Some Notes on Set Theory, Pt 1

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1 Cardinals

An ordinal number α is a *cardinal number* if no $\beta < \alpha$ such that $|\alpha| = |\beta|$. Further, we shall use κ, λ, μ to denote cardinal numbers.

Let W be a well-ordered set, then there exists an ordinal α such that $|W| = |\alpha|$, so we let:

$$|W| = \text{the least ordinal } \alpha \text{ such that } |W| = |\alpha|$$

An *aleph* is an infinite cardinal number.

Let α be an ordinal, then α^+ is the least cardinal bigger than α.

Lemma 1.1.

1. For every α there is a cardinal number κ such that $\kappa > \alpha$.
2. Let X be a set of cardinal, then $\sup X$ is a cardinal.

Proof.

1. Let X be a set, let

   $$h(X) = \text{the least } \alpha \text{ such that no injection from } \alpha \text{ into } X$$

   Consider $X \times X$, so $2^{X \times X}$ is the set of relations on X and there are well-orderings of subsets of X amongst all relations in $2^{X \times X}$, so consider the set

   $$Y = \{R \subseteq Y \times Y \mid Y \subseteq X \;\&\; Y \text{ is a well-ordering }\}$$

   So there is a set of ordinals:

   $$Ord(Y) = \{\alpha \in \text{Ord} \mid \exists R \in Y \; \alpha \text{ is the order type of } Y\}$$

   Note that $Ord(Y)$ is a set and take the least element ordinal β does not belong to $Ord(Y)$. So $h(X) = \beta$. To be more precise, we have:

   $$\beta = \sup Ord(Y)$$

Then $|\alpha| < h(\alpha)$ for each ordinal α.

2. Let $\alpha = \sup X$. Let f be a one-to-one function from α onto some $\beta < \alpha$. Let κ be a cardinal such that $\beta < \kappa \leq \alpha$, then $|\kappa| = |\{f(\xi) \mid \xi < \kappa\}| \leq \beta$, so contradiction and α is a cardinal.

$\square$

The enumeration of all alephs is defined by transfinite induction:

- $\aleph_0 = \omega$
- $\aleph_{\alpha+1} = \aleph_\alpha^+ = \omega_{\alpha+1}$
- If β is a limit ordinal, then $\aleph_\beta = \omega_\beta = \sup\{\omega_\alpha \mid \alpha < \beta\}$.

A cardinal of the form $\aleph_{\alpha+1}$ is a *successor* cardinal, a cardinal $\aleph_\beta$ for limit β is a *limit cardinal*.

1.1 The ordering of $\alpha \times \alpha$

Define a well-ordering of the class $\mathrm{Ord} \times \mathrm{Ord}$ the following way:

$$\begin{gathered}(\alpha, \beta) < (\gamma, \delta) \text{ iff either } \max(\alpha, \beta) < \max(\gamma, \delta) \text{ or} \\ \max(\alpha, \beta) = \max(\gamma, \delta) \text{ and } \alpha < \gamma \text{ or} \\ \max(\alpha, \beta) = \max(\gamma, \delta) \text{ and } \alpha = \gamma \text{ and } \beta < \delta.\end{gathered}$$

Then $<$ is a well-ordering and linear relation on Ord. Moreover, $\alpha \times \alpha$ is the initial segment of $(\mathrm{Ord} \times \mathrm{Ord}, <)$ given by $(0, \alpha)$.

We let:

$$\Gamma(\alpha, \beta) = \text{the order type of } \{(\xi, \eta) \mid (\xi, \eta) < (\alpha, \beta)\}$$

Γ is also one-to-one:

$$(\alpha, \beta) < (\gamma, \delta) \text{ iff } \Gamma(\alpha, \beta) < \Gamma(\gamma, \delta)$$

Γ is increasing and continuous and $\Gamma(\alpha \times \alpha) = \alpha$ for arbitrarily large α.

Theorem 1.1. $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$

Proof. Let us show that $\Gamma(\omega_\alpha \times \omega_\alpha) = \omega_\alpha$.

1. If $\alpha = 0$, then $\Gamma(\omega \times \omega) = \omega$.
2. Let α be the least ordinal such that $\Gamma(\omega_\alpha \times \omega_\alpha) \neq \omega_\alpha$. Let β, γ be ordinals such that $\Gamma(\beta, \gamma) = \omega_\alpha$. Take $\delta < \omega_\alpha$ such that $\delta > \beta, \gamma$. $\delta \times \delta$ is the initial segment of Ord^2 and it contains (β, γ). So $\Gamma(\delta \times \delta) \supset \omega_\alpha = \Gamma(\beta, \gamma)$. Thus $|\delta \times \delta| \geq \aleph_\alpha$. But $|\delta \times \delta| = |\delta| \cdot |\delta| = |\delta|$. But $|\delta| < \aleph_\alpha$ by the assumption of minimality of α. Contradiction.

$\square$

As a corollary:

$$\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = \max(\aleph_\alpha, \aleph_\beta)$$

1.2 Cofinality

Let $\alpha, \beta > 0$ be limit ordinals. An increasing β-sequence $\langle \alpha_\xi : \xi < \beta \rangle$ is *cofinal* in α if $\lim_{\xi \to \beta} \alpha_\xi = \alpha$. A subset $X \subseteq \alpha$ is *cofinal* in α whenever $\sup X = \alpha$.

Let $\alpha > 0$ be a limit ordinal, the *cofinality* of α is:

$$\operatorname{cf} \alpha = \text{the least ordinal } \beta \text{ such that } \exists \ \langle \alpha_\xi : \xi < \beta \rangle \text{ such that } \lim_{\xi \to \beta} \alpha_\xi = \alpha$$

Note that for each α $\operatorname{cf} \alpha$ is a limit ordinal and $\operatorname{cf} \alpha \leq \alpha$.

Lemma 1.2. For each α $\operatorname{cf}(\operatorname{cf} \alpha) \leq \operatorname{cf} \alpha$.

Proof. Let $\langle \alpha_\xi : \xi < \beta \rangle$ be cofinal in α and let $\langle \xi_\nu : \nu < \gamma \rangle$ be cofinal in β.

Consider $\langle \alpha_{\xi_\nu} : \nu < \gamma \rangle$, then

$$\lim_{\nu < \gamma} \alpha_{\xi_\nu} = \alpha$$

since the limit of a subsequence equals the limit of a sequence as in usual real analysis or topology. □

Lemma 1.3. Let α be a non-zero limit ordinal, then

1. If $A \subseteq \alpha$ and $\sup A = \alpha$, the order-type of A is at least $\operatorname{cf} \alpha$.

2. Let $\beta_0 \leq \beta_1 \leq \cdots \leq \beta_\xi \leq \ldots$ for $\xi < \gamma$ be a non-decreasing sequence of ordinals such that $\lim_{\xi \to \gamma} = \alpha$, then $\operatorname{cf} \gamma = \alpha$.

Proof. 1. The order-type of A is the length of the increasing enumeration of A, the limit of which (as an increasing sequence) is α.

2. If $\gamma = \lim_{\nu \to \operatorname{cf} \gamma} \xi_\nu$, then $\alpha = \lim_{\nu \to \operatorname{cf} \gamma} \beta_{\xi_\nu}$, and the non-decreasing sequence $\langle \beta_{\xi_\nu} : \nu < \operatorname{cf} \gamma \rangle$ has an increasing sequence of the length at most $\operatorname{cf} \gamma$ and it has the same limit, so $\operatorname{cf} \alpha \leq \operatorname{cf} \gamma$.

   To show $\operatorname{cf} \gamma \leq \operatorname{cf} \alpha$, assume $\alpha = \lim_{\nu \to \operatorname{cf} \alpha} \alpha_\nu$. Take $\nu < \operatorname{cf} \alpha$, let ξ_ν be the least ξ greater than all ξ_ι for $\iota < \nu$ such that $\beta_\xi > \alpha_\nu$. We have $\alpha = \lim_{\nu \to \operatorname{cf} \alpha} \beta_{\xi_\nu}$, so $\gamma = \lim_{\nu \to \operatorname{cf} \alpha} \xi_\nu$, so the inequation is proved.

□

An infinite cardinal $\aleph_\alpha$ is *regular* if $\operatorname{cf} \omega_\alpha = \omega_\alpha$. $\aleph_\alpha$ is *singular* if $\operatorname{cf} \omega_\alpha < \omega_\alpha$.

Lemma 1.4. Let α be a limit ordinal, then $\operatorname{cf} \alpha$ is a regular cardinal.

Proof. If α is not a cardinal, then there exists an ordinal $\beta < \alpha$ such that $|\beta| = |\alpha|$, then we construct a cofinal sequence in α of length $|\beta|$, then $\operatorname{cf} \alpha = |\beta|$ and $\operatorname{cf} \alpha < \alpha$. □

Let κ be a limit ordinal, a subset $X \subset \kappa$ is *bounded* if $\sup X < \kappa$ and *unbounded* if $\sup X = \kappa$.

Lemma 1.5. Let κ be an aleph, then:

1. If $X \subset \kappa$ and $|X| < \operatorname{cf} \kappa$, then X is bounded.
2. If λ ¡ $\operatorname{cf} \kappa$ and $f : \lambda \to \kappa$, then $\operatorname{Im} f$ is bounded in κ.

Proof. 1. Let X be such subset of κ and assume X is unbounded, so $\sup X = \kappa$. By 1 of Lemma 1.3, the order-type of X is at least $\operatorname{cf} \kappa$, which contradicts to $|X| < \operatorname{cf} \kappa$, so X is bounded.

2. Follows from the first item.

□

Lemma 1.6. (Hausdorff)
Let κ be a cardinal, then the following are equivalent:

1. κ is singular.
2. There is a cardinal $\lambda < \kappa$ and a family $\{S_\xi \,|\, \xi < \lambda\}$ such that each $S_\xi \subset \kappa$, $|S_\xi| < \kappa$ and $\kappa = \bigcup\limits_{\xi<\lambda} S_\xi$.

Proof.

1. (1) $\Rightarrow$ (2).

   If κ is singular, then there is an increasing sequence $\langle \alpha_\xi \,:\, \xi < \operatorname{cf} \kappa \rangle$, so a family of required subsets is actually a family of those α_ξ's and $\lambda = \operatorname{cf} \kappa$ which is strictly less than κ since κ is singular.

2. (2) $\Rightarrow$ (1).

   Let λ be the least cardinal such that $\lambda < \kappa$ and there exists a family $\{S_\xi \mid \xi < \lambda\}$ where each $S_\xi \subset \kappa$, $|S_\xi| < \kappa$ and

   $$\kappa = \bigcup_{\xi<\lambda} S_\xi$$

   For each $\xi < \lambda$, let β_ξ be the order-type of $\cup_{\nu<\xi} S_\nu$. The sequence $\langle \beta_\xi : \xi < \lambda \rangle$ is non-decreasing and each $\beta_\xi < \kappa$ for all $\xi < \lambda$ since λ is minimal.

   Let us show that $\lim\limits_{\xi\to\kappa} \beta_\xi = \kappa$ to show that $\operatorname{cf} \kappa \le \lambda$.

   Assume $\beta = \lim\limits_{\xi\to\kappa} \beta_\xi$. There is a one-to-one mapping $f : \bigcup\limits_{\xi<\beta} S_\xi \to \lambda \times \beta$ such that:

   $$f : \alpha \mapsto (\xi, \gamma)$$

   where ξ is the least ordinal such that $\alpha \in S_\xi$ and γ is the order-type of $S_\xi \cap \gamma$.

   We have $\lambda < \kappa$ and $|\lambda \times \beta| = \lambda \cdot |\beta|$, then $\kappa = \beta$.

□

Theorem 1.2. Let κ be an infinite cardinal, then $\kappa < \kappa^{\operatorname{cf}\kappa}$.

Proof. Let F be a collection of κ functions from $\operatorname{cf}\kappa$ to κ:

$$F = \{f_\alpha : \operatorname{cf}\kappa \to \kappa \mid \alpha < \kappa\}$$

Let us construct f that does not belong to F.
We have $\kappa = \lim_{\xi < \operatorname{cf}\kappa} \alpha_\xi$, for $\xi < \operatorname{cf}\kappa$ we let:

$$f(\xi) = \text{least } \gamma \text{ such that } \gamma \neq \forall \alpha < \alpha_\xi \; f_\alpha \neq \gamma$$

Such γ does exist and f is different from all the f_α. $\square$

An uncountable cardinal κ is *weakly inaccessible* if it is limit and regular, but we cannot prove the existence of weakly inaccessible cardinals in ZFC.

2 Real Numbers and The Baire Space

The *continuum* is the cardinality of $\mathbb{R}$ denoted as $\mathfrak{c}$.

Theorem 2.1. (Cantor)
$\aleph_0 < \mathfrak{c}$.

Proof. One can think of it as a consequence of Theorem 1.2. $\square$

Definition 2.1. The *Continuum Hypothesis* (CH) is the following equation:

$$\aleph_1 = \mathfrak{c}.$$

Let $(P, <)$ be an ordered set, a subset $D \subset P$ is a *dense* subset of P if $a < b$ in P implies $a < d$ and $d < b$ for some $d \in D$.

Theorem 2.2. $(\mathbb{R}, <)$ is the unique complete linear ordering that has a countable dense subset isomorphic to $(\mathbb{Q}, <)$.

Proof. Let C and C' be two complete dense linear orderings and let P and P' be dense in C and C' respectively. Let $f : P \cong P'$, so f can be extended to $f^* : C \cong C'$ by letting:

$$f^* : x \mapsto \sup\{f(p) \mid p \in P \,\&\, p \leq x\}$$

That is, $(.)^*$ is functorial. $\square$

The existence of $(\mathbb{R}, <)$ follows from the following general statement:

Theorem 2.3. Let $(P, <)$ be a dense unbounded linear ordering, then there exists a complete dense unbounded linear ordering $(C, \prec)$ such that:

1. $(P, <)$ embeds to $(C, \prec)$.
2. P is dense in C.

Proof. Recall that a *Dedekind cut* in P is a pair (A, B) of disjoint subsets of P such that:

1. $A \cup B = P$.
2. $\forall a \in A\, \forall b \in B\; a < b$.
3. A has no greatest element.

Let C be the set of all Dedekind cuts in P. We let $(A_1, B_1) \preceq (A_2, B_2)$ if $A_1 \subset A_2$ and $B_2 \subset B_1$. $(C, \preceq)$ is complete.

Let $\{C_i \mid i \in I\} \neq \emptyset$ be a bounded subset of C, then $(\bigcup_i A_i, \bigcap_i B_i)$ is its supremum.

Let $p \in P$, let

$$A_p = \{x \in P \mid x < p\}$$
$$B_p = \{x \in P \mid x \geq p\}$$

Then $(\{(A_p, B_p) \mid p \in P\}, \preceq) \cong (P, <)$ and is dense in C. □

$\mathbb{Q}$ is dense in $\mathbb{R}$, so every open interval (a, b) contains some rational number. Then if S is a disjoint collection of open intervals, then S is at most countable.

Let P be a dense linearly ordered set, if every disjoint collection of open intervals is at most countable, then we say that P satisfies the *countable chain condition.*

(Suslin's Problem) *Let P be a dense linearly ordered set satisfying the countable chain condition. Is P isomorphic to $(\mathbb{R}, <)$?*

Note that neither Suslin's Problem nor its negation can be decided in ZFC.

2.1 Topology of $\mathbb{R}$

The real line is equipped with the natural topology induced by the metric $d(a, b) = |b - a|$ coincides with the order topology on $(\mathbb{R}, <)$. $\mathbb{R}$ is also a complete separable metric space.

Every open set in $\mathbb{R}$ is the union of intervals with rational endpoints, so there are continuum many open sets (and the same observation holds for open sets as well).

A subset P is *perfect* is it has no isolated points.

Theorem 2.4. Every perfect set P has cardinality $\mathfrak{c}$.

Proof. We construct a one-to-one function F from $\{0, 1\}^\omega$ to P. Let S be the set of all finite binary sequences and let $s \in S$.

By induction on $len(s)$ one can find closed intervals I_s such that for each $n < \omega$ and for each $s \in S$ such that $len(s) = n$:

1. $I_s \cap P$ is perfect,
2. the diameter of I_s is $\leq 1/2$,

3. $I_{0:s}, I_{1:s} \subset I_s$ and $I_{0:s} \cap I_{1:s} = \emptyset$

Take $f \in \{0,1\}^\omega$, the set $P \cap \bigcap\limits_{n<\omega} I_{f\restriction n}$ has exactly one element, so let:

$$F : f \mapsto \bigcap_{n<\omega} I_{f\restriction n}$$

□

Theorem 2.5. (**Cantor-Bendixon**)
If F is an uncountable closed set, then $F = P \cup S$, where P is perfect and S is at most countable.

Proof.
Let $F \subset \mathbb{R}$, let

$$F^{'} = \text{the set of all limit points of } F$$

$F^{'}$ is also called the *derived set* of F. F' is closed and obviously a subset of A.
We let:

1. $F_0 = A$.
2. $F_{\alpha+1} = F^{'}_{\alpha}$.
3. $F_\alpha = \bigcap\limits_{\gamma<\alpha} F_\gamma$ if $\alpha > 0$ is a limit ordinal.

Since $F_0 \supset F_1 \supset \cdots \supset F_\alpha \supset$, so we have an ordinal θ such that $F_\theta = F_{\theta+1}$ (otherwise we could map the proper class of ordinals onto some set). We let $P = F_\alpha$. If P is nonempty, then P is also perfect.

Let us show that $F - P$ is at most countable. Let $\langle J_k : k < \omega \rangle$ be an enumeration of rational intervals. We have

$$F - P = \bigcup_{\alpha<\theta} (F_\alpha - F_{\alpha+1})$$

So if $a \in F - P$, then there exists $\alpha < \theta$ such that $a \in F_\alpha - F_{\alpha+1}$, that is, a is an isolated point of F_α. We let k_a be the least k such that a is the only point of F_α in J_k.

If $\alpha \leq \beta$ and $a \neq b$ and b is isolated in F_β, then $b \notin J_{k_a}$, so $k_a \neq k_b$, so the mapping $a \mapsto k_a$ is one-to-one.

□

Corollary 2.1. If $C \subseteq \mathbb{R}$ is closed, then either $|C| = 2^{\aleph_0}$ or $|C| \leq \aleph_0$.

3 The Axiom of Choice

4 Cardinal Arithmetic via the Generalised Continuum Hypothesis