

# Some Notes on Proof Theory and Elements of Ordinal Analysis

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## 1 Provable Recursion in $\mathbf{I}\Delta_0(\text{exp})$

$\mathbf{I}\Delta_0(\text{exp})$  is a theory in first-order logic in the language:

$$\{=, 0, S, P, +, \dot{-}, \cdot, \exp_2\}$$

where  $S$  and  $P$  are successor and predecessor functions respectively. Further, we will denote  $S(x)$  and  $P(x)$  as  $x + 1$  and  $x \dot{-} 1$  respectively.  $2^x$  stands for  $\exp_2(x)$ .

The non-logical axioms of  $\mathbf{I}\Delta_0(\text{exp})$  are the following list:

- $x + 1 \neq 0$
- $0 \dot{-} 1 = 0$
- $x + 0 = x$
- $x \dot{-} 0 = x$
- $x \cdot 0 = 0$
- $2^0 = 1$
- $x + 1 = y + 1 \rightarrow x = y$
- $(x + 1) \dot{-} 1 = x$
- $x + (y + 1) = (x + y) + 1$
- $x \dot{-} (y + 1) = x \dot{-} y \dot{-} 1$
- $x \cdot (y + 1) = x \cdot y + x$
- $2^{x+1} = 2^x + 2^x$

along with the bounded induction scheme:

$$B(0) \wedge \forall x (B(x) \rightarrow B(x + 1)) \rightarrow \forall x B(x)$$

where  $B$  is a  $\Delta$ -formula, that is a formula one of the following forms (with bounded quantifiers only):

- $B \equiv \forall x < t P(x) \equiv \forall x (x < t \rightarrow P(x))$
- $B \equiv \exists x < t P(x) \equiv \exists x (x < t \wedge P(x))$

A  $\Sigma_1$ -formula is a formula of the form:

$$\exists \vec{x} B(\vec{x})$$

where  $B(\vec{x}) \in \Delta_0$ .

**Lemma 1.1.**  $\mathbf{I}\Delta_0(\text{exp})$  proves (the universal closures of):

1.  $x = 0 \vee x = (x \dot{-} 1) + 1$
2.  $x + (y + z) = (x + y) + z$
3.  $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
4.  $x \cdot (y + z) = x \cdot y + x \cdot z$
5.  $x + y = y + x$
6.  $x \cdot y = y \cdot x$
7.  $x \dot{-} (y + z) = (x \dot{-} y) \dot{-} z$
8.  $2^{x+y} = 2^x \cdot 2^y$

*Proof.*

1. This is self-evident.
2. If  $z = 0$ , then  $x + y = x + y$ . If  $z = z' + 1$ , then, by applying the IH and the relevant axioms:

$$(x + (y + (z' + 1))) = (x + ((y + z') + 1)) = (x + (y + z')) + 1 = ((x + y) + z') + 1 = (x + y) + (z' + 1)$$

3. If  $z = 0$ , then  $x \cdot (y \cdot 0) = (x \cdot y) \cdot 0$ . If  $z = z' + 1$ , then:

$$x \cdot (y \cdot (z' + 1)) = x \cdot (y \cdot z' + y) = x \cdot (y \cdot z') + x \cdot y = (x \cdot y) \cdot z' + x \cdot y = (x \cdot y) \cdot (z' + 1)$$

4. The rest of the cases are shown by induction on  $z$ . Consider the exponentiation law. If  $y = 0$ , then

$$2^{x+0} = 2^x = 0 + 2^x = 2^x \cdot 0 + 2^x = 2^x \cdot (0 + 1) = 2^x \cdot 2^0$$

If  $y = y' + 1$ , then:

$$2^{x+(y'+1)} = 2^{(x+y')+1} = 2^x \cdot 2^{y'} + 2^x \cdot 2^{y'} = 2^x \cdot 2^{y'+1}$$

□

**Lemma 1.2.**  $\mathbf{I}\Delta_0(\text{exp})$  proves (the universal closures of):

1.  $\neg x < 0$
2.  $x \leq 0 \leftrightarrow x = 0$
3.  $0 \leq x$
4.  $x \leq x$

5.  $x < x + 1$
6.  $x < y + 1 \leftrightarrow x \leq y$
7.  $x \leq y \leftrightarrow x < y \vee x = y$
8.  $x \leq y \wedge y \leq z \rightarrow x \leq z$
9.  $x < y \wedge y < z \rightarrow x < z$
10.  $x \leq y \vee y < x$
11.  $x < y \rightarrow x + z < y + z$
12.  $x < y \rightarrow x \cdot (z + 1) < y \cdot (z + 1)$
13.  $x < 2^x$
14.  $x < y \rightarrow 2^x < 2^y$

*Proof.* Straightforward induction.  $\square$

**Definition 1.1.** A function  $f : \mathbb{N}^k \rightarrow \mathbb{N}$  is *provably  $\Sigma_1$*  or *provably recursive* in an arithmetical theory if there is a  $\Sigma_1$  formula  $F(\vec{x}, y)$ , a “defining formula” of  $f$ , such that:

1.  $f(\vec{n}) = m$  iff  $\omega \models f(\vec{n}) = m$
2.  $T \vdash \exists y F(\vec{x}, y)$
3.  $T \vdash F(\vec{x}, y) \wedge F(\vec{x}, y') \rightarrow y = y'$

If a defining formula  $F \in \Delta_0$ , then a function  $f$  is *provably bounded* in  $T$  if there is a term  $t(\vec{x})$  such that  $T \vdash F(\vec{x}, y) \rightarrow y < t(\vec{x})$ .

**Theorem 1.1.** Let  $f$  be a provably recursive in  $T$ , then we can conservatively extend  $T$  by adding a new function symbol  $f$  along with the defining axiom  $F(\vec{x}, f(\vec{x}))$ .

*Proof.* Let  $\mathcal{M} \models T$ ,  $\mathcal{M}$  can be made into a model  $(\mathcal{M}, f)$  where we interpret  $f$  as the function which is uniquely determined by the second and third conditions of the definitions above. Let  $\varphi$  be a statement not involving  $f$  such that  $\varphi$  is true in  $(\mathcal{M}, f)$ , so  $\varphi$  is true in  $\mathcal{M}$  as well. By compactness  $T$  proves  $\varphi$ .  $\square$

**Lemma 1.3.** Each term defines a provably bounded function of  $\mathbf{I}\Delta_0(\text{exp})$ .

*Proof.* Let  $f$  be a function defined by some  $\mathbf{I}\Delta_0(\text{exp})$ -term  $t$ , that is,  $f(\vec{x}) = t(\vec{x})$ . Take  $y = t(\vec{x})$  as the defining formula for  $f$  since  $\exists y (y = t(\vec{x}))$  is derivable. If  $y' = t(\vec{x}) \wedge y = t(\vec{x})$ , then  $y = y'$  by transitivity. A formula  $y = t(\vec{x})$  is bounded and  $y = t$  implies  $y < t + 1$ . Thus  $f$  is provably bounded.  $\square$

**Lemma 1.4.** Define  $2_k(x)$  as  $2_0(x) = x$  and  $2_{n+1}(x) = 2^{2^n(x)}$ . Then for every term  $t(x_1, \dots, x_n)$  built up from the constants  $0, S, P, +, -, \cdot, exp_2$  there exists  $k < \omega$  such that:

$$\mathbf{I}\Delta_0(\text{exp}) \vdash t(x_1, \dots, x_n) < 2_k\left(\sum_{k=0}^n x_k\right)$$

*Proof.* Let  $t$  be a term constructed from subterms  $t_0$  and  $t_1$  by using one of the function constants. Assume that inductively  $t_0 < 2_{k_0}(s_0)$  and  $t_1 < 2_{k_1}(s_1)$  are both provable for some  $k_0, k_1 < \omega$ , where  $s_i$  is the sum of the variables of  $t_i$  for  $i = 0, 1$ .

Let  $s$  be the sum of all variables appearing in either  $t_0$  or  $t_1$  and let  $k = \max(k_0, k_1)$ . Then one can prove  $t_0 < 2_k(s)$  and  $t_1 < 2_k(s)$ . So one needs to show the following:

1.  $t_0 + 1 < 2_{k+1}(s)$
2.  $t_0 - 1 < 2_k(s)$
3.  $t_0 - t_1 < 2_k(s)$
4.  $t_0 \cdot t_1 < 2_k(s)$
5.  $t_0 + t_1 < 2_k(s)$
6.  $2^{t_0} < 2_k(s)$

So  $\mathbf{I}\Delta_0(\text{exp}) \vdash t < 2_{k+1}(s)$ . □

**Lemma 1.5.** Let  $f$  be a function defined by composition:

$$f(\vec{x}) = g_0(g_1(\vec{x}), \dots, g_m(\vec{x}))$$

where  $g_0, g_1, \dots, g_m$  are functions each of which is provably bounded in  $\mathbf{I}\Delta_0(\text{exp})$ . Then  $f$  is provably bounded in  $\mathbf{I}\Delta_0(\text{exp})$ .

*Proof.* Each  $g_i$  has a defining formula  $G_i$  and, by Lemma 1.4, there is a number  $k_i < \omega$  such that:

$$\mathbf{I}\Delta_0(\text{exp}) \vdash \exists y < 2_{k_i}(s) G_i(\vec{x}, y)$$

where  $s$  is the sum of elements of  $\vec{x}$ . And for  $i = 0$  one has:

$$\mathbf{I}\Delta_0(\text{exp}) \vdash \exists y < 2_{k_0}(s_0) G_0(y_1, \dots, y_m, y)$$

where  $s_0$  is the sum of  $y_1, \dots, y_m$ .

Let  $k = \max\{k_i < \omega \mid i < m + 1\}$  and let  $F(\vec{x}, y)$  be the bounded formula:

$$\exists y_1 < 2_k(s) \dots \exists y_m < 2_k(s) C(\vec{x}, y_1, \dots, y_m, y)$$

where  $C(\vec{x}, y_1, \dots, y_m, y)$  is the conjunction:

$$G_1(\vec{x}, y_1) \wedge \dots \wedge G_m(\vec{x}, y_m) \wedge G_0(y_1, \dots, y_m, y)$$

$F$  is clearly a defining formula for  $f$  such that  $\mathbf{I}\Delta_0(\text{exp}) \vdash \exists y F(\vec{x}, y)$ .  
Moreover, each  $G_i$  is unique, so  $\mathbf{I}\Delta_0(\text{exp})$  also proves:

$$\begin{aligned} & C(\vec{x}, y_1, \dots, y_m, y) \wedge C(\vec{x}, z_1, \dots, z_m, z) \rightarrow \\ & \rightarrow \bigwedge_{j=1}^m y_j = z_j \wedge G_0(y_1, \dots, y_m, y) \wedge G_0(y_1, \dots, y_m, z) \rightarrow \\ & \rightarrow y = z \end{aligned}$$

so we have (by first order logic):

$$\mathbf{I}\Delta_0(\text{exp}) \vdash F(\vec{x}, y) \wedge F(\vec{x}, z) \rightarrow y = z$$

Thus  $f$  is provably  $\Sigma_1$  in  $\mathbf{I}\Delta_0(\text{exp})$ , so the rest is to find its bounding term.  
 $\mathbf{I}\Delta_0(\text{exp})$  proves the following:

$$C(\vec{x}, y_1, \dots, y_m, y) \rightarrow \bigwedge_{j=1}^m y_j < 2_k(s) \wedge y < 2_k(y_1 + \dots + y_m)$$

and

$$\bigwedge_{j=1}^m y_j < 2_k(s) \rightarrow y_1 + \dots + y_m < 2_k(s) \cdot m$$

Put  $t(\vec{x}) = 2_k(2_k(s) \cdot m)$ , then we obtain

$$\mathbf{I}\Delta_0(\text{exp}) \vdash C(\vec{x}, y_1, \dots, y_m, y) \rightarrow y < t(\vec{x})$$

and so

$$\mathbf{I}\Delta_0(\text{exp}) \vdash F(\vec{x}, y) \rightarrow y < t(\vec{x})$$

□

**Lemma 1.6.** Suppose  $f$  is defined by bounded minimisation

$$f(\vec{n}, m) = \mu_{k < m} (g(\vec{n}, k) = 0)$$

from a function  $g$  which is provably bounded in  $\mathbf{I}\Delta_0(\text{exp})$ . Then  $f$  is provably bounded in  $\mathbf{I}\Delta_0(\text{exp})$ .

*Proof.* Let  $G$  be a defining formula for  $g$ . Let  $F(\vec{x}, z, y)$  be the bounded formula

$$y \leq z \wedge \forall i < y \neg G(\vec{x}, i, 0) \wedge (y = z \vee G(\vec{x}, y, 0))$$

$\omega \models F(\vec{n}, m, k)$  iff either  $k$  is the least number less than  $m$  such that  $g(\vec{n}, k) = 0$  or there is no such and  $k = m$ . Thus it means that  $k$  is the value of  $f(\vec{n}, m)$ , so  $F$  is a defining formula for  $f$ .

Furthermore

$$\mathbf{I}\Delta_0(\text{exp}) \vdash F(\vec{x}, z, y) \rightarrow y < z + 1$$

so  $t(\vec{x}, z) = z + 1$  can be taken as a bounding term for  $f$ .

We can prove:

$$F(\vec{x}, z, y) \wedge F(\vec{x}, z, y') \wedge y < y' \rightarrow G(\vec{x}, y, 0) \wedge \neg G(\vec{x}, y', 0)$$

and similarly for interchanged  $y$  and  $y'$ . So we can prove:

$$F(\vec{x}, z, y) \wedge F(\vec{x}, z, y') \rightarrow \neg y < y' \wedge \neg y' < y$$

As far as  $y < y' \vee y' < y \vee y = y'$ , we have

$$F(\vec{x}, z, y) \wedge F(\vec{x}, z, y') \rightarrow y = y'$$

Now we have to check that  $\mathbf{I}\Delta_0(\text{exp}) \vdash \exists y F(\vec{x}, z, y)$ . We construct such  $y$  by bounded induction on  $z$ .

1.  $z = 0$ .

$F(\vec{x}, 0, 0)$  is provable since  $y = 0 \leftrightarrow y \leq 0$  and  $\neg i < 0$ . So  $\mathbf{I}\Delta_0(\text{exp}) \vdash F(\vec{x}, 0, y)$  is provable.

2. Assume  $\exists y F(\vec{x}, z, y)$  is provable, let show that that  $\exists y F(\vec{x}, z + 1, y)$  is provable.

We can show  $y \leq z \rightarrow y + 1 \leq z + 1$  and, via  $i < y + 1 \leftrightarrow i < y \vee i = y$ ,

$$\begin{aligned} \forall i < y \neg G(\vec{x}, i, 0) \wedge ((y = z) \wedge \neg G(\vec{x}, y, 0)) &\rightarrow \forall i < \\ y + 1 \neg G(\vec{x}, i, 0) \wedge y + 1 = z + 1 & \end{aligned}$$

Therefore

$$F(\vec{x}, z, y) \rightarrow F(\vec{x}, z + 1, y + 1) \vee F(\vec{x}, z + 1, y)$$

and thus:

$$\exists y F(\vec{x}, z, y) \rightarrow \exists y F(\vec{x}, z + 1, y)$$

□

**Theorem 1.2.** Every elementary function is provably bounded in  $\mathbf{I}\Delta_0(\text{exp})$ .

*Proof.* As we know from recursion theory, the class of elementary functions can be characterised as those functions which are definable from 0,  $S$ ,  $P$ ,  $\cdot$ ,  $+$ ,  $\text{exp}_2$ ,  $\dot{-}$  and  $\cdot$  by composition and minimisation. And then we apply above lemmas. □

## 1.1 Proof-theoretic Characterisation

For this section we shall be using a Tait-style formalisation of  $\mathbf{I}\Delta_0(\text{exp})$ . We have the following logical rules:

$$\begin{array}{c} \frac{}{\Gamma, R\vec{t}, \neg R\vec{t}} \mathbf{Ax} \\[10pt] \frac{\Gamma, A_0, A_1}{\Gamma, A_0 \vee A_1} \vee \qquad \frac{\Gamma, A_0 \quad \Gamma, A_1}{\Gamma, A_0 \wedge A_1} \wedge \\[10pt] \frac{\Gamma, A(t)}{\Gamma, \exists x A(x)} \exists \qquad \frac{\Gamma, A}{\Gamma, \forall x A} \forall \end{array}$$

where  $R\vec{t}$  is an atomic formula and  $x$  is not free in  $A$  in the  $\forall$  rule. Here  $\Gamma$  stores all non-logical axioms of  $\mathbf{I}\Delta_0(\text{exp})$  along with its negations. We also have the bounded induction rule:

$$\frac{\Gamma, B(0) \quad \Gamma, \neg B(n), B(n+1)}{\Gamma, B(t)} \mathbf{BInd}$$

where  $B$  is a bounded formula and  $t$  is any term.

Of course, the cut rule is admissible:

$$\frac{\Gamma, A \quad \Gamma, \neg A}{\Gamma} \mathbf{cut}$$

**Definition 1.2.** Let  $\exists \vec{z} B(\vec{z})$  be a closed  $\Sigma_1$ -formula, then it is *true at  $m$* , written as  $m \models \exists \vec{z} B(\vec{z})$ , if there exist natural numbers  $m_1, \dots, m_l$  such that each  $m_i < m$  and  $B(\vec{m})$  is true in the standard model.

A finite set  $\Gamma$  of closed  $\Sigma_1$ -formulas is true at  $m$ , written as  $m \models \Gamma$  if at least one of them is true at  $m$ .

If  $\Gamma(x_1, \dots, x_k)$  is a finite set of  $\Sigma_1$ -formulas whose free variables occur amongst  $x_1, \dots, x_k$ . Let  $f : \mathbb{N}^k \rightarrow \mathbb{N}$ , then  $f \models \Gamma(x_1, \dots, x_k)$  we have  $f(\vec{n}) \models \Gamma(x_1 := n_1, \dots, x_k := n_k)$  for each  $\vec{n} = (n_1, \dots, n_k)$ .

**Fact 1.1. (Persistence)**

1. If  $m \leq m'$ , then  $m \models \exists \vec{z} B(\vec{z})$  implies  $m' \models \exists \vec{z} B(\vec{z})$ .
2. If  $\forall \vec{n} \in \mathbb{N}^k$   $f(\vec{n}) \leq f'(\vec{n})$ , then  $f(\vec{n}) \models \Gamma(x_1 := n_1, \dots, x_k := n_k)$  implies  $f'(\vec{n}) \models \Gamma(x_1 := n_1, \dots, x_k := n_k)$ .

**Lemma 1.7.** Let  $\Gamma(\vec{x})$  be a finite set of  $\Sigma_1$  formulas such that

$$\mathbf{I}\Delta_0(\text{exp}) \vdash \bigvee_{\gamma(\vec{x}) \in \Gamma(\vec{x})} \gamma(\vec{x}).$$

Then there is an elementary function  $f$  such that  $f \models \Gamma(\vec{x})$  and  $f$  is strongly increasing on its variables.

*Proof.* If  $\Gamma$  is provable in  $\mathbf{I}\Delta_0(exp)$ , then it is provable in the Tait-style version of  $\mathbf{I}\Delta_0(exp)$ , where all cut formulas are  $\Sigma_1$ .

If  $\Gamma$  is classically derivable from non-logical axioms  $A_1, \dots, A_s$ , then there is a cut-free proof in the Tait calculus of  $\neg A_1, \Delta, \Gamma$ , where  $\Delta = \neg A_2, \dots, \neg A_s$ . Let us show how to cancel  $\neg A_1$  using a  $\Sigma_1$ -cut.

If  $A_1$  is an induction axiom on some formula  $B$ , then we have a cut-free proof of:

$$B(0) \wedge \forall y(\neg B(y) \vee B(y+1)) \wedge \exists x \neg B(x), \Delta, \Gamma$$

Thus we also have cut-free proofs of  $B(0), \Delta, \Gamma$ ,  $\neg B(y), B(y+1), \Delta, \Gamma$  and  $\exists x \neg B(x), \Delta, \Gamma$ . So we have

$$\frac{\frac{\Delta, \Gamma, B(0) \quad \Delta, \Gamma, \neg B(y), B(y+1)}{\Delta, \Gamma, B(x)} \mathbf{BInd} \quad \frac{\Delta, \Gamma, \forall x B(x)}{\Delta, \Gamma, \forall x B(x)} \forall}{\frac{\Delta, \Gamma, \forall x B(x) \quad \exists x \neg B(x), \Delta, \Gamma}{\Delta, \Gamma} \Sigma_1\text{-cut}}$$

We can similarly cancel each of  $\neg A_2, \dots, \neg A_s$  and so obtain the proof of  $\Gamma$  with  $\Sigma_1$ -cuts only.

Now we choose a proof of  $\Gamma(\vec{x})$  and proceed by induction on the height of the proof and determine an elementary function  $f$  such that  $f \models \Gamma$ .

1. If  $\Gamma(\vec{x})$  is an axiom, then for all  $\vec{n}$   $\Gamma(\vec{n})$  contains a true atom. So for any  $f \models \Gamma$ . Let us choose  $f(\vec{n}) = n_1 + \dots + n_k$ .
2. If  $\Gamma, B_0 \vee B_1$  is derivable, so is  $\Gamma, B_0, B_1$ . Note that  $B_0$  and  $B_1$  are both bounded. Let  $f \models \Gamma, B_0, B_1$ , then  $f \models \Gamma, B_0 \vee B_1$ .
3. Assume  $\Gamma, B_0 \wedge B_1$  is derivable, then  $\Gamma, B_0$  and  $\Gamma, B_1$ . By the induction hypothesis we have  $f_0 \models \Gamma, B_0$  and  $f_1 \models \Gamma, B_1$ , so, by persistence, we have  $\lambda \vec{n}. f_0(\vec{n}) + f_1(\vec{n}) \models \Gamma, B_0 \wedge B_1$ .
4. Assume  $\Gamma, \forall y B(y)$  is derivable, then  $\Gamma, B(y)$  is derivable and  $y$  is not free in  $\Gamma$ . Since all the formulas are  $\Sigma_1$ ,  $\forall x B(y)$  must be bounded, so  $B(y) = \neg(y < t) \vee B'(y)$  for some term  $t$  and for some bounded formula  $B'$ . By the induction hypothesis, assume  $f_0 \models \Gamma, \neg(y < t), B'(y)$  for some increasing elementary function  $f_0$ . Then we have:

$$f_0(\vec{n}, k) \models \Gamma(\vec{n}), \neg(k < t(\vec{n})), B'(\vec{n}, k)$$

Let  $g$  be an increasing elementary function bounding  $t$ , define

$$f(\vec{n}) = \sum_{k < g(\vec{n})} f(\vec{n}, k)$$



We have either  $f(\vec{n}) \models \Gamma(\vec{n})$  or, by persistence,  $B'(\vec{n}, k)$  is true for every  $k < t(\vec{n})$ . So  $f \models \Gamma, \forall y B(y)$  and  $f$  is elementary.

5. Assume  $\Gamma, \exists y A(y, \vec{x})$  is derivable, so  $\Gamma, A(t, \vec{x})$  is derivable for some term  $t$ . By the IH, there is elementary  $f_0$  such that for all  $\vec{n}$  one has

$$f_0(\vec{n}) \models \Gamma(\vec{n}), A(t(\vec{n}), \vec{n})$$

Then either  $f_0(\vec{n}) \models \Gamma(\vec{n})$  or else  $f_0(\vec{n})$  bounds true witnesses for all existential quantifiers in  $A(t(\vec{n}), \vec{n})$ . Choose an elementary function  $g$  which is bounding for  $t$ . Define  $f(\vec{n}) = f_0(\vec{n}) + g(\vec{n})$ , then for all  $\vec{n}$  either  $f(\vec{n}) \models \Gamma(\vec{n})$  or  $f(\vec{n}) \models \exists y A(y, \vec{n})$ .

6. TODO

7. TODO

□

## 2 Primitive Recursion and $\mathbf{I}\Sigma_1$

$\mathbf{I}\Sigma_1$  is an arithmetical theory where the induction scheme is restricted to  $\Sigma_1$  formulas.

**Lemma 2.1.** Every primitive recursion is provably recursive in  $\mathbf{I}\Sigma_1$ .

*Proof.* We have to show represent each primitive recursive function  $f$  with a  $\Sigma_1$  formula  $F(\vec{x}, y) := \exists z C(\vec{x}, y, z)$  such that:

1.  $f(\vec{n}) = m$  iff  $\omega \models F(\vec{x}, y)$ .
2.  $\mathbf{I}\Sigma_1 \vdash \exists y F(\vec{x}, y)$ .
3.  $\mathbf{I}\Sigma_1 \vdash F(\vec{x}, y) \wedge F(\vec{x}, y') \rightarrow y = y'$ .

□