Intuitionistic epistemic logic categorically and algebraically

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Seminar on Programming principles, Logic and Verification
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Slides available at `https://github.com/DanielRrr/pplv-iel-talk`

Introduction

Intro

This talk is based on the author's master thesis defended at Lomonosov Moscow State University (the faculty of philosophy, the department of logic) under supervision by Vladimir Vasyukov and Vladimir Krupski in 2018. See the extended version of the thesis published in Journal of Logic and Computation:

- Daniel Rogozin. Categorical and algebraic aspects of the intuitionistic modal logic $\mathbf{IEL}^{-}$ and its predicate extensions, 2021.

Outline

1. Intuitionistic modal logic: general intro
2. Intuitionistic modality type-theoretically
3. Type-theoretic approach to intuitionistic belief logic (which is rather about functional programming than beliefs)
4. Categorical semantics
5. Kripke-Joyal semantics and locale representation theory
6. Semantic analysis of predicate intuitionistic epistemic logic

Intuitionistic modal logic: a bird's eye view

- As it is well known, modal logic extends classical logic with modal operators.
- Applications: topology, proof theory, formal verification, ontologies, etc.
- Intuitionistic modal logic is a version of modal logic where the underlying logic is the intuitionistic one.
- Possible topics where intuitionistic modal logic is of interest:
    - Constructive necessity, provability in intuitionistic arithmetic, intuitionistic knowledge, etc.
    - Model theory: the finite model property, canonicity à la Salqvist, definability à la Thomason-Goldblatt, etc.
    - Representation theory: general descriptive frames, Esakia duality, etc.

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  - Representation theory: general descriptive frames, Esakia duality, etc.

See this summary paper to have the big picture in more detail

- Frank Wolter and Michael Zakharyaschev. Intuitionistic Modal Logic, 1999.

Modalities type theoretically

- Type theory deals with a computation every value in which is annotated with the corresponding data type. Type theory is closely connected with intuitionistic logic and constructive proofs through the Curry-Howard correspondence.
- One can extend Curry-Howard to intuitionistic modal logic and study modal operators within the “types-as-formulas” and “proofs-as-terms” paradigm.
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- Here we think of modal types as abstract data types of action, which are of interest for functional programming.

See the following:

- Gianluigi Bellin, Valeria De Paiva and Eike Ritter. Extended Curry-Howard Correspondence for a Basic Constructive Modal Logic, 2003
- Frank Pfenning and Rowan Davies. A Judgmental Reconstruction of Modal Logic, 2000.
- Peter Nicholas Benton, Gavin M. Bierman, Valeria de Paiva. Computational types from a logical perspective, 1998.
- David Corfield. Modal homotopy type theory: The prospect of a new logic for philosophy, 2020.

Modal type theory based on IEL^{-}

Bridges with functional programming

We discuss monadic computation first of all.

- Monads in functional programming describe sequential computations categorically,
- In such languages as Haskell, a monad is a method of structuring a computation as a linearly connected sequence of actions (a *pipeline*) within such types as the list or the input/output (`IO`) data types,
- From a logical perspective, one may consider a computational monad as a modality.

See:

- Eugenio Moggi. Notions of computation and monads, 1991

Functors and monads in functional programming

We start with the Haskell type class called `Functor`.

```
class Functor (f :: * -> *) where
  fmap :: (a -> b) -> f a -> f b
```

This type class reflects (roughly) the categorical functor. That allows one to carry a function of type $a \to b$ through a parametrised data type such as the list type.

The type class `Monad` extends `Functor` with `return` and `(>>=)`:

```
class Functor m => Monad m where
  return :: a -> m a
  (>>=) :: m a -> (a -> m b) -> m b
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- A monad provides a uniform tool for such computations such as computation with a mutable state, many-valued computation, side effect input-output computation, etc.
- That is, all these computations are designed uniformly as “pipelines”.
- Side effects are isolated within a monad and code with “side-effects” is purely functional.

Monads logically and type-theoretically

Logically, monads (over CCCs) are axiomatised via lax logic, intuitionistic modal logic of the following form:

- IPC axioms
- $\bigcirc(\varphi \to \psi) \to (\bigcirc\varphi \to \bigcirc\psi)$
- $\bigcirc\bigcirc\varphi \to \bigcirc\psi$
- $\varphi \to \bigcirc\varphi$
- From $\varphi \to \psi$ and φ infer ψ.

One can also think of $\bigcirc$ as a geometric modality, or a Lawvere-Tierney topology, but lax logic is known to be complete w.r.t. elementary sites.

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The intuitionistic epistemic logic $\mathbf{IEL}^{-}$ is the lax logic without the density axiom. It has been introduced by Artemov and Protopopescu to analyse the notion of belief within the BHK semantics. In particular, the axiom $\varphi \to \bigcirc\varphi$ claims that a rational agent believes in constructively proved statements.

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Some references:

- Sergei Artemov and Tudor Protopopescu. Intuitionistic epistemic logic, 2016.
- Robert Goldblatt. Grothendieck Topology as Geometric Modality, 1981.
- Timothy Williamson. On intuitionistic modal epistemic logic, 1992.

$\mathbf{IEL}^{-}$ computationally

Despite being introduced for the phisophical purposes of formal epistemology, $\mathbf{IEL}^{-}$ also admit interpretation in terms of the type class from Haskell called `Applicative`:

```
class Functor f => Applicative f where
  pure :: a -> f a
  (<*>) :: f (a -> b) -> f a -> f b
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This class is about:

- Generalisation of `Functor` for functions of arbitrary arity,
- Applicative is a weaker version of monadic computation where elements of a pipeline do not depend on one another generally.
- Used in practical parsing libraries

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See

- Conor McBride and Ross Paterson. Applicative programming with effects, 2008.
- Haskell library `optparse-applicative` with a framework of parser composing by means of applicative functors, *https://hackage.haskell.org/package/optparse-applicative*

The definition of the type theory

The modal lambda calculus $\lambda_{\mathbf{IEL}^-}$ is axiomatised with the following inference rules. Strictly speaking, it is Curry-Howard isomorphic to the fragment of $\mathbf{IEL}^-$ with $\to$, $\wedge$ and $\bigcirc$.

$$\frac{}{\Gamma, x : \varphi \vdash x : \varphi}\ \mathrm{ax}$$

$$\frac{\Gamma, x : \varphi \vdash M : \psi}{\Gamma \vdash \lambda x.M : \varphi \to \psi} \to_i \qquad \frac{\Gamma \vdash M : \varphi \to \psi \qquad \Gamma \vdash N : \varphi}{\Gamma \vdash MN : \psi} \to_e$$

$$\frac{\Gamma \vdash M : \varphi \qquad \Gamma \vdash N : \psi}{\Gamma \vdash \langle M, N \rangle : \varphi \times \psi} \times_i \qquad \frac{\Gamma \vdash M : \varphi_1 \times \varphi_2}{\Gamma \vdash \pi_i M : \varphi_i} \times_e, i = 1, 2$$

$$\frac{\Gamma \vdash M : \varphi}{\Gamma \vdash \mathbf{pure}\ M : \bigcirc\varphi} \bigcirc_I \qquad \frac{\Gamma \vdash \vec{M} : \bigcirc\vec{\varphi} \qquad \vec{x} : \vec{\varphi} \vdash N : \psi}{\Gamma \vdash \mathbf{let}\ \bigcirc\ \vec{x} = \vec{M}\ \mathbf{in}\ N : \bigcirc\psi}\ \mathrm{let}_{\bigcirc}$$

Reduction rules

The reduction rules are defined with the following rewriting rules:

1. $(\lambda x.M)N \to_\beta M[x := N]$.
2. $\pi_1\langle M, N\rangle \to_\beta M$.
3. $\pi_2\langle M, N\rangle \to_\beta N$.
4. $\textbf{let } \bigcirc \vec{x}, y, \vec{z} = \vec{M}, \textbf{let } \bigcirc \vec{w} = \vec{N} \textbf{ in } Q, \vec{P} \; in \; R \to_\beta \textbf{let } \bigcirc \vec{x}, \vec{w}, \vec{z} = \vec{M}, \vec{N}, \vec{P} \textbf{ in } R[y := Q]$.
5. $\textbf{let } \bigcirc \vec{x} = \textbf{pure } \vec{M} \textbf{ in } N \to_\beta \textbf{pure } N[\vec{x} := \vec{M}]$.
6. $\textbf{let } \bigcirc \_\_ = \_\_ \textbf{ in } M \to_\beta \textbf{pure } M$, where $\_\_$ is an empty sequence of terms.

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The multistep reduction $\twoheadrightarrow_\beta$ is reflexive-transitive closure of $\to_\beta$.

Metatheoretic properties

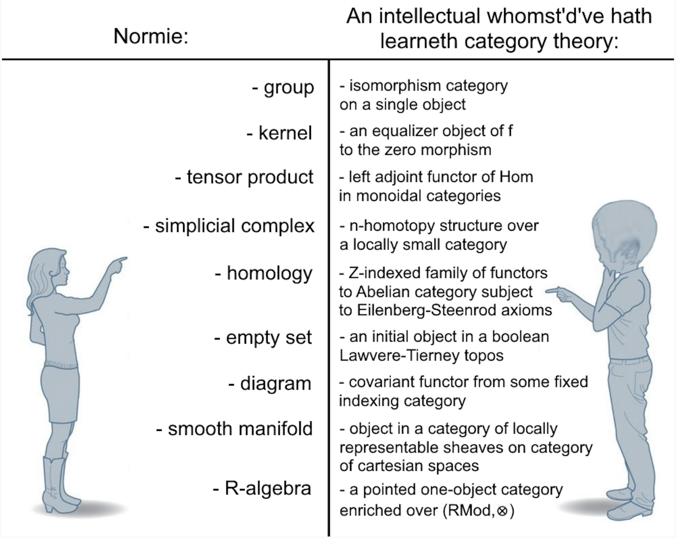
Theorem (D.R. 2018)

1. *(Type preservation)*
   If $\Gamma \vdash M : \varphi$ *and* $M \twoheadrightarrow_\beta N$*, then* $\Gamma \vdash N : \varphi$
2. *(Strong normalisation)*
   Every reduction path terminates, that is, no infinite reduction sequences.
3. *(Church-Rosser)*
   If $M \twoheadrightarrow_\beta N_1, N_2$*, then there exists* P *such that* $N_1, N_2 \twoheadrightarrow_\beta P$*.*

As a corollary, every $\lambda_{\mathbf{IEL}-term}$ has a unique normal form.

Categorical completeness

Now I am going to be like the guy from the right.



General concepts: Category

Recall that a category $\mathcal{C}$ consists of:

- A class of objects $\mathrm{Ob}(\mathcal{C}) = \{A, B, C, \dots\}$,
- A class of morphisms $\mathcal{C}(A, B)$ for each $A, B \in \mathrm{Ob}(\mathcal{C})$, where $f : A \to B$ iff $f \in \mathcal{C}(A, B)$,
- For $f : A \to B$ and $g : B \to C$, then $g \circ f : A \to C$ and $h \circ (g \circ f) = (h \circ g) \circ f$ for each f, g, h having an appropriate domain and codomain,
- For each $A, B \in \mathrm{Ob}(\mathcal{C})$ we have identity morphisms such that for each $f : A \to B$ $f \circ id_A = f$ and $id_B \circ f = f$.

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Some examples:

- **Set**, the category of all sets and all functions betweem them,
- **Top**, the category of all topological spaces and continuous maps,
- $\mathbf{Vect}_k$, the category of vector spaces over a field k and linear maps,
- $(P, \leq)$, any poset where $a \to b$ exists iff $a \leq b$,
- Any monoid (as well as a group) is a category, where $\mathrm{Ob}(\mathcal{C})$ is a singleton set (Cayley's theorem).
- etc.

General concepts: Functor

Intuitively, a functor is a morphism of category. Rigorously, let $\mathcal{C}$ and $\mathcal{D}$ be categories, a functor $\mathbf{F} : \mathcal{C} \to \mathcal{D}$ is a “function” such that:

- Each $A \in \mathrm{Ob}(\mathcal{C})$ maps to $\mathbf{F}A \in \mathrm{Ob}(\mathcal{D})$,
- Each $f : A \to B$ in $\mathcal{C}$ maps to $\mathbf{F}f : \mathbf{F}A \to \mathbf{F}B$ in $\mathcal{D}$,
- $\mathbf{F}(id_A) = id_{\mathbf{F}A}$
- $\mathbf{F}(g \circ f) = \mathbf{F}g \circ \mathbf{F}f$ for each f and g.

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Some examples:

- The powerset functor $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ such that $\mathcal{P} : A \mapsto 2^A$,
- The abelianisation functor $Ab : \mathbf{Group} \to \mathbf{Ab}$ such that $Ab : G \mapsto G/[G, G]$,
- The spectrum functor $\mathrm{Spec} : \mathbf{Ring}^{op} \to \mathbf{Top}$ that maps every commutative ring to its Zariski space,
- $\mathbf{Field} \to \mathbf{Ring}$ such that $k \mapsto k[X]$,
- $\pi_1 : \mathbf{Top}_* \to \mathbf{Group}$ maps every topological space with a base point to its fundamental group, for example, $\pi_1(S) = \mathbb{Z}$ (up to isomorphism).
- etc.

General concepts: Natural transformation

A natural transformation is a functor morphism. Let $\mathcal{C}, \mathcal{D}$ be categories and $\mathbf{F}, \mathbf{G} : \mathcal{C} \to \mathcal{D}$ functors. A natural tranformation $\theta : \mathbf{F} \Rightarrow \mathbf{G}$ is a collection of morphisms $\theta_A : \mathbf{F}A \to \mathbf{G}A$ in $\mathcal{D}$ making the following square commute for each $f : A \to B$ and $A, B \in \mathrm{Ob}(\mathcal{C})$:

$$\begin{tikzcd}
\mathbf{F}A \arrow[r, "\mathbf{F}f"] \arrow[d, "\theta_A"'] & \mathbf{F}B \arrow[d, "\theta_B"] \\
\mathbf{G}A \arrow[r, "\mathbf{G}f"'] & \mathbf{G}B
\end{tikzcd}$$

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An example:

Let det_M be the determinant of an $n \times n$ matrix $M \in \mathbf{GL}_n k$ with entries from a field k and let k^* be the multiplicative group of k. Both $\mathbf{GL}_n$ and $*$ are functors from the category of fields to the category of groups, and $det_M : \mathbf{GL}_n k \to k^*$ is a morphism of groups and it is natural:

$$\begin{tikzcd}
\mathbf{GL}_n k \arrow[r, "f"] \arrow[d, "det_M"'] & \mathbf{GL}_n k' \arrow[d, "det_{M'}"] \\
k^* \arrow[r, "f^*"'] & k'^*
\end{tikzcd}$$

Cartesian closed categories

A category is *cartesian closed* is there are objects $\mathbb{1}$, B^A and $A \times B$ such that:

- $|\mathcal{C}(A, \mathbb{1})| = 1$ for each $A \in \mathrm{Ob}(\mathcal{A})$,
- The following diagrams commute:

$$
\begin{tikzcd}
 & C \arrow[dl, "f"'] \arrow[d, "(f,g)"] \arrow[dr, "g"] & \\
A & A \times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
C^B \times B \arrow[rr, "ev"] & & C \\
A \times B \arrow[u, "\hat{f} \times id_B"] \arrow[urr, "f"'] & &
\end{tikzcd}
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The second diagram can be reformulated as (compare with the definition of implication in Heyting algebras):

$$\mathcal{C}(A \times B, C) \simeq \mathcal{C}(A, C^B)$$

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Some examples:

- **Set**,
- Every Heyting algebra,
- The category of G-sets for a group G (the category of group actions),
- The category of simplicial sets (which are also contravariant functors $\Delta : \omega \to$ **Set**).

Typed lambda calculi type-theoretically

CCCs allow interpreting intuitionistic type theories using the following scheme:

$$\Gamma \models M : A \text{ iff there exists an arrow } [\![M]\!] : [\![\Gamma]\!] \to [\![A]\!].$$

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In particular, simply typed lambda calculus with types $\to$ and $\times$ has the following interpretation in CCCs.

$$\frac{}{[\![\Gamma, x : \varphi \vdash x : \varphi]\!] = \pi_2 : [\![\Gamma]\!] \times [\![\varphi]\!] \to [\![\varphi]\!]}$$

$$\frac{[\![\Gamma, x : \varphi \vdash M : \psi]\!] = [\![M]\!] : [\![\Gamma]\!] \times [\![\varphi]\!] \to [\![\psi]\!]}{[\![\Gamma \vdash (\lambda x.M) : \varphi \to \psi]\!] = \widehat{([\![M]\!])} : [\![\Gamma]\!] \to [\![\psi]\!]^{[\![\varphi]\!]}}$$

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$$\frac{[\![\Gamma \vdash M : \varphi]\!] = [\![M]\!] : [\![\Gamma]\!] \to [\![\varphi]\!] \qquad [\![\Gamma \vdash N : \psi]\!] = [\![N]\!] : [\![\Gamma]\!] \to [\![B]\!]}{[\![\Gamma \vdash (M, N) : \varphi \times \psi]\!] = ([\![M]\!], [\![N]\!]) : [\![\Gamma]\!] \to [\![\varphi]\!] \times [\![\psi]\!]}$$

$$\frac{[\![\Gamma \vdash M : \varphi_1 \times \varphi_2]\!] = [\![M]\!] : [\![\Gamma]\!] \to [\![\varphi_1]\!] \times [\![\varphi_2]\!]}{[\![\Gamma \vdash \pi_i M : \varphi_i]\!] = [\![\Gamma]\!] \xrightarrow{[\![M]\!]} [\![\varphi_1]\!] \times [\![\varphi_2]\!] \xrightarrow{\pi_i} [\![\varphi_i]\!]}\; i \in \{1, 2\}$$

See:

- Samson Abramsky and Nikos Tzevelekos. Introduction to categories and categorical logic, 2011.

Monoidal endofunctors as modalities

We are interested in how to interpret $\Box$-like modality categorically. Recall that one reformulate the **K** axioms of $\Box$ the following way:

- (The multiplicativity axiom)
$$\Box(p \wedge q) \leftrightarrow \Box p \wedge \Box q$$
- (The normality axiom)
$$\Box\top \leftrightarrow \top$$
- (The monotonicity rule)
$$\text{From } \varphi \to \psi \text{ infer } \Box\varphi \to \Box\psi$$

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Categorically, we have an endofunctor $F : \mathcal{C} \to \mathcal{C}$ with the following natural isomorphisms (this is a *strong monoidal endofunctor* over a cartesian closed category):

- $m_{A,B} : \mathbf{F}(A \times B) \cong \mathbf{F}A \times \mathbf{F}B$
- $u : \mathbf{F}\mathbb{1} \cong \mathbb{1}$

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The modal lambda calculus Curry-Howard isomorphic to the intuitionstic modal logic **K** with $\Box$ is known to sound and complete w.r.t. CCCs with strong monoidal endofunctors. See

- Gianluigi Bellin, Valeria De Paiva and Eike Ritter. Extended Curry-Howard Correspondence for a Basic Constructive Modal Logic, 2003
- Y. Kakutani. Call-by-name and call-by-value in normal modal logic, 2007.

$\mathbf{IEL}^-$ as a natural transformation

To interpret the $\mathbf{IEL}^-$ modality, we need the natural transformation $\eta : Id_{\mathcal{C}} \Rightarrow \mathbf{F}$ (where $\mathcal{C}$ is a CCC):

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\eta_A"'] & B \arrow[d, "\eta_B"] \\
\mathbf{F}A \arrow[r, "\mathbf{F}f"'] & \mathbf{F}B
\end{tikzcd}$$

$\mathbf{IEL}^{-}$ as a natural transformation

To interpret the $\mathbf{IEL}^{-}$ modality, we need the natural transformation $\eta : Id_{\mathcal{C}} \Rightarrow \mathbf{F}$ (where $\mathcal{C}$ is a CCC):

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\eta_A"'] & B \arrow[d, "\eta_B"] \\ \mathbf{F}A \arrow[r, "\mathbf{F}f"'] & \mathbf{F}B \end{tikzcd}$$

But F should be a strong monoidal endofunction with the additional principles:

1. $u = \eta_{\mathbb{1}}$
2. For each $A, B \in \mathrm{Ob}(\mathcal{C})$:

$$\begin{tikzcd} A \times B \arrow[r, "\eta_A \times \eta_B"] \arrow[dr, "\eta_{A \times B}"'] & \mathbf{F}A \times \mathbf{F}B \arrow[d, "m_{A,B}"] \\ & \mathbf{F}(A \times B) \end{tikzcd}$$

$\mathbf{IEL}^-$ as a natural transformation

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An $\mathbf{IEL}^-$-*category* is a triple $(\mathcal{C}, \mathbf{F}, \eta)$, where $\mathcal{C}$ is a CCC, $\mathbf{F}$ is a strong monoidal endofunctor and $\eta : Id_{\mathcal{C}} \Rightarrow \mathbf{F}$ is a natural transformation with the additional extra-principles as above.

Categorical semantics

Theorem (D. R. 2018)
If M and N are well-typed and $M =_\beta N$, then $[\![M]\!] = [\![N]\!]$. That is, $\lambda_{\mathbf{IEL}^-}$ is sound and complete w.r.t. $\mathbf{IEL}^-$-categories.

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We skip the complete argument, but we just show how to interpret the modal inference rules in **IEL**$^-$-categories:

$$\frac{[\![\Gamma \vdash M : \bigcirc\varphi]\!] = [\![M]\!] : [\![\Gamma]\!] \to [\![\varphi]\!]}{[\![\Gamma \vdash \mathbf{pure}\ M : \bigcirc\varphi]\!] = [\![M]\!] \circ \eta_{[\![\varphi]\!]} : [\![\Gamma]\!] \to \mathbf{F}[\![\varphi]\!]}$$

$$\frac{[\![\Gamma \vdash \overrightarrow{M} : \bigcirc\overrightarrow{\varphi}]\!] = ([\![M_1]\!], \ldots, [\![M_n]\!]) : [\![\Gamma]\!] \to \prod_{i=1}^{n} \mathbf{F}[\![\varphi_i]\!] \qquad [\![\overrightarrow{x} : \overrightarrow{\varphi} \vdash N : \psi]\!] = [\![N]\!] : \prod_{i=1}^{n}[\![\varphi_i]\!] \to [\![\psi]\!]}{[\![\Gamma \vdash \mathbf{let}\ \bigcirc\ \overrightarrow{x} = \overrightarrow{M}\ \mathbf{in}\ N : \bigcirc\psi]\!] = \mathbf{F}([\![N]\!]) \circ m_{[\![\varphi_1]\!],\ldots,[\![\varphi_n]\!]} \circ ([\![M_1]\!], \ldots, [\![M_n]\!]) : [\![\Gamma]\!] \to \mathbf{F}[\![\psi]\!]}$$

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The interpretation of the second inference rule could be also described via the (slightly modified) quote from Hamlet:

Therefore, since brevity is the soul of wit
and tediousness the limbs and outward flourishes,
I won't be brief.

Kripke-Joyal semantics

Some background: Heyting algebras and locales

Recall that a *Heyting algebra* is a bounded distributive lattice $\mathcal{H} = (H, \wedge, \vee, \rightarrow, 0, 1)$ with the operation $\rightarrow$ satisfying for all $a, b, c \in H$:

$$a \wedge b \leq c \text{ iff } a \leq b \rightarrow c$$

A *locale* (*frame*, *complete Heyting algebra*) is a complete lattice $\mathcal{L} = (L, \wedge, \bigvee)$ satisfying for all $a \in L$ and for each indexed family $(a_i)_{i \in I}$:

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Heyting algebras and locales are about:

- Heyting algebras provide algebraic semantics for intuitionistic and intermediate logics,
- Locales are a lattice-theoretic approximation of topological spaces,
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Some references:

- Francis Borceux. Handbook of Categorical Algebra: Volume 3, Sheaf Theory, 1994.
- Leo Esakia. Heyting algebras: Duality theory, 2019.
- Andre Joyal and Myles Tierney. An extension of the Galois theory of Grothendieck, 1984.
- Steven Vickers. Topology via Logic, 1989.
- Andre Joyal. A crash course in topos theory: the big picture. (A lecture course available on YouTube)

Kripke-Joyal semantics: handwaving motivation

- Cover semantics is based on the notion of local truth in topological/topos-theoretic terms.
- Informally, a statement φ is true at X, where X is, say, an open set, if there exists an open cover of X such that φ is true at each member of it.
- An example of local truth in topology is local equality or the property of being constant (for continuous functions).
- Such notions also arise essentially from presheaf topoi, but we are going to deal with a slighlty simpler kind of structures that allow representing locales.
- Here, intuitionistic logic plays the role of internal logic of topoi (the Mitchell-Bénabou language).

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Some references:

- Olivia Caramello. Theories, Sites, Toposes, 2017.
- Robert Goldblatt. Topoi: The Categorial Analysis of Logic, 1984.
- Saunders MacLane and Ieke Moerdijk. Sheaves in geometry and logic: A first introduction to topos theory, 1992.

Some background: nuclei and prenuclei

A *prenucleus* on a Heyting algebra is a monotone inflationary map that distibutes over finite infima, whereas a *nucleus* is an idempotent prenucleus. If $\mathcal{H}$ is a (locale) Heyting algebra and j is prenucleus, then $(\mathcal{H}, j)$ is a (localic) prenuclear algebra.

(Pre)nuclei are about:

- Heyting subalgebra and sublocale characterisation,
- a Lawvere-Tierney topology that axiomatises the notion of local truth and also allows defining a Grothendieck topology on a presheaf topos equivalently,
- One can think of prenuclei as a weaker version of a Lawvere-Tierney topology,
- Prenuclei are used for characterising lattices of nuclei on a locale.

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Some examples of nuclei (and then prenuclei):

- Let $\mathcal{H}$ be a Heyting algebra, then an operator $j : \mathcal{H} \to \mathcal{H}$ such that $j : a \mapsto \neg\neg a$ is a nucleus,
- More generally, an operator $j : \mathcal{H} \to \mathcal{H}$ s.t. $j : a \mapsto (b \to a) \to a$ is a nucleus on $\mathcal{H}$,
- Actually, one can think of the $\mathbf{IEL}^{-}$ modality as a kind of a prenuclear operator,

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Some references

- Robert Goldblatt. Grothendieck Topology as Geometric Modality, 1981.
- Peter Johnstone. Stone spaces, 1984.
- Martin Escardo. Joins in the frame of nuclei, 2003.

Cover systems

But instead of topoi we will be using a simple kind of structures called *cover systems* introduced by Bell and then developed by Goldblatt.

Let $\mathcal{P} = (P, \leq)$ be a poset, then a cover scheme is a tuple $\mathcal{C} = (\mathcal{P}, \mathrm{Cov})$, where $\mathrm{Cov}(x) \subseteq 2^P$ is the collection of *covers of x* or *x-covers* such that:

- For all $x \in \mathcal{P}$ there exists $C \subseteq \mathcal{P}$ such that $C \in \mathrm{Cov}(x)$ and $C \subseteq \uparrow x$,
- If $C \in \mathrm{Cov}(x)$ and for all $y \in \mathrm{Cov}(C_y)$, then $\bigcup_{y \in C} C_y$,
- If $x \leq y$, then every x-cover can be refined to some y-cover, that is, if $C \in \mathrm{Cov}(x)$, then $C' \in \mathrm{Cov}(y)$ such that $C' \subseteq \uparrow C$,
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An important point, an operator $j : 2^{\mathcal{P}} \to 2^{\mathcal{P}}$:

$$jA = \{x \in \mathcal{P} \mid \exists C \subseteq A \; C \in \mathsf{Cov}(x)\}$$

is a nucleus on a locale $\mathsf{Up}(\mathcal{P})$, so j-closed upward closed subsets of $\mathcal{P}$ form a sublocale of $\mathsf{Up}(\mathcal{P})$. Moreover, every locale is representable as a locale of localised subsets of some cover system (that we shall denote as $\mathsf{Loc}(\mathcal{C})$).

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See:

- John Bell. Cover schemes, frame-valued sets and their potential uses in spacetime physics, 2003.
- Robert Goldblatt. Cover semantics for quantified lax logic, 2011.

Intuitionistic predicate logics

In this section, we consider one-sorted first-order language with no function symbols and constants. Intuitionistic predicate logic is defined standardly with the following axiom schemes and rules:

1. $\varphi \to (\psi \to \varphi)$
2. $(\varphi \to (\psi \to \theta)) \to ((\varphi \to \psi) \to (\varphi \to \theta))$
3. $\varphi_1 \wedge \varphi_2 \to \varphi_i$, for $i = 1, 2$,
4. $\varphi \to (\psi \to \varphi \wedge \psi)$
5. $\varphi_i \to \varphi_1 \vee \varphi_2$, for $i = 1, 2$,
6. $(\varphi \to \theta) \to ((\psi \to \theta) \to (\varphi \vee \psi \to \theta))$
7. $\perp \to \varphi$,
8. $\forall x \varphi \to \varphi(w/x)$,
9. $\varphi(w/x) \to \exists x \varphi$,
10. The inference rules are Modus Ponens and Bernays rules.

Kripke-Joyal semantics for predicate intuitionistic logics

A *cover scheme model* is a structure $\mathcal{M} = (\mathcal{P}, \mathsf{Cov}, U, |.|)$, where $(\mathcal{P}, \mathsf{Cov})$ is a cover scheme, $U \neq \emptyset$ is a set of individuals and $|.|$ is an interpretation such that:

- If x is a free variable, then $|x|_{\mathcal{M}} \in U$,
- If P is an n-ary predicate symbol, then $|P|^{\mathcal{M}} : U^n \to \mathsf{Loc}(\mathcal{P}, \mathsf{Cov})$

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The truth definition is standard (in terms of Kripke-Joyal semantics):

- $\mathcal{M}, x \Vdash P(v_1, \ldots, v_n)$ iff $x \in |P|^{\mathcal{M}}(|v_1|_{\mathcal{M}}, \ldots, |v_n|_{\mathcal{M}})$,
- $\mathcal{M}, x \Vdash \bot$ iff $\emptyset \in \mathrm{Cov}(x)$
- $\mathcal{M}, x \Vdash \varphi \wedge \psi$ iff $\mathcal{M}, x \Vdash \varphi$ and $\mathcal{M}, x \Vdash \psi$,
- $\mathcal{M}, x \Vdash \varphi \vee \psi$ iff there exists $C \in \mathrm{Cov}(x)$ such that for each $y \in C$ $\mathcal{M}, y \Vdash \varphi$ or $\mathcal{M}, y \Vdash \psi$,
- $\mathcal{M}, x \Vdash \varphi \to \psi$ iff for each $y \geq x$ $\mathcal{M}, y \Vdash \varphi$ implies $\mathcal{M}, y \Vdash \psi$,
- $\mathcal{M}, x \Vdash \forall v \varphi$ iff $\mathcal{M}, x \Vdash \varphi(v := d)$ for each individual $u \in U$,
- $\mathcal{M}, x \Vdash \exists v \varphi$ iff there exists $C \in \mathrm{Cov}(x)$ and $u \in U$ such that for each $y \in C$ $\mathcal{M}, y \Vdash \varphi(v := u)$.
- $\mathcal{M} \Vdash \varphi$ iff for each $x \in \mathcal{M}$ $\mathcal{M}, x \Vdash \varphi$.

Kripke-Joyal semantics for predicate intuitionistic logics algebraically

With each formula we can associate its truth set $[\![\varphi]\!]_{\mathcal{M}} = \{x \in \mathcal{P} \mid \mathcal{M}, x \Vdash \varphi\}$, so one can show that $[\![.]\!]$ commutes with algebraic operations on the locale of localised upsets:

- $[\![\bot]\!]_{\mathcal{M}} = j\emptyset$
- $[\![\varphi \wedge \psi]\!]_{\mathcal{M}} = [\![\varphi]\!]_{\mathcal{M}} \cap [\![\psi]\!]_{\mathcal{M}}$
- $[\![\varphi \vee \psi]\!]_{\mathcal{M}} = j([\![\varphi]\!]_{\mathcal{M}} \cup [\![\psi]\!]_{\mathcal{M}})$
- $[\![\varphi \to \psi]\!]_{\mathcal{M}} = [\![\varphi]\!]_{\mathcal{M}} \to [\![\psi]\!]_{\mathcal{M}}$
- $[\![\forall v \varphi]\!]_{\mathcal{M}} = \bigcap\limits_{u \in U} [\![\varphi(v := u)]\!]_{\mathcal{M}}$
- $[\![\exists v \varphi]\!]_{\mathcal{M}} = j(\bigcup\limits_{u \in U} [\![\varphi(v := u)]\!]_{\mathcal{M}})$

Theorem (Cover scheme analogue of Goedel completeness)
(Goldblatt 2011) IPL $\vdash \varphi$ iff $\mathcal{M} \Vdash \varphi$ or, equivalently, $[\![\varphi]\!]_{\mathcal{M}} = \top$ for any model $\mathcal{M}$.

The proof is based on embedding the Lindenbaum-Tarski algebra of *IPL* to a locale, which is isomorphic to the locale of localised upsets of some cover scheme.

Modal cover schemes

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Let $\mathcal{C} = (\mathcal{P}, \mathrm{Cov})$ be a cover scheme and *R* a binary relation of the carrier of $\mathcal{P}$. A structure $(\mathcal{P}, \mathrm{Cov}, R)$ is an **IEL**$^-$-cover scheme if the following holds:

- (Confluence)
  If xRy and $x \leq z$, then there exists w such that $y \leq w$ and zRw,
- (Modal localisation)
  If there exists an x-cover included to $R(X)$, then there exists y such that xRy and there exists y-cover included to X
- R is serial
- If xRy and xRz, then there exists $w \geq y, z$ such that xRw
- xRy implies there exists $z \geq y$ such that xRz and $x \geq z$

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A *modal cover scheme model* is a model on an **IEL**$^-$-cover scheme with the following truth condition for $\bigcirc$:

- $\mathcal{M}, x \Vdash \bigcirc\varphi$ iff there exists $y \in R(x)$ s.t. $\mathcal{M}, y \Vdash \varphi$.

Or, in terms of truth sets,

- $[\![\bigcirc\varphi]\!] = R([\![\varphi]\!])$

The representation theorem for locales with modal operators

Theorem
(D. R. 2020) Every localic prenuclear algebra is isomorphic to the complex algebra of some modal cover scheme.

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These theorems imply the following corollaries:

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(D. R. 2020) The class of prenuclear algebras is closed under Dedekind-MacNeille completitions.

These theorems imply the following corollaries:

Corollary
(D. R. 2020)

1. *Every prenuclear algebra is embeddable to the complex algebra of some modal cover scheme.*
2. $\mathbf{QIEL}^{-}$ *is sound and complete w.r.t. its cover schemes.*

Further directions

- It is also of interest to investigate cover semantics for intuitionistic infinitary and geometric logics,
- There are also non-trivial open questions posed by Goldblatt related to cover scheme completeness for infinitary and geometric predicate logics based on quantales,
- The latter is also could of interest in terms of categorical logic, which could require some non-commutative generalisation of presheaf topoi and Lawvere-Tierney topology (that is, categorical generalisation of quantic nuclei).

See:

- Robert Goldblatt. A Kripke-Joyal Semantics for Noncommutative Logic in Quantales, 2006.
- I also do some open-source notes about geometric logic in orded to tackle some questions related to non-commutative geometric logics, the notes are available at `https://github.com/DanielRrr/geometric-logic`.

Thank you so much indeed!