Some Notes on Proof Theory and Elements of Ordinal Analysis

Daniel Rogozin

1 Provable Recursion in $\mathbf{I}\Delta_0(\exp)$

$\mathbf{I}\Delta_0(\exp)$ is a theory in first-order logic in the language:

$$\{=, 0, S, P, +, \dot{-}, \cdot, exp_2\}$$

where S and P are successor and precessor functions respectively. Further, we will denote $S(x)$ and $P(x)$ as $x+1$ and $x\dot{-}1$ respectively. 2^x stands for $exp_2(x)$.

The non-logical axioms of $\mathbf{I}\Delta_0(\exp)$ are the following list:

- $x + 1 \neq 0$
- $0\dot{-}1 = 0$
- $x + 0 = x$
- $x\dot{-}0 = x$
- $x \cdot 0 = 0$
- $2^0 = 1$
- $x + 1 = y + 1 \to x = y$
- $(x + 1)\dot{-}1 = x$
- $x + (y + 1) = (x + y) + 1$
- $x\dot{-}(y + 1) = x\dot{-}y\dot{-}1$
- $x \cdot (y + 1) = x \cdot y + x$
- $2^{x+1} = 2^x + 2^x$

along with the bounded induction scheme:

$$B(0) \wedge \forall x(B(x) \to B(x + 1)) \to \forall x B(x)$$

where B is a Δ-*formula*, that is a formula one of the following forms (with bounded quantifiers only):

- $B \coloneqq \forall x < tP(x) \equiv \forall x(x < t \to P(x))$
- $B \coloneqq \exists x < tP(x) \equiv \exists x(x < t \wedge P(x))$

A Σ_1-formula is a formula of the form:

$$\exists \vec{x} B(\vec{x})$$

where $B(\vec{x}) \in \Delta_0$.

Lemma 1.1. $\mathbf{I}\Delta_0(\exp)$ proves (the universal closures of):

1. $x = 0 \vee x = (x \dot{-} 1) + 1$
2. $x + (y + z) = (x + y) + z$
3. $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
4. $x \cdot (y + z) = x \cdot y + x \cdot z$
5. $x + y = y + x$
6. $x \cdot y = y \cdot x$
7. $x \dot{-} (y + z) = (x \dot{-} y) \dot{-} z$
8. $2^{x+y} = 2^x \cdot 2^y$

Proof.

1. This is self-evident.
2. If $z = 0$, then $x + y = x + y$. If $z = z' + 1$, then, by applying the IH and the relevant axioms:

$$(x + (y + (z' + 1))) = (x + ((y + z') + 1)) = (x + (y + z')) + 1 = \\ ((x + y) + z') + 1 = (x + y) + (z' + 1)$$

3. If $z = 0$, then $x \cdot (y \cdot 0) = (x \cdot y) \cdot 0$. If $z = z' + 1$, then:

$$x \cdot (y \cdot (z' + 1)) = x \cdot (y \cdot z' + y) = x \cdot (y \cdot z') + x \cdot y = (x \cdot y) \cdot z' + x \cdot y = (x \cdot y) \cdot (z' + 1)$$

4. The rest of the cases are shown by induction on z. Consider the exponentiation law. If $y = 0$, then

$$2^{x+0} = 2^x = 0 + 2^x = 2^x \cdot 0 + 2^x = 2^x \cdot (0 + 1) = 2^x \cdot 2^0$$

If $y = y' + 1$, then:

$$2^{x+(y'+1)} = 2^{(x+y')+1} = 2^x \cdot 2^y + 2^x \cdot 2^y = 2^x \cdot 2^{y+1}$$

□

Lemma 1.2. $\mathbf{I}\Delta_0(\exp)$ proves (the universal closures of):

1. $\neg x < 0$
2. $x \leq 0 \leftrightarrow x = 0$
3. $0 \leq x$
4. $x \leq x$

5. $x < x + 1$
6. $x < y + 1 \leftrightarrow x \leq y$
7. $x \leq y \leftrightarrow x < y \vee x = y$
8. $x \leq y \wedge y \leq z \rightarrow x \leq z$
9. $x < y \wedge y < z \rightarrow x < z$
10. $x \leq y \vee y < x$
11. $x < y \rightarrow x + z < y + z$
12. $x < y \rightarrow x \cdot (z + 1) < y \cdot (z + 1)$
13. $x < 2^x$
14. $x < y \rightarrow 2^x < 2^y$

Proof. Straightforward induction. □

Definition 1.1. A function $f : \mathbb{N}^k \to \mathbb{N}$ is *provably* Σ_1 or *provably recursive* in an arithmetical theory if there is a Σ_1 formula $F(\vec{x}, y)$, a "defining formula" of f, such that:

1. $f(\vec{n}) = m$ iff $\omega \models f(\vec{n}) = m$
2. $T \vdash \exists y F(\vec{x}, y)$
3. $T \vdash F(\vec{x}, y) \wedge F(\vec{x}, y') \rightarrow y = y'$

If a defining formula $F \in \Delta_0$, then a function f is *provably bounded* in T if there is a term $t(\vec{x})$ such that $T \vdash F(\vec{x}, y) \rightarrow y < t(\vec{x})$.

Theorem 1.1. Let f be a provably recursive in T, then we can conservatively extend T by adding a new function symbol f along with the defining axiom $F(\vec{x}, f(\vec{x}))$.

Proof. Let $\mathcal{M} \models T$, $\mathcal{M}$ can be made into a model $(\mathcal{M}, f)$ where we interpret f as the function which is uniquely determined by the second and third conditions of the definitions above. Let φ be a statement not involving f such that φ is true in $(\mathcal{M}, f)$, so φ is true in $\mathcal{M}$ as well. By compactness T proves φ. □

Lemma 1.3. Each term defines a provably bounded function of $\mathbf{I}\Delta_0(\exp)$.

Proof. Let f be a function defined by some $\mathbf{I}\Delta_0(\exp)$-term t, that is, $f(\vec{x}) = t(\vec{x})$. Take $y = t(\vec{x})$ as the defining formula for f since $\exists y\, (y = t(\vec{x}))$ is derivable. If $y' = t(\vec{x}) \wedge y = t(\vec{x})$, then $y = y'$ by transitivity. A formula $y = t(\vec{x})$ is bounded and $y = t$ implies $y < t + 1$. Thus f is provably bounded. □

Lemma 1.4. Define $2_k(x)$ as $2_0(x) = x$ and $2_{n+1}(x) = 2^{2_n(x)}$. Then for every term $t(x_1, \ldots, x_n)$ built up from the constants $0, S, P, +, \dot{-}, \cdot, exp_2$ there exists $k < \omega$ such that:

$$\mathbf{I}\Delta_0(\exp) \vdash t(x_1, \ldots, x_n) < 2_k(\sum_{k=0}^{n} x_k)$$

Proof. Let t be a term constructed from subterms t_0 and t_1 by using one of the function constants. Assume that inductively $t_0 < 2_{k_0}(s_0)$ and $t_1 < 2_{k_1}(s_1)$ are both provable for some $k_0, k_1 < \omega$, where s_i is the sum of the variables of t_i for $i = 0, 1$.

Let s be the sum of all variables appearing in either t_0 or t_1 and let $k = \max(k_0, k_1)$. Then one can prove $t_0 < 2_k(s)$ and $t_1 < 2_k(s)$. So one needs to show the following:

1. $t_0 + 1 < 2_{k+1}(s)$
2. $t_0 \dot{-} 1 < 2_k(s)$
3. $t_0 \dot{-} t_1 < 2_k(s)$
4. $t_0 \cdot t_1 < 2_k(s)$
5. $t_0 + t_1 < 2_k(s)$
6. $2^{t_0} < 2_k(s)$

So $\mathbf{I}\Delta_0(\exp) \vdash t < 2_{k+1}(s)$. □

Lemma 1.5. Let f be a function defined by composition:

$$f(\vec{x}) = g_0(g_1(\vec{x}), \ldots, g_m(\vec{x}))$$

where $g_0, g_1, \ldots, g_m$ are functions each of which is provably bounded in $\mathbf{I}\Delta_0(\exp)$. Then f is provably bounded in $\mathbf{I}\Delta_0(\exp)$.

Proof. Each g_i has a defining formula G_i and, by Lemma 1.4, there is a number $k_i < \omega$ such that:

$$\mathbf{I}\Delta_0(\exp) \vdash \exists y < 2_{k_i}(s)\, G_i(\vec{x}, y)$$

where s is the sum of elements of $\vec{x}$. And for $i = 0$ one has:

$$\mathbf{I}\Delta_0(\exp) \vdash \exists y < 2_{k_0}(s_0)\, G_0(y_1, \ldots, y_m, y)$$

where s_0 is the sum of $y_1, \ldots, y_m$.

Let $k = \max\{k_i < \omega \mid i < m+1\}$ and let $F(\vec{x}, y)$ be the bounded formula:

$$\exists y_1 < 2_k(s)\ \ldots\ \exists y_m < 2_k(s)\, C(\vec{x}, y_1, \ldots, y_m, y)$$

where $C(\vec{x}, y_1, \ldots, y_m, y)$ is the conjunction:

$$G_1(\vec{x}, y_1) \wedge \cdots \wedge G_m(\vec{x}, y_m) \wedge G_0(y_1, \ldots, y_m, y)$$

F is clearly a defining formula for f such that $\mathbf{I}\Delta_0(\exp) \vdash \exists y F(\vec{x}, y)$.
Moreover, each G_i is unique, so $\mathbf{I}\Delta_0(\exp)$ also proves:

$$\begin{aligned}
&C(\vec{x}, y_1, \dots, y_m, y) \wedge C(\vec{x}, z_1, \dots, z_m, z) \rightarrow \\
&\rightarrow \bigwedge_{j=1}^{m} y_j = z_j \wedge G_0(y_1, \dots, y_m, y) \wedge G_0(y_1, \dots, y_m, z) \rightarrow \\
&\rightarrow y = z
\end{aligned}$$

so we have (by first order logic):

$$\mathbf{I}\Delta_0(\exp) \vdash F(\vec{x}, y) \wedge F(\vec{x}, z) \rightarrow y = z$$

Thus f is provably Σ_1 in $\mathbf{I}\Delta_0(\exp)$, so the rest is to find its bounding term. $\mathbf{I}\Delta_0(\exp)$ proves the following:

$$C(\vec{x}, y_1, \dots, y_m, y) \rightarrow \bigwedge_{j=1}^{m} y_j < 2_k(s) \wedge y < 2_k(y_1 + \dots + y_m)$$

and

$$\bigwedge_{j=1}^{m} y_j < 2_k(s) \rightarrow y_1 + \dots + y_m < 2_k(s) \cdot m$$

Put $t(\vec{x}) = 2_k(2_k(s) \cdot m)$, then we obtain

$$\mathbf{I}\Delta_0(\exp) \vdash C(\vec{x}, y_1, \dots, y_m, y) \rightarrow y < t(\vec{x})$$

and so

$$\mathbf{I}\Delta_0(\exp) \vdash F(\vec{x}, y) \rightarrow y < t(\vec{x})$$

□

Lemma 1.6. Suppose f is defined by bounded minimisation

$$f(\vec{n}, m) = \mu_{k<m}(g(\vec{n}, k) = 0)$$

from a function g which is provably bounded in $\mathbf{I}\Delta_0(\exp)$. Then f is provably bounded in $\mathbf{I}\Delta_0(\exp)$.

Proof. Let G be a defining formula for g. Let $F(\vec{x}, z, y)$ be the bounded formula

$$y \leq z \wedge \forall i < y \neg G(\vec{x}, i, 0) \wedge (y = z \vee G(\vec{x}, y, 0))$$

$\omega \models F(\vec{n}, m, k)$ iff either k is the least number less than m such that $g(\vec{n}, k) = 0$ or there is no such and $k = m$. Thus it means that k is the value of $f(\vec{n}, m)$, so F is a defining formula for f.

Furthermore

$$\mathbf{I}\Delta_0(\exp) \vdash F(\vec{x}, z, y) \rightarrow y < z + 1$$

so $t(\vec{x}, z) = z + 1$ can be taken as a bounding term for f.

We can prove:

$$F(\vec{x}, z, y) \wedge F(\vec{x}, z, y') \wedge y < y' \rightarrow G(\vec{x}, y, 0) \wedge \neg G(\vec{x}, y, 0)$$

and similarly for interchanged y and y'. So we can prove:

$$F(\vec{x}, z, y) \wedge F(\vec{x}, z, y') \rightarrow \neg y < y' \wedge \neg y' < y$$

As far as $y < y' \vee y' < y \vee y = y'$, we have

$$F(\vec{x}, z, y) \wedge F(\vec{x}, z, y') \rightarrow y = y'$$

Now we have to check that $\mathbf{I}\Delta_0(\exp) \vdash \exists y F(\vec{x}, z, y)$. We construct such y by bounded induction on z.

1. $z = 0$.

   $F(\vec{x}, 0, 0)$ is provable since $y = 0 \leftrightarrow y \leq 0$ and $\neg i < 0$. So $\mathbf{I}\Delta_0(\exp) \vdash F(\vec{x}, 0, y)$ is provable.

2. Assume $\exists y F(\vec{x}, z, y)$ is provable, let show that that $\exists y F(\vec{x}, z + 1, y)$ is provable.

   We can show $y \leq z \rightarrow y + 1 \leq z + 1$ and, via $i < y + 1 \leftrightarrow i < y \vee i = y$,

   $$\forall i < y \, \neg G(\vec{x}, i, 0) \wedge ((y = z) \wedge \neg G(\vec{x}, y, 0)) \rightarrow \forall i < y + 1 \, \neg G(\vec{x}, i, 0) \wedge y + 1 = z + 1$$

   Therefore

   $$F(\vec{x}, z, y) \rightarrow F(\vec{x}, z + 1, y + 1) \vee F(\vec{x}, z + 1, y)$$

   and thus:

   $$\exists y F(\vec{x}, z, y) \rightarrow \exists y F(\vec{x}, z + 1, y)$$

□

Theorem 1.2. Every elementary function is provably bounded in $\mathbf{I}\Delta_0(\exp)$.

Proof. As we know from recursion theory, the class of elementary functions can be characterised as those functions which are definable from 0, S, P, $\cdot$, $+$, exp_2, $\dot{-}$ and $\cdot$ by composition and minimisation. And then we apply above lemmas. □

1.1 Proof-theoretic Characterisation

For this section we shall be using a Tait-style formalisation of $\mathbf{I}\Delta_0(\exp)$. We have the following logical rules:

$$\frac{}{\Gamma, R\vec{t}, \neg R\vec{t}}\ \mathbf{Ax}$$

$$\frac{\Gamma, A_0, A_1}{\Gamma, A_0 \vee A_1}\ \vee \qquad\qquad \frac{\Gamma, A_0 \qquad \Gamma, A_1}{\Gamma, A_0 \wedge A_1}\ \wedge$$

$$\frac{\Gamma, A(t)}{\Gamma, \exists x A(x)}\ \exists \qquad\qquad \frac{\Gamma, A}{\Gamma, \forall x A}\ \forall$$

where $R\vec{t}$ is an atomic formula and x is not free in A in the $\forall$ rule. Here Γ stores all non-logical axioms of $\mathbf{I}\Delta_0(\exp)$ along with its negations. We also have the bounded induction rule:

$$\frac{\Gamma, B(0) \qquad \Gamma, \neg B(n), B(n+1)}{\Gamma, B(t)}\ \mathbf{BInd}$$

where B is a bounded formula and t is any term.

Of course, the cut rule is admissible:

$$\frac{\Gamma, A \qquad \Gamma, \neg A}{\Gamma}\ \mathbf{cut}$$

Definition 1.2. Let $\exists\vec{z}B(\vec{z})$ be a closed Σ_1-formula, then it is *true at* m, written as $m \models \exists\vec{z}B(\vec{z})$, if there exist natural numbers $m_1, \ldots, m_l$ such that each $m_i < m$ and $B(\vec{m})$ is true in the standard model.

A finite set Γ of closed Σ_1-formulas is true at m, written as $m \models \Gamma$ if at least one of them is true at m.

If $\Gamma(x_1, \ldots, x_k)$ is a finite set of Σ_1-formulas whose free variables occur amongst $x_1, \ldots, x_k$. Let $f : \mathbb{N}^k \to \mathbb{N}$, then $f \models \Gamma(x_1, \ldots, x_k)$ we have $f(\vec{n}) \models \Gamma(x_1 := n_1, \ldots, x_k := n_k)$ for each $\vec{n} = (n_1, \ldots, n_k)$.

Fact 1.1. (Persistence)

1. If $m \leq m'$, then $m \models \exists\vec{z}B(\vec{z})$ implies $m' \models \exists\vec{z}B(\vec{z})$.
2. If $\forall\vec{n} \in \mathbb{N}^k\ f(\vec{n}) \leq f'(\vec{n})$, then $f(\vec{n}) \models \Gamma(x_1 := n_1, \ldots, x_k := n_k)$ implies $f'(\vec{n}) \models \Gamma(x_1 := n_1, \ldots, x_k := n_k)$.

Lemma 1.7. Let $\Gamma(\vec{x})$ be a finite set of Σ_1 formulas such that

$$\mathbf{I}\Delta_0(exp) \vdash \bigvee_{\gamma(\vec{x}) \in \Gamma(\vec{x})} \gamma(\vec{x}).$$

Then there is an elementary function f such that $f \models \Gamma(\vec{x})$ and f is strongly increasing on its variables.

Proof. If Γ is provable in $\mathbf{I}\Delta_0(exp)$, then it is provable in the Tait-style version of $\mathbf{I}\Delta_0(exp)$, where all cut formulas are Σ_1.

If Γ is classically derivable from non-logical axioms $A_1, \ldots, A_s$, then there is a cut-free proof in the Tait calculus of $\neg A_1, \Delta, \Gamma$, where $\Delta = \neg A_2, \ldots, \neg A_s$. Let us show how to cancel $\neg A_1$ using a Σ_1-cut.

If A_1 is an induction axiom on some formula B, then we have a cut-free proof of:

$$B(0) \wedge \forall y(\neg B(y) \vee B(y+1)) \wedge \exists x \neg B(x), \Delta, \Gamma$$

Thus we also have cut-free proofs of $B(0), \Delta, \Gamma$, $\neg B(y), B(y+1), \Delta, \Gamma$ and $\exists x \neg B(x), \Delta, \Gamma$. So we have

$$\dfrac{\dfrac{\dfrac{\Delta, \Gamma, B(0) \qquad \Delta, \Gamma, \neg B(y), B(y+1)}{\Delta, \Gamma, B(x)}\,\mathbf{BInd}}{\Delta, \Gamma, \forall x B(x)}\,\forall \qquad \exists x \neg B(x), \Delta, \Gamma}{\Delta, \Gamma}\,\Sigma_1\text{-}\mathbf{cut}$$

We can similarly cancel each of $\neg A_2, \ldots, \neg A_s$ and so obtain the proof of Γ with Σ_1-cuts only.

Now we choose a proof of $\Gamma(\vec{x})$ and proceed by induction on the height of the proof and determine an elementary function f such that $f \models \Gamma$.

1. If $\Gamma(\vec{x})$ is an axiom, then for all $\vec{n}$ $\Gamma(\vec{n})$ contains a true atom. So for any f $f \models \Gamma$. Let us choose $f(\vec{n}) = n_1 + \cdots + n_k$.

2. If $\Gamma, B_0 \vee B_1$ is derivable, so is Γ, B_0, B_1. Note that B_0 and B_1 are both bounded. Let $f \models \Gamma, B_0, B_1$, then $f \models \Gamma, B_0 \vee B_1$.

3. Assume $\Gamma, B_0 \wedge B_1$ is derivable, then Γ, B_0 and Γ, B_1 By the induction hypothesis we have $f_0 \models \Gamma, B_0$ and $f_1 \models \Gamma, B_1$, so, by persistence, we have $\lambda \vec{n}. f_0(\vec{n}) + f_1(\vec{n}) \models \Gamma, B_0 \wedge B_1$.

4. Assume $\Gamma, \forall y B(y)$ is derivable, then $\Gamma, B(y)$ is derivable and y is not free in Γ. Since all the formulas are Σ_1, $\forall x B(y)$ must be bounded, so $B(y) = \neg(y < t) \vee B'(y)$ for some term t and for some bounded formula B'. By the induction hypothesis, assume $f_0 \models \Gamma, \neg(y < t), B'(y)$ for some increasing elementary function f_0. Then we have:

$$f_0(\vec{n}, k) \models \Gamma(\vec{n}), \neg(k < t(\vec{n})), B'(\vec{n}, k)$$

Let g be an increasing elementary function bounding t, define

$$f(\vec{n}) = \sum_{k < g(\vec{n})} f(\vec{n}, k)$$

We have either $f(\vec{n}) \models \Gamma(\vec{n})$ or, by persistence, $B'(\vec{n}, k)$ is true for every $k < t(\vec{n})$. So $f \models \Gamma, \forall y B(y)$ and f is elementary.

5. Assume $\Gamma, \exists y A(y, \vec{x})$ is derivable, so $\Gamma, A(t, \vec{x})$ is derivable for some term t. By the IH, there is elementary f_0 such that for all $\vec{n}$ one has

$$f_0(\vec{n}) \models \Gamma(\vec{n}), A(t(\vec{n}), \vec{n})$$

Then either $f_0(\vec{n}) \models \Gamma(\vec{n})$ or else $f_0(\vec{n})$ bounds true witnesses for all existential quantifiers in $A(t(\vec{n}), \vec{n})$. Choose an elementary function g which is bounding for t. Define $f(\vec{n}) = f_0(\vec{n}) + g(\vec{n})$, then for all $\vec{n}$ either $f(\vec{n}) \models \Gamma(\vec{n})$ or $f(\vec{n}) \models \exists y A(y, \vec{n})$.

6. Assume Γ comes about by the cut rule with Σ_1 formula $C = \exists \vec{z} B(\vec{z})$, so the premises are $\Gamma, \forall \vec{z} \neg B(\vec{z})$ and $\Gamma, \exists \vec{z} B(\vec{z})$.

   Without increasing the height of a proof, we can invert all universal quantifiers in the first premise. So we have $\neg B(\vec{z})$. B is bounded, so the induction hypothesis can be applied to this formula to obtain an elementary function f_0 such that, for all assignments $[\vec{x} := \vec{n}]$ and $[\vec{z} := \vec{m}]$

$$f_0(\vec{n}, \vec{m}) \models \Gamma(\vec{n}), \neg B(\vec{n}, \vec{m})$$

   Now we apply the induction hypothesis to the second premise of the cut rule, so we have an elementary function f_1 such that for all $\vec{n}$ either $f_1(\vec{n}) \models \Gamma(\vec{n})$ or there are fixed witnesses $\vec{m} < f_1(\vec{n})$ such that $B(\vec{n}, \vec{m})$ is true.

   Define f the following way:

$$f(\vec{n}) = f_0(\vec{n}, f_1(\vec{n}), \ldots, f_1(\vec{n}))$$

   Furthermore $f \models \Gamma$. For otherwise there would be a tuple $\vec{n}$ such that $\Gamma(\vec{n})$ is not true at $f(\vec{n})$, so, by persistence, $\Gamma(\vec{n})$ is not true at $f_1(\vec{n})$. Thus $B(\vec{n}, \vec{m})$ is true for particular numbers $\vec{m} < f_1(\vec{n})$. But then $f_0(\vec{n}, \vec{m}) < f(\vec{n})$, so, by persistence, $\Gamma(\vec{n})$ cannot be true at $f_0(\vec{n}, \vec{m})$. Thus $B(\vec{n}, \vec{m})$ is false, so we have a contradiction.

7. Finally suppose $\Gamma(\vec{x}), B(\vec{x}, t)$ comes from the induction rule on a bounded formula B. The premises of the rule $\Gamma(\vec{x}), B(\vec{x}, 0)$ and $\Gamma(\vec{x}), \neg B(\vec{x}, y), B(\vec{x}, y+1)$.

   Let us apply the induction hypothesis to each of the premises, and then we obtain increasing elementary functions f_0 and f_1 such that for all $\vec{n}$ and for all k

$$f_0(\vec{n}) \models \Gamma(\vec{n}), B(\vec{n}, 0)$$
$$f_1(\vec{n}, k) \models \Gamma(\vec{n}), \neg B(\vec{n}, k), B(\vec{n}, k+1)$$

Now let

$$f(\vec{n}) = f_0(\vec{n}) + \sum_{k<g(\vec{n})} f_1(\vec{n}, k)$$

where g is an increasing elementary function which is bounding for the term t. f is elementary and increasing, and, by persistence for f_0 and f_1, we have either $f(\vec{n}) \models \Gamma(\vec{n})$ or else $B(\vec{n}, 0)$ and $B(\vec{n}, k) \to B(\vec{n}, k+1)$ are true for all $k < t(\vec{n})$. In either case, we have $f \models \Gamma(\vec{x}), B(\vec{x}, t(\vec{x}))$.

□

Theorem 1.3. A number-theoretic function is elementary iff f is provably Σ_1 in $\mathbf{I}\Delta_0(exp)$.

Proof. The only if part is in Theorem 1.2, so we show the if part only. Assume f is provably Σ_1 in $\mathbf{I}\Delta_0(exp)$. Then we have a formula

$$F(\vec{x}, y) = \exists z_1 \ldots \exists z_k B(\vec{x}, y, z_1, \ldots, z_k)$$

which defines f and such that

$$\mathbf{I}\Delta_0(exp) \models \exists y F(\vec{x}, y)$$

By Lemma 1.7, there exists an elementary function g such that for every tuple of arguments $\vec{n}$ there are numbers $m_0, \ldots, m_k$ less that $g(n)$ satisfying the bounded formula $B(\vec{n}, m_0, m_1, \ldots, m_k)$. Apply the elementary sequence coding:

$$h(\vec{n}) = \langle g(\vec{n}), g(\vec{n}), \ldots, g(\vec{n}) \rangle$$

so that if $m = \langle m_0, m_1, \ldots, m_k \rangle$ where $m_i < g(\vec{n})$ for each $i \in n+1$, so $m < h(\vec{n})$.

As far as $f(\vec{n})$ is the unique m_0 for which there are $m_1, \ldots, m_k$ satisfying $B(\vec{n}, m_0, \ldots, m_k)$, we define f as:

$$f(\vec{n}) = (\mu_{m<h(\vec{n})} B(\vec{n}, (m)_0, (m)_1, \ldots, (m)_k))_0.$$

B is a bounded formula of $\mathbf{I}\Delta_0(exp)$, B is elementarily decidable. Moreover, elementary functions are closed under composition and bounded minimisation, so f is elementary. □

2 Primitive Recursion and $\mathbf{I}\Sigma_1$

$\mathbf{I}\Sigma_1$ is an arithmetical theory where the induction scheme is restructed to Σ_1 formulas.

Lemma 2.1. Every primitive recursion is provably recursive in $\mathbf{I}\Sigma_1$.

Proof. We have to show represent each primitive recursive function f with a Σ_1 formula $F(\vec{x}, y) := \exists z C(\vec{x}, y, z)$ such that:

1. $f(\vec{n}) = m$ iff $\omega \models F(\vec{x}, y)$.

2. $\mathbf{I}\Sigma_1 \vdash \exists y F(\vec{x}, y)$.

3. $\mathbf{I}\Sigma_1 \vdash F(\vec{x}, y) \wedge F(\vec{x}, y') \rightarrow y = y'$.

In each case $C(\vec{x}, y, z)$ will be a $\Delta_0(exp)$-formula constructed via sequence encoding in $\mathbf{I}\Delta_0(exp)$. Such a formula expresses that z is a uniquely determined sequence number encoding the computation of $f(\vec{x}) = y$ and containing the output value y as its final element, so $y = \pi_2(z)$.

Condition 1 will hold by the definition of C. Condition 3 will be satisfied by the uniqueness of z. We consider five definitional schemes by which f could be introduced:

1. f is the constant-zero function, that is, $f(x) = 0$, no matter what x is. Then we take $C := y = 0 \wedge z = \langle 0 \rangle$. All the conditions are obviously satisfied.

2. If f is the successor function $f(x) = x + 1$, we let

$$C(x, y, z) := y = x + 1 \wedge z = \langle x + 1 \rangle$$

All the conditions are obvious.

3. Now assume f is the projection function $f(x_0, \ldots, x_n) = x_i$ for some $i \in n + 1$. We let

$$C(\vec{x}, y, z) := y = x_i \wedge z = \langle x_i \rangle$$

4. Now assume f is defined by substitution from previously generated primitive recursive functions f_0, f_1, f_2:

$$f(\vec{x}) = f_0(f_1(\vec{x}), f_2(\vec{x}))$$

By the induction hypothesis, assume thatb f_0, f_1, f_2 are provably recursive and we have $\Delta_0(exp)$-formulas C_0, C_1, C_2 encoding their computations ($\mathrm{len}(z) = 4$). For the function f define:

$$C(\vec{x}, y, z) :=$$
$$\bigwedge_{i \in \{1,2\}} C_i(\vec{x}, \pi_2((z)_i), (z)_i) \wedge C_0(\pi_2((z)_1), \pi_2((z)_2), y, (z)_0) \wedge (z)_3 = y.$$

Let us check the required conditions:

(a) Condition 1 holds since $f(\vec{n}) = m$ iff there are numbers m_1 and m_2 such that $f_1(\vec{n}) = m_1$, $f_2(\vec{n}) = m_2$ and $f_0(m_1, m_2) = m$. These hold if and only if there are number k_1, k_2, k_0 such that $C_1(\vec{n}, m_1, k_1)$, $C_2(\vec{n}, m_2, k_2)$ and $C_0(m_1, m_2, m, k_0)$ are all true. And these hold if and only if $C(\vec{n}, m, \langle k_0, k_1, k_2, m \rangle)$ is true. Thus $f(\vec{n}) = m$ iff and only if $F(\vec{n}, m) = \exists z C(\vec{n}, m, z)$ is true.

(b) Condition 2 holds since from $C_1(\vec{x}, y_1, z_1)$, $C_2(\vec{x}, y_2, z_2)$ and $C(y_1, y_2, y, z_0)$ we can derive $C(\vec{x}, y, \langle z_0, z_1, z_2, y\rangle)$ in $\mathbf{I}\Delta_0$. So provided $\exists y \exists z C_1(\vec{x}, y, z)$, $\exists y \exists z C_2(\vec{x}, y, z)$ and $\forall y_1 \forall y_2 \exists y \exists z C(y_1, y_2, y, z)$, we can prove $\exists y F(\vec{x}, y) := C(\vec{x}, y, z)$.

(c) Condition 3 is self-evident.

5. Now assume that f is defined from f_1 and f_2 by primitive recursion:

$$
\begin{aligned}
f(\vec{v}, 0) &= f_0(\vec{v}) \\
f(\vec{v}, x+1) &= f_1(\vec{v}, x, f(\vec{v}, x))
\end{aligned}
$$

By the induction hypothesis f_0 and f_1 are provably recursive and they have associated Δ_0-formulas C_0 and C_1. Define

$$
\begin{aligned}
C(\vec{v}, x, y, z) := {} & C_0(\vec{v}, \pi_2((z)_0), (z)_0) \wedge \\
& \forall i < x \ (C_i(\vec{v}, i, \pi_2((z)_i), \pi_2((z)_{i+1}))) \wedge \\
& (z)_{x+1} = y \wedge \pi_2((z)_x) = y
\end{aligned}
$$

□

3 ϵ_0 and Peano Arithmetic

4 $\mathbf{RCA}_0$

5 $\mathbf{WKL}_0$

6 $\mathbf{ACA}_0$

7 ATR

8 $\mathbf{\Pi}^1_1$-comprehension