

# Modal type theory based on the intuitionistic epistemic logic

## Abstract

Modal intuitionistic epistemic logic  $\text{IEL}^-$  was proposed by S.Artemov and T. Protopopescu as the formal foundation for the intuitionistic theory of knowledge. We construct a modal simply typed lambda-calculus which is Curry-Howard isomorphic to  $\text{IEL}^-$  as formal theory of calculations with applicative functors in functional programming languages like Haskell or Idris.

## 1 Introduction

Modal intuitionistic epistemic logic  $\text{IEL}$  was proposed by S. Artemov and T. Protopopescu [1].  $\text{IEL}$  provides the epistimology and the theory of knowledge as based on BHK-semantics of intuitionistic logic.  $\text{IEL}^-$  is a variant of  $\text{IEL}$ , that corresponds to intuitionistic belief. Informally,  $\mathbf{K}A$  denotes that  $A$  is verified intuitionistically.

Intuitionistic epistemic logic  $\text{IEL}^-$  is defined by following axioms and derivation rules:

**Definition 1.** *Intuitionistic epistemic logic  $\text{IEL}$ :*

- 1) *IPC axioms;*
  - 2)  $\mathbf{K}(A \rightarrow B) \rightarrow (\mathbf{K}A \rightarrow \mathbf{K}B)$  (*normality*);
  - 3)  $A \rightarrow \mathbf{K}A$  (*co-reflection*);
- Rule: MP.*

V. Krupski and A. Yatmanov provided the sequential calculus for  $\text{IEL}$  and proved that this calculus is PSPACE-complete [2].

Functional programming languages such as Haskell [3], Idris [4], Purescript [5] Elm [6] or Scala [?] have special type classes<sup>1</sup> for calculations with container types like `Functor` and `Applicative`<sup>2</sup>:

```
class Functor f where
  fmap :: (a -> b) -> f a -> f b

class Functor f => Applicative f where
  pure :: a -> f a
  (<*>) :: f (a -> b) -> f a -> f b
```

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<sup>1</sup>Type class in Haskell is a general interface for special group of datatypes.

<sup>2</sup>Reader may read more about container types in the Haskell standard library documentation[7] or in the next one textbook [8]

By *container* (or *computational context*) type we mean some type-operator  $f$ , where  $f$  is a “function” from  $*$  to  $*$ : type operator takes a simple type (kind  $*$ ) and returns another simple type of kind  $*$ . For more detailed description of the type system with kinds used in Haskell see [12].

Applicative functor allows to generalize the action of a functor for functions with arbitrary number of arguments, for instance:

```
liftA2 :: Applicative f => (a -> b -> c) -> f a -> f b -> f c
liftA2 f x y = ((pure f) <*> x) <*> y
```

It's not difficult to see that modal axioms in  $IEL^-$  and types of the methods of Applicative class in Haskell-like languages (which is described below) are syntactically similar and we are going to show that this coincidence has a non-trivial computational meaning.

We investigate the relationship between intuitionistic epistemic logic  $IEL^-$  and applicative programming with side-effects by constructing the type system (which is called  $\lambda_K$ ) which is Curry-Howard isomorphic to  $IEL^-$ . So we will consider  $K$ -modality as an arbitrary applicative functor and we prove that obtained type system is sound and complete for applicative functor on cartesian closed category (using the categorical definition proposed by Paterson [26]).

$\lambda_K$  consists of the rules for simply typed lambda-calculus and special typing rules for lifting types into the applicative functor  $K$ . We assume that our type system will axiomatize the simplest case of computation with effects with one container. We provide a proof-theoretical view at this kind of computations in functional programming and prove strong normalization and confluence.

## 2 Typed lambda-calculus based on $IEL^-$

The first is to define the natural deduction calculus for  $NIEL^-$ :

**Definition 2.** *Natural deduction  $NIEL^-$  for  $IEL^-$  is an extension of intuitionistic natural deduction calculus with additional inference rules for modality:*

$$\frac{\Gamma \vdash A}{\Gamma \vdash KA} K_I \qquad \frac{\Gamma \vdash KA_1, \dots, \Gamma \vdash KA_n \quad A_1, \dots, A_n \vdash B}{\Gamma \vdash KB}$$

The first rule allows to derive co-reflexion. The second modal rule is a counterpart of  $\Box_I$  rule in natural deduction calculus for constructive K (see [25]).

We will denote  $\Gamma \vdash KA_1, \dots, \Gamma \vdash KA_n$  and  $A_1, \dots, A_n \vdash B$  as  $\Gamma \vdash K\vec{A}$  and  $\vec{A} \vdash B$  for brevity.

**Lemma 1.**  $\Gamma \vdash_{NIEL^-} A \Rightarrow IEL^- \vdash \bigwedge \Gamma \rightarrow A$ .

*Proof.* Induction on the derivation.

Let us consider cases with modality.

- 1) If  $\Gamma \vdash_{NIEL^-} A$ , then  $IEL^- \vdash \bigwedge \Gamma \rightarrow KA$ .

- |     |   |                      |
|-----|---|----------------------|
| (1) | $\bigwedge \Gamma \rightarrow A$  | assumption           |
| (2) | $A \rightarrow \mathbf{K}A$   | co-reflection        |
| (3) | $(\bigwedge \Gamma \rightarrow A) \rightarrow ((A \rightarrow \mathbf{K}A) \rightarrow (\bigwedge \Gamma \rightarrow \mathbf{K}A))$ | IPC theorem          |
| (4) | $(A \rightarrow \mathbf{K}A) \rightarrow (\bigwedge \Gamma \rightarrow \mathbf{K}A)$  | from (1), (3) and MP |
| (5) | $\bigwedge \Gamma \rightarrow \mathbf{K}A$  | from (2), (4) and MP |

2) If  $\Gamma \vdash_{\text{NIEL}^-} \mathbf{K}\vec{A}$  and  $\vec{A} \vdash B$ , then  $\text{IEL}^- \vdash \bigwedge \Gamma \rightarrow \mathbf{K}B$ .

- |     |   |                                |
|-----|---|--------------------------------|
| (1) | $\bigwedge \Gamma \rightarrow \bigwedge_{i=1}^n \mathbf{K}A_i$                                      | assumption                     |
| (2) | $\bigwedge_{i=1}^n \mathbf{K}A_i \rightarrow \mathbf{K} \bigwedge_{i=1}^n A_i$                      | $\text{IEL}^-$ theorem         |
| (3) | $\bigwedge \Gamma \rightarrow \mathbf{K} \bigwedge_{i=1}^n A_i$                                     | from (1), (2) and transitivity |
| (4) | $\bigwedge_{i=1}^n A_i \rightarrow B$   | assumption                     |
| (5) | $(\bigwedge_{i=1}^n A_i \rightarrow B) \rightarrow \mathbf{K}(\bigwedge_{i=1}^n A_i \rightarrow B)$ | co-reflection                  |
| (6) | $\mathbf{K}(\bigwedge_{i=1}^n A_i \rightarrow B)$   | from (4), (5) and MP           |
| (7) | $\mathbf{K} \bigwedge_{i=1}^n A_i \rightarrow \mathbf{K}B$  | from (6) and normality         |
| (8) | $\bigwedge \Gamma \rightarrow \mathbf{K}B$  | from (3), (7) and transitivity |

□

**Lemma 2.** *If  $\text{IEL}^- \vdash A$ , then  $\text{NIEL}^- \vdash A$ .*

*Proof.* Straightforward derivation of modal axioms in  $\text{NIEL}^-$ . We consider this derivation below using terms. □

At the next step we build the typed lambda-calculus based on the  $\text{NIEL}^-$  with implication and disjunction by proof-assignment in rules. Obtained fragment is equivalent to  $\text{IEL}^-$  without axioms for negation and disjunction.

At first, we define lambda-terms and types for this lambda-calculus.

**Definition 3.** *The set of terms:*

*Let  $\mathbb{V}$  be the set of variables. The set  $\Lambda_{\mathbf{K}}$  of terms is defined by the grammar:*

$$\Lambda_{\mathbf{K}} ::= \mathbb{V} \mid (\lambda \mathbb{V}. \Lambda_{\mathbf{K}}) \mid (\Lambda_{\mathbf{K}} \Lambda_{\mathbf{K}}) \mid (\Lambda_{\mathbf{K}}, \Lambda_{\mathbf{K}}) \mid (\pi_1 \Lambda_{\mathbf{K}}) \mid (\pi_2 \Lambda_{\mathbf{K}}) \mid (\text{pure } \Lambda_{\mathbf{K}}) \mid (\text{let pure } \mathbb{V}^* = \Lambda_{\mathbf{K}}^* \text{ in } \Lambda_{\mathbf{K}})$$

Where  $\mathbb{V}^*$  and  $\Lambda_{\mathbf{K}}^*$  denote the set of finite sequences of variables  $\bigcup_{i=0}^{\infty} \mathbb{V}^i$  and

the set of finite sequences of terms  $\bigcup_{i=0}^{\infty} \Lambda_{\mathbf{K}}^i$ . Note that the sequence of variables  $\vec{x}$  and the sequence of terms  $\vec{M}$  should have the same length. Otherwise, term is not well-formed.

**Definition 4.** *The set of types:*

*Let  $\mathbb{T}$  be the set of atomic types. The set  $\mathbb{T}_{\mathbf{K}}$  of types with applicative functor  $\mathbf{K}$  is generated by the grammar:*

$$\mathbb{T}_{\mathbf{K}} ::= \mathbb{T} \mid (\mathbb{T}_{\mathbf{K}} \rightarrow \mathbb{T}_{\mathbf{K}}) \mid (\mathbb{T}_{\mathbf{K}} \times \mathbb{T}_{\mathbf{K}}) \mid (\mathbf{K}\mathbb{T}_{\mathbf{K}}) \quad (1)$$

Context, domain of context and range of context are defined standardly [11][12].

Our type system is based on the Curry-style typing rules:

**Definition 5.** *Modal typed lambda calculus  $\lambda_{\mathbf{K}}$  based on  $NIEL_{\wedge, \rightarrow}^-$ :*

$$\begin{array}{c} \frac{}{\Gamma, x : A \vdash x : A} \text{ax} \\[10pt] \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x. M : A \rightarrow B} \rightarrow_i \qquad \frac{\Gamma \vdash M : A \rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B} \rightarrow_e \\[10pt] \frac{\Gamma \vdash M : A \quad \Gamma \vdash N : B}{\Gamma \vdash \langle M, N \rangle : A \times B} \times_i \qquad \frac{\Gamma \vdash M : A_1 \times A_2}{\Gamma \vdash \pi_i M : A_i} \times_e, i \in \{1, 2\} \\[10pt] \frac{\Gamma \vdash M : A}{\Gamma \vdash \mathbf{pure} M : \mathbf{K}A} \mathbf{K}_I \qquad \frac{\Gamma \vdash \vec{M} : \mathbf{K}\vec{A} \quad \vec{x} : \vec{A} \vdash N : B}{\Gamma \vdash \mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} N : \mathbf{K}B} \mathbf{let}_{\mathbf{K}} \end{array}$$

$\mathbf{K}_I$ -typing rule is the same as  $\bigcirc$ -introduction in lax logic (also known as monadic metalanguage [17]) and in typed lambda-calculus which is derived by proof-assignment for lax-logic proofs.  $\mathbf{K}_I$  allows to inject an object of type  $\alpha$  into the functor.  $\mathbf{K}_I$  reflects the Haskell method **pure** for Applicative class. It plays the same role as the **return** method in Monad class.

$\mathbf{let}_{\mathbf{K}}$  is similar to the  $\square$ -rule in typed lambda calculus for intuitionistic normal modal logic  $\mathbf{IK}$ , which is described in [19].

$\Gamma \vdash \vec{M} : \mathbf{K}\vec{A}$  is a syntax sugar for the sequence  $\Gamma \vdash M_1 : \mathbf{K}A_1, \dots, \Gamma \vdash M_n : \mathbf{K}A_n$  and  $\vec{x} : \vec{A} \vdash N : B$  is a short form for  $x_1 : A_1, \dots, x_n : A_n \vdash N : B$ . **let pure**  $\vec{x} = \vec{M}$  **in**  $N$  is a simultaneous local binding in  $N$ . We use this short form instead of **let pure**  $x_1, \dots, x_n = M_1, \dots, M_n$  **in**  $N$ .

In fact, our calculus is the extention of typed lambda calculus for  $\mathbf{IK}$  by  $\mathbf{K}_I$ -rule that is appropriate to co-reflection.

Here are some examples of closed terms:

$$\begin{array}{c} \frac{\frac{x : A \vdash x : A}{x : A \vdash \mathbf{pure} x : \mathbf{K}A}}{\vdash (\lambda x. \mathbf{pure} x) : A \rightarrow \mathbf{K}A} \\[10pt] \frac{\frac{f : \mathbf{K}(A \rightarrow B) \vdash f : \mathbf{K}(A \rightarrow B) \quad x : \mathbf{K}A \vdash x : \mathbf{K}A \quad \frac{g : A \rightarrow B \vdash g : A \rightarrow B \quad y : A \vdash y : A}{g : A \rightarrow B, y : A \vdash gy : B} \rightarrow_e}{\frac{f : \mathbf{K}(A \rightarrow B), x : \mathbf{K}A \vdash \mathbf{let pure} g, y = f, x \mathbf{in} gy : \mathbf{K}B}{f : \mathbf{K}(A \rightarrow B) \vdash \lambda x. \mathbf{let pure} g, y = f, x \mathbf{in} gy : \mathbf{K}A \rightarrow \mathbf{K}B}} \mathbf{let}_{\mathbf{K}} \\[10pt] \vdash \lambda f. \lambda x. \mathbf{let pure} g, y = f, x \mathbf{in} gy : \mathbf{K}(A \rightarrow B) \rightarrow \mathbf{K}A \rightarrow \mathbf{K}B \end{array}$$

Now we define free variables and substitutions.  $\beta$ -reduction, multi-step  $\beta$ -reduction and  $\beta$ -equality are defined standardly:

**Definition 6.** The set  $FV(M)$  of free variables for a term  $M$ :

- 1)  $FV(x) = \{x\}$ ;
- 2)  $FV(\lambda x.M) = FV(M) \setminus \{x\}$ ;
- 3)  $FV(MN) = FV(M) \cup FV(N)$ ;
- 4)  $FV(\langle M, N \rangle) = FV(M) \cup FV(N)$ ;
- 5)  $FV(\pi_i M) \subseteq FV(M)$ ,  $i \in \{1, 2\}$ ;
- 6)  $FV(\text{pure } M) = FV(M)$ ;
- 7)  $FV(\text{let pure } \vec{x} = \vec{M} \text{ in } N) = \bigcup_{i=1}^n FV(M_i)$ , where  $n = |\vec{M}|$ .

**Definition 7.** Substitution:

- 1)  $x[x := N] = N$ ,  $x[y := N] = x$ ;
- 2)  $(MN)[x := N] = M[x := N]N[x := N]$ ;
- 3)  $(\lambda x.M)[x := N] = \lambda x.M[y := N]$ ,  $y \in FV(M)$ ;
- 4)  $(M, N)[x := P] = (M[x := P], N[x := P])$ ;
- 5)  $(\pi_i M)[x := P] = \pi_i(M[x := P])$ ,  $i \in \{1, 2\}$ ;
- 6)  $(\text{pure } M)[x := P] = \text{pure } (M[x := P])$ ;
- 7)  $(\text{let pure } \vec{x} = \vec{M} \text{ in } N)[y := P] = \text{let pure } \vec{x} = (\vec{M}[y := P]) \text{ in } N$ .

Substitution and free variable for terms of kind **let pure** are defined similary to [19].

**Definition 8.** Type substitution

The substitution of type  $C$  for type variable  $B$  in type  $A$  is defined inductively:

- 1)  $B[B := C] = B$  and  $D[B := C] = D$ , if  $B \neq D$ ;
- 2)  $(A_1 \alpha A_2)[B := C] = (A_1[B := C])\alpha(A_2[B := C])$ , where  $\alpha \in \{\rightarrow, \times\}$ ;
- 3)  $(\mathbf{K}A)[B := C] = \mathbf{K}(A[B := C])$ ;
- 4) Let  $\Gamma$  be the context, then  $\Gamma[B := C] = \{x : (A[B := C]) \mid x : A \in \Gamma\}$ .

**Definition 9.**  $\beta$ -reduction and  $\eta$ -reduction rules for  $\lambda_{\mathbf{K}}$ .

- 1)  $(\lambda x.M)N \rightarrow_{\beta} M[x := N]$ ;
- 2)  $\pi_1 \langle M, N \rangle \rightarrow_{\beta} M$ ;
- 3)  $\pi_2 \langle M, N \rangle \rightarrow_{\beta} N$ ;
- 4)  $\text{let pure } \vec{x}, y, \vec{z} = \vec{M}, \text{let pure } \vec{w} = \vec{N} \text{ in } Q, \vec{P} \text{ in } R \rightarrow_{\beta} \text{let pure } \vec{x}, \vec{w}, \vec{z} = \vec{M}, \vec{N}, \vec{P} \text{ in } R[y := Q]$
- 5)  $\text{let pure } \vec{x} = \text{pure } \vec{M} \text{ in } N \rightarrow_{\beta} \text{pure } N[\vec{x} := \vec{M}]$
- 6)  $\text{let pure } \_ = \_ \text{ in } M \rightarrow_{\beta} \text{pure } M$ , where  $\_$  is an empty sequence of terms.
- 7)  $\lambda x.f x \rightarrow_{\eta} f$ ;
- 8)  $\langle \pi_1 P, \pi_2 P \rangle \rightarrow_{\eta} P$ ;
- 9)  $\text{let pure } x = M \text{ in } x \rightarrow_{\eta} M$ ;

By default we use call-by-name evaluation strategy.

This commuting relation denotes that we identify terms of kind **let pure** if they are differ in the order of binding.

Now we will prove standard lemmas for contexts in type systems<sup>3</sup>:

<sup>3</sup>We will not prove cases with  $\rightarrow$ -constructor, they are proved standardly in the same lemmas for simply typed lambda calculus, for example see [11] [12]. We will consider only modal cases

**Lemma 3.** *Generation for  $\mathbf{K}_I$ .*

*Let  $\Gamma \vdash \mathbf{pure} M : \mathbf{K}A$ , then  $\Gamma \vdash M : A$ ;*

*Proof.* Straightforwardly.  $\square$

**Lemma 4.** *Basic lemmas.*

- *If  $\Gamma \vdash M : A$  and  $\Gamma \subseteq \Delta$ , then  $\Delta \vdash M : A$ ;*
- *If  $\Gamma \vdash M : A$ , then  $\Delta \vdash M : A$ , where  $\Delta = \{x_i : A_i \mid (x_i : A_i) \in \Gamma \ \& \ x_i \in FV(M)\}$*
- *If  $\Gamma, x : A \vdash M : B$  and  $\Gamma \vdash N : A$ , then  $\Gamma \vdash M[x := N] : B$ .*
- *If  $\Gamma \vdash M : A$ , then  $\Gamma[B := C] \vdash M : (A[B := C])$ ;*
- *If  $\Gamma \vdash M : A$  and  $\Delta$  is a permutation of  $\Gamma$ , then  $\Delta \vdash M : A$ .*

*Proof.*

1) The derivation ends in

$$\frac{\Gamma \vdash \vec{M} : \mathbf{K}\vec{A} \quad \vec{x} : \vec{A} \vdash N : B}{\Gamma \vdash \mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} N : \mathbf{K}B} \text{let}_{\mathbf{K}}$$

By IH  $\Delta \vdash \vec{M} : \mathbf{K}\vec{A}$ , so  $\Delta \vdash \mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} N : \mathbf{K}B$ .

2) Similary.

3) The derivation ends in

$$\frac{\Gamma, z : C \vdash \vec{M} : \mathbf{K}\vec{A} \quad \vec{x} : \vec{A} \vdash N : B}{\Gamma, z : C \vdash \mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} N : \mathbf{K}B} \text{let}_{\mathbf{K}}$$

Let  $\Gamma \vdash P : C$ .

By IH,  $\Gamma \vdash \vec{M}[z := P] : \mathbf{K}\vec{A}$ . So  $\Gamma \vdash \mathbf{let pure} \vec{x} = \vec{M}[z := P] \mathbf{in} N : \mathbf{K}B$

4-5) Similary.  $\square$

**Theorem 1.** *Subject reduction*

*If  $\Gamma \vdash M : A$  and  $M \rightarrow_{\beta\eta} N$ , then  $\Gamma \vdash N : A$*

*Proof.* Induction on the derivation  $\Gamma \vdash M : A$  and on the generation of  $\rightarrow_{\beta\eta}$ .

For cases with application, abstraction and pairs see [12] [13].

1) If  $\Gamma \vdash \mathbf{let pure} \vec{x}, y, \vec{z} = \vec{M}, \mathbf{let pure} \vec{w} = \vec{N} \mathbf{in} Q, \vec{P} \mathbf{in} R : \mathbf{K}B$ , then  $\Gamma \vdash \mathbf{let pure} \vec{x}, \vec{w}, \vec{z} = \vec{M}, \vec{N}, \vec{P} \mathbf{in} R[y := Q] : \mathbf{K}B$  by rule 4).

2) Let  $\Gamma \vdash \mathbf{let pure} x = M \mathbf{in} x : \mathbf{K}A$ , then  $\Gamma \vdash M : \mathbf{K}A$  by rule 9).

See [19].

3) The derivation ends in

$$\frac{\Gamma \vdash \mathbf{pure} \vec{M} : \mathbf{K}\vec{A} \quad \vec{x} : \vec{A} \vdash N : B}{\Gamma \vdash \mathbf{let pure} \vec{x} = \mathbf{pure} \vec{M} \mathbf{in} N : \mathbf{K}B}$$

So  $\Gamma \vdash \vec{M} : \vec{A}$  by Lemma 4 and  $\Gamma \vdash N[\vec{x} := \vec{M}] : B$  by Lemma 4, part 3. Then we can transform this into the following derivation:

$$\frac{\Gamma \vdash N[\vec{x} := \vec{M}] : B}{\Gamma \vdash \mathbf{pure} N[\vec{x} := \vec{M}] : \mathbf{KB}} \mathbf{K}_I$$

4) The derivation ends in

$$\frac{\vdash M : A}{\vdash \mathbf{let pure} \_ = \_ \mathbf{in} M : \mathbf{KA}}$$

So, if  $\vdash M : A$ , then  $\vdash \mathbf{pure} M : \mathbf{KA}$ .

Note that this part of the lemma works conversly too.  $\square$

**Theorem 2.**

$\rightarrow_\beta$  is strongly normalizing;

*Proof.*

We modify and apply Tait's technique of logical relation for modalities. See [13] [?].

We treat only modal cases below.

**Definition 10.** The set of strongly computable terms:

- $SC_A = \{M : A \mid M \text{ is strongly normalizing}\}$  for  $A \in \mathbb{T}$ ;
- $SC_{A \rightarrow B} = \{M : A \rightarrow B \mid \forall N \in SC_A, MN \in SC_B\}$ , for  $A, B \in \mathbb{T}_{\mathbf{K}}$  for  $A, B \in \mathbb{T}_{\mathbf{K}}$ ;
- $SC_{\mathbf{KA}} = \{M : \mathbf{KA} \mid M \text{ is strongly normalizing}\}$  for  $A \in \mathbb{T}$ ;
- $\forall i \in \{1, \dots, n\}, \prod_{i=1}^n SC_{\mathbf{KA}_i} = \{\vec{M} = (M_1, \dots, M_n) \mid \forall N \in SC_B, FV(N) = \{x_1, \dots, x_n\} \ \& \ \forall i, x_i \in SC_{A_i} \Rightarrow \mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} N \in SC_{\mathbf{KB}}\}$

**Definition 11.** A term  $M$  is neutral, if it has one of the following forms:

- $MN$ ;
- If  $M$  is neutral, then  $\mathbf{pure} M$  is neutral;
- If  $\vec{M}$  and  $N$  are neutral, then  $\mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} N$  is neutral.  $\vec{x}$  is a sequence of free variables of a term  $N$ .

**Lemma 5.**

- If  $M \in SC_A$  and  $A \in \mathbb{T}_{\mathbf{K}}$ , then  $M$  is strongly normalizing;
- If  $M \in SC_A$ ,  $A \in \mathbb{T}_{\mathbf{K}}$  and  $M \rightarrow_\beta N$ , then  $N \in SC_A$ ;
- If  $N$  is neutral,  $N \in SC_A$ . Then, if  $M \rightarrow_\beta N$ , then  $M \in SC_A$ ;

*Proof.*

By induction on the structure of  $A$ .

1)  $A \equiv \mathbf{K}A$ , where  $A \in \mathbb{T}$ .

i-ii-iii) Immediately.

2)

i) Suppose  $\vec{M} = (M_1, \dots, M_n) \in \prod_{i=1}^n SC_{\mathbf{K}A_i}$ .

Let  $N \in SC_B$ , such that  $FV(N) = \{x_1, \dots, x_n\}$  and  $\forall i, x_i \in SC_{A_i}$ .

So **let pure**  $\vec{x} = \vec{M}$  **in**  $N \in SC_{\mathbf{K}B}$  by IH.

So  $\vec{M}$  are strongly normalizing, since **let pure**  $\vec{x} = \vec{M}$  **in**  $N$  is strongly normalizing by IH.

ii) Let  $\vec{M}_1 \in \prod_{i=1}^n SC_{\mathbf{K}A_i}$  and  $\vec{M}_1 \rightarrow_\beta \vec{M}_2$ .

Let  $N \in SC_B$ , such that  $FV(N) = \{x_1, \dots, x_n\}$  and  $\forall i, x_i \in SC_{A_i}$ .

So **let pure**  $\vec{x} = \vec{M}_1$  **in**  $N \rightarrow_\beta$  **let pure**  $\vec{x} = \vec{M}_2$  **in**  $N$

and **let pure**  $\vec{x} = \vec{M}_2$  **in**  $N \in SC_{\mathbf{K}B}$  by assumption.

So  $\vec{M}_2 \in \prod_{i=1}^n SC_{\mathbf{K}A_i}$ .

iii) Let  $M_2$  be neutral,  $M_2 \in \prod_{i=1}^n SC_{\mathbf{K}A_i}$  and  $M_1 \rightarrow_\beta M_2$ .

Let  $N \in SC_B$ , such that  $FV(N) = \{x_1, \dots, x_n\}$  and  $\forall i, x_i \in SC_{A_i}$ .

So **let pure**  $\vec{x} = \vec{M}_2$  **in**  $N \in SC_{\mathbf{K}B}$ .

Thus **let pure**  $\vec{x} = \vec{M}_1$  **in**  $N \rightarrow_\beta$  **let pure**  $\vec{x} = \vec{M}_2$  **in**  $N$ .

Hence **let pure**  $\vec{x} = \vec{M}_1$  **in**  $N \in SC_{\mathbf{K}B}$  by IH, so  $\vec{M}_1 \in \prod_{i=1}^n SC_{\mathbf{K}A_i}$ . □

**Lemma 6.**

If  $M \in SC_A$ , then **pure**  $M \in SC_{\mathbf{K}A}$

*Proof.* Induction on the structure of  $M$ . □

**Lemma 7.**

Let  $x_1 : A_1, \dots, x_n : A_n \vdash M : A$ , then for all  $i, M_i \in SC_{A_i}$ . Then  $M[x_1 := M_1, \dots, x_n := M_n] \in SC_A$ .

*Proof.*

Induction on the derivation of  $x_1 : A_1, \dots, x_n : A_n \vdash M : A$ .

1) The derivation ends in:

$$\frac{x_1 : A_1, \dots, x_n : A_n \vdash M : A}{x_1 : A_1, \dots, x_n : A_n \vdash \mathbf{pure} M : \mathbf{K}A}$$

By assumption  $M[x_1 := M_1, \dots, x_n := M_n] \in SC_A$ , so **pure**  $M[x_1 := M_1, \dots, x_n := M_n] \in SC_{\mathbf{K}A}$ .

2) The derivation ends in:

$$\frac{x_1 : A_1, \dots, x_n : A_n \vdash \vec{M}' : \mathbf{K}\vec{A} \quad \vec{x} : \vec{A} \vdash N : B}{x_1 : A_1, \dots, x_n : A_n \vdash \mathbf{let pure} \vec{x} = \vec{M}' \mathbf{in} N : \mathbf{K}B}$$



By IH forall  $i \in \{1, \dots, \text{length}(\vec{M}')\}$ ,  $M'_i[x_1 := M_1, \dots, x_n := M_n] \in SC_{\mathbf{K}A_i}$ .

So **let pure**  $\vec{x} = \vec{M}'[x_1 := M_1, \dots, x_n := M_n]$  **in**  $N \in SC_{\mathbf{K}B}$ , otherwise we can build infinite reduction path in  $\vec{M}'[x_1 := M_1, \dots, x_n := M_n]$ .  $\square$

**Corollary 1.** *All terms are strongly computable, therefore are strongly normalizing.*

$\square$

**Theorem 3.**

$\rightarrow_\beta$  is confluent.

*Proof.* We modify and apply Barendregt's technique with term underlying. We will consider the grammar without constructors for pairs.

**Definition 12.** *The set of underlined terms.*

- $x \in \mathbb{V} \Rightarrow x \in \underline{\Lambda}$ ;
- $M \in \underline{\Lambda} \Rightarrow (\lambda x.M) \in \underline{\Lambda}$ ;
- $M, N \in \underline{\Lambda} \Rightarrow (MN) \in \underline{\Lambda}$ ;
- $M \in \underline{\Lambda} \Rightarrow (\text{pure } M) \in \underline{\Lambda}$ ;
- $\vec{x} \in \mathbb{V}, \vec{M}, N \in \underline{\Lambda} \Rightarrow \text{let pure } \vec{x} = \vec{M} \text{ in } N \in \underline{\Lambda}$ ;
- $M, N \in \underline{\Lambda} \Rightarrow (\lambda_i x.M)N \in \underline{\Lambda}$ , for all  $i \in \mathbb{N}$ .

**Definition 13.** *Substitution for term with labelled lambda:*

$$((\lambda_i x.M)N)[y := Z] = (\lambda_i x.M[y := Z])(N[y := Z])$$

**Definition 14.** *Index erasing*

Let us define map  $|\cdot| : \underline{\Lambda} \rightarrow \Lambda$  as follows:

- $|x| = x$ ;
- $|\lambda x.M| = \lambda x.|M|$ ;
- $|MN| = |M||N|$ ;
- $|\text{pure } M| = \text{pure } |M|$ ;
- $|\text{let pure } \vec{x} = \vec{M} \text{ in } N| = \text{let pure } \vec{x} = |\vec{M}| \text{ in } |N|$ ;
- $|(\lambda_i x.M)N| = (\lambda x.|M|)|N|$

**Definition 15.** *Reduction rules:*

- $(\lambda x.M)N \rightarrow_{\underline{\beta}} M[x := N]$ ;
- $\text{let pure } \vec{x}, y, \vec{z} = \vec{M}, \text{let pure } \vec{w} = \vec{N} \text{ in } Q, \vec{P} \text{ in } R \rightarrow_{\underline{\beta}} \text{let pure } \vec{x}, \vec{w}, \vec{z} = \vec{M}, \vec{N}, \vec{P} \text{ in } R[y := Q]$  ;
- $\text{let pure } \vec{x} = \text{pure } \vec{M} \text{ in } N \rightarrow_{\underline{\beta}} \text{pure } N[\vec{x} := \vec{M}]$ ;
- $\text{let pure } \_ = \_ \text{ in } M \rightarrow_{\underline{\beta}} \text{pure } M$

- $(\lambda x_i.M)N \rightarrow_{\underline{\beta}} M[x := N]$

$\rightarrow_{\underline{\beta}}$  is a reflexive-transitive closure of  $\rightarrow_{\beta}$ .

**Definition 16.** *Indexed redex erasing:*

Let us define the next map  $\phi : \underline{\Lambda} \rightarrow \Lambda$ :

- $\phi(x) = x$ ;
- $\phi(\lambda x.M) = \lambda x.\phi(M)$ ;
- $\phi(MN) = \phi(M)\phi(N)$ ;
- $\phi(\text{pure } M) = \text{pure } \phi(M)$ ;
- $\phi(\text{let pure } \vec{x} = \vec{M} \text{ in } N) = \text{let pure } \vec{x} = \phi(\vec{M}) \text{ in } \phi(N)$ ;
- $\phi((\lambda_i x.M)N) = \phi(M)[x := \phi(N)]$

**Example 1.**

$$\phi((\lambda_i x.\text{let pure } y = x \text{ in } x)N) = \text{let pure } y = \phi(N) \text{ in } \phi(N)$$

**Lemma 8.**  $\forall \underline{M}, \underline{N} \in \underline{\Lambda} \forall M, N \in \Lambda$ , if  $|\underline{M}| = M, |\underline{N}| = N$ , then

- If  $M \rightarrow_{\beta} N$ , then  $\underline{M} \rightarrow_{\underline{\beta}} \underline{N}$
- Vice versa

*Proof.* Induction on the generation  $\rightarrow_{\beta}$  and  $\rightarrow_{\underline{\beta}}$  correspondently. The general statement follows from transitivity of multi-step reductions of both types.  $\square$

**Lemma 9.**  $\phi(M[x := N]) = \phi(M)[x := \phi(N)]$ .

*Proof.* We treat only cases with **pure** and with **let**. For the rest cases see [15].

- 1)
 

$\phi(\text{pure } (M[x := N])) =$	By the definition of $\phi$
$\text{pure } (\phi(M[x := N])) =$	Induction hypothesis
$\text{pure } (\phi(M)[x := \phi(N)]) =$	Substitution definition
$(\text{pure } \phi(M))[x := \phi(N)]$	
- 2)
 

$\phi((\text{let pure } \vec{x} = \vec{M} \text{ in } N)[y := P]) =$	Substitution definition
$\phi(\text{let pure } \vec{x} = (\vec{M}[y := P]) \text{ in } N) =$	By the definition of $\phi$
$\text{let pure } \vec{x} = \phi(\vec{M}[y := P]) \text{ in } \phi(N) =$	Induction hypothesis
$\text{let pure } \vec{x} = (\phi(\vec{M})[y := \phi(P)]) \text{ in } \phi(N) =$	Substitution definition
$(\text{let pure } \vec{x} = \phi(\vec{M}) \text{ in } \phi(N))[y := \phi(P)]$	

$\square$

**Lemma 10.**

- If  $M \rightarrow_{\underline{\beta}} N$ , then  $\phi(M) \rightarrow_{\beta} \phi(N)$
- If  $|M| = N$  and  $\phi(M) = P$ , then  $N \rightarrow_{\beta} P$ .

*Proof.*

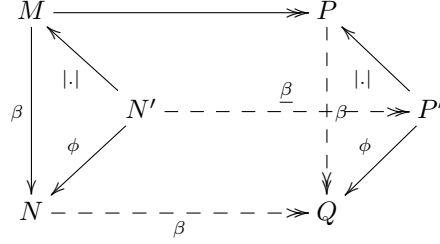
- i) Induction on the generation of  $\rightarrow_{\underline{\beta}}$  using previous lemma.
- ii) Induction on the structure of  $M$ .

$\square$

**Lemma 11.** *Strip lemma.*

If  $M \rightarrow_\beta N$  and  $M \rightarrow_\beta P$ . Then there exists some term  $Q$ , such that  $N \rightarrow_\beta Q$  and  $P \rightarrow_\beta Q$ .

*Proof.* Proof is similar to [15] [18]. We build the following diagram, which commutes by lemmas 8 and 10.



□

**Corollary 2.** If  $M \rightarrow_\beta N$  and  $M \rightarrow_\beta P$ . Then there exists some term  $Q$ , such that  $N \rightarrow_\beta Q$  and  $P \rightarrow_\beta Q$ .

*Proof.* Unfold  $M \rightarrow_\beta N$  as the sequence of one-step reductions and apply strip lemma on the every step. □

□

**Theorem 4.**

Normal form in call-by-name  $\lambda_K$  has the subformula property: if  $M$  is in normal form, then its all subterms are in normal form too.

*Proof.* By induction on the structure of  $M$ . Case with **let pure**  $\vec{x} = \vec{M}$  **in**  $N$  was considered by Kakutani [19] [20].

If **pure**  $M$  is in normal form, so  $M$  is in normal form and its subterms are in normal form too by hypothesis.

Thus if **pure**  $M$  is in normal form, then all its subterms are in normal form too. □

### 3 Categorical semantics

**Definition 17.** *Lax monoidal functor*

Let  $\langle \mathcal{C}, \otimes_1, \mathbb{1} \rangle$  and  $\langle \mathcal{D}, \otimes_2, \mathbb{1}' \rangle$  are monoidal categories.

A lax monoidal functor  $\mathcal{F} : \langle \mathcal{C}, \otimes_1, \mathbb{1} \rangle \rightarrow \langle \mathcal{D}, \otimes_2, \mathbb{1}' \rangle$  is a functor  $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$  with additional natural transformations, which satisfy the well-known conditions described in [23]:

- 1)  $u : \mathbb{1}' \rightarrow \mathcal{F}\mathbb{1}$ ;
- 2)  $*_{A,B} : \mathcal{F}A \otimes_2 \mathcal{F}B \rightarrow \mathcal{F}(A \otimes_1 B)$ .

**Definition 18.** Strong endofunctor on a monoidal category is a with a natural transformation so called tensorial strength

$$\tau_{A,B} : A \otimes \mathcal{K}B \rightarrow \mathcal{K}(A \otimes B) \quad (2)$$

making two diagrams commute:

$$\begin{array}{ccccc}
(A \otimes B) \otimes \mathcal{K}C & \xrightarrow{\tau_{A \otimes B, C}} & \mathcal{K}((A \otimes B) \otimes C) & & \\
\downarrow \alpha_{A, B, \mathcal{K}C} & & \downarrow \mathcal{K}(\alpha_{A, B, C}) & & \\
A \otimes (B \otimes \mathcal{K}C) & \xrightarrow{id_A \otimes \tau_{B, C}} & A \otimes \mathcal{K}(B \otimes C) & \xrightarrow{\tau_{A, (B \otimes C)}} & \mathcal{K}(A \otimes (B \otimes C)) \\
& & \downarrow \mu_{1, A} & & \\
& & \mathbb{1} \otimes \mathcal{K}A & \xrightarrow{\mu_{1, A}} & \mathcal{K}(\mathbb{1} \otimes A) \\
& & \searrow \rho_{\mathcal{K}A} & & \downarrow \mathcal{K}(\rho_A) \\
& & & & \mathcal{K}A
\end{array}$$

**Definition 19.** *Applicative functor*

An applicative functor is a triple  $\langle \mathcal{C}, \mathcal{K}, \eta \rangle$ , where  $\mathcal{C}$  is a monoidal category,  $\mathcal{K}$  is a strong lax monoidal endofunctor and  $\eta : Id_{\mathcal{C}} \Rightarrow \mathcal{K}$  is a natural transformation (similar to unit in monad), such that:

- 1)  $u = \eta_{\mathbb{1}}$ ;
- 2)  $*_{A, B} \circ (\eta_A \otimes \eta_B) = \eta_{A \otimes B}$ , i.e. the following diagram commutes:

$$\begin{array}{ccc}
A \otimes B & \xrightarrow{\eta_A \otimes \eta_B} & \mathcal{K}A \otimes \mathcal{K}B \\
& \searrow \eta_{A \otimes B} & \downarrow *_{A, B} \\
& & \mathcal{K}(A \otimes B)
\end{array}$$

- 3)  $\tau_{A, B} = *_{A, B} \circ \eta_A \otimes id_{\mathcal{K}B}$ .

By default we will consider some strong lax monoidal functor on cartesian closed category.

Note that lax monoidal functor on a cartesian closed category is a special case of an applicative functor [26], but we will define  $\eta$  explicitly for simplicity.

### 3.1 Soundness and completeness

**Theorem 5.** *Soundness*

Let  $\Gamma \vdash M : A$  and  $M =_{\beta\eta} N$ , then  $\llbracket \Gamma \vdash M : A \rrbracket = \llbracket \Gamma \vdash N : A \rrbracket$

*Proof.*

**Definition 20.** *Semantical translation from  $\lambda_{\mathbf{K}}$  to some cartesian closed category  $\mathcal{C}$  with applicative functor  $\mathcal{K}$ :*

- 1) *Interpretation for types:*

$\llbracket A \rrbracket := \hat{A}, A \in \mathbb{T}$ , where  $\hat{A}$  is an object of  $\mathcal{C}$  obtained by some given assignment;

$$\llbracket A \rightarrow B \rrbracket := \llbracket B \rrbracket^{\llbracket A \rrbracket};$$

$$\llbracket A \times B \rrbracket := \llbracket A \rrbracket \times \llbracket B \rrbracket.$$

- 2) *Interpretation for modal types:*  $\llbracket \mathbf{K}A \rrbracket = \mathcal{K}\llbracket A \rrbracket$ ;

- 3) *Interpretation for contexts:*

$$\llbracket \Gamma = \{x_1 : A_1, \dots, x_n : A_n\} \rrbracket := \llbracket \Gamma \rrbracket = \llbracket A_1 \rrbracket \times \dots \times \llbracket A_n \rrbracket;$$

- 4) *Interpretation for typing assignment:*  $\llbracket \Gamma \vdash M : A \rrbracket := \llbracket M \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket$ .

- 5) *Interpretation for typing rules:*

$$\frac{}{\llbracket \Gamma, x : A \vdash x : A \rrbracket = \pi_2 : \llbracket \Gamma \rrbracket \times \llbracket A \rrbracket \rightarrow \llbracket A \rrbracket}$$

$$\begin{array}{c}
\frac{\llbracket \Gamma, x : A \vdash M : B \rrbracket = f : \llbracket \Gamma \rrbracket \times \llbracket A \rrbracket \rightarrow \llbracket B \rrbracket}{\llbracket \Gamma \vdash (\lambda x.M) : A \rightarrow B \rrbracket = \Lambda(f) : \llbracket \Gamma \rrbracket \rightarrow \llbracket B \rrbracket^{\llbracket A \rrbracket}} \\
\\
\frac{\llbracket \Gamma \vdash M : A \rightarrow B \rrbracket = \llbracket M \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket B \rrbracket^{\llbracket A \rrbracket} \quad \llbracket \Gamma \vdash N : A \rrbracket = \llbracket N \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket}{\llbracket \Gamma \vdash (MN) : B \rrbracket = \llbracket \Gamma \rrbracket \xrightarrow{\langle \llbracket M \rrbracket, \llbracket N \rrbracket \rangle} \llbracket B \rrbracket^{\llbracket A \rrbracket} \times \llbracket A \rrbracket \xrightarrow{\epsilon} \llbracket B \rrbracket}} \\
\\
\frac{\llbracket \Gamma \vdash M : A \rrbracket = f : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket \quad \llbracket \Gamma \vdash N : B \rrbracket = g : \llbracket \Gamma \rrbracket \rightarrow \llbracket B \rrbracket}{\llbracket \Gamma \vdash (M, N) : A \times B \rrbracket = \langle f, g \rangle : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket \times \llbracket B \rrbracket}} \\
\\
\frac{\llbracket \Gamma \vdash p : A_1 \times A_2 \rrbracket = f : \llbracket \Gamma \rrbracket \rightarrow \llbracket A_1 \rrbracket \times \llbracket A_2 \rrbracket}{\llbracket \Gamma \vdash \pi_i p : A_i \rrbracket = \llbracket \Gamma \rrbracket \xrightarrow{f} \llbracket A_1 \rrbracket \times \llbracket A_2 \rrbracket \xrightarrow{\pi_i} \llbracket A_i \rrbracket} \quad i \in \{1, 2\} \\
\\
\frac{\llbracket \Gamma \vdash M : A \rrbracket = \llbracket M \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket}{\llbracket \Gamma \vdash \mathbf{pure} M : \mathbf{K}A \rrbracket := \llbracket \Gamma \rrbracket \xrightarrow{\llbracket M \rrbracket} \llbracket A \rrbracket \xrightarrow{\eta_{\llbracket A \rrbracket}} \mathcal{K}[\llbracket A \rrbracket]} \\
\\
\frac{\llbracket \Gamma \vdash \vec{M} : \mathbf{K}\vec{A} \rrbracket = \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle : \llbracket \Gamma \rrbracket \rightarrow \prod_{i=1}^n \mathcal{K}[\llbracket A_i \rrbracket] \quad \llbracket \vec{x} : \vec{A} \vdash N : B \rrbracket = \llbracket N \rrbracket : \prod_{i=1}^n \llbracket A_i \rrbracket \rightarrow \llbracket B \rrbracket}{\llbracket \Gamma \vdash \mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} M : \mathbf{K}B \rrbracket = \mathcal{K}[\llbracket N \rrbracket] \circ *_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle : \llbracket \Gamma \rrbracket \rightarrow \mathcal{K}[\llbracket B \rrbracket]}
\end{array}$$

**Definition 21.** *Simultaneous substitution*

Let  $\Gamma = \{x_1 : A_1, \dots, x_n : A_n\}$ ,  $\Gamma \vdash M : A$  and for all  $i \in \{1, \dots, n\}$ ,  $\Gamma \vdash M_i : A_i$ .

We define simultaneous substitution  $M[\vec{x} := \vec{M}]$  recursively by:

- 1)  $x_i[\vec{x} := \vec{M}] = M_i$ ;
- 2)  $(\lambda x.M)[\vec{x} := \vec{M}] = \lambda x.(M[\vec{x} := \vec{M}])$ ;
- 3)  $(MN)[\vec{x} := \vec{M}] = (M[\vec{x} := \vec{M}])(N[\vec{x} := \vec{M}])$ ;
- 4)  $\langle M, N \rangle = \langle (M[\vec{x} := \vec{M}]), (N[\vec{x} := \vec{M}]) \rangle$ ;
- 5)  $(\pi_i P)[\vec{x} := \vec{M}] = \pi_i(P[\vec{x} := \vec{M}])$ ;
- 6)  $(\mathbf{pure} M)[\vec{x} := \vec{M}] = \mathbf{pure} (M[\vec{x} := \vec{M}])$ ;
- 7)  $(\mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} N)[\vec{y} := \vec{P}] = \mathbf{let pure} \vec{x} = (\vec{M}[\vec{y} := \vec{P}]) \mathbf{in} N$

**Lemma 12.**

$$\llbracket M[x_1 := M_1, \dots, x_n := M_n] \rrbracket = \llbracket M \rrbracket \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle.$$

*Proof.*

$$1) \llbracket \Gamma \vdash (\mathbf{pure} M)[\vec{x} := \vec{M}] : \mathbf{K}A \rrbracket = \llbracket \Gamma \vdash \mathbf{pure} M : \mathbf{K}A \rrbracket \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle.$$

$$\begin{array}{ll}
\llbracket \Gamma \vdash (\mathbf{pure} M)[\vec{x} := \vec{M}] : \mathbf{K}A \rrbracket = \llbracket \Gamma \vdash \mathbf{pure} (M[\vec{x} := \vec{M}]) : \mathbf{K}A \rrbracket & \text{Substitution definition} \\
= \eta_{\llbracket A \rrbracket} \circ \llbracket (M[\vec{x} := \vec{M}]) \rrbracket & \text{Translation for pure} \\
= \eta_{\llbracket A \rrbracket} \circ (\llbracket M \rrbracket \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle) & \text{Induction hypothesis} \\
= (\eta_{\llbracket A \rrbracket} \circ \llbracket M \rrbracket) \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle & \text{Associativity of composition} \\
= \llbracket \Gamma \vdash \mathbf{pure} M : \mathbf{K}A \rrbracket \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle & \text{Translation for pure}
\end{array}$$

$$2) \llbracket \Gamma \vdash (\mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} N)[\vec{y} := \vec{P}] : \mathbf{K}B \rrbracket = \llbracket \Gamma \vdash \mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} N : \mathbf{K}B \rrbracket \circ \langle \llbracket P_1 \rrbracket, \dots, \llbracket P_n \rrbracket \rangle$$

$$\begin{aligned}
\llbracket \Gamma \vdash (\text{let pure } \vec{x} = \vec{M} \text{ in } N)[\vec{y} := \vec{P}] : \mathbf{KB} \rrbracket &= \text{Substitution definition} \\
\llbracket \Gamma \vdash \text{let pure } \vec{x} = (\vec{M}[\vec{y} := \vec{P}]) \text{ in } N : \mathbf{KB} \rrbracket &= \text{Interpretation for } \text{let}_{\mathbf{K}} \\
\mathcal{K}(\llbracket N \rrbracket) \circ *_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ \llbracket \Gamma \vdash (\vec{M}[\vec{y} := \vec{P}]) : \mathbf{KA} \rrbracket &= \text{Induction hypothesis} \\
\mathcal{K}(\llbracket N \rrbracket) \circ *_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ (\llbracket \vec{M} \rrbracket \circ \langle \llbracket P_1 \rrbracket, \dots, \llbracket P_n \rrbracket \rangle) &= \text{Associativity of composition} \\
(\mathcal{K}(\llbracket N \rrbracket) \circ *_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ \llbracket \vec{M} \rrbracket) \circ \langle \llbracket P_1 \rrbracket, \dots, \llbracket P_n \rrbracket \rangle &= \text{By interpretation} \\
\llbracket \Gamma \vdash \text{let pure } \vec{x} = \vec{M} \text{ in } N : \mathbf{KB} \rrbracket \circ \langle \llbracket P_1 \rrbracket, \dots, \llbracket P_n \rrbracket \rangle &
\end{aligned}$$

□

**Lemma 13.**

Let  $\Gamma \vdash M : A$  and  $M \rightarrow_{\beta\eta} N$ , then  $\llbracket \Gamma \vdash M : A \rrbracket = \llbracket \Gamma \vdash N : A \rrbracket$ ;

*Proof.*

Cases with  $\beta$ -reductions for  $\text{let}_{\mathbf{K}}$  are shown in [20]. Let us consider cases with **pure**.

$$1) \llbracket \Gamma \vdash \text{let pure } \vec{x} = \text{pure } \vec{M} \text{ in } N : \mathbf{KB} \rrbracket = \llbracket \Gamma \vdash \text{pure } N[\vec{x} := \vec{M}] : \mathbf{KB} \rrbracket$$

$$\begin{aligned}
&\llbracket \Gamma \vdash \text{let pure } \vec{x} = \text{pure } \vec{M} \text{ in } N : \mathbf{KB} \rrbracket = \\
&\quad \text{By interpretation} \\
&\mathcal{K}(\llbracket N \rrbracket) \circ *_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ \langle \eta_{\llbracket A_1 \rrbracket} \circ \llbracket M_1 \rrbracket, \dots, \eta_{\llbracket A_n \rrbracket} \circ \llbracket M_n \rrbracket \rangle = \\
&\quad \text{By the property of a pair of morphisms} \\
&\mathcal{K}(\llbracket N \rrbracket) \circ *_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ (\eta_{\llbracket A_1 \rrbracket} \times \dots \times \eta_{\llbracket A_n \rrbracket}) \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle = \\
&\quad \text{Associativity of composition} \\
&\mathcal{K}(\llbracket N \rrbracket) \circ (*_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ (\eta_{\llbracket A_1 \rrbracket} \times \dots \times \eta_{\llbracket A_n \rrbracket})) \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle = \\
&\quad \text{By the definition of an applicative functor} \\
&\mathcal{K}(\llbracket N \rrbracket) \circ \eta_{\llbracket A_1 \rrbracket \times \dots \times \llbracket A_n \rrbracket} \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle = \\
&\quad \text{Naturality of } \eta \\
&\eta_{\llbracket B \rrbracket} \circ \llbracket N \rrbracket \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle = \\
&\quad \text{Associativity of composition} \\
&\eta_{\llbracket B \rrbracket} \circ (\llbracket N \rrbracket \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle) = \\
&\quad \text{Simultaneous substitution lemma} \\
&\eta_{\llbracket B \rrbracket} \circ \llbracket N[\vec{x} := \vec{M}] \rrbracket \\
&\quad \text{By interpretation} \\
&\llbracket \Gamma \vdash \text{pure } (N[\vec{x} := \vec{M}]) : \mathbf{KB} \rrbracket
\end{aligned}$$

$$\begin{aligned}
2) \llbracket \vdash \text{let pure } \_ = \_ \text{ in } M : \mathbf{KA} \rrbracket &= \llbracket \vdash \text{pure } M : \mathbf{KA} \rrbracket \\
\llbracket \vdash \text{let pure } \_ = \_ \text{ in } M : \mathbf{KA} \rrbracket &= \text{By interpretation} \\
\mathcal{K}(\llbracket M \rrbracket) \circ u_{\mathbb{1}} &= \text{By the definition of an applicative functor} \\
\mathcal{K}(\llbracket M \rrbracket) \circ \eta_{\mathbb{1}} &= \text{By naturality of } \eta \\
\eta_{\llbracket A \rrbracket} \circ \llbracket M \rrbracket &= \text{By interpretation} \\
\llbracket \vdash \text{pure } M : \mathbf{KA} \rrbracket &
\end{aligned}$$

□

□

**Theorem 6. Completeness**

Let  $\llbracket \Gamma \vdash M : A \rrbracket = \llbracket \Gamma \vdash N : A \rrbracket$ , then  $M =_{\beta\eta} N$ .

*Proof.*

We will consider term model for simply typed lambda calculus  $\times$  and  $\rightarrow$  standardly described in [22]:

**Definition 22.** *Equivalence on term pairs:*

Let us define relation  $\sim_{A,B} \subseteq \mathbb{V} \times \Lambda_{\mathbf{K}}$ , such that:

$$(x, M) \sim_{A,B} (y, N) \Leftrightarrow x : A \vdash M : B \ \& \ y : A \vdash N : A \ \& \ M =_{\beta\eta} N[y := x];$$

We will denote equivalence class as  $[x, M]_{A,B} = \{(y, N) \mid (x, M) \sim_{A,B} (y, N)\}$  (we will drop indices below).

**Definition 23.** *Category  $\mathcal{C}(\lambda)$ :*

- $Ob_{\mathcal{C}} = \{\hat{A} \mid A \in \mathbb{T}\} \cup \{\mathbf{1}\};$
- $Hom_{\mathcal{C}(\lambda)}(\hat{A}, \hat{B}) = (\mathbb{V} \times \Lambda_{\mathbf{K}}) / \sim_{A,B};$
- Let  $[x, M] \in Hom_{\mathcal{C}(\lambda)}(\hat{A}, \hat{B})$  and  $[y, N] \in Hom_{\mathcal{C}(\lambda)}(\hat{B}, \hat{C})$ , then  $[y, M] \circ [x, M] = [x, N[y := M]];$
- Identity morphism  $id_{\hat{A}} = [x, x] \in Hom_{\mathcal{C}(\lambda)}(\hat{A}, \hat{A});$
- $\mathbf{1}$  is a terminal object;
- $\widehat{A \times B} = \hat{A} \times \hat{B};$
- Canonical projection is defined as  $[x, \pi_i x] \in Hom_{\mathcal{C}(\lambda)}(\hat{A}_1 \times \hat{A}_2, \hat{A}_i)$  for  $i \in \{1, 2\};$
- $\widehat{A \rightarrow B} = \hat{B}^{\hat{A}};$
- Evaluation arrow  $\epsilon = [x, (\pi_2 x)(\pi_1 x)] \in Hom_{\mathcal{C}(\lambda)}(\hat{B}^{\hat{A}} \times \hat{A}, \hat{B}).$

It is sufficient to show  $\mathbf{K}$  is an applicative functor on  $\mathcal{C}(\lambda)$ .

**Definition 24.** Let us define an endofunctor  $\mathcal{K} : \mathcal{C}(\lambda) \rightarrow \mathcal{C}(\lambda)$ , such that for all  $[x, M] \in Hom_{\mathcal{C}(\lambda)}(\hat{A}, \hat{B})$ ,  $\mathbf{K}([x, M]) = [y, \text{let pure } x = y \text{ in } M] \in Hom_{\mathcal{C}(\lambda)}(\mathbf{K}\hat{A}, \mathbf{K}\hat{B})$  (denotation:  $fmap$   $f$  for an arbitrary arrow  $f$ ).

**Lemma 14.** *Functoriality*

- $fmap (g \circ f) = fmap (g) \circ fmap (f);$
- $fmap (id_{\hat{A}}) = id_{\mathbf{K}\hat{A}}.$

*Proof.* Easy checking using reduction rules. □

**Definition 25.** Let us define natural transformations:

- 1)  $\eta : Id \Rightarrow \mathcal{K}$ , s. t.  $\forall \hat{A} \in Ob_{\mathcal{C}(\lambda)}, \eta_{\hat{A}} = [x, \text{pure } x] \in Hom_{\mathcal{C}(\lambda)}(\hat{A}, \mathbf{K}\hat{A});$
- 2)  $*_{A,B} : \mathbf{K}\hat{A} \times \mathbf{K}\hat{B} \rightarrow \mathbf{K}(\hat{A} \times \hat{B})$ , s. t.  $\forall \hat{A}, \hat{B} \in Ob_{\mathcal{C}(\lambda)}, *_{\hat{A}, \hat{B}} = [p, \text{let pure } x, y = \pi_1 p, \pi_2 p \text{ in } \langle x, y \rangle] \in Hom_{\mathcal{C}(\lambda)}(\mathbf{K}\hat{A} \times \mathbf{K}\hat{B}, \mathbf{K}(\hat{A} \times \hat{B})).$

Implementation for  $*$  in our term model is a modification of  $\text{let}_{\mathbf{K}}$ -rule:

$$\frac{\frac{p : \mathbf{K}A \times \mathbf{K}B \vdash p : \mathbf{K}A \times \mathbf{K}B}{p : \mathbf{K}A \times \mathbf{K}B \vdash \pi_1 p : \mathbf{K}A} \quad \frac{p : \mathbf{K}A \times \mathbf{K}B \vdash p : \mathbf{K}A \times \mathbf{K}B}{p : \mathbf{K}A \times \mathbf{K}B \vdash \pi_2 p : \mathbf{K}B} \quad \frac{x : A \vdash x : A \quad y : B \vdash y : B}{x : A, y : B \vdash \langle x, y \rangle : A \times B}}{p : \mathbf{K}A \times \mathbf{K}B \vdash \text{let pure } x, y = \pi_1 p, \pi_2 p \text{ in } \langle x, y \rangle : \mathbf{K}(A \times B)}$$

**Lemma 15.**

$\mathbf{K}$  is a monoidal endofunctor

*Proof.*

See [19]

□

**Lemma 16.** *Properties of  $\eta$ :*

- $\text{fmap } f \circ \eta_A = \eta_B \circ f$ ;
- $*_{\hat{A}, \hat{B}} \circ (\eta_A \times \eta_B) = \eta_{\hat{A} \times \hat{B}}$ ;

*Proof.*

i)  $\text{fmap } f \circ \eta_{\hat{A}} = \eta_{\hat{B}} \circ f$

$$\begin{aligned} \eta_{\hat{B}} \circ f &= && \text{By the definition} \\ [y, \mathbf{pure } y] \circ [x, M] &= && \text{By the definition of composition} \\ [x, \mathbf{pure } y[y := M]] &= && \text{By substitution} \\ [x, \mathbf{pure } M] &= && \end{aligned}$$

On the other hand:

$$\begin{aligned} \text{fmap } f \circ \eta_{\hat{A}} &= && \text{By the definition} \\ [z, \mathbf{let pure } x = z \mathbf{ in } M] \circ [x, \mathbf{pure } x] &= && \text{By the definition of composition} \\ [x, \mathbf{let pure } x = z \mathbf{ in } M[z := \mathbf{pure } x]] &= && \text{By substitution} \\ [x, \mathbf{let pure } x = \mathbf{pure } x \mathbf{ in } M] &= && \beta\text{-reduction rule} \\ [x, \mathbf{pure } M[x := x]] &= && \text{By substitution} \\ [x, \mathbf{pure } M] &= && \end{aligned}$$

ii)  $*_{\hat{A}, \hat{B}} \circ (\eta_{\hat{A}} \times \eta_{\hat{B}}) = \eta_{\hat{A} \times \hat{B}}$

$$\begin{aligned} *_{\hat{A}, \hat{B}} \circ (\eta_{\hat{A}} \times \eta_{\hat{B}}) &= \\ \text{By unfolding} & \\ [q, \mathbf{let pure } x, y = \pi_1 q, \pi_2 q \mathbf{ in } \langle x, y \rangle] \circ [p, \langle \mathbf{pure } (\pi_1 p), \mathbf{pure } (\pi_2 p) \rangle] &= \\ \text{Composition} & \\ [p, \mathbf{let pure } x, y = \pi_1 q, \pi_2 q \mathbf{ in } \langle x, y \rangle [q := \langle \mathbf{pure } (\pi_1 p), \mathbf{pure } (\pi_2 p) \rangle]] &= \\ \text{By substitution} & \\ [p, \mathbf{let pure } x, y = \pi_1 (\langle \mathbf{pure } (\pi_1 p), \mathbf{pure } (\pi_2 p) \rangle), \pi_2 (\langle \mathbf{pure } (\pi_1 p), \mathbf{pure } (\pi_2 p) \rangle) \mathbf{ in } \langle x, y \rangle] &= \\ \text{Reduction rules} & \\ [p, \mathbf{let pure } x, y = \mathbf{pure } (\pi_1 p), \mathbf{pure } (\pi_2 p) \mathbf{ in } \langle x, y \rangle] &= \\ \text{Reduction rule} & \\ [p, \mathbf{pure } (\langle x, y \rangle [x := \pi_1 p, y := \pi_2 p])] &= \\ \text{Substitution} & \\ [p, \mathbf{pure } \langle \pi_1 p, \pi_2 p \rangle] &= \\ \eta\text{-reduction} & \\ [p, \mathbf{pure } p] &= \\ \text{By definition} & \\ \eta_{\hat{A} \times \hat{B}} &= \end{aligned}$$

□

**Definition 26.**

$$u_1 = [\blacksquare, \mathbf{let pure } \_ = \_ \mathbf{ in } \blacksquare] \in \text{Hom}_{\mathcal{C}(\lambda)}(\mathbf{1}, \mathbf{K1}).$$



**Lemma 17.**

$$u_1 = \eta_1$$

*Proof.* Immediately.  $\square$

Tensorial strength is defined as follows:

**Definition 27.** *Tensorial strength*

Let  $[p, \langle \mathbf{pure}(\pi_1 p), \pi_2 p \rangle] \in \text{Hom}_{\mathcal{C}(\lambda)}(\hat{A} \times \mathbf{K}\hat{B}, \mathbf{K}\hat{A} \times \mathbf{K}\hat{B})$ .

So tensorial strength is defined as  $\tau_{\hat{A}, \hat{B}} = *_{\hat{A}, \hat{B}} \circ [p, \langle \mathbf{pure}(\pi_1 p), \pi_2 p \rangle]$ .

It is clearly that tensorial strength defined above can be simplified as follows:

$$\begin{aligned} *_{\hat{A}, \hat{B}} \circ [p, \langle \mathbf{pure}(\pi_1 p), \pi_2 p \rangle] &= && \text{By definition} \\ [p', \text{let } \mathbf{pure} \ x, y = \pi_1 p', \pi_2 p' \text{ in } \langle x, y \rangle] \circ [p, \langle \mathbf{pure}(\pi_1 p), \pi_2 p \rangle] &= && \text{By composition} \\ [p, \text{let } \mathbf{pure} \ x, y = \pi_1 p', \pi_2 p' \text{ in } \langle x, y \rangle] [p' := \langle \mathbf{pure}(\pi_1 p), \pi_2 p \rangle] &= && \text{By substitution} \\ [p, \text{let } \mathbf{pure} \ x, y = \pi_1(\langle \mathbf{pure}(\pi_1 p), \pi_2 p \rangle), \pi_2(\langle \pi_1 p, \mathbf{pure}(\pi_2 p) \rangle) \text{ in } \langle x, y \rangle] &= && \text{By } \beta\text{-reduction rules} \\ [p, \text{let } \mathbf{pure} \ x, y = \mathbf{pure}(\pi_1 p), \pi_2 p \text{ in } \langle x, y \rangle] \end{aligned}$$

**Lemma 18.** *Coherence for tensorial strength*

- $fmap \ \alpha_{\hat{A}, \hat{B}, \hat{C}} \circ \tau_{\hat{A} \times \hat{B}, \hat{C}} = \tau_{\hat{A}, \hat{B} \times \hat{C}} \circ (id_{\hat{A}} \times \tau_{\hat{B}, \hat{C}}) \circ \alpha_{\hat{A}, \hat{B}, \mathbf{K}\hat{C}};$
- $\mathbf{K}(\rho_{\hat{A}}) \circ \tau_{1, \hat{A}} = \rho_{\mathbf{K}\hat{A}}.$

where  $\alpha_{\hat{A}, \hat{B}, \hat{C}} = [p, \langle \pi_1(\pi_1 p), \langle \pi_2(\pi_1 p), \pi_2 p \rangle \rangle]$  and  $\rho = \pi_2$ .

*Proof.*

1)

$$\begin{aligned} fmap \ \alpha_{\hat{A}, \hat{B}, \hat{C}} \circ \tau_{\hat{A} \times \hat{B}, \hat{C}} &= \\ [q, \text{let } \mathbf{pure} \ z = q \text{ in } \langle \pi_1(\pi_1 z), \langle \pi_2(\pi_1 z), \pi_2 z \rangle \rangle] \circ [p, \text{let } \mathbf{pure} \ x, y = \mathbf{pure}(\pi_1 p), \pi_2 \text{ in } \langle x, y \rangle] &= \\ [p, \text{let } \mathbf{pure} \ z = q \text{ in } \langle \pi_1(\pi_1 z), \langle \pi_2(\pi_1 z), \pi_2 z \rangle \rangle] [q := \text{let } \mathbf{pure} \ x, y = \mathbf{pure}(\pi_1 p), \pi_2 \text{ in } \langle x, y \rangle] &= \\ [p, \text{let } \mathbf{pure} \ z = (\text{let } \mathbf{pure} \ x, y = \mathbf{pure}(\pi_1 p), \pi_2 \text{ in } \langle x, y \rangle) \text{ in } \langle \pi_1(\pi_1 z), \langle \pi_2(\pi_1 z), \pi_2 z \rangle \rangle] &= \\ [p, \text{let } \mathbf{pure} \ x, y = \mathbf{pure}(\pi_1 p), \pi_2 \text{ in } \langle \pi_1(\pi_1 z), \langle \pi_2(\pi_1 z), \pi_2 z \rangle \rangle] [z := x, y] &= \\ [p, \text{let } \mathbf{pure} \ x, y = \mathbf{pure}(\pi_1 p), \pi_2 \text{ in } \langle \pi_1 x, \langle \pi_2 x, y \rangle \rangle] \end{aligned}$$

$\square$

**Lemma 19.**  $\mathbf{K}$  is an applicative functor

*Proof.* Immediately follows from previous lemmas in the section.  $\square$

Similar to [24], we apply the translation from  $\lambda_{\mathbf{K}}$  to some cartesian closed category with an arbitrary applicative functor  $\mathcal{K}$ , then we have  $\llbracket \Gamma \vdash M : A \rrbracket = [x, M[x_i := \pi_i x]]$ , so  $M =_{\beta\eta} N \Leftrightarrow \llbracket \Gamma \vdash M : A \rrbracket = \llbracket \Gamma \vdash N : A \rrbracket$ .  $\square$

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