

Functional programming, Seminar No. 6

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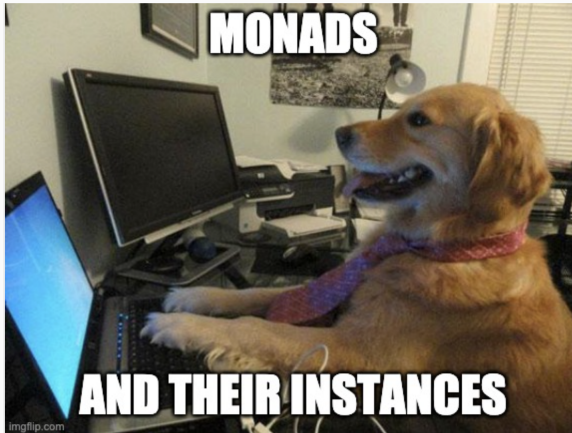
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Today

We will study



Monads

Motivation

We are going to extend pure functions $a \rightarrow b$, we would like to extend them to computations with effects:

- A computation with a possible failure: $a \rightarrow \text{Maybe } b$
- A many-valued computation: $a \rightarrow [b]$
- A computation either succeeds or yields an error: $a \rightarrow \text{Either } e \ b$
- A computation with logs: $a \rightarrow (s, \ b)$
- A computation with reading from an external environment $a \rightarrow (e \rightarrow b)$
- A computation with a mutable state: $a \rightarrow (\text{State } s) \ b$
- An input/output computation: $a \rightarrow \text{IO } b$
- A generalised version of such a function is a Kleisli function $a \rightarrow m \ b$

Motivation

If one needs to provide a uniform interface to deal with Kleisli functions, then this interface should satisfy the following two requirements.

1. One needs to have an opportunity inject a pure value into the computational context
2. Kleisli maps should be composable:

$$(=>=>) :: (a \rightarrow m\ b) \rightarrow (b \rightarrow m\ c) \rightarrow a \rightarrow m\ c$$

3. In general, we **cannot** extract a from $m\ a$

The definition of the Monad class

Let us take a look the full definition of the Monad class

```
class Applicative m => Monad m where
  -- | Sequentially compose two actions,
  -- | passing any value produced
  -- | by the first as an argument to the second.
  (>>=) :: m a -> (a -> m b) -> m b

  -- | Sequentially compose two actions,
  -- | discarding any value produced by the first
  (>>) :: m a -> m b -> m b
  m >> k = m >>= \_ -> k

  -- | Inject a value into the monadic type.
  return :: a -> m a
  return = pure
```

The definition of the `Monad` class

The `Monad` class has the equivalent definition, the following one:

```
class Applicative m => Monad m where  
  join :: m (m a) -> m a
```

The definition of the `Monad` class

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  join :: m (m a) -> m a
```

Moreover, such a definition is closer to the original categorical definition of a monad.

The return function

One may convert any pure function into a Kleisli one:

```
toKleisli :: Monad m => (a -> b) -> a -> m b
toKleisli f = return . f
```

```
cosM :: (Monad m, Floating b) => b -> m b
cosM = toKleisli cos
```

It is clear that $\cos \pi = -1$, but `cosM pi` has the type `(Monad m, Floating b) => m b` and we have several variants:

```
> piM :: Maybe Double
Just (-1.0)
> piM :: [Double]
[-1.0]
> piM :: IO (Double)
-1.0
> piM :: Either String Double
Right (-1.0)
```

The monadic bind operator

Take a look at the monadic type signature closely:

```
(>>=) :: Monad m => m a -> (a -> m b) -> m b
```

In some sense, it is quite close to the reverse application operator.

```
(&) :: a -> (a -> b) -> b
```

```
x & f = f x
```

The monadic bind operator

Let's have a look at this analogy closely:

```
fmap :: Functor f => (a -> b) -> f a -> f b
(<*>) :: Applicative f => f (a -> b) -> f a -> f b
flip (>>=) :: Monad m => (a -> m b) -> m a -> m b
```

Flipped fmap, flipped (<*>) and (>>=):

```
flip fmap :: Functor f => f a -> (a -> b) -> f b
flip (<*>) :: Applicative f => f a -> f (a -> b) -> f b
(>>=) :: Monad m => m a -> (a -> m b) -> m b
```

The very first monad. The Identity type

Let us define the following new type

```
{-# LANGUAGE DeriveFunctor #-}
```

```
newtype Identity a = Identity { runIdentity :: a }  
    deriving (Show, Functor)
```

```
instance Applicative Identity where  
    pure = Identity  
    Identity f <*> Identity x = Identity (f x)
```

```
instance Monad Identity where  
    Identity x >>= k = k x
```

This is a trivial monad.

Playing with the Identity monad

Let us consider a quite trivial example of a Kleisli function

```
cosId, acosId, sinM
  :: Double -> Identity Double
cosId = Identity . cos
acosId = Identity . acos
sinM = Identity . sin
```

An example:

```
> runIdentity $ cosId pi >=> acosId
-1.0
> runIdentity $ cosId pi >=> acosId
3.141592653589793
> runIdentity $ cosId (pi/2) >=> acosId >=> sinM
1.0
```

In fact, `>=>` works similarly to `(&)` in this example.

Some of useful monadic functions

Let us take a look at some widely used monadic operations:

```
(>=>) :: Monad m => (a -> m b) -> (b -> m c) -> (a -> m c)  
f >=> g = \x -> f x >>= g
```

```
join :: Monad m => m (m a) -> m a  
join x = x >>= id
```

```
forever :: Applicative f => f a -> f b  
forever a = let a' = a *> a' in a'
```

Monadic laws

Any monad satisfies the following law:

1. The left identity law:

$$\text{return } a \gg= k = k \ a$$

2. The right identity law:

$$m \gg= \text{return} = m$$

3. The monadic bind operation is associative:

$$m \gg= (\lambda x \rightarrow k \ x \gg= h) = (m \gg= k) \gg= h$$

4. There is the strong connection between the notions of monad and monoid, but we drop this connection.
5. Let us illustrate these laws with the `Identity` monad

The identity laws

According to the identity laws:

```
return a >>= k = k a  
m >>= return = m
```

the return function is a sort of a neutral element:

```
> runIdentity $ cosId (pi / 4)  
0.7071067811865476  
> runIdentity $ return (pi / 4) >>= cosId  
0.7071067811865476  
> runIdentity $ cosId (pi / 4) >>= return  
0.7071067811865476
```


The associativity law

The associative law:

$$m \gg= (\backslash x \rightarrow k \ x \gg= h) \quad = \quad (m \gg= k) \gg= h$$

claims that the monadic bind is associative as follows:

```
> runIdentity $ cosId (pi/2) >>= acosId >>= sinM
1.0
> runIdentity $ cosId (pi/2) >>= (\x -> acosId x >>= sinM)
1.0
```

The associativity law

Let us take a look at these equivalent pipelines:

```
go = cosId (pi/2) >>=  
    acosId      >>=  
    sinM
```

```
go2 = cosId (pi/2) >>= (\x ->  
    acosId x         >>= (\y ->  
    sinM y           >>= \z ->  
    return z))
```

Monads and pseudo-imperative programming

```
go2 = cosId (pi/2) >>= (\x ->  
    acosId x      >>= (\y ->  
    sinM y        >>= \z ->  
    return z))
```

```
go2 = cosId (pi/2) >>= (\x ->  
    acosId x      >>= (\y ->  
    sinM y        >>= \z ->  
    return (x, y, z)))
```

Wow, we have recently invented imperative programming!

Monads and pseudoimperative programming

We may ignore one of the results:

```
go2 = let alpha = pi/2 in
      cosId alpha >>= (\x ->
        acosId x      >>= (\y ->
          sinM y       >>
            return (alpha, x, y)))
```

do-Notation

In Haskell, one has a quite useful syntax sugar to write code within a monad in the “imperative” fashion.

do-expression

```
do { e1; e2 }
```

Unsugared version

```
e1 >> e2
```

do-expression

```
do { p <- e1; e2 }
```

Unsugared version

```
e1 >>= \p -> e2
```

do-expression

```
do { let v = e1; e2 }
```

Unsugared version

```
let v = e1 in do e2
```

do-Notation. Example

The example above:

```
go2 = let alpha = pi/2 in
      cosId alpha  >>= (\x ->
        acosId x    >>= (\y ->
          sinM y      >>
            return (alpha, x, y)))
```

One may rewrite this example using do-notation:

```
go2 = do
  let alpha = pi/2
  x <- cosId alpha
  y <- acosId x
  z <- sinM y
  return (alpha, x, y)
```

do-Notation. Example

Let us consider an example of a monadic function:

```
prodM :: Monad m => (a -> m b) -> (c -> m d)
      -> m (a, c) -> m (b, d)
prodM f g mp =
  mp >>= \ (a,b) -> f a >>= \ c -> g b >>= \ d ->
  return (c, d)
```

The function above might be implemented as follows with the do-notation sugar

```
prodM :: Monad m => (a -> m b) -> (c -> m d)
      -> m (a, c) -> m (b, d)
prodM f g mp = do
  (a, b) <- mp
  c <- f a
  d <- g b
  return (c, d)
```

The Maybe monad

The Maybe monad

The Maybe data type is one of the simplest non-trivial monads.

```
instance Monad Maybe where
```

```
  return = Just
```

```
  Nothing >=> _ = Nothing
```

```
  (Just x) >=> f = f x
```

```
  (Just _) >> a = a
```

```
  Nothing >> _ = Nothing
```

The Maybe monad. Example

```
type Author = String
type Book = String
type Library = [(Author, Book)]
```

```
books :: [Book]
books = ["Faust", "Alice in Wonderland", "The Idiot"]
```

```
authors :: [Author]
authors = ["Goethe", "Carroll", "Dostoevsky"]
```

```
library :: Library
library = zip authors books
```

The Maybe monad. Example

```
library' :: Library
library' = ("Dostoevsky", "Demons") :
  ("Dostoevsky", "White Nights") : library
```

```
getBook :: Author -> Library -> Maybe Book
getBook author library = lookup author library
```

```
getSecondbook, getLastBook :: Author -> Maybe Book
getFirstbook author = do
  let lib' = filter (\p -> fst p == author) library'
  book <- getBook author lib'
  return book
```

```
getLastBook author = do
  let lib' = filter (\p -> fst p == author) library'
  book <- getBook author (reverse lib')
  return book
```

The list monad

The list instance

The `Monad` instance is the following one:

```
instance Monad [] where
  return x = [x]
  xs >>= k = concat (map k xs)
```

List comprehension once more

The following functions are equivalent:

```
cartesianProduct :: [a] -> [b] -> [(a, b)]  
cartesianProduct xs ys =  
  xs >>= \x -> ys >>= \y -> return (x, y)
```

```
cartesianProduct' :: [a] -> [b] -> [(a, b)]  
cartesianProduct' xs ys = do  
  x <- xs  
  y <- ys  
  return (x, y)
```

```
cartesianProduct'' :: [a] -> [b] -> [(a, b)]  
cartesianProduct'' xs ys = [(x, y) | x <- xs, y <- ys]
```