Modal type theory based on the intuitionistic epistemic logic

Abstract

Modal intuitionistic epistemic logic IEL^- was proposed by S.Artemov and T. Protopopescu as the formal foundation for the intuitionistic theory of knowledge. We construct a modal simply typed lambda-calculus which is Curry-Howard isomorphic to IEL^- as formal theory of calculations with applicative functors in functional programming languages like Haskell or Idris. We prove that this typed lambda-calculus has the strong normalization and Church-Rosser properties.

1 Introduction

Modal intutionistic epistemic logic IEL was proposed by S. Artemov and T. Proropopescu [1]. IEL provides the epistimology and the theory of knowledge as based on BHK-semantics of intuitionistic logic. IEL^- is a variant of IEL, that corresponds to intuitionistic belief. Informally, $\mathbf{K}A$ denotes that A is verified intuitionistically.

Intuitionistic epistemic logic IEL^- is defined with by following axioms and derivation rules:

Definition 1. *Intuitionistic epistemic logic IEL:*
1) IPC axioms;
2) $\mathbf{K}(A \to B) \to (\mathbf{K}A \to \mathbf{K}B)$ *(normality);*
3) $A \to \mathbf{K}A$ *(co-reflection);*
Rule: MP.

We have the deduction theorem and necessitation rule which is derivable.

V. Krupski and A. Yatmanov provided the sequential calculus for IEL and proved that this calculus is PSPACE-complete [2].

It's not difficult to see that modal axioms in IEL^- and types of the methods of Applicative class in Haskell-like languages (which is described below) are syntactically similar and we are going to show that this coincidence has a non-trivial computational meaning.

Functional programming languages such as Haskell [3], Idris [4], Purescript [5] or Elm [6] have special type classes[1] for calculations with container types like `Functor` and `Applicative` [2]:

[1] Type class in Haskell is a general interface for special group of datatypes.

[2] Reader may read more about container types in the Haskell standard library documentation[7] or in the next one textbook [8]

```haskell
class Functor f where
  fmap :: (a -> b) -> f a -> f b

class Functor f => Applicative f where
  pure :: a -> f a
  (<*>) :: f (a -> b) -> f a -> f b
```

By *container* (or *computational context*) type we mean some type-operator f, where f is a "function" from $*$ to $*$: type operator takes a simple type (which has kind $*$) and returns another simple type type with kind $*$. For more detailed description of the type system with kinds used in Haskell see [12].

The main goal of our research is a relationship between intuitionistic epistemic logic IEL^- and functional programming with effects. We show that relationship by building the type system (which is called $\lambda_{\mathbf{K}}$) which is Curry-Howard isomorphic to IEL^-. So we will consider **K**-modality as an arbitrary applicative functor.

$\lambda_{\mathbf{K}}$ consists of the rules for simply typed lambda-calculus and special typing rules for lifting types into the applicative functor **K**. We assume that our type system will axiomatize the simplest case of computation with effects with one container. We provide proof-theoretical view on this kind of computations in functional programming and prove strong normalization and confluence.

2 Typed lambda-calculus based on IEL$^-$

At first we define the natural deduction for IEL$^-$ with **K**-modality and binary connectives $\to$ and $\wedge$ (we call that calculus NIEL$^-_{\wedge,\to}$):

Definition 2. *Natural deduction $NIEL^-_{\wedge,\to}$ for IEL^- with $\to$ and $\wedge$:*

$$\dfrac{}{\Gamma, A \vdash A}\ ax$$

$$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} \to_i \qquad \dfrac{\Gamma \vdash A \to B \qquad \Gamma \vdash A}{\Gamma \vdash B} \to_i$$

$$\dfrac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge_i \qquad \dfrac{\Gamma \vdash A_1 \wedge A_2}{\Gamma \vdash A_i} \wedge_e, i \in \{1, 2\}$$

$$\dfrac{\Gamma \vdash A}{\Gamma \vdash \boldsymbol{K}A}\ \boldsymbol{K_I} \qquad \dfrac{\Gamma \vdash \mathbf{K}\vec{A} \qquad \vec{A} \vdash B}{\Gamma \vdash \mathbf{K}B}$$

Where $\Gamma \vdash \mathbf{K}\vec{A}$ is a syntax sugar for $\Gamma \vdash \mathbf{K}A_1, \dots, \Gamma \vdash \mathbf{K}A_n$.

Lemma 1. $\Gamma \vdash_{NIEL^-_{\wedge,\to}} A \Rightarrow IEL^- \vdash \bigwedge \Gamma \to A$.

Proof. Induction on the derivation.

Let us consider cases with modality.

1) If $\Gamma \vdash_{NIEL^-_{\wedge,\to}} A$, then $IEL^- \vdash \bigwedge \Gamma \to \mathbf{K}A$.

(1)	$\bigwedge\Gamma \to A$	assumption
(2)	$A \to \mathbf{K}A$	co-reflection
(3)	$(\bigwedge\Gamma \to A) \to ((A \to \mathbf{K}A) \to (\bigwedge\Gamma \to \mathbf{K}A))$	IPC theorem
(4)	$(A \to \mathbf{K}A) \to (\bigwedge\Gamma \to \mathbf{K}A)$	from (1), (3) and MP
(5)	$\bigwedge\Gamma \to \mathbf{K}A$	from (2), (4) and MP

2) If $\Gamma \vdash_{NIEL^{-}_{\wedge,\to}} \mathbf{K}\vec{A}$ and $\vec{A} \vdash B$, then $IEL^{-} \vdash \bigwedge\Gamma \to \mathbf{K}B$.

(1)	$\bigwedge\Gamma \to \bigwedge\limits_{i=1}^{n} \mathbf{K}A_i$	assumption
(2)	$\bigwedge\limits_{i=1}^{n} \mathbf{K}A_i \to \mathbf{K}\bigwedge\limits_{i=1}^{n} A_i$	IEL theorem
(3)	$\bigwedge\Gamma \to \mathbf{K}\bigwedge\limits_{i=1}^{n} A_i$	from (1), (2) and transitivity
(4)	$\bigwedge\limits_{i=1}^{n} A_i \to B$	assumption
(5)	$(\bigwedge\limits_{i=1}^{n} A_i \to B) \to \mathbf{K}(\bigwedge\limits_{i=1}^{n} A_i \to B)$	co-reflection
(6)	$\mathbf{K}(\bigwedge\limits_{i=1}^{n} A_i \to B)$	from (2), (3) and MP
(7)	$\mathbf{K}\bigwedge\limits_{i=1}^{n} A_i \to \mathbf{K}B$	from (6) and normality
(8)	$\bigwedge\Gamma \to \mathbf{K}B$	from (3), (7) and transitivity

□

Lemma 2. *If $IEL^{-} \vdash A$, then $NIEL^{-} \vdash A$.*

Proof. Straightforward derivation of modal axioms in NIEL^{-}. We consider this derivation below using terms. □

It is clearly that these lemmas could be extended for IEL^{-} with $\vee$ and $\neg$ similary.

At the next step we build the typed lambda-calculus based on NIEL$^{-}_{\wedge,\to}$ by proof-assingment in rules.

At first, we define lambda-terms and types for this lambda-calculus.

Definition 3. *The set of terms:*

Let $\mathbb{V}$ be the set of variables. The set $\Lambda_{\mathbf{K}}$ of terms is defined by the grammar:

$$\Lambda_{\mathbf{K}} ::= \mathbb{V} \mid (\lambda\Lambda.\Lambda_{\mathbf{K}}) \mid (\Lambda_{\mathbf{K}}\Lambda_{\mathbf{K}}) \mid (\Lambda_{\mathbf{K}}, \Lambda_{\mathbf{K}}) \mid (\pi_1\Lambda_{\mathbf{K}}) \mid (\pi_2\Lambda_{\mathbf{K}}) \mid (\mathbf{pure}\ \Lambda_{\mathbf{K}}) \mid (\mathbf{let\ pure}\ \Lambda_{\mathbf{K}} = \Lambda_{\mathbf{K}}\ \mathbf{in}\ \Lambda_{\mathbf{K}})$$

Definition 4. *The set of types:*

Let $\mathbb{T}$ be the set of atomic types. The set $\mathbb{T}_{\mathbf{K}}$ of types with applicative functor $\mathbf{K}$ is generated by the grammar:

$$\mathbb{T}_{\mathbf{K}} ::= \mathbb{T} \mid (\mathbb{T}_{\mathbf{K}} \to \mathbb{T}_{\mathbf{K}}) \mid (\mathbb{T}_{\mathbf{K}} \times \mathbb{T}_{\mathbf{K}}) \mid (\mathbf{K}\mathbb{T}_{\mathbf{K}}) \tag{1}$$

Context, domain of context and range of context are defined standardly [11][12].

Our type system is based on the Curry-style typing rules:

Definition 5. *Modal typed lambda calculus $\lambda_{\mathbf{K}}$ based on NIEL$^{-}_{\wedge,\to}$:*

$$\frac{}{\Gamma, x : A \vdash x : A}\ ax$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B} \to_i \qquad \frac{\Gamma \vdash f : A \to B \qquad \Gamma \vdash x : A}{\Gamma \vdash fx : B} \to_e$$

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash N : B}{\Gamma \vdash \langle x, y \rangle : A \times B} \times_i \qquad \frac{\Gamma \vdash M : A_1 \times A_2}{\Gamma \vdash \pi_i M : A_i} \times_e,\ i \in \{1, 2\}$$

$$\frac{\Gamma \vdash x : A}{\Gamma \vdash \textbf{pure}\ x : \mathbf{K}A}\ \boldsymbol{K_I} \qquad \frac{\Gamma \vdash \vec{M} : \mathbf{K}\vec{A} \qquad \vec{x} : \vec{A} \vdash M : B}{\Gamma \vdash \textbf{let pure}\, \vec{x} = \vec{M}\ \textbf{in}\ M : \mathbf{K}B}\ let_{\mathbf{K}}$$

$\mathbf{K}_I$-typing rule is the same as $\bigcirc$-introduction in lax logic (also known as monadic metalanguage [17]) and in typed lambda-calculus which is derived by proof-assignment for lax-logic proofs. $\mathbf{K}_I$ allows to inject an object of type α into the functor. $\mathbf{K}_I$ reflects the Haskell method **pure** for Applicative class. It plays the same role as the **return** method in Monad class.

$\text{let}_{\mathbf{K}}$ is the same as the $\Box$-rule in typed lambda calculus for intuitionistic normal modal logic **IK**, which is described in [19].

In fact, our calculus is the extention of typed lambda calculus for **IK** with typing rule appropriate to co-reflection.

Here are some examples of derivable terms:
1) $\vdash (\lambda x.\textbf{pure}\, x) : A \to \mathbf{K}A$;
2) $\vdash \lambda f.\lambda x.\textbf{let pure}\, \langle g, y \rangle = \langle f, x \rangle\ \textbf{in}\ gy : \mathbf{K}(A \to B) \to \mathbf{K}A \to \mathbf{K}B$
3) $\lambda f.\lambda x.\textbf{let pure}\, \langle g, y \rangle = \langle \textbf{pure}\, f, x \rangle\ \textbf{in}\ gy : (A \to B) \to \mathbf{K}A \to \mathbf{K}B$

Now we define free variables and substitutions. β-reduction, multi-step β-reduction and β-equality are defined standardly:

Definition 6. *Set $FV(M)$ of free variables for arbitrary term M:*
1) $FV(x) = \{x\}$;
2) $FV(\lambda x.M) = FV(M) \setminus \{x\}$;
3) $FV(MN) = FV(M) \cup FV(N)$;
4) $FV(\langle M, N \rangle) = FV(M) \cup FV(N)$;
5) $FV(\pi_i M) \subseteq FV(M)$, $i \in \{1, 2\}$;
6) $FV(pure\ M) = FV(M)$;
7) $FV(\textbf{let pure}\ \vec{N} = \vec{M}\ \textbf{in}\ M) = \bigcup\limits_{i=1}^{n} FV(M)$, where $n = |\vec{M}|$.

Definition 7. *Substitution:*
1) $x[x := N] = N$, $x[y := N] = x$;
2) $(MN)[x := N] = M[x := N]N[x := N]$;
3) $(\lambda x.M)[x := N] = \lambda x.M[x := N]$;
4) $(M, N)[x := P] = (M[x := P], N[x := P])$;
5) $(\pi_i M)[x := P] = \pi_i(M[x := P])$, $i \in \{1, 2\}$;
6) $(\textbf{pure}\, M)[x := P] = \textbf{pure}\, (M[x := P])$;
7) $(\textbf{let pure}\ \vec{x} = \vec{M}\ \textbf{in}\ M)[y := P] = \textbf{let pure}\ \vec{x} = (\vec{M}[y := P])\ \textbf{in}\ M$.

Definition 8. *β-reduction and η-reduction rules for $\lambda\mathbf{K}$.*

1) $(\lambda x.M)N \to_\beta M[x := N]$*;*

2) $\pi_1\langle M, N\rangle \to_\beta M$*;*

3) $\pi_2\langle M, N\rangle \to_\beta N$*;*

4) $\mathbf{let\,pure}\ \langle\vec{x}, y, \vec{z}\rangle = \langle\vec{M}, \mathbf{let\,pure}\ \vec{w} = \vec{N}\ \mathbf{in}\ Q, \vec{P}\rangle\ in\ R \to_\beta$
$\mathbf{let\,pure}\ \langle\vec{x}, \vec{w}, \vec{z}\rangle = \langle\vec{M}, \vec{N}, \vec{P}\rangle\ \mathbf{in}\ R[y := Q]$

5) $\mathbf{let\,pure}\ \vec{x} = \mathbf{pure}\ \vec{M}\ \mathbf{in}\ N \to_\beta \mathbf{pure}\ N[\vec{x} := \vec{M}]$

6) $\lambda x.fx \to_\eta f$*;*

7) $\langle\pi_1 P, \pi_2 P\rangle \to_\eta P$*;*

8) $\mathbf{let\,pure}\ \_\_ = \_\_\ \mathbf{in}\ N \to_\eta \mathbf{pure}\ N$*;*

9) $\mathbf{let\,pure}\ x = M\ \mathbf{in}\ x \to_\eta M$*;*

10) $M \to_{\beta\eta} N \Rightarrow \mathbf{pure\,M} \to_{\beta\eta} \mathbf{pure\,N}$

3 Basic lemmas

Now we will prove standard lemmas for contexts in type systems[3]:

Lemma 3. *Generation lemma.*

i) Let $\Gamma \vdash \mathbf{pure}\ M : \mathbf{K}A$*, then* $\Gamma \vdash M : A$*;*

ii) Let $\Gamma \vdash \mathbf{let\,pure}\ \vec{x} = \vec{M}\ \mathbf{in}\ N : \mathbf{K}B$*, there are some* $A_1, \dots, A_n \in \mathbb{T}_\mathbf{K}$*, such that* $\Gamma \vdash \vec{M} : \mathbf{K}\vec{A}$ *and* $\vec{x} : \vec{A} \vdash N : B$*.*

Proof.

Induction on $\Gamma \vdash \mathbf{pure}\ M : \mathbf{K}A$ and $\Gamma \vdash \mathbf{let\,pure}\ \vec{x} = \vec{N}\ \mathbf{in}\ N : \mathbf{K}B$ correspondently. □

Lemma 4. *Weakening.*

Let $\Gamma \vdash M : A$ *and* $\Gamma \subseteq \Delta$*, then* $\Delta \vdash M : A$*.*

Proof.

1) Let $\Gamma, x : A \vdash x : A$ and $\Gamma \subseteq \Delta$, then $\Delta, x : A \vdash x : A$ trivially.

2) Let $\Gamma \vdash \mathbf{pure}\ M : \mathbf{K}A$. Then $\Gamma \vdash M : A$ by generation and $\Delta \vdash M : A$ by assumption. So $\Delta \vdash \mathbf{pure}\ M : \mathbf{K}A$ by $\mathbf{K}_I$.

3) Let $\Gamma \vdash \mathbf{let\,pure}\ \vec{x} = \vec{M}\ \mathbf{in}\ N : \mathbf{K}B$ and $\Gamma \subseteq \Delta$. Then $\Gamma \vdash \vec{M} : \mathbf{K}\vec{A}$ and $\vec{x} : \vec{A} \vdash N : B$.

By assumption $\Delta \vdash \vec{M} : \mathbf{K}\vec{A}$. So $\Delta \vdash \mathbf{let\,pure}\ \vec{x} = \vec{N}\ \mathbf{in}\ N : \mathbf{K}B$ by $\mathrm{let}_\mathbf{K}$. □

Definition 9. *Type substituition*

The substituition of type C for type variable B in type A inductively defined as follows:

1) $B[B := C] = B$ *and* $D[B := C] = D$*, if* $B \neq D$*;*

2) $(A_1 \alpha A_2)[B := C] = (A_1[B := C])\alpha(A_2[B := C])$*, where* $\alpha \in \{\to, \times\}$*;*

3) $(\mathbf{K}A)[B := C] = \mathbf{K}(A[B := C])$*.*

4) Let Γ be the context, then $\Gamma[B := C] = \{x : (A[B := C]) \mid x : A \in \Gamma\}$

[3]We will not prove cases with $\to$-constructor, they are proved standardly in the same lemmas for simply typed lambda calculus, for example see [11][12][14]. We will consider only modal cases

Lemma 5. *Substituition lemma.*

i) Let $\Gamma, x : A \vdash M : B$ *and* $\Gamma \vdash N : A$*, then* $\Gamma \vdash M[x := N] : B$.

ii) Let $\Gamma \vdash M : A$*, then* $\Gamma[B := C] \vdash M : (A[B := C])$.

Proof.

i) For term substitution:

1) Let $\Gamma, x : A \vdash x : A$ and $\Gamma \vdash N : A$, but $x[x := N] = N$, so $\Gamma \vdash N : A$.

2) Let $\Gamma, x : A \vdash \mathbf{pure}\, M : \mathbf{K}B$ and $\Gamma \vdash N : A$.

By generation $\Gamma, x : A \vdash M : B$ and by assumption $\Gamma \vdash M[x := N] : B$.

By K_I, $\Gamma \vdash \mathbf{pure}\,(M[x := N]) : \mathbf{K}B$.

3) Let $\Gamma, y : A \vdash \mathbf{let\ pure}\, \vec{x} = \vec{M}\ \mathbf{in}\ N : \mathbf{K}B$ and $\Gamma \vdash N : A$.

By generation, $\Gamma, y : A \vdash \vec{M} : \mathbf{K}\vec{A}$ and $\vec{x} : \vec{A} \vdash N : B$.

By hypothesis, $\Gamma \vdash \vec{M}[x := N] : \mathbf{K}\vec{A}$.

Hence $\Gamma \vdash \mathbf{let\ pure}\, \vec{x} = \vec{M}[x := N]\ \mathbf{in}\ N : \mathbf{K}B$.

ii) For type substitution

1) Let $\Gamma, x : A \vdash x : A$, so $\Gamma[A := C], x : (A[A := C]) \vdash x : (A[A := C])$, or $\Gamma[A := C], x : C \vdash x : C$.

2) Let $\Gamma \vdash \mathbf{pure}\, M : \mathbf{K}A$. By generation $\Gamma \vdash M : A$ and by assumption $\Gamma[B := C] \vdash M : A[B := C]$.

By K_I $\Gamma \vdash \mathbf{pure}\, \mathbf{L}M : \mathbf{K}(A[B := C])$.

3) $\Gamma \vdash \mathbf{let\ pure}\, \vec{x} = \vec{M}\ \mathbf{in}\ N : \mathbf{K}B$. By generation $\Gamma \vdash \vec{M} : \mathbf{K}\vec{A}$ and $\vec{x} : \vec{A} \vdash N : B$.

By assumption $\Gamma[B_1 := C] \vdash \vec{M} : K\vec{A}[B_1 := C]$ and $\vec{x} : \vec{A}[B_1 := C] \vdash N : B[B_1 := C]$.

So by $\mathrm{let}_{\mathbf{K}}$, $\Gamma[B_1 := C] \vdash \mathbf{let\ pure}\, \vec{x} = \vec{M}\ \mathbf{in}\ N : \mathbf{K}(B[B_1 := C])$.

□

Theorem 1. *Subject reduction*

Let $\Gamma \vdash M : A$ *and* $M \twoheadrightarrow_{\beta\eta} N$*, then* $\Gamma \vdash N : A$

For cases with application, abstraction and pairs see [12] [13].

1) Let $\Gamma \vdash \mathbf{let\ pure}\, \langle \vec{x}, y, \vec{z} \rangle = \langle \vec{M}, \mathbf{let\ pure}\, \vec{w} = \vec{N}\ \mathbf{in}\, Q, \vec{P} \rangle\ in\ R : \mathbf{K}B$, then $\Gamma \mathbf{let\ pure}\, \langle \vec{x}, \vec{w}, \vec{z} \rangle = \langle \vec{M}, \vec{N}, \vec{P} \rangle\ \mathbf{in}\ R[y := Q] : \mathbf{K}B$

2) Let $\Gamma \vdash \mathbf{let\ pure}\, x = M\ \mathbf{in}\ x : \mathbf{K}A$, then $\Gamma \vdash M : \mathbf{K}A$.

See [19].

3) Let $\Gamma \vdash \mathbf{let\ pure}\, \vec{x} = \mathbf{pure}\, \vec{M}\ \mathbf{in}\ N : \mathbf{K}B$.

By generation $\Gamma \vdash \mathbf{pure}\, \vec{M} : \mathbf{K}\vec{A}$ and $\vec{x} : \vec{A} \vdash N : B$.

Moreover, $\Gamma \vdash \vec{M} : \vec{A}$. By weakening and substitution lemma $\Gamma \vdash N[\vec{x} = \vec{M}] : B$. By $\mathbf{K}_I$, $\Gamma \vdash \mathbf{pure}\, N[\vec{x} := \vec{M}] : \mathbf{K}B$.

4) Let $\vdash \mathbf{let\ pure}\, \_\_ = \_\_\ \mathbf{in}\ N : \mathbf{K}A$

By generation $\vdash N : A$.

So $\vdash \mathbf{pure}\, N : \mathbf{K}A$ by $\mathbf{K}_I$.

5) Let $\Gamma \vdash \mathbf{pure}\, M : A$ and $M \twoheadrightarrow_{\beta\eta} N$.

By generation $\Gamma \vdash M : A$ and $\Gamma \vdash N : A$ by assumption.

So $\Gamma \vdash \mathbf{pure}\, N : \mathbf{K}A$.

4 Strong normalization

5 Confluence

6 Categorical semantics

Definition 10. *Lax monoidal functor*

Let $\langle \mathcal{C}, \otimes_1, \mathbb{1} \rangle$ and $\langle \mathcal{D}, \otimes_2, \mathbb{1}' \rangle$ are monoidal categories.

A lax monoidal functor $\mathcal{F} : \langle \mathcal{C}, \otimes_1, \mathbb{1} \rangle \to \langle \mathcal{D}, \otimes_2, \mathbb{1}' \rangle$ is a functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ with additional natural transformations:

1) $u : \mathbb{1}' \to \mathcal{F}\mathbb{1}$;

2) $*_{A,B} : \mathcal{F}A \otimes_2 \mathcal{F}B \to \mathcal{F}(A \otimes_1 B)$

Definition 11. *Applicative functor*

An applicative functor is a triple $\langle \mathcal{C}, \mathcal{K}, \eta \rangle$, where $\mathcal{C}$ is a symmetric monoidal category, $\mathcal{K}$ is a lax monoidal endofunctor and η is a natural transformation, such that:

1) $u = \eta_{\mathbb{1}}$;

2) $*_{A,B} \circ (\eta_A \otimes \eta_B) = \eta_{A \otimes B}$.

By default we will consider an arbitrary closed functor on some cartersian closed category, which is the special case of an applicative functor.

We identify terminal objects. So $\mathcal{K}(\mathbb{1}) = \mathbb{1}$ and $\eta_{\mathbb{1}} = id_{\mathbb{1}}$ since $\mathcal{K}$ is an endofunctor.

6.1 Soundness

Theorem 2. *Soundness*

Let $\Gamma \vdash M : A$ and $M =_{\beta\eta} N$, then $[\![\Gamma \vdash M : A]\!] = [\![\Gamma \vdash N : A]\!]$

Proof.

Definition 12. *Semantical translation from $\lambda_{\boldsymbol{K}}$ to CCC with applicative functor $\mathcal{K}$:*

1) Interpretation for types:

$[\![A]\!] := \hat{A}, A \in \mathbb{T}$;

$[\![A \to B]\!] := [\![A]\!] \to [\![B]\!]$;

$[\![A \times B]\!] := [\![A]\!] \times [\![B]\!]$.

2) Interpretation for modal types: $[\![\boldsymbol{K}A]\!] = \mathcal{K}[\![A]\!]$;

3) Interpretaion for contexts:

$[\![\Gamma = \{x_1 : A_1, ..., x_n : A_n\}]\!] := [\![\Gamma]\!] = [\![A_1]\!] \times ... \times [\![A_n]\!]$;

4) Interpretation for typing assignment: $[\![\Gamma \vdash M : A]\!] := [\![M]\!] : [\![\Gamma]\!] \to [\![A]\!]$.

5) Interpretation for typing rules:

$$\frac{}{[\![\Gamma, x : A \vdash x : A]\!] = \pi_2 : [\![\Gamma]\!] \times [\![A]\!] \to [\![A]\!]}$$

$$\frac{[\![\Gamma, x : A \vdash M : B]\!] = f : [\![\Gamma]\!] \times [\![A]\!] \to [\![B]\!]}{[\![\Gamma \vdash (\lambda x.M) : A \to B]\!] = \Lambda(f) : [\![\Gamma]\!] \to [\![B]\!]^{[\![A]\!]}}$$

$$\frac{[\![\Gamma \vdash M : A \to B]\!] = [\![M]\!] : [\![\Gamma]\!] \to [\![B]\!]^{[\![A]\!]} \qquad [\![\Gamma \vdash N : A]\!] = [\![N]\!] : [\![\Gamma]\!] \to [\![A]\!]}{[\![\Gamma \vdash (MN) : B]\!] = [\![\Gamma]\!] \xrightarrow{\langle [\![M]\!], [\![N]\!] \rangle} [\![B]\!]^{[\![A]\!]} \times [\![A]\!] \xrightarrow{\epsilon} [\![B]\!]}$$

$$\frac{[\![\Gamma \vdash M : A]\!] = f : [\![\Gamma]\!] \to [\![A]\!] \qquad [\![\Gamma \vdash N : B]\!] = g : [\![\Gamma]\!] \to [\![B]\!]}{[\![\Gamma \vdash (M, N) : A \times B]\!] = \langle f, g \rangle : [\![\Gamma]\!] \to [\![A]\!] \times [\![B]\!]}$$

$$\frac{[\![\Gamma \vdash p : A_1 \times A_2]\!] = f : [\![\Gamma]\!] \to [\![A_1]\!] \times [\![A_2]\!]}{[\![\Gamma \vdash \pi_i p : A_i]\!] = [\![\Gamma]\!] \xrightarrow{f} [\![A_1]\!] \times [\![A_2]\!] \xrightarrow{\pi_i} [\![A_i]\!]} \ i \in \{1, 2\}$$

$$\frac{[\![\Gamma \vdash M : A]\!] = [\![M]\!] : [\![\Gamma]\!] \to [\![A]\!]}{[\![\Gamma \vdash \textbf{pure}\, M : \boldsymbol{K}A]\!] := [\![\Gamma]\!] \xrightarrow{[\![M]\!]} [\![A]\!] \xrightarrow{\eta_{[\![A]\!]}} \mathcal{K}[\![A]\!]}$$

$$\frac{[\![\Gamma \vdash \vec{M} : \mathbf{K}\vec{A}]\!] = \langle [\![M_1]\!], \ldots, [\![M_n]\!] \rangle : [\![\Gamma]\!] \to \prod_{i=1}^{n} \mathcal{K}[\![A_i]\!] \qquad [\![\vec{x} : \vec{A} \vdash N : B]\!] = [\![N]\!] : \prod_{i=1}^{n} [\![A_i]\!] \to [\![B]\!]}{[\![\Gamma \vdash \textbf{let pure}\, \vec{x} = \vec{M} \textbf{ in } M : \mathbf{K}B]\!] = \mathcal{K}([\![N]\!]) \circ *_{[\![A_1]\!], \ldots, [\![A_n]\!]} \circ \langle [\![M_1]\!], \ldots, [\![M_n]\!] \rangle : [\![\Gamma]\!] \to \mathcal{K}[\![B]\!]}$$

Definition 13. *Simultaneous substitution*

Let $\Gamma = \{x_1 : A_1, ..., x_n : A_n\}$, $\Gamma \vdash M : A$ *and for all* $i \in \{1, ..., n\}$, $\Gamma \vdash M_i : A_i$.

We define simultaneous substitution $M[\vec{x} := \vec{M}]$ *recursively by:*

1) $x_i[\vec{x} := \vec{M}] = M_i$;

2) $(\lambda x.M)[\vec{x} := \vec{M}] = \lambda x.(M[\vec{x} := \vec{M}])$;

3) $(MN)[\vec{x} := \vec{M}] = (M[\vec{x} = \vec{M}])(N[\vec{x} := \vec{M}])$;

4) $\langle M, N \rangle = \langle (M[\vec{x} = \vec{M}]), (N[\vec{x} := \vec{M}]) \rangle$;

5) $(\pi_i P)[\vec{x} := \vec{M}] = \pi_i(P[\vec{x} = \vec{M}])$;

6) $(\textbf{pure}\, M)[\vec{x} := \vec{M}] = \textbf{pure}\,(M[\vec{x} = \vec{M}])$;

7) $(\textbf{let pure}\, \vec{x} = \vec{M} \textbf{ in } N)[\vec{y} := \vec{P}] = \textbf{let pure}\, \vec{x} = (\vec{M}[\vec{y} := \vec{P}]) \textbf{ in } N$

Lemma 6.

$[\![M[x_1 := M_1, \ldots, x_n := M_n]]\!] = [\![M]\!] \circ \langle [\![M_1]\!], \ldots, [\![M_n]\!] \rangle$.

Proof.

1) $[\![\Gamma \vdash (\textbf{pure}\, M)[\vec{x} := \vec{M}] : \mathbf{K}A]\!] = [\![\Gamma \vdash \textbf{pure}\, M : \mathbf{K}A]\!] \circ \langle [\![M_1]\!], \ldots, [\![M_n]\!] \rangle$.

$$\begin{aligned}
[\![\Gamma \vdash (\textbf{pure}\, M)[\vec{x} := \vec{M}] : \mathbf{K}A]\!] &= [\![\Gamma \vdash \textbf{pure}\,(M[\vec{x} := \vec{M}]) : \mathbf{K}A]\!] && \text{Substitution definition}\\
&= \eta_{[\![A]\!]} \circ [\![(M[\vec{x} := \vec{M}])]\!] && \text{Translation for pure}\\
&= \eta_{[\![A]\!]} \circ ([\![M]\!] \circ \langle [\![M_1]\!], \ldots, [\![M_n]\!] \rangle) && \text{Induction hypothesis}\\
&= (\eta_{[\![A]\!]} \circ [\![M]\!]) \circ \langle [\![M_1]\!], \ldots, [\![M_n]\!] \rangle && \text{Associativity of composition}\\
&= [\![\Gamma \vdash \textbf{pure}\, M : \mathbf{K}A]\!] \circ \langle [\![M_1]\!], \ldots, [\![M_n]\!] \rangle && \text{Translation for pure}
\end{aligned}$$

2) $[\![\Gamma \vdash (\textbf{let pure}\, \vec{x} = \vec{M} \textbf{ in } N)[\vec{y} := \vec{P}] : \mathbf{K}B]\!] = [\![\Gamma \vdash \textbf{let pure}\, \vec{x} = \vec{M} \textbf{ in } N : \mathbf{K}B]\!] \circ \langle [\![P_1]\!], \ldots, [\![P_n]\!] \rangle$

$[\![\Gamma \vdash (\textbf{let pure}\ \vec{x} = \vec{M}\ \textbf{in}\ N)[\vec{y} := \vec{P}] : \mathbf{K}B]\!] =$
Substitution definition
$[\![\Gamma \vdash \textbf{let pure}\ \vec{x} = (\vec{M}[\vec{y} := \vec{P}])\ \textbf{in}\ N : \mathbf{K}B]\!] =$
Interpretaion for $let_{\mathbf{K}}$
$\mathcal{K}([\![N]\!]) \circ *_{[\![A_1]\!],\ldots,[\![A_n]\!]} \circ [\![\Gamma \vdash (\vec{M}[\vec{y} := \vec{P}]) \vdash: \mathbf{K}\vec{A}]\!] =$
Induction hypothesis
$\mathcal{K}([\![N]\!]) \circ *_{[\![A_1]\!],\ldots,[\![A_n]\!]} \circ ([\![\vec{M}]\!] \circ \langle [\![P_1]\!], \ldots, [\![P_n]\!]\rangle) =$
Associativity of composition
$(\mathcal{K}([\![N]\!]) \circ *_{[\![A_1]\!],\ldots,[\![A_n]\!]} \circ [\![\vec{M}]\!]) \circ \langle [\![P_1]\!], \ldots, [\![P_n]\!]\rangle =$
By interpretation
$[\![\Gamma \vdash (\textbf{let pure}\ \vec{x} = \vec{M}\ \textbf{in}\ N]\!] \circ \langle [\![P_1]\!], \ldots, [\![P_n]\!]\rangle$

□

Lemma 7.
i) Let $\Gamma \vdash M : A$ *and* $M \twoheadrightarrow_\beta N$*, then* $[\![\Gamma \vdash M : A]\!] = [\![\Gamma \vdash N : A]\!]$*;*
ii) Let $\Gamma \vdash M : A$ *and* $M \twoheadrightarrow_\eta N$*, then* $[\![\Gamma \vdash M : A]\!] = [\![\Gamma \vdash N : A]\!]$*;*

Proof.
i) For β-reduction

Cases with β-reductions for $let_{\mathbf{K}}$ are shown in [20]. Let us consider cases with **pure**.

1) $[\![\Gamma \vdash \textbf{let pure}\ \vec{x} = \textbf{pure}\ \vec{M}\ \textbf{in}\ N : \mathbf{K}B]\!] = [\![\Gamma \vdash \textbf{pure}\ N[\vec{x} := \vec{M}] : \mathbf{K}B]\!]$

$[\![\Gamma \vdash \textbf{let pure}\ \vec{x} = \textbf{pure}\ \vec{M}\ \textbf{in}\ N : \mathbf{K}B]\!] =$
By interpretation
$\mathcal{K}([\![N]\!]) \circ *_{[\![A_1]\!],\ldots,[\![A_n]\!]} \circ \langle \eta_{[\![A_1]\!]} \circ [\![M_1]\!], \ldots, \eta_{[\![A_n]\!]} \circ [\![M_n]\!]\rangle =$
By the property of a pair of morphisms
$\mathcal{K}([\![N]\!]) \circ *_{[\![A_1]\!],\ldots,[\![A_n]\!]} \circ (\eta_{[\![A_1]\!]} \times \cdots \times \eta_{[\![A_n]\!]}) \circ \langle [\![M_1]\!], \ldots, [\![M_n]\!]\rangle =$
Associativity of composition
$\mathcal{K}([\![N]\!]) \circ (*_{[\![A_1]\!],\ldots,[\![A_n]\!]} \circ (\eta_{[\![A_1]\!]} \times \ldots \eta_{[\![A_n]\!]})) \circ \langle [\![M_1]\!], \ldots, [\![M_n]\!]\rangle =$
By the definition of an applicative functor
$\mathcal{K}([\![N]\!]) \circ \eta_{[\![A_1]\!] \times \cdots \times [\![A_n]\!]} \circ \langle [\![M_1]\!], \ldots, [\![M_n]\!]\rangle =$
Naturality of η
$\eta_{[\![B]\!]} \circ [\![N]\!] \circ \langle [\![M_1]\!], \ldots, [\![M_n]\!]\rangle =$
Associativity of composition
$\eta_{[\![B]\!]} \circ ([\![N]\!] \circ \langle [\![M_1]\!], \ldots, [\![M_n]\!]\rangle) =$
Simultaneous substitution lemma
$\eta_{[\![B]\!]} \circ [\![N[\vec{x} := \vec{M}]]\!]$
By interpetation
$[\![\Gamma \vdash \textbf{pure}\ (N[\vec{x} := \vec{M}]) : \mathbf{K}B]\!]$

2)
If $\Gamma \vdash M : A$ and $M \to_{\beta\eta} N$, then $[\![\Gamma \vdash \textbf{pure}\ M : \mathbf{K}A]\!] = [\![\Gamma \vdash \textbf{pure}\ N : \mathbf{K}A]\!]$.

If $\Gamma \vdash M : A$ and $M \to_{\beta\eta} N$, then $\Gamma \vdash N : A$ by subject reduction.
By assumption $[\![\Gamma \vdash M : A]\!] = [\![\Gamma \vdash N : A]\!]$.
So $\eta_{[\![A]\!]} \circ [\![\Gamma \vdash M : A]\!] = \eta_{[\![A]\!]} \circ [\![\Gamma \vdash N : A]\!]$.

Hence $[\![\Gamma \vdash \mathbf{pure}\, M : \mathbf{K}A]\!] = [\![\Gamma \vdash \mathbf{pure}\, N : \mathbf{K}A]\!]$.

ii) For η-reduction.
1) $[\![\vdash \mathbf{let\ pure}\, \_\_ = \_\_ \,\mathbf{in}\, N : KA]\!] = [\![\vdash \mathbf{pure}\, N : \mathbf{KA}]\!]$.

$[\![\vdash \mathbf{let\ pure}\, \_\_ = \_\_ \,\mathbf{in}\, N : KA]\!] =$	By interpetation
$\mathcal{K}([\![N]\!]) \circ \eta_{\mathbb{1}} =$	Naturality for η
$\eta_{[\![A]\!]} \circ [\![N]\!] =$	By interpretation
$[\![\vdash \mathbf{pure}\, N : \mathbf{K}A]\!]$	

□

□

6.2 Completeness

We will consider term model for simply typed lambda calculus $\times$ and $\to$ standardly described in [22] [23].

Definition 14. *Let us define an endofunctor $\mathcal{K} : \mathcal{C}(\lambda) \to \mathcal{C}(\lambda)$, such that:*
1) $\mathbf{K} : A \mapsto \mathbf{K}A$;
2) $\mathbf{K} : [x, M] \in Hom_{\mathcal{C}(\lambda)}(A, B) \mapsto fmap\ f = [y, \mathbf{let\ pure}\, x = y \,\mathbf{in}\, M] \in Hom_{\mathcal{C}(\lambda)}(\mathbf{K}A, \mathbf{K}B)$.

Lemma 8. *Functoriality*
i) $\mathbf{K}(g \circ f) = \mathbf{K}(g) \circ \mathbf{K}(f)$;
ii) $\mathbf{K}(id_A) = id_{\mathbf{K}A}$.

Proof. Easy checking using reduction rules.

□

Definition 15. *Let us define natural transformations:*
1) $\eta : Id \Rightarrow \mathcal{K}$, s. t. $\forall A \in Ob_{\mathcal{C}(\lambda)}$, $\eta_A = [x, \mathbf{pure}\, x] \in Hom_{\mathcal{C}(\lambda)}(A, \mathbf{K}A)$;
2) $*_{A,B} : \mathbf{K}A \times \mathbf{K}B \to \mathbf{K}(A \times B)$, s. t. $\forall A, B \in Ob_{\mathcal{C}(\lambda)}, *_{A,B} = [p, \mathbf{let\ pure}\, x, y = \pi_1 p, \pi_2 p \,\mathbf{in}\, \langle x, y\rangle] \in Hom_{\mathcal{C}(\lambda)}(\mathbf{K}A \times \mathbf{K}B, \mathbf{K}(A \times B))$.

Implementation for $*$ in our term model is a modification of $\text{let}_{\mathbf{K}}$-rule:

$$\dfrac{\dfrac{p : \mathbf{K}A \times \mathbf{K}B \vdash p : \mathbf{K}A \times \mathbf{K}B}{p : \mathbf{K}A \times \mathbf{K}B \vdash \pi_1 p : \mathbf{K}A} \quad \dfrac{p : \mathbf{K}A \times \mathbf{K}B \vdash p : \mathbf{K}A \times \mathbf{K}B}{p : \mathbf{K}A \times \mathbf{K}B \vdash \pi_2 p : \mathbf{K}B} \quad \dfrac{x : A \vdash x : A \qquad y : B \vdash y : B}{x : A, y : B \vdash \langle x, y\rangle : A \times B}}{p : \mathbf{K}A \times \mathbf{K}B \vdash \mathbf{let\ pure}\, \langle x, y\rangle = \langle \pi_1 p, \pi_2 p\rangle \,\mathbf{in}\, \langle x, y\rangle : \mathbf{K}(A \times B)}$$

Lemma 9. *Naturality for η and for $*$*
i) $fmap\ f \circ \eta_A = \eta_B \circ f$;
ii) $fmap\ (f \times g) \circ *_{A,B} = *_{C,D} \circ (fmap\ f) \times (fmap\ g)$.
iii) $*_{A,B} \circ (\eta_A \times \eta_B) = \eta_{A \times B}$;

Proof.
i) fmap $f \circ \eta_A = \eta_B \circ f$

$\eta_B \circ f =$	By the definition
$[y, \textbf{pure}\, y] \circ [x, M] =$	By the definition of composition
$[x, \textbf{pure}\, y[y := M]] =$	By substitution
$[x, \textbf{pure}\, M]$	

On the other hand:

$\text{fmap } f \circ \eta_A =$	By the definiton
$[z, \textbf{let pure}\, x = z \textbf{ in } M] \circ [x, \textbf{pure}\, \mathbf{x}] =$	By the definition of composition
$[x, \textbf{let pure}\, x = z \textbf{ in } M[z := \textbf{pure}\, x]] =$	By substitution
$[x, \textbf{let pure}\, x = \textbf{pure}\, \mathbf{x} \textbf{ in } M] =$	β-reduction rule
$[x, \textbf{pure}\, M[x := x]] =$	By substitution
$[x, \textbf{pure}\, M]$	

ii) fmap $(f \times g) \circ *_{A,B} = *_{C,D} \circ (\text{fmap } f) \times (\text{fmap } g)$

See [19]

iii) $*_{A,B} \circ (\eta_A \times \eta_B) = \eta_{A\times B}$
Follows from i) and ii).

$\square$

Tensorial strength is defined as follows:

Definition 16. *Tensorial strength*
Let $[p, \langle \textbf{pure}\, (\pi_1 p), \pi_2 p\rangle] \in Hom_{\mathcal{C}(\lambda)}(A \times \mathbf{K}B, \mathbf{K}A \times \mathbf{K}B)$.
So tensorial strength is defined as $\tau_{A,B} = *_{A,B} \circ [p, \langle \textbf{pure}\, (\pi_1 p), \pi_2 p\rangle]$.

It is clearly that tensorial strength defined above can be simplified as follows:

$*_{A,B} \circ [p, \langle \textbf{pure}\, (\pi_1 p), \pi_2 p\rangle] =$	By definition
$[p', \textbf{let pure}\, x, y = \pi_1 p', \pi_2 p' \textbf{ in } \langle x, y\rangle] \circ [p, \langle \textbf{pure}\, (\pi_1 p), \pi_2 p\rangle] =$	By composition
$[p, \textbf{let pure}\, x, y = \pi_1 p', \pi_2 p' \textbf{ in } \langle x, y\rangle[p' := \langle \textbf{pure}\, (\pi_1 p), \pi_2 p\rangle]] =$	By substitution
$[p, \textbf{let pure}\, x, y = \pi_1(\langle \textbf{pure}\, (\pi_1 p), \pi_2 p\rangle), \pi_2(\langle \pi_1 p, \textbf{pure}\, (\pi_2 p)\rangle) \textbf{ in } \langle x, y\rangle] =$	By β-reduction rules
$[p, \textbf{let pure}\, x, y = \textbf{pure}\, (\pi_1 p), \pi_2 p \textbf{ in } \langle x, y\rangle]$	

Lemma 10. *Weak commutativity.*
$\mathit{fmap}\ ([p, \langle \pi_2 p, \pi_1 p\rangle]) \circ \tau_{A,B} =$
$*_{B,A} \circ [q, \langle \pi_1 q, \textbf{pure}\, (\pi_2 q)\rangle] \circ [p, \langle \pi_2 p, \pi_1 p\rangle]$

Proof.

$\mathrm{fmap}\ ([r, \langle \pi_2 r, \pi_1 r\rangle]) \circ \tau_{A,B} =$
By the definition of τ
$\mathrm{fmap}\ ([r, \langle \pi_2 r, \pi_1 r\rangle]) \circ [p, \mathbf{let\ pure}\ x, y = \mathbf{pure}\ (\pi_1 p), \pi_2 p\ \mathbf{in}\ \langle x, y\rangle] =$
By the definition of fmap
$[q, \mathbf{let\ pure}\ r = q\ \mathbf{in}\ \langle \pi_2 r, \pi_1 r\rangle] \circ [p, \mathbf{let\ pure}\ x, y = \mathbf{pure}\ (\pi_1 p), \pi_2 p\ \mathbf{in}\ \langle x, y\rangle] =$
Composition
$[p, \mathbf{let\ pure}\ r = q\ \mathbf{in}\ \langle \pi_2 r, \pi_1 r\rangle[q := \mathbf{let\ pure}\ x, y = \mathbf{pure}\ (\pi_1 p), \pi_2 p\ \mathbf{in}\ \langle x, y\rangle]] =$
By β-reduction rules
$[p, \mathbf{let\ pure}\ r = (\mathbf{let\ pure}\ x, y = \mathbf{pure}\ (\pi_1 p), \pi_2 p\ \mathbf{in}\ \langle x, y\rangle)\ \mathbf{in}\ \langle \pi_2 r, \pi_1 r\rangle] =$
By β-reduction rules
$[p, \mathbf{let\ pure}\ x, y = \mathbf{pure}\ (\pi_1 p), \pi_2 p\ \mathbf{in}\ \langle \pi_2 r, \pi_1 r\rangle[r := \langle x, y\rangle]] =$
By substitution
$[p, \mathbf{let\ pure}\ x, y = \mathbf{pure}\ (\pi_1 p), \pi_2 p\ \mathbf{in}\ \langle \pi_2 \langle x, y\rangle, \pi_1 \langle x, y\rangle\rangle] =$
By β-reduction rules
$[p, \mathbf{let\ pure}\ x, y = \mathbf{pure}\ (\pi_1 p), \pi_2 p\ \mathbf{in}\ \langle y, x\rangle] =$

On the other hand
$*_{B,A} \circ [q, \langle \pi_1 q, \mathbf{pure}\ (\pi_2 q)\rangle] \circ [p, \langle \pi_2 p, \pi_1 p\rangle] =$
By the definition of $*$
$[r, \mathbf{let\ pure}\ y, x = \pi_1 r, \pi_2 r\ \mathbf{in}\ \langle y, x\rangle] \circ [q, \langle \pi_1 q, \mathbf{pure}\ (\pi_2 q)\rangle] \circ [p, \langle \pi_2 p, \pi_1 p\rangle] =$
Composition
$[r, \mathbf{let\ pure}\ y, x = \pi_1 r, \pi_2 r\ \mathbf{in}\ \langle y, x\rangle] \circ [p, \langle \pi_1 q, \mathbf{pure}\ (\pi_2 q)\rangle[q := \langle \pi_2 p, \pi_1 p\rangle]] =$
By subsitution and by β-reduction rules
$[r, \mathbf{let\ pure}\ y, x = \pi_1 r, \pi_2 r\ \mathbf{in}\ \langle y, x\rangle] \circ [p, \langle \pi_2 p, \mathbf{pure}\ (\pi_1 p)\rangle]] =$
Composition
$[p, \mathbf{let\ pure}\ y, x = \pi_1 r, \pi_2 r\ \mathbf{in}\ \langle y, x\rangle[r := \langle \pi_2 p, \mathbf{pure}\ (\pi_1 p)\rangle]] =$
By subsitution and by β-reduction rules
$[p, \mathbf{let\ pure}\ y, x = \pi_2 p, \mathbf{pure}\ (\pi_1 p)\ \mathbf{in}\ \langle y, x\rangle] =$
By symmetricity of assingment
$[p, \mathbf{let\ pure}\ x, y = \mathbf{pure}\ (\pi_1 p), \pi_2 p\ \mathbf{in}\ \langle y, x\rangle]$

□

Theorem 3. *Completeness*
i) $\mathbf{K}$ *is an applicative functor;*
ii) Let $[\![\Gamma \vdash M : A]\!] = [\![\Gamma \vdash N : A]\!]$*, then* $M =_{\beta\eta} N$.

Proof. Immediately follows from previous lemmas in the section. □

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