

Modal type theory based on the intuitionistic epistemic logic

Abstract

Modal intuitionistic epistemic logic IEL^- was proposed by S.Artemov and T. Protopopescu as the formal foundation for the intuitionistic theory of knowledge. We construct a modal simply typed lambda-calculus which is Curry-Howard isomorphic to IEL^- as formal theory of calculations with applicative functors in functional programming languages like Haskell or Idris.

1 Introduction

Modal intuitionistic epistemic logic IEL was proposed by S. Artemov and T. Protopopescu [1]. IEL provides the epistimology and the theory of knowledge as based on BHK-semantics of intuitionistic logic. IEL^- is a variant of IEL , that corresponds to intuitionistic belief. Informally, $\mathbf{K}A$ denotes that A is verified intuitionistically.

Intuitionistic epistemic logic IEL^- is defined with by following axioms and derivation rules:

Definition 1. *Intuitionistic epistemic logic IEL :*

- 1) *IPC axioms;*
 - 2) $\mathbf{K}(A \rightarrow B) \rightarrow (\mathbf{K}A \rightarrow \mathbf{K}B)$ (*normality*);
 - 3) $A \rightarrow \mathbf{K}A$ (*co-reflection*);
- Rule: MP.*

We have the deduction theorem and necessitation rule which is derivable.

V. Krupski and A. Yatmanov provided the sequential calculus for IEL and proved that this calculus is PSPACE-complete [2].

It's not difficult to see that modal axioms in IEL^- and types of the methods of Applicative class in Haskell-like languages (which is described below) are syntactically similar and we are going to show that this coincidence has a non-trivial computational meaning.

Functional programming languages such as Haskell [3], Idris [4], Purescript [5] or Elm [6] have special type classes¹ for calculations with container types like `Functor` and `Applicative`²:

```
class Functor f where
  fmap :: (a -> b) -> f a -> f b
```

¹Type class in Haskell is a general interface for special group of datatypes.

²Reader may read more about container types in the Haskell standard library documentation[7] or in the next one textbook [8]

```

class Functor f => Applicative f where
  pure :: a -> f a
  (<*>) :: f (a -> b) -> f a -> f b

```

By *container* (or *computational context*) type we mean some type-operator f , where f is a “function” from $*$ to $*$: type operator takes a simple type (which has kind $*$) and returns another simple type with kind $*$. For more detailed description of the type system with kinds used in Haskell see [12].

The motivation for using an applicative functor is quite natural. Applicative functor allows to generalize the action of a functor for functions with arbitrary number of arguments, for instance:

```

liftA2 :: Applicative f => (a -> b -> c) -> f a -> f b -> f c
liftA2 f x y = pure f <*> x <*> y

```

The main goal of our research is a relationship between intuitionistic epistemic logic IEL^- and functional programming with effects. We show that relationship by building the type system (which is called λ_K) which is Curry-Howard isomorphic to IEL^- . So we will consider K -modality as an arbitrary applicative functor.

λ_K consists of the rules for simply typed lambda-calculus and special typing rules for lifting types into the applicative functor K . We assume that our type system will axiomatize the simplest case of computation with effects with one container. We provide proof-theoretical view on this kind of computations in functional programming and prove strong normalization and confluence.

2 Typed lambda-calculus based on IEL^-

At first we define the natural deduction for IEL^- :

Definition 2. *Natural deduction $NIEL$ for IEL^- is an extension of intuitionistic natural deduction with additional derivation rules for modality:*

$$\frac{\Gamma \vdash A}{\Gamma \vdash KA} K_I \qquad \frac{\Gamma \vdash K\vec{A} \quad \vec{A} \vdash B}{\Gamma \vdash KB}$$

Where $\Gamma \vdash K\vec{A}$ is a syntax sugar for $\Gamma \vdash KA_1, \dots, \Gamma \vdash KA_n$.

Lemma 1. $\Gamma \vdash_{NIEL_{\bigwedge, \rightarrow}^-} A \Rightarrow IEL^- \vdash \bigwedge \Gamma \rightarrow A$.

Proof. Induction on the derivation.

Let us consider cases with modality.

- 1) If $\Gamma \vdash_{NIEL_{\bigwedge, \rightarrow}^-} A$, then $IEL^- \vdash \bigwedge \Gamma \rightarrow KA$.
 - (1) $\bigwedge \Gamma \rightarrow A$ assumption
 - (2) $A \rightarrow KA$ co-reflection
 - (3) $(\bigwedge \Gamma \rightarrow A) \rightarrow ((A \rightarrow KA) \rightarrow (\bigwedge \Gamma \rightarrow KA))$ IPC theorem
 - (4) $(A \rightarrow KA) \rightarrow (\bigwedge \Gamma \rightarrow KA)$ from (1), (3) and MP
 - (5) $\bigwedge \Gamma \rightarrow KA$ from (2), (4) and MP

2) If $\Gamma \vdash_{NIEL_{\wedge, \rightarrow}^-} \mathbf{K}\vec{A}$ and $\vec{A} \vdash B$, then $IEL^- \vdash \bigwedge \Gamma \rightarrow \mathbf{K}B$.

- (1) $\bigwedge \Gamma \rightarrow \bigwedge_{i=1}^n \mathbf{K}A_i$ assumption
- (2) $\bigwedge_{i=1}^n \mathbf{K}A_i \rightarrow \mathbf{K} \bigwedge_{i=1}^n A_i$ IEL theorem
- (3) $\bigwedge \Gamma \rightarrow \mathbf{K} \bigwedge_{i=1}^n A_i$ from (1), (2) and transitivity
- (4) $\bigwedge_{i=1}^n A_i \rightarrow B$ assumption
- (5) $(\bigwedge_{i=1}^n A_i \rightarrow B) \rightarrow \mathbf{K}(\bigwedge_{i=1}^n A_i \rightarrow B)$ co-reflection
- (6) $\mathbf{K}(\bigwedge_{i=1}^n A_i \rightarrow B)$ from (2), (3) and MP
- (7) $\mathbf{K} \bigwedge_{i=1}^n A_i \rightarrow \mathbf{K}B$ from (6) and normality
- (8) $\bigwedge \Gamma \rightarrow \mathbf{K}B$ from (3), (7) and transitivity

□

Lemma 2. *If $IEL^- \vdash A$, then $NIEL^- \vdash A$.*

Proof. Straightforward derivation of modal axioms in $NIEL^-$. We consider this derivation below using terms. □

At the next step we build the typed lambda-calculus based on $NIEL_{\wedge, \rightarrow}^-$ by proof-assigning in rules.

At first, we define lambda-terms and types for this lambda-calculus.

Definition 3. *The set of terms:*

Let \mathbb{V} be the set of variables. The set $\Lambda_{\mathbf{K}}$ of terms is defined by the grammar:

$$\Lambda_{\mathbf{K}} ::= \mathbb{V} \mid (\lambda \Lambda. \Lambda_{\mathbf{K}}) \mid (\Lambda_{\mathbf{K}} \Lambda_{\mathbf{K}}) \mid (\Lambda_{\mathbf{K}}, \Lambda_{\mathbf{K}}) \mid (\pi_1 \Lambda_{\mathbf{K}}) \mid (\pi_2 \Lambda_{\mathbf{K}}) \mid (\text{pure } \Lambda_{\mathbf{K}}) \mid (\text{let pure } \Lambda_{\mathbf{K}} = \Lambda_{\mathbf{K}} \text{ in } \Lambda_{\mathbf{K}})$$

Definition 4. *The set of types:*

Let \mathbb{T} be the set of atomic types. The set $\mathbb{T}_{\mathbf{K}}$ of types with applicative functor \mathbf{K} is generated by the grammar:

$$\mathbb{T}_{\mathbf{K}} ::= \mathbb{T} \mid (\mathbb{T}_{\mathbf{K}} \rightarrow \mathbb{T}_{\mathbf{K}}) \mid (\mathbb{T}_{\mathbf{K}} \times \mathbb{T}_{\mathbf{K}}) \mid (\mathbf{K}\mathbb{T}_{\mathbf{K}}) \quad (1)$$

Context, domain of context and range of context are defined standardly [11][12].

Our type system is based on the Curry-style typing rules:

Definition 5. *Modal typed lambda calculus $\lambda_{\mathbf{K}}$ based on $NIEL_{\wedge, \rightarrow}^-$:*

$$\frac{}{\Gamma, x : A \vdash x : A} \text{ax}$$

$$\begin{array}{c}
\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x. M : A \rightarrow B} \rightarrow_i \qquad \frac{\Gamma \vdash M : A \rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B} \rightarrow_e \\
\\
\frac{\Gamma \vdash M : A \quad \Gamma \vdash N : B}{\Gamma \vdash \langle M, N \rangle : A \times B} \times_i \qquad \frac{\Gamma \vdash M : A_1 \times A_2}{\Gamma \vdash \pi_i M : A_i} \times_e, i \in \{1, 2\} \\
\\
\frac{\Gamma \vdash M : A}{\Gamma \vdash \mathbf{pure} M : \mathbf{K}A} \mathbf{K}_I \qquad \frac{\Gamma \vdash \vec{M} : \mathbf{K}\vec{A} \quad \vec{x} : \vec{A} \vdash M : B}{\Gamma \vdash \mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} M : \mathbf{K}B} \mathbf{let_K}
\end{array}$$

\mathbf{K}_I -typing rule is the same as \bigcirc -introduction in lax logic (also known as monadic metalanguage [17]) and in typed lambda-calculus which is derived by proof-assignment for lax-logic proofs. \mathbf{K}_I allows to inject an object of type α into the functor. \mathbf{K}_I reflects the Haskell method **pure** for Applicative class. It plays the same role as the **return** method in Monad class.

$\mathbf{let_K}$ is similar to the \square -rule in typed lambda calculus for intuitionistic normal modal logic \mathbf{IK} , which is described in [19].

In fact, our calculus is the extention of typed lambda calculus for \mathbf{IK} with typing rule appropriate to co-reflection.

Here are some examples of closed terms:

- $(\lambda x. \mathbf{pure} x) : A \rightarrow \mathbf{K}A$;
- $\lambda f. \lambda x. \mathbf{let pure} g, y = f, x \mathbf{in} gy : \mathbf{K}(A \rightarrow B) \rightarrow \mathbf{K}A \rightarrow \mathbf{K}B$
- $\lambda f. \lambda x. \mathbf{let pure} g, y = \mathbf{pure} f, x \mathbf{in} gy : (A \rightarrow B) \rightarrow \mathbf{K}A \rightarrow \mathbf{K}B$

Now we define free variables and substitutions. β -reduction, multi-step β -reduction and β -equality are defined standardly:

Definition 6. Set $FV(M)$ of free variables for arbitrary term M :

- 1) $FV(x) = \{x\}$;
- 2) $FV(\lambda x. M) = FV(M) \setminus \{x\}$;
- 3) $FV(MN) = FV(M) \cup FV(N)$;
- 4) $FV(\langle M, N \rangle) = FV(M) \cup FV(N)$;
- 5) $FV(\pi_i M) \subseteq FV(M)$, $i \in \{1, 2\}$;
- 6) $FV(\mathbf{pure} M) = FV(M)$;
- 7) $FV(\mathbf{let pure} \vec{N} = \vec{M} \mathbf{in} M) = \bigcup_{i=1}^n FV(M)$, where $n = |\vec{M}|$.

Definition 7. Substitution:

- 1) $x[x := N] = N$, $x[y := N] = x$;
- 2) $(MN)[x := N] = M[x := N]N[x := N]$;
- 3) $(\lambda x. M)[x := N] = \lambda x. M[x := N]$;
- 4) $(M, N)[x := P] = (M[x := P], N[x := P])$;
- 5) $(\pi_i M)[x := P] = \pi_i(M[x := P])$, $i \in \{1, 2\}$;
- 6) $(\mathbf{pure} M)[x := P] = \mathbf{pure}(M[x := P])$;
- 7) $(\mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} M)[y := P] = \mathbf{let pure} \vec{x} = (\vec{M}[y := P]) \mathbf{in} M$.

Definition 8. β -reduction and η -reduction rules for $\lambda\mathbf{K}$.

- 1) $(\lambda x.M)N \rightarrow_\beta M[x := N]$;
- 2) $\pi_1\langle M, N \rangle \rightarrow_\beta M$;
- 3) $\pi_2\langle M, N \rangle \rightarrow_\beta N$;
- 4) $\text{let pure } \langle \vec{x}, y, \vec{z} \rangle = \langle \vec{M}, \text{let pure } \vec{w} = \vec{N} \text{ in } Q, \vec{P} \rangle \text{ in } R \rightarrow_\beta$
 $\text{let pure } \langle \vec{x}, \vec{w}, \vec{z} \rangle = \langle \vec{M}, \vec{N}, \vec{P} \rangle \text{ in } R[y := Q]$
- 5) $\text{let pure } \vec{x} = \text{pure } \vec{M} \text{ in } N \rightarrow_\beta \text{pure } N[\vec{x} := \vec{M}]$
- 6) $\text{let pure } _ = _ \text{ in } N \rightarrow_\beta \text{pure } N$;
- 7) $\lambda x.f x \rightarrow_\eta f$;
- 8) $\langle \pi_1 P, \pi_2 P \rangle \rightarrow_\eta P$;
- 9) $\text{let pure } x = M \text{ in } x \rightarrow_\eta M$;
- 10) $M \rightarrow_{\beta\eta} N \Rightarrow \text{pure } M \rightarrow_{\beta\eta} \text{pure } N$

3 Basic lemmas

Now we will prove standard lemmas for contexts in type systems³:

Definition 9. *Type substitution*

The substitution of type C for type variable B in type A inductively defined as follows:

- 1) $B[B := C] = B$ and $D[B := C] = D$, if $B \neq D$;
- 2) $(A_1 \alpha A_2)[B := C] = (A_1[B := C])\alpha(A_2[B := C])$, where $\alpha \in \{\rightarrow, \times\}$;
- 3) $(\mathbf{K}A)[B := C] = \mathbf{K}(A[B := C])$.
- 4) Let Γ be the context, then $\Gamma[B := C] = \{x : (A[B := C]) \mid x : A \in \Gamma\}$

Lemma 3. *Basic lemmas for \mathbf{K}_I .*

- i) Let $\Gamma \vdash \text{pure } M : \mathbf{K}A$, then $\Gamma \vdash M : A$;
- ii) Let $\Gamma \vdash M : A$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash M : A$;
- iii) Let $\Gamma, x : A \vdash \text{pure } M : \mathbf{K}B$ and $\Gamma \vdash N : A$, then $\Gamma \vdash \text{pure } M[x := N] : \mathbf{K}B$.
- iv) Let $\Gamma \vdash \text{pure } M : \mathbf{K}A$, then $\Gamma[B := C] \vdash \text{pure } M : \mathbf{K}(A[B := C])$.

Proof.

- i) Induction on $\Gamma \vdash \text{pure } M : \mathbf{K}A$ and $\Gamma \vdash \text{let pure } \vec{x} = \vec{N} \text{ in } N : \mathbf{K}B$ correspondently.
- ii) Let $\Gamma \vdash \text{pure } M : \mathbf{K}A$. Then $\Gamma \vdash M : A$ by generation and $\Delta \vdash M : A$ by assumption. So $\Delta \vdash \text{pure } M : \mathbf{K}A$ by \mathbf{K}_I .
- iii) Let $\Gamma, x : A \vdash \text{pure } M : \mathbf{K}B$ and $\Gamma \vdash N : A$.
By generation $\Gamma, x : A \vdash M : B$ and by assumption $\Gamma \vdash M[x := N] : B$.
By K_I , $\Gamma \vdash \text{pure } (M[x := N]) : \mathbf{K}B$.
- iv) Let $\Gamma \vdash \text{pure } M : \mathbf{K}A$. By generation $\Gamma \vdash M : A$ and by assumption $\Gamma[B := C] \vdash M : A[B := C]$.
By K_I $\Gamma \vdash \text{pure } M : \mathbf{K}(A[B := C])$. □

Theorem 1. *Subject reduction*

Let $\Gamma \vdash M : A$ and $M \rightarrow_{\beta\eta} N$, then $\Gamma \vdash N : A$

³We will not prove cases with \rightarrow -constructor, they are proved standardly in the same lemmas for simply typed lambda calculus, for example see [11][12][14]. We will consider only modal cases

Proof. For cases with application, abstraction and pairs see [12] [13].

1) Let $\Gamma \vdash \mathbf{let\ pure} \langle \vec{x}, y, \vec{z} \rangle = \langle \vec{M}, \mathbf{let\ pure} \vec{w} = \vec{N} \mathbf{in} Q, \vec{P} \rangle \mathbf{in} R : \mathbf{KB}$,
then $\Gamma \vdash \mathbf{let\ pure} \langle \vec{x}, \vec{w}, \vec{z} \rangle = \langle \vec{M}, \vec{N}, \vec{P} \rangle \mathbf{in} R[y := Q] : \mathbf{KB}$

2) Let $\Gamma \vdash \mathbf{let\ pure} x = M \mathbf{in} x : \mathbf{KA}$, then $\Gamma \vdash M : \mathbf{KA}$.

See [19].

3) Let $\Gamma \vdash \mathbf{let\ pure} \vec{x} = \mathbf{pure} \vec{M} \mathbf{in} N : \mathbf{KB}$.

By generation $\Gamma \vdash \mathbf{pure} \vec{M} : \mathbf{KA}$ and $\vec{x} : \vec{A} \vdash N : B$.

Moreover, $\Gamma \vdash \vec{M} : \vec{A}$. By weakening and substitution lemma $\Gamma \vdash N[\vec{x} = \vec{M}] : B$.

By \mathbf{K}_I , $\Gamma \vdash \mathbf{pure} N[\vec{x} = \vec{M}] : \mathbf{KB}$.

4) Let $\vdash \mathbf{let\ pure} _ = _ \mathbf{in} N : \mathbf{KA}$

By generation $\vdash N : A$.

So $\vdash \mathbf{pure} N : \mathbf{KA}$ by \mathbf{K}_I .

5) Let $\Gamma \vdash \mathbf{pure} M : A$ and $M \rightarrow_{\beta\eta} N$.

By generation $\Gamma \vdash M : A$ and $\Gamma \vdash N : A$ by assumption.

So $\Gamma \vdash \mathbf{pure} N : \mathbf{KA}$. □

Strong normalization and colfence for \mathbf{IK} was proved by Kakutani for call-by-value and for call-by name [19] [20].

4 Categorical semantics

Definition 10. *Lax monoidal functor*

Let $\langle \mathcal{C}, \otimes_1, \mathbb{1} \rangle$ and $\langle \mathcal{D}, \otimes_2, \mathbb{1}' \rangle$ are monoidal categories.

A monoidal functor $\mathcal{F} : \langle \mathcal{C}, \otimes_1, \mathbb{1} \rangle \rightarrow \langle \mathcal{D}, \otimes_2, \mathbb{1}' \rangle$ is a functor $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ with additional natural transformations, which satisfy the well-known conditions described in [23]:

1) $u : \mathbb{1}' \rightarrow \mathcal{F}\mathbb{1}$;

2) $*_{A,B} : \mathcal{F}A \otimes_2 \mathcal{F}B \rightarrow \mathcal{F}(A \otimes_1 B)$.

Definition 11. *Applicative functor*

An applicative functor is a triple $\langle \mathcal{C}, \mathcal{K}, \eta \rangle$, where \mathcal{C} is a symmetric monoidal category, \mathcal{K} is a monoidal and $\eta : Id_{\mathcal{C}} \Rightarrow \mathcal{K}$ is a natural transformation (similar to unit in monad), such that:

1) $u = \eta_{\mathbb{1}}$;

2) $*_{A,B} \circ (\eta_A \otimes \eta_B) = \eta_{A \otimes B}$;

3) *Weak commutativity condition:*

$$\begin{array}{ccccc}
 A \otimes KB & \xrightarrow{\eta_A \otimes id_{KB}} & KA \otimes KB & \xrightarrow{*_{A,B}} & K(A \otimes B) \\
 \sigma_{A,KB} \downarrow & & & & \downarrow K(\sigma_{A,B}) \\
 KB \otimes A & \xrightarrow{id_{KB} \otimes \eta_A} & KB \otimes KA & \xrightarrow{*_{B,A}} & K(B \otimes A)
 \end{array}$$

By default we will consider an arbitrary closed functor on some cartesian closed category, which is the special case of an applicative functor.

We identify terminal objects. So $K(\mathbb{1}) = \mathbb{1}$ and $\eta_{\mathbb{1}} = id_{\mathbb{1}}$ since \mathcal{K} is an endofunctor.

4.1 Soundness and completeness

Theorem 2. Soundness

Let $\Gamma \vdash M : A$ and $M =_{\beta\eta} N$, then $\llbracket \Gamma \vdash M : A \rrbracket = \llbracket \Gamma \vdash N : A \rrbracket$

Proof.

Definition 12. Semantical translation from λ_K to CCC with applicative functor \mathcal{K} :

1) Interpretation for types:

$$\llbracket A \rrbracket := \hat{A}, A \in \mathbb{T};$$

$$\llbracket A \rightarrow B \rrbracket := \llbracket A \rrbracket \rightarrow \llbracket B \rrbracket;$$

$$\llbracket A \times B \rrbracket := \llbracket A \rrbracket \times \llbracket B \rrbracket.$$

2) Interpretation for modal types: $\llbracket \mathbf{K}A \rrbracket = \mathcal{K}\llbracket A \rrbracket$;

3) Interpretation for contexts:

$$\llbracket \Gamma = \{x_1 : A_1, \dots, x_n : A_n\} \rrbracket := \llbracket \Gamma \rrbracket = \llbracket A_1 \rrbracket \times \dots \times \llbracket A_n \rrbracket;$$

4) Interpretation for typing assignment: $\llbracket \Gamma \vdash M : A \rrbracket := \llbracket M \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket$.

5) Interpretation for typing rules:

$$\begin{array}{c} \frac{}{\llbracket \Gamma, x : A \vdash x : A \rrbracket = \pi_2 : \llbracket \Gamma \rrbracket \times \llbracket A \rrbracket \rightarrow \llbracket A \rrbracket} \\ \frac{\llbracket \Gamma, x : A \vdash M : B \rrbracket = f : \llbracket \Gamma \rrbracket \times \llbracket A \rrbracket \rightarrow \llbracket B \rrbracket}{\llbracket \Gamma \vdash (\lambda x.M) : A \rightarrow B \rrbracket = \Lambda(f) : \llbracket \Gamma \rrbracket \rightarrow \llbracket B \rrbracket^{\llbracket A \rrbracket}} \\ \frac{\llbracket \Gamma \vdash M : A \rightarrow B \rrbracket = \llbracket M \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket B \rrbracket^{\llbracket A \rrbracket} \quad \llbracket \Gamma \vdash N : A \rrbracket = \llbracket N \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket}{\llbracket \Gamma \vdash (MN) : B \rrbracket = \llbracket \Gamma \rrbracket \xrightarrow{\langle \llbracket M \rrbracket, \llbracket N \rrbracket \rangle} \llbracket B \rrbracket^{\llbracket A \rrbracket} \times \llbracket A \rrbracket \xrightarrow{\epsilon} \llbracket B \rrbracket} \\ \frac{\llbracket \Gamma \vdash M : A \rrbracket = f : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket \quad \llbracket \Gamma \vdash N : B \rrbracket = g : \llbracket \Gamma \rrbracket \rightarrow \llbracket B \rrbracket}{\llbracket \Gamma \vdash (M, N) : A \times B \rrbracket = \langle f, g \rangle : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket \times \llbracket B \rrbracket} \\ \frac{\llbracket \Gamma \vdash p : A_1 \times A_2 \rrbracket = f : \llbracket \Gamma \rrbracket \rightarrow \llbracket A_1 \rrbracket \times \llbracket A_2 \rrbracket}{\llbracket \Gamma \vdash \pi_i p : A_i \rrbracket = \llbracket \Gamma \rrbracket \xrightarrow{f} \llbracket A_1 \rrbracket \times \llbracket A_2 \rrbracket \xrightarrow{\pi_i} \llbracket A_i \rrbracket} \quad i \in \{1, 2\} \\ \frac{\llbracket \Gamma \vdash M : A \rrbracket = \llbracket M \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket}{\llbracket \Gamma \vdash \mathbf{pure} M : \mathbf{K}A \rrbracket := \llbracket \Gamma \rrbracket \xrightarrow{\llbracket M \rrbracket} \llbracket A \rrbracket \xrightarrow{\eta_{\llbracket A \rrbracket}} \mathcal{K}\llbracket A \rrbracket} \\ \frac{\llbracket \Gamma \vdash \vec{M} : \mathbf{K}\vec{A} \rrbracket = \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle : \llbracket \Gamma \rrbracket \rightarrow \prod_{i=1}^n \mathcal{K}\llbracket A_i \rrbracket \quad \llbracket \vec{x} : \vec{A} \vdash N : B \rrbracket = \llbracket N \rrbracket : \prod_{i=1}^n \llbracket A_i \rrbracket \rightarrow \llbracket B \rrbracket}{\llbracket \Gamma \vdash \mathbf{let pure} \vec{x} = \vec{M} \mathbf{in} M : \mathbf{K}B \rrbracket = \mathcal{K}(\llbracket N \rrbracket) \circ *_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle : \llbracket \Gamma \rrbracket \rightarrow \mathcal{K}\llbracket B \rrbracket} \end{array}$$

Definition 13. Simultaneous substitution

Let $\Gamma = \{x_1 : A_1, \dots, x_n : A_n\}$, $\Gamma \vdash M : A$ and for all $i \in \{1, \dots, n\}$, $\Gamma \vdash M_i : A_i$.

We define simultaneous substitution $M[\vec{x} := \vec{M}]$ recursively by:

1) $x_i[\vec{x} := \vec{M}] = M_i$;

2) $(\lambda x.M)[\vec{x} := \vec{M}] = \lambda x.(M[\vec{x} := \vec{M}])$;

3) $(MN)[\vec{x} := \vec{M}] = (M[\vec{x} := \vec{M}])(N[\vec{x} := \vec{M}])$;

4) $\langle M, N \rangle = \langle (M[\vec{x} := \vec{M}]), (N[\vec{x} := \vec{M}]) \rangle$;

- 5) $(\pi_i P)[\vec{x} := \vec{M}] = \pi_i(P[\vec{x} = \vec{M}]);$
- 6) $(\mathbf{pure} M)[\vec{x} := \vec{M}] = \mathbf{pure} (M[\vec{x} = \vec{M}]);$
- 7) $(\mathbf{let} \mathbf{pure} \vec{x} = \vec{M} \mathbf{in} N)[\vec{y} := \vec{P}] = \mathbf{let} \mathbf{pure} \vec{x} = (\vec{M}[\vec{y} := \vec{P}]) \mathbf{in} N$

Lemma 4.

$$\llbracket M[x_1 := M_1, \dots, x_n := M_n] \rrbracket = \llbracket M \rrbracket \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle.$$

Proof.

$$1) \llbracket \Gamma \vdash (\mathbf{pure} M)[\vec{x} := \vec{M}] : \mathbf{KA} \rrbracket = \llbracket \Gamma \vdash \mathbf{pure} M : \mathbf{KA} \rrbracket \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle.$$

$$\begin{aligned} \llbracket \Gamma \vdash (\mathbf{pure} M)[\vec{x} := \vec{M}] : \mathbf{KA} \rrbracket &= \llbracket \Gamma \vdash \mathbf{pure} (M[\vec{x} := \vec{M}]) : \mathbf{KA} \rrbracket && \text{Substitution definition} \\ &= \eta_{\llbracket A \rrbracket} \circ \llbracket (M[\vec{x} := \vec{M}]) \rrbracket && \text{Translation for pure} \\ &= \eta_{\llbracket A \rrbracket} \circ (\llbracket M \rrbracket \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle) && \text{Induction hypothesis} \\ &= (\eta_{\llbracket A \rrbracket} \circ \llbracket M \rrbracket) \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle && \text{Associativity of composition} \\ &= \llbracket \Gamma \vdash \mathbf{pure} M : \mathbf{KA} \rrbracket \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle && \text{Translation for pure} \end{aligned}$$

$$2) \llbracket \Gamma \vdash (\mathbf{let} \mathbf{pure} \vec{x} = \vec{M} \mathbf{in} N)[\vec{y} := \vec{P}] : \mathbf{KB} \rrbracket = \llbracket \Gamma \vdash \mathbf{let} \mathbf{pure} \vec{x} = \vec{M} \mathbf{in} N : \mathbf{KB} \rrbracket \circ \langle \llbracket P_1 \rrbracket, \dots, \llbracket P_n \rrbracket \rangle$$

$$\begin{aligned} &\llbracket \Gamma \vdash (\mathbf{let} \mathbf{pure} \vec{x} = \vec{M} \mathbf{in} N)[\vec{y} := \vec{P}] : \mathbf{KB} \rrbracket = \\ &\text{Substitution definition} \\ &\llbracket \Gamma \vdash \mathbf{let} \mathbf{pure} \vec{x} = (\vec{M}[\vec{y} := \vec{P}]) \mathbf{in} N : \mathbf{KB} \rrbracket = \\ &\text{Interpretation for } \mathbf{let}_{\mathbf{K}} \\ &\mathcal{K}(\llbracket N \rrbracket) \circ *_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ \llbracket \Gamma \vdash (\vec{M}[\vec{y} := \vec{P}]) : \mathbf{KA} \rrbracket = \\ &\text{Induction hypothesis} \\ &\mathcal{K}(\llbracket N \rrbracket) \circ *_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ (\llbracket \vec{M} \rrbracket \circ \langle \llbracket P_1 \rrbracket, \dots, \llbracket P_n \rrbracket \rangle) = \\ &\text{Associativity of composition} \\ &(\mathcal{K}(\llbracket N \rrbracket) \circ *_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ \llbracket \vec{M} \rrbracket) \circ \langle \llbracket P_1 \rrbracket, \dots, \llbracket P_n \rrbracket \rangle = \\ &\text{By interpretation} \\ &\llbracket \Gamma \vdash (\mathbf{let} \mathbf{pure} \vec{x} = \vec{M} \mathbf{in} N) \rrbracket \circ \langle \llbracket P_1 \rrbracket, \dots, \llbracket P_n \rrbracket \rangle \end{aligned}$$

□

Lemma 5.

- i) Let $\Gamma \vdash M : A$ and $M \rightarrow_{\beta} N$, then $\llbracket \Gamma \vdash M : A \rrbracket = \llbracket \Gamma \vdash N : A \rrbracket$;
- ii) Let $\Gamma \vdash M : A$ and $M \rightarrow_{\eta} N$, then $\llbracket \Gamma \vdash M : A \rrbracket = \llbracket \Gamma \vdash N : A \rrbracket$;

Proof.

- i) For β -reduction

Cases with β -reductions for $\mathbf{let}_{\mathbf{K}}$ are shown in [20]. Let us consider cases with \mathbf{pure} .

$$1) \llbracket \Gamma \vdash \mathbf{let} \mathbf{pure} \vec{x} = \mathbf{pure} \vec{M} \mathbf{in} N : \mathbf{KB} \rrbracket = \llbracket \Gamma \vdash \mathbf{pure} N[\vec{x} := \vec{M}] : \mathbf{KB} \rrbracket$$

$$\begin{aligned}
& \llbracket \Gamma \vdash \text{let pure } \vec{x} = \text{pure } \vec{M} \text{ in } N : \mathbf{K}B \rrbracket = \\
& \quad \text{By interpretation} \\
& \mathcal{K}(\llbracket N \rrbracket) \circ *_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ \langle \eta_{\llbracket A_1 \rrbracket} \circ \llbracket M_1 \rrbracket, \dots, \eta_{\llbracket A_n \rrbracket} \circ \llbracket M_n \rrbracket \rangle = \\
& \quad \text{By the property of a pair of morphisms} \\
& \mathcal{K}(\llbracket N \rrbracket) \circ *_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ (\eta_{\llbracket A_1 \rrbracket} \times \dots \times \eta_{\llbracket A_n \rrbracket}) \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle = \\
& \quad \text{Associativity of composition} \\
& \mathcal{K}(\llbracket N \rrbracket) \circ (*_{\llbracket A_1 \rrbracket, \dots, \llbracket A_n \rrbracket} \circ (\eta_{\llbracket A_1 \rrbracket} \times \dots \times \eta_{\llbracket A_n \rrbracket})) \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle = \\
& \quad \text{By the definition of an applicative functor} \\
& \mathcal{K}(\llbracket N \rrbracket) \circ \eta_{\llbracket A_1 \rrbracket \times \dots \times \llbracket A_n \rrbracket} \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle = \\
& \quad \text{Naturality of } \eta \\
& \eta_{\llbracket B \rrbracket} \circ \llbracket N \rrbracket \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle = \\
& \quad \text{Associativity of composition} \\
& \eta_{\llbracket B \rrbracket} \circ (\llbracket N \rrbracket \circ \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket \rangle) = \\
& \quad \text{Simultaneous substitution lemma} \\
& \eta_{\llbracket B \rrbracket} \circ \llbracket N[\vec{x} := \vec{M}] \rrbracket \\
& \quad \text{By interpretation} \\
& \llbracket \Gamma \vdash \text{pure } (N[\vec{x} := \vec{M}]) : \mathbf{K}B \rrbracket \\
& 2) \\
& \text{If } \Gamma \vdash M : A \text{ and } M \rightarrow_{\beta\eta} N, \text{ then } \llbracket \Gamma \vdash \text{pure } M : \mathbf{K}A \rrbracket = \llbracket \Gamma \vdash \text{pure } N : \mathbf{K}A \rrbracket.
\end{aligned}$$

If $\Gamma \vdash M : A$ and $M \rightarrow_{\beta\eta} N$, then $\Gamma \vdash N : A$ by subject reduction.

By assumption $\llbracket \Gamma \vdash M : A \rrbracket = \llbracket \Gamma \vdash N : A \rrbracket$.

So $\eta_{\llbracket A \rrbracket} \circ \llbracket \Gamma \vdash M : A \rrbracket = \eta_{\llbracket A \rrbracket} \circ \llbracket \Gamma \vdash N : A \rrbracket$.

Hence $\llbracket \Gamma \vdash \text{pure } M : \mathbf{K}A \rrbracket = \llbracket \Gamma \vdash \text{pure } N : \mathbf{K}A \rrbracket$.

ii) For η -reduction.

1) $\llbracket \vdash \text{let pure } _ = _ \text{ in } N : \mathbf{K}A \rrbracket = \llbracket \vdash \text{pure } N : \mathbf{K}A \rrbracket$.

$$\begin{aligned}
& \llbracket \vdash \text{let pure } _ = _ \text{ in } N : \mathbf{K}A \rrbracket = & \text{By interpretation} \\
& \mathcal{K}(\llbracket N \rrbracket) \circ \eta_{\mathbf{1}} = & \text{Naturality for } \eta \\
& \eta_{\llbracket A \rrbracket} \circ \llbracket N \rrbracket = & \text{By interpretation} \\
& \llbracket \vdash \text{pure } N : \mathbf{K}A \rrbracket
\end{aligned}$$

□

□

Theorem 3. Completeness

Let $\llbracket \Gamma \vdash M : A \rrbracket = \llbracket \Gamma \vdash N : A \rrbracket$, then $M =_{\beta\eta} N$.

Proof.

We will consider term model for simply typed lambda calculus \times and \rightarrow standardly described in [22].

Definition 14. Let us define an endofunctor $\mathcal{K} : \mathcal{C}(\lambda) \rightarrow \mathcal{C}(\lambda)$, such that:

1) $\mathbf{K} : A \mapsto \mathbf{K}A$;

2) $\mathbf{K} : [x, M] \in \text{Hom}_{\mathcal{C}(\lambda)}(A, B) \mapsto \text{fmap } f = [y, \text{let pure } x = y \text{ in } M] \in \text{Hom}_{\mathcal{C}(\lambda)}(\mathbf{K}A, \mathbf{K}B)$.

Lemma 6. *Functoriality*

i) $\text{fmap } (g \circ f) = \text{fmap } (g) \circ \text{fmap } (f)$;

ii) $\text{fmap } (id_A) = id_{\mathbf{K}A}$.

Proof. Easy checking using reduction rules. \square

Definition 15. Let us define natural transformations:

- 1) $\eta : Id \Rightarrow \mathcal{K}$, s. t. $\forall A \in Ob_{\mathcal{C}(\lambda)}$, $\eta_A = [x, \mathbf{pure} x] \in Hom_{\mathcal{C}(\lambda)}(A, \mathbf{K}A)$;
- 2) $*_{A,B} : \mathbf{K}A \times \mathbf{K}B \rightarrow \mathbf{K}(A \times B)$, s. t. $\forall A, B \in Ob_{\mathcal{C}(\lambda)}$, $*_{A,B} = [p, \mathbf{let pure} x, y = \pi_1 p, \pi_2 p \mathbf{ in } \langle x, y \rangle] \in Hom_{\mathcal{C}(\lambda)}(\mathbf{K}A \times \mathbf{K}B, \mathbf{K}(A \times B))$.

Implementation for $*$ in our term model is a modification of $\mathbf{let_K}$ -rule:

$$\frac{\frac{p : \mathbf{K}A \times \mathbf{K}B \vdash p : \mathbf{K}A \times \mathbf{K}B}{p : \mathbf{K}A \times \mathbf{K}B \vdash \pi_1 p : \mathbf{K}A} \quad \frac{p : \mathbf{K}A \times \mathbf{K}B \vdash p : \mathbf{K}A \times \mathbf{K}B}{p : \mathbf{K}A \times \mathbf{K}B \vdash \pi_2 p : \mathbf{K}B} \quad \frac{x : A \vdash x : A \quad y : B \vdash y : B}{x : A, y : B \vdash \langle x, y \rangle : A \times B}}{p : \mathbf{K}A \times \mathbf{K}B \vdash \mathbf{let pure} \langle x, y \rangle = \langle \pi_1 p, \pi_2 p \rangle \mathbf{ in } \langle x, y \rangle : \mathbf{K}(A \times B)}$$

Lemma 7. *Naturality for η and for $*$*

- i) $fmap f \circ \eta_A = \eta_B \circ f$;
- ii) $fmap (f \times g) \circ *_{A,B} = *_{C,D} \circ (fmap f) \times (fmap g)$.
- iii) $*_{A,B} \circ (\eta_A \times \eta_B) = \eta_{A \times B}$;

Proof.

- i) $fmap f \circ \eta_A = \eta_B \circ f$

$$\begin{array}{ll} \eta_B \circ f = & \text{By the definition} \\ [y, \mathbf{pure} y] \circ [x, M] = & \text{By the definition of composition} \\ [x, \mathbf{pure} y[y := M]] = & \text{By substitution} \\ [x, \mathbf{pure} M] & \end{array}$$

On the other hand:

$$\begin{array}{ll} fmap f \circ \eta_A = & \text{By the definition} \\ [z, \mathbf{let pure} x = z \mathbf{ in } M] \circ [x, \mathbf{pure} x] = & \text{By the definition of composition} \\ [x, \mathbf{let pure} x = z \mathbf{ in } M[z := \mathbf{pure} x]] = & \text{By substitution} \\ [x, \mathbf{let pure} x = \mathbf{pure} x \mathbf{ in } M] = & \beta\text{-reduction rule} \\ [x, \mathbf{pure} M[x := x]] = & \text{By substitution} \\ [x, \mathbf{pure} M] & \end{array}$$

- ii) $fmap (f \times g) \circ *_{A,B} = *_{C,D} \circ (fmap f) \times (fmap g)$

See [19].

- iii) $*_{A,B} \circ (\eta_A \times \eta_B) = \eta_{A \times B}$
Follows from i) and ii).

\square

Tensorial strength is defined as follows:

Definition 16. *Tensorial strength*

Let $[p, \langle \mathbf{pure} (\pi_1 p), \pi_2 p \rangle] \in Hom_{\mathcal{C}(\lambda)}(A \times \mathbf{K}B, \mathbf{K}A \times \mathbf{K}B)$.

So tensorial strength is defined as $\tau_{A,B} = *_{A,B} \circ [p, \langle \mathbf{pure} (\pi_1 p), \pi_2 p \rangle]$.

It is clearly that tensorial strength defined above can be simplified as follows:

$$\begin{aligned}
& *_{A,B} \circ [p, \langle \mathbf{pure}(\pi_1 p), \pi_2 p \rangle] = && \text{By definition} \\
& [p', \mathbf{let pure } x, y = \pi_1 p', \pi_2 p' \mathbf{ in } \langle x, y \rangle] \circ [p, \langle \mathbf{pure}(\pi_1 p), \pi_2 p \rangle] = && \text{By composition} \\
& [p, \mathbf{let pure } x, y = \pi_1 p', \pi_2 p' \mathbf{ in } \langle x, y \rangle] [p' := \langle \mathbf{pure}(\pi_1 p), \pi_2 p \rangle] = && \text{By substitution} \\
& [p, \mathbf{let pure } x, y = \pi_1(\langle \mathbf{pure}(\pi_1 p), \pi_2 p \rangle), \pi_2(\langle \pi_1 p, \mathbf{pure}(\pi_2 p) \rangle) \mathbf{ in } \langle x, y \rangle] = && \text{By } \beta\text{-reduction rules} \\
& [p, \mathbf{let pure } x, y = \mathbf{pure}(\pi_1 p), \pi_2 p \mathbf{ in } \langle x, y \rangle]
\end{aligned}$$

Lemma 8. *Weak commutativity.*

$$\begin{aligned}
& fmap ([p, \langle \pi_2 p, \pi_1 p \rangle]) \circ \tau_{A,B} = \\
& *_{B,A} \circ [q, \langle \pi_1 q, \mathbf{pure}(\pi_2 q) \rangle] \circ [p, \langle \pi_2 p, \pi_1 p \rangle]
\end{aligned}$$

Proof.

$$\begin{aligned}
& fmap ([r, \langle \pi_2 r, \pi_1 r \rangle]) \circ \tau_{A,B} = \\
& \text{By the definition of } \tau \\
& fmap ([r, \langle \pi_2 r, \pi_1 r \rangle]) \circ [p, \mathbf{let pure } x, y = \mathbf{pure}(\pi_1 p), \pi_2 p \mathbf{ in } \langle x, y \rangle] = \\
& \text{By the definition of } fmap \\
& [q, \mathbf{let pure } r = q \mathbf{ in } \langle \pi_2 r, \pi_1 r \rangle] \circ [p, \mathbf{let pure } x, y = \mathbf{pure}(\pi_1 p), \pi_2 p \mathbf{ in } \langle x, y \rangle] = \\
& \text{Composition} \\
& [p, \mathbf{let pure } r = q \mathbf{ in } \langle \pi_2 r, \pi_1 r \rangle] [q := \mathbf{let pure } x, y = \mathbf{pure}(\pi_1 p), \pi_2 p \mathbf{ in } \langle x, y \rangle] = \\
& \text{By } \beta\text{-reduction rules} \\
& [p, \mathbf{let pure } r = (\mathbf{let pure } x, y = \mathbf{pure}(\pi_1 p), \pi_2 p \mathbf{ in } \langle x, y \rangle) \mathbf{ in } \langle \pi_2 r, \pi_1 r \rangle] = \\
& \text{By } \beta\text{-reduction rules} \\
& [p, \mathbf{let pure } x, y = \mathbf{pure}(\pi_1 p), \pi_2 p \mathbf{ in } \langle \pi_2 r, \pi_1 r \rangle] [r := \langle x, y \rangle] = \\
& \text{By substitution} \\
& [p, \mathbf{let pure } x, y = \mathbf{pure}(\pi_1 p), \pi_2 p \mathbf{ in } \langle \pi_2 \langle x, y \rangle, \pi_1 \langle x, y \rangle \rangle] = \\
& \text{By } \beta\text{-reduction rules} \\
& [p, \mathbf{let pure } x, y = \mathbf{pure}(\pi_1 p), \pi_2 p \mathbf{ in } \langle y, x \rangle] =
\end{aligned}$$

On the other hand

$$\begin{aligned}
& *_{B,A} \circ [q, \langle \pi_1 q, \mathbf{pure}(\pi_2 q) \rangle] \circ [p, \langle \pi_2 p, \pi_1 p \rangle] = \\
& \text{By the definition of } * \\
& [r, \mathbf{let pure } y, x = \pi_1 r, \pi_2 r \mathbf{ in } \langle y, x \rangle] \circ [q, \langle \pi_1 q, \mathbf{pure}(\pi_2 q) \rangle] \circ [p, \langle \pi_2 p, \pi_1 p \rangle] = \\
& \text{Composition} \\
& [r, \mathbf{let pure } y, x = \pi_1 r, \pi_2 r \mathbf{ in } \langle y, x \rangle] \circ [p, \langle \pi_1 q, \mathbf{pure}(\pi_2 q) \rangle] [q := \langle \pi_2 p, \pi_1 p \rangle] = \\
& \text{By substitution and by } \beta\text{-reduction rules} \\
& [r, \mathbf{let pure } y, x = \pi_1 r, \pi_2 r \mathbf{ in } \langle y, x \rangle] \circ [p, \langle \pi_2 p, \mathbf{pure}(\pi_1 p) \rangle] = \\
& \text{Composition} \\
& [p, \mathbf{let pure } y, x = \pi_1 r, \pi_2 r \mathbf{ in } \langle y, x \rangle] [r := \langle \pi_2 p, \mathbf{pure}(\pi_1 p) \rangle] = \\
& \text{By substitution and by } \beta\text{-reduction rules} \\
& [p, \mathbf{let pure } y, x = \pi_2 p, \mathbf{pure}(\pi_1 p) \mathbf{ in } \langle y, x \rangle] = \\
& \text{By symmetricity of assingment} \\
& [p, \mathbf{let pure } x, y = \mathbf{pure}(\pi_1 p), \pi_2 p \mathbf{ in } \langle y, x \rangle]
\end{aligned}$$

□

Lemma 9. *K is an applicative functor*

Proof. Immediately follows from previous lemmas in the section.

□

□

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