Notes on Geometric logic

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1 Sheaves, Sites and Grothendieck toposes

Let $\mathcal{I} = (I, \theta)$ be a topological space. Consider θ as a poset. A *presheaf* over $\mathcal{I}$ is a contravariant functor from θ to **Set**.

The notion of a presheaf generalises essentially the following construction from set-theoretic topology. First of all, we discuss a set-theoretic examples without referring to topology. Consider an indexed family of disjoint sets:

$$\mathcal{A} = \{A_i \mid i \in I\}.$$

We can associate an obvious map $p : A \to I$ since for every $x \in \mathcal{A}$ there is a unique $i \in I$ such that $x \in A_i$. Take

$$p^{-1}(\{i\}) = \{x \mid p(x) = i\} = A_i$$

Such $p^{-1}(\{i\})$ is called the *fibre* over i, the whole structure is a bundle of sets over the base space I, $\mathcal{A}$ is the stalk space (l'espace etale) of the bundle. More generally, we can extract the bundle from every map $p : A \to I$

A morphism of bundles (A, I) and (B, I) is a commutative triangle of the following form:

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "p_1"'] & & B \arrow[dl, "p_2"] \\
& I &
\end{tikzcd}$$

Topologically, a sheaf is a version of bundles for topological spaces. Let $\mathcal{I} = (I, \theta)$ be a topological space. A sheaf is a tuple $(\mathcal{A}, p)$, where $\mathcal{A}$ is a topological space and $p : A \to I$ is a continuous map, which is also a local homeomorphism, that is, every $x \in \mathcal{A}$ has an open neighbourhood, which mapped homeomorphically by p onto $p(U)$ and $p(U)$ is open in I. The category of all sheaves of I is sometimes called a spatial topos.

We can extract a presheaf from a sheaf (A, f) as a contravariant functor $F_f : \theta \to \mathbf{Set}$ as

$$F_f(V) = \{s : V \to A \mid s \text{ is continuous and } f \circ s = V \hookrightarrow I\}$$

The category of presheafs over I, denoted as $\mathbf{PsC}(I)$, consists of presheafs as objects and natural transformations $\tau : F \Rightarrow G$, that is, a collection of functions $\tau_U : F(U) \to G(U)$ making this square commute whenever $U \subseteq V$

$$\begin{tikzcd}
F(V) \arrow[r, "\tau_V"] \arrow[d, "F_U^V"'] & G(V) \arrow[d, "G_U^V"] \\
F(U) \arrow[r, "\tau_U"'] & G(U)
\end{tikzcd}$$

It is clear that $\mathbf{PsC}(I)$ is equivalent to $\mathbf{Set}^{\theta^{Op}}$.

Let X be an index set and V an open set, an *open cover* of V is a collection of sets $\{V_x\}_{x \in X}$ such that

$$V = \bigcup_{x \in X} V_x$$

Intuitively, a sheaf is a presheaf that preserves open covers.

A *sheaf* is a presheaf F satisfying the following two extra-principles. Let V be an open set and $\{V_x\}_{x \in X}$ an open cover, then:

1. Let $s, t \in F(V)$ be sections such that such that $s|_{V_x} = t|_{V_x}$ for $x \in X$, then $s = t$.
2. Let $\{s_x \in F(V_x)\}_{x \in X}$ be a family of sections. If for all $x, y \in X$ we have $s_x|_{V_x \cap V_y} = s_y|_{V_x \cap V_y}$, then there exists a section $s \in F(V)$ such that $s|_{V_x} = s_x$ for all $x \in X$.

Equivalently, we can reformulate the latter as that $F(V) = \varprojlim_{x \in X} F(V_x)$. The category $\mathbf{Sh}(I)$ is a category of sheaves over I.

1.1 Grothendieck topos

The notion of a Grothendieck topos generalises the aforementioned topological constructions. We start with the definition of a site.

Let $\mathcal{C}$ be a locally small category. A *pretopology* on $\mathcal{C}$ is an assignment of each $A \in \mathbf{Ob}(\mathcal{C})$ of a collection of arrows $\mathrm{Cov}(A)$ (covers of A, or covering sieves) with the following properties:

1. $\{id_A : A \to A\} \in \mathrm{Cov}(A)$
2. If $\{f_x : A_x \to A \mid x \in X\} \in \mathrm{Cov}(A)$ and for each $x \in X$ we have an a_x-cover

$$\{f^x_y : A^x_y \to A_x \mid y \in Y_x\} \in \mathrm{Cov}(A_x)$$

   then

$$\{f_x \circ f^x_y : A^x_y \to A \mid x \in X, y \in Y_x\} \in \mathrm{Cov}(A)$$

3. If $\{f_x : A_x \to A \mid x \in X\} \in \mathrm{Cov}(A)$ and $g : B \to A$ and assume that for each $x \in X$ the pullback of f_x along g exists:

$$\begin{tikzcd} B \times_A A_x \arrow[r] \arrow[d, "g_x"'] & A_x \arrow[d, "f_x"] \\ B \arrow[r, "g"'] & A \end{tikzcd}$$

   then $\{g_x : B \times_A A_x \to B \mid x \in X\} \in \mathrm{Cov}(B)$

A *site* is the pair $(\mathcal{C}, \mathrm{Cov})$ consisting of a category and a pretopology on it.

A Grothendieck topos is a site with extra-conditions that generalise the axioms of topological sheaves in terms of a pretopology. A presheaf of sets over a category $\mathcal{C}$ is a contravariant functor $F : \mathcal{C} \to \mathbf{Set}$

Let Cov be a pretopology on a category $\mathcal{C}$ and $\{f_x : A_x \to A \mid x \in X\} \in \mathrm{Cov}(A)$. Let $x, y \in X$ and we have the pullback of f_x and f_y

$$\begin{tikzcd} A_x \times_A A_y \arrow[r] \arrow[d] & A_y \arrow[d, "f_y"] \\ A_x \arrow[r, "f_x"'] & A \end{tikzcd}$$

If F is a presheaf over $\mathcal{C}$, then we have arrows $F^x_y : F(A_x) \to F(A_x \times_A A_y)$ and $F^y_x : F(A_y) \to F(A_x \times_A A_y)$. Denote F_x as the arrow $F(f_x) : F(A) \to F(A_x)$.

A presheaf F is a sheaf, if for any cover $\{f_x : A_x \to A \mid x \in X\} \in \mathrm{Cov}(A)$, then for all $x, y \in X$ such that for all $s_x \in F(A_x)$ and $s_y \in F(A_y)$ such that $F^x_y(s_x) = F^y_x(y)$, then there exists a unique $s \in F(A)$ such that $F_x(s) = s_x$ for $x \in X$.

Sh(Cov) is the category of sheaves of the site $(\mathcal{C}, \mathrm{Cov})$. A Grothendieck topos is a category of sheaves of some site up to categorical equivalence.

2 Locales and cover systems

3 Kripke-Joyal semantics and quantifiers via adjoint functors

4 Internal logic

5 Geometric morphisms

6 Geometric logic

7 Kripke-Joyal semantics for quantales and non-commutative geometric theories