Completeness theorems for temporal logics extended with Löb and Grzegorczyk formulae

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Definition 1. *A temporal language*

$$\phi, \psi ::= p \,|\, \bot \mid \phi \to \psi \mid \Diamond \phi \mid \Diamond^{-} \phi$$

Here and below, $\neg \phi = \phi \to \bot$, $\Box \phi = \neg \Diamond \neg \phi$, $\Box^{-} \phi = \neg \Diamond^{-} \neg \phi$.
The underlying logic is **K.t**, see [1] or [2]:

Definition 2. *Minimal normal temporal logic*

1. *Classical propositional calculus*
2. $\Box(p \to q) \to (\Box p \to \Box q)$
3. $\Box^{-}(p \to q) \to (\Box^{-} p \to \Box^{-} q)$
4. $\Diamond^{-} \Box p \to p$
5. $\Diamond \Box^{-} p \to p$
6. *Inference rules:*

$$\frac{\phi \qquad \phi \to \psi}{\psi}\ \mathbf{MP} \qquad\qquad \frac{\phi(p_1, \dots, p_n)}{\phi(p_1 := \psi_1, \dots, p_n := \psi_n)}\ \mathbf{Sub}$$

$$\frac{\phi}{\Box \phi}\ \mathbf{Nec} \qquad\qquad \frac{\phi}{\Box^{-} \phi}\ \mathbf{Nec}^{-}$$

Definition 3. *By normal temporal logic (or temporal logic) we mean the set of formulae that contains Kripke axioms for both boxes, $\Diamond^{-} \Box p \to p$, $\Diamond \Box^{-} p \to p$, and closed under both necessitation rules,* **MP**, *and* **Sub**.

Definition 4. *Kripke model*

Let $\mathcal{F} = \langle W, R \rangle$ be a frame, then Kripke model is a tuple $\mathcal{M} = \langle \mathcal{F}, \vartheta \rangle$, where $\vartheta : PV \to 2^W$ is a valuation. A truth condition is defined as follows:

1. $\mathcal{M}, x \models p \Leftrightarrow x \in \vartheta(p)$
2. $\mathcal{M}, x \not\models \bot$
3. $\mathcal{M}, x \models \phi \to \psi \Leftrightarrow \mathcal{M}, x \models \phi \Rightarrow \mathcal{M}, x \models \psi$

4. $\mathcal{M}, x \models \Diamond\phi \Leftrightarrow \exists y \in R(x)\ \mathcal{M}, y \models \phi$
5. $\mathcal{M}, x \models \Diamond^{-}\phi \Leftrightarrow \exists y \in R^{-1}(x)\ \mathcal{M}, y \models \phi$

The truth condition for boxes are defined as:

1. $\mathcal{M}, x \models \Box\phi \Leftrightarrow \forall y \in R(x)\ \mathcal{M}, y \models \phi$
2. $\mathcal{M}, x \models \Box^{-}\phi \Leftrightarrow \forall y \in R^{-1}(x)\ \mathcal{M}, y \models \phi$

Definition 5.

1. $\mathcal{M} \models \varphi \Leftrightarrow \forall x \in W\ \mathcal{M}, x \models \varphi$
2. $\mathcal{F} \models \varphi \Leftrightarrow \forall \vartheta\ \mathcal{M} \models \varphi$, *where* $\mathcal{M} = \langle \mathcal{F}, \vartheta \rangle$
3. *Let* $\mathcal{F}$ *be a Kripke frame, then a temporal logic of* $\mathcal{F}$ *is the set of formulae that valid on* $\mathcal{F}$*, i.e.,* $\mathrm{TL}(\mathcal{F}) = \{\varphi \in \mathrm{Fm} \mid \mathcal{F} \models \varphi\}$
4. *Let* $\mathbb{F}$ *be a class of Kripke frames, then* $\mathrm{TL}(\mathbb{F}) = \bigcap\limits_{\mathcal{F} \in \mathbb{F}} \mathrm{TL}(\mathcal{F})$
5. *Let* $\mathcal{L}$ *be a temporal logic, then* $\mathrm{Frames}(\mathcal{L}) = \{\mathcal{F} \mid \mathcal{F} \models \mathcal{L}\}$

Definition 6. *Let* $\mathcal{F}_1 = \langle W_1, R_1 \rangle$, $\mathcal{F}_2 = \langle W_2, R_1 \rangle$ *be Kripke frames, then a p-morphism is a map* $f : \mathcal{F}_1 \to \mathcal{F}_2$ *with the following data:*

1. $aR_1b \Rightarrow f(a)R_2f(b)$
2. $f(a)R_2c \Rightarrow \exists b \in W_1\ f(b) = c \,\&\, aR_1b$
3. $cR_2f(a) \Rightarrow \exists b \in W_1\ f(b) = c \,\&\, bR_1a$

Definition 7. *Let* $\mathcal{M}_1$*,* $\mathcal{M}_2$ *be Kripke models, then* $f : \mathcal{M}_1 \to \mathcal{M}_2$ *is a temporal p-morphism, if* f *is a temporal p-morphism of underlying frames and the following condition holds:*

$$\mathcal{M}_1, x \models p \Leftrightarrow \mathcal{M}_2, f(x) \models p \text{ for each variable } p$$

Lemma 1.

1. $\mathcal{M}_1, x \models \varphi \Leftrightarrow \mathcal{M}_2, f(x) \models \varphi$.
2. *If* $\mathcal{F}_1 \twoheadrightarrow \mathcal{F}_2$*, then* $\mathrm{TL}(\mathcal{F}_1) \subseteq \mathrm{TL}(\mathcal{F}_2)$.

Definition 8. *Let* $\mathcal{F} = \langle W, R \rangle$ *be a frame, then a formula* ϕ *is* $\mathcal{F}$*-satisfiable, if* $\mathcal{F} \not\models \neg\phi$*, i.e. there exists a valuation* ϑ *such that* $\mathcal{M}, x \models \phi$ *for a model* $\mathcal{M} = \langle \mathcal{F}, \vartheta \rangle$ *and* $x \in W$.

Definition 9. *Let* $\mathcal{L}$ *be a normal temporal logic, then a formula* ϕ *is* $\mathcal{L}$*-consistent, if* $\mathcal{L} \nvdash \neg\phi$

Lemma 2. *Let* $\mathcal{L}$ *be a normal temporal logic, then* $\mathcal{L} = \mathrm{TL}(\mathbb{F})$ *iff every* $\mathbb{F} \models \mathcal{L}$ *and every* $\mathcal{L}$*-consistent formula is* $\mathcal{F}$*-satisfiable.*

Definition 10.

1. $\mathbf{AL}^{+} = \Box(\Box p \to p) \to \Box p = \Diamond p \to \Diamond(p \wedge \neg\Diamond p)$
2. $\mathbf{Grz}^{+} = \Box(\Box(p \to \Box p) \to p) \to p$

Definition 11.

1. $\mathbf{GL.t}^+ = \mathbf{K.t} \oplus \mathbf{AL}^+$
2. $\mathbf{Grz.t}^+ = \mathbf{K.t} \oplus \mathbf{Grz}^+$

Proposition 1. *Let $\mathcal{F} = \langle W, R \rangle$ be a frame, then*

1. $\mathcal{F} \models \mathbf{AL}^+ \Leftrightarrow$ *R is transitive and Noetherian*
2. $\mathcal{F} \models \mathbf{Grz}^+ \Leftrightarrow$ *R is reflexive, transitive and there are no increasing chains $x_0 R x_1 R \ldots$ such that $x_i \neq x_{i+1}$*

Proposition 2. *Let $\mathcal{F} = \langle W, R \rangle$ be a frame, then*

1. *If $\mathcal{F} \models \mathbf{AL}$, then $\mathcal{F} \models \mathbf{K}4.\mathbf{t}$*
2. *If $\mathcal{F} \models \mathbf{Grz}^+$, then $\mathcal{F} \models \mathbf{S}4.\mathbf{t}$*

Proof.

1. If $\mathcal{F} \models \mathbf{AL}$, then $\mathcal{F} \models \Diamond\Diamond p \to \Diamond p$. Hence, $\mathcal{F} \models \Diamond^- \Diamond^- p \to \Diamond^- p$.
2. The argument is similar to the previous one.

$\square$

A selective filtration is defined standardly, e.g. [3].

Definition 12. *Selective filtration*
Let $\mathcal{M} = \langle W, R, \vartheta \rangle$ be a Kripke models, $W' \subseteq W$, $R' \subseteq R$, let Ψ be a set of formulae closed under subformulae. Let us define $\vartheta'(p) = \vartheta(p) \cap W'$ for $p \in \Psi$. Then a submodel $\mathcal{M}' = \langle W', R', \vartheta' \rangle$ is a selective filtration of $\mathcal{M}$ through Ψ, if the following condition holds:

1. $\forall \Diamond \phi \in \Psi \ \forall x \in W' \ \mathcal{M}, x \models \Diamond \phi \Rightarrow \exists y \in R'(x) \ \mathcal{M}, y \models \phi$
2. $\forall \Diamond^- \phi \in \Psi \ \forall x \in W' \ \mathcal{M}, x \models \Diamond^- \phi \Rightarrow \exists y \in R'^{-1}(x) \ \mathcal{M}, y \models \phi$

Lemma 3. *Let $\mathcal{M} = \langle W, R, \vartheta \rangle$ be a Kripke model, Ψ a set of formulae closed under subformulae and $\mathcal{M}'$ is a selective filtration of $\mathcal{M}$ through Ψ, then for each $\phi \in \Psi$ and $x \in W'$:*

$$\mathcal{M}, x \models \phi \Leftrightarrow \mathcal{M}', x \models \phi$$

Theorem 1. $\mathbf{GL.t}^+ = \mathrm{TL}(\mathrm{Frames}(\mathbf{GL.t}^+))$

Proof.
Let us show that every $\mathbf{GL.t}$-consistent formula is satisfiable in some $\mathbf{GL.t}$ frame.

Let φ be $\mathbf{GL.t}$-consistent formula, then there exists $\Gamma \in \mathcal{F}_{\mathbf{GL.t}^+}$ such that $\mathcal{M}_{\mathbf{GL.t}^+}, \Gamma \models \varphi$. It is clear that $\Diamond \phi \to \Diamond(\varphi \wedge \neg \Diamond \varphi)$. Here there exists $\Delta \in R_{\mathbf{GL.t}^+}$ such that $\varphi \in Gamma$ and $(\varphi \wedge \neg \Diamond \varphi)$.

Let us define

$$V_\varphi = V_1 \cap \downarrow V_2$$

where $V_1 = \{y \in \mathcal{F}_{\mathbf{GL.t}^+} \mid \mathcal{M}_{\mathbf{GL.t}^+}, y \models \psi \wedge \neg \Diamond \psi, \psi \in\}$ and $V_2 = \{z \in \mathcal{F}_{\mathbf{GL.t}^+} \mid \mathcal{M}_{\mathbf{GL.t}^+}, y \models \Diamond^- \psi, \Diamond^- \psi \in \mathrm{Sub}(\varphi)\}$ and $\downarrow V_2 = V_2 \cup \{y \in W_{\mathbf{GL.t}^+} \mid \exists x \in W_{\mathbf{GL.t}^+} \ yRx\}$

Lemma 4. $\mathcal{M}_{\mathbf{GL.t}^+} \upharpoonright V_\varphi$ *is a selective filtration through* $\mathrm{Sub}(\varphi)$

Proof. One needs to check that both conditions for diamonds holds. Here we denote $R_{\mathbf{GL.t}^+} \upharpoonright V_\varphi \times V_\varphi$ as R'.

1. Let $\Diamond\psi \in \mathrm{Sub}$ and $\mathcal{M}_{\mathbf{GL.t}^+}, x \models \Diamond\psi$ for $x \in \mathcal{W}_{\mathbf{GL.t}^+} \cap V_\varphi$, then there exists $y \in R_{\mathbf{GL.t}^+}(x)$ such that $\mathcal{M}_{\mathbf{GL.t}^+}, y \models \phi \wedge \neg\Diamond\psi$. Hence, $y \in W_{\mathbf{GL.t}^+} \upharpoonright V_\varphi \times V_\varphi$.

   On the other hand, $\mathcal{M}_{\mathbf{GL.t}^+}, x \models \psi \wedge \neg\Diamond\psi$ and, consequently, $x \in W_{\mathbf{GL.t}^+} \upharpoonright V_\varphi \times V_\varphi$. It is clear that $\mathcal{M}_{\mathbf{GL.t}^+}, y \models \psi$ and xSy.

2. Let $\Diamond^-\psi \in \mathrm{Sub}$ and $\mathcal{M}_{\mathbf{GL.t}^+}, x \models \Diamond^-\psi$ for $x \in \mathcal{W}_{\mathbf{GL.t}^+} \cap V_\varphi$. By construction, there exists $y \in S^{-1}(x)$ such that $\mathcal{M}_{\mathbf{GL.t}^+}, y \models \psi$.

□

Lemma 5. $\mathcal{F}_{\mathbf{GL.t}^+} \upharpoonright V_\varphi \models \mathbf{GL.t}^+$

Proof. By construction. □

□

Theorem 2. $\mathbf{Grz.t}^+ = \mathrm{TL}(\mathrm{Frames}(\mathbf{Grz.t}^+))$

Proof. □

1 Finite model property

Here we introduce the notion of a temporal unravelling [4].

Definition 13. *Let $\mathcal{F} = \langle W, R, R^{-1}\rangle$ be a cone with root r, then the temporal unravelling is the frame $\mathcal{F}^\# = \langle W^\#, R^\#, R^\#_{-1}\rangle$, where $W^\#$ is the set of reduced paths and $\alpha R^\# \beta \Leftrightarrow \beta = \langle \alpha, 1, x\rangle$ for some $x \in \mathcal{W}$ and $\alpha R^\#_{-1}\beta \Leftrightarrow \beta = \langle \alpha, 0, x\rangle$ for some $x \in \mathcal{W}$.*

Lemma 6. *Let $\mathcal{F}^\#$ be a temporal tree, then $\pi : \mathcal{F}^\# \to \mathcal{F}$ is a two-sided p-morphism, where $\pi : \langle \alpha, i, x\rangle \mapsto x$.*

Proof. See, e.g. [4]. □

Theorem 3. $\mathbf{GL.t}^+$ *has finite model property.*

Proof. We showed that $\mathbf{GL.t}^+$ is complete with respect to Noetherian frames of finite height. Let $\mathcal{F} = \langle W, R\rangle$ be a cone and R is a Noetherian and transitive. $\mathcal{F}$ is a p-moprphic image of $\mathcal{F}^\#$.

Let φ be a $\mathbf{GL.t}^+$-consistent formula, then there exists a model on $\mathbf{GL.t}^+$-cone $\mathcal{C}$ and $x \in \mathcal{C}$ such that $\mathcal{M}, x \models \varphi$, where $\mathcal{M} = \langle \mathcal{M}, R, \vartheta\rangle$ for some valuation ϑ.

Thus, φ is satisfiable in $\mathcal{C}^\#$, that is, there exists a path α such that x is the last element of α and $\mathcal{M}^\#, \alpha \models \varphi$, where ϑ' is a valuation on $\mathcal{M}^\#$ and the underlying frame is $\mathcal{C}^\#$.

Let us put $\Psi = \{\Diamond\psi \mid \Diamond\psi \in \mathrm{Sub}(\varphi)\} \cup \{\Diamond^-\psi \mid \Diamond^-\psi \in \mathrm{Sub}(\varphi)\}$.

Suppose $\Psi = \{\Diamond\phi_1, \ldots, \Diamond\phi_m, \Diamond^-\phi'_1, \ldots, \Diamond^-\phi'_n\}$.

Let us define a selective filtration of $\mathcal{M}^\#$ inductively. Let $V_0 = \{\alpha\}$. Let $\beta, \gamma \in V_i$. $\Diamond\psi_i \in \Psi$ and $\Diamond^-\phi'_j \in \Psi$ such that $\mathcal{M}^\#, \beta \Diamond\psi_i$ and $\mathcal{M}^\#, \gamma \models \Diamond^-\phi'_j$.

We choose $\beta' \in R^\#(\beta)$ and $\gamma' \in (R^\#)^{-1}(\gamma)$ such that $\mathcal{M}^\#, \beta' \models \psi_i$ and $\mathcal{M}^\#, \gamma' \models \phi'_j$. Let us denote such a set as $V_{\alpha,\beta}$. It is clear that $|V_{\alpha,\beta}| \leqslant m + n$. Thus, V_{i+1} is defined as follows:

$$V_{i+1} = \bigcup_{\alpha,\beta \in V_i} V_{\alpha,\beta}$$

$\square$

References

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[3] Ilya Shapirovsky and Valentin B. Shehtman, *Chronological future modality in minkowski spacetime*, Advances in Modal Logic 4, papers from the fourth conference on "Advances in Modal logic," held in Toulouse, France, 30 September - 2 October 2002, 2002, pp. 437–460.

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