Some Notes on Proof Theory and Elements of Ordinal Analysis

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Contents

1 Provable Recursion in $\mathbf{I}\Delta_0(\exp)$

$\mathbf{I}\Delta_0(\exp)$ is a theory in first-order logic in the language:

$$\{=, 0, S, P, +, \dot{-}, \cdot, exp_2\}$$

where S and P are successor and precessor functions respectively. Further, we will denote $S(x)$ and $P(x)$ as $x+1$ and $x\dot{-}1$ respectively. 2^x stands for $exp_2(x)$.

The non-logical axioms of $\mathbf{I}\Delta_0(\exp)$ are the following list:

- $x + 1 \neq 0$
- $x + 1 = y + 1 \to x = y$
- $0\dot{-}1 = 0$
- $(x + 1)\dot{-}1 = x$
- $x + 0 = x$
- $x + (y + 1) = (x + y) + 1$
- $x\dot{-}0 = x$
- $x\dot{-}(y + 1) = x\dot{-}y\dot{-}1$
- $x \cdot 0 = 0$
- $x \cdot (y + 1) = x \cdot y + x$
- $2^0 = 1$
- $2^{x+1} = 2^x + 2^x$

along with the bounded induction scheme:

$$B(0) \wedge \forall x(B(x) \to B(x + 1)) \to \forall x B(x)$$

where B is a Δ-*formula*, that is a formula one of the following forms (with bounded quantifiers only):

- $B \coloneqq \forall x < tP(x) \equiv \forall x(x < t \to P(x))$
- $B \coloneqq \exists x < tP(x) \equiv \exists x(x < t \wedge P(x))$

A Σ_1-formula is a formula of the form:

$$\exists \vec{x} B(\vec{x})$$

where $B(\vec{x}) \in \Delta_0$.

Lemma 1.1. $\mathbf{I}\Delta_0(\exp)$ proves (the universal closures of):

1. $x = 0 \vee x = (x\dot{-}1) + 1$
2. $x + (y + z) = (x + y) + z$
3. $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
4. $x \cdot (y + z) = x \cdot y + x \cdot z$
5. $x + y = y + x$
6. $x \cdot y = y \cdot x$
7. $x\dot{-}(y + z) = (x\dot{-}y)\dot{-}z$

8. $2^{x+y} = 2^x \cdot 2^y$

Proof.

1. This is self-evident.

2. If $z = 0$, then $x + y = x + y$. If $z = z' + 1$, then, by applying the IH and the relevant axioms:

$$(x + (y + (z' + 1))) = (x + ((y + z') + 1)) = (x + (y + z')) + 1 = \\ ((x + y) + z') + 1 = (x + y) + (z' + 1)$$

3. If $z = 0$, then $x \cdot (y \cdot 0) = (x \cdot y) \cdot 0$. If $z = z' + 1$, then:

$$x \cdot (y \cdot (z'+1)) = x \cdot (y \cdot z' + y) = x \cdot (y \cdot z') + x \cdot y = (x \cdot y) \cdot z' + x \cdot y = (x \cdot y) \cdot (z'+1)$$

4. The rest of the cases are shown by induction on z. Consider the exponentiation law. If $y = 0$, then

$$2^{x+0} = 2^x = 0 + 2^x = 2^x \cdot 0 + 2^x = 2^x \cdot (0 + 1) = 2^x \cdot 2^0$$

If $y = y' + 1$, then:

$$2^{x+(y'+1)} = 2^{(x+y')+1} = 2^x \cdot 2^y + 2^x \cdot 2^y = 2^x \cdot 2^{y+1}$$

□

Lemma 1.2. $\mathbf{I}\Delta_0(\exp)$ proves (the universal closures of):

1. $\neg x < 0$
2. $x \leq 0 \leftrightarrow x = 0$
3. $0 \leq x$
4. $x \leq x$
5. $x < x + 1$
6. $x < y + 1 \leftrightarrow x \leq y$
7. $x \leq y \leftrightarrow x < y \vee x = y$
8. $x \leq y \wedge y \leq z \rightarrow x \leq z$
9. $x < y \wedge y < z \rightarrow x < z$
10. $x \leq y \vee y < x$
11. $x < y \rightarrow x + z < y + z$

12. $x < y \to x \cdot (z+1) < y \cdot (z+1)$
13. $x < 2^x$
14. $x < y \to 2^x < 2^y$

Proof. Straightforward induction. $\square$

Definition 1.1. A function $f : \mathbb{N}^k \to \mathbb{N}$ is *provably* Σ_1 or *provably recursive* in an arithmetical theory if there is a Σ_1 formula $F(\vec{x}, y)$, a "defining formula" of f, such that:

1. $f(\vec{n}) = m$ iff $\omega \models f(\vec{n}) = m$
2. $T \vdash \exists y F(\vec{x}, y)$
3. $T \vdash F(\vec{x}, y) \wedge F(\vec{x}, y') \to y = y'$

If a defining formula $F \in \Delta_0$, then a function f is *provably bounded* in T if there is a term $t(\vec{x})$ such that $T \vdash F(\vec{x}, y) \to y < t(\vec{x})$.

Theorem 1.1. Let f be a provably recursive in T, then we can conservatively extend T by adding a new function symbol f along with the defining axiom $F(\vec{x}, f(\vec{x}))$.

Proof. Let $\mathcal{M} \models T$, $\mathcal{M}$ can be made into a model $(\mathcal{M}, f)$ where we interpret f as the function which is uniquely determined by the second and third conditions of the definitions above. Let φ be a statement not involving f such that φ is true in $(\mathcal{M}, f)$, so φ is true in $\mathcal{M}$ as well. By compactness T proves φ. $\square$

Lemma 1.3. Each term defines a provably bounded function of $\mathbf{I}\Delta_0(\exp)$.

Proof. Let f be a function defined by some $\mathbf{I}\Delta_0(\exp)$-term t, that is, $f(\vec{x}) = t(\vec{x})$. Take $y = t(\vec{x})$ as the defining formula for f since $\exists y\,(y = t(\vec{x}))$ is derivable. If $y' = t(\vec{x}) \wedge y = t(\vec{x})$, then $y = y'$ by transitivity. A formula $y = t(\vec{x})$ is bounded and $y = t$ implies $y < t + 1$. Thus f is provably bounded. $\square$

Lemma 1.4. Define $2_k(x)$ as $2_0(x) = x$ and $2_{n+1}(x) = 2^{2_n(x)}$. Then for every term $t(x_1, \ldots, x_n)$ built up from the constants $0, S, P, +, \dot{-}, \cdot, exp_2$ there exists $k < \omega$ such that:

$$\mathbf{I}\Delta_0(\exp) \vdash t(x_1, \ldots, x_n) < 2_k(\sum_{k=0}^{n} x_k)$$

Proof. Let t be a term constructed from subterms t_0 and t_1 by using one of the function constants. Assume that inductively $t_0 < 2_{k_0}(s_0)$ and $t_1 < 2_{k_1}(s_1)$ are both provable for some $k_0, k_1 < \omega$, where s_i is the sum of the variables of t_i for $i = 0, 1$.

Let s be the sum of all variables appearing in either t_0 or t_1 and let $k = \max(k_0, k_1)$. Then one can prove $t_0 < 2_k(s)$ and $t_1 < 2_k(s)$. So one needs to show the following:

1. $t_0 + 1 < 2_{k+1}(s)$
2. $t_0 \dot{-} 1 < 2_k(s)$
3. $t_0 \dot{-} t_1 < 2_k(s)$
4. $t_0 \cdot t_1 < 2_k(s)$
5. $t_0 + t_1 < 2_k(s)$
6. $2^{t_0} < 2_k(s)$

So $\mathbf{I}\Delta_0(\exp) \vdash t < 2_{k+1}(s)$. □

Lemma 1.5. Let f be a function defined by composition:

$$f(\vec{x}) = g_0(g_1(\vec{x}), \ldots, g_m(\vec{x}))$$

where $g_0, g_1, \ldots, g_m$ are functions each of which is provably bounded in $\mathbf{I}\Delta_0(\exp)$. Then f is provably bounded in $\mathbf{I}\Delta_0(\exp)$.

Proof. Each g_i has a defining formula G_i and, by Lemma 1.4, there is a number $k_i < \omega$ such that:

$$\mathbf{I}\Delta_0(\exp) \vdash \exists y < 2_{k_i}(s)\, G_i(\vec{x}, y)$$

where s is the sum of elements of $\vec{x}$. And for $i = 0$ one has:

$$\mathbf{I}\Delta_0(\exp) \vdash \exists y < 2_{k_0}(s_0)\, G_0(y_1, \ldots, y_m, y)$$

where s_0 is the sum of $y_1, \ldots, y_m$.

Let $k = \max\{k_i < \omega \mid i < m+1\}$ and let $F(\vec{x}, y)$ be the bounded formula:

$$\exists y_1 < 2_k(s)\ \ldots\ \exists y_m < 2_k(s)\, C(\vec{x}, y_1, \ldots, y_m, y)$$

where $C(\vec{x}, y_1, \ldots, y_m, y)$ is the conjunction:

$$G_1(\vec{x}, y_1) \wedge \cdots \wedge G_m(\vec{x}, y_m) \wedge G_0(y_1, \ldots, y_m, y)$$

F is clearly a defining formula for f such that $\mathbf{I}\Delta_0(\exp) \vdash \exists y F(\vec{x}, y)$.

Moreover, each G_i is unique, so $\mathbf{I}\Delta_0(\exp)$ also proves:

$$\begin{aligned}
&C(\vec{x}, y_1, \ldots, y_m, y) \wedge C(\vec{x}, z_1, \ldots, z_m, z) \to \\
&\to \bigwedge_{j=1}^{m} y_j = z_j \wedge G_0(y_1, \ldots, y_m, y) \wedge G_0(y_1, \ldots, y_m, z) \to \\
&\to y = z
\end{aligned}$$

so we have (by first order logic):

$$\mathbf{I}\Delta_0(\exp) \vdash F(\vec{x}, y) \wedge F(\vec{x}, z) \to y = z$$

Thus f is provably Σ_1 in $\mathbf{I}\Delta_0(\exp)$, so the rest is to find its bounding term. $\mathbf{I}\Delta_0(\exp)$ proves the following:

$$C(\vec{x}, y_1, \ldots, y_m, y) \to \bigwedge_{j=1}^{m} y_j < 2_k(s) \wedge y < 2_k(y_1 + \cdots + y_m)$$

and

$$\bigwedge_{j=1}^{m} y_j < 2_k(s) \to y_1 + \cdots + y_m < 2_k(s) \cdot m$$

Put $t(\vec{x}) = 2_k(2_k(s) \cdot m)$, then we obtain

$$\mathbf{I}\Delta_0(\exp) \vdash C(\vec{x}, y_1, \ldots, y_m, y) \to y < t(\vec{x})$$

and so

$$\mathbf{I}\Delta_0(\exp) \vdash F(\vec{x}, y) \to y < t(\vec{x})$$

$\square$

Lemma 1.6. Suppose f is defined by bounded minimisation

$$f(\vec{n}, m) = \mu_{k<m}(g(\vec{n}, k) = 0)$$

from a function g which is provably bounded in $\mathbf{I}\Delta_0(\exp)$. Then f is provably bounded in $\mathbf{I}\Delta_0(\exp)$.

Proof. Let G be a defining formula for g. Let $F(\vec{x}, z, y)$ be the bounded formula

$$y \leq z \wedge \forall i < y \neg G(\vec{x}, i, 0) \wedge (y = z \vee G(\vec{x}, y, 0))$$

$\omega \models F(\vec{n}, m, k)$ iff either k is the least number less than m such that $g(\vec{n}, k) = 0$ or there is no such and $k = m$. Thus it means that k is the value of $f(\vec{n}, m)$, so F is a defining formula for f.

Furthermore

$$\mathbf{I}\Delta_0(\exp) \vdash F(\vec{x}, z, y) \to y < z + 1$$

so $t(\vec{x}, z) = z + 1$ can be taken as a bounding term for f.

We can prove:

$$F(\vec{x}, z, y) \wedge F(\vec{x}, z, y') \wedge y < y' \to G(\vec{x}, y, 0) \wedge \neg G(\vec{x}, y, 0)$$

and similarly for interchanged y and y'. So we can prove:

$$F(\vec{x}, z, y) \wedge F(\vec{x}, z, y') \to \neg y < y' \wedge \neg y' < y$$

As far as $y < y' \vee y' < y \vee y = y'$, we have

$$F(\vec{x}, z, y) \wedge F(\vec{x}, z, y') \to y = y'$$

Now we have to check that $\mathbf{I}\Delta_0(\exp) \vdash \exists y F(\vec{x}, z, y)$. We construct such y by bounded induction on z.

1. $z = 0$.

   $F(\vec{x}, 0, 0)$ is provable since $y = 0 \leftrightarrow y \leq 0$ and $\neg i < 0$. So $\mathbf{I}\Delta_0(\exp) \vdash F(\vec{x}, 0, y)$ is provable.

2. Assume $\exists y F(\vec{x}, z, y)$ is provable, let show that that $\exists y F(\vec{x}, z+1, y)$ is provable.

   We can show $y \leq z \to y+1 \leq z+1$ and, via $i < y+1 \leftrightarrow i < y \vee i = y$,

$$\forall i < y \, \neg G(\vec{x}, i, 0) \wedge ((y = z) \wedge \neg G(\vec{x}, y, 0)) \to \forall i < y+1 \, \neg G(\vec{x}, i, 0) \wedge y+1 = z+1$$

   Therefore

$$F(\vec{x}, z, y) \to F(\vec{x}, z+1, y+1) \vee F(\vec{x}, z+1, y)$$

   and thus:

$$\exists y F(\vec{x}, z, y) \to \exists y F(\vec{x}, z+1, y)$$

□

Theorem 1.2. Every elementary function is provably bounded in $\mathbf{I}\Delta_0(\exp)$.

Proof. As we know from recursion theory, the class of elementary functions can be characterised as those functions which are definable from 0, S, P, $\cdot$, $+$, exp_2, $\dot{-}$ and $\cdot$ by composition and minimisation. And then we apply above lemmas. □

1.1 Proof-theoretic Characterisation

For this section we shall be using a Tait-style formalisation of $\mathbf{I}\Delta_0(\exp)$. We have the following logical rules:

$$\frac{}{\Gamma, R\vec{t}, \neg R\vec{t}}\ \mathbf{Ax}$$

$$\frac{\Gamma, A_0, A_1}{\Gamma, A_0 \vee A_1}\ \vee \qquad\qquad \frac{\Gamma, A_0 \qquad \Gamma, A_1}{\Gamma, A_0 \wedge A_1}\ \wedge$$

$$\frac{\Gamma, A(t)}{\Gamma, \exists x A(x)}\ \exists \qquad\qquad \frac{\Gamma, A}{\Gamma, \forall x A}\ \forall$$

where $R\vec{t}$ is an atomic formula and x is not free in A in the $\forall$ rule. Here Γ stores all non-logical axioms of $\mathbf{I}\Delta_0(\exp)$ along with its negations. We also have the bounded induction rule:

$$\frac{\Gamma, B(0) \qquad \Gamma, \neg B(n), B(n+1)}{\Gamma, B(t)}\ \mathbf{BInd}$$

where B is a bounded formula and t is any term.

Of course, the cut rule is admissible:

$$\frac{\Gamma, A \qquad \Gamma, \neg A}{\Gamma}\ \mathbf{cut}$$

Definition 1.2. Let $\exists \vec{z} B(\vec{z})$ be a closed Σ_1-formula, then it is *true at* m, written as $m \models \exists \vec{z} B(\vec{z})$, if there exist natural numbers $m_1, \ldots, m_l$ such that each $m_i < m$ and $B(\vec{m})$ is true in the standard model.

A finite set Γ of closed Σ_1-formulas is true at m, written as $m \models \Gamma$ if at least one of them is true at m.

If $\Gamma(x_1, \ldots, x_k)$ is a finite set of Σ_1-formulas whose free variables occur amongst $x_1, \ldots, x_k$. Let $f : \mathbb{N}^k \to \mathbb{N}$, then $f \models \Gamma(x_1, \ldots, x_k)$ we have $f(\vec{n}) \models \Gamma(x_1 := n_1, \ldots, x_k := n_k)$ for each $\vec{n} = (n_1, \ldots, n_k)$.

Fact 1.1. (Persistence)

1. If $m \leq m'$, then $m \models \exists \vec{z} B(\vec{z})$ implies $m' \models \exists \vec{z} B(\vec{z})$.
2. If $\forall \vec{n} \in \mathbb{N}^k\ f(\vec{n}) \leq f'(\vec{n})$, then $f(\vec{n}) \models \Gamma(x_1 := n_1, \ldots, x_k := n_k)$ implies $f'(\vec{n}) \models \Gamma(x_1 := n_1, \ldots, x_k := n_k)$.

Lemma 1.7. Let $\Gamma(\vec{x})$ be a finite set of Σ_1 formulas such that

$$\mathbf{I}\Delta_0(exp) \vdash \bigvee_{\gamma(\vec{x}) \in \Gamma(\vec{x})} \gamma(\vec{x}).$$

Then there is an elementary function f such that $f \models \Gamma(\vec{x})$ and f is strongly increasing on its variables.

Proof. If Γ is provable in $\mathbf{I}\Delta_0(exp)$, then it is provable in the Tait-style version of $\mathbf{I}\Delta_0(exp)$, where all cut formulas are Σ_1.

If Γ is classically derivable from non-logical axioms $A_1, \ldots, A_s$, then there is a cut-free proof in the Tait calculus of $\neg A_1, \Delta, \Gamma$, where $\Delta = \neg A_2, \ldots, \neg A_s$. Let us show how to cancel $\neg A_1$ using a Σ_1-cut.

If A_1 is an induction axiom on some formula B, then we have a cut-free proof of:

$$B(0) \wedge \forall y(\neg B(y) \vee B(y+1)) \wedge \exists x \neg B(x), \Delta, \Gamma$$

Thus we also have cut-free proofs of $B(0), \Delta, \Gamma$, $\neg B(y), B(y+1), \Delta, \Gamma$ and $\exists x \neg B(x), \Delta, \Gamma$. So we have

$$\dfrac{\dfrac{\dfrac{\Delta, \Gamma, B(0) \qquad \Delta, \Gamma, \neg B(y), B(y+1)}{\Delta, \Gamma, B(x)}\ \mathbf{BInd}}{\Delta, \Gamma, \forall x B(x)}\ \forall \qquad \exists x \neg B(x), \Delta, \Gamma}{\Delta, \Gamma}\ \Sigma_1\text{-}\mathbf{cut}$$

We can similarly cancel each of $\neg A_2, \ldots, \neg A_s$ and so obtain the proof of Γ with Σ_1-cuts only.

Now we choose a proof of $\Gamma(\vec{x})$ and proceed by induction on the height of the proof and determine an elementary function f such that $f \models \Gamma$.

1. If $\Gamma(\vec{x})$ is an axiom, then for all $\vec{n}$ $\Gamma(\vec{n})$ contains a true atom. So for any f $f \models \Gamma$. Let us choose $f(\vec{n}) = n_1 + \cdots + n_k$.

2. If $\Gamma, B_0 \vee B_1$ is derivable, so is Γ, B_0, B_1. Note that B_0 and B_1 are both bounded. Let $f \models \Gamma, B_0, B_1$, then $f \models \Gamma, B_0 \vee B_1$.

3. Assume $\Gamma, B_0 \wedge B_1$ is derivable, then Γ, B_0 and Γ, B_1 By the induction hypothesis we have $f_0 \models \Gamma, B_0$ and $f_1 \models \Gamma, B_1$, so, by persistence, we have $\lambda \vec{n}. f_0(\vec{n}) + f_1(\vec{n}) \models \Gamma, B_0 \wedge B_1$.

4. Assume $\Gamma, \forall y B(y)$ is derivable, then $\Gamma, B(y)$ is derivable and y is not free in Γ. Since all the formulas are Σ_1, $\forall x B(y)$ must be bounded, so $B(y) \coloneqq \neg(y < t) \vee B'(y)$ for some term t and for some bounded formula B'. By the induction hypothesis, assume $f_0 \models \Gamma, \neg(y < t), B'(y)$ for some increasing elementary function f_0. Then we have:

   $$f_0(\vec{n}, k) \models \Gamma(\vec{n}), \neg(k < t(\vec{n})), B'(\vec{n}, k)$$

   Let g be an increasing elementary function bounding t, define

   $$f(\vec{n}) = \sum_{k < g(\vec{n})} f(\vec{n}, k)$$

   We have either $f(\vec{n}) \models \Gamma(\vec{n})$ or, by persistence, $B'(\vec{n}, k)$ is true for every $k < t(\vec{n})$. So $f \models \Gamma, \forall y B(y)$ and f is elementary.

5. Assume $\Gamma, \exists y A(y, \vec{x})$ is derivable, so $\Gamma, A(t, \vec{x})$ is derivable for some term t. By the IH, there is elementary f_0 such that for all $\vec{n}$ one has

   $$f_0(\vec{n}) \models \Gamma(\vec{n}), A(t(\vec{n}), \vec{n})$$

   Then either $f_0(\vec{n}) \models \Gamma(\vec{n})$ or else $f_0(\vec{n})$ bounds true witnesses for all existential quantifiers in $A(t(\vec{n}), \vec{n})$. Choose an elementary function g which is bounding for t. Define $f(\vec{n}) = f_0(\vec{n}) + g(\vec{n})$, then for all $\vec{n}$ either $f(\vec{n}) \models \Gamma(\vec{n})$ or $f(\vec{n}) \models \exists y A(y, \vec{n})$.

6. Assume Γ comes about by the cut rule with Σ_1 formula $C \coloneqq \exists \vec{z} B(\vec{z})$, so the premises are $\Gamma, \forall \vec{z} \neg B(\vec{z})$ and $\Gamma, \exists \vec{z} B(\vec{z})$.

   Without increasing the height of a proof, we can invert all universal quantifiers in the first premise. So we have $\neg B(\vec{z})$. B is bounded, so the induction hypothesis can be applied to this formula to obtain an elementary function f_0 such that, for all assignments $[\vec{x} := \vec{n}]$ and $[\vec{z} := \vec{m}]$

   $$f_0(\vec{n}, \vec{m}) \models \Gamma(\vec{n}), \neg B(\vec{n}, \vec{m})$$

   Now we apply the induction hypothesis to the second premise of the cut rule, so we have an elementary function f_1 such that for all $\vec{n}$ either $f_1(\vec{n}) \models \Gamma(\vec{n})$ or there are fixed witnesses $\vec{m} < f_1(\vec{n})$ such that $B(\vec{n}, \vec{m})$ is true.

   Define f the following way:

$$f(\vec{n}) = f_0(\vec{n}, f_1(\vec{n}), \ldots, f_1(\vec{n}))$$

Furthermore $f \models \Gamma$. For otherwise there would be a tuple $\vec{n}$ such that $\Gamma(\vec{n})$ is not true at $f(\vec{n})$, so, by persistence, $\Gamma(\vec{n})$ is not true at $f_1(\vec{n})$. Thus $B(\vec{n}, \vec{m})$ is true for particular numbers $\vec{m} < f_1(\vec{n})$. But then $f_0(\vec{n}, \vec{m}) < f(\vec{n})$, so, by persistence, $\Gamma(\vec{n})$ cannot be true at $f_0(\vec{n}, \vec{m})$. Thus $B(\vec{n}, \vec{m})$ is false, so we have a contradiction.

7. Finally suppose $\Gamma(\vec{x}), B(\vec{x}, t)$ comes from the induction rule on a bounded formula B. The premises of the rule $\Gamma(\vec{x}), B(\vec{x}, 0)$ and $\Gamma(\vec{x}), \neg B(\vec{x}, y), B(\vec{x}, y+1)$.

   Let us apply the induction hypothesis to each of the premises, and then we obtain increasing elementary functions f_0 and f_1 such that for all $\vec{n}$ and for all k

   $$f_0(\vec{n}) \models \Gamma(\vec{n}), B(\vec{n}, 0)$$
   $$f_1(\vec{n}, k) \models \Gamma(\vec{n}), \neg B(\vec{n}, k), B(\vec{n}, k+1)$$

   Now let

   $$f(\vec{n}) = f_0(\vec{n}) + \sum_{k < g(\vec{n})} f_1(\vec{n}, k)$$

   where g is an increasing elementary function which is bounding for the term t. f is elementary and increasing, and, by persistence for f_0 and f_1, we have either $f(\vec{n}) \models \Gamma(\vec{n})$ or else $B(\vec{n}, 0)$ and $B(\vec{n}, k) \to B(\vec{n}, k+1)$ are true for all $k < t(\vec{n})$. In either case, we have $f \models \Gamma(\vec{x}), B(\vec{x}, t(\vec{x}))$.

$\square$

Theorem 1.3. A number-theoretic function is elementary iff f is provably Σ_1 in $\mathbf{I}\Delta_0(exp)$.

Proof. The only if part is in Theorem 1.2, so we show the if part only. Assume f is provably Σ_1 in $\mathbf{I}\Delta_0(exp)$. Then we have a formula

$$F(\vec{x}, y) = \exists z_1 \ldots \exists z_k B(\vec{x}, y, z_1, \ldots, z_k)$$

which defines f and such that

$$\mathbf{I}\Delta_0(exp) \models \exists y F(\vec{x}, y)$$

By Lemma 1.7, there exists an elementary function g such that for every tuple of arguments $\vec{n}$ there are numbers $m_0, \ldots, m_k$ less that $g(n)$ satisfying the bounded formula $B(\vec{n}, m_0, m_1, \ldots, m_k)$. Apply the elementary sequence coding:

$$h(\vec{n}) = \langle g(\vec{n}), g(\vec{n}), \ldots, g(\vec{n}) \rangle$$

so that if $m = \langle m_0, m_1, \dots, m_k\rangle$ where $m_i < g(\vec{n})$ for each $i \in n+1$, so $m < h(\vec{n})$.

As far as $f(\vec{n})$ is the unique m_0 for which there are $m_1, \dots, m_k$ satisfying $B(\vec{n}, m_0, \dots, m_k)$, we define f as:

$$f(\vec{n}) = (\mu_{m<h(\vec{n})} B(\vec{n}, (m)_0, (m)_1, \dots, (m)_k))_0.$$

B is a bounded formula of $\mathbf{I}\Delta_0(exp)$, B is elementarily decidable. Moreover, elementary functions are closed under composition and bounded minimisation, so f is elementary. $\square$

2 Primitive Recursion and $\mathbf{I}\Sigma_1$

$\mathbf{I}\Sigma_1$ is an arithmetical theory where the induction scheme is restructed to Σ_1 formulas.

Lemma 2.1. Every primitive recursion is provably recursive in $\mathbf{I}\Sigma_1$.

Proof. We have to show represent each primitive recursive function f with a Σ_1 formula $F(\vec{x}, y) := \exists z C(\vec{x}, y, z)$ such that:

1. $f(\vec{n}) = m$ iff $\omega \models F(\vec{x}, y)$.
2. $\mathbf{I}\Sigma_1 \vdash \exists y F(\vec{x}, y)$.
3. $\mathbf{I}\Sigma_1 \vdash F(\vec{x}, y) \land F(\vec{x}, y') \to y = y'$.

In each case $C(\vec{x}, y, z)$ will be a $\Delta_0(exp)$-formula constructed via sequence encoding in $\mathbf{I}\Delta_0(exp)$. Such a formula expresses that z is a uniquely determined sequence number encoding the computation of $f(\vec{x}) = y$ and containing the output value y as its final element, so $y = \pi_2(z)$.

Condition 1 will hold by the definition of C. Condition 3 will be satisfied by the uniqueness of z. We consider five definitional schemes by which f could be introduced:

1. f is the constant-zero function, that is, $f(x) = 0$, no matter what x is. Then we take $C := y = 0 \land z = \langle 0\rangle$. All the conditions are obviously satisfied.
2. If f is the successor function $f(x) = x + 1$, we let
$$C(x, y, z) := y = x + 1 \land z = \langle x + 1\rangle$$
All the conditions are obvious.
3. Now assume f is the projection function $f(x_0, \dots, x_n) = x_i$ for some $i \in n + 1$. We let
$$C(\vec{x}, y, z) := y = x_i \land z = \langle x_i\rangle$$

4. Now assume f is defined by substitution from previously generated primitive recursive functions f_0, f_1, f_2:

$$f(\vec{x}) = f_0(f_1(\vec{x}), f_2(\vec{x}))$$

By the induction hypothesis, assume thatb f_0, f_1, f_2 are provably recursive and we have $\Delta_0(exp)$-formulas C_0, C_1, C_2 encoding their computations ($\text{len}(z) = 4$). For the function f define:

$$C(\vec{x}, y, z) := \\ \bigwedge_{i \in \{1,2\}} C_i(\vec{x}, \pi_2((z)_i), (z)_i) \wedge C_0(\pi_2((z)_1), \pi_2((z)_2), y, (z)_0) \wedge (z)_3 = y.$$

Let us check the required conditions:

(a) Condition 1 holds since $f(\vec{n}) = m$ iff there are numbers m_1 and m_2 such that $f_1(\vec{n}) = m_1$, $f_2(\vec{n}) = m_2$ and $f_0(m_1, m_2) = m$. These hold if and only if there are number k_1, k_2, k_0 such that $C_1(\vec{n}, m_1, k_1)$, $C_2(\vec{n}, m_2, k_2)$ and $C_0(m_1, m_2, m, k_0)$ are all true. And these hold if and only if $C(\vec{n}, m, \langle k_0, k_1, k_2, m \rangle)$ is true. Thus $f(\vec{n}) = m$ iff and only if $F(\vec{n}, m) = \exists z C(\vec{n}, m, z)$ is true.

(b) Condition 2 holds since from $C_1(\vec{x}, y_1, z_1)$, $C_2(\vec{x}, y_2, z_2)$ and $C(y_1, y_2, y, z_0)$ we can derive $C(\vec{x}, y, \langle z_0, z_1, z_2, y \rangle)$ in $\mathbf{I}\Delta_0$. So provided $\exists y \exists z C_1(\vec{x}, y, z)$, $\exists y \exists z C_2(\vec{x}, y, z)$ and $\forall y_1 \forall y_2 \exists y \exists z C(y_1, y_2, y, z)$, we can prove $\exists y F(\vec{x}, y) := C(\vec{x}, y, z)$.

(c) Condition 3 is self-evident.

5. Now assume that f is defined from f_1 and f_2 by primitive recursion:

$$f(\vec{v}, 0) = f_0(\vec{v})$$
$$f(\vec{v}, x+1) = f_1(\vec{v}, x, f(\vec{v}, x))$$

By the induction hypothesis f_0 and f_1 are provably recursive and they have associated Δ_0-formulas C_0 and C_1. Define

$$\begin{aligned} C(\vec{v}, x, y, z) := {} & C_0(\vec{v}, \pi_2((z)_0), (z)_0) \wedge \\ & \forall i < x \ (C_i(\vec{v}, i, \pi_2((z)_i), \pi_2((z)_{i+1}))) \wedge \\ & (z)_{x+1} = y \wedge \pi_2((z)_x) = y \end{aligned}$$

Let us check that all the conditions are satisfied:

(a) Condition 1 holds since $f(\vec{l}, n) = m$ if and only if there is a sequence number $k = \langle k_0, \ldots, k_n, m \rangle$ such that k_0 encodes the computation of $f(\vec{l}, 0)$ with the value $\pi_2(k_0)$, and for each $i < n$, k_{i+1} codes the computation of $f(\vec{l}, i+1) = f_1(\vec{l}, i, \pi_2(k_i))$ with values $\pi_2(k_{i+1})$ and $\pi_2(k_n) = m$. This is equivalent to $\models F(\vec{l}, n, m) \leftrightarrow \exists z C(\vec{l}, n, m, z)$.

(b) To show Condition 2 we have to prove the following in $\mathbf{I}\Delta_0$

$$C_0(\vec{v}, y, z) \to C(\vec{v}, 0, y, \langle z, y\rangle)$$

and

$$C(\vec{v}, x, y, z) \wedge C_1(\vec{v}, x, y, y', z') \to C(\vec{v}, x+1, y', t)$$

for a suitable term t which removes the end component y of z and replaces it by z', and then adds the final component y'. More specifically

$$t = \pi(\pi(\pi_1(z), z'), y')$$

Hence from $\exists y \exists z C_0(\vec{v}, y, z)$ we obtain $\exists y \exists z C(\vec{v}, 0, y, z)$, and from $\forall y \exists y' \exists z' C_1(\vec{v}, x, y, y', z')$ one can derive

$$\exists y \exists z C(\vec{v}, x, y, z) \to \exists y \exists z C(\vec{v}, x+1, y, z)$$

We have assumed that f_0 and f_1 are primitive recursive, we can prove $\exists y F(\vec{v}, 0, y)$ and $\exists y F(\vec{v}, x, y) \to \exists y F(\vec{v}, x+1, y)$. Then we derive $\exists y F(\vec{v}, x, y)$ by using Σ_1-induction.

(c) To show Condition 3 assume $C(\vec{v}, x, y, z)$ and $C(\vec{v}, x, y', z')$, where z and z' are sequence numbers of the same length $x+2$. Furthermore we have $C_0(\vec{v}, \pi_2((z)_0), (z)_0)$ and $C_0(\vec{v}, \pi_2((z')_0), (z')_0)$, so we have $(z)_0 = (z')_0$.

Similarly we have $\forall i < x \ C_1(\vec{v}, i, \pi_2((z)_i), \pi_2((z)_{i+1}), (z)_{i+1})$ and the same formula where z is replaced by z'. So if $(z)_i = (z')_i$, and one can deduce $(z)_{i+1} = (z')_{i+1}$ using the uniquness assumption for C_1. By $\Delta_0(exp)$-induction we obtain $\forall i \leq x \ ((z)_i = (z')_i)$.

The final conjuncts in C give $(z)_{x+1} = \pi_2((z)_x) = y$ and the same formulas where z is replaced by z' and where y is replaced by y'. But since $(z)_x = (z')_x$ we have $y = y'$, since all the components are equal, $z = z'$. Thus we have $F(\vec{v}, x, y) \wedge F(\vec{v}, x, y') \to y = y'$.

$\square$

2.1 $\mathbf{I}\Sigma_1$ provable functions are primitive recursive

Definition 2.1. A closed Σ_1-formula $\exists \vec{z} B(z)$ with $B \in \Delta_0(exp)$ is said to be "true at m" (denoted as $m \models \exists \vec{z} B(z)$) if there are numbers $\vec{m} = (m_1, \ldots, m_l)$ such that all $m_i < m$ for $i \in \{1, \ldots, l\}$ such that $B(\vec{m})$ is true in the standard model.

A finite set of formulas Γ of closed Σ_1-formulas is "true at m" (denoted as $m \models \Gamma$) if at least one of them is true at m.

If $\Gamma(x_1, \ldots, x_k)$ is a finite set of Σ_1-formulas all of whose free variables occur amongst $x_1, \ldots, x_k$ and if $f : \mathbb{N}^k \to \mathbb{N}$, then we write $f \models \Gamma$ if for each assignments $\vec{n} = (n_1, \ldots, n_k)$ to the variables $x_1, \ldots, x_k$ we have $f(\vec{n}) \models \Gamma(\vec{n})$.

Note that we have the persistence property for $\models$ which completely repeats persistence for $\mathbf{I}\Delta_0(exp)$.

We shall be using a Tait-style formalisation of $\mathbf{I}\Sigma_0$ where the induction rule

$$\frac{\Gamma, A(0) \qquad \Gamma, \neg A(y), A(y+1)}{\Gamma, A(t)}$$

where y is not free in Γ, t is any term and A is any Σ_1-formula.

Lemma 2.2. (Σ_1-induction) Let $\Gamma(\vec{x})$ be a finite set of Σ_1-formulas such that

$$\mathbf{I}\Sigma_1 \vdash \bigvee \Gamma(\vec{x})$$

then there is a primitive recursive function f such that $f \models \Gamma$ and f is strictly increasing on its variables.

Proof. We note that if Γ is provable in this formalisation, then it has a proof in which all the non-atomic cut formulas are induction Σ_1-formulas. If Γ is classically derivable from non-logical axioms $A_1, \ldots, A_s$, then there is a cut-free proof (à la Tait) of $\neg A_1, \Delta, \Gamma$ where $\Delta = A_2, \ldots, A_s$. Then if A_1 is an induction axiom on a formula F, then we have have a cut-free proof of

$$F(0) \wedge \forall y(\neg F(y) \vee F(y+1)) \wedge \neg F(t), \Delta, \Gamma$$

and thus, by inversion, we have cut-free proofs of $F(0), \Delta, \Gamma,\ \neg F(y), F(y+1), \Delta, \Gamma$ and $\neg F(t), \Delta, \Gamma$.

So we obtain $F(t), \Delta, \Gamma$ by the induction rule and then we obtain Δ, Γ by cutting $F(t)$. One can detach $\neg A_2, \ldots, \neg A_s$, so we finally have a proof of Γ which uses cuts only on Σ_1-induction formulas or on atoms arising from non-logical axioms. Such proofs are said to be "free-cut" free.

Let us choose such a proof for $\Gamma(\vec{x})$ and show by induction on the height of a proof that there exists a primitive recursive function satisfying $f \models \Gamma$.

1. Let $\Gamma(\vec{x})$ be an axiom, the for all $\vec{n}$ $\Gamma(\vec{n})$ contains a true atom. Choose $f(\vec{n}) = n_1 + \cdots + n_k$, and f is clearly primitive recursive, strictly incrasing and $f \models \Gamma$.

2. Assume we have

$$\frac{\Gamma, B_0, B_1}{\Gamma, B_0 \vee B_1} \vee$$

   Then both B_0 and B_1 are both $\Delta_0(exp)$-formulas, so any function f satisfying $f \models \Gamma, B_0, B_1$ also satisfies $\Gamma, B_0 \vee B_1$.

3. Assume we have

$$\frac{\Gamma, B_0 \qquad \Gamma, B_1}{\Gamma, B_0 \wedge B_1} \wedge$$

By the induction hypothesis we have $f_i(\vec{n}) \models \Gamma(\vec{n}), B_i(\vec{n})$ where $i \in \{0,1\}$ for all $\vec{n}$. By the persistence property, $\lambda\vec{n}.f_0(\vec{n}) + f_1(\vec{n}) \models \Gamma, B_0 \wedge B_1$.

4. Assume we have

$$\frac{\Gamma, B(y)}{\Gamma, \forall y B(y)}\ \forall$$

where y is not free in Γ. As far as all formulas are Σ_1, $\forall y B(y)$ must be $\mathbf{I}\Delta_0(exp)$, so $B(y) = \neg(y < t) \vee B'(y)$ for some elemetary or primitive recursive term t. Assume we have $f_0 \models \Gamma, \neg(y < t) \vee B'(y)$ for some increasing primitive recursive function f_0. Then, for any assignments $\vec{x} \mapsto \vec{n}$ and $y \mapsto k$, we have

$$f_0(\vec{n}, k) \models \Gamma(\vec{n}), \neg(k < t(\vec{n})), B'(\vec{n}, k).$$

We let

$$f(\vec{n}) = \sum_{k<g(\vec{n})} f_0(\vec{n}, k)$$

for some function g, which is increasing primitive recursive bounding the values of term t. So we have either $f(\vec{n}) \models \Gamma$ or $B'(\vec{n}, k)$ is true for every $k < t(\vec{n})$. Thus $f \models \Gamma, \forall y B(y)$ as required.

5. Suppose we have

$$\frac{\Gamma, A(t)}{\Gamma, \exists y A(y)}\ \exists$$

where A is a Σ_1-formula. By the induction hypothesis we have a function f_0 such that for all $\vec{n}$

$$f_0(\vec{n}) \models \Gamma(\vec{n}), A(t(\vec{n}), \vec{n})$$

Then either $f_0(\vec{n}) \models \Gamma(\vec{n})$ or otherwise $f_0(\vec{n})$ bounds true witnesses for all the existential quantifiers already in $A(t(\vec{n}, \vec{n}))$. Choose an elementary bounding function g for the term t and define $f(\vec{n}) = f_0(\vec{n}) + g(\vec{n})$, so we have either $f(\vec{n}) \models \Gamma(\vec{n})$ or $f(\vec{n}) \models \exists y A(y, \vec{n})$ for all $\vec{n}$.

6. Assume we have

$$\frac{\Gamma, \forall\vec{z}\neg B(\vec{z}) \qquad \Gamma, \exists\vec{z} B(\vec{z})}{\Gamma}\ \mathbf{cut}$$

where $\exists\vec{z}B(\vec{z})$ is a cut Σ_1-formula.

Note that we have

$$\frac{\Gamma, \neg B(\vec{z})}{\Gamma, \forall \vec{z} \neg B(\vec{z})}\ \forall$$

Note B is a $\Delta_0(\exp)$-formula, so let us apply the induction hypothesis to obtain a primitive recursive function f_0 such that for each assignments $\vec{x} \mapsto \vec{n}$ and $\vec{z} \mapsto \vec{m}$

$$f_0(\vec{n}, \vec{m}) \models \Gamma(\vec{n}), \neg B(\vec{n}, \vec{m}).$$

We apply the induction hypothesis to the second premise to obtain a primitive recursive function f_1 such that for all $\vec{n}$ either $f_1(\vec{n}) \models \Gamma(\vec{n})$ or otherwise there are fixed witnesses $\vec{m} < f_1(\vec{n})$ s.t. $B(\vec{n}, \vec{m})$ is true. Let us define f by substitution:

$$f(\vec{n}) = f_0(\vec{n}, f_1(\vec{n}), \ldots, f_1(\vec{n}))$$

where f is primitive recursive, greater or equal that f_1 (pointwise) and strictly increasing. Furthermore $f \models \Gamma$.

For otherwise, let us suppose there exists a tuple $\vec{n}$ such that $\Gamma(\vec{n})$ is not true $f(\vec{n})$ and, thus, by persistence at $f_1(\vec{n})$. So $B(\vec{n}, \vec{m})$ is true for some $\vec{m} < f_1(\vec{n})$. Thus $f_0(\vec{n}, \vec{m}) < f(\vec{n})$, and then, by persistence, $\Gamma(\vec{n})$ cannot be true at $f_0(\vec{n}, \vec{m})$. Then $B(\vec{n}, \vec{m})$, so we have a contradiction.

7. Suppose we have

$$\frac{\Gamma(\vec{x}), A(\vec{x}, 0) \qquad \Gamma, \neg A(\vec{x}, y), A(\vec{x}, y+1)}{\Gamma, A(\vec{x}, t)}$$

where $A(\vec{x}, y)$ is an induction Σ_1-formula of the form $\exists \vec{z} B(\vec{x}, y, \vec{z})$. Let us invert universal quantifiers in $\neg A(\vec{x}, y)$, the second premise of the rule becomes

$$\Gamma(\vec{x}), \neg B(\vec{x}, y, \vec{z}), A(\vec{x}, y+1)$$

which is now a set Σ_1-formulas. We can apply the induction hypothesis to each of the premises to have primitive recursive function f_0 and f_1 such that for each $\vec{n}$, k and $\vec{m}$

$$\begin{gathered} f_0(\vec{n}) \models \Gamma(\vec{n}), A(\vec{n}, 0) \\ f_1(\vec{n}, k, \vec{m}) \models \Gamma(\vec{n}), \neg B(\vec{n}, k, \vec{m}), A(\vec{n}, k+1) \end{gathered}$$

Define f by primitive recursion from f_0 and f_1 the following way

$$\begin{gathered} f(\vec{n}, 0) = f_0(\vec{n}) \\ f(\vec{n}, k+1) = f_1(\vec{n}, k, f(\vec{n}, k), \ldots, f(\vec{n}, k)) \end{gathered}$$

Then for all $\vec{n}$ and for all $\vec{k}$ one has $f(\vec{n}, k) \models \Gamma(\vec{n}), A(\vec{n}, k)$ which is shown by induction on k. The base case holds by the definition of $f_0(\vec{n})$. For the induction step assume that $f(\vec{n}, k) \models \Gamma(\vec{n}), A(\vec{n}, k)$. If $\Gamma(\vec{n})$ is not true at $f(\vec{n}, k+1)$. By persistence it is not true at $f(\vec{n}, k)$ and thus $f(\vec{n}, k) \models A(\vec{n}, k)$. Therefore there are numbers $\vec{m} < f(\vec{n}, k)$ such that $B(\vec{n}, k, \vec{m})$ is true. Thus $f_1(\vec{n}, k, \vec{m}) \models \Gamma(\vec{n}), A(\vec{n}, k+1)$ and since $f_1(\vec{n}, k, \vec{m}) \leq f(\vec{n}, k+1)$ we have, by persistence, $f(\vec{n}, k+1) \models \Gamma(\vec{n}), A(\vec{n}, k+1)$ as required.

So we substitute for the final argument k in f an elementary (or primitive recursive) function g which bounds the values of t, so that $f'(\vec{n}) = f(\vec{n}, g(\vec{n}))$, and thus we have $f(\vec{n}, t(\vec{n})) \models \Gamma(\vec{n}), A(\vec{n}, t(\vec{n}))$ for all $\vec{n}$ and thus, by persistence, $f' \models \Gamma(\vec{x}), A(\vec{x}, t)$.

□

Theorem 2.1. The provably recursive functions of $\mathbf{I}\Sigma_1$ are exactly primitive recursive functions.

Proof. We have already shown that all primitive recursive functions are provably recursive in $\mathbf{I}\Sigma_1$, so let us show the converse.

Let $g : \mathbb{N}^k \to \mathbb{N}$ is Σ_1 be a function defined by a Σ_1-formula $F(\vec{x}, y) := \exists z C(\vec{x}, y, z)$ where C is $\Delta_0(exp)$ and $\mathbf{I}\Sigma_1 \models \exists y F(\vec{x}, y)$. By the lemma above, there exists a primitive recursive function f such that for all $n \in \mathbb{N}^k$

$$f(\vec{n}) \models \exists y \exists z C(\vec{n}, y, z).$$

That is, for every $\vec{n}$ there is an $m < f(\vec{n})$ and a $k < f(\vec{n})$ such that $C(\vec{n}, m, k)$ is true and this m is the value of $g(\vec{n})$.

g can be defined by primitive recursion from f the following way:

$$g(\vec{n}) = (\mu_{m<h(\vec{n})} C(\vec{n}, (m)_0, (m)_1))$$

where $h(\vec{n}) = \langle f(\vec{n}), f(\vec{n}) \rangle$. □

3 ε_0-recursion in Peano Arithmetic

We show that the provably recursive functions of Peano arithmetic are ε_0-recursive functions, that is, functions definable from the primitive recursive functions by substitutions and recursion over well-orderings of natural numbers with order types strictly less than the ordinal

$$\varepsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \dots\}$$

Equivalently, ε_0 can be defined as the least fixed point of the mapping $\alpha \mapsto \omega^\alpha$ where α is an ordinal.

Let us discuss first how one can represent ordinals below ε_0.

3.1 Ordinals below ε_0

Every ordinal $\alpha < \varepsilon_0$ is either 0 or α can be represented uniquely in *Cantor normal form*:

$$\alpha = \omega^{\gamma_1} \cdot c_1 + \omega^{\gamma^{\gamma_1}} \cdot c_2 + \cdots + \omega^{\gamma_k} \cdot c_k$$

where $k < \omega$, $\gamma_k < \cdots < \gamma_2 < \gamma_1 < \alpha$ and $c_1, \ldots, c_k < \omega$ are coefficients. If $\gamma_k = 0$, then α is a successor ordinal, written $\mathrm{Succ}(\alpha)$, and its predecessor $\alpha - 1$ the representation

$$\alpha = \omega^{\gamma_1} \cdot c_1 + \omega^{\gamma^{\gamma_1}} \cdot c_2 + \cdots + \omega^{\gamma_{k-1}} \cdot c_{k-1}.$$

Otherwise α is a limit ordinal, written $\mathrm{Lim}(\alpha)$, and it has infinitely many possible increasing sequences of smaller ordinals whose limit is α.

We shall pick out one concrete sequence $\{\alpha(n) \,|\, n < \omega\}$ for each limit ordinal α the following way. First write α as $\delta + \omega^\gamma$ where

$$\delta = \omega^{\gamma_1} \cdot c_1 + \cdots + \omega^{\gamma_k} \cdot (c_k - 1)$$
$$\gamma = \gamma_k.$$

By induction we can assume that when γ is a limit ordinal, its fundamental sequence $\{\gamma(n) \mid n < \omega\}$ has been already specified. We let for each $n < \omega$

$$\alpha(n) = \begin{cases} \delta + \omega^{\gamma-1} \cdot (n+1), \text{if } \mathrm{Succ}(\gamma) \\ \delta + \omega^{\gamma(n)}, \text{if } \mathrm{Lim}(\gamma). \end{cases}$$

Clearly

$$\alpha = \lim_{n \to \omega} \alpha(n).$$

Definition 3.1. Let $\alpha < \varepsilon_0$ and $n < \omega$, define a finite set of ordinals $\alpha[n]$ the following way:

$$\alpha[n] = \begin{cases} \emptyset, \text{if } \alpha = 0 \\ (\alpha - 1)[n] \cup \{\alpha - 1\}, \text{if } \mathrm{Succ}(\alpha) \\ \alpha(n)[n], \text{ if } \mathrm{Lim}(\alpha) \end{cases}$$

Lemma 3.1. For each $\alpha = \delta + \omega^\gamma$ and for each $n < \omega$

$$\alpha[n] = \delta[n] \cup \{\delta + \omega^{\gamma_1} \cdot c_1 + \cdots + \omega^{\gamma_k} \cdot c_k \mid \forall i(\gamma_i \in \gamma[n] \wedge c_i \leq n)\}.$$

Proof. Induction on γ.

1. $\gamma = 0$, then $\gamma[n] = \emptyset$ and the right hand side is $\delta[n] \cap \{\delta\}$, which is the same as $\alpha[n] = (\delta + 1)[n]$.

2. If γ is limit, then $\gamma[n] = \gamma(n)[n]$, so the right hand side is the same as the one with $\gamma(n)[n]$ instead of $\gamma[n]$. By the induction hypothesis applied to $\alpha(n) = \delta + \omega^{\gamma(n)}$, which is equal to $\alpha(n)[n]$, which is $\alpha[n]$ by definition.

3. Suppose γ is a successor. Then α is a limit and $\alpha[n] = \alpha(n)[n]$, where $\alpha(n) = \delta + \omega^{\gamma-1} \cdot (n+1)$. So we can write $\alpha(n) = \alpha(n-1) + \omega^{\gamma-1}$, where $\alpha(-1) = \delta$ when $n = 0$. By the induction hypothesis for $\gamma - 1$, the set $\alpha[n]$ equals

   $$\alpha(n-1)[n] \cup \{\alpha(n-1) + \omega^{\gamma_1} \cdot c_1 + \cdots + \omega^{\gamma_k} \cdot c_k \,|\, \forall i(\gamma_1 \in (\gamma-1)[n] \wedge c_i \leq n)\}$$

   and similarly for each $\alpha(n-1)[n], \alpha(n-2)[n], \ldots, \alpha(1)[n]$. For each $m \leq n$, $\alpha(m-q) = \delta + \omega^{\gamma-1} \cdot m$. In turn, this last set is the same as

   $$\delta[n] \cup \{\delta + \omega^{\gamma-1} \cdot m + \omega^{\gamma_1} \cdot c_1 + \cdots + \omega^{\gamma_k} \cdot c_k \,|\, \forall i(\gamma_i \in (\gamma-1)[n] \wedge c_i \leq n) \wedge m \leq n\}$$

   and this is the set since $\gamma[n] = (\gamma-1)[n] \cup \{\gamma - 1\}$.

□

Corollary 3.1. Let $\alpha < \varepsilon_0$ be a limit ordinal, then for every $0 \neq n < \omega$ $\alpha(n) \in \alpha[n+1]$. Furthermore if $\beta \in \gamma[n]$, then $\omega^\beta \in \omega^\gamma[n]$.

Definition 3.2. The *maximum coefficient* of $\beta = \omega^{\beta_1} \cdot b_1 + \cdots + \omega^{\beta_l} \cdot b_l$ is defined by induction to be the maximum of all the b_i's and all the maximum coefficients of the exponents β_i's.

Lemma 3.2. If $\beta < \alpha$ and the maximum coefficient of β is $\leq n$, so $\beta \in \alpha[n]$.

Proof. By induction on α. Let $\alpha = \delta + \omega^\gamma$. If $\beta < \delta$, then $\beta \in \delta[n]$ by the induction hypothesis and $\delta[n] \subseteq \alpha[n]$ by Lemma 3.1. Otherwise

$$\beta = \delta + \omega^{\beta_1} \cdot b_1 + \cdots + \omega^{\beta_k} \cdot b_k$$

for $\alpha > \gamma > \beta_1 > \cdots > \beta_k$ and $b_i \leq n$. By induction hypothesis $\beta_i \in \gamma[n]$, so $\beta \in \alpha[n]$ by Lemma 3.1. □

Definition 3.3. Let $G_\alpha(n)$ denote the cardinality of the finite set $\alpha[n]$. We have

$$G_\alpha(n) = \begin{cases} 0, \text{if } \alpha = 0 \\ G_{\alpha-1}(n+1), \text{ if } \mathrm{Succ}(\alpha) \\ G_{\alpha(n)}(n), \text{if } \mathrm{Lim}(\alpha) \end{cases}$$

The hierarchy of functions G_α is the *slow-growing* hierarchy.

Lemma 3.3. If $\alpha = \delta + \omega^\gamma$, then for all $n < \omega$

$$G_\alpha(n) = G_\delta(n) + (n+1)^{G_\gamma(n)}.$$

Thus for each $\alpha < \varepsilon_0$, $G_\alpha(n)$ is the elementary function which results by substituting $n+1$ for every occurence of ω in the Cantor normal form ω.

Proof. Induction on γ.

1. If $\gamma = 0$, then $\alpha = \delta + 1$, thus

$$G_\alpha(n) = G_\delta(n) + 1 = G_\delta(n) + (n+1)^0.$$

2. If γ is a successor, then $\alpha = \delta + \omega^\gamma$ is limit and $\alpha(n) = \delta + \omega^{\gamma-1} \cdot (n+1)$, so we apply the induction hypothesis for $\gamma - 1$ $n+1$ times and thus we have

$$G_\alpha(n) = G_{\alpha(n)}(n) = G_\delta(n) + (n+1)^{G_{\gamma-1}(n)} \cdot (n+1) = G_\delta(n) + (n+1)^{G_\gamma(n)}$$

since $G_{\gamma-1}(n) + 1 = G_\gamma(n)$.

3. If γ is a limit ordinal, then $\alpha(n) = \delta + \omega^{\gamma(n)}$, so let us apply the induction hypothesis to $\gamma(n)$, then we have

$$G_\alpha(n) = G_{\alpha(n)}(n) = G_\delta(n) + (n+1)^{G_{\gamma(n)}(n)}$$

which gives the result since $\Gamma_{\gamma(n)}(n) = G_\gamma(n)$.

$\square$

Definition 3.4. (Coding ordinals)
Let $\beta = \omega^{\beta_1} \cdot b_1 + \dots \omega^{\beta_l} \cdot b_l$ be an ordinal. A *coding ordinal* is the sequence number $\overline{\beta}$ constructed recursively the following way

$$\overline{\beta} = \langle \langle \overline{\beta_1}, b_1 \rangle, \dots, \langle \overline{\beta_l}, b_l \rangle \rangle.$$

where 0 is coded by the empty sequence number. $\overline{\beta}$ is numerically greater than the maximum coefficient of β and greater than the codes $\overline{\beta_i}$ of all its exponents and their exponents, etc.

Lemma 3.4.

1. There exists an elementary function $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that, for each ordinal $\beta = \omega^{\beta_1} \cdot b_1 + \dots \omega^{\beta_l} \cdot b_l$:

$$h(\overline{\beta}, n) = \begin{cases} 0, \text{if } \beta = 0 \\ \overline{\beta - 1}, \text{if } \mathrm{Succ}(\beta) \\ \overline{\beta(n)}, \text{if } \mathrm{Lim}(\beta) \end{cases}$$

2. For each ordinal $\alpha < \varepsilon_0$ there exists an elementary well-ordering $\prec_\alpha \subset \mathbb{N} \times \mathbb{N}$ such that

$$\forall b, c \in \mathbb{N} \;\; b \prec_\alpha c \leftrightarrow \exists \beta, \gamma < \alpha \;\; \beta < \gamma \,\&\, b = \overline{\beta} \,\&\, c = \overline{\gamma}.$$

Proof.

1. First let

$$h(0, n) = 0$$

for any n. Then let $0 < m < \omega$ be a non-zero sequence number. We first should see if the rightmost component π_2 is a pair (m', n'). If so and $m' = 0$ and $n' \neq 0$, then β is a successor and the code of its predecessor, $h(m, n)$, is defined as the new sequence number that we obtain by reducing n' by one or by removing this final component if $n' = 1$.

If $\pi_2(m) = \langle m', n' \rangle$ where both m' and n' are non-zero, then β is a limit ordinal of then form $\delta + \omega^\gamma \cdot n'$ where $m' = \overline{\gamma}$. Let k be the code of $\delta + \omega^\gamma \cdot (n' - 1)$, which is obtained by reducing n' by one inside m (or by deleting the final component from m when $n' = 1$).

At the "right hand end" of β we have a "spare" ω^γ which must be either reduced to $\omega^{\gamma-1} \cdot (n+1)$ when $\mathrm{Succ}(\gamma)$ or to $\omega^{\gamma(n)}$ if $\mathrm{Lim}(\gamma)$. In either case we are able to produce $\beta(n)$. Thus the required code $h(m, n)$ of $\beta(n)$ will be obtained by tagging on to the end of the sequence number k one additinal pair encoding this additional term.

If we assume inductively that $h(m', n)$ has been already defined for $m' < m$, then such an additional component is either $\langle h(m', n), n+1 \rangle$ if $\mathrm{Succ}(\gamma)$ or $\langle h(m', n), 1 \rangle$ if $\mathrm{Lim}(\gamma)$.

This defines $h(m, n)$, but such a definition is actually primitive recursive so far. Let us chech that h is elementarily bounded, i.e. h is defined by limited recursion from elementary functions. Note that $h(m, n) < m$ whenever m codes a successor ordinal. If m codes a limit ordinal, $h(m, n)$ is obtained from the sequence number $k < m$ by adding a new pair on the end. An extra item i is tagged on the end of a sequence number k by the function $\pi(k, i)$ which is quadratic in both argument. If the item added is the pair $\langle h(m', n), n+1 \rangle$ where $\mathrm{Succ}(\gamma)$, then $h(m', n) < m$, so $h(m, n)$ is numerically bounded by some fixed polynomial in m and n. In the other case, we can say that $h(m, n)$ is numerically bounded by some fixed polynomial of m and $h(m', n)$. Since m' codes an exponent in the Cantor normal form encoded by m, the second polynomial is iterated at most d times, where d is the "exponential height" of the normal form. Thus $h(m, n)$ is bounded by some d-times iterated polynomial of $m+n$. $d < m$, so $h(m, n)$ is bounded by the elementary function $2^{2^{c \cdot (m+n)}}$ for some $c < \omega$. Therefore h is elementary as it is defined by bounded recursion.

2. Let $\alpha < \varepsilon_0$ and let d be the exponential height of its Cantor normal form. We use the function h from the previous part, but we apply it to codes below α only. They have the exponential height $\leq d$, so we can consider h as being bounded by some fixed polynomial of its two arguments. Define $g(0, n) = \overline{\alpha}$ and $g(i+1, n) = h(g(i, n), n)$ and notice that g is therefore bounded by an i-times iterated polynomial, so g is defined by an elementarily limited recursion from h, so it is elementary.

   Define $b \prec_\alpha c$ if and only if $c \neq 0$ and there are i and j such that $0 < i < j \leq G_\alpha(\max(b, c) + 1)$ and $g(i, \max(b, c)) = c$ and $g(j, \max(b, c)) = b$.

The function g and G_α are elementary, so is the relation $\prec_\alpha$ since the quantifiers are bounded. By the properties of h it is clear that if $i < j$ then $g(j, \max(b, c))$ codes an ordinal greater than $g(j, \max(b, c))$. Hence $b \prec_\alpha c$, then $b = \overline{\beta}$ and $c = \overline{\gamma}$ for some $\beta < \gamma < \alpha$.

Now assume $b = \overline{\beta}$, $c = \overline{\gamma}$ and $\beta < \gamma < \alpha$. The code of an ordinal is greater than its maximal coefficient, so we have $\beta \in \alpha[\max(b, c)]$ and $\gamma \in \alpha[\max(b, c)]$. Thus the sequence starting with α and at each stage descending from a δ to either $\delta - 1$ if $\mathrm{Succ}(\delta)$ or $\delta(\max(b, c))$ if $\mathrm{Lim}(\delta)$ necessarily passes through γ and then through β. In turn, it means there are $i, j < \omega$ such that $0 < i < j$, $g(i, \max(b, c)) = c$, $g(j, \max(b, c)) = b$. So $b \prec_\alpha c$ holds if we can show that $j \leq G_\alpha(\max(b, c) + 1)$. In the sequence described above, only the successor stages contribute an element $\delta - 1$ to $\alpha[\max(b, c)]$. At the limit stages $\delta(\max(b, c))$ does not get put it. Although $\delta(n)$ does not belong to $\delta[n]$, it does belong to $\delta[n + 1]$. Therefore all the ordinals in the descending sequence lie in $\alpha[\max(b, c) + 1]$, so j can not be bigger than the cardinality of this set, which is $G_\alpha(\max(b, c) + 1)$.

$\square$

The moral is that the principles of transfinite induction and recursion over the initials segments of ordinals below ε_0 can be expressed by means of $\mathbf{I}\Delta_0(exp)$.

3.2 Introducing the fast-growing hierarchy

Definition 3.5. The *Hardy hierarchy* $\{H_\alpha\}_{\alpha<\varepsilon_0}$ is defined by recursion on α:

$$H_\alpha(n) = \begin{cases} n, \text{ if } \alpha = 0 \\ H_{\alpha-1}(n+1), \text{if } \mathrm{Succ}(\alpha) \\ H_{\alpha(n)}(n), \text{if } \mathrm{Lim}(\alpha) \end{cases}$$

The *fast-growing hierarchy* $\{F_\alpha\}_{\alpha<\varepsilon_0}$ is defined by recursion on α:

$$F_\alpha(n) = \begin{cases} n+1, \text{if } \alpha = 0 \\ F^{n+1}_{\alpha-1}(n), \text{if } \mathrm{Succ}(\alpha) \\ F_{\alpha(n)}(n), \text{if } \mathrm{Lim}(\alpha) \end{cases}$$

where $F^{n+1}_{\alpha-1}(n)$ is the $(n+1)$-times iteration of $F_{\alpha-1}$ on n.

Note that H_α and F_α could be equivalently defined by purely number-theoretic means by working over the well-orderings $\prec_\alpha$ instead of working over ordinals directly. So H_α and F_α are ε_0-recursive.

Lemma 3.5. For all $\alpha, \beta < \varepsilon_0$ and for all $n < \omega$,

1. $H_{\alpha+\beta}(n) = H_\alpha(H_\beta(n))$,
2. $H_{\omega^\alpha}(n) = F_\alpha(n)$.

Proof. The first part is proved by induction on β. If $\beta = 0$, then the equation trivially holds. Assume Succ(β) and the induction hypothesis for $\beta - 1$, then we have:

$$H_{\alpha+\beta}(n) = H_{\alpha+(\beta-1)}(n+1) = H_\alpha(H_{\beta-1}(n+1)) = H_\alpha(H_\beta(n)).$$

If Lim(β), then we have (by using the induction hypothesis for $\beta(n)$):

$$H_{\alpha+\beta}(n) = H_{\alpha+\beta(n)}(n) = H_\alpha(H_{\beta(n)}(n)) = H_\alpha(H_\beta(n)).$$

The second part is proven by induction on α. If $\alpha = 0$, then

$$H_{\omega^0}(n) = H_1(n) = n + 1 = F_0(n)$$

If Succ(α), then

$$H_{\omega^\alpha}(n) = H_{\omega^{\alpha-1}\cdot(n+1)}(n) = H^{n+1}_{\omega^{\alpha-1}}(n) = F^{n+1}_{\alpha-1}(n) = F_\alpha(n).$$

The limit case is immediate. □

Lemma 3.6. For each $\alpha < \varepsilon_0$, H_α is strictly increasing and $H_\beta(n) < H_\alpha(n)$ for $\beta \in \alpha[n]$. The same holds for F_α for $n \neq 0$, for when $n = 0$ we have $F_\alpha(0) = 1$ for each α.

Proof. Induction on α. The case $\alpha = 0$ is trivial since H_0 is the identity function and $0[n] = \emptyset$. If Succ(α), then H_α is $H_{\alpha-1}$ composed with the successor function, it is strictly increasing by the induction hypothesis. Take $\beta \in \alpha[n]$, then either $\beta \in (\alpha - 1)[n]$ or $\beta = \alpha - 1$, thus, by using the induction hypothesis

$$H_\beta(n) \leq H_{\alpha-1}(n) < H_{\alpha-1}(n+1) = H_\alpha(n).$$

If Lim(α) then

$$H_\alpha(n) = H_{\alpha(n)}(n) < H_{\alpha(n)}(n+1)$$

but $\alpha(n) \in \alpha[n+1] = \alpha(n+1)[n+1]$, thus

$$H_{\alpha(n)}(n+1) < H_{\alpha(n+1)}(n+1) = H_\alpha(n+1)$$

Thus $H_\alpha(n) < H_\alpha(n+1)$. Furthermore if $b \in \alpha[n]$, then $\beta \in \alpha(n)[n]$ so $H_\beta(n) < H_{\alpha(n)}(n) = H_\alpha(n)$ by the induction hypothesis for $\alpha(n)$.

The same holds for $F_\alpha = H_{\omega^\alpha}$ since if $\beta \in \alpha[n]$ we then have $\omega^\beta \in \omega^\alpha[n]$. □

Lemma 3.7. If $\beta \in \alpha[n]$, then $F_{\beta+1}(m) \leq F_\alpha(m)$ for all $m \geq n$.

Proof. Induction on α. The zero case is trivial. If Succ(α), then either $\beta \in (\alpha - 1)[n]$ or $\beta = \alpha - 1$. In either case we apply the induction hypothesis. If α is a limit, then we have $\beta \in \alpha(n)[n]$, so by induction hypothesis $F_{\beta+1}(m) \leq F_{\alpha(n)}(m)$, but $F_{\alpha(n)}(m) \leq F_\alpha(m)$. □

3.3 α-recursion and ε_0-recursion

Definition 3.6 (**α-recursion**).

1. An *α-recursion* if a function definition of the following form, defining $f : \mathbb{N}^{k+1} \to \mathbb{N}$ from functions $g_0, g_1, \ldots, g_s$ by the following equations:

$$f(0, \vec{m}) = g_0(\vec{m})$$
$$f(n, \vec{m}) = T(g_1, \ldots, g_s, f_{\prec n}, n, \vec{m}) \text{ provided } n \geq 1.$$

   where $T(g_1, \ldots, g_s, f_{\prec n}, n, \vec{m})$ is a fixed term built up from the number variables n and $\vec{m}$ by applying functions $g_1, \ldots, g_s$ and the function $f_{\prec n}$ defined as

$$f_{\prec n}(n', \vec{m}) = \begin{cases} f(n', \vec{m}), \text{if } n' \prec_\alpha n \\ 0, \text{otherwise} \end{cases}$$

   Note that it is assumed that $\alpha > 0$.

2. An *unnested* α is one of the special form:

$$f(0, \vec{m}) = g_0(\vec{m})$$
$$f(n, \vec{m}) = g_1(n, \vec{m}, f(g_2(n, \vec{m}), \ldots, g_{k+1}(n, \vec{m})))$$

   with a single recursive call of f where $g_2(n, \vec{m}) \prec_\alpha n$ for all n and $\vec{m}$.

3. Let $\varepsilon_0(0) = \omega$ and $\varepsilon_0(i+1) = \omega^{\varepsilon_0(i)}$. For each particular i, a function is $\varepsilon_0(i)$*-recursive* if it can be defined from primitive recursive functions by successive substitutions and α-recursions with $\alpha < \varepsilon_0(i)$. It is *unnested* $\varepsilon_0(i)$*-recursive* if all the α-recursions are unnested. It is ε_0*-recursive* if it is $\varepsilon_0(i)$-recursive for some (any) i.

Lemma 3.8 (**Bounds for α-recursion).** Let f be a function defined from $g_1, \ldots, g_s$ by an α-recursion:

$$f(0, \vec{m}) = g_0(\vec{m})$$
$$f(n, \vec{m}) = T(g_1, \ldots, g_s, f_{\prec n}, n, \vec{m})$$

where for each $i \leq s$ $g_i(\vec{a}) < F_\beta(k + \max \vec{a})$ for all numerical arguments $\vec{a}$. Then there is a constant d such that for all n, $\vec{m}$

$$f(n, \vec{m}) < F_{\alpha+\beta}(k + 2d + \max(n, \vec{m})).$$

Note that β and k are arbitrary constants, but it is assumed that the last exponent in the Cantor normal form of β is $\geq$ the first exponent in the normal form of α, so that $\beta + \alpha$ is in Cantor normal form by default.

Proof. The constant d will be actually the depth of nesting of the term T, where variables have depth have depth 0 and each compositional term $g(T_1, \ldots, T_l)$ has depth greater than the maximum depth of nesting of the subterms T_j.

Assume n lies in the field of the well-ordering $\prec_\alpha$. Then $n = \overline{\gamma}$ for some $\gamma < \alpha$. Let us claim by induction on γ that

$$f(n, \vec{m}) < F_{\beta+\gamma+1}(k + 2d + \max(n, \vec{m})).$$

This is immediate when $n = 0$, because $g_0(\vec{m}) < F_\beta(k + \max \vec{m})$ and F_β is strictly increasing and bounded by $F_{\beta+1}$. Assume $n \neq 0$ and assume the claim for all $n' = \vec{\delta}$ where $\delta < \gamma$.

Let T' be any subterm of $T(g_1, \ldots, g_s, f_{\prec n}, n, \vec{m})$ with depth of nesting d', built up by application of one of the functions $g_1, \ldots, g_s$ or $f_{\prec n}$ to subterms $T_1, \ldots, T_l$. Assume (for a sub-induction on d') that each of these T_j's has numerical value v_j less that $F_{\beta+\gamma}^{2(d'-1)}(k + 2d + \max(n, \vec{m}))$.

If T' is obtained by application of one of the functions g_i then its numerical value will be

$$g_i(v_1, \ldots, v_l) < F_\beta(k + 2^{d'-1}_{\beta+\gamma})(k + 2d + \max(n, \vec{m})) < F^{2d'}_{\beta+\gamma}(k + 2d + \max(n, \vec{m}))$$

since $k < u$ then $F_\beta(k + u) < F_\beta(2u) < F_\beta^2(u)$ provided $\beta \neq 0$. On the other hand, if T' is obtained by application of the function $f_{\prec n}$, its value will be $f(v_1, \ldots, v_l)$ if $v_1 \prec_\alpha n$ or 0 otherwise. Suppose $v_1 = \overline{\delta} \prec_\alpha \overline{\gamma}$. So by the induction hypothesis:

$$f(v_1, \ldots, v_l) < F_{\beta+\delta+1}(k + 2d + \max \vec{v}) \leq F_{\beta+\gamma}(k + 2d + \max \vec{v})$$

because v_1 is greater than the maximum coefficient of δ, so $\delta \in \gamma[v_1]$, so $\beta + \delta \in (\beta + \gamma)[v_1]$ and hence $F_{\beta+\gamma+1}$ is bounded by $F_{\beta+\gamma}$ on arguments $\geq v_1$. TODO: complete the proof $\square$

4 $\mathbf{RCA}_0$

5 $\mathbf{WKL}_0$

6 $\mathbf{ACA}_0$

7 ATR

8 $\mathbf{\Pi^1_1}$-comprehension

9 Kripke-Platek Set Theory