Modal type theory based on the intuitionistic epistemic logic

Abstract

Modal intuitionistic epistemic logic IEL^- was proposed by S.Artemov and T. Protopopescu as the formal foundation for the intuitionistic theory of knowledge. We construct a modal simply typed lambda-calculus which is Curry-Howard isomorphic to IEL^- as formal theory of calculations with applicative functors in functional programming languages like Haskell or Idris.

1 Introduction

Modal intutionistic epistemic logic IEL was proposed by S. Artemov and T. Proropopescu [1]. IEL provides the epistimology and the theory of knowledge as based on BHK-semantics of intuitionistic logic. IEL^- is a variant of IEL, that corresponds to intuitionistic belief. Informally, $\mathbf{K}A$ denotes that A is verified intuitionistically.

Intuitionistic epistemic logic IEL^- is defined with by following axioms and derivation rules:

Definition 1. *Intuitionistic epistemic logic IEL:*
1) IPC axioms;
2) $\mathbf{K}(A \to B) \to (\mathbf{K}A \to \mathbf{K}B)$ *(normality);*
3) $A \to \mathbf{K}A$ *(co-reflection);*
Rule: MP.

We have the deduction theorem and necessitation rule which is derivable.

V. Krupski and A. Yatmanov provided the sequential calculus for IEL and proved that this calculus is PSPACE-complete [2].

Functional programming languages such as Haskell [3], Idris [4], Purescript [5] Elm [6] or Scala [**?**] have special type classes[1] for calculations with container types like `Functor` and `Applicative` [2]:

```
class Functor f where
  fmap :: (a -> b) -> f a -> f b

class Functor f => Applicative f where
  pure :: a -> f a
  (<*>) :: f (a -> b) -> f a -> f b
```

[1]Type class in Haskell is a general interface for special group of datatypes.

[2]Reader may read more about container types in the Haskell standard library documentation[7] or in the next one textbook [8]

By *container* (or *computational context*) type we mean some type-operator f, where f is a "function" from $*$ to $*$: type operator takes a simple type (which has kind $*$) and returns another simple type type with kind $*$. For more detailed description of the type system with kinds used in Haskell see [12].

The motivation for using an applicative functor is quite natural. Applicative functor allows to generalize the action of a functor for functions with arbitrary number of arguments, for instance:

```
liftA2 :: Applicative f => (a -> b -> c) -> f a -> f b -> f c
liftA2 f x y = pure f <*> x <*> y
```

It's not difficult to see that modal axioms in IEL^- and types of the methods of Applicative class in Haskell-like languages (which is described below) are syntactically similar and we are going to show that this coincidence has a non-trivial computational meaning.

The main goal of our research is a relationship between intuitionistic epistemic logic IEL^- and functional programming with effects. We show that relationship by building the type system (which is called $\lambda_{\mathbf{K}}$) which is Curry-Howard isomorphic to IEL^-. So we will consider $\mathbf{K}$-modality as an arbitrary applicative functor.

$\lambda_{\mathbf{K}}$ consists of the rules for simply typed lambda-calculus and special typing rules for lifting types into the applicative functor $\mathbf{K}$. We assume that our type system will axiomatize the simplest case of computation with effects with one container. We provide proof-theoretical view on this kind of computations in functional programming and prove strong normalization and confluence.

2 Typed lambda-calculus based on IEL$^-$

At first we define the natural deduction for IEL$^-$:

Definition 2. *Natural deduction NIEL for IEL$^-$ is an extenstion of intuitionistic natural deduction with additional derivation rules for modality:*

$$\frac{\Gamma \vdash A}{\Gamma \vdash \mathbf{K}A}\,\mathbf{K}_I \qquad\qquad \frac{\Gamma \vdash \mathbf{K}\vec{A} \qquad \vec{A} \vdash B}{\Gamma \vdash \mathbf{K}B}$$

Where $\Gamma \vdash \mathbf{K}\vec{A}$ is a syntax sugar for $\Gamma \vdash \mathbf{K}A_1, \ldots, \Gamma \vdash \mathbf{K}A_n$.

Lemma 1. $\Gamma \vdash_{NIEL^-_{\wedge,\to}} A \Rightarrow IEL^- \vdash \bigwedge \Gamma \to A$.

Proof. Induction on the derivation.

Let us consider cases with modality.

1) If $\Gamma \vdash_{NIEL^-_{\wedge,\to}} A$, then $IEL^- \vdash \bigwedge \Gamma \to \mathbf{K}A$.

(1)	$\bigwedge \Gamma \to A$	assumption
(2)	$A \to \mathbf{K}A$	co-reflection
(3)	$(\bigwedge \Gamma \to A) \to ((A \to \mathbf{K}A) \to (\bigwedge \Gamma \to \mathbf{K}A))$	IPC theorem
(4)	$(A \to \mathbf{K}A) \to (\bigwedge \Gamma \to \mathbf{K}A)$	from (1), (3) and MP
(5)	$\bigwedge \Gamma \to \mathbf{K}A$	from (2), (4) and MP

2) If $\Gamma \vdash_{NIEL^{-}_{\wedge,\rightarrow}} \mathbf{K}\vec{A}$ and $\vec{A} \vdash B$, then $IEL^{-} \vdash \bigwedge \Gamma \rightarrow \mathbf{K}B$.

(1)	$\bigwedge \Gamma \rightarrow \bigwedge_{i=1}^{n} \mathbf{K}A_i$	assumption
(2)	$\bigwedge_{i=1}^{n} \mathbf{K}A_i \rightarrow \mathbf{K} \bigwedge_{i=1}^{n} A_i$	IEL theorem
(3)	$\bigwedge \Gamma \rightarrow \mathbf{K} \bigwedge_{i=1}^{n} A_i$	from (1), (2) and transitivity
(4)	$\bigwedge_{i=1}^{n} A_i \rightarrow B$	assumption
(5)	$(\bigwedge_{i=1}^{n} A_i \rightarrow B) \rightarrow \mathbf{K}(\bigwedge_{i=1}^{n} A_i \rightarrow B)$	co-reflection
(6)	$\mathbf{K}(\bigwedge_{i=1}^{n} A_i \rightarrow B)$	from (4), (5) and MP
(7)	$\mathbf{K} \bigwedge_{i=1}^{n} A_i \rightarrow \mathbf{K}B$	from (6) and normality
(8)	$\bigwedge \Gamma \rightarrow \mathbf{K}B$	from (3), (7) and transitivity

□

Lemma 2. *If $IEL^{-} \vdash A$, then $NIEL^{-} \vdash A$.*

Proof. Straightforward derivation of modal axioms in NIEL^{-}. We consider this derivation below using terms. □

At the next step we build the typed lambda-calculus based on NIEL$^{-}_{\wedge,\rightarrow}$ by proof-assingment in rules.

At first, we define lambda-terms and types for this lambda-calculus.

Definition 3. *The set of terms:*

Let $\mathbb{V}$ be the set of variables. The set $\Lambda_{\mathbf{K}}$ of terms is defined by the grammar:

$$\Lambda_{\mathbf{K}} ::= \mathbb{V} \mid (\lambda\Lambda.\Lambda_{\mathbf{K}}) \mid (\Lambda_{\mathbf{K}}\Lambda_{\mathbf{K}}) \mid (\Lambda_{\mathbf{K}}, \Lambda_{\mathbf{K}}) \mid (\pi_1\Lambda_{\mathbf{K}}) \mid (\pi_2\Lambda_{\mathbf{K}}) \mid (\mathbf{pure}\ \Lambda_{\mathbf{K}}) \mid (\mathbf{let\ pure}\ \Lambda_{\mathbf{K}} = \Lambda_{\mathbf{K}}\ \mathbf{in}\ \Lambda_{\mathbf{K}})$$

Definition 4. *The set of types:*

Let $\mathbb{T}$ be the set of atomic types. The set $\mathbb{T}_{\mathbf{K}}$ of types with applicative functor $\mathbf{K}$ is generated by the grammar:

$$\mathbb{T}_{\mathbf{K}} ::= \mathbb{T} \mid (\mathbb{T}_{\mathbf{K}} \rightarrow \mathbb{T}_{\mathbf{K}}) \mid (\mathbb{T}_{\mathbf{K}} \times \mathbb{T}_{\mathbf{K}}) \mid (\mathbf{K}\mathbb{T}_{\mathbf{K}}) \tag{1}$$

Context, domain of context and range of context are defined standardly [11][12].

Our type system is based on the Curry-style typing rules:

Definition 5. *Modal typed lambda calculus $\lambda_{\mathbf{K}}$ based on $NIEL^{-}_{\wedge,\rightarrow}$:*

$$\frac{}{\Gamma, x : A \vdash x : A}\ ax$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B} \to_i \qquad \frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash MN : B} \to_e$$

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash N : B}{\Gamma \vdash \langle M, N \rangle : A \times B} \times_i \qquad \frac{\Gamma \vdash M : A_1 \times A_2}{\Gamma \vdash \pi_i M : A_i} \times_e,\ i \in \{1, 2\}$$

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash \mathbf{pure}\ M : \mathbf{K}A} \mathbf{K}_I \qquad \frac{\Gamma \vdash \vec{M} : \mathbf{K}\vec{A} \qquad \vec{x} : \vec{A} \vdash N : B}{\Gamma \vdash \mathbf{let\ pure}\ \vec{x} = \vec{M}\ \mathbf{in}\ N : \mathbf{K}B} let_{\mathbf{K}}$$

$\mathbf{K}_I$-typing rule is the same as $\bigcirc$-introduction in lax logic (also known as monadic metalanguage [17]) and in typed lambda-calculus which is derived by proof-assignment for lax-logic proofs. $\mathbf{K}_I$ allows to inject an object of type α into the functor. $\mathbf{K}_I$ reflects the Haskell method **pure** for Applicative class. It plays the same role as the **return** method in Monad class.

let$_{\mathbf{K}}$ is similar to the $\Box$-rule in typed lambda calculus for intuitionistic normal modal logic **IK**, which is described in [19].

In fact, our calculus is the extention of typed lambda calculus for **IK** with typing rule appropriate to co-reflection.

Here are some examples of closed terms:

- $(\lambda x.\mathbf{pure}\, x) : A \to \mathbf{K}A$;
- $\lambda f.\lambda x.\mathbf{let\ pure}\ g, y = f, x\ \mathbf{in}\ gy : \mathbf{K}(A \to B) \to \mathbf{K}A \to \mathbf{K}B$
- $\lambda f.\lambda x.\mathbf{let\ pure}\ g, y = \mathbf{pure}\ f, x\ \mathbf{in}\ gy : (A \to B) \to \mathbf{K}A \to \mathbf{K}B$

Now we define free variables and substitutions. β-reduction, multi-step β-reduction and β-equality are defined standardly:

Definition 6. *Set $FV(M)$ of free variables for arbitrary term M:*
1) $FV(x) = \{x\}$;
2) $FV(\lambda x.M) = FV(M) \backslash \{x\}$;
3) $FV(MN) = FV(M) \cup FV(N)$;
4) $FV(\langle M, N \rangle) = FV(M) \cup FV(N)$;
5) $FV(\pi_i M) \subseteq FV(M)$, $i \in \{1, 2\}$;
6) $FV(pure\ M) = FV(M)$;
7) $FV(\mathbf{let\ pure}\ \vec{N} = \vec{M}\ \mathbf{in}\ M) = \bigcup\limits_{i=1}^{n} FV(M)$, where $n = |\vec{M}|$.

Definition 7. *Substitution:*
1) $x[x := N] = N$, $x[y := N] = x$;
2) $(MN)[x := N] = M[x := N]N[x := N]$;
3) $(\lambda x.M)[x := N] = \lambda x.M[x := N]$;
4) $(M, N)[x := P] = (M[x := P], N[x := P])$;
5) $(\pi_i M)[x := P] = \pi_i(M[x := P])$, $i \in \{1, 2\}$;
6) $(\mathbf{pure}\, M)[x := P] = \mathbf{pure}\,(M[x := P])$;
7) $(\mathbf{let\ pure}\ \vec{x} = \vec{M}\ \mathbf{in}\ N)[y := P] = \mathbf{let\ pure}\ \vec{x} = (\vec{M}[y := P])\ \mathbf{in}\ M$.

Definition 8. *Type substituition*

The substituition of type C for type variable B in type A inductively defined as follows:

1) $B[B := C] = B$ *and* $D[B := C] = D$, *if* $B \neq D$*;*

2) $(A_1 \alpha A_2)[B := C] = (A_1[B := C])\alpha(A_2[B := C])$, *where* $\alpha \in \{\rightarrow, \times\}$*;*

3) $(\mathbf{K}A)[B := C] = \mathbf{K}(A[B := C])$.

4) Let Γ *be the context, then* $\Gamma[B := C] = \{x : (A[B := C]) \mid x : A \in \Gamma\}$

Definition 9. *β-reduction and η-reduction rules for $\lambda_{\mathbf{K}}$.*

1) $(\lambda x.M)N \rightarrow_\beta M[x := N]$*;*

2) $\pi_1\langle M, N\rangle \rightarrow_\beta M$*;*

3) $\pi_2\langle M, N\rangle \rightarrow_\beta N$*;*

4) $\begin{array}{l}\textbf{let pure}\ \vec{x}, y, \vec{z} = \vec{M}, \textbf{let pure}\ \vec{w} = \vec{N}\ \textbf{in}\ Q, \vec{P}\ in\ R \rightarrow_\beta \\ \textbf{let pure}\ \vec{x}, \vec{w}, \vec{z} = \vec{M}, \vec{N}, \vec{P}\ \textbf{in}\ R[y := Q]\end{array}$

5) $\textbf{let pure}\ \vec{x} = \textbf{pure}\ \vec{M}\ \textbf{in}\ N \rightarrow_\beta \textbf{pure}\ N[\vec{x} := \vec{M}]$

6) $\textbf{let pure}\ \_\_ = \_\_\ \textbf{in}\ M \rightarrow_\beta \textbf{pure}\ M$

7) $\lambda x.fx \rightarrow_\eta f$*;*

8) $\langle \pi_1 P, \pi_2 P\rangle \rightarrow_\eta P$*;*

9) $\textbf{let pure}\ x = M\ \textbf{in}\ x \rightarrow_\eta M$*;*

By default we use call-by-name evaluation strategy.

Now we will prove standard lemmas for contexts in type systems[3]:

Lemma 3. *Generation for $\mathbf{K}_I$.*

Let $\Gamma \vdash \textbf{pure}\ M : \mathbf{K}A$, *then* $\Gamma \vdash M : A$*;*

Proof. Induction on the structure of $\textbf{pure}\ M$. □

Lemma 4. *Basic lemmas .*

i) Let $\Gamma \vdash M : A$ *and* $\Gamma \subseteq \Delta$, *then* $\Delta \vdash M : A$*;*

ii) Let $\Gamma, x : A \vdash M : B$ *and* $\Gamma \vdash N : A$, *then* $\Gamma \vdash M[x := N] : B$.

iii) Let $\Gamma \vdash M : A$, *then* $\Gamma[B := C] \vdash M : (A[B := C])$.

Proof.

i-ii-iii) Induction on $\Gamma \vdash M : A$.

□

Theorem 1. *Subject reduction*

Let $\Gamma \vdash M : A$ *and* $M \twoheadrightarrow_{\beta\eta} N$, *then* $\Gamma \vdash N : A$

Proof. For cases with application, abstraction and pairs see [12] [13].

1) Let $\Gamma \vdash \textbf{let pure}\ \vec{x}, y, \vec{z} = \vec{M}, \textbf{let pure}\ \vec{w} = \vec{N}\ \textbf{in}\ Q, \vec{P}\ \textbf{in}\ R : \mathbf{K}B$, then $\Gamma \vdash \textbf{let pure}\ \vec{x}, \vec{w}, \vec{z} = \vec{M}, \vec{N}, \vec{P}\ \textbf{in}\ R[y := Q] : \mathbf{K}B$

2) Let $\Gamma \vdash \textbf{let pure}\ x = M\ \textbf{in}\ x : \mathbf{K}A$, then $\Gamma \vdash M : \mathbf{K}A$.

See [19].

3) If the derivation ends in

$$\frac{\Gamma \vdash \textbf{pure}\ \vec{M} : \mathbf{K}\vec{A} \qquad \vec{x} : \vec{A} \vdash N : B}{\Gamma \vdash \textbf{let pure}\ \vec{x} = \textbf{pure}\ \vec{M}\ \textbf{in}\ N : \mathbf{K}B}$$

[3]We will not prove cases with $\rightarrow$-constructor, they are proved standardly in the same lemmas for simply typed lambda calculus, for example see [11] [12]. We will consider only modal cases

So $\Gamma \vdash \vec{M} : \vec{A}$ by generation and $\Gamma \vdash N[\vec{x} := \vec{M}] : B$ by weakening and substitution.

Then we can transform this into the next derivation:

$$\dfrac{\Gamma \vdash N[\vec{x} := \vec{M}] : B}{\Gamma \vdash \textbf{pure}\, N[\vec{x} := \vec{M}] : \mathbf{K}B}\ \mathbf{K}_I$$

4) If the derivation ends in

$$\dfrac{\vdash M : A}{\vdash \textbf{let pure}\ \_\_ = \_\_\ \textbf{in}\ M : \mathbf{K}A}$$

So, if $\vdash M : A$, then $\vdash \textbf{pure}\, M : \mathbf{K}A$.

Note that this part of the lemma works conversly too. □

Theorem 2.
$\twoheadrightarrow_\beta$ is strongly normalizing;

Proof.
We modify and apply Tait's technique of logical relation for modalities. For strong normalization proof with Tait's method for simply typed lambda calculus see [13].

Definition 10. *The set of strongly computable terms:*

- $SC_A = \{M : A \mid M$ *is strongly normalizing*$\}$ *for* $A \in \mathbb{T}$;
- $SC_{A\to B} = \{M : A \to B \mid \forall N \in SC_A, MN \in SC_B\}$, *for* $A, B \in \mathbb{T}_\mathbf{K}$
- $SC_{\mathbf{K}A} = \{M : \mathbf{K}A \mid M$ *is strongly normalizing*$\}$ *for* $A \in \mathbb{T}$;
- $SC_{\mathbf{K}(A\to B)} = \{M : \mathbf{K}(A \to B) | \forall f \in SC_{A\to B}, \forall x \in SC_A, \forall N \in SC_{\mathbf{K}A}, \textbf{let pure}\ f, x = M, N\ \textbf{in}\ fx \in SC_{\mathbf{K}B}\}$ *for* $A, B \in \mathbb{T}_\mathbf{K}$.

Lemma 5.

- *If* $M \in SC_A$, *then* M *is strongly normalizing;*
- *Let* $M \in SC_A$ *and* $M \to_\beta N$, *then* $N \in SC_A$;
- *Let* N *is non-introduced,* $N \in SC_A$. *Then, if* $M \to_\beta N$, *then* $M \in SC_A$;

Proof.
By induction on the structure of A.

1) $A \equiv \mathbf{K}A$, where $A \in \mathbb{T}$.

i) Follows from the definition;

ii) Immediately;

iii) Let N is non-introduced and $N \in SC_A$, such that $M \to_\beta N$. Any reduction path $M \to_\beta \dots$ passes through $M \to_\beta N$.

N is strongly normalizing, so M too.

2) $A \equiv \mathbf{K}(B \to C)$

i) Suppose $M \in SC_{\mathbf{K}(B\to C)}$. Let $N \in SC_{\mathbf{K}B}$. So $\textbf{let pure}\ f, x = M, N\ \textbf{in}\ fx \in SC_{\mathbf{K}C}$.

So M is strongly normalizing, since **let pure** $f, x = M, N$ **in** fx is strongly normalizing.

ii) Let $M_1 \in SC_{\mathbf{K}(B \to C)}$ and $M_1 \to_\beta M_2$. Fix $N \in SC_{\mathbf{K}B}$.

Then **let pure** $f, x = M_1, N$ **in** $fx \in SC_{\mathbf{K}C}$.

Hence, **let pure** $f, x = M_1, N$ **in** $fx \in SC_{\mathbf{K}C} \to_\beta$ **let pure** $f, x = M_2, N$ **in** fx.

So **let pure** $f, x = M_2, N$ **in** $fx \in SC_{\mathbf{K}C}$. Then $M_2 \in SC_{\mathbf{K}(B \to C)}$.

iii) Let M_2 be non-introduced, $M_2 \in SC_{\mathbf{K}(B \to C)}$ and $M_1 \to_\beta M_2$.

Let $N \in SC_{\mathbf{K}B}$. So **let pure** $f, x = M_2, N$ **in** $fx \in SC_{\mathbf{K}C}$.

So **let pure** $f, x = M_1, N$ **in** $fx \to_\beta$ **let pure** $f, x = M_2, N$ **in** $fx \in SC_{\mathbf{K}C}$.

Thus **let pure** $f, x = M_1, N$ **in** $fx \in SC_{\mathbf{K}C}$ by IH, so $M_1 \in SC_{\mathbf{K}(B \to C)}$. □

Lemma 6.

If $M \in SC_A$, *then* **pure** $M \in SC_{\mathbf{K}A}$

Proof. Induction on the structure of M. □

Lemma 7.

Let $x_1 : A_1, \ldots, x_n : A_n \vdash M : A$, *then forall* $i, M_i \in SC_{A_i}$. *Then* $M[x_1 := M_1, \ldots, x_n := M_n] \in SC_A$.

Proof.

1) Let the derivation ends in:

$$\frac{x_1 : A_1, \ldots, x_n : A_n \vdash M : A}{x_1 : A_1, \ldots, x_n : A_n \vdash \textbf{pure}\, M : \mathbf{K}A}$$

By assumption $M[x_1 := M_1, \ldots, x_n := M_n] \in SC_A$, so **pure** $M[x_1 := M_1, \ldots, x_n := M_n] \in SC_{\mathbf{K}A}$.

2) Let the derivation ends in:

$$\frac{x_1 : A_1, \ldots, x_n : A_n \vdash \vec{M'} : \mathbf{K}\vec{A} \qquad \vec{x} : \vec{A} \vdash N : B}{x_1 : A_1, \ldots, x_n : A_n \vdash \textbf{let pure}\, \vec{x} = \vec{M'}\ \textbf{in}\ N : \mathbf{K}B}$$

By IH forall $i \in \{1, \ldots, \mathrm{length}(\vec{M'})\}$, $M_i'[x_1 := M_1, \ldots, x_n := M_n] \in SC_{\mathbf{K}A_i}$.

So **let pure** $\vec{x} = \vec{M'}[x_1 := M_1, \ldots, x_n := M_n]$ **in** $N \in SC_{\mathbf{K}B}$, otherwise we can build infinite reduction path in $\vec{M'}[x_1 := M_1, \ldots, x_n := M_n]$. □

Corollary 1. *All terms are strongly computable, therefore are strongly normalizing.*

□

Theorem 3.

$\twoheadrightarrow_\beta$ *is confluent.*

Proof. We modify and apply Barendregt's technique with term underlying. We will consider the fragment of the grammar for terms without constructors for pairs for simplicity.

Definition 11. *The set of underlined terms.*

- $x \in \mathbb{V} \Rightarrow x \in \underline{\Lambda}$;

- $M \in \underline{\Lambda} \Rightarrow (\lambda x.M) \in \underline{\Lambda}$;
- $M, N \in \underline{\Lambda} \Rightarrow (MN) \in \underline{\Lambda}$;
- $M \in \underline{\Lambda} \Rightarrow (\mathbf{pure}\, M) \in \underline{\Lambda}$;
- $\vec{x} \in \mathbb{V}, \vec{M}, N \in \underline{\Lambda} \Rightarrow \mathbf{let\ pure}\, \vec{x} = \vec{M}\ \mathbf{in}\ N \in \underline{\Lambda}$;
- $M, N \in \underline{\Lambda} \Rightarrow (\lambda_i x.M)N \in \underline{\Lambda}$, *for all* $i \in \mathbb{N}$.

Definition 12. *Subsitution for term with labelled lambda:*
$((\lambda_i x.M)N)[y := Z] = (\lambda_i x.M[y := Z])(N[y := Z])$

Definition 13. *Index erasing*
Let us define map $|.| : \underline{\Lambda} \to \Lambda$ *as follows:*

- $|x| = x$;
- $|\lambda x.M| = \lambda x.|M|$;
- $|MN| = |M||N|$;
- $|\mathbf{pure}\, M| = \mathbf{pure}\, |M|$;
- $|\mathbf{let\ pure}\, \vec{x} = \vec{M}\ \mathbf{in}\ N| = \mathbf{let\ pure}\, \vec{x} = |\vec{M}|\ \mathbf{in}\ |N|$;
- $|(\lambda_i x.M)N| = (\lambda x.M)N$

Definition 14. *Reduction rules:*

- $(\lambda x.M)N \to_{\underline{\beta}} M[x := N]$;
- $\mathbf{let\ pure}\, \vec{x}, y, \vec{z} = \vec{M}, \mathbf{let\ pure}\, \vec{w} = \vec{N}\ \mathbf{in}\ Q, \vec{P}\ in\ R \to_{\underline{\beta}}$ $\mathbf{let\ pure}\, \vec{x}, \vec{w}, \vec{z} = \vec{M}, \vec{N}, \vec{P}\ \mathbf{in}\ R[y := Q]$;
- $\mathbf{let\ pure}\, \vec{x} = \mathbf{pure}\, \vec{M}\ \mathbf{in}\ N \to_{\underline{\beta}} \mathbf{pure}\, N[\vec{x} := \vec{M}]$;
- $\mathbf{let\ pure}\, \_\_ = \_\_\ \mathbf{in}\ M \to_{\underline{\beta}} \mathbf{pure}\, M$
- $(\lambda x_i.M)N \to_{\underline{\beta}} M[x := N]$

$\twoheadrightarrow_{\underline{\beta}}$ is a reflexive-transitive closure of $\to_{\underline{\beta}}$.

Definition 15. *Indexed redex erasing:*
Let us define the next map $\phi : \underline{\Lambda} \to \Lambda$:

- $\phi(x) = x$;
- $\phi(\lambda x.M) = \lambda x.\phi(M)$;
- $\phi(MN) = \phi(M)\phi(N)$;
- $\phi(\mathbf{pure}\, M) = \mathbf{pure}\, \phi(M)$;
- $\phi(\mathbf{let\ pure}\, \vec{x} = \vec{M}\ \mathbf{in}\ N) = \mathbf{let\ pure}\, \vec{x} = \phi(\vec{M})\ \mathbf{in}\ \phi(N)$;
- $\phi((\lambda_i x.M)N) = \phi(M)[x := \phi(N)]$

Example 1.
$\phi((\lambda_i x.\mathbf{let\ pure}\ y = x\ \mathbf{in}\ x)N) = \mathbf{let\ pure}\ y = \phi(N)\ \mathbf{in}\ \phi(N)$

Lemma 8. *$\forall \underline{M}, \underline{N} \in \underline{\Lambda}\ \forall M, N \in \Lambda$, if $|\underline{M}| = M, |\underline{N}| = N$, then*

- *If $M \twoheadrightarrow_\beta N$, then $\underline{M} \twoheadrightarrow_{\underline{\beta}} \underline{N}$*
- *Vice versa*

Proof. Induction on the generation $\rightarrow_\beta$ and $\rightarrow_{\underline{\beta}}$ correspondently. The general statement follows from transitivity of multi-step reductions of both types. □

Lemma 9. *$\phi(M[x := N]) = \phi(M)[x := \phi(N)]$.*

Proof. We treat only cases with **pure** and with **let**. For the rest cases see [15].
1)

$\phi(\mathbf{pure}\ (M[x := N])) =$	By the definition of ϕ
$\mathbf{pure}\ (\phi(M[x := N])) =$	Induction hypothesis
$\mathbf{pure}\ (\phi(M)[x := \phi(N)]) =$	Substitution definition
$(\mathbf{pure}\ \phi(M))[x := \phi(N)]$	

2)

$\phi((\mathbf{let\ pure}\ \vec{x} = \vec{M}\ \mathbf{in}\ N)[y := P]) =$	Substitution definition
$\phi(\mathbf{let\ pure}\ \vec{x} = (\vec{M}[y := P])\ \mathbf{in}\ N) =$	By the definition of ϕ
$\mathbf{let\ pure}\ \vec{x} = \phi(\vec{M}[y := P])\ \mathbf{in}\ \phi(N) =$	Induction hypothesis
$\mathbf{let\ pure}\ \vec{x} = (\phi(\vec{M})[y := \phi(P)])\ \mathbf{in}\ \phi(N) =$	Substitution definition
$(\mathbf{let\ pure}\ \vec{x} = \phi(\vec{M})\ \mathbf{in}\ \phi(N))[y := \phi(P)]$	

□

Lemma 10.

- *If $M \twoheadrightarrow_{\underline{\beta}} N$, then $\phi(M) \twoheadrightarrow_\beta \phi(N)$*
- *If $|M| = N$ and $\phi(M) = P$, then $N \twoheadrightarrow_\beta P$.*

Proof.
i) Induction on the generation of $\twoheadrightarrow_{\underline{\beta}}$ using previous lemma.
ii) Induction on the structure of M. □

Lemma 11. *Strip lemma.*

If $M \rightarrow_\beta N$ and $M \twoheadrightarrow_\beta P$. Then there exists some term Q, such that $N \twoheadrightarrow_\beta Q$ and $P \twoheadrightarrow_\beta Q$.

Proof. Proof is similar to [15] [18]. We build the following diagram, which commutes by lemmas 8 and 10.

$$\begin{tikzcd}
M \arrow[rr, two heads] \arrow[dd, "\beta"'] & & P \arrow[dd, dashed, "\beta"] & \\
& N' \arrow[ul, "|\cdot|"'] \arrow[dl, "\phi"] \arrow[rr, dashed, two heads, "\underline{\beta}"] & & P' \arrow[ul, "|\cdot|"'] \arrow[dl, "\phi"] \\
N \arrow[rr, dashed, two heads, "\beta"'] & & Q &
\end{tikzcd}$$

□

Corollary 2. *If $M \twoheadrightarrow_\beta N$ and $M \twoheadrightarrow_\beta P$. Then there exists some term Q, such that $N \twoheadrightarrow_\beta Q$ and $P \twoheadrightarrow_\beta Q$.*

Proof. Unfold $M \twoheadrightarrow_\beta N$ as the sequence of one-step reductions and apply strip lemma on the every step. □

□

Theorem 4.

Normal form in call-by-name $\lambda_{\mathbf{K}}$ has the subformula property: if M is in normal formal, then its all subterms are in normal form too.

Proof. By induction on the structure of M. Case with $\mathbf{let\,pure}\,\vec{x} = \vec{M}\,\mathbf{in}\,N$ was considered by Kakutani [19] [20]. Similary, if $\mathbf{pure}\,M$ is in normal form, so M is in normal form to and has subformula property by hypothesis. So $\mathbf{pure}\,M$ has subformula property. □

3 Categorical semantics

Definition 16. *Monoidal functor*

Let $\langle \mathcal{C}, \otimes_1, \mathbb{1} \rangle$ and $\langle \mathcal{D}, \otimes_2, \mathbb{1}' \rangle$ are monoidal categories.

A monoidal functor $\mathcal{F} : \langle \mathcal{C}, \otimes_1, \mathbb{1} \rangle \to \langle \mathcal{D}, \otimes_2, \mathbb{1}' \rangle$ is a functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ with additional natural transformations, which satisfy the well-known conditions described in [23]:

1) $u : \mathbb{1}' \to \mathcal{F}\mathbb{1}$;

2) $*_{A,B} : \mathcal{F}A \otimes_2 \mathcal{F}B \to \mathcal{F}(A \otimes_1 B)$.

Definition 17. *Applicative functor*

An applicative functor is a triple $\langle \mathcal{C}, \mathcal{K}, \eta \rangle$, where $\mathcal{C}$ is a symmetric monoidal category, $\mathcal{K}$ is a monoidal endofunctor and $\eta : Id_{\mathcal{C}} \Rightarrow \mathcal{K}$ is a natural transformation (similar to unit in monad), such that:

1) $u = \eta_{\mathbb{1}}$;

2) $*_{A,B} \circ (\eta_A \otimes \eta_B) = \eta_{A \otimes B}$;

3) Weak commutativity condition:

$$\begin{tikzcd}
A \otimes \mathcal{K}B \arrow[r, "\eta_A \otimes id_{\mathcal{K}B}"] \arrow[d, "\sigma_{A,\mathcal{K}B}"'] & \mathcal{K}A \otimes \mathcal{K}B \arrow[r, "*_{A,B}"] & \mathcal{K}(A \otimes B) \arrow[d, "\mathcal{K}(\sigma_{A,B})"] \\
\mathcal{K}B \otimes A \arrow[r, "id_{\mathcal{K}B} \otimes \eta_A"'] & \mathcal{K}B \otimes \mathcal{K}A \arrow[r, "*_{B,A}"'] & \mathcal{K}(B \otimes A)
\end{tikzcd}$$

3.1 Soundness and completeness

Theorem 5. *Soundness*

Let $\Gamma \vdash M : A$ and $M =_{\beta\eta} N$, then $[\![\Gamma \vdash M : A]\!] = [\![\Gamma \vdash N : A]\!]$

Proof.

Definition 18. *Semantical translation from $\lambda_{\mathbf{K}}$ to some cartesian closed category $\mathcal{C}$ with applicative functor $\mathcal{K}$:*

1) Interpretation for types:

$[\![A]\!] := \hat{A}, A \in \mathbb{T}$, where $\hat{A}$ is an object of $\mathcal{C}$ obtained by some given assignment;

$[\![A \to B]\!] := [\![A]\!] \to [\![B]\!]$;

$[\![A \times B]\!] := [\![A]\!] \times [\![B]\!]$.

2) Interpretation for modal types: $[\![\mathbf{K}A]\!] = \mathcal{K}[\![A]\!]$;

3) Interpretaion for contexts:

$[\![\Gamma = \{x_1 : A_1, ..., x_n : A_n\}]\!] := [\![\Gamma]\!] = [\![A_1]\!] \times ... \times [\![A_n]\!]$;

4) Interpretation for typing assignment: $[\![\Gamma \vdash M : A]\!] := [\![M]\!] : [\![\Gamma]\!] \to [\![A]\!]$.

5) Interpretation for typing rules:

$$\frac{}{[\![\Gamma, x : A \vdash x : A]\!] = \pi_2 : [\![\Gamma]\!] \times [\![A]\!] \to [\![A]\!]}$$

$$\frac{[\![\Gamma, x : A \vdash M : B]\!] = f : [\![\Gamma]\!] \times [\![A]\!] \to [\![B]\!]}{[\![\Gamma \vdash (\lambda x.M) : A \to B]\!] = \Lambda(f) : [\![\Gamma]\!] \to [\![B]\!]^{[\![A]\!]}}$$

$$\frac{[\![\Gamma \vdash M : A \to B]\!] = [\![M]\!] : [\![\Gamma]\!] \to [\![B]\!]^{[\![A]\!]} \qquad [\![\Gamma \vdash N : A]\!] = [\![N]\!] : [\![\Gamma]\!] \to [\![A]\!]}{[\![\Gamma \vdash (MN) : B]\!] = [\![\Gamma]\!] \xrightarrow{\langle [\![M]\!], [\![N]\!] \rangle} [\![B]\!]^{[\![A]\!]} \times [\![A]\!] \xrightarrow{\epsilon} [\![B]\!]}$$

$$\frac{[\![\Gamma \vdash M : A]\!] = f : [\![\Gamma]\!] \to [\![A]\!] \qquad [\![\Gamma \vdash N : B]\!] = g : [\![\Gamma]\!] \to [\![B]\!]}{[\![\Gamma \vdash (M, N) : A \times B]\!] = \langle f, g \rangle : [\![\Gamma]\!] \to [\![A]\!] \times [\![B]\!]}$$

$$\frac{[\![\Gamma \vdash p : A_1 \times A_2]\!] = f : [\![\Gamma]\!] \to [\![A_1]\!] \times [\![A_2]\!]}{[\![\Gamma \vdash \pi_i p : A_i]\!] = [\![\Gamma]\!] \xrightarrow{f} [\![A_1]\!] \times [\![A_2]\!] \xrightarrow{\pi_i} [\![A_i]\!]} \; i \in \{1, 2\}$$

$$\frac{[\![\Gamma \vdash M : A]\!] = [\![M]\!] : [\![\Gamma]\!] \to [\![A]\!]}{[\![\Gamma \vdash \mathbf{pure}\, M : \mathbf{K}A]\!] := [\![\Gamma]\!] \xrightarrow{[\![M]\!]} [\![A]\!] \xrightarrow{\eta_{[\![A]\!]}} \mathcal{K}[\![A]\!]}$$

$$\frac{[\![\Gamma \vdash \vec{M} : \mathbf{K}\vec{A}]\!] = \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle : [\![\Gamma]\!] \to \prod_{i=1}^{n} \mathcal{K}[\![A_i]\!] \qquad [\![\vec{x} : \vec{A} \vdash N : B]\!] = [\![N]\!] : \prod_{i=1}^{n} [\![A_i]\!] \to [\![B]\!]}{[\![\Gamma \vdash \mathbf{let\ pure}\, \vec{x} = \vec{M}\ \mathbf{in}\, M : \mathbf{K}B]\!] = \mathcal{K}([\![N]\!]) \circ *_{[\![A_1]\!], \dots, [\![A_n]\!]} \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle : [\![\Gamma]\!] \to \mathcal{K}[\![B]\!]}$$

Definition 19. *Simultaneous substitution*

Let $\Gamma = \{x_1 : A_1, ..., x_n : A_n\}$, $\Gamma \vdash M : A$ *and for all* $i \in \{1, ..., n\}$, $\Gamma \vdash M_i : A_i$.

We define simultaneous substitution $M[\vec{x} := \vec{M}]$ *recursively by:*

1) $x_i[\vec{x} := \vec{M}] = M_i$;

2) $(\lambda x.M)[\vec{x} := \vec{M}] = \lambda x.(M[\vec{x} := \vec{M}])$;

3) $(MN)[\vec{x} := \vec{M}] = (M[\vec{x} = \vec{M}])(N[\vec{x} := \vec{M}])$;

4) $\langle M, N \rangle = \langle (M[\vec{x} = \vec{M}]), (N[\vec{x} := \vec{M}]) \rangle$;

5) $(\pi_i P)[\vec{x} := \vec{M}] = \pi_i(P[\vec{x} = \vec{M}])$;

6) $(\mathbf{pure}\, M)[\vec{x} := \vec{M}] = \mathbf{pure}\, (M[\vec{x} = \vec{M}])$;

7) $(\mathbf{let\ pure}\, \vec{x} = \vec{M}\ \mathbf{in}\, N)[\vec{y} := \vec{P}] = \mathbf{let\ pure}\, \vec{x} = (\vec{M}[\vec{y} := \vec{P}])\ \mathbf{in}\, N$

Lemma 12.

$[\![M[x_1 := M_1, \dots, x_n := M_n]]\!] = [\![M]\!] \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle$.

Proof.

1) $[\![\Gamma \vdash (\textbf{pure}\, M)[\vec{x} := \vec{M}] : \mathbf{K}A]\!] = [\![\Gamma \vdash \textbf{pure}\, M : \mathbf{K}A]\!] \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle$.

$$\begin{array}{ll}
[\![\Gamma \vdash (\textbf{pure}\, M)[\vec{x} := \vec{M}] : \mathbf{K}A]\!] = [\![\Gamma \vdash \textbf{pure}\,(M[\vec{x} := \vec{M}]) : \mathbf{K}A]\!] & \text{Substitution definition} \\
\quad = \eta_{[\![A]\!]} \circ [\![(M[\vec{x} := \vec{M}])]\!] & \text{Translation for pure} \\
\quad = \eta_{[\![A]\!]} \circ ([\![M]\!] \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle) & \text{Induction hypothesis} \\
\quad = (\eta_{[\![A]\!]} \circ [\![M]\!]) \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle & \text{Associativity of composition} \\
\quad = [\![\Gamma \vdash \textbf{pure}\, M : \mathbf{K}A]\!] \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle & \text{Translation for pure}
\end{array}$$

2) $[\![\Gamma \vdash (\textbf{let pure}\, \vec{x} = \vec{M}\ \textbf{in}\ N)[\vec{y} := \vec{P}] : \mathbf{K}B]\!] = [\![\Gamma \vdash \textbf{let pure}\, \vec{x} = \vec{M}\ \textbf{in}\ N : \mathbf{K}B]\!] \circ \langle [\![P_1]\!], \dots, [\![P_n$

$$\begin{array}{ll}
[\![\Gamma \vdash (\textbf{let pure}\, \vec{x} = \vec{M}\ \textbf{in}\ N)[\vec{y} := \vec{P}] : \mathbf{K}B]\!] = & \text{Substitution definition} \\
[\![\Gamma \vdash \textbf{let pure}\, \vec{x} = (\vec{M}[\vec{y} := \vec{P}])\ \textbf{in}\ N : \mathbf{K}B]\!] = & \text{Interpretaion for } let_{\mathbf{K}} \\
\mathcal{K}([\![N]\!]) \circ *_{[\![A_1]\!], \dots, [\![A_n]\!]} \circ [\![\Gamma \vdash (\vec{M}[\vec{y} := \vec{P}]) : \mathbf{K}\vec{A}]\!] = & \text{Induction hypothesis} \\
\mathcal{K}([\![N]\!]) \circ *_{[\![A_1]\!], \dots, [\![A_n]\!]} \circ ([\![\vec{M}]\!] \circ \langle [\![P_1]\!], \dots, [\![P_n]\!] \rangle) = & \text{Associativity of composition} \\
(\mathcal{K}([\![N]\!]) \circ *_{[\![A_1]\!], \dots, [\![A_n]\!]} \circ [\![\vec{M}]\!]) \circ \langle [\![P_1]\!], \dots, [\![P_n]\!] \rangle = & \text{By interpretation} \\
[\![\Gamma \vdash \textbf{let pure}\, \vec{x} = \vec{M}\ \textbf{in}\ N : \mathbf{K}B]\!] \circ \langle [\![P_1]\!], \dots, [\![P_n]\!] \rangle &
\end{array}$$

□

Lemma 13.
Let $\Gamma \vdash M : A$ *and* $M \twoheadrightarrow_{\beta\eta} N$, *then* $[\![\Gamma \vdash M : A]\!] = [\![\Gamma \vdash N : A]\!]$;

Proof.
Cases with β-reductions for $let_{\mathbf{K}}$ are shown in [20]. Let us consider cases with **pure**.

1) $[\![\Gamma \vdash \textbf{let pure}\, \vec{x} = \textbf{pure}\, \vec{M}\ \textbf{in}\ N : \mathbf{K}B]\!] = [\![\Gamma \vdash \textbf{pure}\, N[\vec{x} := \vec{M}] : \mathbf{K}B]\!]$

$$\begin{array}{l}
[\![\Gamma \vdash \textbf{let pure}\, \vec{x} = \textbf{pure}\, \vec{M}\ \textbf{in}\ N : \mathbf{K}B]\!] = \\
\qquad\qquad \text{By interpretation} \\
\mathcal{K}([\![N]\!]) \circ *_{[\![A_1]\!], \dots, [\![A_n]\!]} \circ \langle \eta_{[\![A_1]\!]} \circ [\![M_1]\!], \dots, \eta_{[\![A_n]\!]} \circ [\![M_n]\!] \rangle = \\
\qquad\qquad \text{By the property of a pair of morphisms} \\
\mathcal{K}([\![N]\!]) \circ *_{[\![A_1]\!], \dots, [\![A_n]\!]} \circ (\eta_{[\![A_1]\!]} \times \dots \times \eta_{[\![A_n]\!]}) \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle = \\
\qquad\qquad \text{Associativity of composition} \\
\mathcal{K}([\![N]\!]) \circ (*_{[\![A_1]\!], \dots, [\![A_n]\!]} \circ (\eta_{[\![A_1]\!]} \times \dots \eta_{[\![A_n]\!]})) \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle = \\
\qquad\qquad \text{By the definition of an applicative functor} \\
\mathcal{K}([\![N]\!]) \circ \eta_{[\![A_1]\!] \times \dots \times [\![A_n]\!]} \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle = \\
\qquad\qquad \text{Naturality of } \eta \\
\eta_{[\![B]\!]} \circ [\![N]\!] \circ \langle [\![M_1]\!], \dots, [\![M_n]\!] \rangle = \\
\qquad\qquad \text{Associativity of composition} \\
\eta_{[\![B]\!]} \circ ([\![N]\!] \circ \langle [\![M_1]\!], \dots, [\![M_n]\!]) \rangle = \\
\qquad\qquad \text{Simultaneous substitution lemma} \\
\eta_{[\![B]\!]} \circ [\![N[\vec{x} := \vec{M}]]\!] \\
\qquad\qquad \text{By interpetation} \\
[\![\Gamma \vdash \textbf{pure}\,(N[\vec{x} := \vec{M}]) : \mathbf{K}B]\!]
\end{array}$$

2) $[\![\vdash \textbf{let pure}\, \_\_ = \_\_\ \textbf{in}\ M : \mathbf{K}A]\!] = [\![\vdash \textbf{pure}\, M : \mathbf{K}A]\!]$

$$
\begin{array}{ll}
[\![\vdash \textbf{let pure}\ \_\_ = \_\_\ \textbf{in}\ M : \mathbf{K}A]\!] = & \text{By interpretation} \\
\mathcal{K}([\![M]\!]) \circ u_{\mathbb{1}} = & \text{By the definition of an applicative functor} \\
\mathcal{K}([\![M]\!]) \circ \eta_{\mathbb{1}} = & \text{By naturality of } \eta \\
\eta_{[\![A]\!]} \circ [\![M]\!] = & \text{By interpretation} \\
[\![\vdash \textbf{pure}\ M : \mathbf{K}A]\!] &
\end{array}
$$

□

□

Theorem 6. *Completeness*
Let $[\![\Gamma \vdash M : A]\!] = [\![\Gamma \vdash N : A]\!]$*, then* $M =_{\beta\eta} N$*.*

Proof.
We will consider term model for simply typed lambda calculus $\times$ and $\rightarrow$ standardly described in [22]:

Definition 20. *Equivalence on term pairs:*
Let us define relation $\sim_{A,B} \subseteq \mathbb{V} \times \Lambda_{\mathbf{K}}$*, such that:*
$(x, M) \sim_{A,B} (y, N) \Leftrightarrow x : A \vdash M : B \ \&\ y : A \vdash N : A \ \&\ M =_{\beta\eta} N[y := x]$*;*

We will denote equivalence class as $[x, M]_{A,B} = \{(y, N) | (x, M) \sim_{A,B} (y, N)\}$ (we will drop indeces below).

Definition 21. *Category* $\mathcal{C}(\lambda)$*:*

- $Ob_{\mathcal{C}} = \{\hat{A} \mid A \in \mathbb{T}\} \cup \{\mathbb{1}\}$*;*
- $Hom_{\mathcal{C}(\lambda)}(\hat{A}, \hat{B}) = (\mathbb{V} \times \Lambda_{\mathbf{K}})/_{\sim_{A,B}}$*;*
- *Let* $[x, M] \in Hom_{\mathcal{C}(\lambda)}(\hat{A}, \hat{B})$ *and* $[y, N] \in Hom_{\mathcal{C}(\lambda)}(\hat{B}, \hat{C})$*, then* $[y, M] \circ [x, M] = [x, N[y := M]]$*;*
- *Identity morphism* $id_{\hat{A}} = [x, x] \in Hom_{\mathcal{C}(\lambda)(\hat{A})}$*;*
- $\mathbb{1}$ *is a terminal object;*
- $\widehat{A \times B} = \hat{A} \times \hat{B}$*;*
- *Canonical projection is defined as* $[x, \pi_i x] \in Hom_{\mathcal{C}(\lambda)}(\hat{A}_1 \times \hat{A}_2, \hat{A}_i)$ *for* $i \in \{1, 2\}$*;*
- $\widehat{A \rightarrow B} = \hat{B}^{\hat{A}}$*;*
- *Evaluation arrow* $\epsilon = [x, (\pi_2 x)(\pi_1 x)] \in Hom_{\mathcal{C}(\lambda)(\hat{B}^{\hat{A}} \times \hat{A}, \hat{B})}$*.*

It is sufficient to show $\mathbf{K}$ is an applicative functor on $\mathcal{C}(\lambda)$.

Definition 22. *Let us define an endofunctor* $\mathcal{K} : \mathcal{C}(\lambda) \rightarrow \mathcal{C}(\lambda)$*, such that forall* $[x, M] \in Hom_{\mathcal{C}(\lambda)}(\hat{A}, \hat{B}), \mathbf{K}([x, M]) = [y, \textbf{let pure}\ x = y\ \textbf{in}\ M] \in Hom_{\mathcal{C}(\lambda)}(\mathbf{K}\hat{A}, \mathbf{K}\hat{B})$ *(denotation: fmap* f *for an arbitrary arrow* f*).*

Lemma 14. *Functoriality*

- *fmap* $(g \circ f) =$ *fmap* $(g) \circ$ *fmap* (f)*;*
- *fmap* $(id_{\hat{A}}) = id_{\mathbf{K}\hat{A}}$*.*

Proof. Easy checking using reduction rules. □

Definition 23. *Let us define natural transformations:*

1) $\eta : Id \Rightarrow \mathcal{K}$, *s. t.* $\forall \hat{A} \in Ob_{\mathcal{C}(\lambda)}$, $\eta_{\hat{A}} = [x, \mathbf{pure}\, x] \in Hom_{\mathcal{C}(\lambda)}(\hat{A}, \mathbf{K}\hat{A})$;

2) $*_{A,B} : \mathbf{K}\hat{A} \times \mathbf{K}\hat{B} \to \mathbf{K}(\hat{A} \times \hat{B})$, *s. t.* $\forall \hat{A}, \hat{B} \in Ob_{\mathcal{C}(\lambda)}, *_{\hat{A},\hat{B}} = [p, \mathbf{let\ pure}\, x, y = \pi_1 p, \pi_2 p\, \mathbf{in}\, \langle x, y\rangle] \in Hom_{\mathcal{C}(\lambda)}(\mathbf{K}A \times \mathbf{K}B, \mathbf{K}(A \times B))$.

Implementation for $*$ in our term model is a modification of $\mathrm{let}_{\mathbf{K}}$-rule:

$$\dfrac{\dfrac{p : \mathbf{K}A \times \mathbf{K}B \vdash p : \mathbf{K}A \times \mathbf{K}B}{p : \mathbf{K}A \times \mathbf{K}B \vdash \pi_1 p : \mathbf{K}A} \quad \dfrac{p : \mathbf{K}A \times \mathbf{K}B \vdash p : \mathbf{K}A \times \mathbf{K}B}{p : \mathbf{K}A \times \mathbf{K}B \vdash \pi_2 p : \mathbf{K}B} \quad \dfrac{x : A \vdash x : A \qquad y : B \vdash y : B}{x : A, y : B \vdash \langle x, y\rangle : A \times B}}{p : \mathbf{K}A \times \mathbf{K}B \vdash \mathbf{let\ pure}\, x, y = \pi_1 p, \pi_2 p\, \mathbf{in}\, \langle x, y\rangle : \mathbf{K}(A \times B)}$$

Lemma 15.

$\mathbf{K}$ *is a monoidal endofunctor*

Proof.

See [19] □

Lemma 16. *Properties of* η*:*

- *fmap* $f \circ \eta_A = \eta_B \circ f$;
- $*_{\hat{A},\hat{B}} \circ (\eta_A \times \eta_B) = \eta_{\hat{A} \times \hat{B}}$;

Proof.

i) fmap $f \circ \eta_{\hat{A}} = \eta_{\hat{B}} \circ f$

$\eta_{\hat{B}} \circ f =$	By the definition
$[y, \mathbf{pure}\, y] \circ [x, M] =$	By the definition of composition
$[x, \mathbf{pure}\, y[y := M]] =$	By substitution
$[x, \mathbf{pure}\, M]$	

On the other hand:

fmap $f \circ \eta_{\hat{A}} =$	By the definiton
$[z, \mathbf{let\ pure}\, x = z\, \mathbf{in}\, M] \circ [x, \mathbf{pure}\, \mathbf{x}] =$	By the definition of composition
$[x, \mathbf{let\ pure}\, x = z\, \mathbf{in}\, M[z := \mathbf{pure}\, x]] =$	By substitution
$[x, \mathbf{let\ pure}\, x = \mathbf{pure}\, x\, \mathbf{in}\, M] =$	β-reduction rule
$[x, \mathbf{pure}\, M[x := x]] =$	By substitution
$[x, \mathbf{pure}\, M]$	

ii) $*_{\hat{A},\hat{B}} \circ (\eta_{\hat{A}} \times \eta_{\hat{B}}) = \eta_{\hat{A} \times \hat{B}}$

$*_{\hat{A},\hat{B}} \circ (\eta_{\hat{A}} \times \eta_{\hat{B}}) =$
By unfolding
$[q, \mathbf{let\ pure}\, x, y = \pi_1 q, \pi_2 q\ \mathbf{in}\ \langle x, y\rangle] \circ [p, \langle \mathbf{pure}\,(\pi_1 p), \mathbf{pure}\,(\pi_2 p)\rangle] =$
Composition
$[p, \mathbf{let\ pure}\, x, y = \pi_1 q, \pi_2 q\ \mathbf{in}\ \langle x, y\rangle[q := \langle \mathbf{pure}\,(\pi_1 p), \mathbf{pure}\,(\pi_2 p)\rangle]] =$
By substitution
$[p, \mathbf{let\ pure}\, x, y = \pi_1(\langle \mathbf{pure}\,(\pi_1 p), \mathbf{pure}\,(\pi_2 p)\rangle), \pi_2(\langle \mathbf{pure}\,(\pi_1 p), \mathbf{pure}\,(\pi_2 p)\rangle)\ \mathbf{in}\ \langle x, y\rangle] =$
Reduction rules
$[p, \mathbf{let\ pure}\, x, y = \mathbf{pure}\,(\pi_1 p), \mathbf{pure}\,(\pi_2 p)\ \mathbf{in}\ \langle x, y\rangle] =$
Reduction rule
$[p, \mathbf{pure}\,(\langle x, y\rangle[x := \pi_1 p, y := \pi_2 p])] =$
Substitution
$[p, \mathbf{pure}\,\langle \pi_1 p, \pi_2 p\rangle] =$
η-reduction
$[p, \mathbf{pure}\, p] =$
By definition
$\eta_{\hat{A} \times \hat{B}}$

□

Definition 24.
$u_{\mathbb{1}} = [\blacksquare, \mathbf{let\ pure}\, \_\_ = \_\_\ \mathbf{in}\ \blacksquare] \in Hom_{\mathcal{C}(\lambda)}(\mathbb{1}, \mathbf{K}\mathbb{1})$.

Lemma 17.
$u_{\mathbb{1}} = \eta_{\mathbb{1}}$

Proof. Immediately. □

Tensorial strength is defined as follows:

Definition 25. *Tensorial strength*
Let $[p, \langle \mathbf{pure}\,(\pi_1 p), \pi_2 p\rangle] \in Hom_{\mathcal{C}(\lambda)}(\hat{A} \times \mathbf{K}\hat{B}, \mathbf{K}\hat{A} \times \mathbf{K}\hat{B})$.
So tensorial strength is defined as $\tau_{\hat{A},\hat{B}} = *_{\hat{A},\hat{B}} \circ [p, \langle \mathbf{pure}\,(\pi_1 p), \pi_2 p\rangle]$.

It is clearly that tensorial strength defined above can be simplified as follows:

$*_{\hat{A},\hat{B}} \circ [p, \langle \mathbf{pure}\,(\pi_1 p), \pi_2 p\rangle] =$ By definition
$[p', \mathbf{let\ pure}\, x, y = \pi_1 p', \pi_2 p'\ \mathbf{in}\ \langle x, y\rangle] \circ [p, \langle \mathbf{pure}\,(\pi_1 p), \pi_2 p\rangle] =$ By composition
$[p, \mathbf{let\ pure}\, x, y = \pi_1 p', \pi_2 p'\ \mathbf{in}\ \langle x, y\rangle[p' := \langle \mathbf{pure}\,(\pi_1 p), \pi_2 p\rangle]] =$ By substitution
$[p, \mathbf{let\ pure}\, x, y = \pi_1(\langle \mathbf{pure}\,(\pi_1 p), \pi_2 p\rangle), \pi_2(\langle \pi_1 p, \mathbf{pure}\,(\pi_2 p)\rangle)\ \mathbf{in}\ \langle x, y\rangle] =$ By β-reduction rules
$[p, \mathbf{let\ pure}\, x, y = \mathbf{pure}\,(\pi_1 p), \pi_2 p\ \mathbf{in}\ \langle x, y\rangle]$

Lemma 18. *Weak commutativity.*
$fmap\ ([p, \langle \pi_2 p, \pi_1 p\rangle]) \circ \tau_{\hat{A},\hat{B}} =$
$*_{\hat{B},\hat{A}} \circ [q, \langle \pi_1 q, \mathbf{pure}\,(\pi_2 q)\rangle] \circ [p, \langle \pi_2 p, \pi_1 p\rangle]$

Proof.

fmap $([r, \langle \pi_2 r, \pi_1 r\rangle]) \circ \tau_{\hat{A},\hat{B}} =$
By the definition of τ
fmap $([r, \langle \pi_2 r, \pi_1 r\rangle]) \circ [p, \textbf{let pure}\, x, y = \textbf{pure}\, (\pi_1 p), \pi_2 p \,\textbf{in}\, \langle x, y\rangle] =$
By the definition of fmap
$[q, \textbf{let pure}\, r = q \,\textbf{in}\, \langle \pi_2 r, \pi_1 r\rangle] \circ [p, \textbf{let pure}\, x, y = \textbf{pure}\, (\pi_1 p), \pi_2 p \,\textbf{in}\, \langle x, y\rangle] =$
Composition
$[p, \textbf{let pure}\, r = q \,\textbf{in}\, \langle \pi_2 r, \pi_1 r\rangle[q := \textbf{let pure}\, x, y = \textbf{pure}\, (\pi_1 p), \pi_2 p \,\textbf{in}\, \langle x, y\rangle]] =$
By β-reduction rules
$[p, \textbf{let pure}\, r = (\textbf{let pure}\, x, y = \textbf{pure}\, (\pi_1 p), \pi_2 p \,\textbf{in}\, \langle x, y\rangle) \,\textbf{in}\, \langle \pi_2 r, \pi_1 r\rangle] =$
By β-reduction rules
$[p, \textbf{let pure}\, x, y = \textbf{pure}\, (\pi_1 p), \pi_2 p \,\textbf{in}\, \langle \pi_2 r, \pi_1 r\rangle[r := \langle x, y\rangle]] =$
By substitution
$[p, \textbf{let pure}\, x, y = \textbf{pure}\, (\pi_1 p), \pi_2 p \,\textbf{in}\, \langle \pi_2\langle x, y\rangle, \pi_1\langle x, y\rangle\rangle] =$
By β-reduction rules
$[p, \textbf{let pure}\, x, y = \textbf{pure}\, (\pi_1 p), \pi_2 p \,\textbf{in}\, \langle y, x\rangle] =$

On the other hand
$*_{\hat{B},\hat{A}} \circ [q, \langle \pi_1 q, \textbf{pure}\, (\pi_2 q)\rangle] \circ [p, \langle \pi_2 p, \pi_1 p\rangle] =$
By the definition of $*$
$[r, \textbf{let pure}\, y, x = \pi_1 r, \pi_2 r \,\textbf{in}\, \langle y, x\rangle] \circ [q, \langle \pi_1 q, \textbf{pure}\, (\pi_2 q)\rangle] \circ [p, \langle \pi_2 p, \pi_1 p\rangle] =$
Composition
$[r, \textbf{let pure}\, y, x = \pi_1 r, \pi_2 r \,\textbf{in}\, \langle y, x\rangle] \circ [p, \langle \pi_1 q, \textbf{pure}\, (\pi_2 q)\rangle[q := \langle \pi_2 p, \pi_1 p\rangle]] =$
By subsitution and by β-reduction rules
$[r, \textbf{let pure}\, y, x = \pi_1 r, \pi_2 r \,\textbf{in}\, \langle y, x\rangle] \circ [p, \langle \pi_2 p, \textbf{pure}\, (\pi_1 p)\rangle]] =$
Composition
$[p, \textbf{let pure}\, y, x = \pi_1 r, \pi_2 r \,\textbf{in}\, \langle y, x\rangle[r := \langle \pi_2 p, \textbf{pure}\, (\pi_1 p)\rangle]] =$
By subsitution and by β-reduction rules
$[p, \textbf{let pure}\, y, x = \pi_2 p, \textbf{pure}\, (\pi_1 p) \,\textbf{in}\, \langle y, x\rangle] =$
By symmetricity of assingment
$[p, \textbf{let pure}\, x, y = \textbf{pure}\, (\pi_1 p), \pi_2 p \,\textbf{in}\, \langle y, x\rangle]$

□

Lemma 19. **K** *is an applicative functor*

Proof. Immediately follows from previous lemmas in the section. □

Similar to [24], we apply the translation from $\lambda_{\mathbf{K}}$ to some cartesian closed category with an abritraty applicative functor $\mathcal{K}$, then we have $[\![\Gamma \vdash M : A]\!] = [x, M[x_i := \pi_i x]]$, so $M =_{\beta\eta} N \Leftrightarrow [\![\Gamma \vdash M : A]\!] = [\![\Gamma \vdash N : A]\!]$.

□

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