

# Some Notes on Set Theory, Pt 1

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## 1 Cardinals

An ordinal number  $\alpha$  is a *cardinal number* if no  $\beta < \alpha$  such that  $|\alpha| = |\beta|$ . Further, we shall use  $\kappa, \lambda, \mu$  to denote cardinal numbers.

Let  $W$  be a well-ordered set, then there exists an ordinal  $\alpha$  such that  $|W| = |\alpha|$ , so we let:

$$|W| = \text{the least ordinal } \alpha \text{ such that } |W| = |\alpha|$$

An *aleph* is an infinite cardinal number.

Let  $\alpha$  be an ordinal, then  $\alpha^+$  is the least cardinal bigger than  $\alpha$ .

**Lemma 1.1.**

1. For every  $\alpha$  there is a cardinal number  $\kappa$  such that  $\kappa > \alpha$ .
2. Let  $X$  be a set of cardinal, then  $\sup X$  is a cardinal.

*Proof.*

1. Let  $X$  be a set, let

$$h(X) = \text{the least } \alpha \text{ such that no injection from } \alpha \text{ into } X$$

Consider  $X \times X$ , so  $2^{X \times X}$  is the set of relations on  $X$  and there are well-orderings of subsets of  $X$  amongst all relations in  $2^{X \times X}$ , so consider the set

$$Y = \{R \subseteq X \times X \mid R \text{ is a well-ordering}\}$$

So there is a set of ordinals:

$$\text{Ord}(Y) = \{\alpha \in \text{Ord} \mid \exists R \in Y \text{ } \alpha \text{ is the order type of } R\}$$

Note that  $\text{Ord}(Y)$  is a set and take the least element ordinal  $\beta$  does not belong to  $\text{Ord}(Y)$ . So  $h(X) = \beta$ . To be more precise, we have:

$$\beta = \sup \text{Ord}(Y)$$

Then  $|\alpha| < h(\alpha)$  for each ordinal  $\alpha$ .

2. Let  $\alpha = \sup X$ . Let  $f$  be a one-to-one function from  $\alpha$  onto some  $\beta < \alpha$ . Let  $\kappa$  be a cardinal such that  $\beta < \kappa \leq \alpha$ , then  $|\kappa| = |\{f(\xi) \mid \xi < \kappa\}| \leq \beta$ , so contradiction and  $\alpha$  is a cardinal.

□

The enumeration of all alephs is defined by transfinite induction:

- $\aleph_0 = \omega$
- $\aleph_{\alpha+1} = \aleph_\alpha^+ = \omega_{\alpha+1}$
- If  $\beta$  is a limit ordinal, then  $\aleph_\beta = \omega_\beta = \sup\{\omega_\alpha \mid \alpha < \beta\}$ .

A cardinal of the form  $\aleph_{\alpha+1}$  is a *successor* cardinal, a cardinal  $\aleph_\beta$  for limit  $\beta$  is a *limit cardinal*.

## 1.1 The ordering of $\alpha \times \alpha$

Define a well-ordering of the class  $\text{Ord} \times \text{Ord}$  the following way:

$$(\alpha, \beta) < (\gamma, \delta) \text{ iff either } \max(\alpha, \beta) < \max(\gamma, \delta) \text{ or} \\ \max(\alpha, \beta) = \max(\gamma, \delta) \text{ and } \alpha < \gamma \text{ or} \\ \max(\alpha, \beta) = \max(\gamma, \delta) \text{ and } \alpha = \gamma \text{ and } \beta < \delta.$$

Then  $<$  is a well-ordering and linear relation on  $\text{Ord}$ . Moreover,  $\alpha \times \alpha$  is the initial segment of  $(\text{Ord} \times \text{Ord}, <)$  given by  $(0, \alpha)$ .

We let:

$$\Gamma(\alpha, \beta) = \text{the order type of } \{(\xi, \eta) \mid (\xi, \eta) < (\alpha, \beta)\}$$

$\Gamma$  is also one-to-one:

$$(\alpha, \beta) < (\gamma, \delta) \text{ iff } \Gamma(\alpha, \beta) < \Gamma(\gamma, \delta)$$

$\Gamma$  is increasing and continuous and  $\Gamma(\alpha \times \alpha) = \alpha$  for arbitrarily large  $\alpha$ .

**Theorem 1.1.**  $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$

*Proof.* Let us show that  $\Gamma(\omega_\alpha \times \omega_\alpha) = \omega_\alpha$ .

1. If  $\alpha = 0$ , then  $\Gamma(\omega \times \omega) = \omega$ .
2. Let  $\alpha$  be the least ordinal such that  $\Gamma(\omega_\alpha \times \omega_\alpha) \neq \omega_\alpha$ . Let  $\beta, \gamma$  be ordinals such that  $\Gamma(\beta, \gamma) = \omega_\alpha$ . Take  $\delta < \omega_\alpha$  such that  $\delta > \beta, \gamma$ .  $\delta \times \delta$  is the initial segment of  $\text{Ord}^2$  and it contains  $(\beta, \gamma)$ . So  $\Gamma(\delta \times \delta) \supset \omega_\alpha = \Gamma(\beta, \gamma)$ . Thus  $|\delta \times \delta| \geq \aleph_\alpha$ . But  $|\delta \times \delta| = |\delta| \cdot |\delta| = |\delta|$ . But  $|\delta| < \aleph_\alpha$  by the assumption of minimality of  $\alpha$ . Contradiction.

□

As a corollary:

$$\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = \max(\aleph_\alpha, \aleph_\beta)$$

## 1.2 Cofinality

Let  $\alpha, \beta > 0$  be limit ordinals. An increasing  $\beta$ -sequence  $\langle \alpha_\xi : \xi < \beta \rangle$  is *cofinal* in  $\alpha$  if  $\lim_{\xi \rightarrow \beta} \alpha_\xi = \alpha$ . A subset  $X \subseteq \alpha$  is *cofinal* in  $\alpha$  whenever  $\sup X = \alpha$ .

Let  $\alpha > 0$  be a limit ordinal, the *cofinality* of  $\alpha$  is:

$\text{cf } \alpha =$  the least ordinal  $\beta$  such that  $\exists \langle \alpha_\xi : \xi < \beta \rangle$  such that  $\lim_{\xi \rightarrow \beta} \alpha_\xi = \alpha$

Note that for each  $\alpha$   $\text{cf } \alpha$  is a limit ordinal and  $\text{cf } \alpha \leq \alpha$ .

**Lemma 1.2.** For each  $\alpha$   $\text{cf}(\text{cf } \alpha) \leq \text{cf } \alpha$ .

*Proof.* Let  $\langle \alpha_\xi : \xi < \beta \rangle$  be cofinal in  $\alpha$  and let  $\langle \xi_\nu : \nu < \gamma \rangle$  be cofinal in  $\beta$ .

Consider  $\langle \alpha_{\xi_\nu} : \nu < \gamma \rangle$ , then

$$\lim_{\nu < \gamma} \alpha_{\xi_\nu} = \alpha$$

since the limit of a subsequence equals the limit of a sequence as in usual real analysis or topology.  $\square$

**Lemma 1.3.** Let  $\alpha$  be a non-zero limit ordinal, then

1. If  $A \subseteq \alpha$  and  $\sup A = \alpha$ , the order-type of  $A$  is at least  $\text{cf } \alpha$ .
2. Let  $\beta_0 \leq \beta_1 \leq \dots \leq \beta_\xi \leq \dots$  for  $\xi < \gamma$  be a non-decreasing sequence of ordinals such that  $\lim_{\xi \rightarrow \gamma} \beta_\xi = \alpha$ , then  $\text{cf } \gamma = \alpha$ .

*Proof.* 1. The order-type of  $A$  is the length of the increasing enumeration of  $A$ , the limit of which (as an increasing sequence) is  $\alpha$ .

2. If  $\gamma = \lim_{\nu \rightarrow \text{cf } \gamma} \xi_\nu$ , then  $\alpha = \lim_{\nu \rightarrow \text{cf } \gamma} \beta_{\xi_\nu}$ , and the non-decreasing sequence  $\langle \beta_{\xi_\nu} : \nu < \text{cf } \gamma \rangle$  has an increasing sequence of the length at most  $\text{cf } \gamma$  and it has the same limit, so  $\text{cf } \alpha \leq \text{cf } \gamma$ .

To show  $\text{cf } \gamma \leq \text{cf } \alpha$ , assume  $\alpha = \lim_{\nu \rightarrow \text{cf } \alpha} \alpha_\nu$ . Take  $\nu < \text{cf } \alpha$ , let  $\xi_\nu$  be the least  $\xi$  greater than all  $\xi_\iota$  for  $\iota < \nu$  such that  $\beta_{\xi_\iota} > \alpha_\nu$ . We have  $\alpha = \lim_{\nu \rightarrow \text{cf } \alpha} \beta_{\xi_\nu}$ , so  $\gamma = \lim_{\nu \rightarrow \text{cf } \alpha} \xi_\nu$ , so the inequation is proved.  $\square$

An infinite cardinal  $\aleph_\alpha$  is *regular* if  $\text{cf } \aleph_\alpha = \aleph_\alpha$ .  $\aleph_\alpha$  is *singular* if  $\text{cf } \aleph_\alpha < \aleph_\alpha$ .

## 2 The Baire Space

## 3 The Axiom of Choice

## 4 Cardinal Arithmetic via the Generalised Continuum Hypothesis