Semantic and Proof-Theoretic Investigation of Noncommutative Geometric Logic

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1 Background

2 Definitions

3 Quantale, Cover Systems and Other Concepts

Definition 3.1. A *quantale* is a structure $\mathcal{Q} = (Q, \bigvee, \cdot, \varepsilon)$ where $(Q, \cdot, \varepsilon)$ is a monoid and $(Q, \bigvee)$ is a sup-lattice such that the following holds for each $a \in Q$ and for each indexed family $\{a_i \in Q \mid i \in I\}$

$$a \cdot \bigvee_{i \in I} a_i = \bigvee_{i \in I} (a \cdot a_i)$$
$$\bigvee_{i \in I} a_i \cdot a = \bigvee_{i \in I} (a_i \cdot a)$$

Definition 3.2. Let $\mathcal{Q}$ be a quantale, a function $j : \mathcal{Q} \to \mathcal{Q}$ is a *quantic nucleus*, if j satisfies the following for each $a, b \in \mathcal{Q}$

- j is order-preserving,
- $a \leq ja$,
- $jja \leq ja$,
- $ja \cdot jb \leq j(a \cdot b)$,
- $j\varepsilon \leq \varepsilon$.

An element $a \in \mathcal{Q}$ is *j-closed* if $ja = a$. Let:

$$\mathcal{Q}_j = \{a \in \mathcal{Q} \mid ja = a\}$$

Proposition 3.1. Let $j : \mathcal{Q} \to \mathcal{Q}$ be a closure operator on a quantale $\mathcal{Q}$, then j is a quantic nucleus iff for each $a, b \in \mathcal{Q}$ $j(a \cdot b) = j(ja \cdot jb)$.

Proof. Take any $a, b \in \mathcal{Q}$. Assume $j(a \cdot b) = j(ja \cdot jb)$, then

$$ja \cdot jb \leq j(ja \cdot jb) = j(a \cdot b).$$

Assume j is a quantic nucleus, then

$$\dfrac{\dfrac{\dfrac{a \leq ja \qquad b \leq jb}{a \cdot b \leq ja \cdot jb}}{j(a \cdot b) \leq j(ja \cdot jb)} \qquad \dfrac{ja \cdot jb \leq j(a \cdot b)}{j(ja \cdot jb) \leq jj(a \cdot b) = j(a \cdot b)}}{j(a \cdot b) = j(a \cdot b)}$$

$\square$

Proposition 3.2. $\mathcal{Q}_j$ forms a subquantale of $\mathcal{Q}$. Moreover, the map $a \mapsto ja$ defined a surjective homomorphism from $\mathcal{Q}$ onto $\mathcal{Q}_j$.

Proof. See the complete proof in [Ros90, Theorem 3.1.1]. Take any elements $a, b \in \mathcal{Q}_j$, any indexed family $\{a_i \in \mathcal{Q}_j \mid i \in I\}$ and define the operations the following way:

- $a \cdot_{\mathcal{Q}_i} b = j(a \cdot_{\mathcal{Q}} b)$
- $\bigvee_{i \in I_{\mathcal{Q}_j}} a_i = j(\bigvee_{i \in I_{\mathcal{Q}}} a_i)$

Note that the identity element is already closed by the definition of a quantic nucleus. $\square$

Let $\mathcal{P} = (P, \leq)$ be a poset and $x \in \mathcal{P}$, the upper cone generated by x is the set $\uparrow x = \{y \in P \mid x \leq y\}$. Let $A \subseteq \mathcal{P}$, define $\uparrow A$ as

$$\uparrow A = \bigcup_{x \in A} \uparrow x$$

A subset set A is upward closed whenever $\uparrow A = A$. We say that y *refines* x if $x \leq y$, or, equivalently, $\uparrow y \subseteq \uparrow x$. We say that a subset Y *refines* if $Y \subseteq \uparrow X$, that is, every element of y refines some element of X. The set $\mathrm{Up}(\mathcal{P})$ is the set of all upward closed subsets of $\mathcal{P}$.

4 Monoidal Grothendieck Topologies and Non-commutative Sites

Let $\mathcal{C}$ be a category and let $X \in \mathcal{C}$, then $\mathcal{C}^{(0)}/_X$ is the full category of $\mathcal{C}/_X$ generated by those maps $U \to X$ which factor through some $V \in \mathcal{C}$.

A *presheaf* is a functor $F : \mathcal{C}^{op} \to \mathrm{Set}$. The *category of all presheaves* on $\mathcal{C}$ is the category of functors $\mathrm{PSh}(\mathcal{C}) = \mathrm{Set}^{\mathcal{C}^{op}}$ and their natural transformations. The Hom-*functor* is a bifunctor $\mathrm{Hom} : \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{Set}$ defined as:

$$\mathrm{Hom}_{\mathcal{C}} : (A, B) \mapsto \{f \in \mathrm{Mor}(\mathcal{C}) \mid f : A \to B\}$$

With each $C \in \mathrm{Ob}(\mathcal{C})$ we can associate a presheaf $\mathbf{y}_C \in \mathrm{PSh}(\mathcal{C})$ on $\mathcal{C}$ defined as:

$$\mathbf{y}_C(D) = \mathrm{Hom}_{\mathcal{C}}(D, C) \text{ for } D \in \mathrm{Ob}(\mathcal{C})(\mathcal{C}).$$
$$\mathbf{y}_C(f) : \mathrm{Hom}_{\mathcal{C}}(D, C) \to \mathrm{Hom}_{\mathcal{C}}(D', C)$$

such that $\mathbf{y}_C(f)(g) = g \circ f$ for $f : D' \to D$ and $g : D \to C$. Functors of the form of $\mathbf{y}_C$ (up to isomorphism) are called *representable* functors.

The *Yoneda embedding* is the following (full and faithful) functor from $\mathcal{C}$ to the category of presheaves:

$$\begin{aligned} \mathbf{y} : \mathcal{C} &\to \mathrm{Set}^{\mathcal{C}^{op}} \\ \mathbf{y}(C) = \mathrm{Hom}_{\mathcal{C}}(-, C) &= \mathbf{y}_C \end{aligned}$$

The following isomorphism

$$\theta : \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(\mathbf{y}_C, P) \cong P(C)$$

is folklore.

We need some basics of coends and the Day convolution, the reader can find more details in [Lor21], [Rie14, Chapter 1] or [Bor94, Chapter 6].

Definition 4.1 (Coend)**.** Let $P : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Set}$ be a profunctor, the coend

$$\int^{X \in \mathcal{C}} P(X, X)$$

is a set with a family of arrows $i_X : P(X, X) \to \int^Z P(Z, Z)$ for each $X \in \mathrm{Ob}(\mathcal{C})$ such that following square commutes for each $f : X \to Y$:

$$\begin{tikzcd} H(Y, X) \arrow[r, "f_*"] \arrow[d, "f^*"'] & H(Y, Y) \arrow[d, "i_Y"] \\ H(X, X) \arrow[r, "i_X"'] & \int^Z P(Z, Z) \end{tikzcd}$$

Equivalently, the coend $\int^Z P(Z, Z)$ can be equivalently defined as the coequaliser of the following diagram:

$$\coprod_f H(\mathrm{cod}(f), \mathrm{dom}(f)) \rightrightarrows \coprod_{X \in \mathrm{Ob}(\mathcal{C})} H(X, X) \dashrightarrow \int^Z H(Z, Z)$$

Definition 4.2 (Day convolution)**.** Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a monoidal category and let $F, G : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ be presheaves. The Day convolution of F and G is the following coend, for each $X \in \mathrm{Ob}(\mathcal{C})$:

$$(F \star G)(X) := \int^{X_1, X_2} \mathrm{Hom}_{\mathcal{C}}(X, X_1 \otimes X_2) \times F(X_1) \times F(X_2).$$

The following is due to Day, see [Day70].

Proposition 4.1. Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a monoidal category, then:

1. $(\mathrm{PSh}(\mathcal{C}), \star, \mathbf{y}(\mathbb{1}))$ is a monoidal category,

2. $\mathcal{C}$ embeds into $(\mathrm{PSh}(\mathcal{C}), \star, \mathbf{y}(\mathbb{1}))$ by the Yoneda embedding.

Definition 4.3 (Monoidal Localisation)**.**

Definition 4.4. Let $\mathcal{C}$ be a category, a collection of morphisms $S \subseteq \mathrm{Hom}(\mathcal{C})$ with a common codomain $X \in \mathrm{Ob}(\mathcal{C})$ is called a *sieve on* X if $f \in S$ implies $g \circ f \in S$ whenever such a composition is well-defined. Let S be a sieve on X and $f : Y \to X$ be an arrow, then the set

$$f^*(S) = \{g \in \mathrm{Hom}(\mathcal{C}) \mid g \circ f \in S\}$$

is a sieve on D.

Let $X \in \mathrm{Ob}(\mathcal{C})$, then *the maximal sieve* generated by X is the following collection:

$$t_X = \{f \mid \mathrm{cod}(f) = X\},$$

Definition 4.5. Let $\{f_i : U_i \to U\}_{i \in I}$ be a family of arrows in a category $\mathcal{C}$. A *refinement* $\{f_j : U_j \to U\}_{j \in J}$ is a family of arrows in $\mathcal{C}$ if for each $j \in J$ there is $i \in I$ such that $f_j : U_j \to U$ factors through $f_i : U_i \to U$:

$$\begin{tikzcd} U_j \arrow[rr, "f_j"] \arrow[dr, dashed] & & U \\ & U_i \arrow[ur, "f_i"'] & \end{tikzcd}$$

There is also a more general definition of refinement for different codomains.

Definition 4.6. Let $U = \{U_i \to X\}_{i \in I}$ be a family of morphisms in $\mathcal{C}$, then $V = \{U_j \to Y\}_{j \in J}$ is a refinement of U is a family of morphisms such that for each $j \in J$ there is $i \in I$ such that the following square commutes

$$\begin{tikzcd} U_j \arrow[r, "f_j"] \arrow[d] & Y \arrow[d] \\ U_i \arrow[r, "f_i"'] & X \end{tikzcd}$$

Definition 4.7. Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a monoidal category, a *monoidal Grothendieck topology* is a function Cov mapping each $X \in \mathrm{Hom}(\mathcal{C})$ to some collections of sieves $\{f : U_i \to X\}_{i \in I}$ called *coverings sieves of* X. Cov satisfies the following properties for each $X, Y \in \mathrm{Ob}(\mathcal{C})$:

1. (*Existence*) $t_X \in \mathrm{Cov}(X)$.

2. (*Transitivity*) Let $S_1 \in \mathrm{Cov}(X)$ and let S_2 be a sieve on X such that $f^*(S_2) \in \mathrm{Cov}(Y)$ for any $f : Y \to X$ in S_1, then $S_2 \in \mathrm{Cov}(X)$.

3. (*Stability*) If $S \in \mathrm{Cov}(X)$, then for any arrow $f : X \to Y$ one $f^*(S) \in \mathrm{Cov}(Y)$.

4. (*Multiplicativity*) If $S_1 \in \mathrm{Cov}(X)$ and $S_2 \in \mathrm{Cov}(Y)$, then there exists a covering sieve $S \in \mathrm{Cov}(X \otimes Y)$ that refines $S_1 \otimes S_2$.

5. (*Identity axiom*) $\mathrm{Cov}(\mathbb{1}) = t_{\mathbb{1}}$.

A monoidal Grothendieck topology is *strong* if it satisfies the following extra-property: let S_1, S_2 be family of arrows and let $X \in \mathrm{Ob}(\mathcal{C})$ such that there exists $S \in \mathrm{Cov}(X)$ such that S refines $S_1 \otimes S_2$, then there exist $X_1, X_2 \in \mathrm{Ob}(\mathcal{C})$ and $f : X_1 \otimes X_2 \to X$ such that there are $S_i' \subseteq S_i$ and $S_i' \in \mathrm{Cov}(X_i)$ for $i \in \{1, 2\}$.

Proposition 4.2. Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a monoidal category and let $\mathrm{Cov} : \mathrm{Hom}(\mathcal{C}) \to \mathrm{Set}$ be a function mapping each object to some collection of sieves, then the following are equivalent:

1. Cov is a monoidal Grothendieck topology,

2. Cov satisfies the axioms of Grothendieck topology, the identity axiom and the following property: if $X \in \mathrm{Ob}(\mathcal{C})$ and $S \in \mathrm{Cov}(X)$, then if $Y \in \mathrm{Ob}(\mathcal{C})$, then $Y \otimes S \in \mathrm{Cov}(Y \otimes X)$ and $S \otimes Y \in \mathrm{Cov}(X \otimes Y)$.

Proof.

1. Assume Cov defines a monoidal Grothendieck topology. Let $X, Y \in \mathrm{Cov}(\mathcal{C})$ and let $S = \{f_i : U_i \to X\}_{i \in I} \in \mathrm{Cov}(X)$, then Consider

$$S \otimes Y = \{f_i \otimes \mathbf{id}_Y : U_i \otimes Y \to X \otimes Y\}$$

then it can be refined to some cover of $X \otimes Y$ by the multiplicativity axiom.

2. Assume Cov defines a Grothendieck topology on $\mathcal{C}$ satisfying the identity axiom and the functors $Y \otimes \_$ and $\_ \otimes Y$ commute with Cov. Take $\{f_i : U_i \to X\}_{i \in I} \in \mathrm{Cov}(X)$ and $\{f_j : U_j \to Y\}_{j \in J} \in \mathrm{Cov}(Y)$, then, in particular one has

$$S = \{f_i \otimes \mathbf{id}_Y : U_i \otimes Y \to X \otimes Y\} \in \mathrm{Cov}(X \otimes Y)$$

Fix any $i \in I$ and $j \in J$ and take $f_i \otimes f_j : U_i \otimes U_j \to X \otimes Y$. The triangle following triangle

$$\begin{tikzcd}
U_i \otimes U_j \arrow[rr, "f_i \otimes f_j"] \arrow[dr, "\mathbf{id}_{U_i} \otimes f_j"'] & & X \otimes Y \\
& U_i \otimes Y \arrow[ur, "f_i \otimes \mathbf{id}_Y"'] &
\end{tikzcd}$$

obviously commutes, so S refines $\{f_i \otimes f_j\}_{i \in I, j \in I}$.

$\square$

Definition 4.8. Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a monoidal category with a monoidal Grothendieck topology Cov. A *sheaf* is a presheaf $F : \mathcal{C}^{op} \to \mathrm{Set}$ such that for every $C \in \mathrm{Ob}(\mathcal{C})$, for every covering sieve $S \in \mathrm{Cov}(C)$ and for every family of the form

$$\{x_f \in F(\mathrm{dom}(f)) \mid f \in S\}$$

such that

$$F(g)(x_f) = x_{g \circ f}$$

for each $f \in S$ and for each arrow g composable with f, there exists a unique $x \in F(C)$ such that $x_f = P(f)(x)$ for each $f \in S$.

The category of monoidal sheaves $\mathbf{MSh}(\mathcal{C}, \mathrm{Cov})$ is the category of sheaves $P : \mathcal{C}^{op} \to \mathrm{Set}$ with natural transformations as morphisms.

Proposition 4.3. Let $(\mathcal{C}, \otimes, \mathbb{1}, \mathrm{Cov})$ be a noncommutative site and let $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$, then the following are equivalent:

1. F is a sheaf,
2. For every covering $\{U_i \to X\}_{i \in I}$, the following diagram

$$F(X) \longrightarrow \prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j \in I} F(U_i \times_X U_j)$$

   is an equaliser.
3. For every covering sieve $\mathcal{C}^0/X \subseteq \mathcal{C}/X$ the map $F(X) \to \mathrm{colim}_{U \in \mathcal{C}^0/X} F(U)$ is a bijection.

Definition 4.9. Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a monoidal category, then $\mathcal{C}$ is a *monoidal topos* if the following axioms are satisfied:

1. $\mathcal{C}$ admits all coproducts and all of them are disjoint,
2. Effective epimorphisms are closed under pullback,
3. Let $f : X \to Y$ in $\mathcal{C}$, the pullback functor $f^* : \mathcal{C}/_Y \to \mathcal{C}/_X$ preserves all coproducts,
4. Every equivalence relation is effective,
5. There exists a set $U \subseteq \mathrm{Ob}(\mathcal{C})$ such that for each $X \in \mathrm{Ob}(\mathcal{C})$ there exists a covering $\{U_i \to X\}_{i \in I}$ where for each $i \in I$ each $U_i \in U$. That is, a covering $\{U_i \to X\}_{i \in I}$ is generated by U.
6. For $X \in \mathrm{Ob}(\mathcal{C})$ and for any indexed family $\{X_i \in \mathrm{Ob}(\mathcal{C}) \mid i \in I\}$ for any index set $I \neq \emptyset$, then there are natural isomorphisms:

$$X \otimes (\operatorname*{colim}_{i \in I} X_i) \cong (\operatorname*{colim}_{i \in I}(X \otimes X_i)),$$
$$(\operatorname*{colim}_{i \in I} X_i) \otimes X \cong (\operatorname*{colim}_{i \in I}(X_i \otimes X)).$$

5 Sheafification and Giraud's Theorem

Fix a category $\mathcal{X}$ such that there is an equivalence $\mathcal{X} \simeq \mathrm{MShv}(\mathcal{C})$ for some monoidal category $\mathcal{C}$.

5.1 Sheafification

First of all, we show the following fact:

Theorem 5.1. Let $(\mathcal{C}, \otimes, \mathbb{1}, \mathrm{Cov})$ be a non-commutative site, then the inclusion function $F : \mathrm{MShv}(\mathcal{C}) \hookrightarrow \mathrm{PSh}(\mathcal{C})$ has a left adjoint $L : \mathrm{PSh}(\mathcal{C}) \to \mathrm{MShv}(\mathcal{C})$. Moreover L is a monoidal functor.

Given a presheaf P, LP is the *sheafification* of P.

Proof. First of all, let us describe the sheafification procedure explicitly. Let $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ be a presheaf, for $X \in \mathrm{Ob}(\mathcal{C})$ we let

$$F^*(X) := \operatorname*{colim}_{\mathcal{C}^0/X} \lim_{U \in \mathcal{C}^0/X} F(U)$$

As far as $\mathcal{C}/X$ is always a covering, we have the following canonical map:

$$F(X) \cong \lim_{U \in \mathcal{C}^0/X} F(U) \xrightarrow{\alpha_F} \operatorname*{colim}_{\mathcal{C}^0/X} \lim_{U \in \mathcal{C}^0/X} \cong F^*(X)$$

We conclude the theorem from the following claim:

Claim 5.1. Let $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ be a presheaf, then the following holds:

1. The composite map of the form
$$F \xrightarrow{\alpha_F} F^* \xrightarrow{\alpha_{F^*}} F^{**}$$
is a sheafification of F. Moreover, the following canonical map is a bijection for every sheaf $G : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$
$$\mathrm{Hom}_{\mathrm{Shv}(\mathcal{C})}(F^**, G) \longrightarrow \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(F, G).$$

2. The functor $F \mapsto F^*$ is left exact and monoidal.

Proof.

1. Let us define a category $\mathrm{Cov}(\mathcal{C})$ the following way:

    - $\mathrm{Ob}(\mathrm{Cov}(\mathcal{C}))$ consists of pairs $(X, \mathcal{C}^0/X)$ where $X \in \mathcal{C}$ and $\mathcal{C}^0/X \subseteq \mathcal{C}/X$ is a covering sieve.
    - Given $(X, \mathcal{C}^0/X)$ and $(Y, \mathcal{C}^0/Y)$, a morphism $(X, \mathcal{C}^0/X) \to (Y, \mathcal{C}^0/Y)$ is a morphism $f : X \to Y$ in $\mathcal{C}$ such that if a morphism $g : U \to X$ belongs to $(X, \mathcal{C}^0/X)$, then $f \circ g$ belongs to $(Y, \mathcal{C}^0/Y)$.

    Let us observe that $\mathrm{Cov}(\mathcal{C})$ is a monoidal category indeed, let

    $$(X, \mathcal{C}^0/X) \otimes (Y, \mathcal{C}^0/Y) := (X \otimes Y, \mathcal{C}^0/(X \otimes Y)).$$

    Note that such a product is well-defined since $\mathcal{C}^0/(X \otimes Y)$ is a covering sieve by the multiplicativity axiom. Further, observe that we have functors $i : \mathcal{C} \to \mathrm{Cov}(\mathcal{C})$ and $\rho : \mathrm{Cov}(\mathcal{C}) \to \mathcal{C}$

2. To show that the functor is (lax) monoidal, let $F, G : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$, then

$$
\begin{gathered}
(F^* \star G^*)(X) = \\
\int^{X_1,X_2} \mathrm{Hom}_{\mathcal{C}}(X, X_1 \otimes X_2) \times F^*(X_1) \times G^*(X_2) = \\
\int^{X_1,X_2} \mathrm{Hom}_{\mathcal{C}}(X, X_1 \otimes X_2) \times \operatorname*{colim}_{\mathcal{C}^0/X_1} \lim_{U_1} F(U_1) \times \operatorname*{colim}_{\mathcal{C}^0/X_2} \lim_{U_2} G(U_2) \cong \\
\int^{X_1,X_2} \mathrm{Hom}_{\mathcal{C}}(X, X_1 \otimes X_2) \times \operatorname*{colim}_{\mathcal{C}^0/_{X_1},\mathcal{C}^0/_{X_2}} (\lim_{U_1} F(U_1) \times \lim_{U_2} G(U_2)) \cong \\
\int^{X_1,X_2} \mathrm{Hom}_{\mathcal{C}}(X, X_1 \otimes X_2) \times \operatorname*{colim}_{\mathcal{C}^0/_{X_1},\mathcal{C}^0/_{X_2}} \lim_{U_1,U_2} (F(U_1) \times G(U_2)) \cong \\
\operatorname*{colim}_{\mathcal{C}^0/_{X_1},\mathcal{C}^0/_{X_2}} \lim_{U_1,U_2} \int^{X_1,X_2} \mathrm{Hom}_{\mathcal{C}}(X, X_1 \otimes X_2) \times F(U_1) \times G(U_2) \cong_? \\
\operatorname*{colim}_{\mathcal{C}^0/_X} \lim_{U} (F \star G)(U) = (F \star G)^*(X).
\end{gathered}
$$

□

□

5.2 Giraud's axioms

Further let us show the following:

Theorem 5.2. $\mathcal{X}$ satisfies:

1. $\mathcal{X}$ has arbitrary colimits and limits.
2. Every equivalence relation in $\mathcal{X}$ is effective.
3. Coproducts in $\mathcal{X}$ are disjoint.
4. Tensors are preserved under arbitrary colimits.
5.

Before proving theorem 5.1, we show the following lemmas.

Lemma 5.1. $\mathcal{X}$ has arbitrary colimits and limits.

Proof. $\mathcal{X}$ has colimits since $\mathrm{PSh}(\mathcal{C})$ has colimits computed pointwise for $X \in \mathrm{Ob}(\mathcal{C})$:

$$(\operatorname{colim} F_\alpha)(X) = \operatorname{colim}(F_\alpha(X))$$

□

Lemma 5.2. Every equivalence relation in $\mathcal{X}$ is effective.

Proof. □

Lemma 5.3. Coproducts in $\mathcal{X}$ are disjoint.

Proof. □

Lemma 5.4. Colimits in $\mathcal{X}$ are universal and there are canonical isomorphisms:

$$\operatorname*{colim}_i (X \otimes X_i) \to X \otimes \operatorname*{colim}_i X_i$$
$$\operatorname*{colim}_i (X_i \otimes X) \to \operatorname*{colim}_i X_i \otimes X$$

Proof. □

6 Proof of Theorem 5.1

Fix a non-commutative site $(\mathcal{C}, \otimes, \mathbb{1}, \mathrm{Cov})$.

Let $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ be a sheaf and let $X \in \mathrm{Ob}(\mathcal{C})$, then we call $x \in F(X)$ a *section of* F *over* X.

Definition 6.1. Let $X \in \mathrm{Ob}(\mathcal{C})$, a *sieve* on X is a full subcategory $\mathcal{C}^{(0)}/_X \subseteq \mathcal{C}/_X$ with the following property: if $U \to V \in \mathcal{C}/_X$ and $V \in \mathcal{C}^{(0)}/_X$, then $U \in \mathcal{C}^{(0)}/_X$.

A sieve $\mathcal{C}^{(0)}$ is a covering if it contains $\{U_i \to X\} \in \mathrm{Cov}(X)$.

Proposition 6.1. Let $\{f_i : U_i \to X\}$ be a family of morphisms with a common codomain X and let $\mathcal{C}^{(0)}/_X$ be a sieve generated by those maps $U \to V$ that factor through some f_i. Then $\mathcal{C}^{(0)}/_X$ is a covering sieve if and only if $\{f_i : U_i \to X\} \in \mathrm{Cov}(X)$.

Proof. □

Proposition 6.2. Let $\{f_i : U_i \to X\}$ be a family of morphisms and let $\mathcal{C}^{(0)}/_X$ be a sieve as in Proposition 6.1. Let $Y \in \mathrm{Ob}(\mathcal{C})$, then $Y \otimes \mathcal{C}^{(0)}/_X$ and $\mathcal{C}^{(0)}/_X \otimes Y$ are covering sieves.

Proof. □

7 Giraud's Theorem

Theorem 7.1. Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a monoidal category, then the following statements are equivalent:

1. $\mathcal{C}$ is equivalent to the category of sheaves with a monoidal Grothendieck topology,

2. There exists a small monoidal category $\mathcal{D}$ such that there is a fully faithful embedding $F : \mathcal{C} \hookrightarrow \mathrm{PSh}(\mathcal{D})$ with the Day convolution monoidal structure such that there is $F^* : \mathrm{PSh}(\mathcal{D}) \to \mathcal{C}$ such that $F^* \dashv F$ and F^* is a monoidal functor preserving finite limits.

3. $\mathcal{C}$ is a monoidal topos.

Proof.

1. (1) $\Rightarrow$ (2)
2. (2) $\Rightarrow$ (3)
3. (3) $\Rightarrow$ (1)

□

The idea of the above lemma is adapted from, for example, [Car18, Theorem 1.1.13].

Proposition 7.1. Let $\mathcal{C}$ be a monoidal topos, let $C, D \in \mathrm{Ob}(\mathcal{C})$ and let $f : C \to D$, then:

1. $\mathrm{Sub}(C)$ forms a quantale.
2. The pullback function $f^* : \mathrm{Sub}(D) \to \mathrm{Sub}(C)$ has a left adjoint $\exists_f : \mathrm{Sub}(C) \to \mathrm{Sub}(D)$ and a right adjoint $\forall_f : \mathrm{Sub}(C) \to \mathrm{Sub}(D)$.

Proof. 1. (a) $\mathrm{Sub}(C)$ is a sup-lattice.

   (b) $(\mathrm{Sub}(\mathcal{C}), \cdot, \varepsilon)$ is a monoid.

   (c) The monoidal distributivity law.

2.

□

Definition 7.1. Let $\mathcal{C}$ be a monoidal topos with a subobject classifier $\Omega \in \mathrm{Ob}(\mathcal{C})$, then a *noncommutative Lawvere-Tierney topology* is an arrow $j : \Omega \to \Omega$ satisfying the following axioms:

1. $j = j \circ \mathrm{true}$:

$$\begin{tikzcd} \top \arrow[r, "\mathrm{true}"] \arrow[dr, "\mathrm{true}"'] & \Omega \arrow[d, "j"] \\ & \Omega \end{tikzcd}$$

2. $j \circ j = j$:

$$\begin{tikzcd} \Omega \arrow[r, "j"] \arrow[dr, "j"'] & \Omega \arrow[d, "j"] \\ & \Omega \end{tikzcd}$$

3. $j \circ \cdot = j \circ \cdot \circ j \otimes j$:

$$\begin{tikzcd}
 & \Omega \otimes \Omega \arrow[dr, "\cdot"] & \\
\Omega \otimes \Omega \arrow[ur, "j\otimes j"] \arrow[d, "\cdot"'] & & \Omega \arrow[d, "j"] \\
\Omega \arrow[rr, "j"'] & & \Omega
\end{tikzcd}$$

4. $j \circ \varepsilon = \varepsilon$:

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "\varepsilon"] \arrow[dr, "\varepsilon"'] & \Omega \arrow[d, "j"] \\
 & \Omega
\end{tikzcd}$$

A *noncommutative site* is a monoidal topos $\mathcal{C}$ equipped with a noncommutative Lawvere-Tierney topology.

Proposition 7.2. Let $\mathcal{C}$ be a monoidal category, then $(\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}, \star, \mathbf{y}(\mathbb{1}))$ has a monoidal subobject classifier.

Proof. As usual, let

$$\Omega(X) = \{S \mid S \text{ is a sieve on } X \text{ in } \mathcal{C}\ \}$$

for $X \in \mathrm{Ob}(\mathcal{C})$. Let $f : Y \to X$ be a morphism, then the morphism $\Omega(f) : \Omega(X) \to \Omega(Y)$ is defined as follows, for S, a sieve on X:

$$\Omega(f)(S) = \{g \mid g \circ f \in S\}.$$

□

Theorem 7.2. Let $(\mathcal{C}, \otimes, I)$ be a monoidal category, then $(\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}, \star, \mathbf{y}(\mathbb{1}))$ is a monoidal topos.

Proof. □

The following is a categorical generalisation of [Gol06, Theorem 5]. One can think of it as a noncommutative generalisation of [MM12, §V.1, Theorem 2].

Theorem 7.3. Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a monoidal category with a monoidal Grothendieck topology Cov, then Cov determines a noncommutative Lawvere-Tierney topology on $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$.

Proof. □

Moreover

Theorem 7.4. Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a monoidal category, then there is a bijection between noncommutative Grothendieck topologies on $\mathcal{C}$ and noncommutative Lawvere-Tierney topologies on the monoidal topos $(\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}, \star, \mathbf{y}(\mathbb{1}))$.

8 Infinitary Substructural Logic

Definition 8.1. A first-order signature is a triple $\Omega = (\mathrm{Sort}, \mathrm{Fn}, \mathrm{Rel})$ where

- Sort is a set of *sorts*,
- Fn is a set of *function symbols*. We associate a type with every $f \in \mathrm{Fn}$ written as

$$f : A_1, A_2, \ldots, A_n \to A$$

  where $A_1, A_2, \ldots, A_n, A \in \mathrm{Sort}$,

Rel is a set of relation symbols. As above, we associate a type with every $R \in \mathrm{Rel}$:

$$R \hookrightarrow A_1, A_2, \ldots, A_n.$$

We associate the set of individual variables $\{v_n : A \mid n < \omega\}$ with each sort $A \in \mathrm{Sort}$, so we define terms the following way:

Definition 8.2. The collection of terms over a signature Σ is defined inductively:

- Every variable $v : A$ is a term of sort A,
- Let $t_1 : A_1, \ldots, t_n : A_n$ be Σ-terms and let $f : A_1, A_2, \ldots, A_n \to A$ be a function symbol, then $f(t_1, \ldots, t_n)$ is a term of sort A.

Let t be a term, then the set of free variables FV is defined by induction on t:

$$\begin{aligned}
&\mathrm{FV}(v : A) = \{v : A\} \\
&\mathrm{FV}(f(t_1, \ldots, t_n)) = \bigcup_{1 \leq k \leq n} \mathrm{FV}(t_k)
\end{aligned}$$

Definition 8.3. Let Σ be a first-order signature, the collection of atomic formulas $\mathrm{At}(\Sigma)$ is defined as follows. Let $R \hookrightarrow A_1, \ldots, A_n$ be a relation symbol and let $t_1 : A_1, \ldots, t_n : A_n$ be terms of the corresponding sorts, then

$$R(t_1, \ldots, t_n)$$

is an atomic formula. The set of free variables of an atomic formula is defined as

$$\mathrm{FV}(R(t_1, \ldots, t_n)) = \bigcup_{1 \leq k \leq n} \mathrm{FV}(t_k).$$

Definition 8.4. Let us define a class F of formulas over a signature Σ is defined by joint induction with the corresponding finite sets of free variables:

1. (*Truth*): $\top \in F$ with $\mathrm{FV}(\top) = \emptyset$,

2. (*Falsity*): $\bot \in F$ with $\mathrm{FV}(\bot) = \emptyset$,

3. (*Identity*): $\mathbf{1} \in F$ with $\mathrm{FV}(\mathbf{1}) = \emptyset$,

4. (*Fusion*): if $\varphi, \psi \in F$, then $\varphi \bullet \psi \in F$ with $\mathrm{FV}(\varphi \bullet \psi) = \mathrm{FV}(\varphi) \cup \mathrm{FV}(\psi)$,

5. (*Residuals*): if $\varphi, \psi \in F$, then $\varphi \backslash \psi, \varphi / \psi \in F$ and $\mathrm{FV}(\varphi \backslash \psi) = \mathrm{FV}(\psi / \varphi) = \mathrm{FV}(\varphi) \cup \mathrm{FV}(\psi)$.

6. (*Universal Quantifier*) Let v_i be a variable and let $\varphi \in F$, then $\forall v_i \varphi$ and $\mathrm{FV}(\forall v_i \varphi) = \mathrm{FV}(\varphi) - \{v_i\}$.

7. (*Existential Quantifier*) Let v_i be a variable and let $\varphi \in F$, then $\exists v_i \varphi$ and $\mathrm{FV}(\exists v_i \varphi) = \mathrm{FV}(\varphi) - \{v_i\}$.

8. (*Infinitary Disjunction and Infinitary Conjunction*) Let $\{\varphi_i \mid i \in I\}$ be an indexed set of formulas such that $| \cup_{i \in I} \mathrm{FV}(\varphi_i)| < \omega$, then

$$\bigvee_{i \in I} \varphi_i, \bigwedge_{i \in I} \varphi_i \in F$$

Fix a monoidal topos $\mathcal{C}$. Let $\alpha : U \to X$ be a generalised element with $\operatorname{Im} \alpha \in \operatorname{Sub}(X)$, let

$$U \Vdash \varphi(\alpha) \text{ iff } \operatorname{Im} \alpha \leq \{x \mid \varphi(x)\}.$$

TODO: draw a proper diagram.

Proposition 8.1. The following holds:

1. (*Monotonicity*) If $U \Vdash \varphi(\alpha)$, then for every $f : U' \to U$ in $\mathcal{C}$, then $U' \Vdash \varphi(\alpha \circ f)$.

2. (*Local character*) If $f : U' \twoheadrightarrow U$ and $U' \Vdash \varphi(f \circ \alpha)$, then $U \Vdash \varphi(\alpha)$.

Theorem 8.1. Let $X \in \mathrm{Ob}(\mathcal{X})$ and $\alpha : U \to X$ a generalised element of X. Let $\varphi(x), \psi(x)$ be formulas with a free variable x of sort X, then

1. $U \Vdash \mathbf{1}$ iff $\mathbf{1}$ ...

2. $U \Vdash \bot$ iff $X = \operatorname{colim} \emptyset$.

3. $U \Vdash \top$ iff $X = \lim \emptyset$

4. $U \Vdash (\varphi \bullet \psi)(\alpha)$ iff there $U_1, U_2 \in \mathrm{Ob}(\mathcal{C})$ such that there is arrow $f : U \to U_1 \otimes U_2$ such that $U_1 \Vdash \varphi(\alpha)$ and $U_2 \Vdash \varphi(\alpha)$. TODO: probably wrong.

5. $U \Vdash (\varphi \setminus \psi)(\alpha)$ iff $U_1 \Vdash \varphi(\alpha)$ implies $U_1 \otimes U \Vdash \psi(\alpha)$.

6. $U \Vdash (\psi / \varphi)(\alpha)$ iff $U_1 \Vdash \varphi(\alpha)$ implies $U \otimes U_1 \Vdash \psi(\alpha)$.

7. $U \Vdash (\bigvee\limits_i \varphi)(\alpha)$ iff there exists $\{f_i : U_i \to U\}_{i \in I} \in \mathrm{Cov}(U)$ such that $\coprod\limits_i U_i \twoheadrightarrow U$ is epic and for each $i \in I$ one has $U_i \Vdash \varphi_i(\alpha \circ f_i)$.

8. $U \Vdash (\bigwedge_i \varphi_i)(\alpha)$ iff for each $U \Vdash \varphi_i(\alpha)$.

9. $U \Vdash \exists y \varphi(y, \alpha)$ iff there is $\{U_i \to U\} \in \mathrm{Cov}(U)$ and there is a generalised element $\beta : \bigsqcup_i U_i \to Y$ ...

10. $U \Vdash \forall y \varphi(y, \alpha)$ iff for every $V \in \mathrm{Ob}(\mathcal{C})$ and $p : V \to U$ and for every generalised element $\beta : V \to Y$ such that $V \Vdash \varphi(p \circ \alpha, \beta)$.

11. $U \Vdash \varphi \Rightarrow \psi$ iff $U \Vdash \varphi$ implies $U \Vdash \psi$.

9 Completeness via Morleyisation

10 On Noncommutative Geometric Logic

10.1 One-sorted Version

Let $\{v_i \mid i < \omega\}$ be a set of individual variables and let $\{P_i^k \mid k, i < \omega\}$ be a set of predicate letters where upper indices are the corresponding arities. The grammar of *atomic formulas* is the set *At* of all words of the form

$$P_i^k(v_{n_1}, \dots, v_{n_k})$$

where $v_{n_1}, \dots, v_{n_k}$ are individual variables and P_i^k is a predicate letter of arity k. A *preformula* is an expression of one of the following form:

- Every atomic formula is a preformula,
- $\mathbf{1}$ is a preformula,
- If φ and ψ are preformulas, so is $\varphi \bullet \psi$,
- Let Φ be *any* set of preformulas, then $\bigvee \Phi$ is a preformula,
- Let v be an individual variable and let φ be a preformula, then $\exists v \varphi$,
- Nothing else is a preformula.

Such definitions as free and bound variables are standard.

A *formula* is a preformula with finitely many free variables.

Definition 10.1. *Noncommutative geometric logic* consists of pairs of formulas $\varphi \Rightarrow \psi$ called *sequents*, where $\Rightarrow$ is the metaimplication sign defined with the following axiom schemes and inference rules:

- $\varphi \Rightarrow \varphi$,
- $(\varphi \bullet \psi) \bullet \theta \Leftrightarrow \varphi \bullet (\psi \bullet \theta)$,
- $\psi \bullet \bigvee_{\varphi \in \Phi} \varphi \Leftrightarrow \bigvee_{\varphi \in \Phi} (\psi \bullet \varphi)$,
- $\varphi \Leftrightarrow \varphi \bullet \mathbf{1} \Leftrightarrow \mathbf{1} \bullet \varphi$,
- $\psi \bullet \exists v \varphi \Leftrightarrow \exists v(\psi \bullet \varphi)$ for $v \notin \mathrm{FV}(\psi)$.

$$\frac{\varphi \Rightarrow \psi \qquad \psi \Rightarrow \theta}{\varphi \Rightarrow \theta}$$

$$\frac{\varphi \Rightarrow \psi}{\theta \bullet \varphi \Rightarrow \theta \bullet \psi}$$

$$\frac{\varphi \Rightarrow \psi \qquad v \notin \mathrm{FV}(\psi)}{\exists v \varphi \Rightarrow \psi}$$

- $\varphi[v := w] \Rightarrow \exists v \varphi$,
- $\varphi \Rightarrow \bigvee \Phi$ for $\varphi \in \Phi$,
- $\bigvee_{\varphi \in \Phi} \varphi \bullet \psi \Leftrightarrow \bigvee_{\varphi \in \Phi} (\varphi \bullet \psi)$,
- $\exists v \varphi \bullet \psi \Leftrightarrow \exists v(\varphi \bullet \psi)$ for $v \notin \mathrm{FV}(\psi)$.

$$\frac{\varphi \Rightarrow \psi}{\varphi \bullet \theta \Rightarrow \psi \bullet \theta}$$

$$\frac{\varphi \Rightarrow \psi \qquad \varphi \in \Phi}{\bigvee \Phi \Rightarrow \psi}$$

$$\frac{\exists v \varphi \Rightarrow \psi \qquad v \notin \mathrm{FV}(\psi)}{\varphi \Rightarrow \psi}$$

Let $\mathcal{Q}$ be a quantale and let D be a domain of individuals. With every predicate letter P of arity $k < \omega$, we associate its interpretation in $\mathcal{Q}$ which is a k-ary function $[\![P]\!] : D^k \to \mathcal{Q}$. Triples of the form $\mathfrak{Q} = (\mathcal{Q}, D, [\![.]\!])$ are called *quantale models*. A *variable valuation* is a function $\sigma : \omega \to D$. The value of a geometric formula φ in a quantale model $\mathfrak{Q}$ under a valuation σ is denoted as $[\![\varphi]\!]^{\mathfrak{Q}}_{\sigma}$ and defined by induction:

- $[\![P^k_i(v_{n_1}, \dots, v_{n_k})]\!]^{\mathfrak{Q}}_{\sigma} = [\![P^k_i]\!]^{\mathfrak{Q}}_{\sigma}(\sigma(n_1), \dots, \sigma(n_k))$,
- $[\![\mathbf{1}]\!] = \varepsilon$,
- $[\![\varphi \bullet \psi]\!]^{\mathfrak{Q}}_{\sigma} = [\![\varphi]\!]^{\mathfrak{Q}}_{\sigma} \cdot [\![\psi]\!]^{\mathfrak{Q}}_{\sigma}$,
- $[\![\bigvee \Phi]\!]^{\mathfrak{Q}}_{\sigma} = \bigvee_{\varphi \in \Phi} [\![\varphi]\!]^{\mathfrak{Q}}_{\sigma}$,
- $[\![\exists v_n \varphi]\!]^{\mathfrak{Q}}_{\sigma} = \bigvee_{d \in D} [\![\varphi]\!]^{\mathfrak{Q}}_{\sigma(n \mapsto d)}$.

A sequent $\varphi \Rightarrow \psi$ is true in a quantale model $\mathfrak{Q}$ if $[\![\varphi]\!]^{\mathfrak{Q}}_{\sigma} \leq [\![\psi]\!]^{\mathfrak{Q}}_{\sigma}$.

Theorem 10.1 (Soundness)**.** If a sequent $\varphi \vdash \psi$ is provable, then $[\![\varphi]\!]^{\mathfrak{Q}}_{\sigma} \leq [\![\psi]\!]^{\mathfrak{Q}}_{\sigma}$ in every quantale model.

Proof. The proof is standard. □

In particular, when $\mathcal{C}$ is an ordered monoid, that is, for each $a, b \in \mathcal{C}$ the set $\mathrm{Hom}_{\mathcal{C}}(a, b)$ is at most singleton, we instantiate the above construction the following way.

Thus we have:

Proposition 10.1. Every ordered monoid is embeddable to some quantale.

The proof of the following is a modification of [Gol06, Theorem 4].

Theorem 10.2 (Completeness)**.** If $[\![\varphi]\!]^{\mathfrak{Q}}_{\sigma} \leq [\![\psi]\!]^{\mathfrak{Q}}_{\sigma}$ in every quantale model $\mathfrak{Q}$, then $\varphi \vdash \psi$ is provable.

Proof. A *fragment* $\mathcal{F}$ is a set of formulas such that:

- $\mathbf{1} \in \mathcal{F}$,
- $\varphi, \psi \in \mathcal{F}$ implies $\varphi \bullet \psi \in \mathcal{F}$,
- $\varphi \in \mathcal{F}$ implies $\exists v_n \varphi$,
- $\varphi \in \mathcal{F}$ implies $\mathrm{Sub}(\varphi) \subseteq \mathcal{F}$,
- $\varphi(x) \in \mathcal{F}$ implies $\varphi(x := v_n) \in \mathcal{F}$.

Any set of formulas F can be extended to a fragment the following way by induction. Construct a sequence of increasing sets:

$$F_0 \subseteq F_1 \subseteq \ldots \subseteq F_n \subseteq F_{n+1} \subseteq \ldots \text{ for } n < \omega.$$

where

- $F_0 = F \cup At$,
- $F_{n+1} = F_n \cup \{\varphi \bullet \psi \mid \varphi, \psi \in F_n\} \cup \{\exists v_n \varphi(v_n) \mid \varphi(v_n) \in F_n\}$.

Then we let

$$\mathcal{F} = \bigcup_{n<\omega} F_n$$

and $\mathcal{F}$ is the smallest fragment extending F.

As usual, we define the following equivalence relation on $\mathcal{F}$

$$\varphi \approx \psi \text{ iff } \varphi \vdash \psi \text{ and } \psi \vdash \varphi.$$

□

Definition 10.2.

TODO: monoidal localisation

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