

# Completeness theorems for temporal logic extended with non-canonical axioms

Daniel Rogozin

Lomonosov Moscow State University

**Definition 1.** *A temporal language*

$$\phi, \psi ::= p \mid \perp \mid \phi \rightarrow \psi \mid \Diamond \phi \mid \Diamond^- \phi$$

Here and below,  $\neg \phi = \phi \rightarrow \perp$ ,  $\Box \phi = \neg \Diamond \neg \phi$ ,  $\Box^- \phi = \neg \Diamond^- \neg \phi$ .

**Definition 2.** *Minimal normal temporal logic*

1. *Classical propositional calculus*
2.  $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$
3.  $\Box^-(p \rightarrow q) \rightarrow (\Box^- p \rightarrow \Box^- q)$
4.  $\Diamond^- \Box p \rightarrow p$
5.  $\Diamond \Box^- p \rightarrow p$
6. *Inference rules:*

$$\frac{\phi \quad \phi \rightarrow \psi}{\psi} \text{MP}$$

$$\frac{\phi(p_1, \dots, p_n)}{\phi(p_1 := \psi_1, \dots, p_n := \psi_n)} \text{Sub}$$

$$\frac{\phi}{\Box \phi} \text{Nec}$$

$$\frac{\phi}{\Box^- \phi} \text{Nec}^-$$

**Definition 3.** *Kripke model*

Let  $\mathcal{F} = \langle W, R \rangle$  be a frame, then Kripke model is a tuple  $\mathcal{M} = \langle \mathcal{F}, \vartheta \rangle$ , where  $\vartheta : PV \rightarrow 2^W$  is a valuation. A truth condition is defined as follows:

1.  $\mathcal{M}, x \models p \Leftrightarrow x \in \vartheta(p)$
2.  $\mathcal{M}, x \not\models \perp$
3.  $\mathcal{M}, x \models \phi \rightarrow \psi \Leftrightarrow \mathcal{M}, x \models \phi \Rightarrow \mathcal{M}, x \models \psi$
4.  $\mathcal{M}, x \models \Diamond \phi \Leftrightarrow \exists y \in R(x) \mathcal{M}, y \models \phi$
5.  $\mathcal{M}, x \models \Diamond^- \phi \Leftrightarrow \exists y \in R^{-1}(x) \mathcal{M}, y \models \phi$

The truth condition for boxes are defined as:

1.  $\mathcal{M}, x \models \Box\phi \Leftrightarrow \forall y \in R(x) \mathcal{M}, y \models \phi$
2.  $\mathcal{M}, x \models \Box^-\phi \Leftrightarrow \forall y \in R^{-1}(x) \mathcal{M}, y \models \phi$

**Definition 4.**

1.  $\mathbf{AL}^+ = \Box(\Box p \rightarrow p) \rightarrow \Box p = \Diamond p \rightarrow \Diamond(p \wedge \neg\Diamond p)$
2.  $\mathbf{Grz}^+ = \Box(\Box(p \rightarrow \Box p) \rightarrow p) \rightarrow p$

**Definition 5.**

1.  $\mathbf{GL.t}^+ = \mathbf{K.t} \oplus \mathbf{AL}^+$
2.  $\mathbf{Grz.t}^+ = \mathbf{K.t} \oplus \mathbf{Grz}^+$

**Proposition 1.** Let  $\mathcal{F} = \langle W, R \rangle$  be a frame, then

1.  $\mathcal{F} \models \mathbf{AL}^+ \Leftrightarrow R$  is transitive and Noetherian
2.  $\mathcal{F} \models \mathbf{Grz}^+ \Leftrightarrow R$  is reflexive, transitive and Noetherian

**Definition 6.** Let  $\mathcal{F} = \langle W, R \rangle$  be a frame, then a formula  $\phi$  is  $\mathcal{F}$ -satisfiable, if  $\mathcal{F} \not\models \neg\phi$ , i.e. there exists a valuation  $\vartheta$  such that  $\mathcal{M}, x \models \phi$  for a model  $\mathcal{M} = \langle \mathcal{F}, \vartheta \rangle$  and  $x \in W$ .

**Definition 7.** Let  $\mathcal{L}$  be a normal temporal logic, then a formula  $\phi$  is  $\mathcal{L}$ -consistent, if  $\mathcal{L} \not\vdash \neg\phi$

**Lemma 1.** Let  $\mathcal{L}$  be a normal temporal logic, then  $\mathcal{L} = TL(\mathbb{F})$  iff every  $\mathbb{F} \models \mathcal{L}$  and every  $\mathcal{L}$ -consistent formula is  $\mathcal{F}$ -satisfiable.

**Definition 8.** Selective filtration

Let  $\mathcal{M} = \langle W, R, \vartheta \rangle$  be a Kripke models,  $W' \subseteq W$ ,  $R' \subseteq R$ , let  $\Psi$  be a set of formulae closed under subformulae. Let us define  $\vartheta'(p) = \vartheta(p) \cap W'$  for  $p \in \Psi$ . Then a submodel  $\mathcal{M}' = \langle W', R', \vartheta' \rangle$  is a selective filtration of  $\mathcal{M}$  through  $\Psi$ , if the following condition holds:

1.  $\forall \Diamond\phi \in \Psi \forall x \in W' \mathcal{M}, x \models \Diamond\phi \Rightarrow \exists y \in R'(x) \mathcal{M}, y \models \phi$
2.  $\forall \Diamond^-\phi \in \Psi \forall x \in W' \mathcal{M}, x \models \Diamond^-\phi \Rightarrow \exists y \in R'^-(x) \mathcal{M}, y \models \phi$

**Lemma 2.** Let  $\mathcal{M} = \langle W, R, \vartheta \rangle$  be a Kripke model,  $\Psi$  a set of formulae closed under subformulae and  $\mathcal{M}'$  is a selective filtration of  $\mathcal{M}$  through  $\Psi$ , then for each  $\phi \in \Psi$  and  $x \in W'$ :

$$\mathcal{M}, x \models \phi \Leftrightarrow \mathcal{M}', x \models \phi$$

**Theorem 1.**  $\mathbf{GL.t}^+ = TL(\text{Frames}(\mathbf{GL.t}^+))$

*Proof.*

□

**Theorem 2.**  $\mathbf{Grz.t}^+ = TL(\text{Frames}(\mathbf{Grz.t}^+))$

*Proof.*

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