Soundness for modal type theory based on the intuitionistic epistemic logic

1 Modal lambda calculus based on IEL$^-$

Definition 1. *The set of terms:*
Let $\mathbb{V}$ is a set of variables. The set $\Lambda_{\boldsymbol{K}}$ of terms is defined by the grammar:

$$\Lambda_{\boldsymbol{K}} ::= \mathbb{V} \mid (\lambda\Lambda.\Lambda_{\boldsymbol{K}}) \mid (\Lambda_{\boldsymbol{K}}\Lambda_{\boldsymbol{K}}) \mid (\Lambda_{\boldsymbol{K}}, \Lambda_{\boldsymbol{K}}) \mid (\pi_i\Lambda_{\boldsymbol{K}}) \mid (\mathit{pure}\ \Lambda_{\boldsymbol{K}}) \mid (\Lambda_{\boldsymbol{K}} \star \Lambda_{\boldsymbol{K}}) \quad (1)$$

where $i \in \{1, 2\}$.

Definition 2. *The set of types:*
Let $\mathbb{T}$ is a set of atomic types. The set $\mathbb{T}_{\boldsymbol{K}}$ of types with applicative functor $\boldsymbol{K}$ is generated by the grammar:

$$\mathbb{T}_{\boldsymbol{K}} ::= \mathbb{T} \mid (\mathbb{T}_{\boldsymbol{K}} \to \mathbb{T}_{\boldsymbol{K}}) \mid (\mathbb{T}_{\boldsymbol{K}} \times \mathbb{T}_{\boldsymbol{K}}) \mid (\boldsymbol{K}\mathbb{T}_{\boldsymbol{K}}) \quad (2)$$

Our type system is based on the Curry-style typing rules:

Definition 3. *Modal typed lambda calculus $\lambda\boldsymbol{K}$ based on NIEL$^-_{\wedge,\to}$:*

$$\frac{}{\Gamma, x : \alpha \vdash x : \alpha}\ ax$$

$$\frac{\Gamma, x : \alpha \vdash M : \beta}{\Gamma \vdash \lambda x.M : \alpha \to \beta} \to_i$$

$$\frac{\Gamma \vdash x : \alpha \qquad \Gamma \vdash y : \beta}{\Gamma \vdash (x, y) : \alpha \times \beta} \times_i$$

$$\frac{\Gamma, \vdash x : \alpha}{\Gamma \vdash \mathit{pure}\ \ x : \boldsymbol{K}\alpha}\ \boldsymbol{K}_I$$

$$\frac{\Gamma \vdash f : \alpha \to \beta \qquad \Gamma \vdash x : \alpha}{\Gamma \vdash fx : \beta} \to_e$$

$$\frac{\Gamma \vdash p : \alpha_1 \times \alpha_2}{\Gamma \vdash \pi_i p : \alpha_i} \times_e,\ i \in \{1, 2\}$$

$$\frac{\Gamma \vdash f : \boldsymbol{K}(\alpha \to \beta) \qquad \Gamma \vdash x : \boldsymbol{K}\alpha}{\Gamma \vdash f \star x : \boldsymbol{K}\beta}\ \boldsymbol{K}_{app}$$

Definition 4. *β-reduction rules:*
1) $(\lambda x.M)N \to_\beta M[x := N]$;
2) $\pi_i\langle M_1, M_2\rangle \to_\beta M_i, i \in \{1, 2\}$;
3) pure $(\lambda x.x) \star M \to_\beta M$;
4) pure $(\lambda fgx.f(gx)) \star M \star N \star P \to_\beta M \star (N \star P)$;
5) (pure $M) \star$ (pure $N) \to_\beta$ pure (MN);
6) $M \star$ pure $N \to_\beta (\lambda f.fN) \star M$;

Definition 5. *η-reduction rules for applicative functor:*
1) pure $(\lambda x.fx) \to_\eta$ pure f;
2) pure $\langle \pi_1 p, \pi_2 p\rangle \to_\eta$ pure p;
3) $\lambda x.f \star x \to_\eta f$.

2 Categorical model.

Let us define monoidal categories and strong lax monoidal functors.

Definition 6. *Monoidal category.*
A monoidal category $\mathcal{C}$ is a category with:
1) A bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the tensor product;
2) An object $\mathbb{1} \in Ob(\mathcal{C})$ called the unit;
3) A natural isomorphism such that for all $A, B, C \in Ob(\mathcal{C})$:

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \xrightarrow{\cong} A \otimes (B \otimes C)$$

where α is called associator.
4) A natural isomorphism (left unitor) for all $A \in Ob(\mathcal{C})$:

$$L_A : (\mathbb{1} \otimes A) \xrightarrow{\cong} A$$

5) A natural isomorphism (right unitor) for all $A \in Ob(\mathcal{C})$:

$$R_A : (A \otimes \mathbb{1}) \xrightarrow{\cong} A$$

6) The next one diagram commutes (the triangle identity):

$$\begin{tikzcd}
 & A \otimes B & \\
(A \otimes \mathbb{1}) \otimes B \arrow[ur, "R_A \otimes id_B"] \arrow[rr, "\alpha_{A,\mathbb{1},B}"'] & & A \otimes (\mathbb{1} \otimes B) \arrow[ul, "id_A \otimes L_B"']
\end{tikzcd}$$

7) The next one diagram commutes too (the pentagon identity):

$$\begin{tikzcd}
& (A \otimes B) \otimes (C \otimes D) \arrow[dr, "\alpha_{A,B,C\otimes D}"] & \\
((A \otimes B) \otimes C) \otimes D \arrow[ur, "\alpha_{A\otimes B,C,D}"] \arrow[d, "\alpha_{A,B,C}\otimes id_D"'] & & A \otimes (B \otimes (C \otimes D)) \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "\alpha_{A,B\otimes C,D}"'] & & A \otimes ((B \otimes C) \otimes D) \arrow[u, "id_A \otimes \alpha_{B,C,D}"']
\end{tikzcd}$$

A monoidal category is symmetrical iff $\forall A, B \in Ob(\mathcal{C}), A \otimes B \cong B \otimes A$.

Definition 7. *A lax monoidal functor between monoinal categories* $\langle \mathcal{C}, \otimes, \mathbb{1} \rangle$ *and* $\langle \mathcal{D}, \otimes', \mathbb{1} \rangle$ *is a functor* $F : \langle \mathcal{C}, \otimes, \mathbb{1} \rangle \to \langle \mathcal{D}, \otimes', \mathbb{1} \rangle$ *with the next natural transformations:*

1) $u : \mathbb{1} \to F\mathbb{1}$ *(unit property);*

2) $* : FA \otimes' FB \to F(A \otimes B)$ *(application property);*

and with the next commuting diagrams:

$$\begin{tikzcd}
\mathbb{1} \otimes' FA \arrow[r, "L_{FA}"'] \arrow[d, "u\otimes id_{FA}"'] & FA \\
F\mathbb{1} \otimes' FA \arrow[r, "*"'] & F(\mathbb{1} \otimes A) \arrow[u, "FL"']
\end{tikzcd}$$

$$\begin{tikzcd}
FA \otimes' \mathbb{1} \arrow[r, "R_{FA}"'] \arrow[d, "id_{FA}\otimes u"'] & FA \\
FA \otimes' F\mathbb{1} \arrow[r, "*"'] & F(A \otimes \mathbb{1}) \arrow[u, "FR"']
\end{tikzcd}$$

$$\begin{tikzcd}
& FA \otimes' F(B \otimes C) \arrow[dr, "*"] & \\
FA \otimes' (FB \otimes' FC) \arrow[ur, "id_{FA}\otimes *"] \arrow[d, "\alpha_{FA,FB,FC}"'] & & F(A \otimes (B \otimes C)) \arrow[d, "F\alpha_{A,B,C}"] \\
(FA \otimes' FB) \otimes' FC \arrow[dr, "*\otimes id_{FC}"'] & & F((A \otimes B) \otimes C) \\
& F(A \otimes B) \otimes' FC \arrow[ur, "*"'] &
\end{tikzcd}$$

Definition 8. *Applicative functor.*

Let $\langle \mathcal{C}, \otimes, \mathbb{1} \rangle$ *is a symmetrcial monoidal category. Applicative functor is an endofunctor* $F : \langle \mathcal{C}, \otimes, \mathbb{1} \rangle \to \langle \mathcal{C}, \otimes, \mathbb{1} \rangle$ *with a natural transformation* $p : Id_{\mathcal{C}} \Rightarrow F$ *with the next properties:*

1) A natural transformation p is defined as follows for all $A \in Ob(\mathcal{C})$ the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "p_A"'] & B \arrow[d, "p_B"] \\
FA \arrow[r, "Ff"'] & FB
\end{tikzcd}$$

2) $p_{\mathbb{1}} = u$:

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "id_{\mathbb{1}}"] \arrow[d, "p_{\mathbb{1}}"'] & \mathbb{1} \arrow[d, "u"] \\
F\mathbb{1} \arrow[r, "id_{F\mathbb{1}}"'] & F\mathbb{1}
\end{tikzcd}$$

3) $p \circ * = * \circ (p \otimes p)$;

4) The following diagram commutes (weak commutativity condition):

$$\begin{tikzcd}
& FA \otimes FB \arrow[dr, "*"] & \\
A \otimes FB \arrow[ur, "p_A \otimes id_{FB}"] \arrow[d, "swap_{A,FB}"'] & & F(A \otimes B) \arrow[d, "F(swap_{A,B})"] \\
FB \otimes A \arrow[dr, "id_{FB} \otimes p_A"'] & & F(B \otimes A) \\
& FB \otimes FA \arrow[ur, "*"'] &
\end{tikzcd}$$

3 Soundness

Definition 9. *Semantical translation from $\lambda_{\boldsymbol{K}}$ to CCC with applicative functor:*

1) Interpretation for types: $[\![A]\!] := \hat{A}, A \in \mathbb{T}$, $[\![A \to B]\!] := [\![A]\!] \to [\![B]\!]$, $[\![A \times B]\!] := [\![A]\!] \times [\![B]\!]$;

2) Interpretation for modal types: $[\![\boldsymbol{K}A]\!] = \mathcal{K}[\![A]\!]$, where $\mathcal{K}$ is an applicative functor;

3) Interpretaion for contexts: $[\![\Gamma = \{x_1 : A_1, ..., x_n : A_n\}]\!] := [\![\Gamma]\!] = [\![A_1]\!] \times ... \times [\![A_n]\!]$;

4) Interpretation for typing assignment: $[\![\Gamma \vdash M : A]\!] := [\![M]\!] : [\![\Gamma]\!] \to [\![A]\!]$, where $[\![M]\!] : [\![\Gamma]\!] \to [\![A]\!] \in \mathcal{C}$;

5) Interpretation for typing rules:

$$\frac{}{[\![\Gamma, x : A \vdash x : A]\!] := \pi_2 : [\![\Gamma]\!] \times [\![A]\!] \to [\![A]\!]}$$

$$\frac{[\![\Gamma, x : A \vdash M : B]\!] := f : [\![\Gamma]\!] \times [\![A]\!] \to [\![B]\!]}{[\![\Gamma \vdash (\lambda x.M) : A \to B]\!] := \Lambda(f) : [\![\Gamma]\!] \to [\![B]\!]^{[\![A]\!]}}$$

$$\frac{[\![\Gamma \vdash M : A \to B]\!] := [\![M]\!] : [\![\Gamma]\!] \to [\![B]\!]^{[\![A]\!]} \qquad [\![\Gamma \vdash N : A]\!] := [\![N]\!] : [\![\Gamma]\!] \to [\![A]\!]}{[\![\Gamma \vdash (MN) : B]\!] := [\![\Gamma]\!] \xrightarrow{\langle [\![M]\!], [\![N]\!] \rangle} [\![B]\!]^{[\![A]\!]} \times [\![A]\!] \xrightarrow{\epsilon} [\![B]\!]}$$

$$\frac{[\![\Gamma \vdash M : A]\!] := f : [\![\Gamma]\!] \to [\![A]\!] \qquad [\![\Gamma \vdash N : B]\!] := g : [\![\Gamma]\!] \to [\![B]\!]}{[\![\Gamma \vdash (M, N) : A \times B]\!] := \langle f, g \rangle : [\![\Gamma]\!] \to [\![A]\!] \times [\![B]\!]}$$

$$\frac{[\![\Gamma \vdash p : A_1 \times A_2]\!] := f : [\![\Gamma]\!] \to [\![A_1]\!] \times [\![A_2]\!]}{[\![\Gamma \vdash \pi_i p : A_i]\!] := [\![\Gamma]\!] \xrightarrow{f} [\![A_1]\!] \times [\![A_2]\!] \xrightarrow{\pi_i} [\![A_i]\!]} \; i \in \{1, 2\}$$

$$\frac{[\![\Gamma \vdash M : A]\!] := [\![M]\!] : [\![\Gamma]\!] \to [\![A]\!]}{[\![\Gamma \vdash \mathit{pure}\ M : \boldsymbol{K}A]\!] := [\![\Gamma]\!] \xrightarrow{[\![M]\!]} [\![A]\!] \xrightarrow{p_A} \mathcal{K}[\![A]\!]}$$

$$\frac{[\![\Gamma \vdash M : \boldsymbol{K}(A \to B)]\!] := [\![M]\!] : [\![\Gamma]\!] \to \mathcal{K}([\![B]\!]^{[\![A]\!]}) \qquad [\![\Gamma \vdash N : \boldsymbol{K}A]\!] := [\![N]\!] : [\![\Gamma]\!] \to \mathcal{K}[\![A]\!]}{[\![\Gamma \vdash M \star N : \boldsymbol{K}B]\!] := [\![\Gamma]\!] \xrightarrow{\mathcal{K}(\epsilon_{A,B}) \circ \cong \circ \langle [\![M]\!], [\![N]\!] \rangle} \mathcal{K}B}$$

Definition 10. *Simultaneous substitution*

Let $\Gamma = \{x_1 : A_1, ..., x_n : A_n\}$*,* $\Gamma \vdash M : A$ *and for all* $i \in \{1, ..., n\}$*,* $\Gamma \vdash M_i : A_i$*.*

We define simultaneous substitution $M[\vec{x} := \vec{M}]$ *recursively by:*

1) $x_i[\vec{x} := \vec{M}] := M_i$*;*

2) $(\lambda x.M)[\vec{x} := \vec{M}] := \lambda x.(M[\vec{x} := \vec{M}])$*;*

3) $(MN)[\vec{x} := \vec{M}] = (M[\vec{x} := \vec{M}])(N[\vec{x} := \vec{M}])$*;*

4) $\langle M, N \rangle = \langle (M[\vec{x} := \vec{M}]), (N[\vec{x} := \vec{M}]) \rangle$*;*

5) $(\pi_i P)[\vec{x} := \vec{M}] = \pi_i(P[\vec{x} := \vec{M}])$*;*

6) $(\mathit{pure}\ M)[\vec{x} := \vec{M}] = \mathit{pure}\ (M[\vec{x} := \vec{M}])$*;*

7) $(M \star N)[\vec{x} := \vec{M}] = (M[\vec{x} := \vec{M}]) \star (N[\vec{x} := \vec{M}])$*.*

Lemma 1.

$[\![M[x_1 := M_1, \ldots, x_n := M_n]]\!] = [\![M]\!] \circ \langle [\![M_1]\!], \ldots, [\![M_n]\!] \rangle$*.*

Proof.

1) $[\![(\text{pure } M)[\vec{x} := \vec{M}]]\!] = [\![\text{pure } M]\!] \circ \langle [\![M_1]\!], \ldots, [\![M_n]\!] \rangle$.

$$\begin{aligned}
[\![(\text{pure } M)[\vec{x} := \vec{M}]]\!] &= [\![\text{pure } (M[\vec{x} := \vec{M}])]\!] && \text{Substitution definition}\\
&= p \circ [\![(M[\vec{x} := \vec{M}])]\!] && \text{Translation for pure}\\
&= p \circ [\![M]\!] \circ \langle [\![M_1]\!], \ldots, [\![M_n]\!] \rangle && \text{Induction hypothesis}\\
&= (p \circ [\![M]\!]) \circ \langle [\![M_1]\!], \ldots, [\![M_n]\!] \rangle && \text{Associativity of composition}\\
&= [\![\text{pure } M]\!] \circ \langle [\![M_1]\!], \ldots, [\![M_n]\!] \rangle && \text{Translation for pure}
\end{aligned}$$

2) $[\![(M \star N)[\vec{x} := \vec{M}]]\!] = [\![M \star N]\!] \circ \langle [\![M_1]\!], \ldots, [\![M_n]\!] \rangle$.

$$
\begin{array}{ll}
[\![(M \star N)[\vec{x} := \vec{M}]]\!] = [\![(M[\vec{x} := \vec{M}]) \star (N[\vec{x} := \vec{M}])]\!] & \text{Definition of substitution} \\
\quad = p_\epsilon \circ * \circ \langle [\![(M[\vec{x} := \vec{M}])]\!], [(N[\vec{x} := \vec{M}])]\!] \rangle & \text{Translation for } \star \\
\quad = p_\epsilon \circ * \circ \langle [\![M]\!] \circ \langle [\![M_1]\!], \ldots, [\![M_n]\!] \rangle, [\![N]\!] \circ \langle [\![M_1]\!], \ldots, [\![M_n]\!] \rangle \rangle & \text{Induction hypothesis} \\
\quad = p_\epsilon \circ * \circ \langle [\![M]\!], [\![N]\!] \rangle \circ \langle [\![M_1]\!], \ldots, [\![M_n]\!] \rangle & \text{Property of morphism product} \\
\quad = (p_\epsilon \circ * \circ \langle [\![M]\!], [\![N]\!] \rangle) \circ \langle [\![M_1]\!], \ldots, [\![M_n]\!] \rangle & \text{Associativity of composition} \\
\quad = [\![M \star N]\!] \circ \langle [\![M_1]\!], \ldots, [\![M_n]\!] \rangle & \text{Translation for } \star
\end{array}
$$

$\square$

Lemma 2.

If $M \to_\beta N$, then $[\![M]\!] = [\![N]\!]$.

1) $[\![\text{pure } (\lambda x.x) \star M]\!] = [\![M]\!]$;

$$
\dfrac{\dfrac{[\![x : A \vdash x : A]\!] = \pi_2 : \mathbb{1} \times [\![A]\!] \to [\![A]\!]}{[\![\vdash \lambda x.x : A \to A]\!] = \Lambda(\pi_2) : \mathbb{1} \to [\![A]\!]^{[\![A]\!]}}}{[\![\vdash \text{pure } (\lambda x.x) : \mathbf{K}(A \to A)]\!] = p_{[\![A]\!]^{[\![A]\!]}} \circ \Lambda(\pi_2) : \mathbb{1} \to \mathcal{K}([\![A]\!]^{[\![A]\!]})}
$$

But by the following diagram:

$$
\begin{tikzcd}
{[\![A]\!]^{[\![A]\!]}} \arrow[r, "p_{[\![A]\!]^{[\![A]\!]}}"] & \mathcal{K}([\![A]\!]^{[\![A]\!]}) \\
\mathbb{1} \arrow[u, "\Lambda\pi_2"] \arrow[r, "p_{\mathbb{1}=id_{\mathbb{1}}}"'] & \mathbb{1} \arrow[u, "\mathcal{K}(\Lambda(\pi_2))"']
\end{tikzcd}
$$

$$
\begin{aligned}
p_{[\![A]\!]^{[\![A]\!]}} \circ \Lambda(\pi_2) &= id_{\mathbb{1}} \circ \mathcal{K}(\Lambda(\pi_2)) \\
&= \mathcal{K}(\Lambda(\pi_2))
\end{aligned}
$$

So:
$[\![\vdash \text{pure } (\lambda x.x) : \mathbf{K}(A \to A)]\!] = \mathcal{K}(\Lambda(\pi_2)) : \mathbb{1} \to \mathcal{K}([\![A]\!]^{[\![A]\!]})$
$[\![M : \mathbf{K}A \vdash M : \mathbf{K}A]\!] = id_{\mathcal{K}A} : \mathcal{K}([\![A]\!]) \to \mathcal{K}([\![A]\!])$, or
$[\![M : \mathbf{K}A \vdash M : \mathbf{K}A]\!] = \pi_2 : \mathbb{1} \times \mathcal{K}([\![A]\!]) \to \mathcal{K}([\![A]\!])$.

Let us consider the next commutative diagram:

$$
\begin{tikzcd}
\mathcal{K}(A^A) \times \mathcal{K}A \arrow[r, "\cong"] & \mathcal{K}(A^A \times A) \arrow[r, "\mathcal{K}(\epsilon)"] & \mathcal{K}A \\
\mathbb{1} \times \mathcal{K}A \arrow[u, "\mathcal{K}(\Lambda(\pi_2)) \times id_{\mathcal{K}A}"] \arrow[r, "\cong"'] & \mathcal{K}(\mathbb{1} \times A) \arrow[u, "\mathcal{K}(\Lambda(\pi_2) \times id_A)"] \arrow[ur, "\mathcal{K}(\pi_2)"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1} \times \mathcal{K}A \arrow[r, "\cong"] \arrow[d, "\pi_2"'] & \mathcal{K}(A^A \times A) \arrow[d, "\mathcal{K}(\pi_2)"] \\
\mathcal{K}A \arrow[r, "id_{\mathcal{K}A}"'] & \mathcal{K}A
\end{tikzcd}
$$

Hence:

$$
\begin{aligned}
[\![M : \mathbf{K}A \vdash \text{pure } (\lambda x.x) \star M]\!] &= (\mathcal{K}(\epsilon) \circ (\cong)) \circ (\mathcal{K}(\Lambda(\pi_2)) \times id_{\mathcal{K}A}) \\
&= \mathcal{K}(\epsilon) \circ \mathcal{K}(\Lambda(\pi_2) \times id_A) \circ (\cong) \\
&= \mathcal{K}(\epsilon) \circ (\cong) \\
&= \pi_2 = [\![M : \mathbf{K}A \vdash M : \mathbf{K}A]\!]
\end{aligned}
$$

2) $[\![(\text{pure } \lambda fgx.f(gx)) \star M \star N \star P]\!] = [\![M \star (N \star P)]\!]$
The first part of equality:

3) $\llbracket (\text{pure } M) \star (\text{pure } N) \rrbracket = \llbracket \text{pure } (MN) \rrbracket$;

1) The left part of the equality:

$$\frac{\llbracket \Gamma \vdash M : A \to B \rrbracket = f : \llbracket \Gamma \rrbracket \to \llbracket B \rrbracket^{[A]}}{\llbracket \Gamma \vdash \text{pure } M : \mathbf{K}(A \to B) \rrbracket = p_{\llbracket B \rrbracket^{[A]}} \circ f : \llbracket \Gamma \rrbracket \to \mathcal{K}(\llbracket B \rrbracket^{[A]})}$$

$$\frac{\llbracket \Delta \vdash N : A \rrbracket = g : \llbracket \Delta \rrbracket \to \llbracket A \rrbracket}{\llbracket \Delta \vdash \text{pure } N : \mathbf{K}A \rrbracket = p_{\llbracket A \rrbracket} \circ g : \llbracket \Delta \rrbracket \to \mathcal{K}\llbracket A \rrbracket}$$

$$\llbracket \Gamma, \Delta \vdash (\text{pure } M) \star (\text{pure } N) : \mathbf{K}B \rrbracket = \mathcal{K}(\epsilon) \circ (\cong) \circ (p_{\llbracket B \rrbracket^{[A]}} \circ f \times p_{\llbracket A \rrbracket} \circ g) : \Gamma \times \Delta \to \mathcal{K}B$$

2) The second part of equality:

$$\frac{\dfrac{\llbracket \Gamma \vdash M : A \to B \rrbracket = f : \llbracket \Gamma \rrbracket \to \llbracket B \rrbracket^{[A]} \qquad \llbracket \Delta \vdash N : A \rrbracket = g : \llbracket \Delta \rrbracket \to \llbracket A \rrbracket}{\llbracket \Gamma, \Delta \vdash MN : B \rrbracket = \epsilon \circ f \times g : \llbracket \Gamma \rrbracket \times \llbracket \Delta \rrbracket \to \llbracket B \rrbracket}}{\llbracket \Gamma, \Delta \vdash \text{pure } (MN) : \mathbf{K}B \rrbracket = p_{\llbracket B \rrbracket} \circ (\epsilon \circ (f \times g)) : \llbracket \Gamma \rrbracket \times \llbracket \Delta \rrbracket \to \mathcal{K}\llbracket B \rrbracket}$$

$$\begin{tikzcd}
 & \llbracket \Gamma \rrbracket \times \llbracket \Delta \rrbracket \arrow[ddl, "p_{\llbracket B \rrbracket^{[A]}} \circ f \times p_{\llbracket A \rrbracket} \circ g"'] \arrow[d, "f \times g"'] \arrow[dr, "\epsilon \circ f \times g"] & \\
 & \llbracket B \rrbracket^{\llbracket A \rrbracket} \times \llbracket A \rrbracket \arrow[r, "\epsilon"] \arrow[dl, "p_{\llbracket B \rrbracket^{[A]}} \times p_{\llbracket A \rrbracket}"] \arrow[dd, "p_{\llbracket B \rrbracket^{[A]} \times \llbracket A \rrbracket}"'] & \llbracket B \rrbracket \arrow[dd, "p_{\llbracket B \rrbracket}"] \\
\mathcal{K}(\llbracket B \rrbracket^{\llbracket A \rrbracket}) \times \mathcal{K}\llbracket A \rrbracket \arrow[dr, "\cong"'] & & \\
 & \mathcal{K}(\llbracket B \rrbracket^{\llbracket A \rrbracket} \times \llbracket A \rrbracket) \arrow[r, "\mathcal{K}(\epsilon)"'] & \mathcal{K}\llbracket B \rrbracket
\end{tikzcd}$$

$$\begin{aligned}
\llbracket \Gamma, \Delta \vdash (\text{pure } M) \star (\text{pure } N) : \mathbf{K}B \rrbracket &= \mathcal{K}(\epsilon) \circ (\cong) \circ (p_{\llbracket B \rrbracket^{[A]}} \circ f \times p_{\llbracket A \rrbracket} \circ g) \\
&= K(\epsilon) \circ (\cong) \circ p_{\llbracket B \rrbracket^{\llbracket A \rrbracket}} \times p_{\llbracket A \rrbracket} \circ f \times g \\
&= K(\epsilon) \circ p_{\llbracket B \rrbracket^{\llbracket A \rrbracket} \times \llbracket A \rrbracket} \circ f \times g \\
&= p_{\llbracket B \rrbracket} \circ \epsilon \circ f \times g \\
&= \llbracket \Gamma, \Delta \vdash \text{pure } (MN) : \mathcal{K}B \rrbracket
\end{aligned}$$

4) $\llbracket N : A, M : \mathbf{K}(A \to B) \vdash M \star \text{pure } N : \mathbf{K}B \rrbracket =$
$\llbracket N : A, M : \mathbf{K}(A \to B) \vdash \text{pure } (\lambda f.fN) \star M : \mathbf{K}B \rrbracket$

It is easy to see that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{K}(\llbracket B\rrbracket^{(\llbracket B\rrbracket^{\llbracket A\rrbracket})}) \times \mathcal{K}(\llbracket B\rrbracket^{\llbracket A\rrbracket}) \arrow[r, "\cong"] & \mathcal{K}(\llbracket B\rrbracket^{(\llbracket B\rrbracket^{\llbracket A\rrbracket})} \times \llbracket B\rrbracket^{\llbracket A\rrbracket}) \arrow[r, "\mathcal{K}(\epsilon)"] & \mathcal{K}\llbracket B\rrbracket \\
\mathcal{K}\llbracket A\rrbracket \times \mathcal{K}(\llbracket B\rrbracket^{\llbracket A\rrbracket}) \arrow[u, "\mathcal{K}(\Lambda(\epsilon\circ\langle\pi_2,\pi_1\rangle))\times id_{\mathcal{K}\llbracket B\rrbracket^{\llbracket A\rrbracket}}"] \arrow[r, "\cong"] & \mathcal{K}(\llbracket A\rrbracket \times \llbracket B\rrbracket^{\llbracket A\rrbracket}) \arrow[u, "\mathcal{K}(\Lambda(\epsilon\circ\langle\pi_2,\pi_1\rangle)\times id_{\llbracket B\rrbracket^{\llbracket A\rrbracket}})"] \arrow[r, "\mathcal{K}(\langle\pi_2,\pi_1\rangle)"] & \mathcal{K}(\llbracket B\rrbracket^{\llbracket A\rrbracket} \times \llbracket A\rrbracket) \arrow[u, "\mathcal{K}(\epsilon)"'] \\
\llbracket A\rrbracket \times \mathcal{K}(\llbracket B\rrbracket^{\llbracket A\rrbracket}) \arrow[u, "p_{\llbracket A\rrbracket}\times id_{\mathcal{K}(\llbracket B\rrbracket^{\llbracket A\rrbracket})}"] \arrow[r, "p_{\llbracket A\rrbracket}\times id_{\mathcal{K}(\llbracket B\rrbracket^{\llbracket A\rrbracket})}"'] & \mathcal{K}\llbracket A\rrbracket \times \mathcal{K}(\llbracket B\rrbracket^{\llbracket A\rrbracket}) \arrow[u, "\cong"'] \arrow[r, "\langle\pi_2,\pi_1\rangle"'] & \mathcal{K}(\llbracket B\rrbracket^{\llbracket A\rrbracket}) \times \mathcal{K}\llbracket A\rrbracket \arrow[u, "\cong"']
\end{tikzcd}$$

$\llbracket N : A, M : \mathbf{K}(A \to B) \vdash \text{pure } (\lambda f.fN) \star M : \mathbf{K}B \rrbracket =$

by interpretation

$\mathcal{K}(\epsilon) \circ (\cong) \circ ((p_{\llbracket B\rrbracket^{(\llbracket B\rrbracket^{\llbracket A\rrbracket})}} \circ \Lambda(\epsilon \circ \langle\pi_2, \pi_1\rangle)) \times id_{\mathcal{K}(\llbracket B\rrbracket^{\llbracket A\rrbracket})}) =$

by the definition of p

$\mathcal{K}(\epsilon) \circ (\cong) \circ (\mathcal{K}(\Lambda(\epsilon \circ \langle\pi_2, \pi_1\rangle)) \circ p_{\llbracket A\rrbracket}) \times id_{\mathcal{K}(\llbracket B\rrbracket^{\llbracket A\rrbracket})} =$

by the definition of identity function

$\mathcal{K}(\epsilon) \circ (\cong) \circ (\mathcal{K}(\Lambda(\epsilon \circ \langle\pi_2, \pi_1\rangle)) \circ p_{\llbracket A\rrbracket}) \times (id_{\mathcal{K}(\llbracket B\rrbracket^{\llbracket A\rrbracket})} \circ id_{\mathcal{K}(\llbracket B\rrbracket^{\llbracket A\rrbracket})}) =$

the property of composition of product morphisms

$\mathcal{K}(\epsilon) \circ (\cong) \circ (\mathcal{K}(\Lambda(\epsilon \circ \langle\pi_2, \pi_1\rangle)) \times id_{\mathcal{K}(\llbracket B\rrbracket^{\llbracket A\rrbracket})}) \circ (p_{\llbracket A\rrbracket} \times id_{\mathcal{K}(\llbracket B\rrbracket^{\llbracket A\rrbracket})})) =$

diagram above

$\mathcal{K}(\epsilon) \circ (\cong) \circ \langle\pi_2, \pi_1\rangle \circ (p_{\llbracket A\rrbracket} \times id_{\mathcal{K}(\llbracket B\rrbracket^{\llbracket A\rrbracket})})) =$

the property of product morphisms

$\mathcal{K}(\epsilon) \circ (\cong) \circ \langle\pi_2 \circ (p_{\llbracket A\rrbracket} \times id_{\mathcal{K}(\llbracket B\rrbracket^{\llbracket A\rrbracket})}, \pi_1 \circ (p_{\llbracket A\rrbracket} \times id_{\mathcal{K}(\llbracket B\rrbracket^{\llbracket A\rrbracket})})\rangle =$

by unfolding the morphism product

$\mathcal{K}(\epsilon) \circ (\cong) \circ \langle\pi_2 \circ \langle p_{\llbracket A\rrbracket} \circ \pi_1, id_{\mathcal{K}(\llbracket B\rrbracket^{\llbracket A\rrbracket})} \circ \pi_2\rangle, \pi_1 \circ \langle p_{\llbracket A\rrbracket} \circ \pi_1, id_{\mathcal{K}(\llbracket B\rrbracket^{\llbracket A\rrbracket})} \circ \pi_2\rangle\rangle =$

by the definition of pair morphism

$\mathcal{K}(\epsilon) \circ (\cong) \circ \langle id_{\mathcal{K}(\llbracket B\rrbracket^{\llbracket A\rrbracket})} \circ \pi_2, p_{\llbracket A\rrbracket} \circ \pi_1\rangle =$

the property of product morphisms

$\mathcal{K}(\epsilon) \circ (\cong) \circ (id_{\mathcal{K}(\llbracket B\rrbracket^{\llbracket A\rrbracket})} \times p_{\llbracket A\rrbracket}) \circ \langle\pi_2, \pi_1\rangle =$

by interpretation

$\llbracket N : A, M : \mathbf{K}(A \to B) \vdash M \star \text{pure } N : \mathbf{K}B \rrbracket$

Lemma 3. *If $M \to_\eta N$, then $\llbracket M\rrbracket = \llbracket N\rrbracket$.*

Proof.
1) $\llbracket \text{pure } (\lambda x.fx)\rrbracket = \llbracket \text{pure } f\rrbracket$.

$$\begin{aligned}
\llbracket \text{pure } (\lambda x.fx) \rrbracket &= p \circ \llbracket \lambda x.fx \rrbracket && \text{Translation for pure} \\
&= p \circ \llbracket f \rrbracket && \eta\text{-reduction rule for application} \\
&= \llbracket \text{pure } f \rrbracket && \text{Translation for pure}
\end{aligned}$$

2) $\llbracket \text{pure } \langle \pi_1 M, \pi_2 M \rangle \rrbracket = \llbracket \text{pure } M \rrbracket$

$$\begin{aligned}
\llbracket \text{pure } \langle \pi_1 M, \pi_2 M \rangle \rrbracket &= p \circ \llbracket \langle \pi_1 M, \pi_2 M \rangle \rrbracket && \text{Translation for pure} \\
&= p \circ \llbracket M \rrbracket && \eta\text{-reduction rule for pair} \\
&= \llbracket \text{pure } M \rrbracket && \text{Translation for pure}
\end{aligned}$$

3)
$\llbracket M : \mathbf{K}(A \times B) \vdash \text{pure } (\lambda x.\lambda y.\langle x, y \rangle) \star (\text{pure } (\lambda x.\pi_1) \star M) \star (\text{pure } (\lambda x.\pi_2) \star M : \mathbf{K}(A \times B) \rrbracket =$
$\llbracket M : \mathbf{K}(A \times B) \vdash M : \mathbf{K}(A \times B) \rrbracket$

i) The first step
Let us consider interpetation for $\vdash$ pure $(\lambda x.\lambda y.\langle x, y \rangle) : \mathbf{K}(A \to B \to A \times B)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\pi_2 : \mathbb{1} \times (\llbracket A \rrbracket \times \llbracket B \rrbracket) \to \llbracket A \rrbracket \times \llbracket B \rrbracket}{\pi_2 \circ \alpha : (\mathbb{1} \times \llbracket A \rrbracket) \times \llbracket B \rrbracket \to \llbracket A \rrbracket \times \llbracket B \rrbracket}}{\Lambda(\pi_2 \circ \alpha) : \mathbb{1} \times \llbracket A \rrbracket \to \llbracket A \rrbracket \times \llbracket B \rrbracket^{\llbracket B \rrbracket}}}{\Lambda(\Lambda(\pi_2 \circ \alpha)) : \mathbb{1} \to \llbracket A \rrbracket \times \llbracket B \rrbracket^{\llbracket B \rrbracket^{\llbracket A \rrbracket}}}}{p_{\llbracket A \rrbracket \times \llbracket B \rrbracket^{\llbracket B \rrbracket^{\llbracket A \rrbracket}}} \circ \Lambda(\Lambda(\pi_2 \circ \alpha)) : \mathbb{1} \to \mathcal{K}(\llbracket A \rrbracket \times \llbracket B \rrbracket^{\llbracket B \rrbracket^{\llbracket A \rrbracket}})}$$

By naturality, $p_{\llbracket A \rrbracket \times \llbracket B \rrbracket^{\llbracket B \rrbracket^{\llbracket A \rrbracket}}} \circ \Lambda(\Lambda(\pi_2 \circ \alpha)) = \mathcal{K}(\Lambda(\Lambda(\pi_2 \circ \alpha))$.
At first let us show that the following diagram commutes in any CCC:

$$\begin{tikzcd}
(\llbracket A \times B \rrbracket^{\llbracket B \rrbracket^{\llbracket A \rrbracket}} \times \llbracket A \rrbracket) \times \llbracket B \rrbracket \arrow[r, "\epsilon \circ (\epsilon \times id_{\llbracket B \rrbracket})"] & \llbracket A \rrbracket \times \llbracket B \rrbracket \\
([1] \times \llbracket A \rrbracket) \times \llbracket B \rrbracket \arrow[u, "(\Lambda(\Lambda(\pi_2 \circ \alpha)) \times id_{\llbracket A \rrbracket}) \times id_{\llbracket B \rrbracket}"] \arrow[ur, "\pi_2 \circ \alpha"'] &
\end{tikzcd}$$

$\epsilon \circ (\epsilon \times id_{\llbracket B \rrbracket}) \circ (\Lambda(\Lambda(\pi_2 \circ \alpha)) \times id_{\llbracket A \rrbracket}) \times id_{\llbracket B \rrbracket} =$
by the definition of morphism product
$\epsilon \circ (\epsilon \times id_{\llbracket B \rrbracket}) \circ \langle \Lambda(\Lambda(\pi_2 \circ \alpha)) \circ \pi_1, id_{\llbracket A \rrbracket} \circ \pi_2 \rangle \times id_{\llbracket B \rrbracket} =$
by the definition of morphism product
$\epsilon \circ (\epsilon \times id_{\llbracket B \rrbracket}) \circ \langle \langle \Lambda(\Lambda(\pi_2 \circ \alpha)) \circ \pi_1, id_{\llbracket A \rrbracket} \circ \pi_2 \rangle \circ \pi_1, id_{\llbracket B \rrbracket} \circ \pi_2 \rangle$
by the property of morphism product
$\epsilon \circ \langle \epsilon \circ \langle \Lambda(\Lambda(\pi_2 \circ \alpha)) \circ \pi_1, id_{\llbracket A \rrbracket} \circ \pi_2 \rangle \circ \pi_1, id_{\llbracket B \rrbracket} \circ id_{\llbracket B \rrbracket} \circ \pi_2 \rangle =$
by the definition of morphism product and by identity
$\epsilon \circ \langle \epsilon \circ (\Lambda(\Lambda(\pi_2 \circ \alpha)) \times id_{\llbracket A \rrbracket}) \circ \pi_1, id_{\llbracket B \rrbracket} \circ \pi_2 \rangle =$
by exponentiation and currying property
$\epsilon \circ \langle \Lambda(\pi_2 \circ \alpha) \circ \pi_1, id_{\llbracket B \rrbracket} \circ \pi_2 \rangle =$
by the definition of morphism product
$\epsilon \circ \Lambda(\pi_2 \circ \alpha) \times id_{\llbracket B \rrbracket}$
by exponentiation and currying property
$\pi_2 \circ \alpha$

2)

$\square$

Lemma 4.

1) $[\![M]\!] = [\![N]\!]$*, if* $[\![\mathit{pure}\ M]\!] = [\![\mathit{pure}\ N]\!]$*;*

2) Let $[\![M]\!] = [\![N]\!]$*, then* $[\![M \star P]\!] = [\![N \star P]\!]$*;*

3) Let $[\![M]\!] = [\![N]\!]$*, then* $[\![P \star M]\!] = [\![P \star N]\!]$*.*

Proof.

1)

i) "only if"-part.

Let $[\![M]\!] : [\![\Gamma]\!] \to [\![A]\!]$, $[\![N]\!] : [\![\Gamma]\!] \to [\![A]\!]$ and $[\![M]\!] = [\![N]\!]$. So $p \circ [\![M]\!] = p \circ [\![N]\!]$, hence $[\![\text{pure } M]\!] = [\![\text{pure } N]\!]$.

$\square$