Intuitionistic epistemic logic categorically and algebraically

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Introduction

Intuitionistic modal logic: the big picture

- As it is well known, modal logic extends classical logic with modal operators.
- Applications: topology, proof theory, formal verification, ontologies, etc.
- Intuitionistic modal logic is a version of modal logic where the underlying logic is the intuitionistic one.
- Possible topics where intuitionistic modal logic is of interest:
  - Constructive necessity, provability in intuitionistic arithmetic, intuitionistic knowledge, etc.
  - Model theory: the finite model property, canonicity à la Salqvist, definability à la Thomason-Goldblatt, etc.
  - Representation theory: general descriptive frames, Esakia duality, etc.

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  - Representation theory: general descriptive frames, Esakia duality, etc.

See this summary paper to have the big picture in more detail

- Frank Wolter and Michael Zakharyaschev. Intuitionistic Modal Logic, 1999.

Modalities type theoretically

- Type theory deals with a computation every value in which is annotated with the corresponding data type. Type theory is closely connected with intuitionistic logic and constructive proofs through the Curry-Howard correspondence.
- One can extend Curry-Howard to intuitionistic modal logic and study modal operators within the “types-as-formulas” and “proofs-as-terms” paradigm.
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- Here we think of modal types as abstract data types of action, which is of interest for functional programming.

See the following:

- Gianluigi Bellin, Valeria De Paiva and Eike Ritter. Extended Curry-Howard Correspondence for a Basic Constructive Modal Logic, 2003
- Frank Pfenning and Rowan Davies. A Judgmental Reconstruction of Modal Logic, 2000.
- Peter Nicholas Benton, Gavin M. Bierman, Valeria de Paiva. Computational types from a logical perspective, 1998.
- David Corfield. Modal homotopy type theory: The prospect of a new logic for philosophy, 2020.

Modal type theory based on IEL^{-}

Bridges with functional programming

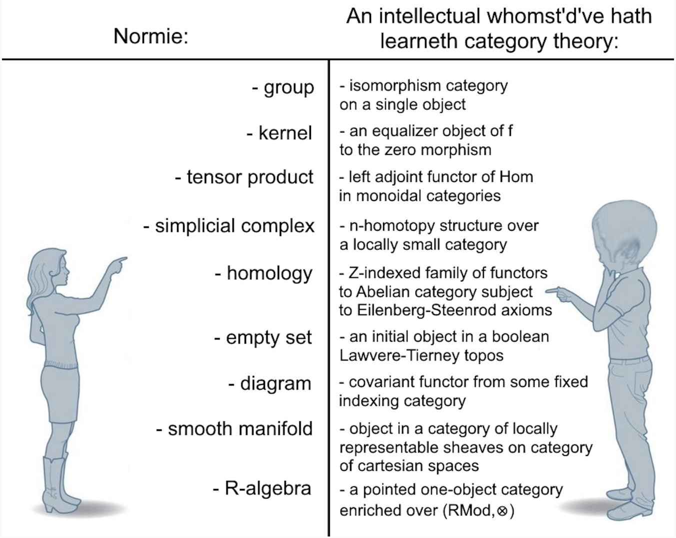
The definition of the type theory

Strong normalisation

The Church-Rosser property

Categorical completeness

Now I am going to be like the guy from the right.



General concepts: Category

Recall that a category $\mathcal{C}$ consists of:

- A class of objects $\mathrm{Ob}(\mathcal{C}) = \{A, B, C, \ldots\}$,
- A class of morphisms $\mathcal{C}(A, B)$ for each $A, B \in \mathrm{Ob}(\mathcal{C})$, where $f : A \to B$ iff $f \in \mathcal{C}(A, B)$,
- For $f : A \to B$ and $g : B \to C$, then $g \circ f : A \to C$ and $h \circ (g \circ f) = (h \circ g) \circ f$ for each f, g, h having an appropriate domain and codomain,
- For each $A, B \in \mathrm{Ob}(\mathcal{C})$ we have identity morphisms such that for each $f : A \to B$ $f \circ id_A = f$ and $id_B \circ g = g$.

Some examples:

- **Set**, the category of all sets and all functions betweem them,
- **Top**, the category of all topological spaces and continuous maps,
- **Vect**$_k$, the category of vector spaces over a field k and linear maps,
- $(P, \leq)$, any poset where $a \to b$ exists iff $a \leq b$,
- Any monoid (as well as a group) is a category, where $\mathrm{Ob}(\mathcal{C})$ is a singleton set (Cayley's theorem).
- etc.

General concepts: Functor

Intuitively, a functor is a morphism of category. Rigorously, let $\mathcal{C}$ and $\mathcal{D}$ be categories, a functor $F : \mathcal{C} \to \mathcal{D}$ is a "function" such that:

- Each $A \in \mathrm{Ob}(\mathcal{C})$ maps to $FA \in \mathrm{Ob}(\mathcal{D})$,
- Each $f : A \to B$ in $\mathcal{C}$ maps to $Ff : FA \to FB$ in $\mathcal{D}$,
- $F(g \circ f) = Fg \circ Ff$ for each f and g.

Some examples:

- The powerset functor $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ such that $\mathcal{P} : A \mapsto 2^A$,
- The abelianisation functor $Ab : \mathbf{Group} \to \mathbf{Ab}$ such that $Ab : G \mapsto G/[G, G]$,
- The spectrum functor $\mathrm{Spec} : \mathbf{Ring}^{op} \to \mathbf{Top}$ that maps every commutative ring to its Zariski space,
- $\mathbf{Field} \to \mathbf{Ring}$ such that $k \mapsto k[X]$,
- $\pi_1 : \mathbf{Top}_* \to \mathbf{Group}$ maps every topological space with a base point to its fundamental group, for example, $\pi_1(S) = \mathbb{Z}$ (up to isomorphism).
- etc.

General concepts: Natural transformation

A natural transformation is a functor morphism. Let $\mathcal{C}, \mathcal{D}$ be categories and $F, G : \mathcal{C} \to \mathcal{D}$ functors. A natural tranformation $\theta : F \Rightarrow G$ is a collection of morphisms $\theta_A : FA \to GA$ in $\mathcal{D}$ making the following square commute for each $f : A \to B$ and $A, B \in \mathrm{Ob}(\mathcal{C})$:

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\theta_A"'] & FB \arrow[d, "\theta_B"] \\ GA \arrow[r, "Gf"'] & GB \end{tikzcd}$$

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An example:

Let det_M be the determinant of the $n \times n$ matrix $M \in \mathrm{GL}_n\, k$ with entries from a field k and k^* is the multiplicative group of k. Both GL_n and $*$ are functors from the category of fields to the category of groups, and $det_M : \mathrm{GL}_n\, k \to k^*$ is a morphism of groups and it is natural:

$$\begin{tikzcd} \mathrm{GL}_n\, k \arrow[r, "f"] \arrow[d, "detM"'] & \mathrm{GL}_n\, k' \arrow[d, "detM'"] \\ k^* \arrow[r, "f^*"'] & k'^* \end{tikzcd}$$

Cartesian closed categories

A category is *cartesian closed* is there are objects $\mathbb{1}$, B^A and $A \times B$ such that:

- $|\mathcal{C}(A, \mathbb{1})| = 1$ for each $A \in \mathrm{Ob}(A)$,
- The following diagrams commute:

$$
\begin{tikzcd}
 & C \arrow[dl, "f"'] \arrow[d, "(f,g)"] \arrow[dr, "g"] & \\
A & A \times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
C^B \times B \arrow[r, "ev"] & C \\
A \times B \arrow[u, "\hat{f} \times id_B"] \arrow[ur, "f"'] &
\end{tikzcd}
$$

The second diagram can be reformulated as (compare with the definition of implication in Heyting algebras):

$$\mathcal{C}(A \times B, C) \simeq \mathcal{C}(A, C^B)$$

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Some examples:

- **Set**,
- Every Heyting algebra,
- The category of G-sets for a group G (the category of group actions),
- The category of simplicial sets (which are also contravariant functors $\Delta : \omega \to$ **Set**).

Typed lambda calculi type-theoretically

Cartesian closed categories allow interpreting intuitionistic type theories using the following scheme:

$$\Gamma \models M : A \text{ iff there exists an arrow } [\![M]\!] : [\![\Gamma]\!] \to [\![A]\!].$$

Typed lambda calculi type-theoretically

Cartesian closed categories allow interpreting intuitionistic type theories using the following scheme:

$$\Gamma \models M : A \text{ iff there exists an arrow } \llbracket M \rrbracket : \llbracket \Gamma \rrbracket \to \llbracket A \rrbracket.$$

In particular, simply typed lambda calculus with types $\to$ and $\times$ has the following interpretation in CCCs.

$$\frac{}{\llbracket \Gamma, x : \varphi \vdash x : \varphi \rrbracket = \pi_2 : \llbracket \Gamma \rrbracket \times \llbracket \varphi \rrbracket \to \llbracket \varphi \rrbracket}$$

$$\frac{\llbracket \Gamma, x : \varphi \vdash M : \psi \rrbracket = \llbracket M \rrbracket : \llbracket \Gamma \rrbracket \times \llbracket \varphi \rrbracket \to \llbracket \psi \rrbracket}{\llbracket \Gamma \vdash (\lambda x.M) : \varphi \to \psi \rrbracket = \widehat{(\llbracket M \rrbracket)} : \llbracket \Gamma \rrbracket \to \llbracket \psi \rrbracket^{\llbracket \varphi \rrbracket}}$$

$$\frac{\llbracket \Gamma \vdash M : \varphi \to \psi \rrbracket = \llbracket M \rrbracket : \llbracket \Gamma \rrbracket \to \llbracket \psi \rrbracket^{\llbracket \varphi \rrbracket} \qquad \llbracket \Gamma \vdash N : \varphi \rrbracket = \llbracket N \rrbracket : \llbracket \Gamma \rrbracket \to \llbracket \varphi \rrbracket}{\llbracket \Gamma \vdash (MN) : \psi \rrbracket = \llbracket \Gamma \rrbracket \xrightarrow{(\llbracket M \rrbracket, \llbracket N \rrbracket)} \llbracket \psi \rrbracket^{\llbracket \varphi \rrbracket} \times \llbracket \varphi \rrbracket \xrightarrow{ev} \llbracket \psi \rrbracket}$$

$$\frac{\llbracket \Gamma \vdash M : \varphi \rrbracket = \llbracket M \rrbracket : \llbracket \Gamma \rrbracket \to \llbracket \varphi \rrbracket \qquad \llbracket \Gamma \vdash N : \psi \rrbracket = \llbracket N \rrbracket : \llbracket \Gamma \rrbracket \to \llbracket B \rrbracket}{\llbracket \Gamma \vdash (M, N) : \varphi \times \psi \rrbracket = (\llbracket M \rrbracket, \llbracket N \rrbracket) : \llbracket \Gamma \rrbracket \to \llbracket \varphi \rrbracket \times \llbracket \psi \rrbracket}$$

$$\frac{\llbracket \Gamma \vdash M : \varphi_1 \times \varphi_2 \rrbracket = \llbracket M \rrbracket : \llbracket \Gamma \rrbracket \to \llbracket \varphi_1 \rrbracket \times \llbracket \varphi_2 \rrbracket}{\llbracket \Gamma \vdash \pi_i M : \varphi_i \rrbracket = \llbracket \Gamma \rrbracket \xrightarrow{\llbracket M \rrbracket} \llbracket \varphi_1 \rrbracket \times \llbracket \varphi_2 \rrbracket \xrightarrow{\pi_i} \llbracket \varphi_i \rrbracket} \; i \in \{1, 2\}$$

Monoidal endofunctors as modalities

We are interested in how to interpret $\Box$-like modality categorically. Recall that one reformulate the **K** axioms of $\Box$ the following way:

- (The multiplicativity axiom)
$$\Box(p \wedge q) \leftrightarrow \Box p \wedge \Box q$$
- (The normality axiom)
$$\Box\top \leftrightarrow \top$$
- (The monotonicity rule)
$$\text{From } \varphi \to \psi \text{ infer } \Box\varphi \to \Box\psi$$

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Categorically, we have an endofunctor $F : \mathcal{C} \to \mathcal{C}$ with the following natural isomorphisms (this is a *strong monoidal endofunctor*):

- $m_{A,B} : F(A \times B) \cong FA \times FB$
- $u : F\mathbb{1} \cong \mathbb{1}$

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The modal lambda calculus Curry-Howard isomorphic to the intuitionstic modal logic **K** with $\Box$ is known to sound and complete w.r.t. CCCs with strong monoidal endofunctors.

See

- Gianluigi Bellin, Valeria De Paiva and Eike Ritter. Extended Curry-Howard Correspondence for a Basic Constructive Modal Logic, 2003
- Y. Kakutani. Call-by-name and call-by-value in normal modal logic, 2007.

IEL$^-$ as a natural transformation

To interpret the **IEL**$^-$ we need the natural transformation $\eta : Id_{\mathcal{C}} \Rightarrow$ **F**, where $\mathcal{C}$ is a CCC and F is a strong monoidal endofunction with the additional principles:

1. $u = \eta_{\mathbb{1}}$
2. For each $A, B \in \mathrm{Ob}(\mathcal{C})$:

$$\begin{tikzcd} A \times B \arrow[r, "\eta_A \times \eta_B"] \arrow[dr, "\eta_{A \times B}"'] & \mathbf{F}A \times \mathbf{F}B \arrow[d, "m_{A,B}"] \\ & \mathbf{F}(A \times B) \end{tikzcd}$$

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An **IEL**$^-$*-category* is a triple $(\mathcal{C}, \mathbf{F}, \eta)$, where $\mathcal{C}$ is a CCC, **F** is a strong monoidal endofunctor and $\eta : Id_{\mathcal{C}} \Rightarrow \mathbf{F}$ is a natural transformation with the additional extra-principles as above.

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Theorem (D. R. 2018)
If M and N are well-typed and $M =_\beta N$, then $[\![M]\!] = [\![N]\!]$. That is, $\lambda_{\mathbf{IEL}^-}$ is sound and complete w.r.t. ***IEL**$^-$-categories.*

Categorical semantics

We skip the complete argument, but we just show how to interpret the modal inference rules in $\mathbf{IEL}^-$-categories:

Kripke-Joyal semantics

Some background: Heyting algebras and locales

Recall that a Heyting algebra is a bounded distributive lattice $\mathcal{H} = (H, \wedge, \vee, \rightarrow, 0, 1)$ with the operation $\rightarrow$ satisfying for all $a, b, c \in H$:

$$a \wedge b \leq c \text{ iff } a \leq b \rightarrow c$$

A locale is a complete lattice $\mathcal{L} = (L, \wedge, \bigvee)$ satisfying for all $a \in L$ and for each indexed family $(a_i)_{i \in I}$:

$$a \wedge \bigvee_{i \in I} a_i = \bigvee_{i \in I} (a \wedge a_i)$$

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Heyting algebras and locales are about:

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- Locales are a lattice-theoretic approximation of a topological spaces,
- Subobject algebras in topoi

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Some references:

- Andre Joyal and Myles Tierney. An extension of the Galois theory of Grothendieck, 1984.
- Francis Borceux. Handbook of Categorical Algebra: Volume 3, Sheaf Theory, 1994.
- Peter Johnstone. Stone spaces, 1984.
- Leo Esakia. Heyting algebras: Duality theory, 2019.

Kripke-Joyal semantics: handwaving motivation

Cover systems

But instead of topoi we will be using a simple kind of structures called *cover systems* introduced by Goldblatt.

Some background: nuclei and prenuclei

The representation theorem for locales with modal operators

Kripke-Joyal semantics for predicate intuitionistic epistemic modal logics