Some Notes on Proof Theory and Elements of Ordinal Analysis

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1 Provable Recursion in $\mathbf{I}\Delta_0(\exp)$

$\mathbf{I}\Delta_0(\exp)$ is a theory in first-order logic in the language:

$$\{=, 0, S, P, +, \dot{-}, \cdot, exp_2\}$$

where S and P are successor and precessor functions respectively. Further, we will denote $S(x)$ and $P(x)$ as $x+1$ and $x\dot{-}1$ respectively. 2^x stands for $exp_2(x)$.

The non-logical axioms of $\mathbf{I}\Delta_0(\exp)$ are the following list:

- $x + 1 \neq 0$
- $x + 1 = y + 1 \to x = y$
- $0\dot{-}1 = 0$
- $(x + 1)\dot{-}1 = x$
- $x + 0 = x$
- $x + (y + 1) = (x + y) + 1$
- $x\dot{-}0 = x$
- $x\dot{-}(y + 1) = x\dot{-}y\dot{-}1$
- $x \cdot 0 = 0$
- $x \cdot (y + 1) = x \cdot y + x$
- $2^0 = 1$
- $2^{x+1} = 2^x + 2^x$

along with the bounded induction scheme:

$$B(0) \land \forall x(B(x) \to B(x + 1)) \to \forall x B(x)$$

where B is a Δ-*formula*, that is a formula one of the following forms (with bounded quantifiers only):

- $B \rightleftharpoons \forall x < tP(x) \equiv \forall x(x < t \to P(x))$
- $B \rightleftharpoons \exists x < tP(x) \equiv \exists x(x < t \land P(x))$

A Σ_1-formula is a formula of the form:

$$\exists \vec{x} B(\vec{x})$$

where $B(\vec{x}) \in \Delta_0$.

Lemma 1.1. $\mathbf{I}\Delta_0(\exp)$ proves (the universal closures of):

1. $x = 0 \vee x = (x \dot{-} 1) + 1$
2. $x + (y + z) = (x + y) + z$
3. $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
4. $x \cdot (y + z) = x \cdot y + x \cdot z$
5. $x + y = y + x$
6. $x \cdot y = y \cdot x$
7. $x \dot{-} (y + z) = (x \dot{-} y) \dot{-} z$
8. $2^{x+y} = 2^x \cdot 2^y$

Proof.

1. This is self-evident.
2. If $z = 0$, then $x + y = x + y$. If $z = z' + 1$, then, by applying the IH and the relevant axioms:
$$(x + (y + (z' + 1))) = (x + ((y + z') + 1)) = (x + (y + z')) + 1 = ((x + y) + z') + 1 = (x + y) + (z' + 1)$$
3. If $z = 0$, then $x \cdot (y \cdot 0) = (x \cdot y) \cdot 0$. If $z = z' + 1$, then:
$$x \cdot (y \cdot (z' + 1)) = x \cdot (y \cdot z' + y) = x \cdot (y \cdot z') + x \cdot y = (x \cdot y) \cdot z' + x \cdot y = (x \cdot y) \cdot (z' + 1)$$
4. The rest of the cases are shown by induction on z. Consider the exponentiation law. If $y = 0$, then
$$2^{x+0} = 2^x = 0 + 2^x = 2^x \cdot 0 + 2^x = 2^x \cdot (0 + 1) = 2^x \cdot 2^0$$
If $y = y' + 1$, then:
$$2^{x+(y'+1)} = 2^{(x+y')+1} = 2^x \cdot 2^y + 2^x \cdot 2^y = 2^x \cdot 2^{y+1}$$

□

Lemma 1.2. $\mathbf{I}\Delta_0(\exp)$ proves (the universal closures of):

1. $\neg x < 0$
2. $x \leq 0 \leftrightarrow x = 0$
3. $0 \leq x$
4. $x \leq x$

5. $x < x + 1$

6. $x < y + 1 \leftrightarrow x \leq y$

7. $x \leq y \leftrightarrow x < y \vee x = y$

8. $x \leq y \wedge y \leq z \rightarrow x \leq z$

9. $x < y \wedge y < z \rightarrow x < z$

10. $x \leq y \vee y < x$

11. $x < y \rightarrow x + z < y + z$

12. $x < y \rightarrow x \cdot (z + 1) < y \cdot (z + 1)$

13. $x < 2^x$

14. $x < y \rightarrow 2^x < 2^y$

Proof. Straightforward induction. □

Definition 1.1. A function $f : \mathbb{N}^k \rightarrow \mathbb{N}$ is *provably* Σ_1 or *provably recursive* in an arithmetical theory if there is a Σ_1 formula $F(\vec{x}, y)$, a "defining formula" of f, such that:

1. $f(\vec{n}) = m$ iff $\omega \models f(\vec{n}) = m$

2. $T \vdash \exists y F(\vec{x}, y)$

3. $T \vdash F(\vec{x}, y) \wedge F(\vec{x}, y') \rightarrow y = y'$

If a defining formula $F \in \Delta_0$, then a function f is *provably bounded* in T if there is a term $t(\vec{x})$ such that $T \vdash F(\vec{x}, y) \rightarrow y < t(\vec{x})$.

Theorem 1.1. Let f be a provably recursive in T, then we can conservatively extend T by adding a new function symbol f along with the defining axiom $F(\vec{x}, f(\vec{x}))$.

Proof. Let $\mathcal{M} \models T$, $\mathcal{M}$ can be made into a model $(\mathcal{M}, f)$ where we interpret f as the function which is uniquely determined by the second and third conditions of the definitions above. Let φ be a statement not involving f such that φ is true in $(\mathcal{M}, f)$, so φ is true in $\mathcal{M}$ as well. By compactness T proves φ. □

Lemma 1.3. Each term defines a provably bounded function of $\mathbf{I}\Delta_0(\exp)$.

Proof. Let f be a function defined by some $\mathbf{I}\Delta_0(\exp)$-term t, that is, $f(\vec{x}) = t(\vec{x})$. Take $y = t(\vec{x})$ as the defining formula for f since $\exists y\, (y = t(\vec{x}))$ is derivable. If $y' = t(\vec{x}) \wedge y = t(\vec{x})$, then $y = y'$ by transitivity. A formula $y = t(\vec{x})$ is bounded and $y = t$ implies $y < t + 1$. Thus f is provably bounded. □

Lemma 1.4. Define $2_k(x)$ as $2_0(x) = x$ and $2_{n+1}(x) = 2^{2_n(x)}$. Then for every term $t(x_1, \ldots, x_n)$ built up from the constants $0, S, P, +, \dot{-}, \cdot, exp_2$ there exists $k < \omega$ such that:

$$\mathbf{I}\Delta_0(\exp) \vdash t(x_1, \ldots, x_n) < 2_k(\sum_{k=0}^{n} x_k)$$

Proof. Let t be a term constructed from subterms t_0 and t_1 by using one of the function constants. Assume that inductively $t_0 < 2_{k_0}(s_0)$ and $t_1 < 2_{k_1}(s_1)$ are both provable for some $k_0, k_1 < \omega$, where s_i is the sum of the variables of t_i for $i = 0, 1$.

Let s be the sum of all variables appearing in either t_0 or t_1 and let $k = \max(k_0, k_1)$. Then one can prove $t_0 < 2_k(s)$ and $t_1 < 2_k(s)$. So one needs to show the following:

1. $t_0 + 1 < 2_{k+1}(s)$
2. $t_0 \dot{-} 1 < 2_k(s)$
3. $t_0 \dot{-} t_1 < 2_k(s)$
4. $t_0 \cdot t_1 < 2_k(s)$
5. $t_0 + t_1 < 2_k(s)$
6. $2^{t_0} < 2_k(s)$

So $\mathbf{I}\Delta_0(\exp) \vdash t < 2_{k+1}(s)$. □

Lemma 1.5. Let f be a function defined by composition:

$$f(\vec{x}) = g_0(g_1(\vec{x}), \ldots, g_m(\vec{x}))$$

where $g_0, g_1, \ldots, g_m$ are functions each of which is provably bounded in $\mathbf{I}\Delta_0(\exp)$. Then f is provably bounded in $\mathbf{I}\Delta_0(\exp)$.

Proof. Each g_i has a defining formula G_i and, by Lemma 1.4, there is a number $k_i < \omega$ such that:

$$\mathbf{I}\Delta_0(\exp) \vdash \exists y < 2_{k_i}(s) G_i(\vec{x}, y)$$

where s is the sum of elements of $\vec{x}$. And for $i = 0$ one has:

$$\mathbf{I}\Delta_0(\exp) \vdash \exists y < 2_{k_0}(s_0) G_0(y_1, \ldots, y_m, y)$$

where s_0 is the sum of $y_1, \ldots, y_m$.

Let $k = \max\{k_i < \omega \mid i < m+1\}$ and let $F(\vec{x}, y)$ be the bounded formula:

$$\exists y_1 < 2_k(s) \ldots \exists y_m < 2_k(s) C(\vec{x}, y_1, \ldots, y_m, y)$$

where $C(\vec{x}, y_1, \ldots, y_m, y)$ is the conjunction:

$$G_1(\vec{x}, y_1) \wedge \cdots \wedge G_m(\vec{x}, y_m) \wedge G_0(y_1, \ldots, y_m, y)$$

F is clearly a defining formula for f such that $\mathbf{I}\Delta_0(\exp) \vdash \exists y F(\vec{x}, y)$.
Moreover, each G_i is unique, so $\mathbf{I}\Delta_0(\exp)$ also proves:

$$\begin{aligned}&C(\vec{x}, y_1, \ldots, y_m, y) \wedge C(\vec{x}, z_1, \ldots, z_m, z) \rightarrow \\ &\rightarrow \bigwedge_{j=1}^{m} y_j = z_j \wedge G_0(y_1, \ldots, y_m, y) \wedge G_0(y_1, \ldots, y_m, z) \rightarrow \\ &\rightarrow y = z\end{aligned}$$

so we have (by first order logic):

$$\mathbf{I}\Delta_0(\exp) \vdash F(\vec{x}, y) \wedge F(\vec{x}, z) \rightarrow y = z$$

Thus f is provably Σ_1 in $\mathbf{I}\Delta_0(\exp)$, so the rest is to find its bounding term. $\mathbf{I}\Delta_0(\exp)$ proves the following:

$$C(\vec{x}, y_1, \ldots, y_m, y) \rightarrow \bigwedge_{j=1}^{m} y_j < 2_k(s) \wedge y < 2_k(y_1 + \cdots + y_m)$$

and

$$\bigwedge_{j=1}^{m} y_j < 2_k(s) \rightarrow y_1 + \cdots + y_m < 2_k(s) \cdot m$$

Put $t(\vec{x}) = 2_k(2_k(s) \cdot m)$, then we obtain

$$\mathbf{I}\Delta_0(\exp) \vdash C(\vec{x}, y_1, \ldots, y_m, y) \rightarrow y < t(\vec{x})$$

and so

$$\mathbf{I}\Delta_0(\exp) \vdash F(\vec{x}, y) \rightarrow y < t(\vec{x})$$

□

Lemma 1.6. Suppose f is defined by bounded minimisation

$$f(\vec{n}, m) = \mu_{k<m}(g(\vec{n}, k) = 0)$$

from a function g which is provably bounded in $\mathbf{I}\Delta_0(\exp)$. Then f is provably bounded in $\mathbf{I}\Delta_0(\exp)$.

Proof. Let G be a defining formula for g. Let $F(\vec{x}, z, y)$ be the bounded formula

$$y \leq z \wedge \forall i < y \neg G(\vec{x}, i, 0) \wedge (y = z \vee G(\vec{x}, y, 0))$$

$\omega \models F(\vec{n}, m, k)$ iff either k is the least number less than m such that $g(\vec{n}, k) = 0$ or there is no such and $k = m$. Thus it means that k is the value of $f(\vec{n}, m)$, so F is a defining formula for f.

Furthermore

$$\mathbf{I}\Delta_0(\exp) \vdash F(\vec{x}, z, y) \rightarrow y < z + 1$$

so $t(\vec{x}, z) = z + 1$ can be taken as a bounding term for f.

We can prove:

$$F(\vec{x}, z, y) \wedge F(\vec{x}, z, y') \wedge y < y' \to G(\vec{x}, y, 0) \wedge \neg G(\vec{x}, y, 0)$$

and similarly for interchanged y and y'. So we can prove:

$$F(\vec{x}, z, y) \wedge F(\vec{x}, z, y') \to \neg y < y' \wedge \neg y' < y$$

As far as $y < y' \vee y' < y \vee y = y'$, we have

$$F(\vec{x}, z, y) \wedge F(\vec{x}, z, y') \to y = y'$$

Now we have to check that $\mathbf{I}\Delta_0(\exp) \vdash \exists y F(\vec{x}, z, y)$. We construct such y by bounded induction on z.

1. $z = 0$.

   $F(\vec{x}, 0, 0)$ is provable since $y = 0 \leftrightarrow y \leq 0$ and $\neg i < 0$. So $\mathbf{I}\Delta_0(\exp) \vdash F(\vec{x}, 0, y)$ is provable.

2. Assume $\exists y F(\vec{x}, z, y)$ is provable, let show that that $\exists y F(\vec{x}, z+1, y)$ is provable.

   We can show $y \leq z \to y + 1 \leq z + 1$ and, via $i < y + 1 \leftrightarrow i < y \vee i = y$,

   $$\forall i < y \, \neg G(\vec{x}, i, 0) \wedge ((y = z) \wedge \neg G(\vec{x}, y, 0)) \to \forall i < y + 1 \, \neg G(\vec{x}, i, 0) \wedge y + 1 = z + 1$$

   Therefore

   $$F(\vec{x}, z, y) \to F(\vec{x}, z+1, y+1) \vee F(\vec{x}, z+1, y)$$

   and thus:

   $$\exists y F(\vec{x}, z, y) \to \exists y F(\vec{x}, z+1, y)$$

□

Theorem 1.2. Every elementary function is provably bounded in $\mathbf{I}\Delta_0(\exp)$.

Proof. As we know from recursion theory, the class of elementary functions can be characterised as those functions which are definable from 0, S, P, $\cdot$, $+$, exp_2, $\dot{-}$ and $\cdot$ by composition and minimisation. And then we apply above lemmas. □

1.1 Proof-theoretic Characterisation

For this section we shall be using a Tait-style formalisation of $\mathbf{I}\Delta_0(\exp)$. We have the following logical rules:

$$\frac{}{\Gamma, R\vec{t}, \neg R\vec{t}}\ \mathbf{Ax}$$

$$\frac{\Gamma, A_0, A_1}{\Gamma, A_0 \vee A_1}\ \vee \qquad\qquad \frac{\Gamma, A_0 \qquad \Gamma, A_1}{\Gamma, A_0 \wedge A_1}\ \wedge$$

$$\frac{\Gamma, A(t)}{\Gamma, \exists x A(x)}\ \exists \qquad\qquad \frac{\Gamma, A}{\Gamma, \forall x A}\ \forall$$

where $R\vec{t}$ is an atomic formula and x is not free in A in the $\forall$ rule. Here Γ stores all non-logical axioms of $\mathbf{I}\Delta_0(\exp)$ along with its negations. We also have the bounded induction rule:

$$\frac{\Gamma, B(0) \qquad \Gamma, \neg B(n), B(n+1)}{\Gamma, B(t)}$$

where B is a bounded formula and t is any term.

Of course, the cut rule is admissible:

$$\frac{\Gamma, A \qquad \Gamma, \neg A}{\Gamma}\ \mathbf{cut}$$

2 Primitive Recursion and $\mathbf{I}\Sigma_1$

$\mathbf{I}\Sigma_1$ is an arithmetical theory where the induction scheme is restructed to Σ_1 formulas.

Lemma 2.1. Every primitive recursion is provably recursive in $\mathbf{I}\Sigma_1$.

Proof. We have to show represent each primitive recursive function f with a Σ_1 formula $F(\vec{x}, y) := \exists z C(\vec{x}, y, z)$ such that:

1. $f(\vec{n}) = m$ iff $\omega \models F(\vec{x}, y)$.
2. $\mathbf{I}\Sigma_1 \vdash \exists y F(\vec{x}, y)$.
3. $\mathbf{I}\Sigma_1 \vdash F(\vec{x}, y) \wedge F(\vec{x}, y') \rightarrow y = y'$.

$\square$