Notes on Geometric logic

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1 Locales, quantales and cover systems

A frame is a complete lattice $\mathcal{L} = (L, \wedge, \bigvee)$ such that, for all $a \in L$ and $A \subseteq L$:

$$a \wedge \bigvee A = \bigvee \{a \wedge b \mid b \in A\}$$

A frame homeomorphism is a map between frames that preserves 0, 1, $\wedge$ and $\bigvee$. The **Frm** is the category of all frames and homeomorphisms, the category of locales **Loc** is said to be the opposite category of the category of frames.

The notion of a quantale generalises frames. A quantale $\mathcal{Q} = (Q, \cdot, \bigvee)$ is a complete lattice-ordered semigroup such that, for all $a \in Q$ and $A \subseteq Q$:

$$\begin{aligned} a \cdot \bigvee A &= \bigvee \{a \cdot b \mid b \in A\} \\ \bigvee A \cdot a &= \bigvee \{b \cdot a \mid b \in A\} \end{aligned}$$

2 Sheaves, Sites and Grothendieck toposes

Let $\mathcal{I} = (I, \theta)$ be a topological space. Consider θ as a poset. A *p*resheaf over $\mathcal{I}$ is a contravariant functor from θ to **Set**.

The notion of a presheaf generalises essentially the following construction from set-theoretic topology. First of all, we discuss a set-theoretic examples without referring to topology. Consider an indexed family of disjoint sets:

$$\mathcal{A} = \{A_i \mid i \in I\}.$$

We can associate an obvious map $p : A \to I$ since for every $x \in \mathcal{A}$ there is a unique $i \in I$ such that $x \in A_i$. Take

$$p^{-1}(\{i\}) = \{x \mid p(x) = i\} = A_i$$

Such $p^{-1}(\{i\})$ is called the *fibre* over i, the whole structure is a bundle of sets over the base space I, $\mathcal{A}$ is the stalk space (l'espace etale) of the bundle. More generally, we can extract the bundle from every map $p : A \to I$

A morphism of bundles (A, I) and (B, I) is a commutative triangle of the following form:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "p_1"'] & & B \arrow[dl, "p_2"] \\ & I & \end{tikzcd}$$

Topologically, a sheaf is a version of bundles for topological spaces. Let $\mathcal{I} = (I, \theta)$ be a topological space. A sheaf is a tuple $(\mathcal{A}, p)$, where $\mathcal{A}$ is a topological space and $p : A \to I$

is a continuous map, which is also a local homeomorphism, that is, every $x \in \mathcal{A}$ has an open neighbourhood, which mapped homeomorphically by p onto $p(U)$ and $p(U)$ is open in I. The category of all sheaves of I is sometimes called a spatial topos.

We can extract a presheaf from a sheaf (A, f) as a contravariant functor $F_f : \theta \to \mathbf{Set}$ as

$$F_f(V) = \{s : V \to A \mid s \text{ is continuous and } f \circ s = V \hookrightarrow I\}$$

The category of presheafs over I, denoted as $\mathbf{PsC}(I)$, consists of presheafs as objects and natural transformations $\tau : F \Rightarrow G$, that is, a collection of functions $\tau_U : F(U) \to G(U)$ making this square commute whenever $U \subseteq V$

$$\begin{tikzcd}
F(V) \arrow[r, "\tau_V"] \arrow[d, "F_U^V"'] & G(V) \arrow[d, "G_U^V"] \\
F(U) \arrow[r, "\tau_U"'] & G(U)
\end{tikzcd}$$

It is clear that $\mathbf{PsC}(I)$ is equivalent to $\mathbf{Set}^{\theta^{Op}}$.

Let X be an index set and V an open set, an *open cover* of V is a collection of sets $\{V_x\}_{x \in X}$ such that

$$V = \bigcup_{x \in X} V_x$$

Intuitively, a sheaf is a presheaf that preserves open covers.

A *sheaf* is a presheaf F satisfying the following two extra-principles. Let V be an open set and $\{V_x\}_{x \in X}$ an open cover, then:

1. Let $s, t \in F(V)$ be sections such that such that $s|_{V_x} = t|_{V_x}$ for $x \in X$, then $s = t$.
2. Let $\{s_x \in F(V_x)\}_{x \in X}$ be a family of sections. If for all $x, y \in X$ we have $s_x|_{V_x \cap V_y} = s_y|_{V_x \cap V_y}$, then there exists a section $s \in F(V)$ such that $s|_{V_x} = s_x$ for all $x \in X$.

Equivalently, we can reformulate the latter as that $F(V) = \varprojlim_{x \in X} F(V_x)$. The category $\mathbf{Sh}(I)$ is a category of sheaves over I.

2.1 Grothendieck topos

The notion of a Grothendieck topos generalises the aforementioned topological constructions. We start with the definition of a site.

Let $\mathcal{C}$ be a locally small category. A *pretopology* on $\mathcal{C}$ is an assignment of each $A \in \mathbf{Ob}(\mathcal{C})$ of a collection of arrows $\mathrm{Cov}(A)$ (covers of A, or covering sieves) with the following properties:

1. $\{id_A : A \to A\} \in \mathrm{Cov}(A)$
2. If $\{f_x : A_x \to A \mid x \in X\} \in \mathrm{Cov}(A)$ and for each $x \in X$ we have an a_x-cover

   $$\{f^x_y : A^x_y \to A_x \mid y \in Y_x\} \in \mathrm{Cov}(A_x)$$

   then

   $$\{f_x \circ f^x_y : A^x_y \to A \mid x \in X, y \in Y_x\} \in \mathrm{Cov}(A)$$

3. If $\{f_x : A_x \to A \mid x \in X\} \in \mathrm{Cov}(A)$ and $g : B \to A$ and assume that for each $x \in X$ the pullback of f_x along g exists:

$$\begin{tikzcd} B \times_A A_x \arrow[r] \arrow[d, "g_x"'] & A_x \arrow[d, "f_x"] \\ B \arrow[r, "g"'] & A \end{tikzcd}$$

then $\{g_x : B \times_A A_x \to B \mid x \in X\} \in \mathrm{Cov}(B)$

A *site* is the pair $(\mathcal{C}, \mathrm{Cov})$ consisting of a category and a pretopology on it.

A Grothendieck topos is a site with extra-conditions that generalise the axioms of topological sheaves in terms of a pretopology. A presheaf of sets over a category $\mathcal{C}$ is a contravariant functor $F : \mathcal{C} \to \mathbf{Set}$

Let Cov be a pretopology on a category $\mathcal{C}$ and $\{f_x : A_x \to A \mid x \in X\} \in \mathrm{Cov}(A)$. Let $x, y \in X$ and we have the pullback of f_x and f_y

$$\begin{tikzcd} A_x \times_A A_y \arrow[r] \arrow[d] & A_y \arrow[d, "f_y"] \\ A_x \arrow[r, "f_x"'] & A \end{tikzcd}$$

If F is a presheaf over $\mathcal{C}$, then we have arrows $F^x_y : F(A_x) \to F(A_x \times_A A_y)$ and $F^y_x : F(A_y) \to F(A_x \times_A A_y)$. Denote F_x as the arrow $F(f_x) : F(A) \to F(A_x)$.

A presheaf F is a sheaf, if for any cover $\{f_x : A_x \to A \mid x \in X\} \in \mathrm{Cov}(A)$, then for all $x, y \in X$ such that for all $s_x \in F(A_x)$ and $s_y \in F(A_y)$ such that $F^x_y(s_x) = F^y_x(y)$, then there exists a unique $s \in F(A)$ such that $F_x(s) = s_x$ for $x \in X$.

$\mathbf{Sh}(\mathrm{Cov})$ is the category of sheaves of the site $(\mathcal{C}, \mathrm{Cov})$. A Grothendieck topos is a category of sheaves of some site up to categorical equivalence.

Alternatively, one can define a Grothendieck topos in terms of a Grothendieck topology as follows. Define a sieve S as family morphisms in a category $\mathcal{C}$ that behaves as a right ideal:

$$f \in S \text{ implies } f \circ g \in S$$

If S is a sieve on $C \in \mathrm{Ob}(\mathcal{C})$ and $h \in Hom(D, C)$ for any $D \in \mathrm{Ob}(\mathcal{C})$, then

$$h^*(S) = \{g \mid cod(g) = D, g \circ h \in S\}$$

A *Grothendieck topology* on a category $\mathcal{C}$ is a function J that maps every $C \in \mathrm{Ob}(\mathcal{C})$, denoted as $J(C)$ such that:

1. the maximal sieve $t_C = \{f \mid cod(f) = C\} \in J(C)$
2. If $S \in J(C)$, then $h^*(S) \in J(D)$
3. If $S \in J(C)$ and R is a sieve of C such that $h^*(R) \in J(D)$ for all $h : D \to C$, then $R \in J(C)$

Also any $J(C)$ is upward closed.

2.2 Examples

We start with some examples of a site.

Let $\mathcal{T}$ be a small category of topological spaces closed under finite limits and under taking open subspaces. Define Cov as:

$$\{f_i : Y_i \to X \mid i \in I\} \text{ iff each } Y_i \text{ is an open subspace of } X \text{ and } \bigcup_{i \in I} Y_i = X$$

The first axiom holds obviously, the second axiom holds since $\mathcal{T}$ is closed under taking subspaces. The third axiom holds because of the closure under finite limits.

Let H be a frame. One can define a pretopology on a frame by putting:

$$\{a_i \mid i \in I\} \in \mathrm{Cov}(c) \text{ iff } c = \bigvee_{i \in I} a_i$$

2.2.1 The Zariski site

Let $f_1, \ldots, f_m \in \mathbb{C}[x_1, \ldots, x_n]$, the locus of $f_1, \ldots, f_m$ is the set

$$V(f_1, \ldots, f_m) = \{(z_1, \ldots, z_n) \in C^n \mid f_i(z_1, \ldots, z_n) = 0, i = 1, \ldots, m\}$$

Such a locus is called a *complex affine variety.* With every variety V we can associate the following ideal in the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$:

$$I_V = \{f \in \mathbb{C}[x_1, \ldots, x_n] \mid \forall \vec{z} \in V f(\vec{z}) = 0\}$$

Conversly, let I be an ideal in the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$, then we can define the variety

$$V_I = \{(z_1, \ldots, z_n) \in C^n \mid f(z_1, \ldots, z_n) = 0, f \in I\}$$

If $I = (f_1, \ldots, f_m)$, then $V_I = V(f_1, \ldots, f_m)$.

With every ideal I we can associate its radical

$$\sqrt{I} = \bigcup_{0 < r < \omega} \{f \in \mathbb{C}[x_1, \ldots, x_n] \mid f^r \in I\}$$

According to the Hilbert Nullstellensatz, $V_J \neq V_I$ whenever $\sqrt{I} \neq \sqrt{J}$.

The maximal ideals in $\mathbb{C}[x_1, \ldots, x_n]$ have the form $(x_1 - a_1, \ldots, x_n - a_n)$, so the corresponding variety is merely the singleton $\{(a_1, \ldots, a_n)\}$, a minimal algebraic variety. A prime ideal P (that is, $fg \in P$ implies $g \in P$) in $\mathbb{C}[x_1, \ldots, x_n]$ is a radical ideal. The corresponding variety of P is irreducible, that is, it cannot be represented as the union of a finite number of smaller ideals. Moreover, every radical ideal can be represented as the intersection of some finite number of prime ideals. Dually, every complex affine variety can be represented as the union of some finite number of irreducible varieties.

3 Kripke-Joyal semantics and quantifiers via adjoint functors

4 Internal logic

5 Geometric morphisms

6 Geometric logic

7 Kripke-Joyal semantics for quantales and non-commutative geometric theories