Some Notes on Set Theory, Pt 1

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1 Cardinals

An ordinal number α is a *cardinal number* if no $\beta < \alpha$ such that $|\alpha| = |\beta|$. Further, we shall use κ, λ, μ to denote cardinal numbers.

Let W be a well-ordered set, then there exists an ordinal α such that $|W| = |\alpha|$, so we let:

$$|W| = \text{the least ordinal } \alpha \text{ such that } |W| = |\alpha|$$

An *aleph* is an infinite cardinal number.

Let α be an ordinal, then α^+ is the least cardinal bigger than α.

Lemma 1.1.

1. For every α there is a cardinal number κ such that $\kappa > \alpha$.
2. Let X be a set of cardinal, then $\sup X$ is a cardinal.

Proof.

1. Let X be a set, let

$$h(X) = \text{the least } \alpha \text{ such that no injection from } \alpha \text{ into } X$$

Consider $X \times X$, so $2^{X \times X}$ is the set of relations on X and there are well-orderings of subsets of X amongst all relations in $2^{X \times X}$, so consider the set

$$Y = \{R \subseteq Y \times Y \mid Y \subseteq X \;\&\; Y \text{ is a well-ordering } \}$$

So there is a set of ordinals:

$$Ord(Y) = \{\alpha \in \mathrm{Ord} \mid \exists R \in Y \; \alpha \text{ is the order type of } Y\}$$

Note that $Ord(Y)$ is a set and take the least element ordinal β does not belong to $Ord(Y)$. So $h(X) = \beta$. To be more precise, we have:

$$\beta = \sup Ord(Y)$$

Then $|\alpha| < h(\alpha)$ for each ordinal α.

2. Let $\alpha = \sup X$. Let f be a one-to-one function from α onto some $\beta < \alpha$. Let κ be a cardinal such that $\beta < \kappa \leq \alpha$, then $|\kappa| = |\{f(\xi) \mid \xi < \kappa\}| \leq \beta$, so contradiction and α is a cardinal.

$\square$

The enumeration of all alephs is defined by transfinite induction:

- $\aleph_0 = \omega$
- $\aleph_{\alpha+1} = \aleph_\alpha^+ = \omega_{\alpha+1}$
- If β is a limit ordinal, then $\aleph_\beta = \omega_\beta = \sup\{\omega_\alpha \mid \alpha < \beta\}$.

A cardinal of the form $\aleph_{\alpha+1}$ is a *successor* cardinal, a cardinal $\aleph_\beta$ for limit β is a *limit cardinal*.

1.1 The ordering of $\alpha \times \alpha$

Define a well-ordering of the class $\mathrm{Ord} \times \mathrm{Ord}$ the following way:

$$\begin{gathered}(\alpha,\beta) < (\gamma,\delta) \text{ iff either } \max(\alpha,\beta) < \max(\gamma,\delta) \text{ or}\\ \max(\alpha,\beta) = \max(\gamma,\delta) \text{ and } \alpha < \gamma \text{ or}\\ \max(\alpha,\beta) = \max(\gamma,\delta) \text{ and } \alpha = \gamma \text{ and } \beta < \delta.\end{gathered}$$

Then $<$ is a well-ordering and linear relation on Ord. Moreover, $\alpha \times \alpha$ is the initial segment of $(\mathrm{Ord} \times \mathrm{Ord}, <)$ given by $(0, \alpha)$.

We let:

$$\Gamma(\alpha,\beta) = \text{the order type of } \{(\xi,\eta) \mid (\xi,\eta) < (\alpha,\beta)\}$$

Γ is also one-to-one:

$$(\alpha,\beta) < (\gamma,\delta) \text{ iff } \Gamma(\alpha,\beta) < \Gamma(\gamma,\delta)$$

Γ is increasing and continuous and $\Gamma(\alpha \times \alpha) = \alpha$ for arbitrarily large α.

Theorem 1.1. $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$

Proof. Let us show that $\Gamma(\omega_\alpha \times \omega_\alpha) = \omega_\alpha$.

1. If $\alpha = 0$, then $\Gamma(\omega \times \omega) = \omega$.
2. Let α be the least ordinal such that $\Gamma(\omega_\alpha \times \omega_\alpha) \neq \omega_\alpha$. Let β, γ be ordinals such that $\Gamma(\beta,\gamma) = \omega_\alpha$. Take $\delta < \omega_\alpha$ such that $\delta > \beta, \gamma$. $\delta \times \delta$ is the initial segment of Ord^2 and it contains (β,γ). So $\Gamma(\delta \times \delta) \supset \omega_\alpha = \Gamma(\beta,\gamma)$. Thus $|\delta \times \delta| \geq \aleph_\alpha$. But $|\delta \times \delta| = |\delta| \cdot |\delta| = |\delta|$. But $|\delta| < \aleph_\alpha$ by the assumption of minimality of α. Contradiction.

$\square$

As a corollary:

$$\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = \max(\aleph_\alpha, \aleph_\beta)$$

1.2 Cofinality

Let $\alpha, \beta > 0$ be limit ordinals. An increasing β-sequence $\langle \alpha_\xi : \xi < \beta \rangle$ is *cofinal* in α if $\lim_{\xi \to \beta} \alpha_\xi = \alpha$. A subset $X \subseteq \alpha$ is *cofinal* in α whenever $\sup X = \alpha$.

Let $\alpha > 0$ be a limit ordinal, the *cofinality* of α is:

$$\operatorname{cf} \alpha = \text{the least ordinal } \beta \text{ such that } \exists \ \langle \alpha_\xi : \xi < \beta \rangle \text{ such that } \lim_{\xi \to \beta} \alpha_\xi = \alpha$$

Note that for each α $\operatorname{cf} \alpha$ is a limit ordinal and $\operatorname{cf} \alpha \leq \alpha$.

Lemma 1.2. For each α $\operatorname{cf}(\operatorname{cf} \alpha) \leq \operatorname{cf} \alpha$.

Proof. Let $\langle \alpha_\xi : \xi < \beta \rangle$ be cofinal in α and let $\langle \xi_\nu : \nu < \gamma \rangle$ be cofinal in β.

Consider $\langle \alpha_{\xi_\nu} : \nu < \gamma \rangle$, then

$$\lim_{\nu < \gamma} \alpha_{\xi_\nu} = \alpha$$

since the limit of a subsequence equals the limit of a sequence as in usual real analysis or topology. □

Lemma 1.3. Let α be a non-zero limit ordinal, then

1. If $A \subseteq \alpha$ and $\sup A = \alpha$, the order-type of A is at least $\operatorname{cf} \alpha$.
2. Let $\beta_0 \leq \beta_1 \leq \cdots \leq \beta_\xi \leq \ldots$ for $\xi < \gamma$ be a non-decreasing sequence of ordinals such that $\lim_{\xi \to \gamma} = \alpha$, then $\operatorname{cf} \gamma = \alpha$.

Proof.

1. The order-type of A is the length of the increasing enumeration of A, the limit of which (as an increasing sequence) is α.
2. If $\gamma = \lim_{\nu \to \operatorname{cf} \gamma} \xi_\nu$, then $\alpha = \lim_{\nu \to \operatorname{cf} \gamma} \beta_{\xi_\nu}$, and the non-decreasing sequence $\langle \beta_{\xi_\nu} : \nu < \operatorname{cf} \gamma \rangle$ has an increasing sequence of the length at most $\operatorname{cf} \gamma$ and it has the same limit, so $\operatorname{cf} \alpha \leq \operatorname{cf} \gamma$.

   To show $\operatorname{cf} \gamma \leq \operatorname{cf} \alpha$, assume $\alpha = \lim_{\nu \to \operatorname{cf} \alpha} \alpha_\nu$. Take $\nu < \operatorname{cf} \alpha$, let ξ_ν be the least ξ greater than all ξ_ι for $\iota < \nu$ such that $\beta_\xi > \alpha_\nu$. We have $\alpha = \lim_{\nu \to \operatorname{cf} \alpha} \beta_{\xi_\nu}$, so $\gamma = \lim_{\nu \to \operatorname{cf} \alpha} \xi_\nu$, so the inequation is proved.

□

An infinite cardinal $\aleph_\alpha$ is *regular* if $\operatorname{cf} \omega_\alpha = \omega_\alpha$. $\aleph_\alpha$ is *singular* if $\operatorname{cf} \omega_\alpha < \omega_\alpha$.

Lemma 1.4. Let α be a limit ordinal, then $\operatorname{cf} \alpha$ is a regular cardinal.

Proof. If α is not a cardinal, then there exists an ordinal $\beta < \alpha$ such that $|\beta| = |\alpha|$, then we construct a cofinal sequence in α of length $|\beta|$, then $\operatorname{cf} \alpha = |\beta|$ and $\operatorname{cf} \alpha < \alpha$. □

Let κ be a limit ordinal, a subset $X \subset \kappa$ is *bounded* if $\sup X < \kappa$ and *unbounded* if $\sup X = \kappa$.

Lemma 1.5. Let κ be an aleph, then:

1. If $X \subset \kappa$ and $|X| < \operatorname{cf} \kappa$, then X is bounded.

2. If λ ¡ $\operatorname{cf} \kappa$ and $f : \lambda \to \kappa$, then $\operatorname{Im} f$ is bounded in κ.

Proof. 1. Let X be such subset of κ and assume X is unbounded, so $\sup X = \kappa$. By 1 of Lemma 1.3, the order-type of X is at least $\operatorname{cf} \kappa$, which contradicts to $|X| < \operatorname{cf} \kappa$, so X is bounded.

2. Follows from the first item.

□

Lemma 1.6. (Hausdorff)
Let κ be a cardinal, then the following are equivalent:

1. κ is singular.

2. There is a cardinal $\lambda < \kappa$ and a family $\{S_\xi \,|\, \xi < \lambda\}$ such that each $S_\xi \subset \kappa$, $|S_\xi| < \kappa$ and $\kappa = \bigcup\limits_{\xi < \lambda} S_\xi$.

Proof.

1. (1) $\Rightarrow$ (2).

   If κ is singular, then there is an increasing sequence $\langle \alpha_\xi \,:\, \xi < \operatorname{cf} \kappa \rangle$, so a family of required subsets is actually a family of those α_ξ's and $\lambda = \operatorname{cf} \kappa$ which is strictly less than κ since κ is singular.

2. (2) $\Rightarrow$ (1).

   Let λ be the least cardinal such that $\lambda < \kappa$ and there exists a family $\{S_\xi \mid \xi < \lambda\}$ where each $S_\xi \subset \kappa$, $|S_\xi| < \kappa$ and

   $$\kappa = \bigcup_{\xi < \lambda} S_\xi$$

   For each $\xi < \lambda$, let β_ξ be the order-type of $\cup_{\nu < \xi} S_\nu$. The sequence $\langle \beta_\xi : \xi < \lambda \rangle$ is non-decreasing and each $\beta_\xi < \kappa$ for all $\xi < \lambda$ since λ is minimal.

   Let us show that $\lim\limits_{\xi \to \kappa} \beta_\xi = \kappa$ to show that $\operatorname{cf} \kappa \leq \lambda$.

   Assume $\beta = \lim\limits_{\xi \to \kappa} \beta_\xi$. There is a one-to-one mapping $f : \bigcup\limits_{\xi < \beta} S_\xi \to \lambda \times \beta$ such that:

   $$f : \alpha \mapsto (\xi, \gamma)$$

   where ξ is the least ordinal such that $\alpha \in S_\xi$ and γ is the order-type of $S_\xi \cap \gamma$.

   We have $\lambda < \kappa$ and $|\lambda \times \beta| = \lambda \cdot |\beta|$, then $\kappa = \beta$.

□

Theorem 1.2. Let κ be an infinite cardinal, then $\kappa < \kappa^{\operatorname{cf}\kappa}$.

Proof. Let F be a collection of κ functions from $\operatorname{cf}\kappa$ to κ:

$$F = \{f_\alpha : \operatorname{cf}\kappa \to \kappa \mid \alpha < \kappa\}$$

Let us construct f that does not belong to F.
We have $\kappa = \lim_{\xi < \operatorname{cf}\kappa} \alpha_\xi$, for $\xi < \operatorname{cf}\kappa$ we let:

$$f(\xi) = \text{least } \gamma \text{ such that } \gamma \neq \forall\alpha < \alpha_\xi \; f_\alpha \neq \gamma$$

Such γ does exist and f is different from all the f_α. □

An uncountable cardinal κ is *weakly inaccessible* if it is limit and regular, but we cannot prove the existence of weakly inaccessible cardinals in ZFC.

2 Real Numbers and The Baire Space

The *continuum* is the cardinality of $\mathbb{R}$ denoted as $\mathfrak{c}$.

Theorem 2.1. (Cantor)
$\aleph_0 < \mathfrak{c}$.

Proof. One can think of it as a consequence of Theorem 1.2. □

Definition 2.1. The *Continuum Hypothesis* (CH) is the following equation:

$$\aleph_1 = \mathfrak{c}.$$

Let $(P, <)$ be an ordered set, a subset $D \subset P$ is a *dense* subset of P if $a < b$ in P implies $a < d$ and $d < b$ for some $d \in D$.

Theorem 2.2. $(\mathbb{R}, <)$ is the unique complete linear ordering that has a countable dense subset isomorphic to $(\mathbb{Q}, <)$.

Proof. Let C and C' be two complete dense linear orderings and let P and P' be dense in C and C' respectively. Let $f : P \cong P'$, so f can be extended to $f^* : C \cong C'$ by letting:

$$f^* : x \mapsto \sup\{f(p) \mid p \in P \;\&\; p \leq x\}$$

That is, $(.)^*$ is functorial. □

The existence of $(\mathbb{R}, <)$ follows from the following general statement:

Theorem 2.3. Let $(P, <)$ be a dense unbounded linear ordering, then there exists a complete dense unbounded linear ordering $(C, \prec)$ such that:

1. $(P, <)$ embeds to $(C, \prec)$.
2. P is dense in C.

Proof. Recall that a *Dedekind cut* in P is a pair (A, B) of disjoint subsets of P such that:

1. $A \cup B = P$.

2. $\forall a \in A \, \forall b \in B \; a < b$.

3. A has no greatest element.

Let C be the set of all Dedekind cuts in P. We let $(A_1, B_1) \preceq (A_2, B_2)$ if $A_1 \subset A_2$ and $B_2 \subset B_1$. $(C, \preceq)$ is complete.

Let $\{C_i \mid i \in I\} \neq \emptyset$ be a bounded subset of C, then $(\bigcup_i A_i, \bigcap_i B_i)$ is its supremum.

Let $p \in P$, let

$$A_p = \{x \in P \mid x < p\}$$
$$B_p = \{x \in P \mid x \geq p\}$$

Then $(\{(A_p, B_p) \mid p \in P\}, \preceq) \cong (P, <)$ and is dense in C. $\square$

$\mathbb{Q}$ is dense in $\mathbb{R}$, so every open interval (a, b) contains some rational number. Then if S is a disjoint collection of open intervals, then S is at most countable.

Let P be a dense linearly ordered set, if every disjoint collection of open intervals is at most countable, then we say that P satisfies the *countable chain condition.*

(Suslin's Problem) *Let P be a dense linearly ordered set satisfying the countable chain condition. Is P isomorphic to $(\mathbb{R}, <)$?*

Note that neither Suslin's Problem nor its negation can be decided in ZFC.

2.1 Topology of $\mathbb{R}$

The real line is equipped with the natural topology induced by the metric $d(a, b) = |b - a|$ coincides with the order topology on $(\mathbb{R}, <)$. $\mathbb{R}$ is also a complete separable metric space.

Every open set in $\mathbb{R}$ is the union of intervals with rational endpoints, so there are continuum many open sets (and the same observation holds for open sets as well).

A subset P is *perfect* is it has no isolated points.

Theorem 2.4. Every perfect set P has cardinality $\mathfrak{c}$.

Proof. We construct a one-to-one function F from $\{0, 1\}^\omega$ to P. Let S be the set of all finite binary sequences and let $s \in S$.

By induction on $len(s)$ one can find closed intervals I_s such that for each $n < \omega$ and for each $s \in S$ such that $len(s) = n$:

1. $I_s \cap P$ is perfect,

2. the diameter of I_s is $\leq 1/2$,

3. $I_{0:s}, I_{1:s} \subset I_s$ and $I_{0:s} \cap I_{1:s} = \emptyset$

Take $f \in \{0,1\}^\omega$, the set $P \cap \bigcap_{n<\omega} I_{f\restriction n}$ has exactly one element, so let:

$$F : f \mapsto \bigcap_{n<\omega} I_{f\restriction n}$$

□

Theorem 2.5. (**Cantor-Bendixon**)
If F is an uncountable closed set, then $F = P \cup S$, where P is perfect and S is at most countable.

Proof.
Let $F \subset \mathbb{R}$, let

$$F^{'} = \text{the set of all limit points of } F$$

$F^{'}$ is also called the *derived set* of F. F' is closed and obviously a subset of A.
We let:

1. $F_0 = A$.
2. $F_{\alpha+1} = F^{'}_{\alpha}$.
3. $F_\alpha = \bigcap_{\gamma<\alpha} F_\gamma$ if $\alpha > 0$ is a limit ordinal.

Since $F_0 \supset F_1 \supset \cdots \supset F_\alpha \supset$, so we have an ordinal θ such that $F_\theta = F_{\theta+1}$ (otherwise we could map the proper class of ordinals onto some set). We let $P = F_\alpha$. If P is nonempty, then P is also perfect.

Let us show that $F - P$ is at most countable. Let $\langle J_k : k < \omega \rangle$ be an enumeration of rational intervals. We have

$$F - P = \bigcup_{\alpha<\theta} (F_\alpha - F_{\alpha+1})$$

So if $a \in F - P$, then there exists $\alpha < \theta$ such that $a \in F_\alpha - F_{\alpha+1}$, that is, a is an isolated point of F_α. We let k_a be the least k such that a is the only point of F_α in J_k.

If $\alpha \leq \beta$ and $a \neq b$ and b is isolated in F_β, then $b \notin J_{k_a}$, so $k_a \neq k_b$, so the mapping $a \mapsto k_a$ is one-to-one.

□

Corollary 2.1. If $C \subseteq \mathbb{R}$ is closed, then either $|C| = 2^{\aleph_0}$ or $|C| \leq \aleph_0$.

A set $A \subset \mathbb{R}$ is *nowhere dense* if $\operatorname{Int}\operatorname{Cl} A = \emptyset$. The following theorem shows that $\mathbb{R}$ is not of the *first category*, that is, $\mathbb{R}$ is not the union of a countable family of nowhere dense sets.

Theorem 2.6. (**The Baire Category Theorem**)
Let $\{D_i \mid i < \omega\}$ be a countable family of dense open subsets of $\mathbb{R}$, then $D = \bigcap_{i<\omega} D_i$ is dense in $\mathbb{R}$.

Proof. We show that $D \cap I \neq \emptyset$ for each open interval I.

Note that each finite intersection $D_0 \cap D_1 \cap \cdots \cap D_n$ is dense and open for each $n < \omega$. Let $\langle J_k : k < \omega \rangle$ be an enumeration of rational intervals.

Let $I_0 := I$ and for each n $I_{n+1} = J_k = (q_k, r_k)$ where k is the smallest index such that $[q_k, r_k] \subset I_n \cap D_n$.

Take $a = \lim_{k\to\infty} q_k$, then $a \in I \cap D$. □

2.2 The Baire Space

The *Baire Space* is the space $\mathcal{N} = \omega^\omega$ of infinite sequences of natural numbers with the topology defined the following way. Let s be a finite sequence $s = \langle a_k : k < n \rangle$, we let:

$$O(s) = \{f \in \mathcal{N} \mid s \subset f\} = \{\langle c_k \mid k < \omega \rangle \mid \forall k < n\, c_k = a_k\}$$

All those $O(s)$'s form the open basis for $\mathcal{N}$.

The Baire space is separable and metrisable. The metric is defined as $d(f, g) = 1/2^{n+1}$ where n is the smallest natural number such that $f(n) \neq g(n)$. We also have separability since the set of all eventually constant sequences is dense in $\mathcal{N}$.

Every infinite sequence $\langle a_k : k < \omega \rangle$ defines a continued fraction $1/(a_0 + 1/(a_1 + 1/(a_2 + \dots)))$, so we have a continuous bijection between infinite sequences and irrational points of the open interval $(0, 1)$. Moreover, the Baire space is homeomorphic to the space of irrational numbers.

Now we describe the characterisation of perfect sets in the Baire space.

Let Seq be the set of all finite sequences in $\mathcal{N}$. A *tree* is a set $T \subset \text{Seq}$ satisfying:

If $t \in T$ and there exists $n < \omega$ such that $s = t \restriction n$, then $s \in T$.

Let T be a tree, let $[T]$ be the set of all infinite paths through T:

$$[T] = \{f \in \mathbb{N} \mid \forall n < \omega\, f \restriction n \in T\}$$

For each T, the set $[T]$ is closed in the Baire space. Let $f \in \mathcal{N}$ such that $f \notin [T]$. Then there exists $n < \omega$ such that $s = f \restriction n \notin T$, so the open neighbourhood of f $O(s) = \{g \in \mathcal{N} \mid g \supset s\}$. Thus $[T]$ is closed.

Conversely, let F be closed in $\mathcal{N}$, then the set

$$T_F = \{s \in \text{Seq} \mid \exists f \in F\, s \subset f\}$$

is a tree and one can verify that $[T_F] = F$. If $f \in \mathcal{N}$ such that $f \restriction n \in T$ for each $n < \omega$, then for each n there is some $g \in F$ such that $g \restriction n = f \restriction n$, so $f \in F$ since F is closed.

If f is an isolated point of a closed set F in $\mathcal{N}$, then there is $n \in \mathbb{N}$ such that no $g \in F$ such that $g \neq f$ and $g \restriction n = f \restriction n$.

3 The Axiom of Choice

4 Cardinal Arithmetic via the Generalised Continuum Hypothesis