Modal type theory based on the intuitionistic epistemic logic

Abstract

Modal intuitionistic epistemic logic IEL^- was proposed by S.Artemov and T. Protopopescu as the formal foundation for the intuitionistic theory of knowledge. We construct a modal simply typed lambda-calculus which is Curry-Howard isomorphic to IEL^- as formal theory of calculations with applicative functors in functional programming languages like Haskell or Idris. We prove that this typed lambda-calculus has the strong normalization and Church-Rosser properties.

1 Introduction

Modal intutionistic epistemic logic IEL was proposed by S. Artemov and T. Proropopescu [1]. IEL provides the epistimology and the theory of knowledge as based on BHK-semantics of intuitionistic logic. IEL^- is a variant of IEL, that corresponds to intuitionistic belief. Informally, $\mathbf{K}A$ denotes that A is verified intuitionistically.

Intuitionistic epistemic logic IEL^- is defined with by following axioms and derivation rules:

Definition 1. *Intuitionistic epistemic logic IEL:*
1) IPC axioms;
2) $\boldsymbol{K}(A \to B) \to (\boldsymbol{K}A \to \boldsymbol{K}B)$ *(normality);*
3) $A \to \boldsymbol{K}A$ *(co-reflection);*
Rule: MP.

We have the deduction theorem and necessitation rule which is derivable.

V. Krupski and A. Yatmanov provided the sequential calculus for IEL and proved that this calculus is PSPACE-complete [2].

It's not difficult to see that modal axioms in IEL^- and types of the methods of Applicative class in Haskell-like languages (which is described below) are syntactically similar and we are going to show that this coincidence has a non-trivial computational meaning.

Functional programming languages such as Haskell [3], Idris [4], Purescript [5] or Elm [6] have special type classes[1] for calculations with container types like `Functor` and `Applicative` [2]:

[1] Type class in Haskell is a general interface for special group of datatypes.

[2] Reader may read more about container types in the Haskell standard library documentation[7] or in the next one textbook [8]

```haskell
class Functor f where
  fmap :: (a -> b) -> f a -> f b

class Functor f => Applicative f where
  pure :: a -> f a
  (<*>) :: f (a -> b) -> f a -> f b
```

By *container* (or *computational context*) type we mean some type-operator f, where f is a "function" from $*$ to $*$: type operator takes a simple type (which has kind $*$) and returns another simple type type with kind $*$. For more detailed description of the type system with kinds used in Haskell see [12].

The main goal of our research is a relationship between intuitionistic epistemic logic IEL^- and functional programming with effects. We show that relationship by building the type system (which is called $\lambda_{\mathbf{K}}$) which is Curry-Howard isomorphic to IEL^-. So we will consider **K**-modality as an arbitrary applicative functor.

λK consists of the rules for simply typed lambda-calculus and special typing rules for lifting types into the applicative functor **K**. We assume that our type system will axiomatize the simplest case of computation with effects with one container. We provide proof-theoretical view on this kind of computations in functional programming and prove strong normalization and confluence.

2 Typed lambda-calculus based on IEL$^-$

At first we define the natural deduction for IEL$^-$ with **K**-modality and binary connectives $\to$ and $\wedge$ (we call that calculus NIEL$^-_{\wedge,\to}$):

Definition 2. *Natural deduction NIEL$^-_{\wedge,\to}$ for IEL$^-$ with $\to$ and $\wedge$:*

$$\dfrac{}{\Gamma, A \vdash A}\ ax$$

$$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ \to_i \qquad \dfrac{\Gamma \vdash A \to B \qquad \Gamma \vdash A}{\Gamma \vdash B}\ \to_i$$

$$\dfrac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge_i \qquad \dfrac{\Gamma \vdash A_1 \wedge A_2}{\Gamma \vdash A_i}\ \wedge_e, i \in \{1,2\}$$

$$\dfrac{\Gamma \vdash A}{\Gamma \vdash \boldsymbol{K}A}\ \boldsymbol{K_I} \qquad \dfrac{\Gamma \vdash \mathbf{K}\vec{A} \qquad \vec{A} \vdash B}{\Gamma \vdash \mathbf{K}B}$$

Where $\Gamma \vdash \mathbf{K}\vec{A}$ is a syntax sugar for $\Gamma \vdash \mathbf{K}A_1, \dots, \Gamma \vdash \mathbf{K}A_n$.

Lemma 1. *$\Gamma \vdash_{NIEL^-_{\wedge,\to}} A \Rightarrow IEL^- \vdash \bigwedge \Gamma \to A$.*

Proof. Induction on the derivation.

Let us consider cases with modality.

1) If $\Gamma \vdash_{NIEL^-_{\wedge,\to}} A$, then $IEL^- \vdash \bigwedge \Gamma \to \mathbf{K}A$.

(1)	$\bigwedge\Gamma \to A$	assumption
(2)	$A \to \mathbf{K}A$	co-reflection
(3)	$(\bigwedge\Gamma \to A) \to ((A \to \mathbf{K}A) \to (\bigwedge\Gamma \to \mathbf{K}A))$	IPC theorem
(4)	$(A \to \mathbf{K}A) \to (\bigwedge\Gamma \to \mathbf{K}A)$	from (1), (3) and MP
(5)	$\bigwedge\Gamma \to \mathbf{K}A$	from (2), (4) and MP

2) If $\Gamma \vdash_{NIEL^-_{\wedge,\to}} \mathbf{K}\vec{A}$ and $\vec{A} \vdash B$, then $IEL^- \vdash \bigwedge\Gamma \to \mathbf{K}B$.

(1)	$\bigwedge\Gamma \to \bigwedge\limits_{i=1}^{n} \mathbf{K}A_i$	assumption
(2)	$\bigwedge\limits_{i=1}^{n} \mathbf{K}A_i \to \mathbf{K}\bigwedge\limits_{i=1}^{n} A_i$	IEL theorem
(3)	$\bigwedge\Gamma \to \mathbf{K}\bigwedge\limits_{i=1}^{n} A_i$	from (1), (2) and transitivity
(4)	$\bigwedge\limits_{i=1}^{n} A_i \to B$	assumption
(5)	$(\bigwedge\limits_{i=1}^{n} A_i \to B) \to \mathbf{K}(\bigwedge\limits_{i=1}^{n} A_i \to B)$	co-reflection
(6)	$\mathbf{K}(\bigwedge\limits_{i=1}^{n} A_i \to B)$	from (2), (3) and MP
(7)	$\mathbf{K}\bigwedge\limits_{i=1}^{n} A_i \to \mathbf{K}B$	from (6) and normality
(8)	$\bigwedge\Gamma \to \mathbf{K}B$	from (3), (7) and transitivity

□

Lemma 2. *If $IEL^- \vdash A$, then $NIEL^- \vdash A$.*

Proof. Straightforward derivation of modal axioms in NIEL$^-$. We consider this derivation below using terms. □

At the next step we build the typed lambda-calculus based on NIEL$^-_{\wedge,\to}$ by proof-assingment in rules.

At first, we define lambda-terms and types for this lambda-calculus.

Definition 3. *The set of terms:*

Let $\mathbb{V}$ be the set of variables. The set $\Lambda_{\boldsymbol{K}}$ of terms is defined by the grammar:

$$\Lambda_{\boldsymbol{K}} ::= \mathbb{V} \mid (\lambda\Lambda.\Lambda_{\boldsymbol{K}}) \mid (\Lambda_{\boldsymbol{K}}\Lambda_{\boldsymbol{K}}) \mid (\Lambda_{\boldsymbol{K}}, \Lambda_{\boldsymbol{K}}) \mid (\pi_1\Lambda_{\boldsymbol{K}}) \mid (\pi_2\Lambda_{\boldsymbol{K}}) \mid (\mathbf{pure}\ \Lambda_{\boldsymbol{K}}) \mid (\mathbf{let\ pure}\ \Lambda_{\boldsymbol{K}} = \Lambda_{\boldsymbol{K}}\ \mathbf{in}\ \Lambda_{\boldsymbol{K}})$$

Definition 4. *The set of types:*

Let $\mathbb{T}$ be the set of atomic types. The set $\mathbb{T}_{\boldsymbol{K}}$ of types with applicative functor $\boldsymbol{K}$ is generated by the grammar:

$$\mathbb{T}_{\boldsymbol{K}} ::= \mathbb{T} \mid (\mathbb{T}_{\boldsymbol{K}} \to \mathbb{T}_{\boldsymbol{K}}) \mid (\mathbb{T}_{\boldsymbol{K}} \times \mathbb{T}_{\boldsymbol{K}}) \mid (\boldsymbol{K}\mathbb{T}_{\boldsymbol{K}}) \tag{1}$$

Context, domain of context and range of context are defined standardly [11][12].

Our type system is based on the Curry-style typing rules:

Definition 5. *Modal typed lambda calculus $\lambda\boldsymbol{K}$ based on NIEL$^-_{\wedge,\to}$:*

$$\frac{}{\Gamma, x : A \vdash x : A}\ ax$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B} \to_i \qquad \frac{\Gamma \vdash f : A \to B \qquad \Gamma \vdash x : A}{\Gamma \vdash fx : B} \to_e$$

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash N : B}{\Gamma \vdash \langle x, y \rangle : A \times B} \times_i \qquad \frac{\Gamma \vdash M : A_1 \times A_2}{\Gamma \vdash \pi_i M : A_i} \times_e,\ i \in \{1, 2\}$$

$$\frac{\Gamma \vdash x : A}{\Gamma \vdash \mathbf{pure}\ x : \mathbf{K}A} \boldsymbol{K_I} \qquad \frac{\Gamma \vdash \vec{M} : \mathbf{K}\vec{A} \qquad \vec{x} : \vec{A} \vdash M : B}{\Gamma \vdash \mathbf{let\ pure}\ \vec{x} = \vec{M}\ \mathbf{in}\ M : \mathbf{K}B} \mathit{let}_{\mathbf{K}}$$

$\mathbf{K}_I$-typing rule is the same as $\bigcirc$-introduction in lax logic (also known as monadic metalanguage [17]) and in typed lambda-calculus which is derived by proof-assignment for lax-logic proofs. $\mathbf{K}_I$ allows to inject an object of type α into the functor. $\mathbf{K}_I$ reflects the Haskell method **pure** for Applicative class. It plays the same role as the **return** method in Monad class.

$\mathrm{let}_{\mathbf{K}}$ is similar to $\Box_I$-rule in typed lambda calculus for intuitionistic normal modal logic **IK**, which is described in [19].

Here are some examples of derivation trees.

$$\dfrac{\dfrac{x : A \vdash x : A}{x : A \vdash \mathbf{pure}\, x : \mathbf{K}A} \mathbf{K}_I}{\vdash (\lambda x.\mathbf{pure}\, x) : A \to \mathbf{K}A} \to_i$$

$$\dfrac{\dfrac{\dfrac{f : \mathbf{K}(A \to B) \vdash f : \mathbf{K}(A \to B) \qquad x : \mathbf{K}A \vdash x : \mathbf{K}A \qquad \dfrac{g : A \to B \qquad y : A}{g : A \to B, y : A \vdash gy : B}}{f : \mathbf{K}(A \to B), x : \mathbf{K}A \vdash \mathbf{let\ pure}\ \langle g, y \rangle = \langle f, x \rangle\ \mathbf{in}\ gy : \mathbf{K}B}}{f : \mathbf{K}(A \to B) \vdash \lambda x.\mathbf{let\ pure}\ \langle g, y \rangle = \langle f, x \rangle\ \mathbf{in}\ gy : \mathbf{K}A \to \mathbf{K}B}}{\vdash \lambda f.\lambda x.\mathbf{let\ pure}\ \langle g, y \rangle = \langle f, x \rangle\ \mathbf{in}\ gy : \mathbf{K}(A \to B) \to \mathbf{K}A \to \mathbf{K}B}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{f : A \to B \vdash f : A \to B}{f : A \to B \vdash \mathbf{pure}\ f : \mathbf{K}(A \to B)} \qquad x : \mathbf{K}A \vdash x : \mathbf{K}A \qquad \dfrac{g : A \to B \qquad y : A}{g : A \to B, y : A \vdash gy : B}}{f : A \to B, x : \mathbf{K}A \vdash \mathbf{let\ pure}\ \langle g, y \rangle = \langle \mathbf{pure}\ f, x \rangle\ \mathbf{in}\ gy : \mathbf{K}B}}{f : A \to B \vdash \lambda x.\mathbf{let\ pure}\ \langle g, y \rangle = \langle \mathbf{pure}\ f, x \rangle\ \mathbf{in}\ gy : \mathbf{K}A \to \mathbf{K}B}}{\lambda f.\lambda x.\mathbf{let\ pure}\ \langle g, y \rangle = \langle \mathbf{pure}\ f, x \rangle\ \mathbf{in}\ gy : (A \to B) \to \mathbf{K}A \to \mathbf{K}B}$$

Now we define free variables and substitutions. β-reduction, multi-step β-reduction and β-equality are defined standardly:

Definition 6. *Set $FV(M)$ of free variables for arbitrary term M:*

1) $FV(x) = \{x\}$;
2) $FV(\lambda x.M) = FV(M) \setminus \{x\}$;
3) $FV(MN) = FV(M) \cup FV(N)$;
4) $FV(\langle M, N \rangle) = FV(M) \cup FV(N)$;
5) $FV(\pi_i M) \subseteq FV(M)$, $i \in \{1, 2\}$;
6) $FV(\mathit{pure}\ M) = FV(M)$;
7) $FV(\mathbf{let\ pure}\ \vec{N} = \vec{M}\ \mathbf{in}\ M) = \bigcup\limits_{i=1}^{n} FV(M)$, where $n = |\vec{M}|$.

Definition 7. *Substitution:*

1) $x[x := N] = N$, $x[y := N] = x$;
2) $(MN)[x := N] = M[x := N]N[x := N]$;
3) $(\lambda x.M)[x := N] = \lambda x.M[x := N]$;
4) $(M, N)[x := P] = (M[x := P], N[x := P])$;
5) $(\pi_i M)[x := P] = \pi_i(M[x := P])$, $i \in \{1, 2\}$;
6) $(\mathbf{pure}\, M)[x := P] = \mathbf{pure}\,(M[x := P])$;
7) $(\mathbf{let\ pure}\ \vec{x} = \vec{M}\ \mathbf{in}\ M)[y := P] = \mathbf{let\ pure}\ \vec{x} = (\vec{M}[y := P])\ \mathbf{in}\ M$.

Definition 8. *β-reduction and η-reduction rules for $\lambda\mathbf{K}$.*

1) $(\lambda x.M)N \to_\beta M[x := N]$;
2) $\pi_1\langle M, N\rangle \to_\beta M$;
3) $\pi_2\langle M, N\rangle \to_\beta N$;
4) $\begin{array}{l}\mathbf{let\ pure}\ \langle\vec{x}, y, \vec{z}\rangle = \langle\vec{M}, \mathbf{let\ pure}\ \vec{w} = \vec{N}\ \mathbf{in}\ Q, \vec{P}\rangle\ in\ R \to_\beta \\ \mathbf{let\ pure}\ \langle\vec{x}, \vec{w}, \vec{z}\rangle = \langle\vec{M}, \vec{N}, \vec{P}\rangle\ \mathbf{in}\ R[y := Q]\end{array}$
5) $\mathbf{pure}\,((\lambda x.M)N) \to_\beta \mathbf{pure}\,(M[x := N])$;
6) $\mathbf{pure}\,(\pi_i\langle M_1, M_2\rangle) \to_\beta \mathbf{pure}\, M_i$, *where* $i \in \{1, 2\}$.
7) $\begin{array}{l}\mathbf{pure}\,(\mathbf{let\ pure}\ \langle\vec{x}, y, \vec{z}\rangle = \langle\vec{M}, \mathbf{let\ pure}\ \vec{w} = \vec{N}\ \mathbf{in}\ Q, \vec{P}\rangle\ in\ R) \to_\beta \\ \mathbf{pure}\,(\mathbf{let\ pure}\ \langle\vec{x}, \vec{w}, \vec{z}\rangle = \langle\vec{M}, \vec{N}, \vec{P}\rangle\ \mathbf{in}\ R[y := Q])\end{array}$
8) $\lambda x.fx \to_\eta f$;
9) $\langle\pi_1 P, \pi_2 P\rangle \to_\eta P$;
10) $\mathbf{let\ pure}\ \_\_ = \_\_\ \mathbf{in}\ N \to_\eta \mathbf{pure}\, N$;
11) $\mathbf{let\ pure}\ x = M\ \mathbf{in}\ x \to_\eta M$;
12) $\mathbf{pure}\,(\lambda x.fx) \to_\eta \mathbf{pure}\, f$;
13) $\mathbf{pure}\,(\langle\pi_1 P, \pi_2 P\rangle) \to_\eta \mathbf{pure}\, P$;
14) $\mathbf{pure}\,(\mathbf{let\ pure}\ x = M\ \mathbf{in}\ x) \to_\eta \mathbf{pure}\, M$;
15) $\mathbf{pure}\,(\mathbf{let\ pure}\ \_\_ = \_\_\ \mathbf{in}\ N) \to_\eta \mathbf{pure}\,(\mathbf{pure}\, N)$.

Let us show the next simple observation, which immeadelty follows from the previous definition.

Lemma 3.
If $M \to_{\beta\eta} N$, then $\mathbf{pure}\, M \to_{\beta\eta} \mathbf{pure}\, N$.

3 Basic lemmas

Now we will prove standard lemmas for contexts in type systems[3]:

Definition 9. *The domain of a context Γ:*
Let $\Gamma = \{x_1 : A_1, ..., x_n : A_n\}$. Then the domain of Γ, or $dom(\Gamma)$, is a set $\{x_1, ..., x_n\}$.

Lemma 4. *If $\Gamma \vdash M : A$, then $FV(M) \subseteq dom(\Gamma)$*

Proof. Induction on the derivation of $\Gamma \vdash M : A$.

□

[3]We will not prove cases with $\to$-constructor, they are proved standardly in the same lemmas for simply typed lambda calculus, for example see [11][12][14]. We will consider only modal cases

Lemma 5. *Generation for $\lambda\boldsymbol{K}$.*

1) $\Gamma \vdash \textbf{pure}\, M : \boldsymbol{K}\alpha$ *implies that* $\Gamma \vdash M : \alpha$*;*

2) $\Gamma \vdash \textbf{let pure}\, \vec{N} = \vec{M}\ \textbf{in}\ M : \mathbf{K}B$ *implies that* $\Gamma \vdash \vec{M} : \mathbf{K}\vec{A}$ *and* $\vec{N} : \vec{A} \vdash M : B$.

Proof.

Induction on the derivation of $\Gamma \vdash \textbf{pure}\, M : \mathbf{K}\alpha$ and $\Gamma \vdash \textbf{let pure}\, \vec{N} = \vec{M}\ \textbf{in}\ M : \mathbf{K}B$ respectively. □

The next one lemma allows that weakening structural rule is admissable.

Lemma 6. *Weakening for $\lambda\boldsymbol{K}$.*

Let $\Gamma \vdash M : A$ *and* $\Gamma \subseteq \Delta$*, then* $\Delta \vdash M : A$.

Proof.

Induction on derivation of $\Gamma \vdash M : A$. Let us assume $\Gamma \subseteq \Delta$.

1) Let $\Gamma \vdash x : A$, such that $\Gamma = \Delta, x : A$ and $\Theta \subseteq \Gamma$. Let $\Sigma = \Theta \setminus \Gamma$, or, which is the same, $\Sigma = \Theta \setminus \Delta, x : A$, then $\Sigma, \Delta, x : A \vdash x : A$, or, $\Theta \vdash x : A$.

2) Let $\Gamma \vdash \textbf{pure}\, M : \mathbf{K}A$ and $\Gamma \subseteq \Theta$.
By generation $\Gamma \vdash M : A$
By hypothesis, $\Theta \vdash M : A$, so $\Theta \vdash \textbf{pure}\, M : \mathbf{K}A$ by applying $\mathbf{K}_I$-rule.

3) Let $\Gamma \vdash \textbf{let pure}\, \vec{x} = \vec{M}\ \textbf{in}\ N : \mathbf{K}B$ and $\Gamma \subseteq \Theta$.
By generation $\Gamma \vdash \vec{M} : \mathbf{K}\vec{A}$ and $\vec{x} : \vec{A} \vdash N : B$.
By assumption $\Theta \vdash \vec{M} : \mathbf{K}\vec{A}$.
Hence $\Theta \vdash \textbf{let pure}\, \vec{x} = \vec{M}\ \textbf{in}\ N : \mathbf{K}B$.

□

Lemma 7. *Considering for $\lambda\boldsymbol{K}$.*

If $\Gamma \vdash M : \alpha$*, then* $\Gamma \uparrow FV(M) \vdash M : \alpha$*, where* $\Gamma \uparrow FV(M)$ *is a subcontext of* Γ*, such that* $dom(\Gamma \uparrow FV(M)) = dom(\Gamma) \cap FV(M)$.

Proof.

1) Let $\Gamma \vdash x : A$, where $\Gamma = \Delta, x : A$, $x \in \mathbb{V}$.

$FV(x) = \{x\}$, then $dom(\Gamma) \cap \{x\} = \{x\}$. So $(\Delta, x : A) \uparrow FV(x) = \{x : A\}$, then $x : A \vdash x : A$ by axiom.

2) Let $\Gamma \vdash \textbf{pure}\, M : \mathbf{K}A$.
By generation, $\Gamma \vdash M : A$ and $\Gamma \uparrow FV(M) \vdash M : A$ by hypothesis.
So $\Gamma \uparrow FV(M) \vdash \textbf{pure}\, M : \mathbf{K}A$ by $\mathbf{K}_I$.

3) Let $\Gamma \vdash \textbf{let pure}\, \vec{x} = \vec{M}\ \textbf{in}\ N : \mathbf{K}B$.
By generation, $\Gamma \vdash \vec{M} : \mathbf{K}\vec{A}$ and $\vec{x} : \vec{A} \vdash N : B$.
By assumption, $\Gamma \uparrow FV(\vec{M}) \vdash \vec{M} : \vec{A}$.
By $\text{let}_{\mathbf{K}}$, $\Gamma \uparrow FV(\vec{M}) \vdash \textbf{let pure}\, \vec{x} = \vec{M}\ \textbf{in}\ N : \mathbf{K}B$

□

Lemma 8. *If* $\Gamma, x : A \vdash M : B$ *and* $\Gamma \vdash N : A$*, then* $\Gamma \vdash (M[x := N]) : B$

Proof.

1) Let $\Gamma, x : A \vdash \textbf{pure}\, M : \mathbf{K}B$ and $\Gamma \vdash N : A$.

By generation, $\Gamma, x : A \vdash M : B$.

By assumption, $\Gamma \vdash (M[x := N]) : B$

Then $\Gamma \vdash \textbf{pure}\,(M[x := N]) : \mathbf{K}B$ by $\mathbf{K}_I$.

But $\textbf{pure}\,(M[x := N]) = (\textbf{pure}\, M[x := N])$ by substitution definition, so $\Gamma \vdash (\textbf{pure}\, M[x := N]) : \mathbf{K}B$

2) Let $\Gamma, y : A \vdash \textbf{let pure}\, \vec{x} = \vec{M}\, \textbf{in}\, N : \mathbf{K}B$ and $\Gamma \vdash N : A$.

By generation, $\Gamma, y : A \vdash \vec{M} : \mathbf{K}\vec{A}$ and $\vec{x} : \vec{A} \vdash N : B$.

By hypothesis, $\Gamma \vdash \vec{M}[x := N] : \mathbf{K}\vec{A}$.

Hence $\Gamma \vdash \textbf{let pure}\, \vec{x} = \vec{M}[x := N]\, \textbf{in}\, N : \mathbf{K}B$.

$\square$

Theorem 1. *Subject reduction*

i) Let $\Gamma \vdash M : A$ and $M \twoheadrightarrow_\beta N$, then $\Gamma \vdash N : A$
ii) Let $\Gamma \vdash M : A$ and $M \twoheadrightarrow_\eta N$, then $\Gamma \vdash N : A$

We consider only modal β-reduction rules. The general statement for $\twoheadrightarrow_\beta$ follows from transitivity of multi-step β-reduction.

Proof.

i) For multistep β-reduction:

1) Let $\Gamma \vdash \textbf{let pure}\, \langle \vec{x}, y, \vec{z} \rangle = \langle \vec{M}, \textbf{let pure}\, \vec{w} = \vec{N}\, \textbf{in}\, Q, \vec{P} \rangle\, \textbf{in}\, R : \mathbf{K}B$

By generation we have $\Gamma \vdash \vec{M} : \mathbf{K}\vec{A_1}$, $\Gamma \vdash \textbf{let pure}\, \vec{w} = \vec{N}\, \textbf{in}\, Q : \mathbf{K}\vec{A_2}$, $\Gamma \vdash \vec{P} : \mathbf{K}\vec{A_3}$ and $\vec{x} : \vec{A_1}, y : A_2, \vec{z} : \vec{A_3} \vdash R : B$.

If $\Gamma \vdash \textbf{let pure}\, \vec{w} = \vec{N}\, \textbf{in}\, Q : \mathbf{K}\vec{A_2}$, then

$\Gamma \vdash \vec{N} : \mathbf{K}\vec{A_4}$ and $\vec{w} : \vec{A_4} \vdash Q : A_2$.

Then $\vec{x} : \vec{A_1}, \vec{w} : \vec{A_4}, \vec{z} : \vec{A_3} \vdash R[y := Q] : B$ by substitution lemma and weakening.

Hence $\Gamma \vdash \textbf{let pure}\, \langle \vec{x}, \vec{w}, \vec{z} \rangle = \langle \vec{M}, \vec{N}, \vec{P} \rangle\, \textbf{in}\, R[y := Q] : \mathbf{K}B$ by $let_{\mathbf{K}}$.

2) Let $\Gamma \vdash \textbf{pure}((\lambda x.M)N) : \mathbf{K}B$.

By generation $\Gamma \vdash (\lambda x.M)N : B$, but $\Gamma \vdash M[x := N] : B$, then, by $\mathbf{K}_I$, $\Gamma \vdash \textbf{pure}\,(M[x := N]) : \mathbf{K}B$.

3) Let $\Gamma \vdash \textbf{pure}(\pi_i \langle M_1, M_2 \rangle) : \mathbf{K}A_i$, where $i \in \{1, 2\}$.

By generation $\Gamma \vdash \pi_i \langle M_1, M_2 \rangle : A_i$ and $\Gamma \vdash M_i : A_i$.

Hence $\Gamma \vdash \textbf{pure}\, M_i : \mathbf{K}A_i$ by $\mathbf{K}_I$.

4) Let $\Gamma \vdash \textbf{let pure}\, (\langle \vec{x}, y, \vec{z} \rangle = \langle \vec{M}, \textbf{let pure}\, \vec{w} = \vec{N}\, \textbf{in}\, Q, \vec{P} \rangle\, \textbf{in}\, R) : \mathbf{K}^2B$.

By generation $\Gamma \vdash \textbf{let pure}\, \langle \vec{x}, y, \vec{z} \rangle = \langle \vec{M}, \textbf{let pure}\, \vec{w} = \vec{N}\, \textbf{in}\, Q, \vec{P} \rangle\, \textbf{in}\, R : \mathbf{K}B$,

hence $\Gamma \vdash \textbf{let pure}\, \langle \vec{x}, \vec{w}, \vec{z} \rangle = \langle \vec{M}, \vec{N}, \vec{P} \rangle\, \textbf{in}\, R[y := Q] : \mathbf{K}B$ by the first case.

So $\Gamma \vdash \textbf{pure}\,(\textbf{let pure}\, \langle \vec{x}, \vec{w}, \vec{z} \rangle = \langle \vec{M}, \vec{N}, \vec{P} \rangle\, \textbf{in}\, R[y := Q]) : \mathbf{K}^2B$ by $\mathbf{K}_I$.

ii) For multistep η-reduction:

1) Let $\vdash \textbf{let pure}\,\_\_ = \_\_\,\textbf{in}\, N : \mathbf{K}A$.
Then by generation $\vdash N : A$, so $\vdash \textbf{pure}\, N : \mathbf{K}A$ by $\mathbf{K}_I$.

2) Let $\Gamma \vdash \textbf{let pure}\, x = M\, \textbf{in}\, x : \mathbf{K}A$.
By generation $\Gamma \vdash M : \mathbf{K}A$ and $x : A \vdash x : A$, hence $\Gamma \vdash M : \mathbf{K}A$.

3) Let $\Gamma \vdash \textbf{pure}\,(\lambda x.fx) : \mathbf{K}(A \to B)$.
By generation $\Gamma \vdash \lambda x.fx : A \to B$, so $\Gamma \vdash f : A \to B$, then $\Gamma \vdash \textbf{pure}\, f : \mathbf{K}(A \to B)$ by $\mathbf{K}_I$.

4) Let $\Gamma \vdash \textbf{pure}\,(\langle \pi_1 P, \pi_2 P\rangle) : \mathbf{K}(A \times B)$.
By generation $\Gamma \vdash \langle \pi_1 P, \pi_2 P\rangle : A \times B$, then $\Gamma \vdash P : A \times B$.
By $\mathbf{K}_I$, $\Gamma \vdash \textbf{pure}\, P : \mathbf{K}(A \times B)$.

5) $\Gamma \vdash \textbf{pure}\,(\textbf{let pure}\, x = M\, \textbf{in}\, x) : \mathbf{K}^2 A$.
Then $\Gamma \vdash M : \mathbf{K}^2 A$ and $x : \mathbf{K}A \vdash x : \mathbf{K}A$, so $\Gamma \vdash M : \mathbf{K}^2 A$.

6) Let $\vdash \textbf{pure}\,(\textbf{let pure}\,\_\_ = \_\_\,\textbf{in}\, N) : \mathbf{K}^2 A$.
By generation $\textbf{let pure}\,\_\_ = \_\_\,\textbf{in}\, N : \mathbf{K}A$, so $\vdash N : A$, then $\vdash \textbf{pure}\, N : \mathbf{K}$.
$\square$

4 Strong normalization

We modify and apply Tait's technique of logical relation for modalities. Strong normalization proof with Tait's method for simply typed lambda calculus is described here [13].

Strong normalization for **IK** is proved in [21] [19]. So we consider simply typed lambda calculus with $\mathbf{K}_I$ rule and show that $\lambda_{\to,\times} + \mathbf{K}_I$ is strongly normalizable.

Theorem 2. *Let $M \in \Lambda_{\mathbf{K}}$, then any sequence of reduction $M \to_\beta M_1 \ldots$ terminates.*

Proof.
We build the subset of strongly normalazing terms and show that an arbitrary term belongs to this subset.

Definition 10. *The set of strongly computable terms for every type $T \in \mathbb{T}_{\mathbf{K}}$.*

- *Let $A \in \mathbb{T}$, then $SC_A = \{M : A \mid M \text{ is strongly normalizing}\}$;*
- *$SC_{A \to B} = \{M : A \to B \mid \forall A \in SC_A, MN \in SC_B\}$;*
- *$SC_{A_1 \times A_2} = \{M : A \times B \mid \pi_i M \in SC_{A_i}, i \in \{1, 2\}\}$;*
- *$SC_{\mathbf{K}A} = \{\textbf{pure}\, M : \mathbf{K}A \mid M \in SC_A\}$*

Strong normalization proof reduces to the proof of the next lemma:

Lemma 9.

i) If $M \in SC_A$, then M is stronly normalizing;

ii) If $M \to_\beta M^{'}$ and $M \in SC_A$, then $M^{'}$;

iii) Let $M \to_\beta M^{'}$ and $M^{'} \in SC_A$, then, if M is a neutral term, then $M \in SC_A$.

iv) Let $x_1 : A_1, \dots, x_n : A_n \vdash M : B$ and $\forall i \in \{1, \dots, n\}, N_i \in SC_{A_i}$, then $M[\vec{x} := \vec{N}] \in SC_B$.

Proof.

i)

The base case follows from the definition.

Let us consider case with $SC_{\mathbf{K}A}$. If **pure** $M \in SC_{\mathbf{K}A}$, then $M \in SC_A$ and M is strongly normalizible. So **pure** M is strongly normalizible, otherwise there would be an infinite reduction path in **pure** M.

ii)

The base case is trivial.

Let **pure** $M \to_\beta$ **pure** $M^{'}$ and **pure** $M \in SN_{\mathbf{K}A}$. By assumption, $M \in SN_A$ and $M \to_\beta M^{'}$, so $M^{'} \in SN_A$. Hence **pure** $M^{'} \in SC_{\mathbf{K}A}$ by the first statement of the lemma.

iii)

The base case is trivial.

Let **pure** $M \to_\beta$ **pure** $M^{'}$ and **pure** $M^{'} \in SN_{\mathbf{K}A}$.

pure $M^{'}$ is a neutral by the definition. By assumption M is a strongly normalizing. So **pure** M is a strongly normalizing by the first part of the current lemma.

iv)

Let $x_1 : A_1, \dots, x_n : A_n \vdash$ **pure** $M : \mathbf{K}A$ and $\forall i \in \{1, \dots, n\}, N_i \in SC_{A_i}$.

By generation $x_1 : A_1, \dots, x_n : A_n \vdash M : A$ and by assumption $M[\vec{x} := \vec{N}] \in SC_B$.

Hence, by the first part of lemma, **pure** $(M[\vec{x} := \vec{N}]) \in SC_{\mathbf{K}B}$. □

Corollary 1.

Let $\vdash N : A$, then N is strongly normalizing.

Proof.

If $\vdash N : A$, then $N \in SC_A$, hence N is strongly normalizing. □

□

5 Confluence

6 Categorical semantics

Definition 11. *Lax monoidal functor*

Let $\langle \mathcal{C}, \oplus_1, \mathbb{1} \rangle$ and $\langle \mathcal{D}, \oplus_2, \mathbb{1}' \rangle$ are monoidal categories.

A lax monoidal functor $\mathcal{F} : \langle \mathcal{C}, \oplus_1, \mathbb{1} \rangle \to \langle \mathcal{D}, \oplus_2, \mathbb{1}' \rangle$ is a functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ with additional natural transformations:

1) $u : \mathbb{1}' \to \mathcal{F}\mathbb{1}$;

2) $*_{A,B} \mathcal{F}A \otimes_2 \mathcal{F}B \to \mathcal{F}(A \otimes_1 B)$

Definition 12. *Applicative functor*

An applicative functor is a triple $\langle \mathcal{C}, \mathcal{K}, \eta \rangle$, where $\mathcal{C}$ is a symmetric monoidal category, $\mathcal{K}$ is a lax monoidal endofunctor and η is a natural transformation, such that:

1) $u = \eta_{\mathbb{1}}$;

2) $*_{A,B} \circ (\eta_A \otimes \eta_B) = \eta_{A \otimes B}$;

3) Weak commutativity condition holds:

$$
\begin{array}{ccc}
A \otimes \mathcal{K}B & \mathcal{K}A \otimes \mathcal{K}B & \mathcal{K}(A \otimes B) \\
 & & \\
 & & \\
\mathcal{K}B \otimes A & \mathcal{K}B \otimes \mathcal{K}A & \mathcal{K}(B \otimes A)
\end{array}
$$

By default we will consider an arbitrary closed functor on some cartersian closed category, which is the special case of an applicative functor.

6.1 Soundness

Definition 13. *Semantical translation from $\lambda_{\boldsymbol{K}}$ to CCC with applicative functor $\mathcal{K}$:*

1) Interpretation for types:

$[\![A]\!] := \hat{A}, A \in \mathbb{T}$;

$[\![A \to B]\!] := [\![A]\!] \to [\![B]\!]$;

$[\![A \times B]\!] := [\![A]\!] \times [\![B]\!]$.

2) Interpretation for modal types: $[\![\boldsymbol{K}A]\!] = \mathcal{K}[\![A]\!]$;

3) Interpretaion for contexts:

$[\![\Gamma = \{x_1 : A_1, ..., x_n : A_n\}]\!] := [\![\Gamma]\!] = [\![A_1]\!] \times ... \times [\![A_n]\!]$;

4) Interpretation for typing assignment: $[\![\Gamma \vdash M : A]\!] := [\![M]\!] : [\![\Gamma]\!] \to [\![A]\!]$.

5) Interpretation for typing rules:

$$\frac{}{[\![\Gamma, x : A \vdash x : A]\!] = \pi_2 : [\![\Gamma]\!] \times [\![A]\!] \to [\![A]\!]}$$

$$\frac{[\![\Gamma, x : A \vdash M : B]\!] = f : [\![\Gamma]\!] \times [\![A]\!] \to [\![B]\!]}{[\![\Gamma \vdash (\lambda x.M) : A \to B]\!] = \Lambda(f) : [\![\Gamma]\!] \to [\![B]\!]^{[\![A]\!]}}$$

$$\frac{[\![\Gamma \vdash M : A \to B]\!] = [\![M]\!] : [\![\Gamma]\!] \to [\![B]\!]^{[\![A]\!]} \qquad [\![\Gamma \vdash N : A]\!] = [\![N]\!] : [\![\Gamma]\!] \to [\![A]\!]}{[\![\Gamma \vdash (MN) : B]\!] = [\![\Gamma]\!] \xrightarrow{\langle [\![M]\!], [\![N]\!] \rangle} [\![B]\!]^{[\![A]\!]} \times [\![A]\!] \xrightarrow{\epsilon} [\![B]\!]}$$

$$\frac{[\![\Gamma \vdash M : A]\!] = f : [\![\Gamma]\!] \to [\![A]\!] \qquad [\![\Gamma \vdash N : B]\!] = g : [\![\Gamma]\!] \to [\![B]\!]}{[\![\Gamma \vdash (M, N) : A \times B]\!] = \langle f, g \rangle : [\![\Gamma]\!] \to [\![A]\!] \times [\![B]\!]}$$

$$\frac{[\![\Gamma \vdash p : A_1 \times A_2]\!] = f : [\![\Gamma]\!] \to [\![A_1]\!] \times [\![A_2]\!]}{[\![\Gamma \vdash \pi_i p : A_i]\!] = [\![\Gamma]\!] \xrightarrow{f} [\![A_1]\!] \times [\![A_2]\!] \xrightarrow{\pi_i} [\![A_i]\!]} \; i \in \{1, 2\}$$

$$\frac{[\![\Gamma \vdash M : A]\!] = [\![M]\!] : [\![\Gamma]\!] \to [\![A]\!]}{[\![\Gamma \vdash \mathbf{pure}\, M : \boldsymbol{K}A]\!] := [\![\Gamma]\!] \xrightarrow{[\![M]\!]} [\![A]\!] \xrightarrow{\eta_{[\![A]\!]}} \mathcal{K}[\![A]\!]}$$

$$\frac{[\![\Gamma \vdash \vec{M} : \mathbf{K}\vec{A}]\!] = \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle : [\![\Gamma]\!] \to \prod_{i=1}^{n} \mathcal{K}[\![A_i]\!] \qquad [\![\vec{x} : \vec{A} \vdash N : B]\!] = [\![N]\!] : \prod_{i=1}^{n} [\![A_i]\!] \to [\![B]\!]}{[\![\Gamma \vdash \mathbf{let\ pure}\, \vec{x} = \vec{M}\ \mathbf{in}\ M : \mathbf{K}B]\!] = \mathcal{K}([\![N]\!]) \circ *_{[\![A_1]\!],\dots,[\![A_n]\!]} \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle : [\![\Gamma]\!] \to \mathcal{K}[\![B]\!]}$$

Definition 14. *Simultaneous substitution*

Let $\Gamma = \{x_1 : A_1, ..., x_n : A_n\}$, $\Gamma \vdash M : A$ *and for all* $i \in \{1, ..., n\}$, $\Gamma \vdash M_i : A_i$.

We define simultaneous substitution $M[\vec{x} := \vec{M}]$ *recursively by:*

1) $x_i[\vec{x} := \vec{M}] = M_i$;

2) $(\lambda x.M)[\vec{x} := \vec{M}] = \lambda x.(M[\vec{x} := \vec{M}])$;

3) $(MN)[\vec{x} := \vec{M}] = (M[\vec{x} = \vec{M}])(N[\vec{x} := \vec{M}])$;

4) $\langle M, N \rangle = \langle (M[\vec{x} = \vec{M}]), (N[\vec{x} := \vec{M}]) \rangle$;

5) $(\pi_i P)[\vec{x} := \vec{M}] = \pi_i(P[\vec{x} = \vec{M}])$;

6) $(\mathbf{pure}\, M)[\vec{x} := \vec{M}] = \mathbf{pure}\, (M[\vec{x} = \vec{M}])$;

7) $(\mathbf{let\ pure}\, \vec{x} = \vec{M}\ \mathbf{in}\ N)[\vec{y} := \vec{P}] = \mathbf{let\ pure}\, \vec{x} = (\vec{M}[\vec{y} := \vec{P}])\ \mathbf{in}\ N$

Lemma 10.

$[\![M[x_1 := M_1, \dots, x_n := M_n]]\!] = [\![M]\!] \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle$.

Proof.

1) $[\![\Gamma \vdash (\mathbf{pure}\, M)[\vec{x} := \vec{M}] : \mathbf{K}A]\!] = [\![\Gamma \vdash \mathbf{pure}\, M : \mathbf{K}A]\!] \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle$.

$[\![\Gamma \vdash (\mathbf{pure}\, M)[\vec{x} := \vec{M}] : \mathbf{K}A]\!] = [\![\Gamma \vdash \mathbf{pure}\, (M[\vec{x} := \vec{M}]) : \mathbf{K}A]\!]$	Substitution definition
$= \eta_{[\![A]\!]} \circ [\![(M[\vec{x} := \vec{M}])]\!]$	Translation for pure
$= \eta_{[\![A]\!]} \circ ([\![M]\!] \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle)$	Induction hypothesis
$= (\eta_{[\![A]\!]} \circ [\![M]\!]) \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle$	Associativity of composition
$= [\![\Gamma \vdash \mathbf{pure}\, M : \mathbf{K}A]\!] \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle$	Translation for pure

2) $[\![\Gamma \vdash (\mathbf{let\ pure}\, \vec{x} = \vec{M}\ \mathbf{in}\ N)[\vec{y} := \vec{P}] : \mathbf{K}B]\!] = [\![\Gamma \vdash \mathbf{let\ pure}\, \vec{x} = \vec{M}\ \mathbf{in}\ N : \mathbf{K}B]\!] \circ \langle [\![P_1]\!], \dots, [\![P_n]\!] \rangle$

$[\![\Gamma \vdash (\mathbf{let\ pure}\, \vec{x} = \vec{M}\ \mathbf{in}\ N)[\vec{y} := \vec{P}] : \mathbf{K}B]\!] =$
Substitution definition
$[\![\Gamma \vdash \mathbf{let\ pure}\, \vec{x} = (\vec{M}[\vec{y} := \vec{P}])\ \mathbf{in}\ N : \mathbf{K}B]\!] =$
Interpretaion for $let_{\mathbf{K}}$
$\mathcal{K}([\![N]\!]) \circ *_{[\![A_1]\!],\dots,[\![A_n]\!]} \circ [\![\Gamma \vdash (\vec{M}[\vec{y} := \vec{P}]) \vdash: \mathbf{K}\vec{A}]\!] =$
Induction hypothesis
$\mathcal{K}([\![N]\!]) \circ *_{[\![A_1]\!],\dots,[\![A_n]\!]} \circ ([\![\vec{M}]\!] \circ \langle [\![P_1]\!], \dots, [\![P_n]\!] \rangle) =$
Associativity of composition
$(\mathcal{K}([\![N]\!]) \circ *_{[\![A_1]\!],\dots,[\![A_n]\!]} \circ [\![\vec{M}]\!]) \circ \langle [\![P_1]\!], \dots, [\![P_n]\!] \rangle =$
By interpretation
$[\![\Gamma \vdash (\mathbf{let\ pure}\, \vec{x} = \vec{M}\ \mathbf{in}\ N]\!] \circ \langle [\![P_1]\!], \dots, [\![P_n]\!] \rangle$

□

Lemma 11.

i) Let $\Gamma \vdash M : A$ *and* $M \twoheadrightarrow_\beta N$, *then* $[\![\Gamma \vdash M : A]\!] = [\![\Gamma \vdash N : A]\!]$;

ii) Let $\Gamma \vdash M : A$ *and* $M \twoheadrightarrow_\eta N$, *then* $[\![\Gamma \vdash M : A]\!] = [\![\Gamma \vdash N : A]\!]$;

Proof. i) For β-reduction

Cases with β-reductions for $let_{\mathbf{K}}$ are shown in [20]. Let us consider cases with **pure**.

1) $[\![\Gamma \vdash \mathbf{pure}\,((\lambda x.M)N) : \mathbf{K}B]\!] = [\![\Gamma \vdash \mathbf{pure}\,(M[x := N]]) : \mathbf{K}B]\!]$

$$
\begin{array}{ll}
[\![\Gamma \vdash \mathbf{pure}\,(\lambda x.M)N : \mathbf{K}B]\!] = & \text{By interpretation} \\
\eta_{[\![B]\!]} \circ (\epsilon \circ \langle \Lambda([\![M]\!]), [\![N]\!]\rangle) = & \text{Property of } \times \\
\eta_{[\![B]\!]} \circ (\epsilon \circ (\Lambda([\![M]\!]) \times id_{[\![A]\!]}) \circ \langle id_{[\![\Gamma]\!]}, [\![N]\!]\rangle) = & \text{Associativity of composition} \\
\eta_{[\![B]\!]} \circ ((\epsilon \circ (\Lambda([\![M]\!]) \times id_{[\![A]\!]})) \circ \langle id_{[\![\Gamma]\!]}, [\![N]\!]\rangle) = & \text{Exponentiation property} \\
\eta_{[\![B]\!]} \circ ([\![M]\!] \circ \langle id_{[\![\Gamma]\!]}, [\![N]\!]\rangle) = & \text{Substitution lemma} \\
\eta_{[\![B]\!]} \circ [\![M[\vec{x}, x := \vec{x}, N]]\!] = & \text{By interpretation} \\
[\![\Gamma \vdash \mathbf{pure}\,(M[x := N]]) : \mathbf{K}B]\!] &
\end{array}
$$

2) $[\![\Gamma \vdash \mathbf{pure}\,(\pi_i\langle [\![M_1]\!], [\![M_2]\!]\rangle) : \mathbf{K}A_i]\!] = [\![\Gamma \vdash\ \mathbf{pure}M_i : \mathbf{K}A_i]\!]$

$$
\begin{array}{ll}
[\![\Gamma \vdash \mathbf{pure}\,(\pi_i\langle M_1, M_2]\rangle) : \mathbf{K}A_i]\!] = & \text{By interpretation} \\
\eta_{[\![A_i]\!]} \circ \pi_i \circ \langle [\![M_1]\!], [\![M_2]\!]\rangle = & \text{Property of } \times \\
\eta_{[\![A_i]\!]} \circ [\![M_i]\!] = & \text{By interpetation} \\
[\![\Gamma \vdash \mathbf{pure}\,M_i : \mathbf{K}A_i]\!] &
\end{array}
$$

□

Theorem 3. *Soundness*
Let $\Gamma \vdash M : A$ *and* $M =_{\beta\eta} N$*, then* $[\![\Gamma \vdash M : A]\!] = [\![\Gamma \vdash N : A]\!]$

Proof. Straightforwardly follows from two previous lemmas. □

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