Completeness theorems for temporal logic extended with non-canonical axioms

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Definition 1. *A temporal language*

$$\phi, \psi ::= p \,|\, \bot \mid \phi \to \psi \mid \Diamond\phi \mid \Diamond^-\phi$$

Here and below, $\neg\phi = \phi \to \bot$, $\Box\phi = \neg\Diamond\neg\phi$, $\Box^-\phi = \neg\Diamond^-\neg\phi$.

Definition 2. *Minimal normal temporal logic*

1. *Classical propositional calculus*
2. $\Box(p \to q) \to (\Box p \to \Box q)$
3. $\Box^-(p \to q) \to (\Box^- p \to \Box^- q)$
4. $\Diamond^-\Box p \to p$
5. $\Diamond\Box^- p \to p$
6. *Inference rules:*

$$\frac{\phi \qquad \phi \to \psi}{\psi}\ \textbf{MP} \qquad\qquad \frac{\phi(p_1, \dots, p_n)}{\phi(p_1 := \psi_1, \dots, p_n := \psi_n)}\ \textbf{Sub}$$

$$\frac{\phi}{\Box\phi}\ \textbf{Nec} \qquad\qquad \frac{\phi}{\Box^-\phi}\ \textbf{Nec}^-$$

Definition 3. *Kripke model*
Let $\mathcal{F} = \langle W, R\rangle$ be a frame, then Kripke model is a tuple $\mathcal{M} = \langle \mathcal{F}, \vartheta\rangle$, where $\vartheta : PV \to 2^W$ is a valuation. A truth condition is defined as follows:

1. $\mathcal{M}, x \models p \Leftrightarrow x \in \vartheta(p)$
2. $\mathcal{M}, x \not\models \bot$
3. $\mathcal{M}, x \models \phi \to \psi \Leftrightarrow \mathcal{M}, x \models \phi \Rightarrow \mathcal{M}, x \models \psi$
4. $\mathcal{M}, x \models \Diamond\phi \Leftrightarrow \exists y \in R(x)\ \ \mathcal{M}, y \models \phi$
5. $\mathcal{M}, x \models \Diamond^-\phi \Leftrightarrow \exists y \in R^{-1}(x)\ \ \mathcal{M}, y \models \phi$

The truth condition for boxes are defined as:

1. $\mathcal{M}, x \models \Box\phi \Leftrightarrow \forall y \in R(x)\ \mathcal{M}, y \models \phi$

2. $\mathcal{M}, x \models \Box^{-}\phi \Leftrightarrow \forall y \in R^{-1}(x)\ \mathcal{M}, y \models \phi$

Definition 4.

Definition 5.

1. $\mathbf{AL}^{+} = \Box(\Box p \to p) \to \Box p = \Diamond p \to \Diamond(p \wedge \neg\Diamond p)$

2. $\mathbf{Grz}^{+} = \Box(\Box(p \to \Box p) \to p) \to p$

Definition 6.

1. $\mathbf{GL.t}^{+} = \mathbf{K.t} \oplus \mathbf{AL}^{+}$

2. $\mathbf{Grz.t}^{+} = \mathbf{K.t} \oplus \mathbf{Grz}^{+}$

Proposition 1. *Let $\mathcal{F} = \langle W, R\rangle$ be a frame, then*

1. $\mathcal{F} \models \mathbf{AL}^{+} \Leftrightarrow$ *R is transitive and Noetherian*

2. $\mathcal{F} \models \mathbf{Grz}^{+} \Leftrightarrow$ *R is reflexive, transitive and Noetherian*

Definition 7. *Let $\mathcal{F} = \langle W, R\rangle$ be a frame, then a formula ϕ is $\mathcal{F}$-satisfiable, if $\mathcal{F} \not\models \neg\phi$, i.e. there exists a valuation ϑ such that $\mathcal{M}, x \models \phi$ for a model $\mathcal{M} = \langle \mathcal{F}, \vartheta\rangle$ and $x \in W$.*

Definition 8. *Let $\mathcal{L}$ be a normal temporal logic, then a formula ϕ is $\mathcal{L}$-consistent, if $\mathcal{L} \not\vdash \neg\phi$*

Lemma 1. *Let $\mathcal{L}$ be a normal temporal logic, then $\mathcal{L} = TL(\mathbb{F})$ iff every $\mathbb{F} \models \mathcal{L}$ and every $\mathcal{L}$-consistent formula is $\mathcal{F}$-satisfiable.*

Definition 9. *Selective filtration*

Let $\mathcal{M} = \langle W, R, \vartheta\rangle$ be a Kripke models, $W' \subseteq W$, $R' \subseteq R$, let Ψ be a set of formulae closed under subformulae. Let us define $\vartheta'(p) = \vartheta(p) \cap W'$ for $p \in \Psi$. Then a submodel $\mathcal{M}' = \langle W', R', \vartheta'\rangle$ is a selective filtration of $\mathcal{M}$ through Ψ, if the following condition holds:

1. $\forall \Diamond\phi \in \Psi\ \forall x \in W'\ \mathcal{M}, x \models \Diamond\phi \Rightarrow \exists y \in R'(x)\ \mathcal{M}, y \models \phi$

2. $\forall \Diamond^{-}\phi \in \Psi\ \forall x \in W'\ \mathcal{M}, x \models \Diamond^{-}\phi \Rightarrow \exists y \in R'^{-}(x)\ \mathcal{M}, y \models \phi$

Lemma 2. *Let $\mathcal{M} = \langle W, R, \vartheta\rangle$ be a Kripke model, Ψ a set of formulae closed under subformulae and $\mathcal{M}'$ is a selective filtration of $\mathcal{M}$ through Ψ, then for each $\phi \in \Psi$ and $x \in W'$:*

$$\mathcal{M}, x \models \phi \Leftrightarrow \mathcal{M}', x \models \phi$$

Definition 10. *Gentzen sequent calculus for $\mathbf{GL.t}^{+}$ ($\mathbf{GL.t}^{+}_{G}$)*

$$\frac{}{\Gamma, \phi \Rightarrow \Theta, \phi}\ \mathbf{ax} \qquad \frac{}{\Gamma, \bot \Rightarrow \Theta}\ \bot$$

$$\frac{\Gamma \Rightarrow \Delta, \phi \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \phi \to \psi \Rightarrow \Delta}\ \to\Rightarrow \qquad \frac{\Gamma, \phi \Rightarrow \psi, \Delta}{\Gamma, \Rightarrow \phi \to \psi, \Delta}\ \Rightarrow\to$$

$$\frac{\Box\Gamma, \Gamma, \Box\phi \Rightarrow \phi}{\Box\Gamma \Rightarrow \Box\phi}\ \Box_I \qquad \frac{\Gamma \Rightarrow \phi}{\Box^{-}\Gamma \Rightarrow \Box^{-}\phi}\ \Box_I$$

$$\frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma, \Diamond^{-}\Box\phi \Rightarrow \Delta}\ \Diamond^{-}\Box \qquad \frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma, \Diamond\Box^{-}\phi \Rightarrow \Delta}\ \Diamond\Box^{-}$$

Lemma 3. *If* $\mathbf{GL.t}^+_G \vdash \Gamma \Rightarrow \Delta$*, then* $\mathbf{GL.t}^+ \vdash \bigwedge \Gamma \to \bigvee \Delta$

Proof. 1. Let $\mathbf{GL.t}^+_G \vdash \Gamma, \phi \Rightarrow \Delta$

(1) $\mathbf{GL.t}^+ \vdash (\bigwedge \Gamma \wedge \phi) \to \bigvee \Delta$
Induction hypothesis

(2) $\mathbf{GL.t}^+ \vdash \Diamond^- \Box \phi \to \phi$
$\mathbf{GL.t}^+$ axiom

(3) $\mathbf{GL.t}^+ \vdash ((\phi \wedge \bigwedge \Gamma) \to \bigvee \Delta) \to (\phi \to (\bigwedge \Gamma \to \bigvee \Delta))$
Boolean tautology

(4) $\mathbf{GL.t}^+ \vdash \phi \to (\bigwedge \Gamma \to \bigvee \Delta)$
(1), (3), **MP**

(5) $\mathbf{GL.t}^+ \vdash \Diamond^- \Box \phi \to (\bigwedge \Gamma \to \bigvee \Delta)$
(2), (4), transitivity

(6) $\mathbf{GL.t}^+ \vdash (\Diamond^- \Box \phi \to (\bigwedge \Gamma \to \bigvee \Delta)) \to ((\Diamond^- \Box \phi \wedge \bigwedge \Gamma) \to \bigvee \Delta)$
Boolean tautology

(7) $\mathbf{GL.t}^+ \vdash (\Diamond^- \Box \phi \wedge \bigwedge \Gamma) \to \bigvee \Delta$
(5), (6), **MP**

2. The case with $\Diamond \Box^-$ is similar to the previous one.

$\Box$

Lemma 4. *If* $\mathbf{GL.t}^+ \vdash \phi$*, then* $\mathbf{GL.t}^+_G \vdash \Rightarrow \phi$

Proof.

1. $\mathbf{GL.t}^+_G \vdash \Rightarrow \Diamond^- \Box p \to p$

$$\dfrac{\dfrac{p \Rightarrow p}{\Diamond^- \Box p \Rightarrow p}\ \Diamond^- \Box}{\Rightarrow \Diamond^- \Box p \to p}\ \Rightarrow\to$$

2. $\mathbf{GL.t}^+_G \vdash \Rightarrow \Diamond \Box^- p \to p$

$$\dfrac{\dfrac{p \Rightarrow p}{\Diamond \Box^- p \Rightarrow p}\ \Diamond \Box^-}{\Rightarrow \Diamond \Box^- p \to p}\ \Rightarrow\to$$

$\Box$

Theorem 1. $\mathbf{GL.t}^+ = TL(Frames(\mathbf{GL.t}^+))$

Proof. $\Box$

Theorem 2. $\mathbf{Grz.t}^+ = TL(Frames(\mathbf{Grz.t}^+))$

Proof. $\Box$