Functional programming, Seminar No. 6

Danya Rogozin
Institute for Information Transmission Problems, RAS
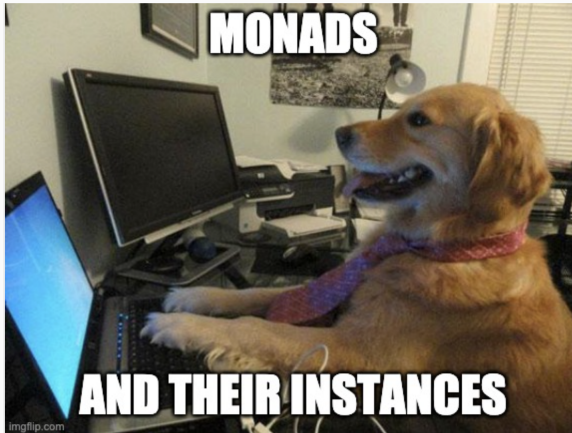Serokell OÜ
Higher School of Economics
The Department of Computer Science

We will study

Monads

Motivation

We are going to extend pure functions `a -> b`, we would like to extend them to computations with effects:

- A computation with a possible failure: `a -> Maybe b`
- A many-valued computation: `a -> [b]`
- A computation either succeeds or yields an error: `a -> Either e b`
- A computation with logs: `a -> (s, b)`
- A computation with reading from an external environment `a -> (e -> b)`
- A computation with a mutable state: `a -> (State s) b`
- An input/output computation: `a -> IO b`
- A generalised version of such a function is a Kleisli function `a -> m b`

If one needs to provide a uniform interface to deal with Kreisli functions, then this interface should satisfy the following two requirements.

1. One needs to have an opportunity inject a pure value into the computational context
2. Kleisli maps should be composable:

    `(>=>) :: (a -> m b) -> (b -> m c) -> a -> m c`

3. In general, we cannot extract a from m a

The definition of the `Monad` class

Let us take a look the full definition of the `Monad` class

```haskell
class Applicative m => Monad m where
  -- | Sequentially compose two actions,
  -- | passing any value produced
  -- | by the first as an argument to the second.
  (>>=) :: m a -> (a -> m b) -> m b

  -- | Sequentially compose two actions,
  -- |discarding any value produced by the first
  (>>) :: m a -> m b -> m b
  m >> k = m >>= \_ -> k

  -- | Inject a value into the monadic type.
  return :: a -> m a
  return = pure
```

The Monad class has the equivalent definition, the following one:

```
class Applicative m => Monad m where
  join :: m (m a) -> m a
```

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```
class Applicative m => Monad m where
  join :: m (m a) -> m a
```

Moreover, such a definition is closer to the original categorical definition of a monad.

One may convert any pure function into a Kleisli one:

```haskell
toKleisli :: Monad m => (a -> b) -> a -> m b
toKleisli f = return . f

cosM :: (Monad m, Floating b) => b -> m b
cosM = toKleisli cos
```

It is clear that $\cos \pi = -1$, but `cosM pi` has the type
`(Monad m, Floating b) => m b` and we have several variants:

```haskell
> cosM pi :: Maybe Double
Just (-1.0)
> cosM pi :: [Double]
[-1.0]
> cosM pi :: IO (Double)
-1.0
> cosM pi :: Either String Double
Right (-1.0)
```

Take a look at the monadic type signature closely:

```
(>>=) :: Monad m => m a -> (a -> m b) -> m b
```

In some sense, it is quite close to the reverse application operator.

```
(&) :: a -> (a -> b) -> b
x & f = f x
```

The monadic bind operator

Let's have a look at this analogy closely:

```
fmap :: Functor f => (a -> b) -> f a -> f b
(<*>) :: Applicative f => f (a -> b) -> f a -> f b
flip (>>=) :: Monad m => (a -> m b) -> m a -> m b
```

Flipped fmap, flipped (<*>) and (>>=):

```
flip fmap :: Functor f => f a -> (a -> b) -> f b
flip (<*>) :: Applicative f => f a -> f (a -> b) -> f b
(>>=) :: Monad m => m a -> (a -> m b) -> m b
```

The very first monad. The `Identity` type

Let us define the following new type

```haskell
{-# LANGUAGE DeriveFunctor #-}

newtype Identity a = Identity { runIdentity :: a }
  deriving (Show, Functor)

instance Applicative Identity where
  pure = Identity
  Identity f <*> Identity x = Identity (f x)

instance Monad Identity where
  Identity x >>= k = k x
```

This is a trivial monad.

Playing with the `Identity` **monad**

Let us consider a quite trivial example of a Kleisli function

```
cosId, acosId, sinM
  :: Double -> Identity Double
cosId = Identity . cos
acosId = Identity . acos
sinM = Identity . sin
```

An example:

```
> runIdentity $ cosId pi >>= acosId
-1.0
> runIdentity $ cosId pi >>= acosId
3.141592653589793
> runIdentity $ cosId (pi/2) >>= acosId >>= sinM
1.0
```

In fact, >>= works similarly to (&) in this example.

Some of useful monadic functions

Let us take a look at some widely used monadic operations:

```
(>=>) :: Monad m => (a -> m b) -> (b -> m c) -> (a -> m c)
f >=> g = \x -> f x >>= g

join :: Monad m => m (m a) -> m a
join x = x >>= id

forever :: Applicative f => f a -> f b
forever a = let a' = a *> a' in a'
```

Monadic laws

Any monad satisfies the following law:

1. The left identity law:

   ```
   return a >>= k  = k a
   ```

2. The right identity law:

   ```
   m >>= return  = m
   ```

3. The monadic bind operation is associative:

   ```
   m >>= (\x -> k x >>= h)  =  (m >>= k) >>= h
   ```

4. There is the strong connection between the notions of monad and monoid, but we drop this connection.

5. Let us illustrate these laws with the `Identity` monad

The identity laws

According to the identity laws:

```
return a >>= k = k a
m >>= return = m
```

the return function is a sort of a neutral element:

```
> runIdentity $ cosId (pi / 4)
0.7071067811865476
> runIdentity $ return (pi / 4) >>= cosId
0.7071067811865476
> runIdentity $ cosId (pi / 4) >>= return
0.7071067811865476
```

The associative law:

```
m >>= (\x -> k x >>= h)  =  (m >>= k) >>= h
```

claims that the monadic bind is associative as follows:

```
> runIdentity $ cosId (pi/2) >>= acosId >>= sinM
1.0
> runIdentity $ cosId (pi/2) >>= (\x -> acosId x >>= sinM)
1.0
```

The associativity law

Let us take a look at these equivalent pipelines:

```
go = cosId (pi/2) >>=
     acosId       >>=
     sinM

go2 = cosId (pi/2) >>= (\x ->
      acosId x      >>= (\y ->
      sinM y        >>= \z ->
      return z))
```

Monads and pseudo-imperative programming

```
go2 = cosId (pi/2) >>= (\x ->
      acosId x      >>= (\y ->
      sinM y        >>= \z ->
      return z))

go2 = cosId (pi/2) >>= (\x ->
      acosId x      >>= (\y ->
      sinM y        >>= \z ->
      return (x, y, z)))
```

Wow, we have recently invented imperative programming!

Monads and pseudoimperative programming

We may ignore one of the results:

```
go2 = let alpha = pi/2 in
      cosId alpha  >>= (\x ->
      acosId x     >>= (\y ->
      sinM y       >>
      return (alpha, x, y)))
```

do-Notation

In Haskell, one has a quite useful syntax sugar to write code within a monad in the "imperative" fashion.

do-expression

```
do { e1; e2 }
```

Unsugared version

```
e1 >> e2
```

do-expression

```
do { p <- e1; e2 }
```

Unsugared version

```
e1 >>= \p -> e2
```

do-expression

```
do { let v = e1; e2 }
```

Unsugared version

```
let v = e1 in do e2
```

do-Notation. Example

The example above:

```
go2 = let alpha = pi/2 in
      cosId alpha  >>= (\x ->
      acosId x     >>= (\y ->
      sinM y       >>
      return (alpha, x, y)))
```

One may rewrite this example using do-notation:

```
go2 = do
  let alpha = pi/2
  x <- cosId alpha
  y <- acosId x
  z <- sinM y
  return (alpha, x, y)
```

do-Notation. Example

Let us consider an example of a monadic function:

```
prodM :: Monad m => (a -> m b) -> (c -> m d)
  -> m (a, c) -> m (b, d)
prodM f g mp =
  mp >>= \(a,b) -> f a >>= \c -> g b >>= \d ->
  return (c, d)
```

The function above might be implemented as follows with the do-notation sugar

```
prodM :: Monad m => (a -> m b) -> (c -> m d)
  -> m (a, c) -> m (b, d)
prodM f g mp = do
  (a, b) <- mp
  c <- f a
  d <- g b
  return (c, d)
```

The Maybe monad

The Maybe monad

The Maybe data type is one of the simplest non-trivial monads.

```
instance Monad Maybe where
  return = Just

  Nothing  >>= _ = Nothing
  (Just x) >>= f = f x

  (Just _) >> a = a
  Nothing >> _ = Nothing
```

The Maybe monad. Example

```haskell
type Author = String
type Book = String
type Library = [(Author, Book)]

books :: [Book]
books = ["Faust", "Alice in Wonderland", "The Idiot"]

authors :: [Author]
authors = ["Goethe", "Carroll", "Dostoevsky"]

library :: Library
library = zip authors books
```

The Maybe monad. Example

```haskell
library' :: Library
library' = ("Dostoevsky", "Demons") :
  ("Dostoevsky", "White Nights") : library

getBook :: Author -> Library -> Maybe Book
getBook author library = lookup author library

getSecondbook, getLastBook :: Author -> Maybe Book
getFirstbook author = do
  let lib' = filter (\p -> fst p == author) library'
  book <- getBook author lib'
  return book

getLastBook author = do
  let lib' = filter (\p -> fst p == author) library'
  book <- getBook author (reverse lib')
  return book
```

The list monad

The list instance

The Monad instance is the following one:

```
instance Monad [] where
  return x = [x]
  xs >>= k = concat (map k xs)
```

List compeherension once more

The following functions are equivalent:

```haskell
cartesianProduct :: [a] -> [b] -> [(a, b)]
cartesianProduct xs ys =
  xs >>= \x -> ys >>= \y -> return (x, y)

cartesianProduct' :: [a] -> [b] -> [(a, b)]
cartesianProduct' xs ys = do
  x <- xs
  y <- ys
  return (x, y)

cartesianProduct'' :: [a] -> [b] -> [(a, b)]
cartesianProduct'' xs ys = [(x, y) | x <- xs, y <- ys]
```