Intuitionistic epistemic logic categorically and algebraically

Daniel Rogozin
University College London

Seminar on Programming principles, Logic and Verification
The 24th of March 2023

Introduction

Intuitionistic modal logic: the big picture

- As it is well known, modal logic extends classical logic with modal operators.
- Applications: topology, proof theory, formal verification, ontologies, etc.
- Intuitionistic modal logic is a version of modal logic where the underlying logic is the intuitionistic one.
- Possible topics where intuitionistic modal logic is of interest:
    - Constructive necessity, provability in intuitionistic arithmetic, intuitionistic knowledge, etc.
    - Model theory: the finite model property, canonicity à la Salqvist, definability à la Thomason-Goldblatt, etc.
    - Representation theory: general descriptive frames, Esakia duality, etc.

Intuitionistic modal logic: the big picture

- As it is well known, modal logic extends classical logic with modal operators.
- Applications: topology, proof theory, formal verification, ontologies, etc.
- Intuitionistic modal logic is a version of modal logic where the underlying logic is the intuitionistic one.
- Possible topics where intuitionistic modal logic is of interest:
  - Constructive necessity, provability in intuitionistic arithmetic, intuitionistic knowledge, etc.
  - Model theory: the finite model property, canonicity à la Salqvist, definability à la Thomason-Goldblatt, etc.
  - Representation theory: general descriptive frames, Esakia duality, etc.

See this summary paper to have the big picture in more detail

- Frank Wolter and Michael Zakharyaschev. Intuitionistic Modal Logic, 1999.

Modalities type theoretically

- Type theory deals with a computation every value in which is annotated with the corresponding data type. Type theory is closely connected with intuitionistic logic and constructive proofs through the Curry-Howard correspondence.
- One can extend Curry-Howard to intuitionistic modal logic and study modal operators within the “types-as-formulas” and “proofs-as-terms” paradigm.
- Here we think of modal types as abstract data types of action, which is of interest for functional programming.

Modalities type theoretically

- Type theory deals with a computation every value in which is annotated with the corresponding data type. Type theory is closely connected with intuitionistic logic and constructive proofs through the Curry-Howard correspondence.
- One can extend Curry-Howard to intuitionistic modal logic and study modal operators within the “types-as-formulas” and “proofs-as-terms” paradigm.
- Here we think of modal types as abstract data types of action, which is of interest for functional programming.

See the following:

- Gianluigi Bellin, Valeria De Paiva and Eike Ritter. Extended Curry-Howard Correspondence for a Basic Constructive Modal Logic, 2003
- Frank Pfenning and Rowan Davies. A Judgmental Reconstruction of Modal Logic, 2000.
- Peter Nicholas Benton, Gavin M. Bierman, Valeria de Paiva. Computational types from a logical perspective, 1998.
- David Corfield. Modal homotopy type theory: The prospect of a new logic for philosophy, 2020.

Modal type theory based on IEL^-

Bridges with functional programming

The definition of the type theory

$$\frac{}{\Gamma, x : \varphi \vdash x : \varphi}\ \text{ax}$$

$$\frac{\Gamma, x : \varphi \vdash M : \psi}{\Gamma \vdash \lambda x.M : \varphi \to \psi} \to_i \qquad \frac{\Gamma \vdash M : \varphi \to \psi \qquad \Gamma \vdash N : \varphi}{\Gamma \vdash MN : \psi} \to_e$$

$$\frac{\Gamma \vdash M : \varphi \qquad \Gamma \vdash N : \psi}{\Gamma \vdash \langle M, N \rangle : \varphi \times \psi} \times_i \qquad \frac{\Gamma \vdash M : \varphi_1 \times \varphi_2}{\Gamma \vdash \pi_i M : \varphi_i} \times_e, i = 1, 2$$

$$\frac{\Gamma \vdash M : \varphi}{\Gamma \vdash \textbf{pure}\ M : \bigcirc\varphi} \bigcirc_I \qquad \frac{\Gamma \vdash \overrightarrow{M} : \bigcirc\overrightarrow{\varphi} \qquad \overrightarrow{x} : \overrightarrow{\varphi} \vdash N : \psi}{\Gamma \vdash \textbf{let}\ \bigcirc\ \overrightarrow{x} = \overrightarrow{M}\ \textbf{in}\ N : \bigcirc\psi} \text{let}_\bigcirc$$

Metatheoretic properties

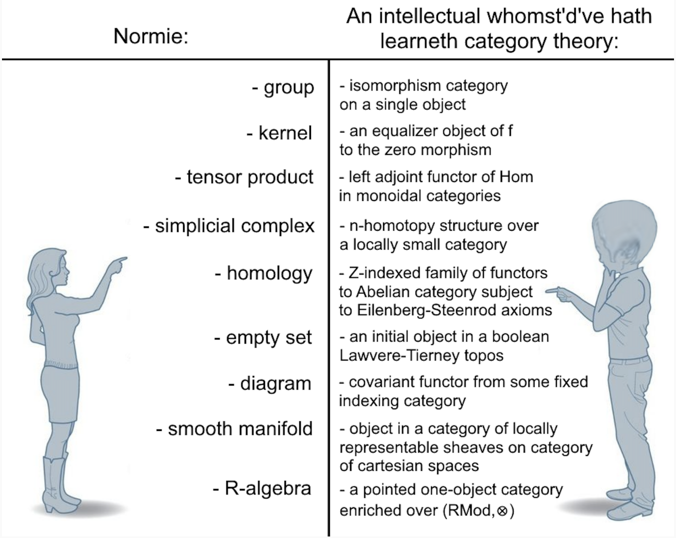
Theorem (D.R. 2018)

1. *(Type preservation)*
   If $\Gamma \vdash M : \varphi$ *and* $M \twoheadrightarrow_\beta N$*, then* $\Gamma \vdash N : \varphi$
2. *(Strong normalisation)*
   Every reduction path terminates, that is, no infinite reduction sequences.
3. *(Church-Rosser)*
   If $M \twoheadrightarrow_\beta N_1, N_2$*, then there exists* P *such that* $N_1, N_2 \twoheadrightarrow_\beta P$*.*

As a corollary, every $\lambda_{\mathbf{IEL}-term}$ has a unique normal form.

Categorical completeness

Now I am going to be like the guy from the right.



General concepts: Category

Recall that a category $\mathcal{C}$ consists of:

- A class of objects $\text{Ob}(\mathcal{C}) = \{A, B, C, \ldots\}$,
- A class of morphisms $\mathcal{C}(A, B)$ for each $A, B \in \text{Ob}(\mathcal{C})$, where $f : A \to B$ iff $f \in \mathcal{C}(A, B)$,
- For $f : A \to B$ and $g : B \to C$, then $g \circ f : A \to C$ and $h \circ (g \circ f) = (h \circ g) \circ f$ for each f, g, h having an appropriate domain and codomain,
- For each $A, B \in \text{Ob}(\mathcal{C})$ we have identity morphisms such that for each $f : A \to B$ $f \circ id_A = f$ and $id_B \circ g = g$.

Some examples:

- **Set**, the category of all sets and all functions betweem them,
- **Top**, the category of all topological spaces and continuous maps,
- $\mathbf{Vect}_k$, the category of vector spaces over a field k and linear maps,
- $(P, \leq)$, any poset where $a \to b$ exists iff $a \leq b$,
- Any monoid (as well as a group) is a category, where $\text{Ob}(\mathcal{C})$ is a singleton set (Cayley's theorem).
- etc.

General concepts: Functor

Intuitively, a functor is a morphism of category. Rigorously, let $\mathcal{C}$ and $\mathcal{D}$ be categories, a functor $\mathbf{F} : \mathcal{C} \to \mathcal{D}$ is a "function" such that:

- Each $A \in \mathrm{Ob}(\mathcal{C})$ maps to $\mathbf{F}A \in \mathrm{Ob}(\mathcal{D})$,
- Each $f : A \to B$ in $\mathcal{C}$ maps to $\mathbf{F}f : \mathbf{F}A \to \mathbf{F}B$ in $\mathcal{D}$,
- $\mathbf{F}(id_A) = id_{\mathbf{F}A}$
- $\mathbf{F}(g \circ f) = \mathbf{F}g \circ \mathbf{F}f$ for each f and g.

Some examples:

- The powerset functor $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ such that $\mathcal{P} : A \mapsto 2^A$,
- The abelianisation functor $Ab : \mathbf{Group} \to \mathbf{Ab}$ such that $Ab : G \mapsto G/[G, G]$,
- The spectrum functor $\mathrm{Spec} : \mathbf{Ring}^{op} \to \mathbf{Top}$ that maps every commutative ring to its Zariski space,
- $\mathbf{Field} \to \mathbf{Ring}$ such that $k \mapsto k[X]$,
- $\pi_1 : \mathbf{Top}_* \to \mathbf{Group}$ maps every topological space with a base point to its fundamental group, for example, $\pi_1(S) = \mathbb{Z}$ (up to isomorphism).
- etc.

General concepts: Natural transformation

A natural transformation is a functor morphism. Let $\mathcal{C}, \mathcal{D}$ be categories and $\mathbf{F}, \mathbf{G} : \mathcal{C} \to \mathcal{D}$ functors. A natural tranformation $\theta : \mathbf{F} \Rightarrow \mathbf{G}$ is a collection of morphisms $\theta_A : \mathbf{F}A \to \mathbf{G}A$ in $\mathcal{D}$ making the following square commute for each $f : A \to B$ and $A, B \in \mathrm{Ob}(\mathcal{C})$:

$$\begin{tikzcd}
\mathbf{F}A \arrow[r, "\mathbf{F}f"] \arrow[d, "\theta_A"'] & \mathbf{F}B \arrow[d, "\theta_B"] \\
\mathbf{G}A \arrow[r, "\mathbf{G}f"'] & \mathbf{G}B
\end{tikzcd}$$

A natural transformation is a functor morphism. Let $\mathcal{C}, \mathcal{D}$ be categories and $\mathbf{F}, \mathbf{G} : \mathcal{C} \to \mathcal{D}$ functors. A natural tranformation $\theta : \mathbf{F} \Rightarrow \mathbf{G}$ is a collection of morphisms $\theta_A : \mathbf{F}A \to \mathbf{G}A$ in $\mathcal{D}$ making the following square commute for each $f : A \to B$ and $A, B \in \mathrm{Ob}(\mathcal{C})$:

$$\begin{tikzcd}
\mathbf{F}A \arrow[r, "\mathbf{F}f"] \arrow[d, "\theta_A"'] & \mathbf{F}B \arrow[d, "\theta_B"] \\
\mathbf{G}A \arrow[r, "\mathbf{G}f"'] & \mathbf{G}B
\end{tikzcd}$$

An example:

Let det_M be the determinant of the $n \times n$ matrix $M \in \mathrm{GL}_n\, k$ with entries from a field k and k^* is the multiplicative group of k. Both $\mathbf{GL}_n$ and $*$ are functors from the category of fields to the category of groups, and $det_M : \mathbf{GL}_n k \to k^*$ is a morphism of groups and it is natural:

$$\begin{tikzcd}
\mathbf{GL}_n k \arrow[r, "f"] \arrow[d, "det_M"'] & \mathbf{GL}_n k' \arrow[d, "det_{M'}"] \\
k^* \arrow[r, "f^*"'] & k'^*
\end{tikzcd}$$

Cartesian closed categories

A category is *cartesian closed* is there are objects $\mathbb{1}$, B^A and $A \times B$ such that:

- $|\mathcal{C}(A, \mathbb{1})| = 1$ for each $A \in \mathrm{Ob}(A)$,
- The following diagrams commute:

$$
\begin{tikzcd}
 & C \arrow[dl, "f"'] \arrow[d, "{(f,g)}"] \arrow[dr, "g"] & \\
A & A \times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
C^B \times B \arrow[r, "ev"] & C \\
A \times B \arrow[u, "\hat{f} \times id_B"] \arrow[ur, "f"'] &
\end{tikzcd}
$$

The second diagram can be reformulated as (compare with the definition of implication in Heyting algebras):

$$\mathcal{C}(A \times B, C) \simeq \mathcal{C}(A, C^B)$$

Cartesian closed categories

A category is *cartesian closed* is there are objects $\mathbb{1}$, B^A and $A \times B$ such that:

- $|\mathcal{C}(A, \mathbb{1})| = 1$ for each $A \in \mathrm{Ob}(A)$,
- The following diagrams commute:

$$\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[d, "(f,g)"] \arrow[dr, "g"] & \\ A & A \times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B \end{tikzcd} \qquad \begin{tikzcd} C^B \times B \arrow[r, "ev"] & C \\ A \times B \arrow[u, "\hat{f} \times id_B"] \arrow[ur, "f"'] & \end{tikzcd}$$

The second diagram can be reformulated as (compare with the definition of implication in Heyting algebras):

$$\mathcal{C}(A \times B, C) \simeq \mathcal{C}(A, C^B)$$

Some examples:

- **Set**,
- Every Heyting algebra,
- The category of G-sets for a group G (the category of group actions),
- The category of simplicial sets (which are also contravariant functors $\Delta : \omega \to$ **Set**).

Typed lambda calculi type-theoretically

Cartesian closed categories allow interpreting intuitionistic type theories using the following scheme:

$$\Gamma \models M : A \text{ iff there exists an arrow } [\![M]\!] : [\![\Gamma]\!] \to [\![A]\!].$$

Typed lambda calculi type-theoretically

Cartesian closed categories allow interpreting intuitionistic type theories using the following scheme:

$$\Gamma \models M : A \text{ iff there exists an arrow } [\![M]\!] : [\![\Gamma]\!] \to [\![A]\!].$$

In particular, simply typed lambda calculus with types $\to$ and $\times$ has the following interpretation in CCCs.

$$\frac{}{[\![\Gamma, x : \varphi \vdash x : \varphi]\!] = \pi_2 : [\![\Gamma]\!] \times [\![\varphi]\!] \to [\![\varphi]\!]}$$

$$\frac{[\![\Gamma, x : \varphi \vdash M : \psi]\!] = [\![M]\!] : [\![\Gamma]\!] \times [\![\varphi]\!] \to [\![\psi]\!]}{[\![\Gamma \vdash (\lambda x.M) : \varphi \to \psi]\!] = \widehat{([\![M]\!])} : [\![\Gamma]\!] \to [\![\psi]\!]^{[\![\varphi]\!]}}$$

$$\frac{[\![\Gamma \vdash M : \varphi \to \psi]\!] = [\![M]\!] : [\![\Gamma]\!] \to [\![\psi]\!]^{[\![\varphi]\!]} \qquad [\![\Gamma \vdash N : \varphi]\!] = [\![N]\!] : [\![\Gamma]\!] \to [\![\varphi]\!]}{[\![\Gamma \vdash (MN) : \psi]\!] = [\![\Gamma]\!] \xrightarrow{([\![M]\!], [\![N]\!])} [\![\psi]\!]^{[\![\varphi]\!]} \times [\![\varphi]\!] \xrightarrow{ev} [\![\psi]\!]}$$

$$\frac{[\![\Gamma \vdash M : \varphi]\!] = [\![M]\!] : [\![\Gamma]\!] \to [\![\varphi]\!] \qquad [\![\Gamma \vdash N : \psi]\!] = [\![N]\!] : [\![\Gamma]\!] \to [\![B]\!]}{[\![\Gamma \vdash (M, N) : \varphi \times \psi]\!] = ([\![M]\!], [\![N]\!]) : [\![\Gamma]\!] \to [\![\varphi]\!] \times [\![\psi]\!]}$$

$$\frac{[\![\Gamma \vdash M : \varphi_1 \times \varphi_2]\!] = [\![M]\!] : [\![\Gamma]\!] \to [\![\varphi_1]\!] \times [\![\varphi_2]\!]}{[\![\Gamma \vdash \pi_i M : \varphi_i]\!] = [\![\Gamma]\!] \xrightarrow{[\![M]\!]} [\![\varphi_1]\!] \times [\![\varphi_2]\!] \xrightarrow{\pi_i} [\![\varphi_i]\!]} \; i \in \{1, 2\}$$

Monoidal endofunctors as modalities

We are interested in how to interpret $\Box$-like modality categorically. Recall that one reformulate the **K** axioms of $\Box$ the following way:

- (The multiplicativity axiom)
$$\Box(p \wedge q) \leftrightarrow \Box p \wedge \Box q$$
- (The normality axiom)
$$\Box\top \leftrightarrow \top$$
- (The monotonicity rule)
$$\text{From } \varphi \to \psi \text{ infer } \Box\varphi \to \Box\psi$$

Monoidal endofunctors as modalities

We are interested in how to interpret $\Box$-like modality categorically. Recall that one reformulate the **K** axioms of $\Box$ the following way:

- (The multiplicativity axiom)
$$\Box(p \wedge q) \leftrightarrow \Box p \wedge \Box q$$
- (The normality axiom)
$$\Box\top \leftrightarrow \top$$
- (The monotonicity rule)
$$\text{From } \varphi \to \psi \text{ infer } \Box\varphi \to \Box\psi$$

Categorically, we have an endofunctor $F : \mathcal{C} \to \mathcal{C}$ with the following natural isomorphisms (this is a *strong monoidal endofunctor*):

- $m_{A,B} : \mathbf{F}(A \times B) \cong \mathbf{F}A \times \mathbf{F}B$
- $u : \mathbf{F}\mathbb{1} \cong \mathbb{1}$

Monoidal endofunctors as modalities

We are interested in how to interpret $\Box$-like modality categorically. Recall that one reformulate the **K** axioms of $\Box$ the following way:

- (The multiplicativity axiom)
$$\Box(p \wedge q) \leftrightarrow \Box p \wedge \Box q$$
- (The normality axiom)
$$\Box\top \leftrightarrow \top$$
- (The monotonicity rule)
$$\text{From } \varphi \to \psi \text{ infer } \Box\varphi \to \Box\psi$$

Categorically, we have an endofunctor $F : \mathcal{C} \to \mathcal{C}$ with the following natural isomorphisms (this is a *strong monoidal endofunctor*):

- $m_{A,B} : \mathbf{F}(A \times B) \cong \mathbf{F}A \times \mathbf{F}B$
- $u : \mathbf{F}\mathbb{1} \cong \mathbb{1}$

The modal lambda calculus Curry-Howard isomorphic to the intuitionstic modal logic **K** with $\Box$ is known to sound and complete w.r.t. CCCs with strong monoidal endofunctors.

See

- Gianluigi Bellin, Valeria De Paiva and Eike Ritter. Extended Curry-Howard Correspondence for a Basic Constructive Modal Logic, 2003
- Y. Kakutani. Call-by-name and call-by-value in normal modal logic, 2007.

$\mathbf{IEL}^{-}$ as a natural transformation

To interpret the $\mathbf{IEL}^{-}$ we need the natural transformation $\eta : Id_{\mathcal{C}} \Rightarrow \mathbf{F}$, where $\mathcal{C}$ is a CCC and F is a strong monoidal endofunction with the additional principles:

1. $u = \eta_{\mathbb{1}}$
2. For each $A, B \in \mathrm{Ob}(\mathcal{C})$:

$$\begin{tikzcd} A \times B \arrow[r, "\eta_A \times \eta_B"] \arrow[dr, "\eta_{A \times B}"'] & \mathbf{F}A \times \mathbf{F}B \arrow[d, "m_{A,B}"] \\ & \mathbf{F}(A \times B) \end{tikzcd}$$

IEL$^-$ as a natural transformation

To interpret the **IEL**$^-$ we need the natural transformation $\eta : Id_{\mathcal{C}} \Rightarrow \mathbf{F}$, where $\mathcal{C}$ is a CCC and F is a strong monoidal endofunction with the additional principles:

1. $u = \eta_{\mathbb{1}}$
2. For each $A, B \in \mathrm{Ob}(\mathcal{C})$:

$$\begin{tikzcd} A \times B \arrow[r, "\eta_A \times \eta_B"] \arrow[dr, "\eta_{A \times B}"'] & \mathbf{F}A \times \mathbf{F}B \arrow[d, "m_{A,B}"] \\ & \mathbf{F}(A \times B) \end{tikzcd}$$

An **IEL**$^-$*-category* is a triple $(\mathcal{C}, \mathbf{F}, \eta)$, where $\mathcal{C}$ is a CCC, **F** is a strong monoidal endofunctor and $\eta : Id_{\mathcal{C}} \Rightarrow \mathbf{F}$ is a natural transformation with the additional extra-principles as above.

IEL$^-$ as a natural transformation

To interpret the **IEL**$^-$ we need the natural transformation $\eta : Id_{\mathcal{C}} \Rightarrow \mathbf{F}$, where $\mathcal{C}$ is a CCC and F is a strong monoidal endofunction with the additional principles:

1. $u = \eta_{\mathbb{1}}$
2. For each $A, B \in \mathrm{Ob}(\mathcal{C})$:

$$\begin{tikzcd} A \times B \arrow[r, "\eta_A \times \eta_B"] \arrow[dr, "\eta_{A\times B}"'] & \mathbf{F}A \times \mathbf{F}B \arrow[d, "m_{A,B}"] \\ & \mathbf{F}(A \times B) \end{tikzcd}$$

An ***IEL**$^-$-category* is a triple $(\mathcal{C}, \mathbf{F}, \eta)$, where $\mathcal{C}$ is a CCC, **F** is a strong monoidal endofunctor and $\eta : Id_{\mathcal{C}} \Rightarrow \mathbf{F}$ is a natural transformation with the additional extra-principles as above.

Theorem (D. R. 2018)
If M and N are well-typed and $M =_\beta N$, then $[\![M]\!] = [\![N]\!]$. That is, $\lambda_{\mathbf{IEL}^-}$ is sound and complete w.r.t. ***IEL**$^-$-categories.*

We skip the complete argument, but we just show how to interpret the modal inference rules in $\mathbf{IEL}^-$-categories:

Kripke-Joyal semantics

Some background: Heyting algebras and locales

Recall that a Heyting algebra is a bounded distributive lattice $\mathcal{H} = (H, \wedge, \vee, \rightarrow, 0, 1)$ with the operation $\rightarrow$ satisfying for all $a, b, c \in H$:

$$a \wedge b \leq c \text{ iff } a \leq b \rightarrow c$$

A locale is a complete lattice $\mathcal{L} = (L, \wedge, \bigvee)$ satisfying for all $a \in L$ and for each indexed family $(a_i)_{i \in I}$:

$$a \wedge \bigvee_{i \in I} a_i = \bigvee_{i \in I} (a \wedge a_i)$$

Some background: Heyting algebras and locales

Recall that a Heyting algebra is a bounded distributive lattice $\mathcal{H} = (H, \wedge, \vee, \rightarrow, 0, 1)$ with the operation $\rightarrow$ satisfying for all $a, b, c \in H$:

$$a \wedge b \leq c \text{ iff } a \leq b \rightarrow c$$

A locale is a complete lattice $\mathcal{L} = (L, \wedge, \bigvee)$ satisfying for all $a \in L$ and for each indexed family $(a_i)_{i \in I}$:

$$a \wedge \bigvee_{i \in I} a_i = \bigvee_{i \in I} (a \wedge a_i)$$

Heyting algebras and locales are about:

- Heyting algebras provide algebraic semantics for intuitionistic and intermediate logics,
- Locales are a lattice-theoretic approximation of a topological spaces,
- Subobject algebras in topoi

Some background: Heyting algebras and locales

Recall that a Heyting algebra is a bounded distributive lattice $\mathcal{H} = (H, \wedge, \vee, \rightarrow, 0, 1)$ with the operation $\rightarrow$ satisfying for all $a, b, c \in H$:

$$a \wedge b \leq c \text{ iff } a \leq b \rightarrow c$$

A locale is a complete lattice $\mathcal{L} = (L, \wedge, \bigvee)$ satisfying for all $a \in L$ and for each indexed family $(a_i)_{i \in I}$:

$$a \wedge \bigvee_{i \in I} a_i = \bigvee_{i \in I} (a \wedge a_i)$$

Heyting algebras and locales are about:

- Heyting algebras provide algebraic semantics for intuitionistic and intermediate logics,
- Locales are a lattice-theoretic approximation of a topological spaces,
- Subobject algebras in topoi

Some references:

- Andre Joyal and Myles Tierney. An extension of the Galois theory of Grothendieck, 1984.
- Francis Borceux. Handbook of Categorical Algebra: Volume 3, Sheaf Theory, 1994.
- Peter Johnstone. Stone spaces, 1984.
- Leo Esakia. Heyting algebras: Duality theory, 2019.

Kripke-Joyal semantics: handwaving motivation

Cover systems

But instead of topoi we will be using a simple kind of structures called *cover systems* introduced by Bell and then developed by Goldblatt.

Cover systems

But instead of topoi we will be using a simple kind of structures called *cover systems* introduced by Bell and then developed by Goldblatt.

See:

- John Bell. Cover schemes, frame-valued sets and their potential uses in spacetime physics, 2003.
- Robert Goldblatt. Cover semantics for quantified lax logic, 2011.
- Robert Goldblatt. A Kripke-Joyal Semantics for Noncommutative Logic in Quantales, 2006.

Some background: nuclei and prenuclei

The representation theorem for locales with modal operators

Kripke-Joyal semantics for predicate intuitionistic epistemic modal logics