

Modal type theory based on the intuitionistic epistemic logic

Abstract

Modal intuitionistic epistemic logic IEL^- was proposed by S.Artemov and T. Protopopescu as the formal foundation for the intuitionistic theory of knowledge. We construct a modal simply typed lambda-calculus which is Curry-Howard isomorphic to IEL^- as formal theory of calculations with applicative functors in functional programming languages like Haskell or Idris. We prove that this typed lambda-calculus has the strong normalization and Church-Rosser properties.

1 Introduction

Modal intuitionistic epistemic logic IEL was proposed by S. Artemov and T. Protopopescu [1]. IEL provides the epistimology and the theory of knowledge as based on BHK-semantics of intuitionistic logic. IEL^- is a variant of IEL , that corresponds to intuitionistic belief. Informally, $\mathbf{K}A$ denotes that A is verified intuitionistically.

Intuitionistic epistemic logic IEL^- is defined with by following axioms and derivation rules:

Definition 1. *Intuitionistic epistemic logic IEL :*

- 1) *IPC axioms;*
 - 2) $\mathbf{K}(A \rightarrow B) \rightarrow (\mathbf{K}A \rightarrow \mathbf{K}B)$ (*normality*);
 - 3) $A \rightarrow \mathbf{K}A$ (*co-reflection*);
- Rule: MP.*

We have the deduction theorem and necessitation rule which is derivable.

V. Krupski and A. Yatmanov provided the sequential calculus for IEL and proved that this calculus is PSPACE-complete [2].

It's not difficult to see that modal axioms in IEL^- and types of the methods of Applicative class in Haskell-like languages (which is described below) are syntactically similar and we are going to show that this coincidence has a non-trivial computational meaning.

Functional programming languages such as Haskell [3], Idris [4], Purescript [5] or Elm [6] have special type classes¹ for calculations with container types like `Functor` and `Applicative`²:

¹Type class in Haskell is a general interface for special group of datatypes.

²Reader may read more about container types in the Haskell standard library documentation[7] or in the next one textbook [8]

```

class Functor f where
  fmap :: (a -> b) -> f a -> f b

class Functor f => Applicative f where
  pure :: a -> f a
  (<*>) :: f (a -> b) -> f a -> f b

```

By *container* (or *computational context*) type we mean some type-operator f , where f is a “function” from $*$ to $*$: type operator takes a simple type (which has kind $*$) and returns another simple type type with kind $*$. For more detailed description of the type system with kinds used in Haskell see [12].

The main goal of our research is a relationship between intuitionistic epistemic logic IEL^- and functional programming with effects. We show that relationship by building the type system (which is called λ_K) which is Curry-Howard isomorphic to IEL^- . So we will consider K -modality as an arbitrary applicative functor.

λK consists of the rules for simply typed lambda-calculus and special typing rules for lifting types into the applicative functor K . We assume that our type system will axiomatize the simplest case of computation with effects with one container. We provide proof-theoretical view on this kind of computations in functional programming and prove strong normalization and confluence.

2 Typed lambda-calculus based on IEL^-

At first we define the natural deduction for IEL^- with K -modality and binary connectives \rightarrow and \wedge (we call that calculus $NIEL_{\wedge, \rightarrow}^-$):

Definition 2. *Natural deduction $NIEL_{\wedge, \rightarrow}^-$ for IEL^- with \rightarrow and \wedge :*

$$\begin{array}{c}
\frac{}{\Gamma, \alpha \vdash A} \text{ax} \\
\\
\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \rightarrow_i \qquad \frac{\Gamma \vdash A \rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B} \rightarrow_i \\
\\
\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge_i \qquad \frac{\Gamma \vdash A_1 \wedge A_2}{\Gamma \vdash A_i} \wedge_e, i \in \{1, 2\} \\
\\
\frac{\Gamma \vdash A}{\Gamma \vdash KA} K_I \qquad \frac{\Gamma \vdash K\vec{A} \quad \vec{A} \vdash B}{\Gamma \vdash KB}
\end{array}$$

Where $\Gamma \vdash K\vec{A}$ is a syntax sugar for $\Gamma \vdash KA_1, \dots, \Gamma \vdash KA_n$.

Lemma 1. $\Gamma \vdash_{NIEL_{\wedge, \rightarrow}^-} A \Rightarrow IEL^- \vdash \bigwedge \Gamma \rightarrow A$.

Proof. Induction on the derivation.

Let us consider cases with modality.

1) If $\Gamma \vdash_{NIEL_{\wedge, \rightarrow}^-} A$, then $IEL^- \vdash \bigwedge \Gamma \rightarrow KA$.

- | | | |
|-----|---|----------------------|
| (1) | $\bigwedge \Gamma \rightarrow A$ | assumption |
| (2) | $A \rightarrow \mathbf{K}A$ | co-reflection |
| (3) | $(\bigwedge \Gamma \rightarrow A) \rightarrow ((A \rightarrow \mathbf{K}A) \rightarrow (\bigwedge \Gamma \rightarrow \mathbf{K}A))$ | IPC theorem |
| (4) | $(A \rightarrow \mathbf{K}A) \rightarrow (\bigwedge \Gamma \rightarrow \mathbf{K}A)$ | from (1), (3) and MP |
| (5) | $\bigwedge \Gamma \rightarrow \mathbf{K}A$ | from (2), (4) and MP |

2) If $\Gamma \vdash_{NIEL_{\wedge, \rightarrow}^-} \mathbf{K}\vec{A}$ and $\vec{A} \vdash B$, then $IEL^- \vdash \bigwedge \Gamma \rightarrow \mathbf{K}B$.

- | | | |
|-----|---|--------------------------------|
| (1) | $\bigwedge \Gamma \rightarrow \bigwedge_{i=1}^n \mathbf{K}A_i$ | assumption |
| (2) | $\bigwedge_{i=1}^n \mathbf{K}A_i \rightarrow \mathbf{K} \bigwedge_{i=1}^n A_i$ | IEL theorem |
| (3) | $\bigwedge \Gamma \rightarrow \mathbf{K} \bigwedge_{i=1}^n A_i$ | from (1), (2) and transitivity |
| (4) | $\bigwedge_{i=1}^n A_i \rightarrow B$ | assumption |
| (5) | $(\bigwedge_{i=1}^n A_i \rightarrow B) \rightarrow \mathbf{K}(\bigwedge_{i=1}^n A_i \rightarrow B)$ | co-reflection |
| (6) | $\mathbf{K}(\bigwedge_{i=1}^n A_i \rightarrow B)$ | from (2), (3) and MP |
| (7) | $\mathbf{K} \bigwedge_{i=1}^n A_i \rightarrow \mathbf{K}B$ | from (6) and normality |
| (8) | $\bigwedge \Gamma \rightarrow \mathbf{K}B$ | from (3), (7) and transitivity |

□

Lemma 2. *If $IEL^- \vdash A$, then $NIEL^- \vdash A$.*

Proof. Straightforward derivation of modal axioms in $NIEL^-$. We consider this derivation below using terms. □

At the next step we build the typed lambda-calculus based on $NIEL_{\wedge, \rightarrow}^-$ by proof-assignment in rules.

At first, we define lambda-terms and types for this lambda-calculus.

Definition 3. *The set of terms:*

Let \mathbb{V} be the set of variables. The set Λ_K of terms is defined by the grammar:

$$\Lambda_K ::= \mathbb{V} \mid (\lambda \Lambda. \Lambda_K) \mid (\Lambda_K \Lambda_K) \mid (\Lambda_K, \Lambda_K) \mid (\pi_1 \Lambda_K) \mid (\pi_2 \Lambda_K) \mid (\text{pure } \Lambda_K) \mid (\text{let pure } \Lambda_K = \Lambda_K \text{ in } \Lambda_K)$$

Definition 4. *The set of types:*

Let \mathbb{T} be the set of atomic types. The set \mathbb{T}_K of types with applicative functor K is generated by the grammar:

$$\mathbb{T}_K ::= \mathbb{T} \mid (\mathbb{T}_K \rightarrow \mathbb{T}_K) \mid (\mathbb{T}_K \times \mathbb{T}_K) \mid (K\mathbb{T}_K) \quad (1)$$

Context, domain of context and range of context are defined standardly [11][12].

Our type system is based on the Curry-style typing rules:

Definition 5. *Modal typed lambda calculus λK based on $NIEL_{\wedge, \rightarrow}^-$:*

$$\frac{}{\Gamma, x : A \vdash x : A} \text{ ax}$$

$$\begin{array}{c}
\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \rightarrow B} \rightarrow_i \qquad \frac{\Gamma \vdash f : A \rightarrow B \quad \Gamma \vdash x : A}{\Gamma \vdash fx : B} \rightarrow_e \\
\\
\frac{\Gamma \vdash M : A \quad \Gamma \vdash N : B}{\Gamma \vdash \langle x, y \rangle : A \times B} \times_i \qquad \frac{\Gamma \vdash M : A_1 \times A_2}{\Gamma \vdash \pi_i M : A_i} \times_e, i \in \{1, 2\} \\
\\
\frac{\Gamma \vdash x : A}{\Gamma \vdash \mathbf{pure} \ x : \mathbf{KA}} \mathbf{K}_I \qquad \frac{\Gamma \vdash \vec{M} : \mathbf{KA} \quad \vec{x} : \vec{A} \vdash M : B}{\Gamma \vdash \mathbf{let} \ \mathbf{pure} \ \vec{x} = \vec{M} \ \mathbf{in} \ M : \mathbf{KB}} \mathbf{let}_{\mathbf{K}}
\end{array}$$

\mathbf{K}_I -typing rule is the same as \bigcirc -introduction in lax logic (also known as monadic metalanguage [17]) and in typed lambda-calculus which is derived by proof-assignment for lax-logic proofs. \mathbf{K}_I allows to inject an object of type α into the functor. \mathbf{K}_I reflects the Haskell method **pure** for Applicative class. It plays the same role as the **return** method in Monad class.

$\mathbf{let}_{\mathbf{K}}$ is similar to \square_I -rule in typed lambda calculus for intuitionistic normal modal logic \mathbf{IK} , which is described in [19].

Here are some examples of derivation trees.

$$\begin{array}{c}
\frac{x : A \vdash x : A}{x : A \vdash \mathbf{pure} \ x : \mathbf{KA}} \mathbf{K}_I \\
\hline
\vdash (\lambda x. \mathbf{pure} \ x) : A \rightarrow \mathbf{KA} \rightarrow_i \\
\\
\frac{f : \mathbf{K}(A \rightarrow B) \vdash f : \mathbf{K}(A \rightarrow B) \quad x : \mathbf{KA} \vdash x : \mathbf{KA} \quad \frac{g : A \rightarrow B \quad y : A}{g : A \rightarrow B, y : A \vdash gy : B}}{f : \mathbf{K}(A \rightarrow B), x : \mathbf{KA} \vdash \mathbf{let} \ \mathbf{pure} \ \langle g, y \rangle = \langle f, x \rangle \ \mathbf{in} \ gy : \mathbf{KB}} \\
\hline
\frac{f : \mathbf{K}(A \rightarrow B) \vdash \lambda x. \mathbf{let} \ \mathbf{pure} \ \langle g, y \rangle = \langle f, x \rangle \ \mathbf{in} \ gy : \mathbf{KA} \rightarrow \mathbf{KB}}{\vdash \lambda f. \lambda x. \mathbf{let} \ \mathbf{pure} \ \langle g, y \rangle = \langle f, x \rangle \ \mathbf{in} \ gy : \mathbf{K}(A \rightarrow B) \rightarrow \mathbf{KA} \rightarrow \mathbf{KB}} \\
\\
\frac{f : A \rightarrow B \vdash f : A \rightarrow B \quad \frac{g : A \rightarrow B \quad y : A}{g : A \rightarrow B, y : A \vdash gy : B}}{f : A \rightarrow B \vdash \mathbf{pure} \ f : \mathbf{K}(A \rightarrow B) \quad x : \mathbf{KA} \vdash x : \mathbf{KA}} \\
\hline
\frac{f : A \rightarrow B, x : \mathbf{KA} \vdash \mathbf{let} \ \mathbf{pure} \ \langle g, y \rangle = \langle \mathbf{pure} \ f, x \rangle \ \mathbf{in} \ gy : \mathbf{KB}}{f : A \rightarrow B \vdash \lambda x. \mathbf{let} \ \mathbf{pure} \ \langle g, y \rangle = \langle \mathbf{pure} \ f, x \rangle \ \mathbf{in} \ gy : \mathbf{KA} \rightarrow \mathbf{KB}} \\
\hline
\vdash \lambda f. \lambda x. \mathbf{let} \ \mathbf{pure} \ \langle g, y \rangle = \langle \mathbf{pure} \ f, x \rangle \ \mathbf{in} \ gy : (A \rightarrow B) \rightarrow \mathbf{KA} \rightarrow \mathbf{KB}
\end{array}$$

Now we define free variables and substitutions. β -reduction, multi-step β -reduction and β -equality are defined standardly:

Definition 6. Set $FV(M)$ of free variables for arbitrary term M :

- 1) $FV(x) = \{x\}$;
- 2) $FV(\lambda x.M) = FV(M) \setminus \{x\}$;
- 3) $FV(MN) = FV(M) \cup FV(N)$;
- 4) $FV((M, N)) = FV(M) \cup FV(N)$;
- 5) $FV(\pi_i p) \subseteq FV(p)$, $i \in \{1, 2\}$;
- 6) $FV(\mathbf{pure} \ M) = FV(M)$;
- 7) $FV(\mathbf{let} \ \mathbf{pure} \ \vec{N} = \vec{M} \ \mathbf{in} \ M) = \bigcup_{i=1}^n FV(M)$, where $n = |\vec{M}|$.

Definition 7. *Substitution:*

- 1) $x[x := N] = N$, $x[y := N] = x$;
- 2) $(MN)[x := N] = M[x := N]N[x := N]$;
- 3) $(\lambda x.M)[x := N] = \lambda x.M[x := N]$;
- 4) $(M, N)[x := P] = (M[x := P], N[x := P])$;
- 5) $(\pi_i M)[x := P] = \pi_i(M[x := P])$, $i \in \{1, 2\}$;
- 6) $(\mathbf{pure} M)[x := P] = \mathbf{pure}(M[x := P])$;
- 7) $(\mathbf{let pure} \vec{N} = \vec{M} \mathbf{in} M)[x := P] = \mathbf{let pure} \vec{N} = (\vec{M}[x := P]) \mathbf{in} M$.

Definition 8. *β -reduction and η -reduction rules for $\lambda\mathbf{K}$.*

- 1) $(\lambda x.M)N \rightarrow_\beta M[x := N]$;
- 2) $\pi_1 \langle M, N \rangle \rightarrow_\beta M$;
- 3) $\pi_2 \langle M, N \rangle \rightarrow_\beta N$;
- 4) $\mathbf{let pure} \langle \vec{x}, y, \vec{z} \rangle = \langle \vec{M}, \mathbf{let pure} \vec{w} = \vec{N} \mathbf{in} Q, \vec{P} \rangle \mathbf{in} R \rightarrow_\beta \mathbf{let pure} \langle \vec{x}, \vec{w}, \vec{z} \rangle = \langle \vec{M}, \vec{N}, \vec{P} \rangle \mathbf{in} R[y := Q]$
- 5) $\mathbf{pure} ((\lambda x.M)N) \rightarrow_\beta \mathbf{pure}(M[x := N])$;
- 6) $\mathbf{pure} (\pi_i \langle M_1, M_2 \rangle) \rightarrow_\beta \mathbf{pure} M_i$, where $i \in \{1, 2\}$.
- 7) $\mathbf{pure} (\mathbf{let pure} \langle \vec{x}, y, \vec{z} \rangle = \langle \vec{M}, \mathbf{let pure} \vec{w} = \vec{N} \mathbf{in} Q, \vec{P} \rangle \mathbf{in} R) \rightarrow_\beta \mathbf{pure} (\mathbf{let pure} \langle \vec{x}, \vec{w}, \vec{z} \rangle = \langle \vec{M}, \vec{N}, \vec{P} \rangle \mathbf{in} R[y := Q])$
- 8) $\lambda x.f x \rightarrow_\eta f$;
- 9) $\langle \pi_1 P, \pi_2 P \rangle \rightarrow_\eta P$;
- 10) $\mathbf{let pure} _ = _ \mathbf{in} N \rightarrow_\eta \mathbf{pure} N$;
- 11) $\mathbf{let pure} x = M \mathbf{in} x \rightarrow_\eta M$;
- 12) $\mathbf{pure} (\lambda x.f x) \rightarrow_\eta \mathbf{pure} f$;
- 13) $\mathbf{pure} (\langle \pi_1 P, \pi_2 P \rangle) \rightarrow_\eta \mathbf{pure} P$;
- 14) $\mathbf{pure} (\mathbf{let pure} x = M \mathbf{in} x) \rightarrow_\eta \mathbf{pure} M$;
- 15) $\mathbf{pure} (\mathbf{let pure} _ = _ \mathbf{in} N) \rightarrow_\eta \mathbf{pure} (\mathbf{pure} N)$.

3 Basic lemmas

Now we will prove standard lemmas for contexts in type systems³:

Definition 9. *The domain of a context Γ :*

Let $\Gamma = \{x_1 : A_1, \dots, x_n : A_n\}$. Then the domain of Γ , or $\text{dom}(\Gamma)$, is a set $\{x_1, \dots, x_n\}$.

Lemma 3. *If $\Gamma \vdash M : A$, then $FV(M) \subseteq \text{dom}(\Gamma)$*

Proof. Induction on the derivation of $\Gamma \vdash M : A$.

□

Lemma 4. *Generation for $\lambda\mathbf{K}$.*

- 1) $\Gamma \vdash \mathbf{pure} M : \mathbf{K}\alpha$ implies that $\Gamma \vdash M : \alpha$;
- 2) $\Gamma \vdash \mathbf{let pure} \vec{N} = \vec{M} \mathbf{in} M : \mathbf{K}B$ implies that $\Gamma \vdash \vec{M} : \mathbf{K}\vec{A}$ and $\vec{N} : \vec{A} \vdash M : B$.

³We will not prove cases with \rightarrow -constructor, they are proved standardly in the same lemmas for simply typed lambda calculus, for example see [11][12][14]. We will consider only modal cases

Proof.

Induction on the derivation of $\Gamma \vdash \mathbf{pure} M : \mathbf{K}\alpha$ and $\Gamma \vdash \mathbf{let pure} \vec{N} = \vec{M} \mathbf{in} M : \mathbf{K}B$ respectively. \square

The next one lemma allows that weakening structural rule is admissable.

Lemma 5. *Weakening for $\lambda\mathbf{K}$.*

Let $\Gamma \vdash M : A$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash M : A$.

Proof.

Induction on derivation of $\Gamma \vdash M : A$. Let us assume $\Gamma \subseteq \Delta$.

1) Let $\Gamma \vdash x : A$, such that $\Gamma = \Delta, x : A$ and $\Theta \subseteq \Gamma$. Let $\Sigma = \Theta \setminus \Gamma$, or, which is the same, $\Sigma = \Theta \setminus \Delta, x : A$, then $\Sigma, \Delta, x : A \vdash x : A$, or, $\Theta \vdash x : A$.

2) Let $\Gamma \vdash \mathbf{pure} M : \mathbf{K}A$ and $\Gamma \subseteq \Theta$.

By generation $\Gamma \vdash M : A$

By hypothesis, $\Theta \vdash M : A$, so $\Theta \vdash \mathbf{pure} M : \mathbf{K}A$ by applying \mathbf{K}_I -rule.

3) Let $\Gamma \vdash \mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} N : \mathbf{K}B$ and $\Gamma \subseteq \Theta$.

By generation $\Gamma \vdash \vec{M} : \mathbf{K}\vec{A}$ and $\vec{x} : \vec{A} \vdash N : B$.

By assumption $\Theta \vdash \vec{M} : \mathbf{K}\vec{A}$.

Hence $\Theta \vdash \mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} N : \mathbf{K}B$. \square

Lemma 6. *Considering for $\lambda\mathbf{K}$.*

If $\Gamma \vdash M : \alpha$, then $\Gamma \uparrow FV(M) \vdash M : \alpha$, where $\Gamma \uparrow FV(M)$ is a subcontext of Γ , such that $\text{dom}(\Gamma \uparrow FV(M)) = \text{dom}(\Gamma) \cap FV(M)$.

Proof.

1) Let $\Gamma \vdash x : A$, where $\Gamma = \Delta, x : A$, $x \in \mathbb{V}$.

$FV(x) = \{x\}$, then $\text{dom}(\Gamma) \cap \{x\} = \{x\}$. So $(\Delta, x : A) \uparrow FV(x) = \{x : A\}$, then $x : A \vdash x : A$ by axiom.

2) Let $\Gamma \vdash \mathbf{pure} M : \mathbf{K}A$.

By generation, $\Gamma \vdash M : A$ and $\Gamma \uparrow FV(M) \vdash M : A$ by hypothesis.

So $\Gamma \uparrow FV(M) \vdash \mathbf{pure} M : \mathbf{K}A$ by \mathbf{K}_I .

3) Let $\Gamma \vdash \mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} N : \mathbf{K}B$.

By generation, $\Gamma \vdash \vec{M} : \mathbf{K}\vec{A}$ and $\vec{x} : \vec{A} \vdash N : B$.

By assumption, $\Gamma \uparrow FV(\vec{M}) \vdash \vec{M} : \vec{A}$.

By $\mathbf{let_K}$, $\Gamma \uparrow FV(\vec{M}) \vdash \mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} N : \mathbf{K}B$ \square

Lemma 7. *If $\Gamma, x : A \vdash M : B$ and $\Gamma \vdash N : A$, then $\Gamma \vdash (M[x := N]) : B$*

Proof.

1) Let $\Gamma, x : A \vdash \mathbf{pure} M : \mathbf{K}B$ and $\Gamma \vdash N : A$.

By generation, $\Gamma, x : A \vdash M : B$.

By assumption, $\Gamma \vdash (M[x := N]) : B$

Then $\Gamma \vdash \mathbf{pure}(M[x := N]) : \mathbf{KB}$ by \mathbf{K}_I .
 But $\mathbf{pure}(M[x := N]) = (\mathbf{pure} M[x := N])$ by substitution definition, so
 $\Gamma \vdash (\mathbf{pure} M[x := N]) : \mathbf{KB}$

2) Let $\Gamma, y : A \vdash \mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} N : \mathbf{KB}$ and $\Gamma \vdash N : A$.
 By generation, $\Gamma, y : A \vdash \vec{M} : \mathbf{KA}$ and $\vec{x} : \vec{A} \vdash N : B$.
 By hypothesis, $\Gamma \vdash \vec{M}[x := N] : \mathbf{KA}$.
 Hence $\Gamma \vdash \mathbf{let pure} \vec{x} = \vec{M}[x := N] \mathbf{in} N : \mathbf{KB}$.

□

Theorem 1. *Subject reduction*

Let $\Gamma \vdash M : \alpha$ and $M \rightarrow_\beta N$, then $\Gamma \vdash N : \alpha$

We consider only modal β -reduction rules. The general statement for \rightarrow_β follows from transitivity of multi-step β -reduction.

Proof.

1) Let $\Gamma \vdash \mathbf{let pure} \langle \vec{x}, y, \vec{z} \rangle = \langle \vec{M}, \mathbf{let pure} \vec{w} = \vec{N} \mathbf{in} Q, \vec{P} \rangle \mathbf{in} R : \mathbf{KB}$
 By generation we have $\Gamma \vdash \vec{M} : \mathbf{KA}_1$, $\Gamma \vdash \mathbf{let pure} \vec{w} = \vec{N} \mathbf{in} Q : \mathbf{KA}_2$,
 $\Gamma \vdash \vec{P} : \mathbf{KA}_3$ and $\vec{x} : \vec{A}_1, y : A_2, \vec{z} : \vec{A}_3 \vdash R : B$.
 If $\Gamma \vdash \mathbf{let pure} \vec{w} = \vec{N} \mathbf{in} Q : \mathbf{KA}_2$, then
 $\Gamma \vdash \vec{N} : \mathbf{KA}_4$ and $\vec{w} : \vec{A}_4 \vdash Q : A_2$.
 Then $\vec{x} : \vec{A}_1, \vec{w} : \vec{A}_4, \vec{z} : \vec{A}_3 \vdash R[y := Q] : B$ by substitution lemma and
 weakening.
 Hence $\Gamma \vdash \mathbf{let pure} \langle \vec{x}, \vec{w}, \vec{z} \rangle = \langle \vec{M}, \vec{N}, \vec{P} \rangle \mathbf{in} R[y := Q] : \mathbf{KB}$ by $\mathbf{let_K}$.

2) Let $\Gamma \vdash \mathbf{pure}((\lambda x.M)N) : \mathbf{KB}$.
 By generation $\Gamma \vdash (\lambda x.M)N : B$, but $\Gamma \vdash M[x := N] : B$, then, by \mathbf{K}_I ,
 $\Gamma \vdash \mathbf{pure}(M[x := N]) : \mathbf{KB}$.

3) Let $\Gamma \vdash \mathbf{pure}(\pi_i \langle M_1, M_2 \rangle) : \mathbf{KA}_i$, where $i \in \{1, 2\}$.
 By generation $\Gamma \vdash \pi_i \langle M_1, M_2 \rangle : A_i$ and $\Gamma \vdash M_i : A_i$.
 Hence $\Gamma \vdash \mathbf{pure} M_i : \mathbf{KA}_i$ by \mathbf{K}_I .

4) Let $\Gamma \vdash \mathbf{let pure} (\langle \vec{x}, y, \vec{z} \rangle = \langle \vec{M}, \mathbf{let pure} \vec{w} = \vec{N} \mathbf{in} Q, \vec{P} \rangle \mathbf{in} R) : \mathbf{K}^2 B$.
 By generation $\Gamma \vdash \mathbf{let pure} \langle \vec{x}, y, \vec{z} \rangle = \langle \vec{M}, \mathbf{let pure} \vec{w} = \vec{N} \mathbf{in} Q, \vec{P} \rangle \mathbf{in} R : \mathbf{KB}$,
 hence $\Gamma \vdash \mathbf{let pure} \langle \vec{x}, \vec{w}, \vec{z} \rangle = \langle \vec{M}, \vec{N}, \vec{P} \rangle \mathbf{in} R[y := Q] : \mathbf{KB}$ by the first
 case.
 So $\Gamma \vdash \mathbf{pure} (\mathbf{let pure} \langle \vec{x}, \vec{w}, \vec{z} \rangle = \langle \vec{M}, \vec{N}, \vec{P} \rangle \mathbf{in} R[y := Q]) : \mathbf{K}^2 B$ by \mathbf{K}_I .

□

4 Strong normalization

We modify and apply Tait's technique of logical relation for modalities. Strong normalization proof with Tait's method for simply typed lambda calculus is described here [13].

Theorem 2. *Let $M \in \Lambda_K$, then any sequence of reduction $M \rightarrow_\beta M_1 \dots$ terminates.*

Proof. We build the smallest of subset of strongly normalizing terms of modal types and show that an arbitrary term belongs to this subset.

Definition 10. *The set of strongly computable terms of type $\phi \in \mathbb{T}_K$, SC_ϕ :*

- Let $\phi = K\alpha$ and $\alpha \in \mathbb{T}$, then:

$$SC_{K\alpha} = \{M : K\alpha \mid M \text{ is strongly normalizing}\} \quad (2)$$

- Let $\phi = K(\tau \rightarrow \psi)$ and $\tau, \psi \in \mathbb{T}_K$, then:

$$SC_{K(\tau \rightarrow \psi)} = \{M : K(\tau \rightarrow \psi) \mid \forall N \in SC_{K\tau}, M \star N \in SC_{K\psi}\} \quad (3)$$

- Let $\phi = K(\tau_1 \times \tau_2)$ and $\tau_1, \tau_2 \in \mathbb{T}_K$, then:

$$SC_{K(\tau_1 \times \tau_2)} = \{P : K(\tau_1 \times \tau_2) \mid \mathbf{pure}(\lambda x. \pi_i x) \star P \in SC_{K\tau_i}, i \in \{1, 2\}\} \quad (4)$$

Lemma 8.

If $M \in SC_\alpha$, then M is strongly normalizing.

Proof.

1) If $M \in SC_{K\alpha}$ and $\alpha \in \mathbb{T}$, then M is strongly normalizing by the definition of $SC_{K\alpha}$.

2) Let $M \in SC_{K(\tau \rightarrow \psi)}$, so by every $N \in SC_{K\tau}$, $M \star N \in SC_{K\psi}$, which is strongly normalizing by hypothesis. So M is strongly normalizing.

3) Let $M \in SC_{K(\tau_1 \times \tau_2)}$, so $\mathbf{pure}(\lambda x. \pi_i x) \star M \in SC_{K\tau_i}$, $i \in \{1, 2\}$, which are strongly normalizing. So M is strongly normalizing. □

Lemma 9.

Let $M \rightarrow_\beta M'$ and $M \in SC_\alpha$, then $M' \in SC_\alpha$.

Proof.

1) Let $M \rightarrow_\beta M'$ and $M \in SC_{K\alpha}$, where $\alpha \in \mathbb{T}$.

M has the longest reduction path (which we denote as $p(M)$). So $p(M') < p(M)$, then $M' \in SC_{K\alpha}$.

2) Let $M \in SC_{K(\alpha \rightarrow \beta)}$ and $M \rightarrow_\beta M'$. Let $N \in SC_{K\alpha}$. So $M \star N \in SC_{K\beta}$.

If $M \rightarrow_\beta M'$, then $M \star N \rightarrow_\beta M' \star N$ by reduction rule, so $M' \star N \in SC_{K\beta}$ and $M' \in SC_{K(\alpha \rightarrow \beta)}$ by hypothesis.

3) Let $M \in SC_{K(\tau_1 \times \tau_2)}$ and $M \rightarrow_\beta M'$.

So $\mathbf{pure}(\lambda x. \pi_i x) \star M \rightarrow_\beta \mathbf{pure}(\lambda x. \pi_i x) \star M'$, $i \in \{1, 2\}$ by reduction rule.

So $\mathbf{pure}(\lambda x. \pi_i x) \star M' \in SC_{K\tau_i}$ and $M' \in SC_{K(\tau_1 \times \tau_2)}$. □

Definition 11. *Neutral term:*

We define a term M to be neutral if it has of the next forms:

- 1) $M = x$, where $x \in \mathbb{V}$;
- 2) $M = (PQ)$;
- 3) $M = \pi_i M$, $i \in \{1, 2\}$;
- 4) $M = P \star Q$;
- 5) If M is a neutral, then **pure** M is a neutral.

Lemma 10. *Let $M \rightarrow_\beta M'$ and $M' \in SC_\alpha$ for every one-step reduction. So if M' is a neutral, then $M \in SC_\alpha$.*

Proof.

Simple induction on the structure of M' . □

Lemma 11.

Let $x_1 : \phi_1, \dots, x_n : \phi_n \vdash M : \phi$ and for all $i \in \{1, \dots, n\}$, $N_i \in SC_{\phi_i}$, then $(M[x_1 := N_1, \dots, x_n := N_n]) \in SC_\phi$.

Proof.

1) If ϕ is an atomic and M is a variable, then this condition holds straightforwardly.

2) Let $\Gamma = \{x_1 : \phi_1, \dots, x_n : \phi_n\}$, $\Gamma \vdash \mathbf{pure} M : \mathbf{K}\alpha$ and for all $i \in \{1, \dots, n\}$, $N_i \in SC_{\phi_i}$.

Then by $\Gamma \vdash M : \alpha$ by generation and $(M[x_1 := N_1, \dots, x_n := N_n]) \in SC_\alpha$ by induction hypothesis.

Hence, $\Gamma \vdash \mathbf{pure} M : \mathbf{K}\alpha$ and $(\mathbf{pure} M([x_1 := N_1, \dots, x_n := N_n])) \in SC_{\mathbf{K}\alpha}$ by definition of $SC_{\mathbf{K}\alpha}$.

3) Let $\Gamma = \{x_1 : \phi_1, \dots, x_n : \phi_n\}$, $\Gamma : \phi_n \vdash M \star P : \mathbf{K}\beta$ and for all $i \in \{1, \dots, n\}$, $N_i \in SC_{\phi_i}$.

Then $\Gamma \vdash M : \mathbf{K}(\alpha \rightarrow \beta)$, $\Gamma \vdash P : \mathbf{K}\alpha$ by generation.

But by induction hypothesis $M[x_1 := N_1, \dots, x_n := N_n] \in SC_{\mathbf{K}(\alpha \rightarrow \beta)}$ and $P[x_1 := N_1, \dots, x_n := N_n] \in SC_{\mathbf{K}\alpha}$.

Then, by definition of $SC_{\mathbf{K}\beta}$, $((M[x_1 := N_1, \dots, x_n := N_n]) \star (P[x_1 := N_1, \dots, x_n := N_n])) \in SC_{\mathbf{K}\beta}$, i.e. $(M \star N([x_1 := N_1, \dots, x_n := N_n])) \in SC_{\mathbf{K}\beta}$. □

Corollary 1.

If $\vdash M : \alpha$, then M is strongly normalizing.

Proof. $M \in SC_\alpha$ by Lemma 10, so M is strongly normalizing. □

□

5 Confluence

In the confluence proof (below) we treat the cases with **pure** and \star similar to [15] [18].

Definition 12. *Alphabet for the labelled terms:*

variables: $x, y, z, x_1, y_1, z_1, \dots$;
lambdas: $\lambda, \lambda_0, \lambda_1, \lambda_2, \dots$;
constructors for an applicative functor: **pure**, \star ;
parentheses $(,)$.

Definition 13. *The set of labelled terms Λ'_K inductively defined as a set of words on the alphabet described above:*

- 1) $x \in \Lambda'$;
- 2) If $M \in \Lambda'_K$, then $(\lambda x.M) \in \Lambda'_K$;
- 3) If $M, N \in \Lambda'_K$, then $(MN) \in \Lambda'_K$;
- 4) If $M \in \Lambda'_K$, then **pure** $M \in \Lambda'_K$;
- 5) If $M, N \in \Lambda'_K$, then $M \star N \in \Lambda'_K$;
- 6) If $M, N \in \Lambda'_K$, then for all $i \in \mathbb{N}$, $((\lambda_i x.M)N) \in \Lambda'_K$.

Definition 14. *Erasing map*

Erasing map is a map $|\cdot| : \Lambda'_K \rightarrow \Lambda_K$, such that:

- 1) $|x| = x$;
- 2) $|(\lambda x.M)| = \lambda x.|M|$;
- 3) $|(MN)| = |M||N|$;
- 4) $|(\text{pure } M)| = \text{pure } |M|$;
- 5) $|M \star N| = |M| \star |N|$;
- 6) $|((\lambda_i x.M)N)| = (\lambda x.|M|)|N|$

Example 1.

$$|\text{pure } ((\lambda_i x.M)N) \star P| = \text{pure } (\lambda x.|M|)|N| \star |P|$$

Definition 15. *Substitution for Λ'_K :*

- 1) $x[x := N] = N$, $x[y := N] = x$;
- 2) $(MN)[x := N] = M[x := N]N[x := N]$;
- 3) $(\lambda x.M)[x := N] = \lambda x.M[x := N]$;
- 4) $(\text{pure } M)[x := P] = \text{pure } (M[x := P])$;
- 5) $(M \star N)[x := P] = (M[x := P]) \star (N[x := P])$;
- 6) $(\lambda_i x.M)N[y := P] = (\lambda_i x.M[y := P])(N[y := P])$.

Definition 16. *One-step reduction $\rightarrow_{\beta'}$ for Λ'_K :*

- 1) $(\lambda x.M)N \rightarrow_{\beta'} M[x := N]$;
- 2) **pure** $(\lambda x.x) \star M \rightarrow_{\beta'} M$;
- 3) **pure** $(\lambda f g x.f(gx)) \star M \star N \star P \rightarrow_{\beta'} M \star (N \star P)$;
- 4) **pure** $M \star (\text{pure } N) \rightarrow_{\beta'} \text{pure } (MN)$;
- 5) $M \star (\text{pure } N) \rightarrow_{\beta'} \text{pure } (\lambda f.fN) \star M$;
- 6) $(\lambda_i x.M)N \rightarrow_{\beta'} M[x := N]$.

Multi-step reduction $\rightarrow_{\beta'}$ is a reflexive-transitive closure of $\rightarrow_{\beta'}$.

Definition 17. *Let us define a map $\phi : \Lambda'_K \rightarrow \Lambda_K$ inductively as follows:*

- 1) $\phi(x) = x$;
- 2) $\phi(MN) = \phi(M)\phi(N)$;

- 3) $\phi(\lambda x.M) = \lambda x.\phi(M)$;
- 4) $\phi(\mathbf{pure} M) = \mathbf{pure}(\phi(M))$;
- 5) $\phi(M \star N) = \phi(M) \star \phi(N)$;
- 6) $\phi((\lambda_i x.M)N) = \phi(M)[x := \phi(N)]$.

Example 2.

$$\phi(\mathbf{pure}((\lambda_i x.M)N) \star P) = \mathbf{pure}(\phi(M)[x := \phi(N)]) \star \phi(P)$$

Lemma 12.

- 1) Let $M, N \in \Lambda'_K$ and $|M| \rightarrow_\beta |N|$, then $M \rightarrow_{\beta'} N$.
- 2) Let $M, N \in \Lambda'_K$ and $M \rightarrow_{\beta'} N$, then $|M| \rightarrow_\beta |N|$.

Proof.

Induction on the generation of \rightarrow_β ($\rightarrow_{\beta'}$).

1) Let us consider homomorphism rule. The rest applicative reduction rules are considered similary.

Let $(\mathbf{pure} M') \star (\mathbf{pure} N'), \mathbf{pure}(M'N') \in \Lambda'_K$.

So $|(\mathbf{pure} M') \star (\mathbf{pure} N')| = (\mathbf{pure} |M'|) \star (\mathbf{pure} |N'|)$ and $|\mathbf{pure}(M'N')| = \mathbf{pure}(|M'| |N'|)$.

By reduction rule, $(\mathbf{pure} |M'|) \star (\mathbf{pure} |N'|) \rightarrow_\beta \mathbf{pure}(|M'| |N'|)$.

But $(\mathbf{pure} M') \star (\mathbf{pure} N') \rightarrow_{\beta'} \mathbf{pure}(M'N')$ by reduction rule for $\rightarrow_{\beta'}$.

2) Let us consider interchange rule.

Let $M \star (\mathbf{pure} N), \mathbf{pure}(\lambda f.fN) \star M \in \Lambda'_K$ and $M \star (\mathbf{pure} N) \rightarrow_{\beta'} \mathbf{pure}(\lambda f.fN) \star M$.

But $|M \star (\mathbf{pure} N)| = |M| \star (\mathbf{pure} |N|)$ and $|\mathbf{pure}(\lambda f.fN) \star M| = \mathbf{pure}(\lambda f.f|N|) \star |M|$.

So $|M| \star (\mathbf{pure} |N|) \rightarrow_\beta \mathbf{pure}(\lambda f.f|N|) \star |M|$ by β -reduction rule.

It is easy to see, that the statement for $\rightarrow_{\beta'}$ and \rightarrow_β immedeatly follows from transitivity of multi-step rediction for labelled terms and for usual terms respectively.

□

Lemma 13.

$$\phi(M[x := N]) = \phi(M)[x := \phi(N)].$$

Proof. Induction on M .

1) Let $M = x$. Then $\phi(x[x := N]) = \phi(N)$.

On the other hand, $\phi(x)[x := \phi(N)] = x[x := \phi(N)] = \phi(N)$.

So $\phi(x[x := N]) = \phi(x)[x := \phi(N)]$.

2) Let $M = y$ and $y \neq x$. Then $\phi(y[x := N]) = \phi(y) = y$.

But $\phi(y)[x := \phi(N)] = y[x := \phi(N)] = y$.

Therefore $\phi(y[x := N]) = \phi(y)[x := \phi(N)]$.

3) Let $M = \mathbf{pure} M'$. Then $\phi(\mathbf{pure} M'[x := N]) = \mathbf{pure} \phi(M'[x := N])$.

By hypothesis, $\mathbf{pure}(\phi(M'[x := N])) = \mathbf{pure}(\phi(M')[x := \phi(N)])$, which is $(\mathbf{pure} \phi(M'))[x := \phi(N)]$ by substitution definition.

4) Let $M = M' \star N'$. So $\phi((M' \star N')[x := N]) = \phi(M'[x := N] \star N'[x := N])$.

By definition of ϕ ,

$$\phi(M'[x := N] \star N'[x := N]) = \phi(M'[x := N]) \star \phi(N'[x := N]).$$

But by induction hypothesis,

$$\phi(M'[x := N]) = \phi(M')[x := \phi(N)] \text{ and}$$

$$\phi(N'[x := N]) = \phi(N')[x := \phi(N)].$$

Hence,

$$\phi(M'[x := N]) \star \phi(N'[x := N]) = \phi(M')[x := \phi(N)] \star \phi(N')[x := \phi(N)].$$

So,

$$\phi(M'[x := \phi(N)] \star \phi(N')[x := \phi(N)]) = (\phi(M') \star \phi(N'))[x := \phi(N)].$$

And by definition of ϕ , $(\phi(M') \star \phi(N'))[x := \phi(N)] = \phi(M' \star N')[x := \phi(N)]$. \square

Lemma 14.

Let $M, N \in \Lambda'_K$ and $M \rightarrow_{\beta'} N$, then $\phi(M) \rightarrow_{\beta} \phi(N)$.

Proof.

1) Let $\mathbf{pure}(\lambda x.x) \star M, M \in \Lambda'_K$ and $\mathbf{pure}(\lambda x.x) \star M \rightarrow_{\beta'} M$.

But $\phi(\mathbf{pure}(\lambda x.x) \star M) = \mathbf{pure}(\lambda x.x) \star \phi(M)$.

So $\mathbf{pure}(\lambda x.x) \star \phi(M) \rightarrow_{\beta} \phi(M)$ by β -reduction rule.

2) Let $\mathbf{pure}(\lambda f g x.f(gx)) \star M \star N \star P, M \star (N \star P) \in \Lambda'_K$ and $\mathbf{pure}(\lambda f g x.f(gx)) \star M \star N \star P \rightarrow_{\beta'} M \star (N \star P)$.

By the definition of ϕ :

$$\phi(\mathbf{pure}(\lambda f g x.f(gx)) \star M \star N \star P) = \mathbf{pure}(\lambda f g x.f(gx)) \star \phi(M) \star \phi(N) \star \phi(P);$$

$$M \star (N \star P) = \phi(M) \star (\phi(N) \star \phi(P)).$$

Hence, $\mathbf{pure}(\lambda f g x.f(gx)) \star \phi(M) \star \phi(N) \star \phi(P) \rightarrow_{\beta} \phi(M) \star (\phi(N) \star \phi(P))$ by β -reduction rule.

3) Let $(\mathbf{pure} M) \star (\mathbf{pure} N), \mathbf{pure}(MN) \in \Lambda'_K$ and $(\mathbf{pure} M) \star (\mathbf{pure} N) \rightarrow_{\beta} \mathbf{pure}(MN)$.

By the definition of ϕ :

$$\phi((\mathbf{pure} M) \star (\mathbf{pure} N)) = (\mathbf{pure} \phi(M)) \star (\mathbf{pure} \phi(N));$$

$$\phi(\mathbf{pure}(MN)) = \mathbf{pure}(\phi(M)\phi(N)).$$

So, by reduction rule, $(\mathbf{pure} \phi(M)) \star (\mathbf{pure} \phi(N)) \rightarrow_{\beta} \mathbf{pure}(\phi(M)\phi(N))$.

4) Let $M \star (\mathbf{pure} N), \mathbf{pure}(\lambda f.fN) \star M$ and $M \star (\mathbf{pure} N) \rightarrow_{\beta'} (\lambda f.fN) \star M$.

$$\phi(M \star (\mathbf{pure} N)) = \phi(M) \star (\mathbf{pure} \phi(N))$$

$$\phi((\lambda f.fN) \star M) = (\lambda f.f\phi(N)) \star \phi(M).$$

So, $\phi(M) \star (\mathbf{pure} \phi(N)) \rightarrow_{\beta} \mathbf{pure}(\lambda f.f\phi(N)) \star \phi(M)$. \square

Lemma 15.

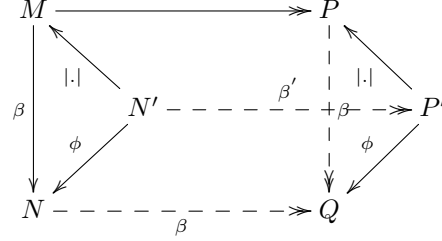
Let $M \in \Lambda'_K$. Then $|M| \rightarrow_{\beta} \phi(M)$.

Proof. Induction on the structure of M . \square

Lemma 16. *Strip lemma.*

If $M \rightarrow_{\beta} N$ and $M \rightarrow_{\beta} P$. Then there exists some term Q , such that $N \rightarrow_{\beta} Q$ and $P \rightarrow_{\beta} Q$.

Proof. Proof is similar to [15] [18]. We build the following diagram



□

which commutes by lemmas 11 – 14.

Theorem 3. *Confluence.*

If $M \twoheadrightarrow_{\beta} N$ and $M \twoheadrightarrow_{\beta} P$. Then there exists some term Q , such that $N \twoheadrightarrow_{\beta} Q$ and $P \twoheadrightarrow_{\beta} Q$.

Proof.

By unfolding $M \twoheadrightarrow_{\beta} N$ as the sequence of one-step reductions $M \rightarrow_{\beta} M_1 \rightarrow_{\beta} \dots \rightarrow_{\beta} M_n \rightarrow_{\beta} N$ and applying strip lemma on every step.

□

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References

- [1] Artemov S. and Protopopescu T., “Intuitionistic Epistemic Logic”, *The Review of Symbolic Logic*, 2016, vol. 9, no 2. pp. 266-298.
- [2] Krupski V. N. and Yatmanov A., “Sequent Calculus for Intuitionistic Epistemic Logic IEL”, *Logical Foundations of Computer Science: International Symposium, LFCS 2016, Deerfield Beach, FL, USA, January 4-7, 2016. Proceedings*, 2016, pp. 187-201.
- [3] Haskell Language. // URL: <https://www.haskell.org>. (Date: 1.08.2017)
- [4] Idris. A Language with Dependent Types.// URL:<https://www.idris-lang.org>. (Date: 1.08.2017)
- [5] Purescript. A strongly-typed functional programming language that compiles to JavaScript. URL: <http://www.purescript.org>. (Date: 1.08.2017)
- [6] Elm. A delightful language for reliable webapps. // URL: <http://elm-lang.org>. (Date: 1.08.2017)
- [7] Hackage, “The base package” // URL: <https://hackage.haskell.org/package/base-4.10.0.0> (Date: 1.08.2017)

- [8] Lipovaca M, “Learn you a Haskell for Great Good!”. //URL: <http://learnyouahaskell.com/chapters> (Date: 1.08.2017)
- [9] McBride C. and Paterson R., “Applicative programming with effects”, *Journal of Functional Programming*, 2008, vol. 18, no 01. pp 1-13.
- [10] McBride C. and Paterson R, “Functional Pearl. Idioms: applicative programming with effects”, *Journal of Functional Programming*, 2005. vol. 18, no 01. pp 1-20.
- [11] R. Nederpelt and H. Geuvers, “Type Theory and Formal Proof: An Introduction”. *Cambridge University Press*, New York, NY, USA, 2014. pp. 436.
- [12] Sorensen M. H. and Urzyczyn P, “Lectures on the Curry-Howard isomorphism”, *Studies in Logic and the Foundations of Mathematics*, vol. 149, *Elsevier Science*, 1998. pp 261.
- [13] Pierce B. C., “Types and Programming Languages”. *Cambridge, Mass: The MIT Press*, 2002. pp. 605.
- [14] Girard J.-Y., Taylor P. and Lafont Y, “Proofs and Types”, *Cambridge University Press*, New York, NY, USA, 1989. pp. 175.
- [15] Barendregt. H. P., “Lambda calculi with types” // Abramsky S., Gabbay Dov M., and S. E. Maibaum, “Handbook of logic in computer science (vol. 2), Osborne Handbooks Of Logic In Computer Science”, Vol. 2. *Oxford University Press, Inc.*, New York, NY, USA, 1993. pp 117-309.
- [16] Hindley J. Roger, “Basic Simple Type Theory”. *Cambridge University Press*, New York, NY, USA, 1997. pp. 185.
- [17] Pfenning F. and Davies R., “A judgmental reconstruction of modal logic”, *Mathematical Structures in Computer Science*, vol. 11, no 4, 2001, pp. 511-540.
- [18] H.P. Barendregt. The Lambda Calculus — Its Syntax and Semantics. *Studies in Logic and the Foundations of Mathematics*, vol. 103. Amsterdam: North-Holland, 1985.
- [19] Yoshihiko KAKUTANI, A Curry-Howard Correspondence for Intuitionistic Normal Modal Logic, Computer Software, Released February 29, 2008, Online ISSN , Print ISSN 0289-6540.