Cylindric notes

Daniel Rogozin

May 2023

1 Cylindric algebras: background

1.1 Atomic representations of Boolean algebras

Let B be a Boolean algebra, an element a is an *atom* if for every $b \in B$ $b < a$ implies $b = 0$. $\mathrm{At}(B)$ is the set of all atoms of B.

Definition 1. *Let B be a Boolean algebra and F a field of sets such that $h : B \to F$ is a representation of B, then B is a complete representation of B, if for every $A \subseteq B$ such that ΣA is defined the following holds:*

$$h(\Sigma A) = \bigcup_{a \in A} h(a) \tag{1}$$

A representation h is called atomic, if $x \in h(1)$, then there exists $b \in \mathrm{At}(\mathcal{B})$ such that $x \in h(b)$.

Theorem 1. *Let $\mathcal{B}$ be a Boolean algebra, then $\mathcal{B}$ is atomic iff $\mathcal{B}$ is completely representable. See [HH97, Corollary 6].*

1.2 Proper cylindric algebras

Let $X \neq \emptyset$ along and $X^\omega = \{f \mid f : \omega \to X\}$. Let $x \in X^\omega$, x_i stands for $x(i)$ for $i < \omega$. A subset of X^ω is an ω-ry relation on U. For $i, j < \omega$, the i, j*-diagonal* D_{ij} is the set of all elements of X^ω such that $y_i = y_j$.

If $i < \omega$ and Y is an ω-ry relation on X, then the i-th cylindrification $C_i Y$ is the set of all elements of U that agree with some element of Y on each coordinate except, perhaps, the i-th one:

$$C_i Y = \{y \in X^\omega \mid \exists x \in Y \forall i < \omega \, (i \neq j \Rightarrow y_j = x_j)\}.$$

We define the following equivalence relation for $i < \alpha$ and $x, y \in X^\omega$:

$$x \equiv_i y \Leftrightarrow \forall j \in \alpha \, (i \neq j \Rightarrow x(i) = y(j))$$

Then one may reformulate the definition of the i-th cylindrification the following way:

$$C_iY = \{y \in X^\omega \mid \exists x \in X \ \ x \equiv_i y\}$$

According to this version of the definition, one can think of cylindrification operators as **S5** modalities.

Definition 2. *A cylindic set algebra of dimension ω is an algebra consisting of a set S of ω-ry relation on some base set X with the constants and operations $0 = \emptyset$, $1 = X^\omega$, $\cap$, $-$, the diagonal elements $(D_{ij})_{i,j<\omega}$, the cylindrifications $(C_i)_{i<\omega}$. A* generalised cylindric set algebra of dimension ω *is a subdirect of cylindric algebras that have dimension ω.* $\mathbf{Cs}_\omega$ *denotes the class of all cylindric set algebras of dimension ω.*

Definition 3. *A cylindric algebra of dimension omega is an algebra $C = (B, (c_i)_{i<\omega}, (d_{ij})_{i,j<\omega})$ such that B is a Boolean algebra, each $d_{ij} \in \mathcal{B}$ and for all $i, j, k < \omega$ and for all $a, b \in B$:*

1. $c_i 0 = 0,$
2. $c_i(a + b) = c_i a + c_i b,$
3. $a \leq c_i a,$
4. $c_i(a \cdot c_i b) = c_i a \cdot c_i b$
5. $d_{ii} = 1,$
6. $c_i c_j a = c_j c_i a,$
7. *If* $k \neq i, j$, *then* $d_{ij} = c_k(d_{ij} \cdot d_{jk}),$
8. *If* $i \neq j$, *then* $c_i(d_{ij} \cdot a) \cdot c_i(d_{ij} \cdot -a) = 0.$

$\mathbf{CA}_\omega$ *is the class of all cylindric algebras of dimension ω.*

One can define a representation of a cylindric algebra explicitly the following way:

Definition 4. *Let $\mathcal{A}$ be a cylindric algebra of dimension ω. A* representation *of $\mathcal{A}$ over the non-empty domain X is a one-to-one map $f : \mathcal{A} \hookrightarrow 2^{X^\omega}$ such that:*

1. $f(1) = \bigcup\limits_{i \in I} X_i^\omega$ *for some disjoint family* $\{X_i\}_{i \in I}$ *where each* $X_i \subseteq X$
2. $h : \mathcal{A} \to 2^{f(1)}$ *is a representation of a Boolean reduct*
3. *for all* $i, k < \omega$, $x \in h(d_{ik})$ *iff* $x_i = x_k$
4. *for all* $i < \omega$ *and* $a \in \mathcal{A}$, $x \in h(c_i(a))$ *iff there is* $y \in X$ *such that* $x[i \mapsto y] \in h(a)$

Let C be a cylindric algebra, C is *representable* if there exists a representation of C. $\mathbf{RCA}_\omega$ is the class of all representable cylindric algebras. Alternatively, $\mathbf{RCA}_\omega$ can be defined as the closure of $\mathbf{Cs}_\omega$ under isomorphism:

$$\mathbf{RCA}_\omega = \mathbf{ICs}_\omega.$$

It is well known that $\mathbf{RCA}_\omega$ is a variety, $\mathbf{RCA}_n$ is finitely axiomatisable for $n \leq 2$ and $\mathbf{RCA}_\alpha$ $(2 < \alpha < \omega)$ has no finite axiomatisation, see [HMT88].

Let $C \in \mathbf{RCA}_\omega$, $\mathcal{A}$ has a *complete representation* if its representation preserves all existing suprema as in Definition 1. In other words, $\mathcal{A}$ is *completely representable.*

Proposition 1. *Let $A \in \mathbf{CA}_\omega$, then A is completely representation iff it has an atomic representation.*

Proof. Follows from Theorem 1. □

2 Atom structures and canonical extensions

First of all, we introduce the following operations on classes of algebras or frames. Let $\mathcal{A}$ be a class of algebras and $\mathcal{F}$ a class of frames, then:

- $\mathbf{I}\mathcal{K}$ is the closure of $\mathcal{K}$ under isomorphic copies,
- $\mathbf{Ud}\mathcal{F}$ is the closure of $\mathcal{F}$ under disjoint unions,
- $\mathbf{Ub}\mathcal{F}$ is the closure of $\mathcal{F}$ under bounded unions,
- $\mathcal{F}^+$ is the class of all complex algebras generated from elements of $\mathcal{F}$,
- $\mathcal{A}_+$ is the class of all ultrafilter frames generated from elements of $\mathcal{A}$,
- $\mathbf{Pu}\mathcal{K}$ is the closure of $\mathcal{K}$ under ultraproducts,
- $\mathbf{Pw}\mathcal{K}$ is the closure of $\mathcal{K}$ under ultrapowers,
- $\mathbf{S}\mathcal{A}$ is the closure of $\mathcal{A}$ under subalgebras,
- $\mathbb{S}\mathcal{F}$ is the closure of $\mathcal{F}$ under generated subframes,
- $\mathbb{H}\mathcal{F}$ is the closure $\mathcal{F}$ under p-morphic images.

The following definition of an ω-frame is taken from [Ven13].

Definition 5. *A cylindric ω-frame is a structure $F = (W, (R_i)_{i<\omega}, (D_{ij})_{i,j<\omega})$ where $(R_i)_{i<\omega}$ are binary relations and $(D_{ij})_{i,j<\omega}$ are unary relations such that, for all $i, j, k < \omega$:*

1. *Every R_i is an equivalence relation on W,*
2. *$R_i \circ R_j = R_j \circ R_i$,*
3. *For all $x \in W$, $D_{ii} = W$.*
4. *For all $x, y, z \in W$, if xR_iy, xR_iz, $D_{ij} = W$ and $D_{ij} = W$, then $y = z$.*

5. *For all $x \in W$, $D_{ij} = W$ iff there exists $y \in W$ such that $xR_k y$, $D_{ik} = W$ and $D_{kj} = W$.*

$\mathbf{CFrames}_\omega$ *is the class of all ω-frames.*

Remark 1.

Observe that the conditions of cylindric ω-frames can be expressed as first-order formulas. Therefore, $\mathbf{CFrames}_\omega$ is an elementary class.

We can associate a complete atomic cylindric algebra of dimension ω with every cylindric ω-frame $F = (W, (R_i)_{i<\omega}, (D_{ij})_{i,j<\omega})$ by taking its *complex algebra*, which is the algebra $F^+ = (2^W, \cup, -, (C_i)_{i<\omega}, \emptyset, W, (D_{ij})_{i,j<\omega})$ where each C_i is an operator $C_i : 2^W \to 2^W$ defined as:

$$C_i A = \{w \in W \mid \exists a \in A \, wRa\} = R_i^{-1}(A).$$

If $F \in \mathbf{CFrames}_\omega$ and $x \in F$, then F^x is a *generated subframe* generated by x. Generally, F_1 is a generated subframe of F_2, if $W_1 \subseteq W_2$, $R_{i1} \subseteq R_{i2}$ and for all $x \in W_1$ $y \in R_{i_2}(x)$ implies $y \in W_1$ for every $i < \omega$. That is, for all $i < \omega$ and $x \in F_1$, we have $R_{i_2}(x) \subseteq F_1$ and, thus, $R_{i_1}(x) = R_{i_2}(x)$.

Let $F_1 = (W_1, (R_{i_1})_{i<\omega}, (D_{ij_1})_{i,j<\omega})$ and $F_2 = (W_2, (R_{i_2})_{i<\omega}, (D_{ij_2})_{i,j<\omega})$ be cylindric ω-frames. A *bounded morphism* is a function $f : F_1 \to F_2$ such that, for each $i, j < \omega$:

1. (Monotonicity) $xR_{i_1}y$ implies $f(x)R_{i_2}f(y)$ for all $x, y \in W_1$,
2. (The lifting property) If $f(x)R_{i_2}z$, then there exists $y \in R_{i_1}(x)$ such that $f(y) = z$,
3. $x \in D_{ij_1}$ iff $f(x) \in D_{ij_2}$.

A bounded morphism is a *p-morphism* if it is onto. Notation: $F_1 \twoheadrightarrow F_2$. In this case, we say that F_1 is a *p-morphic image* of F_2.

We have the following connection between ω-frames and their generated subframes, which is standard for modal logic:

Proposition 2. *Let $F \in \mathbf{CFrames}_\omega$, then*

1. $F = \coprod\limits_{x \in F} F^x$,
2. $F^+ \cong \prod\limits_{x \in F} (F^x)^+$,
3. $(F^x)^+$ *is subdirectly irreducible.*

Let F be a cylindric ω-frame and let $(F_j)_{j\in J}$ be a family of cylindric ω-frames such that each F_j is a generated subframe of F. Then $G = (W, R_i, D_{ij})$ is the *bounded union* of $(F_j)_{j\in J}$, where $W = \bigcup\limits_{j\in J} W_j$ and R_i and D_{ij} are defined by corresponding relations in F_j's.

The following fact connects cylindric frames and cylindric algebras through complex algebras, see [Ven13, Proposition 2.1.5]:

Proposition 3. *A structure F is a cylindric ω-frame iff F^+ is a cylindric algebra of dimension ω.*

Let $(F_j)_{j\in J}$ be a disjoint family of cylindric ω-frames, the *disjoint sum* of $(F_i)_{i\in I}$ is $F = \coprod\limits_{i\in I} F_i$, where each $R_i = \bigcup\limits_{j\in J} R_{ij}$ and $D_{ik} = \bigcup\limits_{j\in J} D_{ikj}$. Disjoint sums and direct products are connected with one another through complex algebras as follows (see [Gol89, Lemma 3.4.1]):

$$(\coprod_{j\in J} F_j)^+ \cong \prod_{j\in J} F_j^+ \tag{2}$$

We define a particular frame of cylindric ω-frames. Let X be a non-empty set, the *full Cartesian structure over X of dimension ω* is a cylindric ω-frame $\mathfrak{C}(X) = (X^\omega, (\equiv_i)_{i<\omega}, D_{ij}{}_{i,j<\omega})$. $\mathcal{F}\mathfrak{ct}_\omega$ is the class of all full Cartesian structures of dimension ω. Observe that

$$\mathbf{Cs}_\omega = (\mathcal{F}\mathfrak{ct}_\omega)^+, \tag{3}$$

$$\mathbf{ICs}_\omega = \mathbf{S}(\mathcal{F}\mathfrak{ct}_\omega)^+. \tag{4}$$

The class of *generalised cylindric set algebras* of dimension ω, $\mathbf{Gs}_\omega$, consists of complex algebras of the closure of $\mathcal{F}\mathfrak{ct}_\omega$ under disjoint unions:

$$\mathbf{Gs}_\omega = (\mathbf{Ud}(\mathcal{F}\mathfrak{ct}_\omega))^+ \tag{5}$$

or, by (2):

$$\mathbf{Gs}_\omega = \mathbf{P}(\mathcal{F}\mathfrak{ct}_\omega^+) \tag{6}$$

$\mathbf{RCA}_\omega$ is the closure of $\mathbf{Gs}_\omega$ under isomorphism:

$$\mathbf{RCA}_\omega = \mathbf{IGs}_\omega \tag{7}$$

or, assuming (5) and (6):

$$\mathbf{RCA}_\omega = \mathbf{IGs}_\omega = \mathbf{S}((\mathbf{Ud}(\mathcal{F}\mathfrak{ct}_\omega))^+) = \mathbf{SP}(\mathcal{F}\mathfrak{ct}_\omega^+). \tag{8}$$

If $C \in \mathbf{CA}_\omega$ is atomic, then we can associate a cylindric omega frame with it. Let C be an atomic cylindric algebra of dimension ω, its *atom structure* is the structure $\mathbf{At}(C) = (\mathrm{At}(C), (R_i)_{i<\omega}, (D_{ij})_{i,j<\omega})$ such that each $D_{ij} \subseteq \mathbf{At}(C)$ and for all $i < \omega$ and for all $a, b \in \mathrm{At}(C)$:

$$aR_ib \text{ iff } c_ib \leq a.$$

As a corollary from Proposition 3:

Proposition 4. *If $C \in \mathbf{CA}_\omega$ is atomic, then $\mathbf{At}(C)$ is a cylindric ω-frame.*

3 Canonical extensions

Let B be a Boolean algebra, a proper subset $F \subsetneq B$ is an *filter* if the following holds:

1. $a \in B$ and $a \leq b$ imply $b \in B$,
2. If $a, b \in B$, then $a \cdot b \in B$.

A filter U is an *ultrafilter* if either $a \in U$ or $-a \in U$, or, equivalently, $U \subseteq U'$ implies $U = U'$. $\mathbf{Spec}(B)$ is the *spectum* of B, that is, the set of all ultrafilters of B.

Let C be a cylindric algebra of dimension ω, the ultrafilter frame of C is a structure $C_+ = (\mathbf{Spec}(C), (R_i)_{i<\omega}, (D_{ij})_{i,j<\omega})$ such that, for all $U_1, U_2 \in \mathbf{Spec}(C)$ and for all $i, j < \omega$:

1. $U_1 R_i U_2$ iff $\{c_i a \mid a \in U_2\} \subseteq U_1$,
2. $D_{ij} \subseteq \mathbf{Spec}(C)$.

From Proposition 3 we have:

Proposition 5. *If C is a cylindric algebra, then C_+ is a cylindric ω-frame.*

The canonical extension of C is the algebra $(C_+)^+$, that is, the complex algebra of the ultrafilter frame.

Theorem 2. *(See [JT51])*
$C \in \mathbf{CA}_\omega$ embeds to $(C_+)^+$ by mapping $a \mapsto \{U \in \mathbf{Spec}(C) \mid a \in U\}$.

4 Canonicity of $\mathbf{RCA}_\omega$

In this section, we reproduce the results related to characterisation $\mathbf{RCA}_\omega$. The following results are due to Goldblatt [Gol95]. This denotes that a cylindric algebra of dimension algebra is representable iff it is isomorphic to a subalgebra of the complex algebra of disjoint sum of some full ω-dimensional Cartesian structure.

The following characterisation result is known from [Ven13, Theorem 2.2.3].

Theorem 3. $\mathbf{RCA}_\omega = \mathbf{HSP}(\mathcal{F}\mathfrak{ct}_\omega^+)$

That is, the class of representable cylindric algebras of dimension ω is a variety generated by complex algebras of full Cartesian structures of dimension ω. If we consider the equational theory of $\mathbf{RCA}_\omega$ as a polymodal logic, we could say that it is Kripke complete with respect to the class of all full Cartesian structures of dimension ω.

To show that $\mathbf{RCA}_\omega$ is canonical we have got to show the following inclusion:

$$(\mathbf{RCA}_{\omega+})^+ \subseteq \mathbf{RCA}_\omega.$$

Definition 6. *The weak Cartesian space with base set X and dimension ω determined by $x \in X^\omega$ is the set:*

$$X^{\omega(x)} = \{y \in X^\omega \mid \mathrm{card}(\{k < \omega \mid x_k \neq y_k\}) < \aleph_0\}$$

$\mathfrak{S}_\omega(X^{\omega(x)})$ *is a weak Cartesian structure of dimension ω. $\mathcal{W}\mathfrak{ct}_\omega$ is the class of all weak Cartesian structure of dimension ω up to isomorphism.*

Note that we have $\mathcal{W}\mathfrak{ct}_\omega \subseteq \mathbf{CFrames}_\omega$.

Define also the class $\mathcal{S}\mathfrak{ct}_\omega$ of *sub-Cartesian structures of dimension ω* consisting of $\mathfrak{S}_\omega(V)$ for $V \subseteq X^\omega$, where X is a non-empty base set. Note that $\mathfrak{S}_\omega(V)$ does not have to be a cylindric ω-frame.

Let F be a generated subframe of a full Cartesian structure of dimension ω $\mathfrak{C}(X)$, then

$$F \cong \coprod_{x \in F} F^x \tag{9}$$

or, by (2):

$$F^+ \cong \prod_{x \in F} (F^x)^+ \tag{10}$$

The latter implies the inclusion:

$$(\mathbb{S}\mathcal{F}\mathfrak{ct}_\omega)^+ \subseteq \mathbf{P}(\mathcal{W}\mathfrak{ct}_\omega^+). \tag{11}$$

Note that (follows from [HMT+81, p. 118]):

Fact 1. $\mathcal{W}\mathfrak{ct}_\omega^+ \subseteq \mathbf{RCA}_\omega$

Complex algebras based on $\mathfrak{S}_\omega(X^x)$ form the class $\mathbf{Ws}_\omega$ of *weak cylindric set algebras of dimension ω*. The class $\mathbf{Gws}_\omega$ of *generalised weak cylindric set algebras of dimension ω* consists of complex algebras based on the closure of $\mathbf{Ws}_\omega$ under disjoint unions:

$$\mathbf{IWs}_\omega = \mathbf{S}(\mathcal{W}\mathfrak{ct}_\omega^+) \tag{12}$$

$$\mathbf{IGws}_\omega = \mathbf{S}((\mathbf{Ud}\mathcal{W}\mathfrak{ct}_\omega)^+) = \mathbf{SP}(\mathcal{W}\mathfrak{ct}_\omega^+) \tag{13}$$

The following is by Goldblatt, see [Gol95, Lemma 3.4]:

Lemma 1. $\mathbf{RCA}_\omega = \mathbf{S}\left((\mathbb{S}\mathbf{Ud}\mathcal{F}\mathfrak{ct}_\omega)^+\right) = \mathbf{S}\left((\mathbb{S}\mathbf{Ud}(\mathcal{W}\mathfrak{ct}_\omega))^+\right) = \mathbf{IGws}_\omega$

4.1 Ultraproducts of full Cartesian structures

Let $(F_j)_{j \in J}$ be an indexed family of full Cartesian structures of dimension ω, where each F_j is of the form

$$F_j = (W_j, (R_{i_j})_{i<\omega}, (D_{ik_j})_{i,k<\omega})$$

and let U be an ultrafilter on U. Define the following equivalence relation on $\prod\limits_{j\in J} W_j$ for $f, g \in \prod\limits_{j\in J} W_j$:

$$f \sim_U g \text{ iff } \{j \in J \mid f(j) = g(j)\} \in J$$

The *ultraproduct* of $(F_j)_{j\in J}$ is an algebra $\prod\limits_J F_j/U = (W, (R_i)_{i<\omega}, (D_{ik})_{i,k<\omega})$, where $W = \prod\limits_{j\in J} W_j$ and

1. $f_U R_i g_U$ iff $\{j \in J \mid R_{i_j}(f_U(j), g_U(j))\} \in U$,
2. $f_U \in D_{ik}$ iff $\{j \in J \mid f_U(j) \in D_{ik\,j}\} \in U$.

where f_U and g_U are equivalence classes of f and g modulo U.

See [Gol95, Lemma 3.5], a similar construction for modal logics could be found in [Fin75]:

Lemma 2.

Let $(F_j)_{j\in J}$ be an indexed family of full Cartesian structures of dimension ω and U an ultrafilter on J. There exists a p-morphism:

$$\varphi : \prod_J F_j/U \twoheadrightarrow \mathfrak{S}_\omega((\prod_J W_j/U))$$

that restricts to an isomorphism $F^x \cong I^{\varphi(x)}$ of generated subframes generated by $x \in F$.

Proof. Consider the equation:

$$f_i(j) = f(j)_i \tag{14}$$

If $j \in \prod\limits_{j\in J} W_j^\omega$, then the equation defines a function $f_i \in \prod_{j\in J} W_j$ for each $i < \omega$. Then a sequence $(f_i)_{i<\omega}$ defines a function by Equation 14. Clearly $f_U = g_U$ implies $f_{iU} = g_{iU}$ for $i < \omega$. So define φ as:

$$\varphi(f_U) = (f_{iU})_{i<\omega} \tag{15}$$

It is readily checked that:

1. $f_U \in D_{kl}$ iff $f_{kU} = f_{lU}$ iff $\varphi(f_U) \in E^\omega_{kl}$,
2. $(f_U)R_k(g_U)$ implies $f_{lU} = g_{lU}$ whenever $k \neq l < \omega$, so $(f_U)R^\omega_k(g_U)$, so φ is monotone.

Let us show that φ has the lifting property. Assume that $\varphi(f_U)R^\omega_k z$ where $z = (g_k)_{k<\omega}$. We have got to show that there exists h_U such that $\varphi(h_U) = z$ and $(f_U)R_k(h_U)$. Put $h_k = g_k$ and $h_l = f_l$ for $k \neq l < \omega$, so for $k \neq l$ one has $P(f_l)_U = (g_l)_U$ since $\varphi(f_U)R^\omega_k z$, so $(g_l)_U = (h_l)_U$, so $z = (h_U)$ are the same sequence. Moreover, $\{j \mid h(j)R_{k\,j}f(j)\} = J \in U$, since $h(j)_l = f(j)_l$ for $l \neq k$, so $(f_U)R_k(h_U)$ in the ultraproduct.

Let us show that φ acts isomorphically on every generated subframe F^x of the ultraproduct. Take $f_U, g_U \in F^x$, then there are $i_0, \ldots, i_n < \omega$ such that

$$f_U(R_{i_0} \circ \cdots \circ R_{i_n})g_U.$$

By Łoś's theorem we have

$$J_{fg} = \{j \in J \mid f(j)(R_{i_{0_j}} \circ \cdots \circ R_{i_{n_j}})g(j)\}$$

So for J_{fg}, the ω-sequences $f(j)$ and $g(j)$ agree except possibly on $i_0, \ldots, i_n$. If $\varphi(f_U) = \varphi(g_U)$, then for each $k < \omega$, $f_{k_U} = g_{k_U}$ and then:

$$J_k = \{j \in J \mid f_k(j) = g_k(j)\} \in U$$

But f, g are identical on the set

$$J_k \cap J_{i_0} \cap \cdots \cap J_{i_n} \in F$$

and thus $f_U = g_U$, so φ is injective on F^x. □

Theorem 4. $\mathbf{Pu}\mathcal{F}\mathfrak{ct}_\omega \subseteq \mathbf{Ub}\mathcal{F}\mathfrak{ct}_\omega$.

Proof. Let $F = \prod_J F_j/U$ be an ultraproduct of full Cartesian structures of dimension ω. To show $F \in \mathbf{Ub}\mathcal{F}\mathfrak{ct}_\omega$ one needs to show that for each point $x \in F$ there exists a generated subframe that contains x and is isomorphic to $I = \mathfrak{S}_\omega((\prod_J F_j/U))$.

Let Z be a choice set that contains exactly one element from each weak Cartesian substructure of I. But I is the disjoint union of all its weak substructures, so we have:

$$I = \coprod_{z \in Z} I^z$$

Fix $x \in F$, for each $z \in Z$ choose $\psi(z)$ to be any member of F such that $\varphi(\psi(z)) = z$ and I^z is the weak substructure containing $\varphi(x)$, where φ is a p-morphism from Lemma 2. By the previous lemma, we have

$$F^{\psi(z)} = I^z.$$

If z and z' are different elements of Z, so I^z and $I^{z'}$ are disjoint, so $F^{\psi(z)}$ and $F^{\psi(z')}$ are also disjoint.

$F(x)$ is defined to be the union of the collection of $\{F^{\psi(z)} \mid z \in Z\}$ and forms a generated subframe of F which is isomorphic of I^z's, so $F^x \cong I$, but $x = \psi(z)$ for some z, so $x \in F(x)$. □

Corollary 1. $\mathbf{Ub}\mathcal{F}\mathfrak{ct}_\omega$ *is closed under ultraproducts.*

Theorem 5.

1. $\mathbf{Pu}\mathcal{W}\mathfrak{ct}_\omega \subseteq \mathbf{Ub}\mathcal{W}\mathfrak{ct}_\omega$,

2. $\mathbf{Pu}\mathcal{S}\mathfrak{ct}_\omega \subseteq \mathcal{S}\mathfrak{ct}_\omega$.

Proof.

1. Let $F^* = \prod\limits_J F^*_j/U$ be an ultraproduct of weak Cartesian structures of dimension ω. Each F^*_j is a generated subframe of some full Cartesian structure F_j, so F^* is isomorphic to a generated subframe of the ultraproduct $F = \prod\limits_J F_j/U$ and we identify F^* with this generated subframe. But the ultraproduct F is a cylindric ω-frame since each $F_j \in \mathbf{CFrames}_\omega$, but $\mathbf{CFrames}_\omega$ is elementary and then closed under ultraproducts. But F^* can be defined as

$$F^* = \coprod_{x \in F^*} F^x$$

But each F^x is isomorphic to each $I^{\varphi(x)}$.

2. Let $F^* = \prod\limits_J F^*_j/U$ be an ultraproduct of sub-Cartesian structures of dimension ω, then each F^*_j is a substructure of some full Cartesian structure F_j of dimension ω. So F^* is isomorphic to a substructure of the ultraproduct $F = \prod\limits_J F_j/U$. As in the previous item, $F \in \mathbf{CFrames}_\omega$ of its all point-generated substructures F^x, each of which is isomorphic to some sub-Cartesian structure of dimension ω. Then

$$F = \coprod_{x \in F} F^x \in \mathbf{Ud}\mathcal{S}\mathfrak{ct}_\omega = \mathcal{S}\mathfrak{ct}_\omega$$

That makes $F^* \in \mathcal{S}\mathfrak{ct}_\omega$.

□

Theorem 6. $\mathcal{W}\mathfrak{ct}^+_\omega \subseteq \mathbf{S}((\mathbf{Pw}(\mathcal{W}\mathfrak{ct}_\omega))^+)$.

Proof. Let J be the set of finite subsets ω and let U be an ultrafilter on J that contains, for each $i \in J$, the set

$$J_i = \{j \in J \mid i \subseteq j\}.$$

In particular $J_k = \{j \in J \mid k \in j\}$ for all $k < \omega$. Now take $\mathfrak{S}_\omega(X^{(x)}) \in \mathcal{W}\mathfrak{ct}_\omega$. For each $y \in X^{\omega(x)}$, let $f_y \in \left(X^{\omega(x)}\right)^J$ be the constant function $f_y(j) = y$. Then $\psi : y \mapsto f_y/U$ is the isomorphic embedding

$$\psi : \mathfrak{S}_\omega(X^{(x)}) \hookrightarrow \mathfrak{S}_\omega(X^{(x)})^J/U$$

of $\mathfrak{S}_\omega(X^{(x)})$ to its ultrapower $\mathfrak{S}_\omega(X^{(x)})^J/U$ with respect to U. We have got to show:

Claim 1. *There exists a bounded morphism $\varphi : \mathfrak{S}_\omega(X)^J/U \to \mathfrak{S}_\omega(X^{(x)})/U$ such that its image contains the image of ψ:*

$$\psi : \mathfrak{S}_\omega(X^{(x)}) \hookrightarrow \operatorname{Im}\psi \subseteq \operatorname{Im}\phi \subseteq \mathfrak{S}_\omega(X^{(x)})^J/U$$

By duality ϕ induces a homomorphism:

$$\phi^+ : (\mathfrak{S}_\omega(X^{(x)})/U)^+ \to (\mathfrak{S}_\omega(X)^J/U)^+$$

ϕ^+ composes with the homomorphism:

$$(\mathfrak{S}_\omega(X^{(x)}))^+ \to (\mathfrak{S}_\omega(X^{(x)})^J/U)^+$$

that gives a homomorphism:

$$\theta : (\mathfrak{S}_\omega(X^{(x)}))^+ \to (\mathfrak{S}_\omega(X)^J/U)^+$$

Let us describe the action of θ, take $f \in (X^\omega)^J$ and choose any $f^\bullet \in (X^{\omega(x)})^J$ such that $\varphi(f/U) = f^\bullet/U$, so for any $Y \subseteq X^{\omega(x)}$:

$$\theta(Y) = \{f/U \in (X^\omega)^J/U \mid \{j \mid f^\bullet(j) \in Y\} \in U\}.$$

So for $y \in Y$, then $\psi(y)$ (that is f_y/U) is equal to $\varphi(f/U)$ for some f, so then $f^\bullet/U = f_y/U$ and then $\{j \,|\, f^\bullet(j) = y \in Y\} \in U$ showing that $f/U \in \theta(Y)$. As far as $\mathfrak{S}_\omega(X)^J/U$ is an ultrapower of a full Cartesian structure of dimension ω, so $(\mathfrak{S}_\omega(X^{(x)}))^+ \in \mathbf{S}((\mathbf{Pw}(\mathcal{F}\mathfrak{ct}_\omega))^+)$, so the theorem is proved.

Now let us prove Claim 1:

Proof. Take $f \in (X^\omega)^J$, define $f^\bullet \in (X^{\omega(x)})^J$ as:

$$f^\bullet(j)_k = \begin{cases} f(j)_k & \text{if } k \in j \\ x_k & \text{otherwise} \end{cases} \tag{16}$$

Each $f^\bullet(j)$ differs from x at most on the finite set j. Clearly that $f(j) = g(j)$ implies $f^\bullet(j) = g^\bullet(j)$, so $f_U = g_U$ in $(X^\omega)^J/U$ implies $f^\bullet_U = g^\bullet_U$ in $(X^{\omega(x)})^J/U$. So the mapping $\varphi : f_U \mapsto f^\bullet_U$ is well-defined.

Let us show that $\operatorname{Im}\psi \subseteq \operatorname{Im}\varphi$. Take $f_{yU} \in \operatorname{Im}\psi$ with $y \in X^{\omega(x)}$. We also have $f_{yU} \in (X^\omega)^J$, so that is enough to show that $f^\bullet_{y\,U} = f_{yU}$ in $\operatorname{Im}\psi$. Put $i = \{k < \omega \mid x_k \neq y_k\} \in J$, so for $j \in J_j$:

$$f^\bullet_y(j)_k = \begin{cases} f_y(j)_k & \text{if } k \in j \\ x_k = f_y(j)_k & \text{otherwise} \end{cases} \tag{17}$$

since $f_y(j)$ agrees with x outside i. Thus $f^\bullet_y(j) = f_y(j)$ for each $j \in J_j \in U$, so $f^\bullet_{y\,U} = f_{yU} = \psi(y)$. $\square$

We skip the proof φ is a bounded morphism. $\square$

Theorem 7. $\mathcal{W}\mathfrak{ct}^+_\omega \subseteq \mathbf{RCA}_\omega$.

Proof.

$\mathcal{W}\mathfrak{ct}_\omega^+ \subseteq$

By Theorem 6

$\subseteq \mathbf{SCmPw}(\mathcal{F}\mathfrak{ct}_\omega)$

By Theorem 4

$\subseteq \mathbf{SCmUb}(\mathcal{F}\mathfrak{ct}_\omega)$

Since $\mathbf{Ub}\mathcal{F}\mathfrak{ct}_\omega \subseteq \mathbb{H}\mathbf{Ud}\mathcal{F}\mathfrak{ct}_\omega$

$\subseteq \mathbf{SCm}\mathbb{H}\mathbf{Ud}\mathcal{F}\mathfrak{ct}_\omega$

Since $\mathbb{H}\mathbf{Ud}\mathcal{F}\mathfrak{ct}_\omega = \mathbf{Ud}\mathcal{F}\mathfrak{ct}_\omega$

$= \mathbf{SCmUd}\mathcal{F}\mathfrak{ct}_\omega$

$= \mathbf{RCA}_\omega$

□

Theorem 8. $\mathbf{IWs}_\omega \subseteq \mathbf{ICs}_\omega$.

Proof. Let $F_1 = \mathfrak{S}_\omega(X)^J/U$ and $F_2 = \mathfrak{S}_\omega(X^{(x)})^J/U$ be as in the proof of Theorem 6. Let $\varphi : F_1 \to F_2$ be a p-morphism as in the same proof.

We transform φ to a p-morphism $\mathfrak{S}_\omega(X^J/U) \to F_2$, so we have embedding of $\mathfrak{S}_\omega(X^{(x)})^J/U^+$ into the cylindric set algebra $\mathfrak{S}_\omega(X^J/U)^+$.

As in Theorem 4, each point of F_1 belongs to a generated subframe of F_1, which is isomorphism to $\mathfrak{S}_\omega(X^J/U)$. In particular, there is an injection:

$$\varphi_x : \mathfrak{S}_\omega(X^J/U) \hookrightarrow F_1$$

whose image is a generated subframe of F_1 containing the point f_x/U. Composing φ_x with φ given a morphism:

$$\varphi \circ \varphi_x : \mathfrak{S}_\omega(X^J/U) \to F_2$$

By duality, we have a homomorphism:

$$F_2^+ \to (\mathfrak{S}_\omega(X^J/U))^+$$

which composes with the map $(\mathfrak{S}_\omega(X^{(x)}))^+ \to F_2^+$ that gives a homomorphism

$$(\mathfrak{S}_\omega(X^{(x)}))^+ \to (\mathfrak{S}_\omega(X^J/U))^+. \tag{18}$$

The last homomorphism is injective whenever the image of $\varphi \circ \varphi_x$ contains the image of the embedding $\psi : \mathfrak{S}_\omega(X^{(x)}) \hookrightarrow \mathfrak{S}_\omega(X^{(x)})^J/U$.

Take $y \in X^{\omega(x)}$, then $x(R_{i_0} \circ \cdots \circ R_{i_n})y$ in $\mathfrak{S}_\omega(X)$ for some $i_0, \ldots, i_n < \omega$, so $f_x/U(R_{i_0} \circ \cdots \circ R_{i_n})f_y/U$, so f_y/U belongs to the generated subframe $\operatorname{Im}\varphi_x$ of F_1. Then $\operatorname{Im}(\varphi \circ \varphi_x)$ contains $\varphi(f_y/U) = f_y^\bullet/U$, but $f_y^\bullet/U = \psi(y)$ in F_2.

Thus $\operatorname{Im}\psi \subseteq \operatorname{Im}(\varphi \circ \varphi_x)$, so (18) is an injection. So we have

$$(\mathcal{W}\mathfrak{ct}_\omega)^+ \subseteq \mathbf{S}((\mathcal{F}\mathfrak{ct}_\omega)^+) = \mathbf{ICs}_\omega$$

Then

$$\mathbf{IWs}_\omega = \mathbf{S}((\mathcal{W}\mathfrak{ct}_\omega)^+) \subseteq \mathbf{ICs}_\omega.$$

□

4.2 Elementary generating classes and proof of canonicity

Lemma 3. *Let $\mathcal{A}$ be a class of BAOs and let $\mathcal{F}$ be a class of Kripke frames, then:*

1. $(\mathbf{HS}\mathcal{A})_+ \subseteq \mathbb{SH}(\mathcal{A}_+)$,
2. $(\mathbf{HS}(\mathcal{F}^+))_+ \subseteq \mathbb{SH}\mathbf{Pw}\mathcal{A}$,
3. $\mathbf{HS}(\mathcal{F}^+) \subseteq \mathbf{S}(\mathbb{S}\mathbf{Pw}\mathcal{F}^+)$.

Proof.

1. Let $A \in \mathbf{HS}\mathcal{A}$, then there are A_1 and $A_2 \in \mathcal{A}$ such that $A \twoheadleftarrow A_1 \hookrightarrow A_2$, then by duality we have $A_+ \hookrightarrow A_{1+} \twoheadleftarrow A_{2+}$ so $A_+ \in \mathbb{SH}(\mathcal{A}_+)$.
2. $(\mathbf{HS}(\mathcal{F}^+))_+ \subseteq \mathbb{SH}((\mathcal{A}_+)^+) \subseteq \mathbb{SH}\mathbf{Pw}\mathcal{A}$.
3. $\mathbf{HS}(\mathcal{F}^+) \subseteq \mathbf{S}(((\mathbf{HS}(\mathcal{F}^+))_+)^+) \subseteq \mathbf{S}((\mathbb{HS}\mathbf{Pw}\mathcal{F})^+) \subseteq \mathbf{S}((\mathbb{SH}\mathbf{Pw}\mathcal{F})^+) \subseteq \mathbf{S}((\mathbb{S}\mathbf{Pw}\mathcal{F})^+)$

$\square$

Theorem 9. *Let $\mathcal{F}$ be a class of frames closed under generated subframes and ultrapowers, then $\mathbf{S}(\mathcal{F}^+)$ is closed under homomorphic images. If $\mathcal{F}$ is also closed under disjoint unions, then $\mathbf{S}(\mathcal{F}^+)$ is a canonical variety.*

Proof. If $\mathcal{F} = \mathbb{S}\mathbf{Pw}\mathcal{F}$, then by Lemma 3 $\mathbf{HS}((\mathcal{F})^+) = \mathbf{S}((\mathcal{F})^+)$.

If $\mathbf{Ud}\mathcal{F} = \mathcal{F}$, then $\mathbf{PS}(\mathcal{F}^+) \subseteq \mathbf{SP}(\mathcal{F}^+) = \mathbf{S}((\mathbf{Ud}\mathcal{F})^+) = \mathbf{S}((\mathcal{F})^+)$, so $\mathbf{S}((\mathcal{F})^+)$ is closed under products, so $\mathbf{S}((\mathcal{F})^+)$ is variety.

If $\mathcal{F}$ is also closed under ultrapowers, then $\mathbf{S}((\mathcal{F})^+)$ is canonical, see [Gol89, Corollary 3.6.3]. $\square$

Lemma 4. *Let $\mathcal{F}$ be a class of frames, then:*

1. $\mathbf{HS}(\mathbb{S}\mathcal{F})^+ \subseteq \mathbf{S}((\mathbb{S}\mathbf{Pw}\mathcal{F})^+)$.
2. *If* $\mathbf{Pw}\mathcal{F} \subseteq \mathbb{HS}\mathcal{F}$, *then* $\mathbf{HS}((\mathbb{S}\mathcal{F})^+) = \mathbf{S}((\mathbb{S}\mathcal{F})^+)$.
3. $((\mathbf{S}(\mathcal{F}^+))_+)^+ \subseteq \mathbf{S}(\mathcal{F}^+{}_+)^+ \subseteq \mathbf{S}(\mathbf{Pw}\mathcal{F})^+$.
4. *If* $\mathbf{Pw}\mathcal{F} \subseteq \mathbb{H}\mathcal{F}$, *then* $\mathbf{S}\mathcal{F}^+$ *is canonical.*
5. *If* $\mathbf{Pw}\mathcal{F} \subseteq \mathbb{HS}\mathcal{F}$, *then* $\mathbf{S}(\mathbb{S}\mathcal{F})^+$ *is canonical.*
6. *If* $\mathbf{Pu}\mathcal{F} \subseteq \mathbb{HS}\mathbf{Ud}\mathcal{F}$, *then* $\mathbf{PuUd}\mathcal{F} \subseteq \mathbb{HS}\mathbf{Ud}\mathcal{F}$.

Proof.

1. $\mathbf{HS}(\mathbb{S}\mathcal{F})^+ \subseteq \mathbf{S}(\mathbb{S}\mathbf{Pw}\mathbb{S}\mathcal{F})^+ \subseteq \mathbf{S}(\mathbb{SS}\mathbf{Pw}\mathcal{F})^+ \subseteq \mathbf{S}(\mathbb{S}\mathbf{Pw}\mathcal{F})^+$.
2. $\mathbf{HS}((\mathbb{S}\mathcal{F})^+) \subseteq \mathbf{S}((\mathbb{S}\mathbf{Pw}\mathcal{F})^+) \subseteq \mathbf{S}((\mathbb{SHS}\mathcal{F})^+) \subseteq \mathbf{S}((\mathbb{HSS}\mathcal{F})^+) = \mathbf{S}((\mathbb{S}\mathcal{F})^+)$.

3. Let $A \in \mathbf{S}\mathcal{F}^+$, then $A \hookrightarrow F^+$ for some $F \in \mathcal{F}$, then $(A_+)^+ \hookrightarrow ((F^+)_+)^+$. To show the second inclusion, recall that $(\mathcal{F}^+)_+ \subseteq \mathbb{H}\mathbf{Pw}\mathcal{F}$, so we have $\mathbf{S}((\mathcal{F}^+)_+)^+ \subseteq \mathbf{S}(\mathbb{H}\mathbf{Pw}\mathcal{F})^+ = \mathbf{S}(\mathbf{Pw}\mathcal{F})^+$.

4. $(\mathbf{S}(\mathcal{F}^+)_+)^+ \subseteq \mathbf{S}((\mathbf{Pw}\mathcal{F})^+) \subseteq \mathbf{S}((\mathbb{H}\mathcal{F})^+) = \mathbf{S}(\mathcal{F}^+)$.

5. If $\mathbf{Pw}\mathcal{F} \subseteq \mathbb{H}\mathbb{S}\mathcal{F}$, then:

$$\mathbf{Pw}\mathbb{S}\mathcal{F} \subseteq \mathbb{S}\mathbf{Pw}\mathcal{F} \subseteq \mathbb{S}\mathbb{H}\mathbb{S}\mathcal{F} \subseteq \mathbb{H}\mathbb{S}\mathbb{S}\mathcal{F} = \mathbb{H}\mathbb{S}\mathcal{F}.$$

   Therefore, $((\mathbf{S}(\mathbb{S}\mathcal{F})^+)_+)^+ = \mathbf{S}(\mathbb{S}\mathcal{F})^+$.

6. $\mathbf{PwUd}\mathcal{F} \subseteq \mathbf{PuUb}\mathcal{F} \subseteq \mathbf{UbPu}\mathcal{F} \subseteq \mathbf{Ub}\mathbb{H}\mathbb{S}\mathbf{Ud}\mathcal{F} \subseteq \mathbb{H}\mathbf{Ud}\mathbb{S}\mathbf{Ud}\mathcal{F} \subseteq \mathbb{H}\mathbb{S}\mathbf{UdUd}\mathcal{F} = \mathbb{H}\mathbb{S}\mathbf{Ud}\mathcal{F}$.

□

Theorem 10. *If* $\mathbf{Pu}\mathcal{F} \subseteq \mathbb{H}\mathbb{S}\mathbf{Ud}\mathcal{F}$, *then* $\mathbf{S}((\mathbb{S}\mathbf{Ud}\mathcal{F})^+)$ *is a canonical variety.*

Proof. By the previous lemma, $\mathbf{S}((\mathbb{S}\mathbf{Ud}\mathcal{F})^+)$ is canonical and closed under homomorphic images. It is also closed under products since:

$$\mathbf{PS}((\mathbb{S}\mathbf{Ud}\mathcal{F})^+) \subseteq \mathbf{SP}((\mathbb{S}\mathbf{Ud}\mathcal{F})^+) = \mathbf{S}((\mathbf{Ud}\mathbb{S}\mathbf{Ud}\mathcal{F})^+) = \mathbf{S}((\mathbb{S}\mathbf{Ud}\mathcal{F})^+).$$

□

Theorem 11. $\mathbf{RCA}_\omega$ *and* $\mathbf{ICrs}_\omega$ *are canonical varieties.*

Proof. By Lemma 1, $\mathbf{RCA}_\omega = \mathbf{S}(\mathbb{S}\mathbf{Ud}\mathcal{F}\mathfrak{ct}_\omega)^+$. By the previous theorem, this is enough to show that $\mathbf{Pu}\mathcal{F}\mathfrak{ct}_\omega \subseteq \mathbb{H}\mathbb{S}\mathbf{Ud}\mathcal{F}\mathfrak{ct}_\omega$, which holds by Theorem 4.

As regards, $\mathbf{ICrs}_\omega = \mathbf{S}((\mathcal{S}\mathfrak{ct}_\omega)^+)$. We have $\mathbf{ICrs}_\omega = \mathbf{S}((\mathbb{S}\mathbf{Ud}\mathcal{S}\mathfrak{ct}_\omega)^+)$, and also, by Theorem 5, $\mathbf{Pu}\mathcal{S}\mathfrak{ct}_\omega \subseteq \mathcal{S}\mathfrak{ct}_\omega \subseteq \mathbb{H}\mathbb{S}\mathbf{Ud}\mathcal{S}\mathfrak{ct}_\omega$. □

5 Representability via games

5.1 Monk's theorem for $\mathbf{RCA}_n$ via saturation

In this section we consider classes $\mathbf{RCA}_n$ for $n < \omega$.

We provide the complete proof of the following theorem [HH13, Theorem 3.4.3].

Theorem 12. *Let* $\mathcal{A} \in \mathbf{CA}_n$, *then* $\mathcal{A}$ *is representable iff* $(\mathcal{A}_+)^+$ *is completely representable.*

For that we need such model-theoretic notions as saturation and types, see [Hod93, Section 6.3].

Definition 7. *Let* $\mathcal{M}$ *be a first-order structure of a signature* L *and* $S \subseteq \mathcal{M}$. *Let* $L(S)$ *be an extension of* L *with copies of elements from* S *as additional constants. We assume that* $Cnst(L)$ *and* S *are disjoint.*

1. *Let $n < \omega$, an n-type over S is a set $\mathcal{T}$ of $L(S)$ formulas $A(\overline{x})$, where $\overline{x}$ is a fixed n-tuple of elements from S. Notation: $\mathcal{T}(\overline{x})$. A type is an n-type for some $n < \omega$.*

2. *An n-type $\mathcal{T}(\overline{x})$ is realised in $\mathcal{M}$, if there exists $\overline{m} \in \mathcal{M}^n$ such that $\mathcal{M} \models A(\overline{m})$ for every $A \in \mathcal{T}(\overline{x})$. $\mathcal{M}$ omits $\mathcal{T}(\overline{x})$, if $\mathcal{T}(\overline{x})$ is not realised in $\mathcal{M}$.*

3. *$\mathcal{T}(\overline{x})$ is finitely satisfied in $\mathcal{M}$, if every finite subtype $\mathcal{T}_0(\overline{x}) \subseteq \mathcal{T}(\overline{x})$ is realised in $\mathcal{M}$. We can reformulate that as $\mathcal{M} \models \exists \overline{a} \bigwedge\limits_{A \in \mathcal{T}_0} A(\overline{a})$.*

4. *Let T be a theory, then a type $\mathcal{T}$ over the empty set of constants is T-consistent, if there exists a model $\mathcal{M} \models T$ such that $\mathcal{T}$ is finitely satisfied in $\mathcal{M}$.*

5. *Let κ be a cardinal, then $\mathcal{M}$ is κ-saturated, if for every $S \subseteq \mathcal{M}$ with $|S| < \kappa$ every finitely satisfied 1-type $\mathcal{T}$ is realised in $\mathcal{M}$.*

By default, a *saturated* model is an ω-saturated model for us.
A couple of useful facts from [CK90] and [Hod93]:

Fact 2. *Let $\mathcal{M}$ be an FO-structue and κ a cardinal, then:*

1. *$\mathcal{M}$ is κ-saturated, iff every finitely satisfiable α-type (an arbitrary $\alpha \leq \kappa$) with fewer than κ parameters is realised in $\mathcal{M}$.*

2. *If $\mathcal{M}$ is κ-saturated, then $\mathcal{M}$ is λ-saturated for every $\lambda < \kappa$.*

3. *Every consistent theory has a κ-saturated model and every model has an elementary κ-saturated extension.*

4. *Let $(\mathcal{M}_i)_{i<\omega}$ a family of structures of the (at most) countable signature and D a non-principal ultrafilter over ω, then $\Pi_D \mathcal{M}_i$ is ω_1-saturated.*

5.2 Proof of Theorem 12

Let $\mathcal{A} \in \mathbf{CA}_n$, then if $\mathcal{A}$ is completely representable, then h, a complete representation of $\mathcal{A}$, is atomic. That is, $(a_1, \ldots, a_n) \in h(1)$, then $(a_1, \ldots, a_n) \in h(y)$ for some $y \in \mathrm{At}(\mathcal{A})$.

Definition 8. *Let $\mathcal{A}$ be a cylindric algebra of dimension $n < \omega$. $L(\mathcal{A})$ is the first-order language that consists of equality plus n-ary predicate letters $(R_a^n)_{a \in \mathcal{A}}$. The $L(\mathcal{A})$-theory $T_{\mathcal{A}}$ consists of the following sentences:*

1. *$A_+(a, b, c) := \forall x_1, \ldots, x_n (R_a(x_1, \ldots, x_n) \leftrightarrow R_b(x_1, \ldots, x_n) \vee R_c(x_1, \ldots, x_n))$. Informally, that means $\mathcal{A} \models a = b + c$.*

2. *$A_-(a, b) := \forall x_1, \ldots, x_n \ (R_a(x_1, \ldots, x_n) \leftrightarrow \neg R_b(x_1, \ldots, x_n))$. That is, $\mathcal{A} \models a = -b$.*

3. *$A_{\neq 0}(a) := \exists x_1, \ldots, x_n R_a(x_1, \ldots, x_n)$. That is, $\mathcal{A} \models a \neq 0$.*

4. $A_{c_i}(a) := \forall x_1, \ldots, x_n(R_{c_i a}(x_1, \ldots, x_n) \leftrightarrow \exists y_1, \ldots y_n(R_a(y_1, \ldots, y_n) \wedge x_i = y_j))$, *for* $i < n$ *and* $j < n$ *such that* $i \neq j$*. Informally,* $\mathcal{A} \models c_i a = 1$*.*

5. $A_{d_{ij}} := \forall x_1, \ldots, x_n(R_{d_{ij}}(x_1, \ldots, x_n) \leftrightarrow x_i = x_j)$, *for* $i, j < n$*.*

Assume that $\mathcal{A}$ is representable, then the theory $T(\mathcal{A})$ is satisfiable, then it has an ω-saturated model $\mathcal{M}$ by Fact 3. We have the following claim:

Claim 2. *The set* $U_{x_1,\ldots,x_n} = \{a \in \mathcal{A} \,|\, \mathcal{M} \models R_a(x_1, \ldots, x_n)\}$ *is an ultrafilter of* $\mathcal{A}$*, for* $x_1, \ldots, x_n \in \mathcal{M}$ *with* $R_1(x_1, \ldots, x_n)$*.*

Those $U_{x_1,\ldots,x_n}$'s allow us to represent atoms of $\mathcal{A}^+$.
We define a representation of $\mathcal{A}^+$ as a map $h : \mathcal{A}^+ \to 2^{\mathcal{M}^n}$ such that:

$$h : S \mapsto \{(x_1, \ldots, x_n) \in 1^{\mathcal{M}} \mid U_{x_1,\ldots,x_n} \in S\}, \text{ for } S \in \mathbf{Spec}(\mathcal{A}).$$

Claim 3. *Let* $A_1, A_2 \in \mathbf{Spec}(\mathcal{A})$

1. $h(0^{\mathcal{A}^+}) = \emptyset$
2. $h(-A_1) = -h(A_1)$
3. $h(1^{\mathcal{A}^+}) = 1^{\mathcal{M}}$
4. *If* $S \subseteq \mathbf{Spec}(\mathcal{A})$*, then* $h(\bigcup S) = \bigcup\limits_{U \in S} h(U)$

In particular, h *is a Boolean homomorphism.*

Proof.

1. $h(0^{\mathcal{A}^+}) = h(\emptyset) = \emptyset$.
2. From the definiton of h.
3. $h(-A_1) = -h(A_1)$
   Let $x_1, \ldots, x_n \in 1^{\mathcal{M}}$, then we have:

$$(x_1, \ldots, x_n) \in h(-A_1) \text{ iff } U_{x_1,\ldots,x_n} \in -A_1 \text{ iff } U_{x_1,\ldots,x_n} \notin A_1 \text{ iff } (x_1, \ldots, x_n) \notin h(A_1)$$

4. Let $S = \bigcup\limits_{i \in I} S_i$, where $S_i \in \mathbf{Spec}(\mathcal{A})$ for every $i \in I$. Let $(x_1, \ldots, x_n) \in 1^{\mathcal{M}}$, then we have:

$$(x_1, \ldots, x_n) \in h(\bigcup_{i \in I} S_i) \text{ iff } f_{x_1,\ldots,x_n} \in \bigcup_{i \in I} S_i \text{ iff } \exists i \in I \;\; f_{x_1,\ldots,x_n} \in S_i \text{ iff } \exists i \in I \;\; (x_1, \ldots, x_n) \in h(S_i) \text{ iff } (x_1, \ldots, x_n) \in \bigcup_{i \in I} S_i$$

$\square$

Claim 4. *h is injective.*

Proof. Let $U \in \mathbf{Spec}(\mathcal{A})$. The first is to show that $h(U)$ is non-empty. The following n-type:

$$T(x_1, \ldots, x_n) = \{R_a(x_1, \ldots, x_n) \mid a \in U\}$$

if finitely satisfied in $\mathcal{M}$.

Consider $T_0 = \{R_{a_1}(x_1, \ldots, x_n), \ldots, R_{a_k}(x_1, \ldots, x_n)\} \subseteq T$. Then $a_1, \ldots, a_k \in U$ and $a = a_1 \cdot \cdots \cdot a_k \in U$. By the instance of the $A_{\neq 0}(a)$-axiom, we have $\mathcal{M} \models \exists x_1, \ldots, x_n R_a(x_1, \ldots, x_n)$. $a \leq a_i$ for $i \leq k$, so we have $\mathcal{M} \models \exists x_1, \ldots, x_n R_{a_i}(x_1, \ldots, x_n)$ for every a_i with $i \leq k$ by the instance of the $A_+(a_i, a, a)$-axiom. That makes every finite subtype of T satisfiable, thus the whole type is finitely satisfiable in $\mathcal{M}$. $\mathcal{M}$ is ω-saturated, then T is realised in $\mathcal{M}$ by some $(x_1, \ldots, x_n) \in \mathcal{M}^n$ and, moreover, $\mathcal{M} \models 1(x_1, \ldots, x_n)$. As we have already said, $U_{x_1,\ldots,x_n}$ is an ultrafilter, but $U_{x_1,\ldots,x_n} \subseteq U$, thus $U = U_{x_1,\ldots,x_n}$, so $(x_1, \ldots, x_n) \in h(U)$.

That makes h one-to-one. □

Claim 5.

1. $h(c_i^{\mathcal{A}^+} U) = C_i(h(U))$
2. $h(d_{ij}^{\mathcal{A}^+}) = D_{ij} \subseteq \mathbf{Spec}(\mathcal{A})$

Proof.

1. Assume $(x_1, \ldots, x_n) \in h(c_i^{\mathcal{A}^+} S)$.

   Let us show that $\overline{x} \in C_i(h(S))$, that is, there exists $\overline{y} = (y_1, \ldots, y_n) \in h(S)$ such that $\overline{x} \equiv_i \overline{y}$.

   Then $\mathcal{M} \models 1(x_1, \ldots, x_n)$ and $U_{x_1,\ldots x_n} \in c_i^{\mathcal{A}^+} S$. But $\mathcal{A}^+$ is the complex algebra of the ultrafilter frame $\mathcal{F}_{\mathcal{A}}$. Then we have:

   $$c_i^{\mathcal{A}^+} S = \{U_1 \in \mathbf{Spec}(\mathcal{A}) \mid \exists U' \in S \; U_1 R_i U'\}$$

   Then there must be an ultrafilter $U' \in S$ such that $U_{x_1,\ldots x_n} R_i U'$, that is, $c_i a \in U_{x_1,\ldots x_n}$ whenever $a \in U'$. Hence $\mathcal{M} \models R_{c_i}(x_1, \ldots x_n)$. By the $A_{c_i}(a)$-axiom, we have

   $\mathcal{M} \models \exists z_1, \ldots, z_n(R_a(z_1, \ldots, z_n) \wedge x_i = z_j)$ for $i < n$ and $j < n$ such that $i \neq j$.

   Consider the following n-type with free variables $z_1, \ldots, z_n$ and parameters $x_1, \ldots, x_n \in \mathcal{M}$:

   $$T(z_1, \ldots, z_n) = \{R_a(z_1, \ldots, z_n) \wedge x_i = z_j \mid i < n, j < n, i \neq j, a \in U'\}.$$

   Let us show that $T(z_1, \ldots, z_n)$ is finitely satifsiable in $\mathcal{M}$. Consider a finite subset of T, say $T_0 = \{R_{b_k}(z_1, \ldots, z_n) \wedge x_i = y_j \mid i < n, j < n, i \neq j, b_k \in U', k < \omega\}$. We put $p = p_1 \cdot \cdots \cdot p_k$ and $p \in U'$ since U' is a filter. Then we have:

$\mathcal{M} \models \exists z_1, \ldots, z_n (R_b(z_1, \ldots, z_n) \wedge x_i = z_j)$ for $i < n$ and $j < n$ such that $i \neq j$

Thus, we have, as required:

$$\mathcal{M} \models \exists z_1, \ldots, z_n \bigwedge_{i=1}^{k} (R_{b_k}(z_1, \ldots, z_n) \wedge x_i = z_j) \text{ for } i < n \text{ and } j < n \text{ such that } i \neq j.$$

As above, using ω-saturation, we conclude that T is realised in $\mathcal{M}$ at an n-tuple $(y_1, \ldots, y_n) = \overline{y}$. Then we have:

$$\mathcal{M} \models 1(\overline{y}),\ \overline{x} \equiv_i \overline{y},\ U_{\overline{y}} \supseteq U'$$

Then $U_{\overline{y}} = U'$, then $\overline{y} \in h(S)$. Then $\overline{x} \in C_i(h(S))$.

Suppose for the converse, $\overline{x} = (x_1, \ldots, x_n) \in C_i(h(S))$. We need $\overline{x} \in h(c_i(S))$. Then there exists $\overline{y} = (y_1, \ldots, y_n)$ such that $\overline{x} \equiv_i \overline{y}$ and $\overline{y} \in h(S)$. Then there exists an ultrafilter $U_{y_1,\ldots,y_n} \in S$. Let us show that $\mathcal{M} \models 1(x_1, \ldots, x_n)$ and $U_{x_1,\ldots,x_n} \in c_i U_{y_1,\ldots,y_n}$. Let $a \in U_{y_1,\ldots,y_n}$. Then we have $\mathcal{M} \models R_a(y_1, \models, y_n)$. By the $A_{c_i}(a)$ axiom, we have $\mathcal{M} \models R_{c_i a}(x_1, \ldots, x_n)$. Then $\mathcal{M} \models 1(x_1, \ldots, x_n)$ and $c_i a \in U_{x_1,\ldots,x_n}$, thus, $\overline{x} \in h(c_i(S))$.

2. Let us show that h preserves cylindrifications.

   Let $(x_1, \ldots, x_n) \in \mathcal{M}^n$. Then $(x_1, \ldots, x_n) \in D_{ij}$ iff $\mathcal{M} \models 1(x_1, \ldots, x_n)$ and $x_i = x_j$ iff $U_{x_1,\ldots,x_n} \in d_{ij}^{\mathcal{A}^+} = \{U \in \mathbf{Spec}(\mathcal{A}) \mid d_{ij} \in U\}$ iff $\mathcal{M} \models d_{ij}^{\mathcal{M}}(x_1, \ldots, x_n)$.

$\square$

5.3 Game-theoretic approach

Definition 9. *Network*

Theorem 13. *Completely representable iff $\exists$ has a ws.*

Definition 10. *Ultrafilter network*

Theorem 14. $\mathbf{RCA}_n$ *is a pseudoelementary class for $3 \leq n < \omega$.*

Theorem 15. *$\exists$ has a ws for the canonical extension.*

5.4 Dimension ω

Question 1. *Can we characterise $\mathbf{RCA}_\omega$ as an enumerably axiomatisable pseudo-elementary class in three-sorted logic with sorts* **b** *(Boolean part),* **c** *(cylindric part) and* **r** *(representation part)?*

Definition 11. *Network*

Theorem 16. *Completely representable iff $\exists$ has a ws.*

Definition 12. *Ultrafilter network*

Theorem 17. *$\exists$ has a ws for the canonical extension.*

5.5 Counterexamples

References

[AGN98] Hajnal Andréka, Robert Goldblatt, and István Németi. Relativised quantification: Some canonical varieties of sequence-set algebras. *The Journal of Symbolic Logic*, 63(1):163–184, 1998.

[Bez06] Nick Bezhanishvili. *Lattices of intermediate and cylindric modal logics*. University of Amsterdam, 2006.

[BH13] Jannis Bulian and Ian Hodkinson. Bare canonicity of representable cylindric and polyadic algebras. *Annals of Pure and Applied Logic*, 164(9):884–906, 2013.

[CK90] Chen Chung Chang and H Jerome Keisler. *Model theory*. Elsevier, 1990.

[Fin75] Kit Fine. Some connections between elementary and modal logic. In *Studies in Logic and the Foundations of Mathematics*, volume 82, pages 15–31. Elsevier, 1975.

[Gol89] Robert Goldblatt. Varieties of complex algebras. *Annals of pure and applied logic*, 44(3):173–242, 1989.

[Gol95] Robert Goldblatt. Elementary generation and canonicity for varieties of boolean algebras with operators. *Algebra Universalis*, 34(4):551–607, 1995.

[HA14] Robin Hirsch and Tarek Sayed Ahmed. The neat embedding problem for algebras other than cylindric algebras and for infinite dimensions. *The Journal of Symbolic Logic*, 79(1):208–222, 2014.

[HH97] Robin Hirsch and Ian Hodkinson. Complete representations in algebraic logic. *Journal of Symbolic Logic*, pages 816–847, 1997.

[HH09] Robin Hirsch and Ian Hodkinson. Strongly representable atom structures of cylindric algebras. *The Journal of Symbolic Logic*, 74(3):811–828, 2009.

[HH13] Robin Hirsch and Ian Hodkinson. Completions and complete representations. *Cylindric-like Algebras and Algebraic Logic*, pages 61–89, 2013.

[HMT+81] Leon Henkin, J Donald Monk, Alfred Tarski, Hajnalka Andréka, and István Németi. *Cylindric set algebras and related structures*. Springer, 1981.

[HMT88] Leon Henkin, J. Donald Monk, and Alfred Tarski. Cylindric algebras. part ii. *Journal of Symbolic Logic*, 53(2):651–653, 1988.

[Hod93] Wilfrid Hodges. *Model theory.* Cambridge University Press, 1993.

[HV05] Ian Hodkinson and Yde Venema. Canonical varieties with no canonical axiomatisation. *Transactions of the American Mathematical Society*, 357(11):4579–4605, 2005.

[JT51] Bjarni Jónsson and Alfred Tarski. Boolean algebras with operators. part i. *American journal of mathematics*, 73(4):891–939, 1951.

[Mon00] J Donald Monk. An introduction to cylindric set algebras. *Logic Journal of the IGPL*, 8(4):451–496, 2000.

[Ven95] Yde Venema. Cylindric modal logic. *The Journal of Symbolic Logic*, 60(2):591–623, 1995.

[Ven13] Yde Venema. *Cylindric Modal Logic*, pages 249–269. Springer Berlin Heidelberg, Berlin, Heidelberg, 2013.