Soundness for modal type theory based on the intuitionistic epistemic logic

1 Modal lambda calculus based on $\mathbf{IEL}^-$

Definition 1. *The set of terms:*

Let $\mathbb{V}$ is a set of variables. The set $\Lambda_{\boldsymbol{K}}$ of terms is defined by the grammar:

$$\Lambda_{\boldsymbol{K}} ::= \mathbb{V} \mid (\lambda\Lambda.\Lambda_{\boldsymbol{K}}) \mid (\Lambda_{\boldsymbol{K}}\Lambda_{\boldsymbol{K}}) \mid (\Lambda_{\boldsymbol{K}}, \Lambda_{\boldsymbol{K}}) \mid (\pi_i \Lambda_{\boldsymbol{K}}) \mid (pure\, \Lambda_{\boldsymbol{K}}) \mid (\Lambda_{\boldsymbol{K}} \star \Lambda_{\boldsymbol{K}}) \quad (1)$$

where $i \in \{1, 2\}$.

Definition 2. *The set of types:*

Let $\mathbb{T}$ is a set of atomic types. The set $\mathbb{T}_{\boldsymbol{K}}$ of types with applicative functor $\boldsymbol{K}$ is generated by the grammar:

$$\mathbb{T}_{\boldsymbol{K}} ::= \mathbb{T} \mid (\mathbb{T}_{\boldsymbol{K}} \to \mathbb{T}_{\boldsymbol{K}}) \mid (\mathbb{T}_{\boldsymbol{K}} \times \mathbb{T}_{\boldsymbol{K}}) \mid (\boldsymbol{K}\mathbb{T}_{\boldsymbol{K}}) \quad (2)$$

Our type system is based on the Curry-style typing rules:

Definition 3. *Modal typed lambda calculus $\lambda\boldsymbol{K}$ based on $NIEL^-_{\wedge,\to}$:*

$$\frac{}{\Gamma, x : \alpha \vdash x : \alpha}\ ax$$

$$\frac{\Gamma, x : \alpha \vdash M : \beta}{\Gamma \vdash \lambda x.M : \alpha \to \beta} \to_i$$

$$\frac{\Gamma \vdash x : \alpha \qquad \Gamma \vdash y : \beta}{\Gamma \vdash (x, y) : \alpha \times \beta} \times_i$$

$$\frac{\Gamma, \vdash x : \alpha}{\Gamma \vdash pure\ \ x : \boldsymbol{K}\alpha}\ \boldsymbol{K}_I$$

$$\frac{\Gamma \vdash f : \alpha \to \beta \qquad \Gamma \vdash x : \alpha}{\Gamma \vdash fx : \beta} \to_e$$

$$\frac{\Gamma \vdash p : \alpha_1 \times \alpha_2}{\Gamma \vdash \pi_i p : \alpha_i} \times_e,\ i \in \{1, 2\}$$

$$\frac{\Gamma \vdash f : \boldsymbol{K}(\alpha \to \beta) \qquad \Gamma \vdash x : \boldsymbol{K}\alpha}{\Gamma \vdash f \star x : \boldsymbol{K}\beta}\ \boldsymbol{K}_{app}$$

Definition 4. *β-reduction rules:*

1) $(\lambda x.M)N \to_\beta M[x := N]$;

2) $\pi_i\langle M_1, M_2\rangle \to_\beta M_i, i \in \{1,2\}$;

3) pure $(\lambda x.x) \star M \to_\beta M$;

4) pure $(\lambda fgx.f(gx)) \star M \star N \star P \to_\beta M \star (N \star P)$;

5) (pure M) $\star$ (pure N) $\to_\beta$ pure (MN);

6) $M \star$ pure $N \to_\beta (\lambda f.fN) \star M$;

Definition 5. *η-reduction rules for applicative functor:*

1) pure $(\lambda x.fx) \to_\eta$ pure f;

2) pure $\langle \pi_1 p, \pi_2 p\rangle \to_\eta$ pure p;

3) $\lambda x.f \star x \to_\eta f$.

2 Categorical model.

3 Soundness

Definition 6. *Semantical translation from $\lambda_{\boldsymbol{K}}$ to CCC with applicative functor:*

1) Interpretation for types: $[\![A]\!] := \hat{A}, A \in \mathbb{T}$, $[\![A \to B]\!] := [\![A]\!] \to [\![B]\!]$, $[\![A \times B]\!] := [\![A]\!] \times [\![B]\!]$;

2) Interpretation for modal types: $[\![\boldsymbol{K}A]\!] = \mathcal{K}[\![A]\!]$, where $\mathcal{K}$ is an applicative functor;

3) Interpretaion for contexts: $[\![\Gamma = \{x_1 : A_1, ..., x_n : A_n\}]\!] := [\![\Gamma]\!] = [\![A_1]\!] \times ... \times [\![A_n]\!]$;

4) Interpretation for typing assignment: $[\![\Gamma \vdash M : A]\!] := [\![M]\!] : [\![\Gamma]\!] \to [\![A]\!]$, where $[\![M]\!] : [\![\Gamma]\!] \to [\![A]\!] \in \mathcal{C}$;

5) Interpretation for typing rules:

$$\frac{}{[\![\Gamma, x : A \vdash x : A]\!] := \pi_2 : [\![\Gamma]\!] \times [\![A]\!] \to [\![A]\!]}$$

$$\frac{[\![\Gamma, x : A \vdash M : B]\!] := f : [\![\Gamma]\!] \times [\![A]\!] \to [\![B]\!]}{[\![\Gamma \vdash (\lambda x.M) : A \to B]\!] := \Lambda(f) : [\![\Gamma]\!] \to [\![B]\!]^{[\![A]\!]}}$$

$$\frac{[\![\Gamma \vdash M : A \to B]\!] := [\![M]\!] : [\![\Gamma]\!] \to [\![B]\!]^{[\![A]\!]} \qquad [\![\Gamma \vdash N : A]\!] := [\![N]\!] : [\![\Gamma]\!] \to [\![A]\!]}{[\![\Gamma \vdash (MN) : B]\!] := [\![\Gamma]\!] \xrightarrow{\langle [\![M]\!], [\![N]\!]\rangle} [\![B]\!]^{[\![A]\!]} \times [\![A]\!] \xrightarrow{\epsilon} [\![B]\!]}$$

$$\frac{[\![\Gamma \vdash M : A]\!] := f : [\![\Gamma]\!] \to [\![A]\!] \qquad [\![\Gamma \vdash N : B]\!] := g : [\![\Gamma]\!] \to [\![B]\!]}{[\![\Gamma \vdash (M, N) : A \times B]\!] := \langle f, g\rangle : [\![\Gamma]\!] \to [\![A]\!] \times [\![B]\!]}$$

$$\frac{[\![\Gamma \vdash p : A_1 \times A_2]\!] := f : [\![\Gamma]\!] \to [\![A_1]\!] \times [\![A_2]\!]}{[\![\Gamma \vdash \pi_i p : A_i]\!] := [\![\Gamma]\!] \xrightarrow{f} [\![A_1]\!] \times [\![A_2]\!] \xrightarrow{\pi_i} [\![A_i]\!]} \; i \in \{1,2\}$$

$$\frac{[\![\Gamma \vdash M : A]\!] := [\![M]\!] : [\![\Gamma]\!] \to [\![A]\!]}{[\![\Gamma \vdash \textit{pure } M : \boldsymbol{K}A]\!] := [\![\Gamma]\!] \xrightarrow{[\![M]\!]} [\![A]\!] \xrightarrow{p_A} \mathcal{K}[\![A]\!]}$$

$$\frac{[\![\Gamma \vdash M : \boldsymbol{K}(A \to B)]\!] := [\![M]\!] : [\![\Gamma]\!] \to \mathcal{K}([\![B]\!]^{[\![A]\!]}) \qquad [\![\Gamma \vdash N : \boldsymbol{K}A]\!] := [\![N]\!] : [\![\Gamma]\!] \to \mathcal{K}[\![A]\!]}{[\![\Gamma \vdash M \star N : \boldsymbol{K}B]\!] := [\![\Gamma]\!] \xrightarrow{\mathcal{K}(\epsilon_{A,B}) \circ \cong \circ \langle [\![M]\!], [\![N]\!]\rangle} \mathcal{K}B}$$

Definition 7. *Simultaneous substitution*

Let $\Gamma = \{x_1 : A_1, ..., x_n : A_n\}$, $\Gamma \vdash M : A$ *and for all* $i \in \{1, ..., n\}$, $\Gamma \vdash M_i : A_i$.

We define simultaneous substitution $M[\vec{x} := \vec{M}]$ *recursively by:*

1) $x_i[\vec{x} := \vec{M}] := M_i$;

2) $(\lambda x.M)[\vec{x} := \vec{M}] := \lambda x.(M[\vec{x} := \vec{M}])$;

3) $(MN)[\vec{x} := \vec{M}] = (M[\vec{x} := \vec{M}])(N[\vec{x} := \vec{M}])$;

4) $\langle M, N\rangle = \langle (M[\vec{x} := \vec{M}]), (N[\vec{x} := \vec{M}])\rangle$;

5) $(\pi_i P)[\vec{x} := \vec{M}] = \pi_i(P[\vec{x} := \vec{M}])$;

6) $(\mathit{pure}\ M)[\vec{x} := \vec{M}] = \mathit{pure}\ (M[\vec{x} := \vec{M}])$;

7) $(M \star N)[\vec{x} := \vec{M}] = (M[\vec{x} := \vec{M}]) \star (N[\vec{x} := \vec{M}])$.

Lemma 1.

$[\![M[x_1 := M_1, \ldots, x_n := M_n]]\!] = [\![M]\!] \circ \langle [\![M_1]\!], \ldots, [\![M_n]\!]\rangle$.

Proof.

1) $[\![(\text{pure } M)[\vec{x} := \vec{M}]]\!] = [\![\text{pure } M]\!] \circ \langle [\![M_1]\!], \ldots, [\![M_n]\!]\rangle$.

$$\begin{aligned}
[\![(\text{pure } M)[\vec{x} := \vec{M}]]\!] &= [\![\text{pure } (M[\vec{x} := \vec{M}])]\!] && \text{Substitution definition}\\
&= p \circ [\![(M[\vec{x} := \vec{M}])]\!] && \text{Translation for pure}\\
&= p \circ [\![M]\!] \circ \langle [\![M_1]\!], \ldots, [\![M_n]\!]\rangle && \text{Induction hypothesis}\\
&= (p \circ [\![M]\!]) \circ \langle [\![M_1]\!], \ldots, [\![M_n]\!]\rangle && \text{Associativity of composition}\\
&= [\![\text{pure } M]\!] \circ \langle [\![M_1]\!], \ldots, [\![M_n]\!]\rangle && \text{Translation for pure}
\end{aligned}$$

2) $[\![(M \star N)[\vec{x} := \vec{M}]]\!] = [\![M \star N]\!] \circ \langle [\![M_1]\!], \ldots, [\![M_n]\!]\rangle$.

$$\begin{aligned}
&[\![(M \star N)[\vec{x} := \vec{M}]]\!] = [\![(M[\vec{x} := \vec{M}]) \star (N[\vec{x} := \vec{M}])]\!] && \text{Definition of substitution}\\
&\quad = p_\epsilon \circ * \circ \langle [\![(M[\vec{x} := \vec{M}])]\!], [\![(N[\vec{x} := \vec{M}])]\!]\rangle && \text{Translation for } \star\\
&\quad = p_\epsilon \circ * \circ \langle [\![M]\!] \circ \langle [\![M_1]\!], \ldots, [\![M_n]\!]\rangle, [\![N]\!] \circ \langle [\![M_1]\!], \ldots, [\![M_n]\!]\rangle\rangle && \text{Induction hypothesis}\\
&\quad = p_\epsilon \circ * \circ \langle [\![M]\!], [\![N]\!]\rangle \circ \langle [\![M_1]\!], \ldots, [\![M_n]\!]\rangle && \text{Property of morphism product}\\
&\quad = (p_\epsilon \circ * \circ \langle [\![M]\!], [\![N]\!]\rangle) \circ \langle [\![M_1]\!], \ldots, [\![M_n]\!]\rangle && \text{Associativity of composition}\\
&\quad = [\![M \star N]\!] \circ \langle [\![M_1]\!], \ldots, [\![M_n]\!]\rangle && \text{Translation for } \star
\end{aligned}$$

□

Lemma 2.

If $M \to_\beta N$, *then* $[\![M]\!] = [\![N]\!]$.

1) $[\![\text{pure } (\lambda x.x) \star M]\!] = [\![M]\!]$;

$$\dfrac{\dfrac{[\![x : A \vdash x : A]\!] = \pi_2 : \mathbb{1} \times [\![A]\!] \to [\![A]\!]}{[\![\vdash \lambda x.x : A \to A]\!] = \Lambda(\pi_2) : \mathbb{1} \to [\![A]\!]^{[\![A]\!]}}}{[\![\vdash \text{pure } (\lambda x.x) : \mathbf{K}(A \to A)]\!] = p_{[\![A]\!]^{[\![A]\!]}} \circ \Lambda(\pi_2) : \mathbb{1} \to \mathcal{K}([\![A]\!]^{[\![A]\!]})}$$

But by the following diagram:

$$\begin{tikzcd}
{[\![A]\!]^{[\![A]\!]}} \arrow[r, "p_{[\![A]\!]^{[\![A]\!]}}"] & \mathcal{K}([\![A]\!]^{[\![A]\!]}) \\
\mathbb{1} \arrow[u, "\Lambda\pi_2"] \arrow[r, "p_{\mathbb{1}=id_{\mathbb{1}}}"'] & \mathbb{1} \arrow[u, "\mathcal{K}(\Lambda(\pi_2))"']
\end{tikzcd}$$

$$
\begin{aligned}
p_{[\![A]\!]^{[\![A]\!]}} \circ \Lambda(\pi_2) &= id_{\mathbb{1}} \circ \mathcal{K}(\Lambda(\pi_2)) \\
&= \mathcal{K}(\Lambda(\pi_2))
\end{aligned}
$$

So:
$[\![\vdash \text{pure } (\lambda x.x) : \mathbf{K}(A \to A)]\!] = \mathcal{K}(\Lambda(\pi_2)) : \mathbb{1} \to \mathcal{K}([\![A]\!]^{[\![A]\!]})$
$[\![M : \mathbf{K}A \vdash M : \mathbf{K}A]\!] = id_{\mathcal{K}A} : \mathcal{K}([\![A]\!]) \to \mathcal{K}([\![A]\!])$, or
$[\![M : \mathbf{K}A \vdash M : \mathbf{K}A]\!] = \pi_2 : \mathbb{1} \times \mathcal{K}([\![A]\!]) \to \mathcal{K}([\![A]\!])$.

Let us consider the next commutative diagram:

$$
\begin{tikzcd}
\mathcal{K}(A^A) \times \mathcal{K}A \arrow[r, "\cong"] & \mathcal{K}(A^A \times A) \arrow[r, "\mathcal{K}(\epsilon)"] & \mathcal{K}A \\
\mathbb{1} \times \mathcal{K}A \arrow[u, "\mathcal{K}(\Lambda(\pi_2)) \times id_{\mathcal{K}A}"] \arrow[r, "\cong"'] & \mathcal{K}(\mathbb{1} \times A) \arrow[u, "\mathcal{K}(\Lambda(\pi_2) \times id_A)"] \arrow[ur, "\mathcal{K}(\pi_2)"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1} \times \mathcal{K}A \arrow[r, "\cong"] \arrow[d, "\pi_2"'] & \mathcal{K}(A^A \times A) \arrow[d, "\mathcal{K}(\pi_2)"] \\
\mathcal{K}A \arrow[r, "id_{\mathcal{K}A}"'] & \mathcal{K}A
\end{tikzcd}
$$

Hence:

$$
\begin{aligned}
[\![M : \mathbf{K}A \vdash \text{pure } (\lambda x.x) \star M]\!] &= (\mathcal{K}(\epsilon) \circ (\cong)) \circ (\mathcal{K}(\Lambda(\pi_2)) \times id_{\mathcal{K}A}) \\
&= \mathcal{K}(\epsilon) \circ \mathcal{K}(\Lambda(\pi_2) \times id_A) \circ (\cong) \\
&= \mathcal{K}(\epsilon) \circ (\cong) \\
&= \pi_2 = [\![M : \mathbf{K}A \vdash M : \mathbf{K}A]\!]
\end{aligned}
$$

2) $[\![(\text{pure } \lambda fgx.f(gx)) \star M \star N \star P]\!] = [\![M \star (N \star P)]\!]$

i) The first step.

Let us consider interpretation for $\vdash \text{pure } \lambda fgx.f(gx) : \mathbf{K}((B \to C) \to (A \to B) \to A \to C)$:

$$\begin{array}{c}
\pi_2 : \mathbb{1} \times [\![B]\!] \Rightarrow [\![C]\!] \to [\![B]\!] \Rightarrow [\![C]\!] \qquad \epsilon : [\![A]\!] \Rightarrow [\![B]\!] \times [\![A]\!] \to [\![B]\!] \\
\hline
\pi_2 \times \epsilon : (\mathbb{1} \times [\![B]\!] \Rightarrow [\![C]\!]) \times ([\![A]\!] \Rightarrow [\![B]\!] \times [\![A]\!]) \to [\![B]\!] \Rightarrow [\![C]\!] \times [\![B]\!] \\
\hline
\epsilon \circ (\pi_2 \times \epsilon) : (\mathbb{1} \times [\![B]\!] \Rightarrow [\![C]\!]) \times ([\![A]\!] \Rightarrow [\![B]\!] \times [\![A]\!]) \to [\![C]\!] \\
\hline
\epsilon \circ (\pi_2 \times \epsilon) \circ \alpha : (((\mathbb{1} \times [\![B]\!] \Rightarrow [\![C]\!]) \times [\![A]\!] \Rightarrow [\![B]\!]) \times [\![A]\!] \to [\![C]\!] \\
\hline
\Lambda(\epsilon \circ (\pi_2 \times \epsilon) \circ \alpha) : (\mathbb{1} \times [\![B]\!] \Rightarrow [\![C]\!]) \times [\![A]\!] \Rightarrow [\![B]\!] \to [\![A]\!] \Rightarrow [\![C]\!] \\
\hline
\Lambda(\Lambda(\epsilon \circ (\pi_2 \times \epsilon) \circ \alpha)) : \mathbb{1} \times [\![B]\!] \Rightarrow [\![C]\!] \to ([\![A]\!] \Rightarrow [\![B]\!]) \Rightarrow [\![A]\!] \Rightarrow [\![C]\!] \\
\hline
\Lambda(\Lambda(\Lambda(\epsilon \circ (\pi_2 \times \epsilon) \circ \alpha))) : \mathbb{1} \to ([\![B]\!] \Rightarrow [\![C]\!]) \Rightarrow ([\![A]\!] \Rightarrow [\![B]\!]) \Rightarrow [\![A]\!] \Rightarrow [\![C]\!] \\
\hline
\mathcal{K}(\Lambda(\Lambda(\Lambda(\epsilon \circ (\pi_2 \times \epsilon) \circ \alpha)))) : \mathbb{1} \to \mathcal{K}(([\![B]\!] \Rightarrow [\![C]\!]) \Rightarrow ([\![A]\!] \Rightarrow [\![B]\!]) \Rightarrow [\![A]\!] \Rightarrow [\![C]\!])
\end{array}$$

2) The second step:

3) $[\![(\text{pure } M) \star (\text{pure } N)]\!] = [\![\text{pure } (MN)]\!]$;

1) The left part of the equality:

$$\frac{[\![\Gamma \vdash M : A \to B]\!] = f : [\![\Gamma]\!] \to [\![B]\!]^{[A]}}{[\![\Gamma \vdash \text{pure } M : \mathbf{K}(A \to B)]\!] = p_{[\![B]\!]^{[A]}} \circ f : [\![\Gamma]\!] \to \mathcal{K}([\![B]\!]^{[A]})}$$

$$\frac{[\![\Delta \vdash N : A]\!] = g : [\![\Delta]\!] \to [\![A]\!]}{[\![\Delta \vdash \text{pure } N : \mathbf{K}A]\!] = p_{[\![A]\!]} \circ g : [\![\Delta]\!] \to \mathcal{K}[\![A]\!]}$$

$$[\![\Gamma, \Delta \vdash (\text{pure } M) \star (\text{pure } N) : \mathbf{K}B]\!] = \mathcal{K}(\epsilon) \circ (\cong) \circ (p_{[\![B]\!]^{[A]}} \circ f \times p_{[\![A]\!]} \circ g) : \Gamma \times \Delta \to \mathcal{K}B$$

2) The second part of equality:

$$\frac{\dfrac{[\![\Gamma \vdash M : A \to B]\!] = f : [\![\Gamma]\!] \to [\![B]\!]^{[A]} \qquad [\![\Delta \vdash N : A]\!] = g : [\![\Delta]\!] \to [\![A]\!]}{[\![\Gamma, \Delta \vdash MN : B]\!] = \epsilon \circ f \times g : [\![\Gamma]\!] \times [\![\Delta]\!] \to [\![B]\!]}}{[\![\Gamma, \Delta \vdash \text{pure } (MN) : \mathbf{K}B]\!] = p_{[\![B]\!]} \circ (\epsilon \circ (f \times g)) : [\![\Gamma]\!] \times [\![\Delta]\!] \to \mathcal{K}[\![B]\!]}$$

$$\begin{tikzcd}
 & {[\![\Gamma]\!] \times [\![\Delta]\!]} \arrow[ddl, "p_{[\![B]\!]^{[A]}} \circ f \times p_{[\![A]\!]} \circ g"'] \arrow[d, "f \times g"'] \arrow[dr, "\epsilon \circ f \times g"] & \\
 & {[\![B]\!]^{[\![A]\!]} \times [\![A]\!]} \arrow[r, "\epsilon"] \arrow[dl, "p_{[\![B]\!]^{[A]}} \times p_{[\![A]\!]}"] \arrow[dd, "p_{[\![B]\!]^{[A]} \times [\![A]\!]}"'] & {[\![B]\!]} \arrow[dd, "p_{[\![B]\!]}"] \\
{\mathcal{K}([\![B]\!]^{[\![A]\!]}) \times \mathcal{K}[\![A]\!]} \arrow[dr, "\cong"'] & & \\
 & {\mathcal{K}([\![B]\!]^{[\![A]\!]} \times [\![A]\!])} \arrow[r, "\mathcal{K}(\epsilon)"'] & {\mathcal{K}[\![B]\!]}
\end{tikzcd}$$

$$\begin{aligned}
[\![\Gamma, \Delta \vdash (\text{pure } M) \star (\text{pure } N) : \mathbf{K}B]\!] &= \mathcal{K}(\epsilon) \circ (\cong) \circ (p_{[\![B]\!]^{[A]}} \circ f \times p_{[\![A]\!]} \circ g) \\
&= K(\epsilon) \circ (\cong) \circ p_{[\![B]\!]^{[\![A]\!]}} \times p_{[\![A]\!]} \circ f \times g \\
&= K(\epsilon) \circ p_{[\![B]\!]^{[\![A]\!]} \times [\![A]\!]} \circ f \times g \\
&= p_{[\![B]\!]} \circ \epsilon \circ f \times g \\
&= [\![\Gamma, \Delta \vdash \text{pure } (MN) : \mathcal{K}B]\!]
\end{aligned}$$

4) $\llbracket N : A, M : \mathbf{K}(A \to B) \vdash M \star \text{pure } N : \mathbf{K}B \rrbracket =$
$\llbracket N : A, M : \mathbf{K}(A \to B) \vdash \text{pure } (\lambda f.fN) \star M : \mathbf{K}B \rrbracket$

It is easy to see that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{K}(\llbracket B \rrbracket^{(\llbracket B \rrbracket^{\llbracket A \rrbracket})}) \times \mathcal{K}(\llbracket B \rrbracket^{\llbracket A \rrbracket}) \arrow[r, "\cong"] & \mathcal{K}(\llbracket B \rrbracket^{(\llbracket B \rrbracket^{\llbracket A \rrbracket})} \times \llbracket B \rrbracket^{\llbracket A \rrbracket}) \arrow[r, "\mathcal{K}(\epsilon)"] & \mathcal{K}\llbracket B \rrbracket \\
\mathcal{K}\llbracket A \rrbracket \times \mathcal{K}(\llbracket B \rrbracket^{\llbracket A \rrbracket}) \arrow[u, "\mathcal{K}(\Lambda(\epsilon \circ \langle \pi_2, \pi_1 \rangle))) \times id_{\mathcal{K}\llbracket B \rrbracket^{\llbracket A \rrbracket}}"] \arrow[r, "\cong"] & \mathcal{K}(\llbracket A \rrbracket \times \llbracket B \rrbracket^{\llbracket A \rrbracket}) \arrow[u, "\mathcal{K}(\Lambda(\epsilon \circ \langle \pi_2, \pi_1 \rangle) \times id_{\llbracket B \rrbracket^{\llbracket A \rrbracket}})"] \arrow[r, "\mathcal{K}(\langle \pi_2, \pi_1 \rangle)"] & \mathcal{K}(\llbracket B \rrbracket^{\llbracket A \rrbracket} \times \llbracket A \rrbracket) \arrow[u, "\mathcal{K}(\epsilon)"'] \\
\llbracket A \rrbracket \times \mathcal{K}(\llbracket B \rrbracket^{\llbracket A \rrbracket}) \arrow[u, "p_{\llbracket A \rrbracket} \times id_{\mathcal{K}(\llbracket B \rrbracket^{\llbracket A \rrbracket})}"] \arrow[r, "p_{\llbracket A \rrbracket} \times id_{\mathcal{K}(\llbracket B \rrbracket^{\llbracket A \rrbracket})}"'] & \mathcal{K}\llbracket A \rrbracket \times \mathcal{K}(\llbracket B \rrbracket^{\llbracket A \rrbracket}) \arrow[u, "\cong"'] \arrow[r, "\langle \pi_2, \pi_1 \rangle"'] & \mathcal{K}(\llbracket B \rrbracket^{\llbracket A \rrbracket}) \times \mathcal{K}\llbracket A \rrbracket \arrow[u, "\cong"']
\end{tikzcd}$$

$\llbracket N : A, M : \mathbf{K}(A \to B) \vdash \text{pure } (\lambda f.fN) \star M : \mathbf{K}B \rrbracket =$
by interpretation
$\mathcal{K}(\epsilon) \circ (\cong) \circ ((p_{\llbracket B \rrbracket^{(\llbracket B \rrbracket^{\llbracket A \rrbracket})}} \circ \Lambda(\epsilon \circ \langle \pi_2, \pi_1 \rangle)) \times id_{\mathcal{K}(\llbracket B \rrbracket^{\llbracket A \rrbracket})}) =$
by the definition of p
$\mathcal{K}(\epsilon) \circ (\cong) \circ (\mathcal{K}(\Lambda(\epsilon \circ \langle \pi_2, \pi_1 \rangle)) \circ p_{\llbracket A \rrbracket}) \times id_{\mathcal{K}(\llbracket B \rrbracket^{\llbracket A \rrbracket})} =$
by the definition of identity function
$\mathcal{K}(\epsilon) \circ (\cong) \circ (\mathcal{K}(\Lambda(\epsilon \circ \langle \pi_2, \pi_1 \rangle)) \circ p_{\llbracket A \rrbracket}) \times (id_{\mathcal{K}(\llbracket B \rrbracket^{\llbracket A \rrbracket})} \circ id_{\mathcal{K}(\llbracket B \rrbracket^{\llbracket A \rrbracket})}) =$
the property of composition of product morphisms
$\mathcal{K}(\epsilon) \circ (\cong) \circ (\mathcal{K}(\Lambda(\epsilon \circ \langle \pi_2, \pi_1 \rangle)) \times id_{\mathcal{K}(\llbracket B \rrbracket^{\llbracket A \rrbracket})}) \circ (p_{\llbracket A \rrbracket} \times id_{\mathcal{K}(\llbracket B \rrbracket^{\llbracket A \rrbracket})})) =$
diagram above
$\mathcal{K}(\epsilon) \circ (\cong) \circ \langle \pi_2, \pi_1 \rangle \circ (p_{\llbracket A \rrbracket} \times id_{\mathcal{K}(\llbracket B \rrbracket^{\llbracket A \rrbracket})})) =$
the property of product morphisms
$\mathcal{K}(\epsilon) \circ (\cong) \circ \langle \pi_2 \circ (p_{\llbracket A \rrbracket} \times id_{\mathcal{K}(\llbracket B \rrbracket^{\llbracket A \rrbracket})}, \pi_1 \circ (p_{\llbracket A \rrbracket} \times id_{\mathcal{K}(\llbracket B \rrbracket^{\llbracket A \rrbracket})} \rangle =$
by unfolding the morphism product
$\mathcal{K}(\epsilon) \circ (\cong) \circ \langle \pi_2 \circ \langle p_{\llbracket A \rrbracket} \circ \pi_1, id_{\mathcal{K}(\llbracket B \rrbracket^{\llbracket A \rrbracket})} \circ \pi_2 \rangle, \pi_1 \circ \langle p_{\llbracket A \rrbracket} \circ \pi_1, id_{\mathcal{K}(\llbracket B \rrbracket^{\llbracket A \rrbracket})} \circ \pi_2 \rangle \rangle =$
by the definition of pair morphism
$\mathcal{K}(\epsilon) \circ (\cong) \circ \langle id_{\mathcal{K}(\llbracket B \rrbracket^{\llbracket A \rrbracket})} \circ \pi_2, p_{\llbracket A \rrbracket} \circ \pi_1 \rangle =$
the property of product morphisms
$\mathcal{K}(\epsilon) \circ (\cong) \circ (id_{\mathcal{K}(\llbracket B \rrbracket^{\llbracket A \rrbracket})} \times p_{\llbracket A \rrbracket}) \circ \langle \pi_2, \pi_1 \rangle =$
by interpretation
$\llbracket N : A, M : \mathbf{K}(A \to B) \vdash M \star \text{pure } N : \mathbf{K}B \rrbracket$

Lemma 3. *If $M \to_\eta N$, then $\llbracket M \rrbracket = \llbracket N \rrbracket$.*

Proof.
1) $\llbracket \text{pure } (\lambda x.fx) \rrbracket = \llbracket \text{pure } f \rrbracket$.

$$
\begin{aligned}
\llbracket \text{pure } (\lambda x.fx) \rrbracket &= p \circ \llbracket \lambda x.fx \rrbracket && \text{Translation for pure} \\
&= p \circ \llbracket f \rrbracket && \eta\text{-reduction rule for application} \\
&= \llbracket \text{pure } f \rrbracket && \text{Translation for pure}
\end{aligned}
$$

2) $\llbracket \text{pure } \langle \pi_1 M, \pi_2 M \rangle \rrbracket = \llbracket \text{pure } M \rrbracket$

$$
\begin{aligned}
\llbracket \text{pure } \langle \pi_1 M, \pi_2 M \rangle \rrbracket &= p \circ \llbracket \langle \pi_1 M, \pi_2 M \rangle \rrbracket && \text{Translation for pure} \\
&= p \circ \llbracket M \rrbracket && \eta\text{-reduction rule for pair} \\
&= \llbracket \text{pure } M \rrbracket && \text{Translation for pure}
\end{aligned}
$$

3)
$\llbracket M : \mathbf{K}(A \times B) \vdash \text{pure } (\lambda x.\lambda y.\langle x, y \rangle) \star (\text{pure } (\lambda x.\pi_1) \star M) \star (\text{pure } (\lambda x.\pi_2) \star M : \mathbf{K}(A \times B) \rrbracket =$
$\llbracket M : \mathbf{K}(A \times B) \vdash M : \mathbf{K}(A \times B) \rrbracket$

i) The first step
Let us consider interpetation for $\vdash \text{pure } (\lambda x.\lambda y.\langle x, y \rangle) : \mathbf{K}(A \to B \to A \times B)$:

$$
\cfrac{\cfrac{\cfrac{\cfrac{\pi_2 : \mathbb{1} \times \llbracket A \rrbracket \to \llbracket A \rrbracket \qquad id_{\llbracket B \rrbracket} : \llbracket B \rrbracket \to \llbracket B \rrbracket}{\pi_2 \times id_{\llbracket B \rrbracket} : (\mathbb{1} \times \llbracket A \rrbracket) \times \llbracket B \rrbracket \to \llbracket A \rrbracket \times \llbracket B \rrbracket}}{\Lambda(\pi_2 \times id_{\llbracket B \rrbracket}) : \mathbb{1} \times \llbracket A \rrbracket \to \llbracket A \rrbracket \times \llbracket B \rrbracket^{\llbracket B \rrbracket}}}{\Lambda(\Lambda(\pi_2 \times id_{\llbracket B \rrbracket})) : \mathbb{1} \to \llbracket A \rrbracket \times \llbracket B \rrbracket^{\llbracket B \rrbracket^{\llbracket A \rrbracket}}}}{p_{\llbracket A \rrbracket \times \llbracket B \rrbracket^{\llbracket B \rrbracket^{\llbracket A \rrbracket}}} \circ \Lambda(\Lambda(\pi_2 \times id_{\llbracket B \rrbracket})) : \mathbb{1} \to \mathcal{K}(\llbracket A \rrbracket \times \llbracket B \rrbracket^{\llbracket B \rrbracket^{\llbracket A \rrbracket}})}
$$

By naturality, $p_{\llbracket A \rrbracket \times \llbracket B \rrbracket^{\llbracket B \rrbracket^{\llbracket A \rrbracket}}} \circ \Lambda(\Lambda(\pi_2 \circ \alpha)) = \mathcal{K}(\Lambda(\Lambda(\pi_2 \times id_{\llbracket B \rrbracket}))$.
At first let us show that the following diagram commutes in any CCC:

$$
\begin{tikzcd}
(\llbracket A \times B \rrbracket^{\llbracket B \rrbracket^{\llbracket A \rrbracket}} \times \llbracket A \rrbracket) \times \llbracket B \rrbracket \arrow[r, "\epsilon \circ (\epsilon \times id_{\llbracket B \rrbracket})"] & \llbracket A \rrbracket \times \llbracket B \rrbracket \\
([1] \times \llbracket A \rrbracket) \times \llbracket B \rrbracket \arrow[u, "(\Lambda(\Lambda(\pi_2 \times id_{\llbracket B \rrbracket}) \times id_{\llbracket A \rrbracket}) \times id_{\llbracket B \rrbracket}"] \arrow[ur, "\pi_2 \times id_{\llbracket B \rrbracket}"'] &
\end{tikzcd}
$$

$\epsilon \circ (\epsilon \times id_{\llbracket B \rrbracket}) \circ (\Lambda(\Lambda(\pi_2 \times id_{\llbracket B \rrbracket})) \times id_{\llbracket A \rrbracket}) \times id_{\llbracket B \rrbracket} =$
by the definition of morphism product
$\epsilon \circ (\epsilon \times id_{\llbracket B \rrbracket}) \circ \langle \Lambda(\Lambda(\pi_2 \times id_{\llbracket B \rrbracket})) \circ \pi_1, id_{\llbracket A \rrbracket} \circ \pi_2 \rangle \times id_{\llbracket B \rrbracket} =$
by the definition of morphism product
$\epsilon \circ (\epsilon \times id_{\llbracket B \rrbracket}) \circ \langle \langle \Lambda(\Lambda(\pi_2 \times id_{\llbracket B \rrbracket})) \circ \pi_1, id_{\llbracket A \rrbracket} \circ \pi_2 \rangle \circ \pi_1, id_{\llbracket B \rrbracket} \circ \pi_2 \rangle$
by the property of morphism product
$\epsilon \circ \langle \epsilon \circ \langle \Lambda(\Lambda(\pi_2 \times id_{\llbracket B \rrbracket})) \circ \pi_1, id_{\llbracket A \rrbracket} \circ \pi_2 \rangle \circ \pi_1, id_{\llbracket B \rrbracket} \circ id_{\llbracket B \rrbracket} \circ \pi_2 \rangle =$
by the definition of morphism product and by identity
$\epsilon \circ \langle \epsilon \circ (\Lambda(\Lambda(\pi_2 \times id_{\llbracket B \rrbracket})) \times id_{\llbracket A \rrbracket}) \circ \pi_1, id_{\llbracket B \rrbracket} \circ \pi_2 \rangle =$
by exponentiation and currying property
$\epsilon \circ \langle \Lambda(\pi_2 \times id_{\llbracket B \rrbracket}) \circ \pi_1, id_{\llbracket B \rrbracket} \circ \pi_2 \rangle =$
by the definition of morphism product
$\epsilon \circ \Lambda(\pi_2 \times id_{\llbracket B \rrbracket}) \times id_{\llbracket B \rrbracket}$
by exponentiation and currying property
$\pi_2 \times id_{\llbracket B \rrbracket}$

2)

$\square$

Lemma 4.

1) $[\![M]\!] = [\![N]\!]$*, if* $[\![\mathit{pure}\ M]\!] = [\![\mathit{pure}\ N]\!]$*;*

2) Let $[\![M]\!] = [\![N]\!]$*, then* $[\![M \star P]\!] = [\![N \star P]\!]$*;*

3) Let $[\![M]\!] = [\![N]\!]$*, then* $[\![P \star M]\!] = [\![P \star N]\!]$*.*

Proof.

1)

i) "only if"-part.

Let $[\![M]\!] : [\![\Gamma]\!] \to [\![A]\!]$, $[\![N]\!] : [\![\Gamma]\!] \to [\![A]\!]$ and $[\![M]\!] = [\![N]\!]$. So $p \circ [\![M]\!] = p \circ [\![N]\!]$, hence $[\![\text{pure } M]\!] = [\![\text{pure } N]\!]$.

$\square$