Intuitionistic epistemic logic categorically and algebraically

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Introduction

Intuitionistic modal logic: the big picture

- As it is well known, modal logic extends classical logic with modal operators.
- Applications: topology, proof theory, formal verification, ontologies, etc.
- Intuitionistic modal logic is a version of modal logic where the underlying logic is the intuitionistic one.
- Possible topics where intuitionistic modal logic is of interest:
  - Constructive necessity, provability in intuitionistic arithmetic, intuitionistic knowledge, etc.
  - Model theory: the finite model property, canonicity à la Salqvist, definability à la Thomason-Goldblatt, etc.
  - Representation theory: general descriptive frames, Esakia duality, etc.

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  - Representation theory: general descriptive frames, Esakia duality, etc.

See this summary paper to have the big picture in more detail

- Frank Wolter and Michael Zakharyaschev. Intuitionistic Modal Logic, 1999.

Modalities type theoretically

- Type theory deals with a computation every value in which is annotated with the corresponding data type. Type theory is closely connected with intuitionistic logic and constructive proofs through the Curry-Howard correspondence.
- One can extend Curry-Howard to intuitionistic modal logic and study modal operators within the “types-as-formulas” and “proofs-as-terms” paradigm.
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- Here we think of modal types as abstract data types of action, which are of interest for functional programming.

See the following:

- Gianluigi Bellin, Valeria De Paiva and Eike Ritter. Extended Curry-Howard Correspondence for a Basic Constructive Modal Logic, 2003
- Frank Pfenning and Rowan Davies. A Judgmental Reconstruction of Modal Logic, 2000.
- Peter Nicholas Benton, Gavin M. Bierman, Valeria de Paiva. Computational types from a logical perspective, 1998.
- David Corfield. Modal homotopy type theory: The prospect of a new logic for philosophy, 2020.

Modal type theory based on IEL^{-}

Bridges with functional programming

The definition of the type theory

The modal lambda calculus $\lambda_{\mathbf{IEL}^-}$ is axiomatised with the following inference rules.

$$\frac{}{\Gamma, x : \varphi \vdash x : \varphi}\ \text{ax}$$

$$\frac{\Gamma, x : \varphi \vdash M : \psi}{\Gamma \vdash \lambda x.M : \varphi \to \psi}\ \to_i \qquad \frac{\Gamma \vdash M : \varphi \to \psi \qquad \Gamma \vdash N : \varphi}{\Gamma \vdash MN : \psi}\ \to_e$$

$$\frac{\Gamma \vdash M : \varphi \qquad \Gamma \vdash N : \psi}{\Gamma \vdash \langle M, N \rangle : \varphi \times \psi}\ \times_i \qquad \frac{\Gamma \vdash M : \varphi_1 \times \varphi_2}{\Gamma \vdash \pi_i M : \varphi_i}\ \times_e, i = 1, 2$$

$$\frac{\Gamma \vdash M : \varphi}{\Gamma \vdash \textbf{pure } M : \bigcirc\varphi}\ \bigcirc_I \qquad \frac{\Gamma \vdash \vec{M} : \bigcirc\vec{\varphi} \qquad \vec{x} : \vec{\varphi} \vdash N : \psi}{\Gamma \vdash \textbf{let } \bigcirc \vec{x} = \vec{M} \textbf{ in } N : \bigcirc\psi}\ \text{let}_{\bigcirc}$$

Reduction rules

The reduction rules are defined with the following rewriting rules:

1. $(\lambda x.M)N \to_\beta M[x := N]$.
2. $\pi_1\langle M, N\rangle \to_\beta M$.
3. $\pi_2\langle M, N\rangle \to_\beta N$.
4. $\textbf{let} \bigcirc \vec{x}, y, \vec{z} = \vec{M}, \textbf{let} \bigcirc \vec{w} = \vec{N} \textbf{ in } Q, \vec{P}\ in\ R \to_\beta \textbf{let} \bigcirc \vec{x}, \vec{w}, \vec{z} = \vec{M}, \vec{N}, \vec{P} \textbf{ in } R[y := Q]$.
5. $\textbf{let} \bigcirc \vec{x} = \textbf{pure}\ \vec{M} \textbf{ in } N \to_\beta \textbf{pure}\ N[\vec{x} := \vec{M}]$.
6. $\textbf{let} \bigcirc \_\_ = \_\_ \textbf{ in } M \to_\beta \textbf{pure}\ M$, where __ is an empty sequence of terms.

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The multistep reduction $\twoheadrightarrow_\beta$ is reflexive-transitive closure of $\to_\beta$.

Metatheoretic properties

Theorem (D.R. 2018)

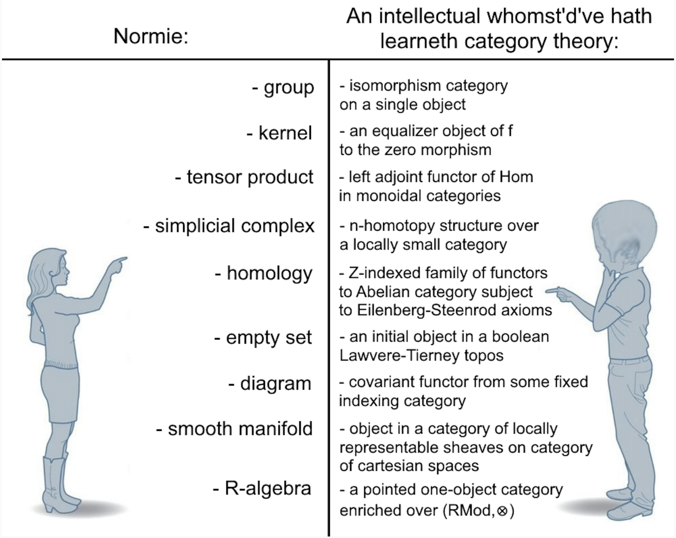
1. *(Type preservation)*
   If $\Gamma \vdash M : \varphi$ *and* $M \twoheadrightarrow_{\beta} N$*, then* $\Gamma \vdash N : \varphi$
2. *(Strong normalisation)*
   Every reduction path terminates, that is, no infinite reduction sequences.
3. *(Church-Rosser)*
   If $M \twoheadrightarrow_{\beta} N_1, N_2$*, then there exists* P *such that* $N_1, N_2 \twoheadrightarrow_{\beta} P$*.*

As a corollary, every $\lambda_{\mathbf{IEL}-term}$ has a unique normal form.

Categorical completeness

Category theory

Now I am going to be like the guy from the right.



General concepts: Category

Recall that a category $\mathcal{C}$ consists of:

- A class of objects $\mathrm{Ob}(\mathcal{C}) = \{A, B, C, \ldots\}$,
- A class of morphisms $\mathcal{C}(A, B)$ for each $A, B \in \mathrm{Ob}(\mathcal{C})$, where $f : A \to B$ iff $f \in \mathcal{C}(A, B)$,
- For $f : A \to B$ and $g : B \to C$, then $g \circ f : A \to C$ and $h \circ (g \circ f) = (h \circ g) \circ f$ for each f, g, h having an appropriate domain and codomain,
- For each $A, B \in \mathrm{Ob}(\mathcal{C})$ we have identity morphisms such that for each $f : A \to B$ $f \circ id_A = f$ and $id_B \circ f = f$.

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Some examples:

- **Set**, the category of all sets and all functions betweem them,
- **Top**, the category of all topological spaces and continuous maps,
- $\mathbf{Vect}_k$, the category of vector spaces over a field k and linear maps,
- $(P, \leq)$, any poset where $a \to b$ exists iff $a \leq b$,
- Any monoid (as well as a group) is a category, where $\mathrm{Ob}(\mathcal{C})$ is a singleton set (Cayley's theorem).
- etc.

General concepts: Functor

Intuitively, a functor is a morphism of category. Rigorously, let $\mathcal{C}$ and $\mathcal{D}$ be categories, a functor $\mathbf{F} : \mathcal{C} \to \mathcal{D}$ is a "function" such that:

- Each $A \in \mathrm{Ob}(\mathcal{C})$ maps to $\mathbf{F}A \in \mathrm{Ob}(\mathcal{D})$,
- Each $f : A \to B$ in $\mathcal{C}$ maps to $\mathbf{F}f : \mathbf{F}A \to \mathbf{F}B$ in $\mathcal{D}$,
- $\mathbf{F}(id_A) = id_{\mathbf{F}A}$
- $\mathbf{F}(g \circ f) = \mathbf{F}g \circ \mathbf{F}f$ for each f and g.

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Some examples:

- The powerset functor $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ such that $\mathcal{P} : A \mapsto 2^A$,
- The abelianisation functor $Ab : \mathbf{Group} \to \mathbf{Ab}$ such that $Ab : G \mapsto G/[G, G]$,
- The spectrum functor $\mathrm{Spec} : \mathbf{Ring}^{op} \to \mathbf{Top}$ that maps every commutative ring to its Zariski space,
- $\mathbf{Field} \to \mathbf{Ring}$ such that $k \mapsto k[X]$,
- $\pi_1 : \mathbf{Top}_* \to \mathbf{Group}$ maps every topological space with a base point to its fundamental group, for example, $\pi_1(S) = \mathbb{Z}$ (up to isomorphism).
- etc.

General concepts: Natural transformation

A natural transformation is a functor morphism. Let $\mathcal{C}, \mathcal{D}$ be categories and $\mathbf{F}, \mathbf{G} : \mathcal{C} \to \mathcal{D}$ functors. A natural tranformation $\theta : \mathbf{F} \Rightarrow \mathbf{G}$ is a collection of morphisms $\theta_A : \mathbf{F}A \to \mathbf{G}A$ in $\mathcal{D}$ making the following square commute for each $f : A \to B$ and $A, B \in \mathrm{Ob}(\mathcal{C})$:

$$\begin{tikzcd}
\mathbf{F}A \arrow[r, "\mathbf{F}f"] \arrow[d, "\theta_A"'] & \mathbf{F}B \arrow[d, "\theta_B"] \\
\mathbf{G}A \arrow[r, "\mathbf{G}f"'] & \mathbf{G}B
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An example:

Let det_M be the determinant of an $n \times n$ matrix $M \in \mathbf{GL}_n k$ with entries from a field k and let k^* be the multiplicative group of k. Both $\mathbf{GL}_n$ and $*$ are functors from the category of fields to the category of groups, and $det_M : \mathbf{GL}_n k \to k^*$ is a morphism of groups and it is natural:

$$\begin{tikzcd} \mathbf{GL}_n k \arrow[r, "f"] \arrow[d, "det_M"'] & \mathbf{GL}_n k' \arrow[d, "det_{M'}"] \\ k^* \arrow[r, "f^*"'] & k'^* \end{tikzcd}$$

Cartesian closed categories

A category is *cartesian closed* is there are objects $\mathbb{1}$, B^A and $A \times B$ such that:

- $|\mathcal{C}(A, \mathbb{1})| = 1$ for each $A \in \mathrm{Ob}(A)$,
- The following diagrams commute:

$$
\begin{tikzcd}
 & C \arrow[dl, "f"'] \arrow[d, "(f,g)"] \arrow[dr, "g"] & \\
A & A \times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
C^B \times B \arrow[rr, "ev"] & & C \\
A \times B \arrow[u, "\hat{f} \times id_B"] \arrow[urr, "f"'] & &
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$$

The second diagram can be reformulated as (compare with the definition of implication in Heyting algebras):

$$\mathcal{C}(A \times B, C) \simeq \mathcal{C}(A, C^B)$$

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Some examples:

- **Set**,
- Every Heyting algebra,
- The category of G-sets for a group G (the category of group actions),
- The category of simplicial sets (which are also contravariant functors $\Delta : \omega \to$ **Set**).

Typed lambda calculi type-theoretically

Cartesian closed categories allow interpreting intuitionistic type theories using the following scheme:

$$\Gamma \models M : A \text{ iff there exists an arrow } [\![M]\!] : [\![\Gamma]\!] \to [\![A]\!].$$

Typed lambda calculi type-theoretically

Cartesian closed categories allow interpreting intuitionistic type theories using the following scheme:

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In particular, simply typed lambda calculus with types $\to$ and $\times$ has the following interpretation in CCCs.

$$\frac{}{[\![\Gamma, x : \varphi \vdash x : \varphi]\!] = \pi_2 : [\![\Gamma]\!] \times [\![\varphi]\!] \to [\![\varphi]\!]}$$

$$\frac{[\![\Gamma, x : \varphi \vdash M : \psi]\!] = [\![M]\!] : [\![\Gamma]\!] \times [\![\varphi]\!] \to [\![\psi]\!]}{[\![\Gamma \vdash (\lambda x.M) : \varphi \to \psi]\!] = \widehat{([\![M]\!])} : [\![\Gamma]\!] \to [\![\psi]\!]^{[\![\varphi]\!]}}$$

$$\frac{[\![\Gamma \vdash M : \varphi \to \psi]\!] = [\![M]\!] : [\![\Gamma]\!] \to [\![\psi]\!]^{[\![\varphi]\!]} \qquad [\![\Gamma \vdash N : \varphi]\!] = [\![N]\!] : [\![\Gamma]\!] \to [\![\varphi]\!]}{[\![\Gamma \vdash (MN) : \psi]\!] = [\![\Gamma]\!] \xrightarrow{([\![M]\!], [\![N]\!])} [\![\psi]\!]^{[\![\varphi]\!]} \times [\![\varphi]\!] \xrightarrow{ev} [\![\psi]\!]}$$

$$\frac{[\![\Gamma \vdash M : \varphi]\!] = [\![M]\!] : [\![\Gamma]\!] \to [\![\varphi]\!] \qquad [\![\Gamma \vdash N : \psi]\!] = [\![N]\!] : [\![\Gamma]\!] \to [\![B]\!]}{[\![\Gamma \vdash (M, N) : \varphi \times \psi]\!] = ([\![M]\!], [\![N]\!]) : [\![\Gamma]\!] \to [\![\varphi]\!] \times [\![\psi]\!]}$$

$$\frac{[\![\Gamma \vdash M : \varphi_1 \times \varphi_2]\!] = [\![M]\!] : [\![\Gamma]\!] \to [\![\varphi_1]\!] \times [\![\varphi_2]\!]}{[\![\Gamma \vdash \pi_i M : \varphi_i]\!] = [\![\Gamma]\!] \xrightarrow{[\![M]\!]} [\![\varphi_1]\!] \times [\![\varphi_2]\!] \xrightarrow{\pi_i} [\![\varphi_i]\!]} \; i \in \{1, 2\}$$

Monoidal endofunctors as modalities

We are interested in how to interpret $\Box$-like modality categorically. Recall that one reformulate the **K** axioms of $\Box$ the following way:

- (The multiplicativity axiom)
$$\Box(p \wedge q) \leftrightarrow \Box p \wedge \Box q$$
- (The normality axiom)
$$\Box\top \leftrightarrow \top$$
- (The monotonicity rule)
$$\text{From } \varphi \to \psi \text{ infer } \Box\varphi \to \Box\psi$$

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Categorically, we have an endofunctor $F : \mathcal{C} \to \mathcal{C}$ with the following natural isomorphisms (this is a *strong monoidal endofunctor*):

- $m_{A,B} : \mathbf{F}(A \times B) \cong \mathbf{F}A \times \mathbf{F}B$
- $u : \mathbf{F}\mathbb{1} \cong \mathbb{1}$

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The modal lambda calculus Curry-Howard isomorphic to the intuitionstic modal logic **K** with $\Box$ is known to sound and complete w.r.t. CCCs with strong monoidal endofunctors. See

- Gianluigi Bellin, Valeria De Paiva and Eike Ritter. Extended Curry-Howard Correspondence for a Basic Constructive Modal Logic, 2003
- Y. Kakutani. Call-by-name and call-by-value in normal modal logic, 2007.

$\mathbf{IEL}^{-}$ as a natural transformation

To interpret the $\mathbf{IEL}^{-}$ modality, we need the natural transformation $\eta : Id_{\mathcal{C}} \Rightarrow \mathbf{F}$ (where $\mathcal{C}$ is a CCC):

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\eta_A"'] & B \arrow[d, "\eta_B"] \\ \mathbf{F}A \arrow[r, "\mathbf{F}f"'] & \mathbf{F}B \end{tikzcd}$$

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But F should be a strong monoidal endofunction with the additional principles:

1. $u = \eta_{\mathbb{1}}$
2. For each $A, B \in \mathrm{Ob}(\mathcal{C})$:

$$\begin{tikzcd} A \times B \arrow[r, "\eta_A \times \eta_B"] \arrow[dr, "\eta_{A \times B}"'] & \mathbf{F}A \times \mathbf{F}B \arrow[d, "m_{A,B}"] \\ & \mathbf{F}(A \times B) \end{tikzcd}$$

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An $\mathbf{IEL}^-$*-category* is a triple $(\mathcal{C}, \mathbf{F}, \eta)$, where $\mathcal{C}$ is a CCC, $\mathbf{F}$ is a strong monoidal endofunctor and $\eta : Id_\mathcal{C} \Rightarrow \mathbf{F}$ is a natural transformation with the additional extra-principles as above.

Categorical semantics

Theorem (D. R. 2018)
If M and N are well-typed and $M =_\beta N$, then $[\![M]\!] = [\![N]\!]$. That is, $\lambda_{\mathbf{IEL}^-}$ is sound and complete w.r.t. $\mathbf{IEL}^-$*-categories.*

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We skip the complete argument, but we just show how to interpret the modal inference rules in $\mathbf{IEL}^-$-categories:

$$\frac{[\![\Gamma \vdash M : \bigcirc\varphi]\!] = [\![M]\!] : [\![\Gamma]\!] \to [\![\varphi]\!]}{[\![\Gamma \vdash \textbf{pure } M : \bigcirc\varphi]\!] = [\![M]\!] \circ \eta_{[\![\varphi]\!]} : [\![\Gamma]\!] \to \mathbf{F}[\![\varphi]\!]}$$

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The interpretation of the second inference rule could be also described via the (slightly modified) quote from Hamlet:

Therefore, since brevity is the soul of wit
and tediousness the limbs and outward flourishes,
I won't be brief.

Kripke-Joyal semantics

Some background: Heyting algebras and locales

Recall that a *Heyting algebra* is a bounded distributive lattice $\mathcal{H} = (H, \wedge, \vee, \rightarrow, 0, 1)$ with the operation $\rightarrow$ satisfying for all $a, b, c \in H$:

$$a \wedge b \leq c \text{ iff } a \leq b \rightarrow c$$

A *locale* (*frame*, *complete Heyting algebra*) is a complete lattice $\mathcal{L} = (L, \wedge, \bigvee)$ satisfying for all $a \in L$ and for each indexed family $(a_i)_{i \in I}$:

$$a \wedge \bigvee_{i \in I} a_i = \bigvee_{i \in I} (a \wedge a_i)$$

Some background: Heyting algebras and locales

Recall that a *Heyting algebra* is a bounded distributive lattice $\mathcal{H} = (H, \wedge, \vee, \to, 0, 1)$ with the operation $\to$ satisfying for all $a, b, c \in H$:

$$a \wedge b \leq c \text{ iff } a \leq b \to c$$

A *locale* (*frame*, *complete Heyting algebra*) is a complete lattice $\mathcal{L} = (L, \wedge, \bigvee)$ satisfying for all $a \in L$ and for each indexed family $(a_i)_{i \in I}$:

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Heyting algebras and locales are about:

- Heyting algebras provide algebraic semantics for intuitionistic and intermediate logics,
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Some references:

- Andre Joyal and Myles Tierney. An extension of the Galois theory of Grothendieck, 1984.
- Francis Borceux. Handbook of Categorical Algebra: Volume 3, Sheaf Theory, 1994.
- Leo Esakia. Heyting algebras: Duality theory, 2019.

Kripke-Joyal semantics: handwaving motivation

Some background: nuclei and prenuclei

A *prenucleus* on a Heyting algebra is a monotone inflationary map that distibutes over finite infima, whereas a *nucleus* is an idempotent prenucleus.

(Pre)nuclei are about:

- Heyting subalgebra and sublocale characterisation,
- a Lawvere-Tierney topology that axiomatises the notion of local truth and also allows defining a Grothendieck topology on a presheaf topos equivalently,
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Some examples of nuclei (and then prenuclei):

- Let $\mathcal{H}$ be a Heyting algebra, then an operator $j : \mathcal{H} \to \mathcal{H}$ such that $j : a \mapsto \neg\neg a$ is a nucleus,
- More generally, an operator $j : \mathcal{H} \to \mathcal{H}$ s.t. $j : a \mapsto (b \to a) \to a$ is a nucleus on $\mathcal{H}$,
- Actually, one can think of the $\mathbf{IEL}^{-}$ modality as a kind of a prenuclear operator,
- We will see a couple of more examples of nuclei further.

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Some references

- Robert Goldblatt. Grothendieck Topology as Geometric Modality, 1981.
- Peter Johnstone. Stone spaces, 1984.
- Martin Escardo. Joins in the frame of nuclei, 2003.

Cover systems

But instead of topoi we will be using a simple kind of structures called *cover systems* introduced by Bell and then developed by Goldblatt.

Let $\mathcal{P} = (P, \leq)$ be a poset, then a cover scheme is a tuple $\mathcal{C} = (\mathcal{P}, \mathrm{Cov})$, where $\mathrm{Cov}(x) \subseteq 2^P$ is the collection of *covers of x* or *x-covers* such that:

- For all $x \in \mathcal{P}$ there exists $C \subseteq \mathcal{P}$ such that $C \in \mathrm{Cov}(x)$ and $C \subseteq \uparrow x$,
- If $C \in \mathrm{Cov}(x)$ and for all $y \in \mathrm{Cov}(C_y)$, then $\bigcup_{y \in C} C_y$,
- If $x \leq y$, then every x-cover can be refined to some y-cover, that is, if $C \in \mathrm{Cov}(x)$, then $C' \in \mathrm{Cov}(y)$ such that $C' \subseteq \uparrow C$,
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An important point, an operator $j : 2^{\mathcal{P}} \to 2^{\mathcal{P}}$:

$$jA = \{x \in \mathcal{P} \mid \exists C \subseteq \mathcal{P}\ C \in \mathrm{Cov}(x) \ \&\ C \subseteq A\}$$

is a nucleus on a locale $\mathrm{Up}(\mathcal{P})$, so j-closed upward closed subsets of $\mathcal{P}$ form a sublocale of $\mathrm{Up}(\mathcal{P})$. Moreover, every locale is representable as a locale of localised subsets of some cover system (that we shall denote as $\mathrm{Loc}(\mathcal{C})$).

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See:

- John Bell. Cover schemes, frame-valued sets and their potential uses in spacetime physics, 2003.
- Robert Goldblatt. Cover semantics for quantified lax logic, 2011.

Intuitionistic predicate logics

In this section, we consider one-sorted first-order language with no function symbols. Intuitionistic predicate logic is defined standardly with the following axiom schemes and rules:

1. $\varphi \to (\psi \to \varphi)$
2. $(\varphi \to (\psi \to \theta)) \to ((\varphi \to \psi) \to (\varphi \to \theta))$
3. $\varphi_1 \wedge \varphi_2 \to \varphi_i$, for $i = 1, 2$,
4. $\varphi \to (\psi \to \varphi \wedge \psi)$
5. $\varphi_i \to \varphi_1 \vee \varphi_2$, for $i = 1, 2$,
6. $(\varphi \to \theta) \to ((\psi \to \theta) \to (\varphi \vee \psi \to \theta))$
7. $\perp \to \varphi$,
8. $\forall x \varphi \to \varphi(t/x)$,
9. $\varphi(t/x) \to \exists x \varphi$,
10. The inference rules are Modus Ponens and Bernays rules.

Kripke-Joyal semantics for predicate intuitionistic logics

A *cover scheme model* is a structure $\mathcal{M} = (\mathcal{P}, \mathrm{Cov}, U, |.|)$, where $(\mathcal{P}, \mathrm{Cov})$ is a cover scheme, $U \neq \emptyset$ is a set of individuals and $|.|$ is an interpretation such that:

- If x is a free variable, then $|x|_{\mathcal{M}} \in U$,
- If P is an n-ary predicate symbol, then $|P|^{\mathcal{M}} : U^n \to \mathrm{Loc}(\mathcal{P}, \mathrm{Cov})$

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The truth definition is standard (in terms of Kripke-Joyal semantics):

- $\mathcal{M}, x \Vdash P(v_1, \ldots, v_n)$ iff $x \in |P|^{\mathcal{M}}(|v_1|_{\mathcal{M}}, \ldots, |v_n|_{\mathcal{M}})$,
- $\mathcal{M}, x \Vdash \bot$ iff $\emptyset \in \mathrm{Cov}(x)$
- $\mathcal{M}, x \Vdash \varphi \wedge \psi$ iff $\mathcal{M}, x \Vdash \varphi$ and $\mathcal{M}, x \Vdash \psi$,
- $\mathcal{M}, x \Vdash \varphi \vee \psi$ iff there exists $C \in \mathrm{Cov}(x)$ such that for each $y \in C$ $\mathcal{M}, y \Vdash \varphi$ or $\mathcal{M}, y \Vdash \psi$,
- $\mathcal{M}, x \Vdash \varphi \to \psi$ iff for each $y \geq x$ $\mathcal{M}, y \Vdash \varphi$ implies $\mathcal{M}, y \Vdash \psi$,
- $\mathcal{M}, x \Vdash \forall v \varphi$ iff $\mathcal{M}, x \Vdash \varphi(v := d)$ for each individual $u \in U$,
- $\mathcal{M}, x \Vdash \exists v \varphi$ iff there exists $C \in \mathrm{Cov}(x)$ and $u \in U$ such that for each $y \in C$ $\mathcal{M}, y \Vdash \varphi(v := u)$.
- $\mathcal{M} \Vdash \varphi$ iff for each $x \in \mathcal{M}$ $\mathcal{M}, x \Vdash \varphi$.

Kripke-Joyal semantics for predicate intuitionistic logics algebraically

With each formula we can associate its truth set $[\![\varphi]\!]_{\mathcal{M}} = \{x \in \mathcal{P} \mid \mathcal{M}, x \Vdash \varphi\}$, so one can show that $[\![.]\!]$ commutes with algebraic operations on the locale of localised upsets:

- $[\![\bot]\!]_{\mathcal{M}} = j\emptyset$
- $[\![\varphi \wedge \psi]\!]_{\mathcal{M}} = [\![\varphi]\!]_{\mathcal{M}} \cap [\![\psi]\!]_{\mathcal{M}}$
- $[\![\varphi \vee \psi]\!]_{\mathcal{M}} = j([\![\varphi]\!]_{\mathcal{M}} \cup [\![\psi]\!]_{\mathcal{M}})$
- $[\![\varphi \to \psi]\!]_{\mathcal{M}} = [\![\varphi]\!]_{\mathcal{M}} \to [\![\psi]\!]_{\mathcal{M}}$
- $[\![\forall v \varphi]\!]_{\mathcal{M}} = \bigcap_{u \in U} [\![\varphi(v := u)]\!]_{\mathcal{M}}$
- $[\![\exists v \varphi]\!]_{\mathcal{M}} = j(\bigcup_{u \in U} [\![\varphi(v := u)]\!]_{\mathcal{M}})$

Theorem (Cover scheme analogue of Goedel completeness)
IPL $\vdash \varphi$ iff $\mathcal{M} \Vdash \varphi$ or, equivalently, $[\![\varphi]\!]_{\mathcal{M}} = \top$ for any model $\mathcal{M}$.

The proof is based on embedding the Lindenbaum-Tarski algebra of *IPL* to a locale, which is isomorphic to the locale of localised upsets of some cover scheme.

The representation theorem for locales with modal operators

Theorem
(D. R. 2020) Every localic prenuclear algebra is isomorphic to the complex algebra of some modal cover scheme.

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These theorems imply the following corollaries:

Corollary
(D. R. 2020)

1. *Every prenuclear algebra is embeddable to the complex algebra of some modal cover scheme.*
2. $\mathbf{QIEL}^{-}$ *is sound and complete w.r.t. its cover schemes.*

Thank you so much indeed!