

# Cylindric notes

Daniel Rogozin

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## 1 Cylindric algebras: background

### 1.1 Atomic representations of Boolean algebras

Let  $B$  be a Boolean algebra, an element  $a$  is an *atom* if for every  $b \in B$   $b < a$  implies  $b = 0$ .  $\text{At}(B)$  is the set of all atoms of  $B$ .

**Definition 1.** Let  $B$  be a Boolean algebra and  $F$  a field of sets such that  $h : B \rightarrow F$  is a representation of  $B$ , then  $B$  is a *complete representation* of  $B$ , if for every  $A \subseteq B$  such that  $\Sigma A$  is defined the following holds:

$$h(\Sigma A) = \bigcup_{a \in A} h(a) \quad (1)$$

A representation  $h$  is called *atomic*, if  $x \in h(1)$ , then there exists  $b \in \text{At}(B)$  such that  $x \in h(b)$ .

**Theorem 1.** Let  $\mathcal{B}$  be a Boolean algebra, then  $\mathcal{B}$  is atomic iff  $\mathcal{B}$  is completely representable. See [HH97, Corollary 6].

### 1.2 Proper cylindric algebras

Let  $X \neq \emptyset$  along and  $X^\omega = \{f \mid f : \omega \rightarrow X\}$ . Let  $x \in X^\omega$ ,  $x_i$  stands for  $x(i)$  for  $i < \omega$ . A subset of  $X^\omega$  is an  $\omega$ -ry relation on  $U$ . For  $i, j < \omega$ , the  $i, j$ -diagonal  $D_{ij}$  is the set of all elements of  $X^\omega$  such that  $y_i = y_j$ .

If  $i < \omega$  and  $Y$  is an  $\omega$ -ry relation on  $X$ , then the  $i$ -th cylindrification  $C_i Y$  is the set of all elements of  $U$  that agree with some element of  $Y$  on each coordinate except, perhaps, the  $i$ -th one:

$$C_i Y = \{y \in X^\omega \mid \exists x \in Y \forall i < \omega (i \neq j \Rightarrow y_j = x_j)\}.$$

We define the following equivalence relation for  $i < \alpha$  and  $x, y \in X^\omega$ :

$$x \equiv_i y \Leftrightarrow \forall j \in \alpha (i \neq j \Rightarrow x(i) = y(j))$$

Then one may reformulate the definition of the  $i$ -th cylindrification the following way:

$$C_i Y = \{y \in X^\omega \mid \exists x \in X \ x \equiv_i y\}$$

According to this version of the definition, one can think of cylindrification operators as **S5** modalities.

**Definition 2.** A cylindric set algebra of dimension  $\omega$  is an algebra consisting of a set  $S$  of  $\omega$ -ry relation on some base set  $X$  with the constants and operations  $0 = \emptyset$ ,  $1 = X^\omega$ ,  $\cap$ ,  $-$ , the diagonal elements  $(D_{ij})_{i,j < \omega}$ , the cylindrifications  $(C_i)_{i < \omega}$ . A generalised cylindric set algebra of dimension  $\omega$  is a subdirect of cylindric algebras that have dimension  $\omega$ .  $\mathbf{Cs}_\omega$  denotes the class of all cylindric set algebras of dimension  $\omega$ .

**Definition 3.** A cylindric algebra of dimension  $\omega$  is an algebra  $C = (B, (c_i)_{i < \omega}, (d_{ij})_{i,j < \omega})$  such that  $B$  is a Boolean algebra, each  $d_{ij} \in \mathcal{B}$  and for all  $i, j, k < \omega$  and for all  $a, b \in B$ :

1.  $c_i 0 = 0$ ,
2.  $c_i(a + b) = c_i a + c_i b$
3.  $a \leq c_i a$ ,
4.  $c_i(a \cdot c_i b) = c_i a \cdot c_i b$
5.  $d_{ii} = 1$ ,
6.  $c_i c_j a = c_j c_i a$ ,
7. If  $k \neq i, j$ , then  $d_{ij} = c_k(d_{ij} \cdot d_{jk})$ ,
8. If  $i \neq j$ , then  $c_i(d_{ij} \cdot a) \cdot c_i(d_{ij} \cdot -a) = 0$ .

$\mathbf{CA}_\omega$  is the class of all cylindric algebras of dimension  $\omega$ .

One can define a representation of a cylindric algebra explicitly the following way:

**Definition 4.** Let  $\mathcal{A}$  be a cylindric algebra of dimension  $\omega$ . A representation of  $\mathcal{A}$  over the non-empty domain  $X$  is a one-to-one map  $f : \mathcal{A} \hookrightarrow 2^{X^\omega}$  such that:

1.  $f(1) = \bigcup_{i \in I} X_i^\omega$  for some disjoint family  $\{X_i\}_{i \in I}$  where each  $X_i \subseteq X$
2.  $h : \mathcal{A} \rightarrow 2^{f(1)}$  is a representation of a Boolean reduct
3. for all  $i, k < \omega$ ,  $x \in h(d_{ik})$  iff  $x_i = x_k$
4. for all  $i < \omega$  and  $a \in \mathcal{A}$ ,  $x \in h(c_i(a))$  iff there is  $y \in X$  such that  $x[i \mapsto y] \in h(a)$

Let  $C$  be a cylindric algebra,  $C$  is *representable* if there exists a representation of  $C$ .  $\mathbf{RCA}_\omega$  is the class of all representable cylindric algebras. Alternatively,  $\mathbf{RCA}_\omega$  can be defined as the closure of  $\mathbf{Cs}_\omega$  under isomorphism:

$$\mathbf{RCA}_\omega = \mathbf{ICS}_\omega.$$

It is well known that  $\mathbf{RCA}_\omega$  is a variety,  $\mathbf{RCA}_\alpha$  ( $\alpha \leq 2$ ) is finitely axiomatisable and  $\mathbf{RCA}_\alpha$  ( $2 < \alpha < \omega$ ) has no finite axiomatisation, see [HMT88].

Let  $C \in \mathbf{RCA}_\omega$ ,  $\mathcal{A}$  has a *complete representation* if its representation preserves all existing suprema as in Definition 1. In other words,  $\mathcal{A}$  is *completely representable*.

**Proposition 1.** *Let  $A \in \mathbf{CA}_\omega$ , then  $A$  is completely representation iff it has an atomic representation.*

*Proof.* Follows from Theorem 1. □

## 2 Atom structures and canonical extensions

First of all, we introduce the following operations on classes of algebras or frames. Let  $\mathcal{A}$  be a class of algebras and  $\mathcal{F}$  a class of frames, then:

- $\mathbf{IK}$  is the closure of  $\mathcal{K}$  under isomorphic copies,
- $\mathbf{UdF}$  is the closure of  $\mathcal{F}$  under disjoint unions,
- $\mathbf{UbF}$  is the closure of  $\mathcal{F}$  under bounded unions,
- $\mathcal{F}^+$  is the class of all complex algebras generated from elements of  $\mathcal{F}$ ,
- $\mathbf{PuK}$  is the closure of  $\mathcal{K}$  under ultraproducts,
- $\mathbf{PwK}$  is the closure of  $\mathcal{K}$  under ultrapowers,
- $\mathbf{SA}$  is the closure of  $\mathcal{A}$  under subalgebras,
- $\mathbf{SF}$  is the closure of  $\mathcal{F}$  under generated subframes,
- $\mathbf{HF}$  is the closure  $\mathcal{F}$  under  $p$ -morphic images.

The following definition of an  $\omega$ -frame is taken from [Ven13].

**Definition 5.** *A cylindric  $\omega$ -frame is a structure  $F = (W, (R_i)_{i < \omega}, (E_{ij})_{i, j < \omega})$  where  $(R_i)_{i < \omega}$  are binary relations and  $(E_{ij})_{i, j < \omega}$  are unary relations such that, for all  $i, j, k < \omega$ :*

1. *Every  $R_i$  is an equivalence relation on  $W$ ,*
2.  *$R_i \circ R_j = R_j \circ R_i$ ,*
3. *For all  $x \in W$ ,  $E_{ii} = W$ .*
4. *For all  $x, y, z \in W$ , if  $xR_iy$ ,  $xR_iz$ ,  $E_{ij} = W$  and  $E_{ij} = W$ , then  $y = z$ .*
5. *For all  $x \in W$ ,  $E_{ij} = W$  iff there exists  $y \in W$  such that  $xR_ky$ ,  $E_{ik} = W$ , and  $E_{kj} = W$ .*

$\mathbf{CFrames}_\omega$  is the class of all  $\omega$ -frames.

*Remark 1.*

Observe that the conditions of cylindric  $\omega$ -frames can be expressed as first-order formulas. Therefore,  $\mathbf{CFrames}_\omega$  is an elementary class.

We can associate a complete atomic cylindric algebra of dimension  $\omega$  with every cylindric  $\omega$ -frame  $F = (W, (R_i)_{i < \omega}, (E_{ij})_{i,j < \omega})$  by taking its *complex algebra*, which is the algebra  $F^+ = (2^W, \cup, -, (C_i)_{i < \omega}, \emptyset, W, (E_{ij})_{i,j < \omega})$  where each  $C_i$  is an operator  $C_i : 2^W \rightarrow 2^W$  defined as:

$$C_i A = \{w \in W \mid \exists a \in A \, w R_i a\} = R_i^{-1}(A).$$

If  $F \in \mathbf{CFrames}_\omega$  and  $x \in F$ , then  $F^x$  is a *generated subframe* generated by  $x$ . Generally,  $F_1$  is a generated subframe of  $F_2$ , if  $W_1 \subseteq W_2$ ,  $R_{i1} \subseteq R_{i2}$  and for all  $x \in W_1$   $y \in R_{i2}(x)$  implies  $y \in W_1$  for every  $i < \omega$ . That is, for all  $i < \omega$  and  $x \in F_1$ , we have  $R_{i2}(x) \subseteq F_1$  and, thus,  $R_{i1}(x) = R_{i2}(x)$ .

Given cylindric  $\omega$ -frames  $F_1 = (W_1, (R_{i1})_{i < \omega}, (E_{ij1})_{i,j < \omega})$  and  $F_2 = (W_2, (R_{i2})_{i < \omega}, (E_{ij2})_{i,j < \omega})$ . A *bounded morphism* is a function  $f : F_1 \rightarrow F_2$  such that, for each  $i, j < \omega$ :

1. (Monotonicity)  $x R_{i1} y$  implies  $f(x) R_{i2} f(y)$  for all  $x, y \in W_1$ ,
2. (The lifting property) If  $f(x) R_{i2} z$ , then there exists  $y \in R_{i1}(x)$  such that  $f(y) = z$ ,
3.  $x \in E_{ij1}$  iff  $f(x) \in E_{ij2}$ .

A bounded morphism is a *p-morphism* if it is onto. Notation:  $F_1 \twoheadrightarrow F_2$ . In this case, we say that  $F_1$  is a *p-morphic image* of  $F_2$ .

We have the following connection between  $\omega$ -frames and their generated subframes, which is standard for modal logic:

**Proposition 2.** *Let  $F \in \mathbf{CFrames}_\omega$ , then*

1.  $F = \prod_{x \in F} F^x$ ,
2.  $F^+ \cong \prod_{x \in F} (F^x)^+$ ,
3.  $(F^x)^+$  is subdirectly irreducible.

Let  $F$  be a cylindric  $\omega$ -frame and let  $(F_j)_{j \in J}$  be a family of cylindric  $\omega$ -frames such that each  $F_j$  is a generated subframe of  $F$ . Then  $G = (W, R_i, E_{ij})$  is the *bounded union* of  $(F_j)_{j \in J}$ , where  $W = \bigcup_{j \in J} W_j$  and  $R_i$  and  $E_{ij}$  are defined

by corresponding relations in  $F_j$ 's.

The following fact connects cylindric frames and cylindric algebras through complex algebras, see [Ven13, Proposition 2.1.5]:

**Proposition 3.** *A structure  $F$  is a cylindric  $\omega$ -frame iff  $F^+$  is a cylindric algebra of dimension  $\omega$ .*

Let  $(F_j)_{j \in J}$  be a disjoint family of cylindric  $\omega$ -frames, the *disjoint sum* of  $(F_i)_{i \in I}$  is  $F = \coprod_{i \in I} F_i$ , where each  $R_i = \bigcup_{j \in J} R_{ij}$  and  $E_{ik} = \bigcup_{j \in J} E_{ikj}$ . Disjoint sums and direct products are connected with one another through complex algebras as follows (see [Gol89, Lemma 3.4.1]):

$$\left(\prod_{j \in J} F_j\right)^+ \cong \prod_{j \in J} F_j^+ \quad (2)$$

We define a particular frame of cylindric  $\omega$ -frames. Let  $X$  be a non-empty set, the *full Cartesian structure over  $X$  of dimension  $\omega$*  is a cylindric  $\omega$ -frame  $\mathfrak{C}(X) = (X^\omega, (\equiv_i)_{i < \omega}, (\text{Id}_{ij})_{i, j < \omega})$ .  $\mathcal{Fct}_\omega$  is the class of all full Cartesian structures of dimension  $\omega$ . Observe that

$$\mathbf{Cs}_\omega = (\mathcal{Fct}_\omega)^+, \quad (3)$$

$$\mathbf{ICs}_\omega = \mathbf{S}(\mathcal{Fct}_\omega)^+. \quad (4)$$

The class of *generalised cylindric set algebras* of dimension  $\omega$ ,  $\mathbf{Gs}_\omega$ , consists of complex algebras of the closure of  $\mathcal{Fct}_\omega$  under disjoint unions:

$$\mathbf{Gs}_\omega = (\mathbf{Ud}(\mathcal{Fct}_\omega))^+ \quad (5)$$

or, by (2):

$$\mathbf{Gs}_\omega = \mathbf{P}(\mathcal{Fct}_\omega^+) \quad (6)$$

$\mathbf{RCA}_\omega$  is the closure of  $\mathbf{Gs}_\omega$  under isomorphism:

$$\mathbf{RCA}_\omega = \mathbf{IGs}_\omega \quad (7)$$

or, assuming (5) and (6):

$$\mathbf{RCA}_\omega = \mathbf{IGs}_\omega = \mathbf{S}((\mathbf{Ud}(\mathcal{Fct}_\omega))^+) = \mathbf{SP}(\mathcal{Fct}_\omega^+). \quad (8)$$

If  $C \in \mathbf{CA}_\omega$  is atomic, then we can associate a cylindric omega frame with it. Let  $C$  be an atomic cylindric algebra of dimension  $\omega$ , its *atom structure* is the structure  $\mathbf{At}(C) = (\text{At}(C), (R_i)_{i < \omega}, (E_{ij})_{i, j < \omega})$  such that each  $E_{ij} \subseteq \mathbf{At}(C)$  and for all  $i < \omega$  and for all  $a, b \in \text{At}(C)$ :

$$aR_ib \text{ iff } c_ib \leq a.$$

As a corollary from Proposition 3:

**Proposition 4.** *If  $C \in \mathbf{CA}_\omega$  is atomic, then  $\mathbf{At}(C)$  is a cylindric  $\omega$ -frame.*

### 3 Canonical extensions

Let  $B$  be a Boolean algebra, a proper subset  $F \subsetneq B$  is an *filter* if the following holds:

1.  $a \in B$  and  $a \leq b$  imply  $b \in B$ ,
2. If  $a, b \in B$ , then  $a \cdot b \in B$ .

A filter  $U$  is an *ultrafilter* if either  $a \in U$  or  $-a \in U$ , or, equivalently,  $U \subseteq U'$  implies  $U = U'$ .  $\mathbf{Spec}(B)$  is the *spectrum* of  $B$ , that is, the set of all ultrafilters of  $B$ .

Let  $C$  be a cylindric algebra of dimension  $\omega$ , the ultrafilter frame of  $C$  is a structure  $C_+ = (\mathbf{Spec}(C), (R_i)_{i < \omega}, (E_{ij})_{i, j < \omega})$  such that, for all  $U_1, U_2 \in \mathbf{Spec}(C)$  and for all  $i, j < \omega$ :

1.  $U_1 R_i U_2$  iff  $\{c_i a \mid a \in U_2\} \subseteq U_1$ ,
2.  $E_{ij} \subseteq \mathbf{Spec}(C)$ .

From Proposition 3 we have:

**Proposition 5.** *If  $C$  is a cylindric algebra, then  $C_+$  is a cylindric  $\omega$ -frame.*

The *canonical extension* of  $C$  is the algebra  $(C_+)^+$ , that is, the complex algebra of the ultrafilter frame.

**Theorem 2.** (See [JT51])

$C \in \mathbf{CA}_\omega$  embeds to  $(C_+)^+$  by mapping  $a \mapsto \{U \in \mathbf{Spec}(C) \mid a \in U\}$ .

### 4 Canonicity of $\mathbf{RCA}_\omega$

In this section, we reproduce the results related to characterisation  $\mathbf{RCA}_\omega$ . The following results are due to Goldblatt [Gol95]. This denotes that a cylindric algebra of dimension  $\omega$  is representable iff it is isomorphic to a subalgebra of the complex algebra of disjoint sum of some full  $\omega$ -dimensional Cartesian structure.

The following characterisation result is known from [Ven13, Theorem 2.2.3].

**Theorem 3.**  $\mathbf{RCA}_\omega = \mathbf{HSP}(\mathcal{Fct}_\omega^+)$

That is, the class of representable cylindric algebras of dimension  $\omega$  is a variety generated by complex algebras of full Cartesian structures of dimension  $\omega$ . If we consider the equational theory of  $\mathbf{RCA}_\omega$  as a polymodal logic, we could say that it is Kripke complete with respect to the class of all full Cartesian structures of dimension  $\omega$ .

To show that  $\mathbf{RCA}_\omega$  is canonical we have got to show the following inclusion:

$$(\mathbf{RCA}_{\omega+})^+ \subseteq \mathbf{RCA}_\omega.$$

**Definition 6.** The weak Cartesian space with base set  $X$  and dimension  $\omega$  determined by  $x \in X^\omega$  is the set:

$$X^{\omega(x)} = \{y \in X^\omega \mid \text{card}(\{k < \omega \mid x_k \neq y_k\}) < \aleph_0\}$$

$\mathfrak{S}_\omega(X^{\omega(x)})$  is a weak Cartesian structure of dimension  $\omega$ .  $\mathcal{Wct}_\omega$  is the class of all weak Cartesian structure of dimension  $\omega$  up to isomorphism.

Note that we have  $\mathcal{Wct}_\omega \subseteq \mathbf{CFrames}_\omega$ .

Define also the class  $\mathcal{Sct}_\omega$  of sub-Cartesian structures of dimension  $\omega$  consisting of  $\mathfrak{S}_\omega(V)$  for  $V \subseteq X^\omega$ , where  $X$  is a non-empty base set. Note that  $\mathfrak{S}_\omega(V)$  does not have to be a cylindric  $\omega$ -frame.

Let  $F$  be a generated subframe of a full Cartesian structure of dimension  $\omega$   $\mathfrak{C}(X)$ , then

$$F \cong \coprod_{x \in F} F^x \quad (9)$$

or, by (2):

$$F^+ \cong \prod_{x \in F} (F^x)^+ \quad (10)$$

The latter implies the inclusion:

$$(\mathcal{Sct}_\omega)^+ \subseteq \mathbf{P}(\mathcal{Wct}_\omega^+). \quad (11)$$

Note that (follows from [HMT<sup>+</sup>81, p. 118])

**Fact 1.**  $\mathcal{Wct}_\omega^+ \subseteq \mathbf{RCA}_\omega$

Complex algebras based on  $\mathfrak{S}_\omega(X^x)$  form the class  $\mathbf{Ws}_\omega$  of weak cylindric set algebras of dimension  $\omega$ . The class  $\mathbf{Gws}_\omega$  of generalised weak cylindric set algebras of dimension  $\omega$  consists of complex algebras based on the closure of  $\mathbf{Ws}_\omega$  under disjoint unions:

$$\mathbf{IWs}_\omega = \mathbf{S}(\mathcal{Wct}_\omega^+) \quad (12)$$

$$\mathbf{IGws}_\omega = \mathbf{S}((\mathbf{Ud}\mathcal{Wct}_\omega)^+) = \mathbf{SP}(\mathcal{Wct}_\omega^+) \quad (13)$$

The following is by Goldblatt, see [Gol95, Lemma 3.4]:

**Lemma 1.**  $\mathbf{RCA}_\omega = \mathbf{S}((\mathcal{S}\mathbf{Ud}\mathcal{Fct}_\omega)^+) = \mathbf{S}((\mathcal{S}\mathbf{Ud}(\mathcal{Wct}_\omega))^+) = \mathbf{IGws}_\omega$

#### 4.1 Ultraproducts of full Cartesian structures

Let  $(F_j)_{j \in J}$  be an indexed family of full Cartesian structures of dimension  $\omega$ , where each  $F_j$  is of the form

$$F_j = (W_j, (R_{ij})_{i < \omega}, (E_{ikj})_{i, k < \omega})$$

and let  $U$  be an ultrafilter on  $U$ . Define the following equivalence relation on  $\prod_{j \in J} W_j$  for  $f, g \in \prod_{j \in J} W_j$ :

$$f \sim_U g \text{ iff } \{j \in J \mid f(j) = g(j)\} \in U$$

The *ultraproduct* of  $(F_j)_{j \in J}$  is an algebra  $\prod_j F_j/U = (W, (R_i)_{i < \omega}, (E_{ik})_{i, k < \omega})$ , where  $W = \prod_{j \in J} W_j$  and

1.  $f_U R_i g_U$  iff  $\{j \in J \mid R_{ij}(f_U(j), g_U(j))\} \in U$ ,
2.  $f_U \in E_{ik}$  iff  $\{j \in J \mid f_U(j) \in E_{ikj}\} \in U$ .

where  $f_U$  and  $g_U$  are equivalence classes of  $f$  and  $g$  modulo  $U$ .

See [Gol95, Lemma 3.5], a similar construction for modal logics could be found in [Fin75]:

**Lemma 2.** *Let  $(F_j)_{j \in J}$  be an indexed family of full Cartesian structures of dimension  $\omega$  and  $U$  an ultrafilter on  $J$ . There exists a  $p$ -morphism:*

$$\varphi : \prod_j F_j/U \rightarrow \mathfrak{S}_\omega((\prod_j W_j/U))$$

that restricts to an isomorphism  $F^x \cong I^{\varphi(x)}$  of generated subframes generated by  $x \in F$ .

*Proof.* Consider the equation:

$$f_i(j) = f(j)_i \tag{14}$$

If  $j \in \prod_{j \in J} W_j^\omega$ , then the equation defines a function  $f_i \in \prod_{j \in J} W_j$  for each  $i < \omega$ .

Then a sequence  $(f_i)_{i < \omega}$  defines a function by Equation 14. Clearly  $f_U = g_U$  implies  $f_{iU} = g_{iU}$  for  $i < \omega$ . So define  $\varphi$  as:

$$\varphi(f_U) = (f_{iU})_{i < \omega} \tag{15}$$

It is readily checked that:

1.  $f_U \in E_{kl}$  iff  $f_{kU} = f_{lU}$  iff  $\varphi(f_U) \in E_{kl}^\omega$ ,
2.  $(f_U)R_k(g_U)$  implies  $f_{lU} = g_{lU}$  whenever  $k \neq l < \omega$ , so  $(f_U)R_k^\omega(g_U)$ , so  $\varphi$  is monotone.

Let us show that  $\varphi$  has the lifting property. Assume that  $\varphi(f_U)R_k^\omega z$  where  $z = (g_k)_{k < \omega}$ . We have got to show that there exists  $h_U$  such that  $\varphi(h_U) = z$  and  $(f_U)R_k(h_U)$ . Put  $h_k = g_k$  and  $h_l = f_l$  for  $k \neq l < \omega$ , so for  $k \neq l$  one has  $(f_l)_U = (g_l)_U$  since  $\varphi(f_U)R_k^\omega z$ , so  $(g_l)_U = (h_l)_U$ , so  $z = (h_U)$  are the same sequence. Moreover,  $\{j \mid h(j)R_{kj}f(j)\} = J \in U$ , since  $h(j)_l = f(j)_l$  for  $l \neq k$ , so  $(f_U)R_k(h_U)$  in the ultraproduct.

Let us show that  $\varphi$  acts isomorphically on every generated subframe  $F^x$  of the ultraproduct. Take  $f_U, g_U \in F^x$ , then there are  $i_0, \dots, i_n < \omega$  such that



$$f_U(R_{i_0} \circ \dots \circ R_{i_n})g_U.$$

By Łoś's theorem we have

$$J_{fg} = \{j \in J \mid f(j)(R_{i_0j} \circ \dots \circ R_{i_nj})g(j)\}$$

So for  $J_{fg}$ , the  $\omega$ -sequences  $f(j)$  and  $g(j)$  agree except possibly on  $i_0, \dots, i_n$ . If  $\varphi(f_U) = \varphi(g_U)$ , then for each  $k < \omega$ ,  $f_{kU} = g_{kU}$  and then:

$$J_k = \{j \in J \mid f_k(j) = g_k(j)\} \in U$$

But  $f, g$  are identical on the set

$$J_k \cap J_{i_0} \cap \dots \cap J_{i_n} \in F$$

and thus  $f_U = g_U$ , so  $\varphi$  is injective on  $F^x$ .  $\square$

**Theorem 4.**  $\mathbf{PuFct}_\omega \subseteq \mathbf{UbFct}_\omega$ .

*Proof.* Let  $F = \prod_j F_j/U$  be an ultraproduct of full Cartesian structures of dimension  $\omega$ . To show  $F \in \mathbf{UbFct}_\omega$  one needs to show that for each point  $x \in F$  there exists a generated subframe that contains  $x$  and is isomorphic to  $I = \mathfrak{S}_\omega((\prod_j F_j/U))$ .

Let  $Z$  be a choice set that contains exactly one element from each weak Cartesian substructure of  $I$ . But  $I$  is the disjoint union of all its weak substructures, so we have:

$$I = \coprod_{z \in Z} I^z$$

Fix  $x \in F$ , for each  $z \in Z$  choose  $\psi(z)$  to be any member of  $F$  such that  $\varphi(\psi(z)) = z$  and  $I^z$  is the weak substructure containing  $\varphi(x)$ , where  $\varphi$  is a  $p$ -morphism from Lemma 2. By the previous lemma, we have

$$F^{\psi(z)} = I^z.$$

If  $z$  and  $z'$  are different elements of  $Z$ , so  $I^z$  and  $I^{z'}$  are disjoint, so  $F^{\psi(z)}$  and  $F^{\psi(z')}$  are also disjoint.

$F(x)$  is defined to be the union of the collection of  $\{F^{\psi(z)} \mid z \in Z\}$  and forms a generated subframe of  $F$  which is isomorphic of  $I^{z'}$ s, so  $F^x \cong I$ , but  $x = \psi(z)$  for some  $z$ , so  $x \in F(x)$ .  $\square$

**Corollary 1.**  $\mathbf{UbFct}_\omega$  is closed under ultraproducts.

**Theorem 5.**

$$1. \mathbf{PuWct}_\omega \subseteq \mathbf{UbWct}_\omega,$$

$$2. \mathbf{PuSct}_\omega \subseteq \mathbf{Sct}_\omega.$$

*Proof.*

1. Let  $F^* = \prod_j F_j^*/U$  be an ultraproduct of weak Cartesian structures of dimension  $\omega$ . Each  $F_j^*$  is a generated subframe of some full Cartesian structure  $F_j$ , so  $F^*$  is isomorphic to a generated subframe of the ultraproduct  $F = \prod_j F_j/U$  and we identify  $F^*$  with this generated subframe. But the ultraproduct  $F$  is a cylindric  $\omega$ -frame since each  $F_j \in \mathbf{CFrames}_\omega$ , but  $\mathbf{CFrames}_\omega$  is elementary and then closed under ultraproducts. But  $F^*$  can be defined as

$$F^* = \coprod_{x \in F^*} F^x$$

But each  $F^x$  is isomorphic to each  $I^{\varphi(x)}$ .

2. Let  $F^* = \prod_j F_j^*/U$  be an ultraproduct of sub-Cartesian structures of dimension  $\omega$ , then each  $F_j^*$  is a substructure of some full Cartesian structure  $F_j$  of dimension  $\omega$ . So  $F^*$  is isomorphic to a substructure of the ultraproduct  $F = \prod_j F_j/U$ . As in the previous item,  $F \in \mathbf{CFrames}_\omega$  of its all point-generated substructures  $F^x$ , each of which is isomorphic to some sub-Cartesian structure of dimension  $\omega$ . Then

$$F = \coprod_{x \in F} F^x \in \mathbf{UdSct}_\omega = \mathbf{Sct}_\omega$$

That makes  $F^* \in \mathbf{Sct}_\omega$ .

□

**Theorem 6.**  $\mathcal{Wct}_\omega^+ \subseteq \mathbf{S}((\mathbf{Pw}(\mathcal{Wct}_\omega))^+)$ .

*Proof.* Let  $J$  be the set of finite subsets  $\omega$  and let  $U$  be an ultrafilter on  $J$  that contains, for each  $i \in J$ , the set

$$J_i = \{j \in J \mid i \subseteq j\}.$$

In particular  $J_k = \{j \in J \mid k \in j\}$  for all  $k < \omega$ . Now take  $\mathfrak{S}_\omega(X^{(x)}) \in \mathcal{Wct}_\omega$ . For each  $y \in X^{\omega(x)}$ , let  $f_y \in (X^{\omega(x)})^J$  be the constant function  $f_y(j) = y$ . Then  $\psi : y \mapsto f_y/U$  is the isomorphic embedding

$$\psi : \mathfrak{S}_\omega(X^{(x)}) \hookrightarrow \mathfrak{S}_\omega(X^{(x)})^J/U$$

of  $\mathfrak{S}_\omega(X^{(x)})$  to its ultrapower  $\mathfrak{S}_\omega(X^{(x)})^J/U$  with respect to  $U$ . We have got to show:

**Claim 1.** *There exists a bounded morphism  $\varphi : \mathfrak{S}_\omega(X)^J/U \rightarrow \mathfrak{S}_\omega(X^{(x)})/U$  such that its image contains the image of  $\psi$ :*

$$\psi : \mathfrak{S}_\omega \hookrightarrow \text{Im } \psi \subseteq \text{Im } \phi \subseteq \mathfrak{S}_\omega(X^{(x)})^J/U$$

By duality  $\phi$  induces a homomorphism:

$$\phi^+ : (\mathfrak{S}_\omega(X^{(x)})/U)^+ \rightarrow (\mathfrak{S}_\omega(X)^J/U)^+$$

$\phi^+$  composes with the homomorphism:

$$(\mathfrak{S}_\omega(X^{(x)}))^+ \rightarrow (\mathfrak{S}_\omega(X^{(x)})^J/U)^+$$

that gives a homomorphism:

$$\theta : (\mathfrak{S}_\omega(X^{(x)}))^+ \rightarrow (\mathfrak{S}_\omega(X)^J/U)^+$$

Let us describe the action of  $\theta$ , take  $f \in (X^\omega)^J$  and choose any  $f^+ \in f \in (X^{\omega(x)})^J$  such that  $\varphi(f/U) = f^+/U$ , so for any  $Y \subseteq X^{\omega(x)}$ :

$$\theta(Y) = \{f/U \in (X^\omega)^J/U \mid \{j \mid f^+(j) \in Y\} \in U\}.$$

So for  $y \in Y$ , then  $\psi(y)$  (that is  $f_y/U$ ) is equal to  $\varphi(f/U)$  for some  $f$ , so then  $f^+/U = f_y/U$  and then  $\{j \mid f^+(j) = y \in Y\} \in U$  showing that  $f/U \in \theta(Y)$ . As far as  $\mathfrak{S}_\omega(X)^J/U$  is an ultrapower of a full Cartesian structure of dimension  $\omega$ , so  $(\mathfrak{S}_\omega(X^{(x)}))^+ \in \mathbf{S}((\mathbf{Pw}(\mathcal{Fct}_\omega))^+)$ , so the theorem is proved.

Now let us prove Claim 1:

*Proof.* Take  $f \in (X^\omega)^J$ , define  $f^+ \in (X^{\omega(x)})^J$  as:

$$f^+(j)_k = \begin{cases} f(j)_k & \text{if } k \in j \\ x_k & \text{otherwise} \end{cases} \quad (16)$$

Each  $f^+(j)$  differs from  $x$  at most on the finite set  $j$ . Clearly that  $f(j) = g(j)$  implies  $f^+(j) = g^+(j)$ , so  $f_U = g_U$  in  $(X^\omega)^J/U$  implies  $f_U^+ = g_U^+$  in  $(X^{\omega(x)})^J/U$ . So the mapping  $\varphi : f_U \mapsto f_U^+$  is well-defined.

Let us show that  $\text{Im } \psi \subseteq \text{Im } \varphi$ . Take  $f_{y_U} \in \text{Im } \psi$  with  $y \in X^{\omega(x)}$ . We also have  $f_{y_U} \in (X^\omega)^J$ , so that is enough to show that  $f_{y_U}^+ = f_{y_U}$  in  $\text{Im } \psi$ . Put  $i = \{k < \omega \mid x_k \neq y_k\} \in J$ , so for  $j \in J_j$ :

$$f_{y_U}^+(j)_k = \begin{cases} f_y(j)_k & \text{if } k \in j \\ x_k = f_y(j)_k & \text{otherwise} \end{cases} \quad (17)$$

since  $f_y(j)$  agrees with  $x$  outside  $i$ . Thus  $f_y^+(j) = f_y(j)$  for each  $j \in J_j \in U$ , so  $f_{y_U}^+ = f_{y_U} = \psi(y)$ .  $\square$

We skip the proof  $\varphi$  is a bounded morphism.  $\square$

**Theorem 7.**  $\mathcal{Wct}_\omega^+ \subseteq \mathbf{RCA}_\omega$ .

*Proof.*

$$\begin{aligned}
\mathcal{W}\mathbf{ct}_\omega^+ &\subseteq \\
&\text{By Theorem 6} \\
&\subseteq \mathbf{SCmPw}(\mathcal{F}\mathbf{ct}_\omega) \\
&\text{By Theorem 4} \\
&\subseteq \mathbf{SCmUb}(\mathcal{F}\mathbf{ct}_\omega) \\
&\text{Since } \mathbf{Ub} \leq \mathbf{H}\mathbf{Ud} \quad \square \\
&\subseteq \mathbf{SCmH}\mathbf{Ud}\mathcal{F}\mathbf{ct}_\omega \\
&\text{Since } \mathbf{H}\mathbf{Ud} = \mathbf{Ud} \\
&= \mathbf{SCmUd}\mathcal{F}\mathbf{ct}_\omega \\
&= \mathbf{RCA}_\omega
\end{aligned}$$

**Theorem 8.**  $\mathbf{IWs}_\omega \subseteq \mathbf{ICs}_\omega$ .

*Proof.*  $\square$

## 4.2 The proof of canonicity

Here we use the following fact related to canonical varieties generated by some class of complex algebras. Let  $\mathbf{K}$  be an elementary class of relational structures, then:

If  $\mathcal{K}$  is closed under  $p$ -morphic images, generated subframes, and disjoint unions, then  $\mathbf{S}(\mathcal{K}^+)$  is a canonical variety.

One may think of this fact as a more abstract version of Fine's theorem which claims that every elementary modal logic is canonical [Fin75]. This version denotes the same fact, but it is formulated in terms of varieties BAOs generated by complex algebras of some atom structures. We provide a more precise formulation of the fact above.

**Proposition 6.** *Let  $\mathcal{K}$  be a class of frames, then  $\mathbf{PuK} \subseteq \mathbf{HSUdK}$  implies that  $\mathbf{S}((\mathbf{SUdK})^+)$  is a canonical variety.*

This is a special case of [Gol95, Theorem 4.4] for dimension  $\omega$ .

**Theorem 9.**  $\mathbf{RCA}_\omega$  is a canonical variety.

*Proof.* We have  $\mathbf{RCA}_\omega = \mathbf{S}((\mathbf{SUd}\mathcal{F}\mathbf{ct}_\omega)^+)$ . That's enough to show that  $\mathbf{Pu}\mathcal{F}\mathbf{ct}_\omega \subseteq \mathbf{HSUd}\mathcal{F}\mathbf{ct}_\omega$ . For that, we need the following claim:  $\square$

## 5 Representability via games

### 5.1 Monk's theorem for $\mathbf{RCA}_n$ via saturation

In this section we consider classes  $\mathbf{RCA}_n$  for  $n < \omega$ .

We provide the complete proof of the following theorem [HH13, Theorem 3.4.3].

**Theorem 10.** *Let  $\mathcal{A} \in \mathbf{CA}_n$ , then  $\mathcal{A}$  is representable iff  $(\mathcal{A}_+)^+$  is completely representable.*

For that we need such model-theoretic notions as saturation and types, see [Hod93, Section 6.3].

**Definition 7.** *Let  $\mathcal{M}$  be a first-order structure of a signature  $L$  and  $S \subseteq \mathcal{M}$ . Let  $L(S)$  be an extension of  $L$  with copies of elements from  $S$  as additional constants. We assume that  $\text{Cnst}(L)$  and  $S$  are disjoint.*

1. *Let  $n < \omega$ , an  $n$ -type over  $S$  is a set  $\mathcal{T}$  of  $L(S)$  formulas  $A(\bar{x})$ , where  $\bar{x}$  is a fixed  $n$ -tuple of elements from  $S$ . Notation:  $\mathcal{T}(\bar{x})$ . A type is an  $n$ -type for some  $n < \omega$ .*
2. *An  $n$ -type  $\mathcal{T}(\bar{x})$  is realised in  $\mathcal{M}$ , if there exists  $\bar{m} \in \mathcal{M}^n$  such that  $\mathcal{M} \models A(\bar{m})$  for every  $A \in \mathcal{T}(\bar{x})$ .  $\mathcal{M}$  omits  $\mathcal{T}(\bar{x})$ , if  $\mathcal{T}(\bar{x})$  is not realised in  $\mathcal{M}$ .*
3.  *$\mathcal{T}(\bar{x})$  is finitely satisfied in  $\mathcal{M}$ , if every finite subtype  $\mathcal{T}_0(\bar{x}) \subseteq \mathcal{T}(\bar{x})$  is realised in  $\mathcal{M}$ . We can reformulate that as  $\mathcal{M} \models \exists \bar{a} \bigwedge_{A \in \mathcal{T}_0} A(\bar{a})$ .*
4. *Let  $T$  be a theory, then a type  $\mathcal{T}$  over the empty set of constants is  $T$ -consistent, if there exists a model  $\mathcal{M} \models T$  such that  $\mathcal{T}$  is finitely satisfied in  $\mathcal{M}$ .*
5. *Let  $\kappa$  be a cardinal, then  $\mathcal{M}$  is  $\kappa$ -saturated, if for every  $S \subseteq \mathcal{M}$  with  $|S| < \kappa$  every finitely satisfied 1-type  $\mathcal{T}$  is realised in  $\mathcal{M}$ .*

By default, a saturated model is an  $\omega$ -saturated model for us.

A couple of useful facts from [CK90] and [Hod93]:

**Fact 2.** *Let  $\mathcal{M}$  be an FO-structure and  $\kappa$  a cardinal, then:*

1.  *$\mathcal{M}$  is  $\kappa$ -saturated, iff every finitely satisfiable  $\alpha$ -type (an arbitrary  $\alpha \leq \kappa$ ) with fewer than  $\kappa$  parameters is realised in  $\mathcal{M}$ .*
2. *If  $\mathcal{M}$  is  $\kappa$ -saturated, then  $\mathcal{M}$  is  $\lambda$ -saturated for every  $\lambda < \kappa$ .*
3. *Every consistent theory has a  $\kappa$ -saturated model and every model has an elementary  $\kappa$ -saturated extension.*
4. *Let  $(\mathcal{M}_i)_{i < \omega}$  a family of structures of the (at most) countable signature and  $D$  a non-principal ultrafilter over  $\omega$ , then  $\Pi_D \mathcal{M}_i$  is  $\omega_1$ -saturated.*

## 5.2 Proof of Theorem 10

Let  $\mathcal{A} \in \mathbf{CA}_n$ , then if  $\mathcal{A}$  is completely representable, then  $h$ , a complete representation of  $\mathcal{A}$ , is atomic. That is,  $(a_1, \dots, a_n) \in h(1)$ , then  $(a_1, \dots, a_n) \in h(y)$  for some  $y \in \text{At}(\mathcal{A})$ .

**Definition 8.** Let  $\mathcal{A}$  be a cylindric algebra of dimension  $n < \omega$ .  $L(\mathcal{A})$  is the first-order language that consists of equality plus  $n$ -ary predicate letters  $(R_a^n)_{a \in \mathcal{A}}$ . The  $L(\mathcal{A})$ -theory  $T_{\mathcal{A}}$  consists of the following sentences:

1.  $A_+(a, b, c) := \forall x_1, \dots, x_n (R_a(x_1, \dots, x_n) \leftrightarrow R_b(x_1, \dots, x_n) \vee R_c(x_1, \dots, x_n))$ .  
Informally, that means  $\mathcal{A} \models a = b + c$ .
2.  $A_-(a, b) := \forall x_1, \dots, x_n (R_a(x_1, \dots, x_n) \leftrightarrow \neg R_b(x_1, \dots, x_n))$ . That is,  $\mathcal{A} \models a = -b$ .
3.  $A_{\neq 0}(a) := \exists x_1, \dots, x_n R_a(x_1, \dots, x_n)$ . That is,  $\mathcal{A} \models a \neq 0$ .
4.  $A_{c_i}(a) := \forall x_1, \dots, x_n (R_{c_i a}(x_1, \dots, x_n) \leftrightarrow \exists y_1, \dots, y_n (R_a(y_1, \dots, y_n) \wedge x_i = y_j))$ , for  $i < n$  and  $j < n$  such that  $i \neq j$ . Informally,  $\mathcal{A} \models c_i a = 1$ .
5.  $A_{d_{ij}} := \forall x_1, \dots, x_n (R_{d_{ij}}(x_1, \dots, x_n) \leftrightarrow x_i = x_j)$ , for  $i, j < n$ .

In fact, we need to show the following implication:

If  $\mathcal{A}$  is representable, then  $\mathcal{A}^+$  is completely representable.

Assume that  $\mathcal{A}$  is representable, then the theory  $T(\mathcal{A})$  is consistent, then it has an  $\omega$ -saturated model  $\mathcal{M}$  by Fact 3. We have the following claim:

**Claim 2.** The set  $U_{x_1, \dots, x_n} = \{a \in \mathcal{A} \mid \mathcal{M} \models R_a(x_1, \dots, x_n)\}$  is an ultrafilter of  $\mathcal{A}$ , for  $x_1, \dots, x_n \in \mathcal{M}$  with  $R_1(x_1, \dots, x_n)$ .

Those  $U_{x_1, \dots, x_n}$ 's allow us to represent atoms of  $\mathcal{A}^+$ .

We define a representation of  $\mathcal{A}^+$  as a map  $h : \mathcal{A}^+ \rightarrow 2^{\mathcal{M}^n}$  such that:

$$h : S \mapsto \{(x_1, \dots, x_n) \in 1^{\mathcal{M}} \mid U_{x_1, \dots, x_n} \in S\}, \text{ for } S \in \mathbf{Spec}(\mathcal{A}).$$

**Claim 3.** Let  $A_1, A_2 \in \mathbf{Spec}(\mathcal{A})$

1.  $h(0^{\mathcal{A}^+}) = \emptyset$
2.  $h(-A_1) = -h(A_1)$
3.  $h(1^{\mathcal{A}^+}) = 1^{\mathcal{M}}$
4. If  $S \subseteq \mathbf{Spec}(\mathcal{A})$ , then  $h(\bigcup S) = \bigcup_{U \in S} h(U)$

In particular,  $h$  is a Boolean homomorphism.

*Proof.*

1.  $h(0^{\mathcal{A}^+}) = h(\emptyset) = \emptyset$ .
2. From the definition of  $h$ .
3.  $h(-A_1) = -h(A_1)$

Let  $x_1, \dots, x_n \in 1^{\mathcal{M}}$ , then we have:

$$(x_1, \dots, x_n) \in h(-A_1) \text{ iff } U_{x_1, \dots, x_n} \in -A_1 \text{ iff } U_{x_1, \dots, x_n} \notin A_1 \text{ iff } (x_1, \dots, x_n) \notin h(A_1)$$

4. Let  $S = \bigcup_{i \in I} S_i$ , where  $S_i \in \mathbf{Spec}(\mathcal{A})$  for every  $i \in I$ . Let  $(x_1, \dots, x_n) \in 1^{\mathcal{M}}$ , then we have:

$$(x_1, \dots, x_n) \in h\left(\bigcup_{i \in I} S_i\right) \text{ iff } f_{x_1, \dots, x_n} \in \bigcup_{i \in I} S_i \text{ iff } \exists i \in I \text{ } f_{x_1, \dots, x_n} \in S_i \text{ iff } \exists i \in I (x_1, \dots, x_n) \in h(S_i) \text{ iff } (x_1, \dots, x_n) \in \bigcup_{i \in I} S_i$$

□

**Claim 4.**  $h$  is injective.

*Proof.* Let  $U \in \mathbf{Spec}(\mathcal{A})$ . The first is to show that  $h(U)$  is non-empty. The following  $n$ -type:

$$T(x_1, \dots, x_n) = \{R_a(x_1, \dots, x_n) \mid a \in U\}$$

is finitely satisfied in  $\mathcal{M}$ .

Consider  $T_0 = \{R_{a_1}(x_1, \dots, x_n), \dots, R_{a_k}(x_1, \dots, x_n)\} \subseteq T$ . Then  $a_1, \dots, a_k \in U$  and  $a = a_1 \cdots a_k \in U$ . By the instance of the  $A_{\neq 0}(a)$ -axiom, we have  $\mathcal{M} \models \exists x_1, \dots, x_n R_a(x_1, \dots, x_n)$ .  $a \leq a_i$  for  $i \leq k$ , so we have  $\mathcal{M} \models \exists x_1, \dots, x_n R_{a_i}(x_1, \dots, x_n)$  for every  $a_i$  with  $i \leq k$  by the instance of the  $A_+(a_i, a, a)$ -axiom. That makes every finite subtype of  $T$  satisfiable, thus the whole type is finitely satisfiable in  $\mathcal{M}$ .  $\mathcal{M}$  is  $\omega$ -saturated, then  $T$  is realised in  $\mathcal{M}$  by some  $(x_1, \dots, x_n) \in \mathcal{M}^n$  and, moreover,  $\mathcal{M} \models 1(x_1, \dots, x_n)$ . As we have already said,  $U_{x_1, \dots, x_n}$  is an ultrafilter, but  $U_{x_1, \dots, x_n} \subseteq U$ , thus  $U = U_{x_1, \dots, x_n}$ , so  $(x_1, \dots, x_n) \in h(U)$ .

That makes  $h$  one-to-one. □

**Claim 5.**

1.  $h(c_i^{\mathcal{A}^+} U) = C_i(h(U))$
2.  $h(d_{ij}^{\mathcal{A}^+}) = D_{ij} \subseteq \mathbf{Spec}(\mathcal{A})$

*Proof.*

1. Assume  $(x_1, \dots, x_n) \in h(c_i^{\mathcal{A}^+} S)$ .

Let us show that  $\bar{x} \in C_i(h(S))$ , that is, there exists  $\bar{y} = (y_1, \dots, y_n) \in h(S)$  such that  $\bar{x} \equiv_i \bar{y}$ .

Then  $\mathcal{M} \models 1(x_1, \dots, x_n)$  and  $U_{x_1, \dots, x_n} \in c_i^{\mathcal{A}^+} S$ . But  $\mathcal{A}^+$  is the complex algebra of the ultrafilter frame  $\mathcal{F}_{\mathcal{A}}$ . Then we have:

$$c_i^{\mathcal{A}^+} S = \{U_1 \in \mathbf{Spec}(\mathcal{A}) \mid \exists U' \in S \text{ } U_1 R_i U'\}$$

Then there must be an ultrafilter  $U' \in S$  such that  $U_{x_1, \dots, x_n} R_i U'$ , that is,  $c_i a \in U_{x_1, \dots, x_n}$  whenever  $a \in U'$ . Hence  $\mathcal{M} \models R_{c_i}(x_1, \dots, x_n)$ . By the  $A_{c_i}(a)$ -axiom, we have

$$\mathcal{M} \models \exists z_1, \dots, z_n (R_a(z_1, \dots, z_n) \wedge x_i = z_j) \text{ for } i < n \text{ and } j < n \text{ such that } i \neq j.$$

Consider the following  $n$ -type with free variables  $z_1, \dots, z_n$  and parameters  $x_1, \dots, x_n \in \mathcal{M}$ :

$$T(z_1, \dots, z_n) = \{R_a(z_1, \dots, z_n) \wedge x_i = z_j \mid i < n, j < n, i \neq j, a \in U'\}.$$

Let us show that  $T(z_1, \dots, z_n)$  is finitely satisfiable in  $\mathcal{M}$ . Consider a finite subset of  $T$ , say  $T_0 = \{R_{b_k}(z_1, \dots, z_n) \wedge x_i = y_j \mid i < n, j < n, i \neq j, b_k \in U', k < \omega\}$ . We put  $p = p_1 \cdot \dots \cdot p_k$  and  $p \in U'$  since  $U'$  is a filter. Then we have:

$$\mathcal{M} \models \exists z_1, \dots, z_n (R_b(z_1, \dots, z_n) \wedge x_i = z_j) \text{ for } i < n \text{ and } j < n \text{ such that } i \neq j$$

Thus, we have, as required:

$$\mathcal{M} \models \exists z_1, \dots, z_n \bigwedge_{i=1}^k (R_{b_k}(z_1, \dots, z_n) \wedge x_i = z_j) \text{ for } i < n \text{ and } j < n \text{ such that } i \neq j.$$

As above, using  $\omega$ -saturation, we conclude that  $T$  is realised in  $\mathcal{M}$  at an  $n$ -tuple  $(y_1, \dots, y_n) = \bar{y}$ . Then we have:

$$\mathcal{M} \models 1(\bar{y}), \bar{x} \equiv_i \bar{y}, U_{\bar{y}} \supseteq U'$$

Then  $U_{\bar{y}} = U'$ , then  $\bar{y} \in h(S)$ . Then  $\bar{x} \in C_i(h(S))$ .

Suppose for the converse,  $\bar{x} = (x_1, \dots, x_n) \in C_i(h(S))$ . We need  $\bar{x} \in h(c_i(S))$ . Then there exists  $\bar{y} = (y_1, \dots, y_n)$  such that  $\bar{x} \equiv_i \bar{y}$  and  $\bar{y} \in h(S)$ . Then there exists an ultrafilter  $U_{y_1, \dots, y_n} \in S$ . Let us show that  $\mathcal{M} \models 1(x_1, \dots, x_n)$  and  $U_{x_1, \dots, x_n} \in c_i U_{y_1, \dots, y_n}$ . Let  $a \in U_{y_1, \dots, y_n}$ . Then we have  $\mathcal{M} \models R_a(y_1, \dots, y_n)$ . By the  $A_{c_i}(a)$  axiom, we have  $\mathcal{M} \models R_{c_i a}(x_1, \dots, x_n)$ . Then  $\mathcal{M} \models 1(x_1, \dots, x_n)$  and  $c_i a \in U_{x_1, \dots, x_n}$ , thus,  $\bar{x} \in h(c_i(S))$ .

2. Let us show that  $h$  preserves cylindrifications.

Let  $(x_1, \dots, x_n) \in \mathcal{M}^n$ . Then  $(x_1, \dots, x_n) \in D_{ij}$  iff  $\mathcal{M} \models 1(x_1, \dots, x_n)$  and  $x_i = x_j$  iff  $U_{x_1, \dots, x_n} \in d_{ij}^{\mathcal{A}^+} = \{U \in \mathbf{Spec}(\mathcal{A}) \mid d_{ij} \in U\}$  iff  $\mathcal{M} \models d_{ij}^{\mathcal{M}}(x_1, \dots, x_n)$ .

□



### 5.3 Finite dimensions

**Definition 9.** *Network*

**Theorem 11.** *Completely representable iff  $\exists$  has a ws.*

**Definition 10.** *Ultrafilter network*

**Theorem 12.**  $\mathbf{RCA}_n$  is a pseudoelementary class for  $3 \leq n < \omega$ .

**Theorem 13.**  $\exists$  has a ws for the canonical extension.

### 5.4 Dimension $\omega$

**Question 1.** *Can we characterise  $\mathbf{RCA}_\omega$  as an enumerably axiomatisable pseudo-elementary class in three-sorted logic with sorts **b** (Boolean part), **c** (cylindric part) and **r** (representation part)?*

**Definition 11.** *Network*

**Theorem 14.** *Completely representable iff  $\exists$  has a ws.*

**Definition 12.** *Ultrafilter network*

**Theorem 15.**  $\exists$  has a ws for the canonical extension.

### 5.5 Counterexamples

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