

Notes on Geometric logic

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1 Locales and quantales

A frame is a complete lattice $\mathcal{L} = (L, \wedge, \vee)$ such that, for all $a \in L$ and $A \subseteq L$:

$$a \wedge \bigvee A = \bigvee \{a \wedge b \mid b \in A\}$$

A frame homomorphism is a map between frames that preserves 0, 1, \wedge and \vee . The **Frm** is the category of all frames and homeomorphisms, the category of locales **Loc** is said to be the opposite category of the category of frames.

The notion of a quantale generalises frames. A quantale $\mathcal{Q} = (Q, \cdot, \vee)$ is a complete lattice-ordered semigroup such that, for all $a \in Q$ and $A \subseteq Q$:

$$\begin{aligned} a \cdot \bigvee A &= \bigvee \{a \cdot b \mid b \in A\} \\ \bigvee A \cdot a &= \bigvee \{b \cdot a \mid b \in A\}. \end{aligned}$$

1.1 Examples of quantales

Let $\text{Sub}(\mathcal{R})$ be the set of all additive subgroups of a ring \mathcal{R} . Let $A \subseteq \text{Sub}(\mathcal{R})$ and $G, H \in \text{Sub}(\mathcal{R})$, define supremum and product as follows:

$$\begin{aligned} \bigvee A &= \{\sum A' \mid A' \subseteq_{fin} A\}, \\ G \cdot H &= \{\sum_{i=0}^n a_i b_i \mid a_i \in G, b_i \in H, n < \omega\} \end{aligned}$$

1.1.1 Quantales from C^* -algebras

The first is to define C^* -algebras. Let K be a field, an *associative K -algebra* is a vector space A over K with a bilinear map $\times : A \times A \rightarrow A$, which is also associative. A Banach algebra is a K -algebra (where K is either \mathbb{R} or \mathbb{C}), which is also a Banach space, i.e., a complete normed vector space and the norm satisfies the inequation, for all $x, y \in A$:

$$\|x \times y\| \leq \|x\| \cdot \|y\|$$

A C^* -algebra is a Banach algebra A over the field of complex numbers with the involution operation $*$: $A \rightarrow A$ such that, for all $x, y \in A$:

- $(x + y)^* = x^* + y^*$
- $(x \times y)^* = y^* \times x^*$
- $\forall \lambda \in \mathbb{C} \ (\lambda x)^* = \bar{\lambda} \cdot x^*$
- $\|x^* \times x\| = \|x\|^2$

Let $I \subseteq A$ be a vector subspace of a C^* -algebra A , then I is an (two-sided) ideal, if:

- $a \in A$ and $b \in I$ implies $a \times b \in I$,
- $a \in I$ and $b \in A$ implies $a \times b \in I$.

An ideal is closed, if it is closed subset of A in the norm topology. One can show that any ideal is closed under involution.

TODO: Let A be a C^* -algebra, define the *spectrum* of A as

2 Quantalic cover systems for quantale representation

Let $\mathcal{P} = (P, \leq)$ be a poset and $x \in \mathcal{P}$, the upper cone generated by x is the set $\uparrow x = \{y \in P \mid x \leq y\}$. Let $A \subseteq \mathcal{P}$, define $\uparrow A$ as

$$\uparrow A = \bigcup_{x \in A} \uparrow x$$

A subset set A is upward closed whenever $\uparrow A = A$. We say that y *refines* x if $x \leq y$, or, equivalently, $\uparrow y \subseteq \uparrow x$. We say that a subset Y *refines* if $Y \subseteq \uparrow X$, that is, every element of y refines some element of X . The set $\text{Up}(\mathcal{P})$ is the set of all upward closed subsets of \mathcal{P} .

Let \mathcal{Q} be a quantale, a function $j : \mathcal{Q} \rightarrow \mathcal{Q}$ is a quantic nucleus, if f is a closure operator such that $ja \cdot jb \leq j(a \cdot b)$. An element a is j -closed iff $ja = a$.

Let $\mathcal{S} = (S, \cdot, \leq)$ be a partially ordered semigroup and $\text{Cov} : \mathcal{S} \rightarrow 2^{2^S}$ a function that assigns every $x \in S$ to the collection of subsets $\text{Cov}(x)$ called x -covers. A *quantalic cover system* is a structure $\mathcal{C} = (\mathcal{S}, \text{Cov})$, where \mathcal{S} is a partially ordered semigroup and Cov is a covering function such that:

1. For all $x \in \mathcal{P}$ there exists $C \subseteq \mathcal{P}$ such that $C \in \text{Cov}(x)$ and $C \subseteq \uparrow x$,
2. If $C \in \text{Cov}(x)$ and for all $y \in C$, then $\bigcup_{y \in C} C_y$,
3. If $x \leq y$, then every x -cover can be refined to some y -cover, that is, if $C \in \text{Cov}(x)$, then $C' \in \text{Cov}(y)$ such that $C' \subseteq \uparrow C$,
4. If $C \in \text{Cov}(x)$ and $D \in \text{Cov}(y)$, then $C \cdot D$ can be refined by an $x \cdot y$ -cover,
5. If $C \in \text{Cov}(x)$ such that C refines $C \cdot D$, then there are $x', y' \in \mathcal{S}$ and $C' \subseteq C$ and $D' \subseteq D$ such that $x' \cdot y' \leq x$ and $C' \in \text{Cov}(x')$ and $D' \in \text{Cov}(y')$.

Given a partially ordered semigroup $\mathcal{S} = (S, \cdot, \leq)$, we can associate the quantale of upsets $\text{Up}(\mathcal{S}) = (\text{Up}(S), \bullet, \subseteq)$, where product is defined by the upward closure of pointwise product.

Given a cover system $\mathcal{C} = (\mathcal{S}, \text{Cov})$. Define a function $j : \text{Up}(\mathcal{S}) \rightarrow \text{Up}(\mathcal{S})$:

$$jX = \{x \in \mathcal{S} \mid \exists C \in \text{Cov}(x) \mid C \subseteq X\}$$

Lemma 1. $j(X \bullet Y) \subseteq jX \bullet jY$, and, therefore, j is a quantic nucleus on the quantale of upsets $\text{Up}(\mathcal{S}) = (\text{Up}(S), \bullet, \subseteq)$.

Proof. $z \in j(X \bullet Y)$, then there are x, y and there are $C \subseteq X$ and $D \subseteq Y$ such that $C \in \text{Cov}(x)$ and $D \in \text{Cov}(y)$ by the definition of j . By the forth axiom, $C \cdot D$ can be refined by an $x \cdot y$ -cover $A \subseteq \uparrow (C \cdot D)$, so by the third axiom there exists a z -cover $E \subseteq \uparrow (C \cdot D) = C \bullet D$, so $z \in E$, and then $z \in j(C \bullet D)$ \square

Let $\mathcal{C} = (\mathcal{S}, \text{Cov})$ be a cover system, the complex algebra of \mathcal{C} , is the quantale $\mathcal{C}^+ = (\text{Up}(\mathcal{S})_j, \bullet, \bigvee)$, where

$$\bigvee A = j(\bigcup A)$$

Dually, given a quantale \mathcal{Q} , define $\mathcal{Q}_+ = (\mathcal{Q}, \leq, \text{Cov})$, where $x \leq y$ iff $y \leq x$ and $C \in \text{Cov}(x)$ iff $x \leq \bigvee C$. Product is defined by product in \mathcal{Q} .

Lemma 2. \mathcal{Q}_+ satisfies the quantalic cover system axioms

Theorem 1. Let \mathcal{Q} be a quantale, then $\mathcal{Q} \cong (\mathcal{Q}_+)^+$.

3 Localic cover systems for locale representation

A *localic cover system* is a tuple $\mathcal{C} = (\mathcal{P}, \text{Cov})$, where \mathcal{P} is a poset and $\text{Cov} : \mathcal{P} \rightarrow 2^{2^{\mathcal{P}}}$ (we call the $\text{Cov}(x)$ *covers of x* or *x -covers*) such that:

- For all $x \in \mathcal{P}$ there exists $C \subseteq \mathcal{P}$ such that $C \in \text{Cov}(x)$ and $C \subseteq \uparrow x$,
- If $C \in \text{Cov}(x)$ and for all $y \in C$, then $\bigcup_{y \in C} C_y$,
- If $x \leq y$, then every x -cover can be refined to some y -cover, that is, if $C \in \text{Cov}(x)$, then $C' \in \text{Cov}(y)$ such that $C' \subseteq \uparrow C$,
- Every x -cover is included in $\uparrow x$.

Goldblatt uses the term "localic cover system" for cover systems where the forth condition is weaker than ours. According to Goldblatt's terminology, a cover system as above is called a *strict localic cover system*. In this subsection, we are not going to consider other cover systems except for localic ones, so we will be omitting the word "localic" most of the times.

Let $\mathcal{C} = (\mathcal{P}, \text{Cov})$ be a localic cover system. Define an operator $j : 2^{\mathcal{P}} \rightarrow 2^{\mathcal{P}}$:

$$jA = \{x \in \mathcal{P} \mid \exists C \subseteq \mathcal{P} \ C \in \text{Cov}(x) \ \& \ C \subseteq A\}$$

An upward closed subset of a cover system is localised if $jA = A$. $\text{Up}(\mathcal{P})_j$ is the set of all j -localised subsets of \mathcal{P}

Lemma 3. *Let $\mathcal{C} = (\mathcal{P}, \text{Cov})$ be a cover system and A an upset, then*

1. jA is an upset,
2. j is a closure operator on $\text{Up}(\mathcal{P})$,
3. $j(A \cap B) = jA \cap jB$.

Proof.

1. Let $x \in jA$ and $x \leq y$. We need $y \in jA$.

By the definition of j , there exists $C \in \text{Cov}(x)$ such that $C \subseteq A$. By the refinement axiom, we have $C' \in \text{Cov}(y)$ such that $C' \subseteq \uparrow C$. But $\uparrow C \subseteq \uparrow A = A$, so $y \in jA$, that makes jA upward closed.

2. Let $x \in A$, then by the first axiom, we have $C \in \text{Cov}(x)$ with $C \subseteq \uparrow x \subseteq A$, so $x \in jA$. Idempotence follows from transitivity (the third axiom).

3. To show multiplicativity, this is enough to show $A \cap jB \subseteq j(A \cap B)$. Let $x \in A \cap jB$. Then there exists an x -cover $C \subseteq B$, but every x -cover is included in $\uparrow x$, but $\uparrow x \subseteq A$ and also $\uparrow C \subseteq B$, so $x \in j(A \cap B)$

□

Let $\mathcal{C} = (\mathcal{P}, \text{Cov})$ be a cover system, the complex algebra of a cover system is a structure $\mathcal{C}^+ = (\text{Up}(\mathcal{P})_j, \wedge, \bigvee)$, where

- $A \wedge B = A \cap B$,
- $\bigvee_{i \in I} A_i = j(\bigcup_{i \in I} A_i)$

Lemma 4. \mathcal{C}^+ is well-defined, moreover, \mathcal{C}^+ is a frame.

Proof. $A \wedge \bigvee_{i \in I} A_i = A \cap j(\bigcup_{i \in I} A_i) = jA \cap j(\bigcup_{i \in I} A_i) = j(A \cap \bigcup_{i \in I} A_i) = j(\bigcup_{i \in I} A \cap A_i) = \bigvee_{i \in I} (A \wedge A_i)$ \square

Dually, let $\mathcal{L} = (L, \wedge, \bigvee)$ be a frame, then its dual cover system $\mathcal{L}_+ = (L, \leq, \text{Cov})$, where (L, \leq) is a dual poset of L , that is, $x \leq y$ iff $y \leq x$ and $C \in \text{Cov}(x)$ iff $x = \bigvee C$. Clearly the upper cone generated by x in (L, \leq) is the same as the lower cone generated by x in (L, \leq) . We denote it as $(a]$

Lemma 5. Let \mathcal{L} be a frame, then \mathcal{L}_+ is a localic cover system

Proof. We have to verify four axioms.

- Let $x \in \mathcal{L}$, then $x = \bigvee(x]$, so $\bigvee(x]$ is an x -cover that obviously contains $\uparrow x$.
- Let $x = \bigvee C$ and for all $y \in C$ we have $y = \bigvee_{y \in C} C_y$, then

$$x = \bigvee_{y \in C} \bigvee C_y = \bigvee_{y \in C} C_y = \bigvee_{y \in C} (\bigcup_{y \in C} C_y), \text{ so } x \text{ is also covered by } \bigcup_{y \in C} C_y.$$

- Let $x \leq y$ and $x = \bigvee C$. Let $C' = \{y \wedge c \mid c \in C\}$, then $y = \bigvee C'$ is a y -cover and C' refines C since $y \wedge c \leq c$.
- Let $x = \bigvee C$, so $C \subseteq \uparrow x$

\square

Theorem 2. (Representation theorem)
 $\mathcal{L} \cong (\mathcal{L}_+)^+$

Proof. By mapping $x \mapsto (x]$. \square

4 Geometric logic and geometric categories

The first order signature Σ that we are going to consider in this section consists of the following data:

- A set Σ – Sort of sorts,
- A set Σ –Fun of function symbols having the form $f : A_1 \times \cdots \times A_n \rightarrow B$ where A_1, \dots, A_n, B are sorts,
- A set Σ –Fun of predicate symbols having the form $R \hookrightarrow A_1 \times \cdots \times A_n$, where A_1, \dots, A_n are sorts,

The collection of terms is defined inductively:

- A variable x^A of sort A is a term of sort A ,
- If $t_1 : A_1, \dots, t_n : A_n$ are terms of sorts A_1, \dots, A_n and $f : A_1 \times \cdots \times A_n \rightarrow B$ is a function symbol, then $f(t_1, \dots, t_n) : B$ is a term of sort B .

The set of formulas is also defined inductively:

- Let $t_1 : A_1, \dots, t_n : A_n$ be terms of sorts A_1, \dots, A_n and let $R \hookrightarrow A_1 \times \dots \times A_n$ be a predicate symbol, then $R(t_1, \dots, t_n)$ is an atomic formula,
- Let t_1, t_2 be terms of sort A , then $t_1 = t_2$ is an atomic formula,
- If φ and ψ are formulas, so is $(\varphi \wedge \psi)$.
- Let I be an index set and $\{\varphi_i \mid i \in I\}$ a family of formulas such that $|\text{FV}((\varphi_i)_{i \in I})| < \omega$, then $\bigvee_{i \in I} \varphi_i$ is a formula
- If φ is a formula and $x \in \text{FV}(\varphi)$, then $\exists x \varphi$ is a formula and $\text{FV}(\exists x \varphi) = \text{FV}(\varphi) \setminus \{x\}$.

$$\begin{array}{c}
\overline{\varphi \rightarrow \varphi} \\
\\
\begin{array}{cc}
\overline{\varphi \vdash \top} & \overline{\varphi \vdash \bigvee \Phi} \quad \varphi \in \Phi \\
\\
\overline{\varphi_1 \wedge \varphi_2 \vdash \varphi_i} & \frac{\varphi \vdash \psi}{\bigvee \Phi \vdash \psi} \quad \varphi \in \Phi \\
\\
\frac{\varphi \vdash \psi \quad \varphi \vdash \theta}{\varphi \vdash \psi \wedge \theta} & \frac{}{\varphi \wedge \bigvee \Phi \vdash \bigvee_{\phi \in \Phi} (\varphi \wedge \phi)} \\
\\
\frac{\varphi \vdash \psi}{\exists x \varphi \vdash \psi} \quad x \notin \text{FV}(\varphi) & \frac{\exists x \varphi \vdash \psi}{\varphi \vdash \psi} \quad x \notin \text{FV}(\varphi) \\
\\
\frac{\varphi \vdash \psi \quad \psi \vdash \theta}{\varphi \vdash \theta} & \frac{}{\varphi \wedge \exists x \psi \vdash \exists x (\varphi \wedge \psi)} \quad x \notin \text{FV}(\psi)
\end{array}
\end{array}$$

5 Non-commutative geometric logic, non-commutative geometric categories and completeness

We define the non-commutative geometric logic with the following axioms and inference rules.

The definition of a language is identical to the corresponding definition from the previous section, but we replace \wedge with \bullet .

$$\overline{\varphi \rightarrow \varphi}$$

$$\begin{array}{c}
\frac{\varphi \vdash \psi}{\varphi \bullet \theta \vdash \psi \bullet \theta} \\
\\
\frac{\varphi \vdash \psi}{\theta \bullet \varphi \vdash \theta \bullet \psi} \\
\\
\frac{}{\bigvee \Phi \bullet \varphi \vdash \bigvee_{\phi \in \Phi} (\phi \bullet \varphi)} \\
\\
\frac{\varphi \vdash \psi}{\exists x \varphi \vdash \psi} \quad x \notin FV(\varphi) \\
\\
\frac{}{\exists x \psi \bullet \varphi \vdash \exists x (\psi \bullet \varphi)} \quad x \notin FV(\varphi) \\
\\
\frac{\varphi \vdash \psi \quad \psi \vdash \theta}{\varphi \vdash \theta}
\end{array}
\qquad
\begin{array}{c}
\frac{}{\varphi \vdash \bigvee \Phi} \quad \varphi \in \Phi \\
\\
\frac{\varphi \vdash \psi}{\bigvee \Phi \vdash \psi} \quad \varphi \in \Phi \\
\\
\frac{}{\varphi \bullet \bigvee \Phi \vdash \bigvee_{\phi \in \Phi} (\varphi \bullet \phi)} \\
\\
\frac{}{\varphi \bullet (\psi \bullet \theta) \dashv\vdash (\varphi \bullet \psi) \bullet \theta} \\
\\
\frac{}{\varphi \bullet \exists x \psi \vdash \exists x (\varphi \bullet \psi)} \quad x \notin FV(\varphi)
\end{array}$$

6 Elementary topoi

Let $A, B \in \text{Ob}(\mathcal{C})$. A is said to be a subobject of B iff there exists a monic arrow $h : A \hookrightarrow B$. The inclusion relation on objects is defined by “inclusion” on monic arrows.

Let $f : A \hookrightarrow D$ and $g : B \hookrightarrow D$, then $f \subseteq g$ iff there exists an arrow $h : A \rightarrow B$ making this triangle commute:

$$\begin{array}{ccc}
& B & \\
& \curvearrowright g & \\
h \uparrow & & \searrow \\
A & & D \\
& \curvearrowleft f &
\end{array}$$

that is, $f = g \circ h$

Proposition 1. *Such an h is monic.*

Proposition 2. $f \subseteq f$.

Proof. The following triangle obviously commutes:

$$\begin{array}{ccc}
& A & \\
& \curvearrowleft f & \\
\text{id}_A \uparrow & & \searrow \\
A & & D \\
& \curvearrowleft f &
\end{array}$$

□

Proposition 3. $f \subseteq g$ and $g \subseteq k$ imply $f \subseteq k$.

Proof. The following triangle commutes:

$$\begin{array}{ccc}
 & C & \\
 i \uparrow & \searrow k & \\
 B & \xrightarrow{g} & D \\
 h \uparrow & \nearrow f & \\
 A & &
 \end{array}$$

Because of obvious calculations:

$$k \circ i \circ h = k \circ (i \circ h) = k \circ g = f$$

□

We also say that $f \simeq g$, whenever $f \subseteq g$ and $g \subseteq f$. In such case we shall identify them and omit metamathematical nitpicking.

Assume that a category \mathcal{C} has a terminal object $\mathbf{1}$, an arrow $x : \mathbf{1} \rightarrow A$ is called an *element* of A .

Let \mathcal{C} be a category with a terminal object $\mathbf{1}$. A *subobject classifier* for \mathcal{C} is an object Ω with an arrow $\top : \mathbf{1} \rightarrow \Omega$ such that for each $f : A \hookrightarrow D$ there exists a unique arrow $\chi_f : D \rightarrow \Omega$ (the characteristic arrow of f) making the following square a pullback:

$$\begin{array}{ccc}
 A & \xrightarrow{f} & D \\
 ! \downarrow & & \downarrow \exists! \chi_f \\
 \mathbf{1} & \xrightarrow{\top} & \Omega
 \end{array}$$

Consider **Top**, the category of topological spaces. Although it is not an elementary topos (**Top** has no exponentiation objects), however it has an open subspace classifier. The Sierpinski space Σ is the space on the two-element set $2 = \{0, 1\}$ with opens \emptyset , 2 and $\{1\}$. Let D be a topological space and A its open subspace, so the following diagram

$$\begin{array}{ccc}
 A & \hookrightarrow & D \\
 ! \downarrow & & \downarrow \chi_A \\
 \mathbf{1} & \xrightarrow{\top} & \Sigma
 \end{array}$$

says that there is a unique continuous function χ_A defined as

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

such that $A = \chi_A^{-1}(\mathbf{1})$.

Also the Sierpinski space also allows us classifying closed subspaces. Let D be a topological space and A a closed subspace of D . The following diagram

$$\begin{array}{ccc}
 A & \hookrightarrow & D \\
 ! \downarrow & & \downarrow \chi_A \\
 \mathbf{0} & \xrightarrow{\perp} & \Sigma
 \end{array}$$

says that there is a unique continuous function χ_A such that $A = \chi_A^{-1}(\mathbf{0})$.

As in the example with the Sierpinski space, we shall write χ_A for characteristic arrows instead of χ_f to make notation closer to usual set-theoretic mathematics.

One can establish the criterion of subobject equality in term of a subobject classifier.

Lemma 6. *Let \mathcal{C} be a category with a subobject classifier Ω , then for all $f : A \hookrightarrow D$ and $g : B \hookrightarrow D$, then*

$$f \simeq g \text{ iff } \chi_A = \chi_B$$

Proof. Suppose $\chi_A = \chi_B$. But the axioms of a subobject classifier, the following square is a pullback:

$$\begin{array}{ccc} B & \xrightarrow{g} & X \\ \downarrow k & \searrow f & \downarrow \chi_A \\ A & \xrightarrow{f} & X \\ \downarrow ! & & \downarrow \chi_A \\ 1 & \xrightarrow{\top} & \Omega \end{array}$$

By assumption, the outer square commutes, the inner square is a pullback, so we have a unique arrow k that factors g through h , so $g \subseteq f$. One can show that $f \subseteq g$ similarly.

If $f \simeq g$, then k is iso, so the outer square is a pullback, and that implies that χ_A is the unique character of B , so $\chi_A = \chi_B$. \square

Moreover, we have:

Lemma 7. *Let \mathcal{C} be a locally small category with a terminal object and a subobject classifier Ω . Let $A \in \text{Ob}(\mathcal{C})$, define the set:*

$$\text{Sub}(A) = \{B \in \text{Ob}(\mathcal{C}) \mid B \hookrightarrow A\}$$

$$\text{then } \text{Sub}(A) \cong \text{Hom}_{\mathcal{C}}(A, \Omega)$$

Proof. Follows from the lemma above. The map $f : B \mapsto \chi_B$ establishes a bijection. \square

A category \mathcal{C} is called an *elementary topos* if:

- \mathcal{C} is Cartesian closed,
- \mathcal{C} has coproducts,
- \mathcal{C} has an initial object $\mathbf{0}$,
- \mathcal{C} has a subobject classifier Ω .

Moreover, every $\text{Sub}(A)$ is a Heyting algebra, so one can extract truth arrows $\neg, \wedge, \vee, \Rightarrow$ and also the binary relation \leq that reflect Heyting algebra operations and ordering. Let us define them explicitly. Let \mathcal{E} be an elementary topos.

- $\neg : \Omega \rightarrow \Omega$ is the arrow making this square pullback:

$$\begin{array}{ccc} 1 & \xrightarrow{\perp} & \Omega \\ id_A \downarrow & & \downarrow \neg \\ 1 & \xrightarrow{\top} & \Omega \end{array}$$

where \perp is the character of $\mathbf{0}$ considered as a subobject of $\mathbf{1}$:

$$\begin{array}{ccc}
0 & \longrightarrow & 1 \\
\downarrow ! & & \downarrow \perp \\
1 & \xrightarrow{\top} & \Omega
\end{array}$$

- $\wedge : \Omega \times \Omega \rightarrow \Omega$ is the character of the product arrow $(\top, \top) : 1 \rightarrow \Omega \times \Omega$:

$$\begin{array}{ccc}
1 & \xrightarrow{(\top, \top)} & \Omega \times \Omega \\
id_1 \downarrow & & \downarrow \wedge \\
1 & \xrightarrow{\top} & \Omega
\end{array}$$

- Ordering relation is a monomorphism $e : \leq \hookrightarrow \Omega \times \Omega$, which is the equaliser of

$$\Omega \times \Omega \begin{array}{c} \xrightarrow{\wedge} \\ \xrightarrow{\pi_1} \end{array} \Omega$$

- $\vee : \Omega \times \Omega \rightarrow \Omega$ is the character of the arrow $[(\top, id_\Omega), (id_\Omega, \top)] : \Omega + \Omega \rightarrow \Omega \times \Omega$

Lemma 8. Let \mathcal{E} be an elementary topos with a subobject classifier $\top : 1 \rightarrow \Omega$ and let $D \in \text{Ob}(\mathcal{E})$, then the structure $(\text{Hom}_{\mathcal{E}}(D, \Omega), \wedge, \vee, \Rightarrow, \neg)$ is a Heyting algebra, where operations are defined by:

- $\neg p = \neg \circ p$,
- $p \wedge q = \wedge \circ (p, q)$,
- $p \vee q = \vee \circ (p, q)$,
- $p \Rightarrow q = \Rightarrow \circ (p, q)$.

6.1 Sites over elementary topoi

Let \mathcal{E} be an elementary topos. A Lawvere-Tierney topology on \mathcal{E} is an arrow $j : \Omega \rightarrow \Omega$ making the following diagrams commute:

$$\begin{array}{ccc}
\begin{array}{ccc} 1 & & \\ \top \downarrow & \searrow \top & \\ \Omega & \xrightarrow{j} & \Omega \end{array} &
\begin{array}{ccc} \Omega & & \\ j \downarrow & \searrow j & \\ \Omega & \xrightarrow{j} & \Omega \end{array} &
\begin{array}{ccc} \Omega \times \Omega & \xrightarrow{\wedge} & \Omega \\ j \times j \downarrow & & \downarrow j \\ \Omega \times \Omega & \xrightarrow{\wedge} & \Omega \end{array}
\end{array}$$

A tuple $\mathcal{E}_j = (\mathcal{E}, j)$ is called an elementary site.

Proposition 4. The map $n : h \mapsto j \circ h$ is a nucleus on $\text{Hom}_{\mathcal{E}}(D, \Omega)$

Lemma 9. Let \mathcal{P} be a poset, then one define an elementary site on the category of set-valued functors $\text{Set}^{\mathcal{P}}$.

Proof. Define F_{pq} as an arrow $F(p) \rightarrow F(q)$, an image of $p \leq q$ in Set .

First of all, let us show that $\text{Set}^{\mathcal{P}}$ is an elementary topos.

- Subobject classifier
- Pullbacks

- Exponentiation
- Terminal object: The terminal object is the constant functor $\mathbf{1} : \mathcal{P} \rightarrow \mathbf{Set}$ such that $\mathbf{1} : _ \mapsto \{\emptyset\}$.

□

7 Grothendieck topology

7.1 Sheaves and presheaves

Let $\mathcal{I} = (I, \theta)$ be a topological space. Consider θ as a poset. A *presheaf* over \mathcal{I} is a contravariant functor from θ to **Set**.

The notion of a presheaf generalises essentially the following construction from set-theoretic topology. First of all, we discuss a set-theoretic examples without referring to topology. Consider an indexed family of disjoint sets:

$$\mathcal{A} = \{A_i \mid i \in I\}.$$

We can associate an obvious map $p : A \rightarrow I$ since for every $x \in \mathcal{A}$ there is a unique $i \in I$ such that $x \in A_i$. Take

$$p^{-1}(\{i\}) = \{x \mid p(x) = i\} = A_i$$

Such $p^{-1}(\{i\})$ is called the *fibre* over i , the whole structure is a bundle of sets over the base space I , \mathcal{A} is the stalk space (l'espace etale) of the bundle. More generally, we can extract the bundle from every map $p : A \rightarrow I$

A morphism of bundles (A, I) and (B, I) is a commutative triangle of the following form:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ p_1 \searrow & & \swarrow p_2 \\ & I & \end{array}$$

Topologically, a sheaf is a version of bundles for topological spaces. Let $\mathcal{I} = (I, \theta)$ be a topological space. A sheaf is a tuple (\mathcal{A}, p) , where \mathcal{A} is a topological space and $p : \mathcal{A} \rightarrow I$ is a continuous map, which is also a local homeomorphism, that is, every $x \in \mathcal{A}$ has an open neighbourhood, which mapped homeomorphically by p onto $p(U)$ and $p(U)$ is open in I . The category of all sheaves of I is sometimes called a spatial topos.

We can extract a presheaf from a sheaf (\mathcal{A}, f) as a contravariant functor $F_f : \theta \rightarrow \mathbf{Set}$ as

$$F_f(V) = \{s : V \rightarrow \mathcal{A} \mid s \text{ is continuous and } f \circ s = V \hookrightarrow I\}$$

The category of presheaves over I , denoted as $\mathbf{PsC}(I)$, consists of presheaves as objects and natural transformations $\tau : F \Rightarrow G$, that is, a collection of functions $\tau_U : F(U) \rightarrow G(U)$ making this square commute whenever $U \subseteq V$

$$\begin{array}{ccc} F(V) & \xrightarrow{\tau_V} & G(V) \\ F_U^V \downarrow & & \downarrow G_U^V \\ F(U) & \xrightarrow{\tau_U} & G(U) \end{array}$$

It is clear that $\mathbf{PsC}(I)$ is equivalent to $\mathbf{Set}^{\theta^{Op}}$.

Let $F : \theta \rightarrow \mathbf{Set}$ be a presheaf. Let $x \in X$ and let U, V be open neighbourhoods of x . Let $s \in F(U)$ and $t \in F(V)$. S and T are said to have the *same germ* at x , whenever there exists an open set $W \subseteq U \cap V$ with $x \in W$ such that $s|_W = t|_W \in P(W)$. Having the same germ is an equivalence relation, and the *germ* of s in x in an equivalence class of s denoted as $germ_x s$.

The *stalk* of F in x is the set of all germs of s

$$F_x = \{germ_x s \mid s \in F(U), U \in \theta\}$$

Let X be an index set and V an open set, an *open cover* of V is a collection of sets $\{V_x\}_{x \in X}$ such that

$$V = \bigcup_{x \in X} V_x$$

Intuitively, a sheaf is a presheaf that preserves open covers.

A *sheaf* is a presheaf F satisfying the following two extra-principles. Let V be an open set and $\{V_x\}_{x \in X}$ an open cover, then:

1. Let $s, t \in F(V)$ be sections such that $s|_{V_x} = t|_{V_x}$ for $x \in X$, then $s = t$.
2. Let $\{s_x \in F(V_x)\}_{x \in X}$ be a family of sections. If for all $x, y \in X$ we have $s_x|_{V_x \cap V_y} = s_y|_{V_x \cap V_y}$, then there exists a section $s \in F(V)$ such that $s|_{V_x} = s_x$ for all $x \in X$.

Equivalently, we can reformulate the latter as that $F(V) = \varprojlim_{x \in X} F(V_x)$. The category $\mathbf{Sh}(I)$ is a category of sheaves over I .

7.2 Grothendieck topos

The notion of a Grothendieck topos generalises the aforementioned topological constructions. We start with the definition of a site.

Let \mathcal{C} be a locally small category. A *pretopology* on \mathcal{C} is an assignment of each $A \in \mathbf{Ob}(\mathcal{C})$ of a collection of arrows $\text{Cov}(A)$ (covers of A , or covering sieves) with the following properties:

1. $\{id_A : A \rightarrow A\} \in \text{Cov}(A)$
2. If $\{f_x : A_x \rightarrow A \mid x \in X\} \in \text{Cov}(A)$ and for each $x \in X$ we have an a_x -cover

$$\{f_y^x : A_y^x \rightarrow A_x \mid y \in Y_x\} \in \text{Cov}(A_x)$$

then

$$\{f_x \circ f_y^x : A_y^x \rightarrow A \mid x \in X, y \in Y_x\} \in \text{Cov}(A)$$

3. If $\{f_x : A_x \rightarrow A \mid x \in X\} \in \text{Cov}(A)$ and $g : B \rightarrow A$ and assume that for each $x \in X$ the pullback of f_x along g exists:

$$\begin{array}{ccc} B \times_A A_x & \xrightarrow{\quad} & A_x \\ g_x \downarrow & & \downarrow f_x \\ B & \xrightarrow{\quad g \quad} & A \end{array}$$

then $\{g_x : B \times_A A_x \rightarrow B \mid x \in X\} \in \text{Cov}(B)$

A *site* is the pair $(\mathcal{C}, \text{Cov})$ consisting of a category and a pretopology on it.

A Grothendieck topos is a site with extra-conditions that generalise the axioms of topological sheaves in terms of a pretopology. A presheaf of sets over a category \mathcal{C} is a contravariant functor $F : \mathcal{C} \rightarrow \mathbf{Set}$

Let Cov be a pretopology on a category \mathcal{C} and $\{f_x : A_x \rightarrow A \mid x \in X\} \in \text{Cov}(A)$. Let $x, y \in X$ and we have the pullback of f_x and f_y

$$\begin{array}{ccc} A_x \times_A A_y & \longrightarrow & A_y \\ \downarrow & & \downarrow f_y \\ A_x & \xrightarrow{f_x} & A \end{array}$$

If F is a presheaf over \mathcal{C} , then we have arrows $F_y^x : F(A_x) \rightarrow F(A_x \times_A A_y)$ and $F_x^y : F(A_y) \rightarrow F(A_x \times_A A_y)$. Denote F_x as the arrow $F(f_x) : F(A) \rightarrow F(A_x)$.

A presheaf F is a sheaf, if for any cover $\{f_x : A_x \rightarrow A \mid x \in X\} \in \text{Cov}(A)$, then for all $x, y \in X$ such that for all $s_x \in F(A_x)$ and $s_y \in F(A_y)$ such that $F_y^x(s_x) = F_x^y(s_y)$, then there exists a unique $s \in F(A)$ such that $F_x(s) = s_x$ for $x \in X$.

$\mathbf{Sh}(\text{Cov})$ is the category of sheaves of the site $(\mathcal{C}, \text{Cov})$. A Grothendieck topos is a category of sheaves of some site up to categorical equivalence.

Alternatively, one can define a Grothendieck topos in terms of a Grothendieck topology as follows. Define a sieve S as family morphisms in a category \mathcal{C} that behaves as a right ideal:

$$f \in S \text{ implies } f \circ g \in S$$

If S is a sieve on $C \in \text{Ob}(\mathcal{C})$ and $h \in \text{Hom}(D, C)$ for any $D \in \text{Ob}(\mathcal{C})$, then

$$h^*(S) = \{g \mid \text{cod}(g) = D, g \circ h \in S\}$$

A *Grothendieck topology* on a category \mathcal{C} is a function J that maps every $C \in \text{Ob}(\mathcal{C})$, denoted as $J(C)$ such that:

1. the maximal sieve $t_C = \{f \mid \text{cod}(f) = C\} \in J(C)$
2. If $S \in J(C)$, then $h^*(S) \in J(D)$
3. If $S \in J(C)$ and R is a sieve of C such that $h^*(R) \in J(D)$ for all $h : D \rightarrow C$, then $R \in J(C)$

Also any $J(C)$ is upward closed.

7.3 Lawvere-Tierney topology

7.4 Examples

We start with some examples of a site.

Let \mathcal{T} be a small category of topological spaces closed under finite limits and under taking open subspaces. Define Cov as:

$$\{f_i : Y_i \rightarrow X \mid i \in I\} \text{ iff each } Y_i \text{ is an open subspace of } X \text{ and } \bigcup_{i \in I} Y_i = X$$

The first axiom holds obviously, the second axiom holds since \mathcal{T} is closed under taking subspaces. The third axiom holds because of the closure under finite limits.

Let H be a frame. One can define a pretopology on a frame by putting:

$$\{a_i \mid i \in I\} \in \text{Cov}(c) \text{ iff } c = \bigvee_{i \in I} a_i$$

7.4.1 The Zariski site

Let $f_1, \dots, f_m \in \mathbb{C}[x_1, \dots, x_n]$, the locus of f_1, \dots, f_m is the set

$$V(f_1, \dots, f_m) = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid f_i(z_1, \dots, z_n) = 0, i = 1, \dots, m\}$$

Such a locus is called a *complex affine variety*. With every variety V we can associate the following ideal in the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$:

$$I_V = \{f \in \mathbb{C}[x_1, \dots, x_n] \mid \forall \vec{z} \in V f(\vec{z}) = 0\}$$

Conversely, let I be an ideal in the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$, then we can define the variety

$$V_I = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid f(z_1, \dots, z_n) = 0, f \in I\}$$

If $I = (f_1, \dots, f_m)$, then $V_I = V(f_1, \dots, f_m)$.

With every ideal I we can associate its radical

$$\sqrt{I} = \bigcup_{0 < r < \omega} \{f \in \mathbb{C}[x_1, \dots, x_n] \mid f^r \in I\}$$

According to the Hilbert Nullstellensatz, $V_J \neq V_I$ whenever $\sqrt{I} \neq \sqrt{J}$.

The maximal ideals in $\mathbb{C}[x_1, \dots, x_n]$ have the form $(x_1 - a_1, \dots, x_n - a_n)$, so the corresponding variety is merely the singleton $\{(a_1, \dots, a_n)\}$, a minimal algebraic variety. A prime ideal P (that is, $fg \in P$ implies $g \in P$) in $\mathbb{C}[x_1, \dots, x_n]$ is a radical ideal. The corresponding variety of P is irreducible, that is, it cannot be represented as the union of a finite number of smaller ideals. Moreover, every radical ideal can be represented as the intersection of some finite number of prime ideals. Dually, every complex affine variety can be represented as the union of some finite number of irreducible varieties.

The *Zariski topology* on \mathbb{C}^n is a topology defined on irreducible varieties as a closed subbasis. An algebraic hypersurface is the locus of a single polynomial $f(x_1, \dots, x_n) = 0$. The complements of hypersurfaces form the open subbasis for the Zariski topology. An example of a cover of \mathbb{C}^n can be defined by t polynomials $f_1, \dots, f_t \in \mathbb{C}[x_1, \dots, x_n]$ such that $f_1 + \dots + f_t = 1$. Their hypersurfaces have no common points, so their complements are an open cover of \mathbb{C}^n .

With the Zariski topology, we can associate a *structure sheaf*. Consider a field $\mathbb{F} = \mathbb{C}(x_1, \dots, x_n)$ consisting of rational functions, where $g \neq 0$. A rational function h is defined at $Q \in (a_1, \dots, a_n) \in \mathbb{C}^n$, whenever there is a Zariski open set W , an open neighbourhood of Q , such that h has the form f/g for some $f, g \in \mathbb{C}[x_1, \dots, x_n]$ such that $g(\vec{z}) \neq 0$ for all $\vec{z} \in W$. In other words, h yields a function $W \rightarrow \mathbb{C}$.

Now let U be an open set in the Zariski topology, define $\mathcal{O}(U)$ as

$$\mathcal{O}(U) = \{h \in \mathbb{F} \mid \forall Q \in U h(Q) \text{ is defined}\}$$

The set $\mathcal{O}(U)$ is a ring, a moreover, a subring of a field \mathbb{F} whenever U is non-empty. \mathcal{O} is also a contravariant functor (if we consider the Zariski topology on \mathbb{C} as a category). Given $U \subseteq U'$, then $\mathcal{O}(U') \rightarrow \mathcal{O}(U)$ is a ring homomorphism that restricts each $h \in \mathcal{O}(U')$ to points of U . That is, \mathcal{O} is a presheaf of rings for the Zariski topology on \mathbb{C}^n . It is also can be showed that \mathcal{O} is a sheaf.

The stalk of \mathcal{O} at a point $p \in \mathbb{C}^n$ consists of germs of all those rational functions defined in some open neighbourhood of p . Such a stalk is a ring with a unique maximal proper ideal, the ideal of all those germs that vanish at p . A *local ring* is a ring of such form.

More generally, we consider the category of all affine varieties of $V \subseteq \mathbb{C}^n$ (for various $n \geq 0$), where morphisms $\phi : V \rightarrow W$ are defined as follows. Clearly $V = V(I)$ and $W = V(I')$ for some ideals I and I' . ϕ is a function defined by an m -tuple $\phi = (h_1, \dots, h_m)$ of rational functions

of x_1, \dots, x_n such that each h_i is defined at every point of V and (h_1, \dots, h_m) as a function $V \rightarrow \mathbb{C}^m$ maps V into W .

This category can be equipped with the open cover topology, which is the Grothendieck topology defined by covering families of Zariski open sets.

This construction can be generalised as the Zariski site over a commutative ring K .

Let K be a commutative ring with unit, define a ring $K[a^{-1}]$ of quotients for $a \in K$ that extends K with fractions b/a^n for each $b \in K$ and $n < \omega$.

As above, one can define the n -dimensional affine space K^n . Let I be an ideal in the polynomial ring $K[x_1, \dots, x_n]$, we can define the variety V_I defined similarly to the complex case. Generally, the Nullstellensatz is not the case for an abstract $K[x_1, \dots, x_n]$, so we shall work with quotient rings $K[x_1, \dots, x_n]/I$. Let $f_1, \dots, f_m \in K[x_1, \dots, x_n]$, the *finitely presented K -algebra* is the quotient ring $K[x_1, \dots, x_n]/(f_1, \dots, f_m)$. $(K - \text{Alg})_{fp}$ is the category all finitely presented K -algebra. There is a contravariant adjointness between $(K - \text{Alg})_{fp}$ and the corresponding varieties.

7.4.2 Sheaves of rings on a locale

This example is an example of both sheaves and quantales.

Let \mathcal{L} be a frame and $\mathcal{O} : L \rightarrow \mathbf{Ring}$ a sheaf of rings on \mathcal{L} . We show that $\text{Idl}(\mathcal{O})$, a sheaf of ideals of \mathcal{O} is a quantale.

A presheaf I on \mathcal{L} is a *presheaf of ideals* of \mathcal{O} if for all $a \in \mathcal{L}$ $I(a)$ is an ideal $\mathcal{O}(a)$. If I is a sheaf, then I is a sheaf of ideals, or ideal of \mathcal{O} . The set $\text{Idl}(\mathcal{O})$ is the set of all ideals of \mathcal{O} .

Let I be a presheaf of ideals of \mathcal{O} , then the *sheafification* $\langle I \rangle$ of I is defined as, for all $a \in L$:

$$r \in \langle I \rangle(a) \text{ iff there exists a cover } \{a_i\}_{i \in I} \text{ of } L \text{ with } r|_{a_i} \text{ for all } i \in I$$

$\langle I \rangle$ can be also defined explicitly as:

$$\langle I \rangle = \bigcap \{J \in \text{Idl}(\mathcal{O}) \mid I \subseteq J\}$$

Consider a family $\{I_j\}_{j \in J}$ of ideals where each $I_j \in \text{Idl}(\mathcal{O})$, then the assignment $a \mapsto \sum_{j \in J} I_j(a)$ is a presheaf, but it does not have to be a sheaf, but supremum in the lattice $\text{Idl}(\mathcal{O})$ is defined as $\langle \sum_{j \in J} I_j \rangle$.

Now we define an example of such a sheaf from the Zariski space, the dual space of a commutative ring.

TODO: complete this subsection

8 Kripke-Joyal semantics and quantifiers via adjoint functors

9 Geometric morphisms

10 Kripke-Joyal semantics for quantales and non-commutative geometric theories

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