

Modal type theory based on the intuitionistic epistemic logic

Abstract

Modal intuitionistic epistemic logic IEL^- was proposed by S.Artemov and T. Protopopescu as the formal foundation for the intuitionistic theory of knowledge. We construct a modal simply typed lambda-calculus which is Curry-Howard isomorphic to IEL^- as formal theory of calculations with applicative functors in functional programming languages like Haskell or Idris. We prove that this typed lambda-calculus has the strong normalization and Church-Rosser properties.

1 Introduction

Modal intuitionistic epistemic logic IEL was proposed by S. Artemov and T. Protopopescu [1]. IEL provides the epistimology and the theory of knowledge as based on BHK-semantics of intuitionistic logic. IEL^- is a variant of IEL , that corresponds to intuitionistic belief. Informally, $\mathbf{K}A$ denotes that A is verified intuitionistically.

Intuitionistic epistemic logic IEL^- is defined with by following axioms and derivation rules:

Definition 1. *Intuitionistic epistemic logic IEL :*

- 1) *IPC axioms;*
 - 2) $\mathbf{K}(A \rightarrow B) \rightarrow (\mathbf{K}A \rightarrow \mathbf{K}B)$ (*normality*);
 - 3) $A \rightarrow \mathbf{K}A$ (*co-reflection*);
- Rule: MP.*

We have the deduction theorem and necessitation rule which is derivable.

V. Krupski and A. Yatmanov provided the sequential calculus for IEL and proved that this calculus is PSPACE-complete [2].

It's not difficult to see that modal axioms in IEL^- and types of the methods of Applicative class in Haskell-like languages (which is described below) are syntactically similar and we are going to show that this coincidence has a non-trivial computational meaning.

Functional programming languages such as Haskell [3], Idris [4], Purescript [5] or Elm [6] have special type classes¹ for calculations with container types like `Functor` and `Applicative`²:

¹Type class in Haskell is a general interface for special group of datatypes.

²Reader may read more about container types in the Haskell standard library documentation[7] or in the next one textbook [8]

```

class Functor f where
  fmap :: (a -> b) -> f a -> f b

class Functor f => Applicative f where
  pure :: a -> f a
  (<*>) :: f (a -> b) -> f a -> f b

```

By *container* (or *computational context*) type we mean some type-operator f , where f is a “function” from $*$ to $*$: type operator takes a simple type (which has kind $*$) and returns another simple type type with kind $*$. For more detailed description of the type system with kinds used in Haskell see [12].

The main goal of our research is a relationship between intuitionistic epistemic logic IEL^- and functional programming with effects. We show that relationship by building the type system (which is called $\lambda_{\mathbf{K}}$) which is Curry-Howard isomorphic to IEL^- . So we will consider \mathbf{K} -modality as an arbitrary applicative functor.

$\lambda_{\mathbf{K}}$ consists of the rules for simply typed lambda-calculus and special typing rules for lifting types into the applicative functor \mathbf{K} . We assume that our type system will axiomatize the simplest case of computation with effects with one container. We provide proof-theoretical view on this kind of computations in functional programming and prove strong normalization and confluence.

2 Typed lambda-calculus based on IEL^-

At first we define the natural deduction for IEL^- with \mathbf{K} -modality and binary connectives \rightarrow and \wedge (we call that calculus $NIEL_{\wedge, \rightarrow}^-$):

Definition 2. *Natural deduction $NIEL_{\wedge, \rightarrow}^-$ for IEL^- with \rightarrow and \wedge :*

$$\begin{array}{c}
\frac{}{\Gamma, A \vdash A} \text{ax} \\
\\
\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \rightarrow_i \qquad \frac{\Gamma \vdash A \rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B} \rightarrow_i \\
\\
\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge_i \qquad \frac{\Gamma \vdash A_1 \wedge A_2}{\Gamma \vdash A_i} \wedge_e, i \in \{1, 2\} \\
\\
\frac{\Gamma \vdash A}{\Gamma \vdash \mathbf{K}A} \mathbf{K}_I \qquad \frac{\Gamma \vdash \mathbf{K}\vec{A} \quad \vec{A} \vdash B}{\Gamma \vdash \mathbf{K}B}
\end{array}$$

Where $\Gamma \vdash \mathbf{K}\vec{A}$ is a syntax sugar for $\Gamma \vdash \mathbf{K}A_1, \dots, \Gamma \vdash \mathbf{K}A_n$.

Lemma 1. $\Gamma \vdash_{NIEL_{\wedge, \rightarrow}^-} A \Rightarrow IEL^- \vdash \bigwedge \Gamma \rightarrow A$.

Proof. Induction on the derivation.

Let us consider cases with modality.

1) If $\Gamma \vdash_{NIEL_{\wedge, \rightarrow}^-} A$, then $IEL^- \vdash \bigwedge \Gamma \rightarrow \mathbf{K}A$.

- (1) $\bigwedge \Gamma \rightarrow A$ assumption
- (2) $A \rightarrow \mathbf{K}A$ co-reflection
- (3) $(\bigwedge \Gamma \rightarrow A) \rightarrow ((A \rightarrow \mathbf{K}A) \rightarrow (\bigwedge \Gamma \rightarrow \mathbf{K}A))$ IPC theorem
- (4) $(A \rightarrow \mathbf{K}A) \rightarrow (\bigwedge \Gamma \rightarrow \mathbf{K}A)$ from (1), (3) and MP
- (5) $\bigwedge \Gamma \rightarrow \mathbf{K}A$ from (2), (4) and MP

2) If $\Gamma \vdash_{NIEL_{\wedge, \rightarrow}^-} \mathbf{K}\vec{A}$ and $\vec{A} \vdash B$, then $IEL^- \vdash \bigwedge \Gamma \rightarrow \mathbf{K}B$.

- (1) $\bigwedge \Gamma \rightarrow \bigwedge_{i=1}^n \mathbf{K}A_i$ assumption
- (2) $\bigwedge_{i=1}^n \mathbf{K}A_i \rightarrow \mathbf{K} \bigwedge_{i=1}^n A_i$ IEL theorem
- (3) $\bigwedge \Gamma \rightarrow \mathbf{K} \bigwedge_{i=1}^n A_i$ from (1), (2) and transitivity
- (4) $\bigwedge_{i=1}^n A_i \rightarrow B$ assumption
- (5) $(\bigwedge_{i=1}^n A_i \rightarrow B) \rightarrow \mathbf{K}(\bigwedge_{i=1}^n A_i \rightarrow B)$ co-reflection
- (6) $\mathbf{K}(\bigwedge_{i=1}^n A_i \rightarrow B)$ from (2), (3) and MP
- (7) $\mathbf{K} \bigwedge_{i=1}^n A_i \rightarrow \mathbf{K}B$ from (6) and normality
- (8) $\bigwedge \Gamma \rightarrow \mathbf{K}B$ from (3), (7) and transitivity

□

Lemma 2. *If $IEL^- \vdash A$, then $NIEL^- \vdash A$.*

Proof. Straightforward derivation of modal axioms in $NIEL^-$. We consider this derivation below using terms. □

It is clearly that these lemmas could be extended for IEL^- with \vee and \neg similary.

At the next step we build the typed lambda-calculus based on $NIEL_{\wedge, \rightarrow}^-$ by proof-assingment in rules.

At first, we define lambda-terms and types for this lambda-calculus.

Definition 3. *The set of terms:*

Let \mathbb{V} be the set of variables. The set $\Lambda_{\mathbf{K}}$ of terms is defined by the grammar:

$$\Lambda_{\mathbf{K}} ::= \mathbb{V} \mid (\lambda \Lambda. \Lambda_{\mathbf{K}}) \mid (\Lambda_{\mathbf{K}} \Lambda_{\mathbf{K}}) \mid (\Lambda_{\mathbf{K}}, \Lambda_{\mathbf{K}}) \mid (\pi_1 \Lambda_{\mathbf{K}}) \mid (\pi_2 \Lambda_{\mathbf{K}}) \mid (\text{pure } \Lambda_{\mathbf{K}}) \mid (\text{let pure } \Lambda_{\mathbf{K}} = \Lambda_{\mathbf{K}} \text{ in } \Lambda_{\mathbf{K}})$$

Definition 4. *The set of types:*

Let \mathbb{T} be the set of atomic types. The set $\mathbb{T}_{\mathbf{K}}$ of types with applicative functor \mathbf{K} is generated by the grammar:

$$\mathbb{T}_{\mathbf{K}} ::= \mathbb{T} \mid (\mathbb{T}_{\mathbf{K}} \rightarrow \mathbb{T}_{\mathbf{K}}) \mid (\mathbb{T}_{\mathbf{K}} \times \mathbb{T}_{\mathbf{K}}) \mid (\mathbf{K}\mathbb{T}_{\mathbf{K}}) \quad (1)$$

Context, domain of context and range of context are defined standardly [11][12].

Our type system is based on the Curry-style typing rules:

Definition 5. *Modal typed lambda calculus $\lambda_{\mathbf{K}}$ based on $NIEL_{\wedge, \rightarrow}^-$:*

$$\overline{\Gamma, x : A \vdash x : A}^{ax}$$

$$\begin{array}{c} \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x. M : A \rightarrow B} \rightarrow_i \qquad \frac{\Gamma \vdash f : A \rightarrow B \quad \Gamma \vdash x : A}{\Gamma \vdash fx : B} \rightarrow_e \\[10pt] \frac{\Gamma \vdash M : A \quad \Gamma \vdash N : B}{\Gamma \vdash \langle x, y \rangle : A \times B} \times_i \qquad \frac{\Gamma \vdash M : A_1 \times A_2}{\Gamma \vdash \pi_i M : A_i} \times_e, i \in \{1, 2\} \\[10pt] \frac{\Gamma \vdash x : A}{\Gamma \vdash \mathbf{pure} \ x : \mathbf{KA}} \mathbf{K}_I \qquad \frac{\Gamma \vdash \vec{M} : \mathbf{KA} \quad \vec{x} : \vec{A} \vdash M : B}{\Gamma \vdash \mathbf{let} \ \mathbf{pure} \ \vec{x} = \vec{M} \ \mathbf{in} \ M : \mathbf{KB}} \mathbf{let}_{\mathbf{K}} \end{array}$$

\mathbf{K}_I -typing rule is the same as \bigcirc -introduction in lax logic (also known as monadic metalanguage [17]) and in typed lambda-calculus which is derived by proof-assignment for lax-logic proofs. \mathbf{K}_I allows to inject an object of type α into the functor. \mathbf{K}_I reflects the Haskell method **pure** for Applicative class. It plays the same role as the **return** method in Monad class.

$\mathbf{let}_{\mathbf{K}}$ is the same as the \square -rule in typed lambda calculus for intuitionistic normal modal logic \mathbf{IK} , which is described in [19].

In fact, our calculus is the extension of typed lambda calculus for \mathbf{IK} with typing rule appropriate to co-reflection.

Here are some examples of derivation trees.

$$\begin{array}{c} \frac{\frac{x : A \vdash x : A}{x : A \vdash \mathbf{pure} \ x : \mathbf{KA}} \mathbf{K}_I}{\vdash (\lambda x. \mathbf{pure} \ x) : A \rightarrow \mathbf{KA}} \rightarrow_i \\[20pt] \frac{\frac{f : \mathbf{K}(A \rightarrow B) \vdash f : \mathbf{K}(A \rightarrow B) \quad x : \mathbf{KA} \vdash x : \mathbf{KA} \quad \frac{g : A \rightarrow B \quad y : A}{g : A \rightarrow B, y : A \vdash gy : B}}{f : \mathbf{K}(A \rightarrow B), x : \mathbf{KA} \vdash \mathbf{let} \ \mathbf{pure} \ \langle g, y \rangle = \langle f, x \rangle \ \mathbf{in} \ gy : \mathbf{KB}}}{f : \mathbf{K}(A \rightarrow B) \vdash \lambda x. \mathbf{let} \ \mathbf{pure} \ \langle g, y \rangle = \langle f, x \rangle \ \mathbf{in} \ gy : \mathbf{KA} \rightarrow \mathbf{KB}}}{\vdash \lambda f. \lambda x. \mathbf{let} \ \mathbf{pure} \ \langle g, y \rangle = \langle f, x \rangle \ \mathbf{in} \ gy : \mathbf{K}(A \rightarrow B) \rightarrow \mathbf{KA} \rightarrow \mathbf{KB}} \\[20pt] \frac{\frac{f : A \rightarrow B \vdash f : A \rightarrow B}{f : A \rightarrow B \vdash \mathbf{pure} \ f : \mathbf{K}(A \rightarrow B)} \quad x : \mathbf{KA} \vdash x : \mathbf{KA} \quad \frac{g : A \rightarrow B \quad y : A}{g : A \rightarrow B, y : A \vdash gy : B}}{f : A \rightarrow B, x : \mathbf{KA} \vdash \mathbf{let} \ \mathbf{pure} \ \langle g, y \rangle = \langle \mathbf{pure} \ f, x \rangle \ \mathbf{in} \ gy : \mathbf{KB}}}{f : A \rightarrow B \vdash \lambda x. \mathbf{let} \ \mathbf{pure} \ \langle g, y \rangle = \langle \mathbf{pure} \ f, x \rangle \ \mathbf{in} \ gy : \mathbf{KA} \rightarrow \mathbf{KB}}}{\lambda f. \lambda x. \mathbf{let} \ \mathbf{pure} \ \langle g, y \rangle = \langle \mathbf{pure} \ f, x \rangle \ \mathbf{in} \ gy : (A \rightarrow B) \rightarrow \mathbf{KA} \rightarrow \mathbf{KB}} \end{array}$$

Now we define free variables and substitutions. β -reduction, multi-step β -reduction and β -equality are defined standardly:

Definition 6. Set $FV(M)$ of free variables for arbitrary term M :

- 1) $FV(x) = \{x\}$;
- 2) $FV(\lambda x. M) = FV(M) \setminus \{x\}$;

- 3) $FV(MN) = FV(M) \cup FV(N)$;
- 4) $FV(\langle M, N \rangle) = FV(M) \cup FV(N)$;
- 5) $FV(\pi_i M) \subseteq FV(M)$, $i \in \{1, 2\}$;
- 6) $FV(\text{pure } M) = FV(M)$;
- 7) $FV(\text{let pure } \vec{N} = \vec{M} \text{ in } M) = \bigcup_{i=1}^n FV(M)$, where $n = |\vec{M}|$.

Definition 7. *Substitution:*

- 1) $x[x := N] = N$, $x[y := N] = x$;
- 2) $(MN)[x := N] = M[x := N]N[x := N]$;
- 3) $(\lambda x.M)[x := N] = \lambda x.M[x := N]$;
- 4) $(M, N)[x := P] = (M[x := P], N[x := P])$;
- 5) $(\pi_i M)[x := P] = \pi_i(M[x := P])$, $i \in \{1, 2\}$;
- 6) $(\text{pure } M)[x := P] = \text{pure } (M[x := P])$;
- 7) $(\text{let pure } \vec{x} = \vec{M} \text{ in } M)[y := P] = \text{let pure } \vec{x} = (\vec{M}[y := P]) \text{ in } M$.

Definition 8. β -reduction and η -reduction rules for $\lambda\mathbf{K}$.

- 1) $(\lambda x.M)N \rightarrow_\beta M[x := N]$;
- 2) $\pi_1 \langle M, N \rangle \rightarrow_\beta M$;
- 3) $\pi_2 \langle M, N \rangle \rightarrow_\beta N$;
- 4) $\text{let pure } \langle \vec{x}, y, \vec{z} \rangle = \langle \vec{M}, \text{let pure } \vec{w} = \vec{N} \text{ in } Q, \vec{P} \rangle \text{ in } R \rightarrow_\beta \text{let pure } \langle \vec{x}, \vec{w}, \vec{z} \rangle = \langle \vec{M}, \vec{N}, \vec{P} \rangle \text{ in } R[y := Q]$
- 5) $\text{let pure } \vec{x} = \text{pure } \vec{M} \text{ in } N \rightarrow_\beta \text{pure } N[\vec{x} := \vec{M}]$
- 6) $\lambda x.f x \rightarrow_\eta f$;
- 7) $\langle \pi_1 P, \pi_2 P \rangle \rightarrow_\eta P$;
- 8) $\text{let pure } _ = _ \text{ in } N \rightarrow_\eta \text{pure } N$;
- 9) $\text{let pure } x = M \text{ in } x \rightarrow_\eta M$;
- 10) $M \rightarrow_{\beta\eta} N \Rightarrow \text{pure } \mathbf{M} \rightarrow_{\beta\eta} \text{pure } \mathbf{N}$

3 Basic lemmas

Now we will prove standard lemmas for contexts in type systems³:

Lemma 3. *Generation lemma.*

- i) Let $\Gamma \vdash \text{pure } M : \mathbf{K}A$, then $\Gamma \vdash M : A$;
- ii) Let $\Gamma \vdash \text{let pure } \vec{x} = \vec{M} \text{ in } N : \mathbf{K}B$, there are some $A_1, \dots, A_n \in \mathbb{T}_{\mathbf{K}}$, such that $\Gamma \vdash \vec{M} : \mathbf{K}\vec{A}$ and $\vec{x} : \vec{A} \vdash N : B$.

Proof.

Induction on $\Gamma \vdash \text{pure } M : \mathbf{K}A$ and $\Gamma \vdash \text{let pure } \vec{x} = \vec{M} \text{ in } N : \mathbf{K}B$ correspondently. \square

Lemma 4. *Weakening.*

Let $\Gamma \vdash M : A$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash M : A$.

³We will not prove cases with \rightarrow -constructor, they are proved standardly in the same lemmas for simply typed lambda calculus, for example see [11][12][14]. We will consider only modal cases

Proof.

- 1) Let $\Gamma, x : A \vdash x : A$ and $\Gamma \subseteq \Delta$, then $\Delta, x : A \vdash x : A$ trivially.
- 2) Let $\Gamma \vdash \mathbf{pure} M : \mathbf{KA}$. Then $\Gamma \vdash M : A$ by generation and $\Delta \vdash M : A$ by assumption. So $\Delta \vdash \mathbf{pure} M : \mathbf{KA}$ by \mathbf{K}_I .
- 3) Let $\Gamma \vdash \mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} N : \mathbf{KB}$ and $\Gamma \subseteq \Delta$. Then $\Gamma \vdash \vec{M} : \mathbf{KA}$ and $\vec{x} : \vec{A} \vdash N : B$.
By assumption $\Delta \vdash \vec{M} : \mathbf{KA}$. So $\Delta \vdash \mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} N : \mathbf{KB}$ by $\mathbf{let_K}$. \square

Definition 9. *Type substitution*

The substitution of type C for type variable B in type A inductively defined as follows:

- 1) $B[B := C] = B$ and $D[B := C] = D$, if $B \neq D$;
- 2) $(A_1 \alpha A_2)[B := C] = (A_1[B := C])\alpha(A_2[B := C])$, where $\alpha \in \{\rightarrow, \times\}$;
- 3) $(\mathbf{KA})[B := C] = \mathbf{K}(A[B := C])$.
- 4) Let Γ be the context, then $\Gamma[B := C] = \{x : (A[B := C]) \mid x : A \in \Gamma\}$

Lemma 5. *Substitution lemma.*

- i) Let $\Gamma, x : A \vdash M : B$ and $\Gamma \vdash N : A$, then $\Gamma \vdash M[x := N] : B$.
- ii) Let $\Gamma \vdash M : A$, then $\Gamma[B := C] \vdash M : (A[B := C])$.

Proof.

i) For term substitution:

- 1) Let $\Gamma, x : A \vdash x : A$ and $\Gamma \vdash N : A$, but $x[x := N] = N$, so $\Gamma \vdash N : A$.
- 2) Let $\Gamma, x : A \vdash \mathbf{pure} M : \mathbf{KB}$ and $\Gamma \vdash N : A$.
By generation $\Gamma, x : A \vdash M : B$ and by assumption $\Gamma \vdash M[x := N] : B$.
By \mathbf{K}_I , $\Gamma \vdash \mathbf{pure} (M[x := N]) : \mathbf{KB}$.
- 3) Let $\Gamma, y : A \vdash \mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} N : \mathbf{KB}$ and $\Gamma \vdash N : A$.
By generation, $\Gamma, y : A \vdash \vec{M} : \mathbf{KA}$ and $\vec{x} : \vec{A} \vdash N : B$.
By hypothesis, $\Gamma \vdash \vec{M}[x := N] : \mathbf{KA}$.
Hence $\Gamma \vdash \mathbf{let pure} \vec{x} = \vec{M}[x := N] \mathbf{in} N : \mathbf{KB}$.

ii) For type substitution

- 1) Let $\Gamma, x : A \vdash x : A$, so $\Gamma[A := C], x : (A[A := C]) \vdash x : (A[A := C])$, or $\Gamma[A := C], x : C \vdash x : C$.
- 2) Let $\Gamma \vdash \mathbf{pure} M : \mathbf{KA}$. By generation $\Gamma \vdash M : A$ and by assumption $\Gamma[B := C] \vdash M : A[B := C]$.
By \mathbf{K}_I $\Gamma \vdash \mathbf{pure} M : \mathbf{K}(A[B := C])$.
- 3) $\Gamma \vdash \mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} N : \mathbf{KB}$. By generation $\Gamma \vdash \vec{M} : \mathbf{KA}$ and $\vec{x} : \vec{A} \vdash N : B$.
By assumption $\Gamma[B_1 := C] \vdash \vec{M} : \mathbf{K}\vec{A}[B_1 := C]$ and $\vec{x} : \vec{A}[B_1 := C] \vdash N : B[B_1 := C]$.
So by $\mathbf{let_K}$, $\Gamma[B_1 := C] \vdash \mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} N : \mathbf{K}(B[B_1 := C])$. \square

Theorem 1. *Subject reduction*

Let $\Gamma \vdash M : A$ and $M \rightarrow_{\beta\eta} N$, then $\Gamma \vdash N : A$

For cases with application, abstraction and pairs see [12] [13].

1) Let $\Gamma \vdash \mathbf{let\ pure} \langle \vec{x}, y, \vec{z} \rangle = \langle \vec{M}, \mathbf{let\ pure} \vec{w} = \vec{N} \text{ in } Q, \vec{P} \rangle \text{ in } R : \mathbf{KB}$,
then $\Gamma \vdash \mathbf{let\ pure} \langle \vec{x}, \vec{w}, \vec{z} \rangle = \langle \vec{M}, \vec{N}, \vec{P} \rangle \text{ in } R[y := Q] : \mathbf{KB}$

2) Let $\Gamma \vdash \mathbf{let\ pure} x = M \text{ in } x : \mathbf{KA}$, then $\Gamma \vdash M : \mathbf{KA}$.

See [19].

3) Let $\Gamma \vdash \mathbf{let\ pure} \vec{x} = \mathbf{pure} \vec{M} \text{ in } N : \mathbf{KB}$.

By generation $\Gamma \vdash \mathbf{pure} \vec{M} : \mathbf{KA}$ and $\vec{x} : \vec{A} \vdash N : B$.

Moreover, $\Gamma \vdash \vec{M} : \vec{A}$. By weakening and substitution lemma $\Gamma \vdash N[\vec{x} = \vec{M}] : B$.

By \mathbf{K}_I , $\Gamma \vdash \mathbf{pure} N[\vec{x} := \vec{M}] : \mathbf{KB}$.

4) Let $\vdash \mathbf{let\ pure} _ = _ \text{ in } N : \mathbf{KA}$

By generation $\vdash N : A$.

So $\vdash \mathbf{pure} N : \mathbf{KA}$ by \mathbf{K}_I .

5) Let $\Gamma \vdash \mathbf{pure} M : A$ and $M \rightarrow_{\beta\eta} N$.

By generation $\Gamma \vdash M : A$ and $\Gamma \vdash N : A$ by assumption.

So $\Gamma \vdash \mathbf{pure} N : \mathbf{KA}$.

4 Strong normalization

5 Confluence

6 Categorical semantics

Definition 10. *Lax monoidal functor*

Let $\langle \mathcal{C}, \otimes_1, \mathbb{1} \rangle$ and $\langle \mathcal{D}, \otimes_2, \mathbb{1}' \rangle$ are monoidal categories.

A lax monoidal functor $\mathcal{F} : \langle \mathcal{C}, \otimes_1, \mathbb{1} \rangle \rightarrow \langle \mathcal{D}, \otimes_2, \mathbb{1}' \rangle$ is a functor $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ with additional natural transformations:

1) $u : \mathbb{1}' \rightarrow \mathcal{F}\mathbb{1}$;

2) $*_{A,B} : \mathcal{F}A \otimes_2 \mathcal{F}B \rightarrow \mathcal{F}(A \otimes_1 B)$

Definition 11. *Applicative functor*

An applicative functor is a triple $\langle \mathcal{C}, \mathcal{K}, \eta \rangle$, where \mathcal{C} is a symmetric monoidal category, \mathcal{K} is a lax monoidal endofunctor and η is a natural transformation, such that:

1) $u = \eta_{\mathbb{1}}$;

2) $*_{A,B} \circ (\eta_A \otimes \eta_B) = \eta_{A \otimes B}$;

3) Weak commutativity condition holds:

$$A \otimes \mathcal{K}B \quad \mathcal{K}A \otimes \mathcal{K}B \quad \mathcal{K}(A \otimes B)$$

$$\mathcal{K}B \otimes A \quad \mathcal{K}B \otimes \mathcal{K}A \quad \mathcal{K}(B \otimes A)$$

By default we will consider an arbitrary closed functor on some cartesian closed category, which is the special case of an applicative functor.

We identify terminal objects. So $\mathcal{K}(\mathbb{1}) = \mathbb{1}$ and $\eta_{\mathbb{1}} = id_{\mathbb{1}}$ since \mathcal{K} is an endofunctor.

6.1 Soundness

Theorem 2. Soundness

Let $\Gamma \vdash M : A$ and $M =_{\beta\eta} N$, then $\llbracket \Gamma \vdash M : A \rrbracket = \llbracket \Gamma \vdash N : A \rrbracket$

Proof.

Definition 12. Semantical translation from λ_K to CCC with applicative functor \mathcal{K} :

1) Interpretation for types:

$\llbracket A \rrbracket := \hat{A}, A \in \mathbb{T};$

$\llbracket A \rightarrow B \rrbracket := \llbracket A \rrbracket \rightarrow \llbracket B \rrbracket;$

$\llbracket A \times B \rrbracket := \llbracket A \rrbracket \times \llbracket B \rrbracket.$

2) Interpretation for modal types: $\llbracket \mathbf{K}A \rrbracket = \mathcal{K}\llbracket A \rrbracket;$

3) Interpretation for contexts:

$\llbracket \Gamma = \{x_1 : A_1, \dots, x_n : A_n\} \rrbracket := \llbracket \Gamma \rrbracket = \llbracket A_1 \rrbracket \times \dots \times \llbracket A_n \rrbracket;$

4) Interpretation for typing assignment: $\llbracket \Gamma \vdash M : A \rrbracket := \llbracket M \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket.$

5) Interpretation for typing rules:

$$\begin{array}{c}
\frac{}{\llbracket \Gamma, x : A \vdash x : A \rrbracket = \pi_2 : \llbracket \Gamma \rrbracket \times \llbracket A \rrbracket \rightarrow \llbracket A \rrbracket} \\
\frac{\llbracket \Gamma, x : A \vdash M : B \rrbracket = f : \llbracket \Gamma \rrbracket \times \llbracket A \rrbracket \rightarrow \llbracket B \rrbracket}{\llbracket \Gamma \vdash (\lambda x.M) : A \rightarrow B \rrbracket = \Lambda(f) : \llbracket \Gamma \rrbracket \rightarrow \llbracket B \rrbracket^{\llbracket A \rrbracket}} \\
\frac{\llbracket \Gamma \vdash M : A \rightarrow B \rrbracket = \llbracket M \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket B \rrbracket^{\llbracket A \rrbracket} \quad \llbracket \Gamma \vdash N : A \rrbracket = \llbracket N \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket}{\llbracket \Gamma \vdash (MN) : B \rrbracket = \llbracket \Gamma \rrbracket \xrightarrow{\langle \llbracket M \rrbracket, \llbracket N \rrbracket \rangle} \llbracket B \rrbracket^{\llbracket A \rrbracket} \times \llbracket A \rrbracket \xrightarrow{\epsilon} \llbracket B \rrbracket} \\
\frac{\llbracket \Gamma \vdash M : A \rrbracket = f : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket \quad \llbracket \Gamma \vdash N : B \rrbracket = g : \llbracket \Gamma \rrbracket \rightarrow \llbracket B \rrbracket}{\llbracket \Gamma \vdash (M, N) : A \times B \rrbracket = \langle f, g \rangle : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket \times \llbracket B \rrbracket} \\
\frac{\llbracket \Gamma \vdash p : A_1 \times A_2 \rrbracket = f : \llbracket \Gamma \rrbracket \rightarrow \llbracket A_1 \rrbracket \times \llbracket A_2 \rrbracket}{\llbracket \Gamma \vdash \pi_i p : A_i \rrbracket = \llbracket \Gamma \rrbracket \xrightarrow{f} \llbracket A_1 \rrbracket \times \llbracket A_2 \rrbracket \xrightarrow{\pi_i} \llbracket A_i \rrbracket} \quad i \in \{1, 2\} \\
\frac{\llbracket \Gamma \vdash M : A \rrbracket = \llbracket M \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket}{\llbracket \Gamma \vdash \mathbf{pure} M : \mathbf{K}A \rrbracket := \llbracket \Gamma \rrbracket \xrightarrow{\llbracket M \rrbracket} \llbracket A \rrbracket \xrightarrow{\eta_{\llbracket A \rrbracket}} \mathcal{K}\llbracket A \rrbracket} \\
\frac{\llbracket \Gamma \vdash \vec{M} : \mathbf{K}\vec{A} \rrbracket = \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle : \llbracket \Gamma \rrbracket \rightarrow \prod_{i=1}^n \mathcal{K}\llbracket A_i \rrbracket \quad \llbracket \vec{x} : \vec{A} \vdash N : B \rrbracket = \llbracket N \rrbracket : \prod_{i=1}^n \llbracket A_i \rrbracket \rightarrow \llbracket B \rrbracket}{\llbracket \Gamma \vdash \mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} M : \mathbf{K}B \rrbracket = \mathcal{K}(\llbracket N \rrbracket) \circ *_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle : \llbracket \Gamma \rrbracket \rightarrow \mathcal{K}\llbracket B \rrbracket}
\end{array}$$

Definition 13. Simultaneous substitution

Let $\Gamma = \{x_1 : A_1, \dots, x_n : A_n\}$, $\Gamma \vdash M : A$ and for all $i \in \{1, \dots, n\}$, $\Gamma \vdash M_i : A_i$.

We define simultaneous substitution $M[\vec{x} := \vec{M}]$ recursively by:

1) $x_i[\vec{x} := \vec{M}] = M_i;$

2) $(\lambda x.M)[\vec{x} := \vec{M}] = \lambda x.(M[\vec{x} := \vec{M}]);$

3) $(MN)[\vec{x} := \vec{M}] = (M[\vec{x} := \vec{M}])(N[\vec{x} := \vec{M}]);$

4) $\langle M, N \rangle = \langle (M[\vec{x} := \vec{M}]), (N[\vec{x} := \vec{M}]) \rangle;$

- 5) $(\pi_i P)[\vec{x} := \vec{M}] = \pi_i(P[\vec{x} = \vec{M}]);$
- 6) $(\mathbf{pure} M)[\vec{x} := \vec{M}] = \mathbf{pure} (M[\vec{x} = \vec{M}]);$
- 7) $(\mathbf{let} \mathbf{pure} \vec{x} = \vec{M} \mathbf{in} N)[\vec{y} := \vec{P}] = \mathbf{let} \mathbf{pure} \vec{x} = (\vec{M}[\vec{y} := \vec{P}]) \mathbf{in} N$

Lemma 6.

$$\llbracket M[x_1 := M_1, \dots, x_n := M_n] \rrbracket = \llbracket M \rrbracket \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle.$$

Proof.

$$1) \llbracket \Gamma \vdash (\mathbf{pure} M)[\vec{x} := \vec{M}] : \mathbf{KA} \rrbracket = \llbracket \Gamma \vdash \mathbf{pure} M : \mathbf{KA} \rrbracket \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle.$$

$$\begin{aligned} \llbracket \Gamma \vdash (\mathbf{pure} M)[\vec{x} := \vec{M}] : \mathbf{KA} \rrbracket &= \llbracket \Gamma \vdash \mathbf{pure} (M[\vec{x} := \vec{M}]) : \mathbf{KA} \rrbracket && \text{Substitution definition} \\ &= \eta_{\llbracket A \rrbracket} \circ \llbracket (M[\vec{x} := \vec{M}]) \rrbracket && \text{Translation for pure} \\ &= \eta_{\llbracket A \rrbracket} \circ (\llbracket M \rrbracket \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle) && \text{Induction hypothesis} \\ &= (\eta_{\llbracket A \rrbracket} \circ \llbracket M \rrbracket) \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle && \text{Associativity of composition} \\ &= \llbracket \Gamma \vdash \mathbf{pure} M : \mathbf{KA} \rrbracket \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle && \text{Translation for pure} \end{aligned}$$

$$2) \llbracket \Gamma \vdash (\mathbf{let} \mathbf{pure} \vec{x} = \vec{M} \mathbf{in} N)[\vec{y} := \vec{P}] : \mathbf{KB} \rrbracket = \llbracket \Gamma \vdash \mathbf{let} \mathbf{pure} \vec{x} = \vec{M} \mathbf{in} N : \mathbf{KB} \rrbracket \circ \langle \llbracket P_1 \rrbracket, \dots, \llbracket P_n \rrbracket \rangle$$

$$\begin{aligned} &\llbracket \Gamma \vdash (\mathbf{let} \mathbf{pure} \vec{x} = \vec{M} \mathbf{in} N)[\vec{y} := \vec{P}] : \mathbf{KB} \rrbracket = \\ &\text{Substitution definition} \\ &\llbracket \Gamma \vdash \mathbf{let} \mathbf{pure} \vec{x} = (\vec{M}[\vec{y} := \vec{P}]) \mathbf{in} N : \mathbf{KB} \rrbracket = \\ &\text{Interpretation for } \mathbf{let}_{\mathbf{K}} \\ &\mathcal{K}(\llbracket N \rrbracket) \circ *_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ \llbracket \Gamma \vdash (\vec{M}[\vec{y} := \vec{P}]) : \mathbf{KA} \rrbracket = \\ &\text{Induction hypothesis} \\ &\mathcal{K}(\llbracket N \rrbracket) \circ *_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ (\llbracket \vec{M} \rrbracket \circ \langle \llbracket P_1 \rrbracket, \dots, \llbracket P_n \rrbracket \rangle) = \\ &\text{Associativity of composition} \\ &(\mathcal{K}(\llbracket N \rrbracket) \circ *_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ \llbracket \vec{M} \rrbracket) \circ \langle \llbracket P_1 \rrbracket, \dots, \llbracket P_n \rrbracket \rangle = \\ &\text{By interpretation} \\ &\llbracket \Gamma \vdash (\mathbf{let} \mathbf{pure} \vec{x} = \vec{M} \mathbf{in} N) \rrbracket \circ \langle \llbracket P_1 \rrbracket, \dots, \llbracket P_n \rrbracket \rangle \end{aligned}$$

□

Lemma 7.

- i) Let $\Gamma \vdash M : A$ and $M \rightarrow_{\beta} N$, then $\llbracket \Gamma \vdash M : A \rrbracket = \llbracket \Gamma \vdash N : A \rrbracket$;
- ii) Let $\Gamma \vdash M : A$ and $M \rightarrow_{\eta} N$, then $\llbracket \Gamma \vdash M : A \rrbracket = \llbracket \Gamma \vdash N : A \rrbracket$;

Proof.

- i) For β -reduction

Cases with β -reductions for $\mathbf{let}_{\mathbf{K}}$ are shown in [20]. Let us consider cases with \mathbf{pure} .

$$1) \llbracket \Gamma \vdash \mathbf{let} \mathbf{pure} \vec{x} = \mathbf{pure} \vec{M} \mathbf{in} N : \mathbf{KB} \rrbracket = \llbracket \Gamma \vdash \mathbf{pure} N[\vec{x} := \vec{M}] : \mathbf{KB} \rrbracket$$

$$\begin{aligned}
& \llbracket \Gamma \vdash \text{let pure } \vec{x} = \text{pure } \vec{M} \text{ in } N : \mathbf{K}B \rrbracket = \\
& \quad \text{By interpretation} \\
& \mathcal{K}(\llbracket N \rrbracket) \circ *_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ \langle \eta_{\llbracket A_1 \rrbracket} \circ \llbracket M_1 \rrbracket, \dots, \eta_{\llbracket A_n \rrbracket} \circ \llbracket M_n \rrbracket \rangle = \\
& \quad \text{By the property of a pair of morphisms} \\
& \mathcal{K}(\llbracket N \rrbracket) \circ *_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ (\eta_{\llbracket A_1 \rrbracket} \times \dots \times \eta_{\llbracket A_n \rrbracket}) \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle = \\
& \quad \text{Associativity of composition} \\
& \mathcal{K}(\llbracket N \rrbracket) \circ (*_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ (\eta_{\llbracket A_1 \rrbracket} \times \dots \times \eta_{\llbracket A_n \rrbracket})) \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle = \\
& \quad \text{By the definition of an applicative functor} \\
& \mathcal{K}(\llbracket N \rrbracket) \circ \eta_{\llbracket A_1 \rrbracket \times \dots \times \llbracket A_n \rrbracket} \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle = \\
& \quad \text{Naturality of } \eta \\
& \eta_{\llbracket B \rrbracket} \circ \llbracket N \rrbracket \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle = \\
& \quad \text{Associativity of composition} \\
& \eta_{\llbracket B \rrbracket} \circ (\llbracket N \rrbracket \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle) = \\
& \quad \text{Simultaneous substitution lemma} \\
& \eta_{\llbracket B \rrbracket} \circ \llbracket N[\vec{x} := \vec{M}] \rrbracket \\
& \quad \text{By interpretation} \\
& \llbracket \Gamma \vdash \text{pure } (N[\vec{x} := \vec{M}]) : \mathbf{K}B \rrbracket
\end{aligned}$$

2)

If $\Gamma \vdash M : A$ and $M \rightarrow_\beta N$, then $\llbracket \Gamma \vdash \text{pure } M : \mathbf{K}A \rrbracket = \llbracket \Gamma \vdash \text{pure } N : \mathbf{K}A \rrbracket$.

If $\Gamma \vdash M : A$ and $M \rightarrow_\beta N$, then $\Gamma \vdash N : A$ by subject reduction.

By assumption $\llbracket \Gamma \vdash M : A \rrbracket = \llbracket \Gamma \vdash N : A \rrbracket$.

So $\eta_{\llbracket A \rrbracket} \circ \llbracket \Gamma \vdash M : A \rrbracket = \eta_{\llbracket A \rrbracket} \circ \llbracket \Gamma \vdash N : A \rrbracket$.

Hence $\llbracket \Gamma \vdash \text{pure } M : \mathbf{K}A \rrbracket = \llbracket \Gamma \vdash \text{pure } N : \mathbf{K}A \rrbracket$.

ii) For η -reduction.

1) $\llbracket \vdash \text{let pure } _ = _ \text{ in } N : \mathbf{K}A \rrbracket = \llbracket \vdash \text{pure } N : \mathbf{K}A \rrbracket$.

$$\begin{aligned}
& \llbracket \vdash \text{let pure } _ = _ \text{ in } N : \mathbf{K}A \rrbracket = & \text{By interpretation} \\
& \mathcal{K}(\llbracket N \rrbracket) \circ \eta_{\mathbf{1}} = & \text{Naturality for } \eta \\
& \eta_{\llbracket A \rrbracket} \circ \llbracket N \rrbracket = & \text{By interpretation} \\
& \llbracket \vdash \text{pure } N : \mathbf{K}A \rrbracket
\end{aligned}$$

2)

If $\Gamma \vdash M : A$ and $M \rightarrow_\eta N$, then $\llbracket \Gamma \vdash \text{pure } M : \mathbf{K}A \rrbracket = \llbracket \Gamma \vdash \text{pure } N : \mathbf{K}A \rrbracket$.

Similar to case with β -reduction.

□

□

6.2 Completeness

We will consider term model for simply typed lambda calculus \times and \rightarrow standardly described in [22] [23].

Definition 14. Let us define an endofunctor $\mathcal{K} : \mathcal{C}(\lambda) \rightarrow \mathcal{C}(\lambda)$, such that:

- $\mathbf{K} : A \mapsto \mathbf{K}A$;
- $\mathbf{K} : f : A \rightarrow B \mapsto fmap f : \mathbf{K}A \rightarrow \mathbf{K}B$;

Lemma 8. *Functoriality*

- i) $\mathbf{K}(g \circ f) = \mathbf{K}(g) \circ \mathbf{K}(f)$;
- ii) $\mathbf{K}(id_A) = id_{\mathbf{K}A}$.

Proof.

- i-ii) See [24];
- iii)

□

Definition 15. *Let us define natural transformations:*

- $\eta : Id_{\mathcal{C}(\lambda)} \rightarrow \mathcal{K}$;
- $*_{A,B} : \mathbf{K}A \times \mathbf{K}B \rightarrow \mathbf{K}(A \times B)$

Lemma 9. *Naturality for η and for $*$*

Lemma 10. *Tensorial strength*

Lemma 11. *\mathbf{K} is an applicative functor.*

Theorem 3. *Completeness*

Let $\llbracket \Gamma \vdash M : A \rrbracket = \llbracket \Gamma \vdash N : A \rrbracket$, then $M =_{\beta\eta} N$.

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