb{1}} =$	Naturality for η
$\eta_{[\![A]\!]} \circ [\![N]\!] =$	By interpretation
$[\![\vdash \mathbf{pure}\ N : \mathbf{K}A]\!]$	

3)
If $\Gamma \vdash M : A$ and $M \to_{\beta\eta} N$, then $[\![\Gamma \vdash \mathbf{pure}\ M : \mathbf{K}A]\!] = [\![\Gamma \vdash \mathbf{pure}\ N : \mathbf{K}A]\!]$.

If $\Gamma \vdash M : A$ and $M \to_{\beta\eta} N$, then $\Gamma \vdash N : A$ by subject reduction.
By assumption $[\![\Gamma \vdash M : A]\!] = [\![\Gamma \vdash N : A]\!]$.
So $\eta_{[\![A]\!]} \circ [\![\Gamma \vdash M : A]\!] = \eta_{[\![A]\!]} \circ [\![\Gamma \vdash N : A]\!]$.
Hence $[\![\Gamma \vdash \mathbf{pure}\ M : \mathbf{K}A]\!] = [\![\Gamma \vdash \mathbf{pure}\ N : \mathbf{K}A]\!]$.

□

□

Theorem 5. *Completeness*
Let $[\![\Gamma \vdash M : A]\!] = [\![\Gamma \vdash N : A]\!]$, then $M =_{\beta\eta} N$.

Proof.
We will consider term model for simply typed lambda calculus $\times$ and $\to$ standardly described in [22].

Definition 14. *Let us define an endofunctor $\mathcal{K} : \mathcal{C}(\lambda) \to \mathcal{C}(\lambda)$, such that forall $[x, M] \in Hom_{\mathcal{C}(\lambda)}(A, B), \mathbf{K}([x, M]) = [y, \mathbf{let\ pure}\ x = y\ \mathbf{in}\ M] \in Hom_{\mathcal{C}(\lambda)}(\mathbf{K}A, \mathbf{K}B)$ (denotation: fmap f for an arbitrary arrow f).*

Lemma 6. *Functoriality*
i) fmap $(g \circ f) =$ fmap $(g) \circ$ fmap (f);
ii) fmap $(id_A) = id_{\mathbf{K}A}$.

Proof. Easy checking using reduction rules.

□

Definition 15. *Let us define natural transformations:*

1) $\eta : Id \Rightarrow \mathcal{K}$, *s. t.* $\forall A \in Ob_{\mathcal{C}(\lambda)}$, $\eta_A = [x, \mathbf{pure}\, x] \in Hom_{\mathcal{C}(\lambda)}(A, \mathbf{K}A)$;

2) $*_{A,B} : \mathbf{K}A \times \mathbf{K}B \to \mathbf{K}(A \times B)$, *s. t.* $\forall A, B \in Ob_{\mathcal{C}(\lambda)}, *_{A,B} = [p, \mathbf{let\, pure}\, x, y = \pi_1 p, \pi_2 p\, \mathbf{in}\, \langle x, y\rangle] \in Hom_{\mathcal{C}(\lambda)}(\mathbf{K}A \times \mathbf{K}B, \mathbf{K}(A \times B))$.

Implementation for $*$ in our term model is a modification of $\text{let}_{\mathbf{K}}$-rule:

$$\dfrac{\dfrac{p : \mathbf{K}A \times \mathbf{K}B \vdash p : \mathbf{K}A \times \mathbf{K}B}{p : \mathbf{K}A \times \mathbf{K}B \vdash \pi_1 p : \mathbf{K}A} \quad \dfrac{p : \mathbf{K}A \times \mathbf{K}B \vdash p : \mathbf{K}A \times \mathbf{K}B}{p : \mathbf{K}A \times \mathbf{K}B \vdash \pi_2 p : \mathbf{K}B} \quad \dfrac{x : A \vdash x : A \qquad y : B \vdash y : B}{x : A, y : B \vdash \langle x, y\rangle : A \times B}}{p : \mathbf{K}A \times \mathbf{K}B \vdash \mathbf{let\, pure}\, \langle x, y\rangle = \langle \pi_1 p, \pi_2 p\rangle\, \mathbf{in}\, \langle x, y\rangle : \mathbf{K}(A \times B)}$$

Lemma 7. *Naturality for* η *and for* $*$

i) *fmap* $f \circ \eta_A = \eta_B \circ f$;

ii) *fmap* $(f \times g) \circ *_{A,B} = *_{C,D} \circ (\textit{fmap}\ f) \times (\textit{fmap}\ g)$.

iii) $*_{A,B} \circ (\eta_A \times \eta_B) = \eta_{A \times B}$;

Proof.

i) fmap $f \circ \eta_A = \eta_B \circ f$

$\eta_B \circ f =$	By the definition
$[y, \mathbf{pure}\, y] \circ [x, M] =$	By the definition of composition
$[x, \mathbf{pure}\, y[y := M]] =$	By substitution
$[x, \mathbf{pure}\, M]$	

On the other hand:

fmap $f \circ \eta_A =$	By the definiton
$[z, \mathbf{let\, pure}\, x = z\, \mathbf{in}\, M] \circ [x, \mathbf{pure}\, \mathbf{x}] =$	By the definition of composition
$[x, \mathbf{let\, pure}\, x = z\, \mathbf{in}\, M[z := \mathbf{pure}\, x]] =$	By substitution
$[x, \mathbf{let\, pure}\, x = \mathbf{pure}\, \mathbf{x}\, \mathbf{in}\, M] =$	β-reduction rule
$[x, \mathbf{pure}\, M[x := x]] =$	By substitution
$[x, \mathbf{pure}\, M]$	

ii) fmap $(f \times g) \circ *_{A,B} = *_{C,D} \circ (\text{fmap } f) \times (\text{fmap } g)$

See [19].

iii) $*_{A,B} \circ (\eta_A \times \eta_B) = \eta_{A \times B}$
Follows from i) and ii).

□

Tensorial strength is defined as follows:

Definition 16. *Tensorial strength*

Let $[p, \langle \mathbf{pure}\, (\pi_1 p), \pi_2 p\rangle] \in Hom_{\mathcal{C}(\lambda)}(A \times \mathbf{K}B, \mathbf{K}(A \times B))$.

So tensorial strength is defined as $\tau_{A,B} = *_{A,B} \circ [p, \langle \mathbf{pure}\, (\pi_1 p), \pi_2 p\rangle]$.

It is clearly that tensorial strength defined above can be simplified as follows:

$*_{A,B} \circ [p, \langle \mathbf{pure}\ (\pi_1 p), \pi_2 p\rangle] =$	By definition
$[p', \mathbf{let\ pure}\ x, y = \pi_1 p', \pi_2 p'\ \mathbf{in}\ \langle x, y\rangle] \circ [p, \langle \mathbf{pure}\ (\pi_1 p), \pi_2 p\rangle] =$	By composition
$[p, \mathbf{let\ pure}\ x, y = \pi_1 p', \pi_2 p'\ \mathbf{in}\ \langle x, y\rangle [p' := \langle \mathbf{pure}\ (\pi_1 p), \pi_2 p\rangle]] =$	By substitution
$[p, \mathbf{let\ pure}\ x, y = \pi_1(\langle \mathbf{pure}\ (\pi_1 p), \pi_2 p\rangle), \pi_2(\langle \pi_1 p, \mathbf{pure}\ (\pi_2 p)\rangle)\ \mathbf{in}\ \langle x, y\rangle] =$	By β-reduction rules
$[p, \mathbf{let\ pure}\ x, y = \mathbf{pure}\ (\pi_1 p), \pi_2 p\ \mathbf{in}\ \langle x, y\rangle]$	

Lemma 8. *Weak commutativity.*

$fmap\ ([p, \langle \pi_2 p, \pi_1 p\rangle]) \circ \tau_{A,B} =$
$*_{B,A} \circ [q, \langle \pi_1 q, \mathbf{pure}\ (\pi_2 q)\rangle] \circ [p, \langle \pi_2 p, \pi_1 p\rangle]$

Proof.

$\text{fmap}\ ([r, \langle \pi_2 r, \pi_1 r\rangle]) \circ \tau_{A,B} =$
By the definition of τ
$\text{fmap}\ ([r, \langle \pi_2 r, \pi_1 r\rangle]) \circ [p, \mathbf{let\ pure}\ x, y = \mathbf{pure}\ (\pi_1 p), \pi_2 p\ \mathbf{in}\ \langle x, y\rangle] =$
By the definition of fmap
$[q, \mathbf{let\ pure}\ r = q\ \mathbf{in}\ \langle \pi_2 r, \pi_1 r\rangle] \circ [p, \mathbf{let\ pure}\ x, y = \mathbf{pure}\ (\pi_1 p), \pi_2 p\ \mathbf{in}\ \langle x, y\rangle] =$
Composition
$[p, \mathbf{let\ pure}\ r = q\ \mathbf{in}\ \langle \pi_2 r, \pi_1 r\rangle[q := \mathbf{let\ pure}\ x, y = \mathbf{pure}\ (\pi_1 p), \pi_2 p\ \mathbf{in}\ \langle x, y\rangle]] =$
By β-reduction rules
$[p, \mathbf{let\ pure}\ r = (\mathbf{let\ pure}\ x, y = \mathbf{pure}\ (\pi_1 p), \pi_2 p\ \mathbf{in}\ \langle x, y\rangle)\ \mathbf{in}\ \langle \pi_2 r, \pi_1 r\rangle] =$
By β-reduction rules
$[p, \mathbf{let\ pure}\ x, y = \mathbf{pure}\ (\pi_1 p), \pi_2 p\ \mathbf{in}\ \langle \pi_2 r, \pi_1 r\rangle[r := \langle x, y\rangle]] =$
By substitution
$[p, \mathbf{let\ pure}\ x, y = \mathbf{pure}\ (\pi_1 p), \pi_2 p\ \mathbf{in}\ \langle \pi_2\langle x, y\rangle, \pi_1\langle x, y\rangle\rangle] =$
By β-reduction rules
$[p, \mathbf{let\ pure}\ x, y = \mathbf{pure}\ (\pi_1 p), \pi_2 p\ \mathbf{in}\ \langle y, x\rangle] =$

On the other hand
$*_{B,A} \circ [q, \langle \pi_1 q, \mathbf{pure}\ (\pi_2 q)\rangle] \circ [p, \langle \pi_2 p, \pi_1 p\rangle] =$
By the definition of $*$
$[r, \mathbf{let\ pure}\ y, x = \pi_1 r, \pi_2 r\ \mathbf{in}\ \langle y, x\rangle] \circ [q, \langle \pi_1 q, \mathbf{pure}\ (\pi_2 q)\rangle] \circ [p, \langle \pi_2 p, \pi_1 p\rangle] =$
Composition
$[r, \mathbf{let\ pure}\ y, x = \pi_1 r, \pi_2 r\ \mathbf{in}\ \langle y, x\rangle] \circ [p, \langle \pi_1 q, \mathbf{pure}\ (\pi_2 q)\rangle[q := \langle \pi_2 p, \pi_1 p\rangle]] =$
By subsitution and by β-reduction rules
$[r, \mathbf{let\ pure}\ y, x = \pi_1 r, \pi_2 r\ \mathbf{in}\ \langle y, x\rangle] \circ [p, \langle \pi_2 p, \mathbf{pure}\ (\pi_1 p)\rangle]] =$
Composition
$[p, \mathbf{let\ pure}\ y, x = \pi_1 r, \pi_2 r\ \mathbf{in}\ \langle y, x\rangle[r := \langle \pi_2 p, \mathbf{pure}\ (\pi_1 p)\rangle]] =$
By subsitution and by β-reduction rules
$[p, \mathbf{let\ pure}\ y, x = \pi_2 p, \mathbf{pure}\ (\pi_1 p)\ \mathbf{in}\ \langle y, x\rangle] =$
By symmetricity of assingment
$[p, \mathbf{let\ pure}\ x, y = \mathbf{pure}\ (\pi_1 p), \pi_2 p\ \mathbf{in}\ \langle y, x\rangle]$

□

Lemma 9. **K** *is an applicative functor*

Proof. Immediately follows from previous lemmas in the section. □

□

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