

# Some Notes on Set Theory, Pt 1

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# 1 Cardinals

An ordinal number  $\alpha$  is a *cardinal number* if no  $\beta < \alpha$  such that  $|\alpha| = |\beta|$ . Further, we shall use  $\kappa, \lambda, \mu$  to denote cardinal numbers.

Let  $W$  be a well-ordered set, then there exists an ordinal  $\alpha$  such that  $|W| = |\alpha|$ , so we let:

$$|W| = \text{the least ordinal } \alpha \text{ such that } |W| = |\alpha|$$

An *aleph* is an infinite cardinal number.

Let  $\alpha$  be an ordinal, then  $\alpha^+$  is the least cardinal bigger than  $\alpha$ .

**Lemma 1.1.**

1. For every  $\alpha$  there is a cardinal number  $\kappa$  such that  $\kappa > \alpha$ .
2. Let  $X$  be a set of cardinal, then  $\sup X$  is a cardinal.

*Proof.*

1. Let  $X$  be a set, let

$$h(X) = \text{the least } \alpha \text{ such that no injection from } \alpha \text{ into } X$$

Consider  $X \times X$ , so  $2^{X \times X}$  is the set of relations on  $X$  and there are well-orderings of subsets of  $X$  amongst all relations in  $2^{X \times X}$ , so consider the set

$$Y = \{R \subseteq X \times X \mid R \text{ is a well-ordering}\}$$

So there is a set of ordinals:

$$\text{Ord}(Y) = \{\alpha \in \text{Ord} \mid \exists R \in Y \text{ } \alpha \text{ is the order type of } R\}$$

Note that  $\text{Ord}(Y)$  is a set and take the least element ordinal  $\beta$  does not belong to  $\text{Ord}(Y)$ . So  $h(X) = \beta$ . To be more precise, we have:

$$\beta = \sup \text{Ord}(Y)$$

Then  $|\alpha| < h(\alpha)$  for each ordinal  $\alpha$ .

2. Let  $\alpha = \sup X$ . Let  $f$  be a one-to-one function from  $\alpha$  onto some  $\beta < \alpha$ . Let  $\kappa$  be a cardinal such that  $\beta < \kappa \leq \alpha$ , then  $|\kappa| = |\{f(\xi) \mid \xi < \kappa\}| \leq \beta$ , so contradiction and  $\alpha$  is a cardinal.

□

The enumeration of all alephs is defined by transfinite induction:

- $\aleph_0 = \omega$
- $\aleph_{\alpha+1} = \aleph_\alpha^+ = \omega_{\alpha+1}$
- If  $\beta$  is a limit ordinal, then  $\aleph_\beta = \omega_\beta = \sup\{\omega_\alpha \mid \alpha < \beta\}$ .

A cardinal of the form  $\aleph_{\alpha+1}$  is a *successor* cardinal, a cardinal  $\aleph_\beta$  for limit  $\beta$  is a *limit cardinal*.

## 1.1 The ordering of $\alpha \times \alpha$

Define a well-ordering of the class  $\text{Ord} \times \text{Ord}$  the following way:

$$\begin{aligned} (\alpha, \beta) < (\gamma, \delta) \text{ iff either } \max(\alpha, \beta) < \max(\gamma, \delta) \text{ or} \\ \max(\alpha, \beta) = \max(\gamma, \delta) \text{ and } \alpha < \gamma \text{ or} \\ \max(\alpha, \beta) = \max(\gamma, \delta) \text{ and } \alpha = \gamma \text{ and } \beta < \delta. \end{aligned}$$

Then  $<$  is a well-ordering and linear relation on  $\text{Ord}$ . Moreover,  $\alpha \times \alpha$  is the initial segment of  $(\text{Ord} \times \text{Ord}, <)$  given by  $(0, \alpha)$ .

We let:

$$\Gamma(\alpha, \beta) = \text{the order type of } \{(\xi, \eta) \mid (\xi, \eta) < (\alpha, \beta)\}$$

$\Gamma$  is also one-to-one:

$$(\alpha, \beta) < (\gamma, \delta) \text{ iff } \Gamma(\alpha, \beta) < \Gamma(\gamma, \delta)$$

$\Gamma$  is increasing and continuous and  $\Gamma(\alpha \times \alpha) = \alpha$  for arbitrarily large  $\alpha$ .

**Theorem 1.1.**  $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$

*Proof.* Let us show that  $\Gamma(\omega_\alpha \times \omega_\alpha) = \omega_\alpha$ .

1. If  $\alpha = 0$ , then  $\Gamma(\omega \times \omega) = \omega$ .
2. Let  $\alpha$  be the least ordinal such that  $\Gamma(\omega_\alpha \times \omega_\alpha) \neq \omega_\alpha$ . Let  $\beta, \gamma$  be ordinals such that  $\Gamma(\beta, \gamma) = \omega_\alpha$ . Take  $\delta < \omega_\alpha$  such that  $\delta > \beta, \gamma$ .  $\delta \times \delta$  is the initial segment of  $\text{Ord}^2$  and it contains  $(\beta, \gamma)$ . So  $\Gamma(\delta \times \delta) \supset \omega_\alpha = \Gamma(\beta, \gamma)$ . Thus  $|\delta \times \delta| \geq \aleph_\alpha$ . But  $|\delta \times \delta| = |\delta| \cdot |\delta| = |\delta|$ . But  $|\delta| < \aleph_\alpha$  by the assumption of minimality of  $\alpha$ . Contradiction.

□

As a corollary:

$$\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = \max(\aleph_\alpha, \aleph_\beta)$$

## 1.2 Cofinality

Let  $\alpha, \beta > 0$  be limit ordinals. An increasing  $\beta$ -sequence  $\langle \alpha_\xi : \xi < \beta \rangle$  is *cofinal* in  $\alpha$  if  $\lim_{\xi \rightarrow \beta} \alpha_\xi = \alpha$ . A subset  $X \subseteq \alpha$  is *cofinal* in  $\alpha$  whenever  $\sup X = \alpha$ .

Let  $\alpha > 0$  be a limit ordinal, the *cofinality* of  $\alpha$  is:

$$\text{cf } \alpha = \text{the least ordinal } \beta \text{ such that } \exists \langle \alpha_\xi : \xi < \beta \rangle \text{ such that } \lim_{\xi \rightarrow \beta} \alpha_\xi = \alpha$$

Note that for each  $\alpha$   $\text{cf } \alpha$  is a limit ordinal and  $\text{cf } \alpha \leq \alpha$ .

**Lemma 1.2.** For each  $\alpha$   $\text{cf}(\text{cf } \alpha) \leq \text{cf } \alpha$ .

*Proof.* Let  $\langle \alpha_\xi : \xi < \beta \rangle$  be cofinal in  $\alpha$  and let  $\langle \xi_\nu : \nu < \gamma \rangle$  be cofinal in  $\beta$ .

Consider  $\langle \alpha_{\xi_\nu} : \nu < \gamma \rangle$ , then

$$\lim_{\nu < \gamma} \alpha_{\xi_\nu} = \alpha$$

since the limit of a subsequence equals the limit of a sequence as in usual real analysis or topology.  $\square$

**Lemma 1.3.** Let  $\alpha$  be a non-zero limit ordinal, then

1. If  $A \subseteq \alpha$  and  $\sup A = \alpha$ , the order-type of  $A$  is at least  $\text{cf } \alpha$ .
2. Let  $\beta_0 \leq \beta_1 \leq \dots \leq \beta_\xi \leq \dots$  for  $\xi < \gamma$  be a non-decreasing sequence of ordinals such that  $\lim_{\xi \rightarrow \gamma} \beta_\xi = \alpha$ , then  $\text{cf } \gamma = \alpha$ .

*Proof.* 1. The order-type of  $A$  is the length of the increasing enumeration of  $A$ , the limit of which (as an increasing sequence) is  $\alpha$ .

2. If  $\gamma = \lim_{\nu \rightarrow \text{cf } \gamma} \xi_\nu$ , then  $\alpha = \lim_{\nu \rightarrow \text{cf } \gamma} \beta_{\xi_\nu}$ , and the non-decreasing sequence  $\langle \beta_{\xi_\nu} : \nu < \text{cf } \gamma \rangle$  has an increasing sequence of the length at most  $\text{cf } \gamma$  and it has the same limit, so  $\text{cf } \alpha \leq \text{cf } \gamma$ .

To show  $\text{cf } \gamma \leq \text{cf } \alpha$ , assume  $\alpha = \lim_{\nu \rightarrow \text{cf } \alpha} \alpha_\nu$ . Take  $\nu < \text{cf } \alpha$ , let  $\xi_\nu$  be the least  $\xi$  greater than all  $\xi_\iota$  for  $\iota < \nu$  such that  $\beta_\xi > \alpha_\nu$ . We have  $\alpha = \lim_{\nu \rightarrow \text{cf } \alpha} \beta_{\xi_\nu}$ , so  $\gamma = \lim_{\nu \rightarrow \text{cf } \alpha} \xi_\nu$ , so the inequation is proved.  $\square$

An infinite cardinal  $\aleph_\alpha$  is *regular* if  $\text{cf } \omega_\alpha = \omega_\alpha$ .  $\aleph_\alpha$  is *singular* if  $\text{cf } \omega_\alpha < \omega_\alpha$ .

**Lemma 1.4.** Let  $\alpha$  be a limit ordinal, then  $\text{cf } \alpha$  is a regular cardinal.

*Proof.* If  $\alpha$  is not a cardinal, then there exists an ordinal  $\beta < \alpha$  such that  $|\beta| = |\alpha|$ , then we construct a cofinal sequence in  $\alpha$  of length  $|\beta|$ , then  $\text{cf } \alpha = |\beta|$  and  $\text{cf } \alpha < \alpha$ .  $\square$

Let  $\kappa$  be a limit ordinal, a subset  $X \subset \kappa$  is *bounded* if  $\sup X < \kappa$  and *unbounded* if  $\sup X = \kappa$ .

**Lemma 1.5.** Let  $\kappa$  be an aleph, then:

1. If  $X \subset \kappa$  and  $|X| < \text{cf } \kappa$ , then  $X$  is bounded.
2. If  $\lambda \vdash \text{cf } \kappa$  and  $f : \lambda \rightarrow \kappa$ , then  $\text{Im } f$  is bounded in  $\kappa$ .

*Proof.* 1. Let  $X$  be such subset of  $\kappa$  and assume  $X$  is unbounded, so  $\sup X = \kappa$ . By 1 of Lemma 1.3, the order-type of  $X$  is at least  $\text{cf } \kappa$ , which contradicts to  $|X| < \text{cf } \kappa$ , so  $X$  is bounded.

2. Follows from the first item.  $\square$

**Lemma 1.6. (Hausdorff)**

Let  $\kappa$  be a cardinal, then the following are equivalent:

1.  $\kappa$  is singular.

2. There is a cardinal  $\lambda < \kappa$  and a family  $\{S_\xi \mid \xi < \lambda\}$  such that each  $S_\xi \subset \kappa$ ,  $|S_\xi| < \kappa$  and  $\kappa = \bigcup_{\xi < \lambda} S_\xi$ .

*Proof.*

1. (1)  $\Rightarrow$  (2).

If  $\kappa$  is singular, then there is an increasing sequence  $\langle \alpha_\xi : \xi < \text{cf } \kappa \rangle$ , so a family of required subsets is actually a family of those  $\alpha_\xi$ 's and  $\lambda = \text{cf } \kappa$  which is strictly less than  $\kappa$  since  $\kappa$  is singular.

2. (2)  $\Rightarrow$  (1).

Let  $\lambda$  be the least cardinal such that  $\lambda < \kappa$  and there exists a family  $\{S_\xi \mid \xi < \lambda\}$  where each  $S_\xi \subset \kappa$ ,  $|S_\xi| < \kappa$  and

$$\kappa = \bigcup_{\xi < \lambda} S_\xi$$

For each  $\xi < \lambda$ , let  $\beta_\xi$  be the order-type of  $\bigcup_{\nu < \xi} S_\nu$ . The sequence  $\langle \beta_\xi : \xi < \lambda \rangle$  is non-decreasing and each  $\beta_\xi < \kappa$  for all  $\xi < \lambda$  since  $\lambda$  is minimal.

Let us show that  $\lim_{\xi \rightarrow \kappa} \beta_\xi = \kappa$  to show that  $\text{cf } \kappa \leq \lambda$ .

Assume  $\beta = \lim_{\xi \rightarrow \kappa} \beta_\xi$ . There is a one-to-one mapping  $f : \bigcup_{\xi < \beta} S_\xi \rightarrow \lambda \times \beta$  such that:

$$f : \alpha \mapsto (\xi, \gamma)$$

where  $\xi$  is the least ordinal such that  $\alpha \in S_\xi$  and  $\gamma$  is the order-type of  $S_\xi \cap \gamma$ .

We have  $\lambda < \kappa$  and  $|\lambda \times \beta| = \lambda \cdot |\beta|$ , then  $\kappa = \beta$ .

□

**Theorem 1.2.** Let  $\kappa$  be an infinite cardinal, then  $\kappa < \kappa^{\text{cf } \kappa}$ .

*Proof.* Let  $F$  be a collection of  $\kappa$  functions from  $\text{cf } \kappa$  to  $\kappa$ :

$$F = \{f_\alpha : \text{cf } \kappa \rightarrow \kappa \mid \alpha < \kappa\}$$

Let us construct  $f$  that does not belong to  $F$ .

We have  $\kappa = \lim_{\xi < \text{cf } \kappa} \alpha_\xi$ , for  $\xi < \text{cf } \kappa$  we let:

$$f(\xi) = \text{least } \gamma \text{ such that } \gamma \neq \forall \alpha < \alpha_\xi f_\alpha \neq \gamma$$

Such  $\gamma$  does exist and  $f$  is different from all the  $f_\alpha$ .

□

An uncountable cardinal  $\kappa$  is *weakly inaccessible* if it is limit and regular, but we cannot prove the existence of weakly inaccessible cardinals in ZFC.

## 2 Real Numbers and The Baire Space

The *continuum* is the cardinality of  $\mathbb{R}$  denoted as  $\mathfrak{c}$ .

**Theorem 2.1. (Cantor)**

$$\aleph_0 < \mathfrak{c}.$$

*Proof.* One can think of it as a consequence of Theorem 1.2.  $\square$

**Definition 2.1.** The *Continuum Hypothesis* (CH) is the following statement:

$$\aleph_1 = \mathfrak{c}.$$

Let  $(P, <)$  be an ordered set, a subset  $D \subset P$  is a *dense* subset of  $P$  if  $a < b$  in  $P$  implies  $a < d$  and  $d < b$  for some  $d \in D$ .

**Theorem 2.2.**  $(\mathbb{R}, <)$  is the unique complete linear ordering that has a countable dense subset isomorphic to  $(\mathbb{Q}, <)$ .

*Proof.* Let  $C$  and  $C'$  be two complete dense linear orderings and let  $P$  and  $P'$  be dense in  $C$  and  $C'$  respectively. Let  $f : P \cong P'$ , so  $f$  can be extended to  $f^* : C \cong C'$  by letting:

$$f^* : x \mapsto \sup\{f(p) \mid p \in P \text{ \& } p \leq x\}$$

That is,  $(.)^*$  is functorial.  $\square$

The existence of  $(\mathbb{R}, <)$  follows from the following general statement:

**Theorem 2.3.** Let  $(P, <)$  be a dense unbounded linear ordering, then there exists a complete dense unbounded linear ordering  $(C, \prec)$  such that:

1.  $(P, <)$  embeds to  $(C, \prec)$ .
2.  $P$  is dense in  $C$ .

*Proof.* Recall that a *Dedekind cut* in  $P$  is a pair  $(A, B)$  of disjoint subsets of  $P$  such that:

1.  $A \cup B = P$ .
2.  $\forall a \in A \forall b \in B \ a < b$ .
3.  $A$  has no greatest element.

Let  $C$  be the set of all Dedekind cuts in  $P$ . We let  $(A_1, B_1) \preceq (A_2, B_2)$  if  $A_1 \subset A_2$  and  $B_2 \subset B_1$ .  $(C, \preceq)$  is complete.

Let  $\{C_i \mid i \in I\} \neq \emptyset$  be a bounded subset of  $C$ , then  $(\bigcup_i A_i, \bigcap_i B_i)$  is its supremum.

Let  $p \in P$ , let

$$\begin{aligned} A_p &= \{x \in P \mid x < p\} \\ B_p &= \{x \in P \mid x \geq p\} \end{aligned}$$

Then  $(\{(A_p, B_p) \mid p \in P\}, \preceq) \cong (P, <)$  and is dense in  $C$ .  $\square$

$\mathbb{Q}$  is dense in  $\mathbb{R}$ , so every open interval  $(a, b)$  contains some rational number. Then if  $S$  is a disjoint collection of open intervals, then  $S$  is at most countable.

Let  $P$  be a dense linearly ordered set, if every disjoint collection of open intervals is at most countable, then we say that  $P$  satisfies the *countable chain condition*.

**(Suslin's Problem)** *Let  $P$  be a dense linearly ordered set satisfying the countable chain condition. Is  $P$  isomorphic to  $(\mathbb{R}, <)$ ?*

Note that neither Suslin's Problem nor its negation can be decided in ZFC.

## 2.1 Topology of $\mathbb{R}$

The real line is equipped with the natural topology induced by the metric  $d(a, b) = |b - a|$  coincides with the order topology on  $(\mathbb{R}, <)$ .  $\mathbb{R}$  is also a complete separable metric space.

Every open set in  $\mathbb{R}$  is the union of intervals with rational endpoints, so there are continuum many open sets (and the same observation holds for open sets as well).

A subset  $P$  is *perfect* if it has no isolated points.

**Theorem 2.4.** Every perfect set  $P$  has cardinality  $\mathfrak{c}$ .

*Proof.* We construct a one-to-one function  $F$  from  $\{0, 1\}^\omega$  to  $P$ . Let  $S$  be the set of all finite binary sequences and let  $s \in S$ .

By induction on  $\text{len}(s)$  one can find closed intervals  $I_s$  such that for each  $n < \omega$  and for each  $s \in S$  such that  $\text{len}(s) = n$ :

1.  $I_s \cap P$  is perfect,
2. the diameter of  $I_s$  is  $\leq 1/2$ ,
3.  $I_{0:s}, I_{1:s} \subset I_s$  and  $I_{0:s} \cap I_{1:s} = \emptyset$

Take  $f \in \{0, 1\}^\omega$ , the set  $P \cap \bigcap_{n < \omega} I_{f \upharpoonright n}$  has exactly one element, so let:

$$F : f \mapsto \bigcap_{n < \omega} I_{f \upharpoonright n}$$

$\square$

**Theorem 2.5. (Cantor-Bendixon)**

If  $F$  is an uncountable closed set, then  $F = P \cup S$ , where  $P$  is perfect and  $S$  is at most countable.

*Proof.*

Let  $F \subset \mathbb{R}$ , let

$F'$  = the set of all limit points of  $F$

$F'$  is also called the *derived set* of  $F$ .  $F'$  is closed and obviously a subset of  $A$ .  
We let:

1.  $F_0 = A$ .
2.  $F_{\alpha+1} = F'_\alpha$ .
3.  $F_\alpha = \bigcap_{\gamma < \alpha} F_\gamma$  if  $\alpha > 0$  is a limit ordinal.

Since  $F_0 \supset F_1 \supset \cdots \supset F_\alpha \supset$ , so we have an ordinal  $\theta$  such that  $F_\theta = F_{\theta+1}$  (otherwise we could map the proper class of ordinals onto some set). We let  $P = F_\alpha$ . If  $P$  is nonempty, then  $P$  is also perfect.

Let us show that  $F - P$  is at most countable. Let  $\langle J_k : k < \omega \rangle$  be an enumeration of rational intervals. We have

$$F - P = \bigcup_{\alpha < \theta} (F_\alpha - F_{\alpha+1})$$

So if  $a \in F - P$ , then there exists  $\alpha < \theta$  such that  $a \in F_\alpha - F_{\alpha+1}$ , that is,  $a$  is an isolated point of  $F_\alpha$ . We let  $k_a$  be the least  $k$  such that  $a$  is the only point of  $F_\alpha$  in  $J_k$ .

If  $\alpha \leq \beta$  and  $a \neq b$  and  $b$  is isolated in  $F_\beta$ , then  $b \notin J_{k_a}$ , so  $k_a \neq k_b$ , so the mapping  $a \mapsto k_a$  is one-to-one.

□

**Corollary 2.1.** If  $C \subseteq \mathbb{R}$  is closed, then either  $|C| = 2^{\aleph_0}$  or  $|C| \leq \aleph_0$ .

A set  $A \subset \mathbb{R}$  is *nowhere dense* if  $\text{Int Cl } A = \emptyset$ . The following theorem shows that  $\mathbb{R}$  is not of the *first category*, that is,  $\mathbb{R}$  is not the union of a countable family of nowhere dense sets.

**Theorem 2.6. (The Baire Category Theorem)**

Let  $\{D_i \mid i < \omega\}$  be a countable family of dense open subsets of  $\mathbb{R}$ , then  $D = \bigcap_{i < \omega} D_i$  is dense in  $\mathbb{R}$ .

*Proof.* We show that  $D \cap I \neq \emptyset$  for each open interval  $I$ .

Note that each finite intersection  $D_0 \cap D_1 \cap \cdots \cap D_n$  is dense and open for each  $n < \omega$ . Let  $\langle J_k : k < \omega \rangle$  be an enumeration of rational intervals.

Let  $I_0 := I$  and for each  $n$   $I_{n+1} = J_k = (q_k, r_k)$  where  $k$  is the smallest index such that  $[q_k, r_k] \subset I_n \cap D_n$ .

Take  $a = \lim_{k \rightarrow \infty} q_k$ , then  $a \in I \cap D$ .

□



## 2.2 The Baire Space

The *Baire Space* is the space  $\mathcal{N} = \omega^\omega$  of infinite sequences of natural numbers with the topology defined the following way. Let  $s$  be a finite sequence  $s = \langle a_k : k < n \rangle$ , we let:

$$O(s) = \{f \in \mathcal{N} \mid s \subset f\} = \{\langle c_k \mid k < \omega \rangle \mid \forall k < n \ c_k = a_k\}$$

All those  $O(s)$ 's form the open basis for  $\mathcal{N}$ .

The Baire space is separable and metrisable. The metric is defined as  $d(f, g) = 1/2^{n+1}$  where  $n$  is the smallest natural number such that  $f(n) \neq g(n)$ . We also have separability since the set of all eventually constant sequences is dense in  $\mathcal{N}$ .

Every infinite sequence  $\langle a_k : k < \omega \rangle$  defines a continued fraction  $1/(a_0 + 1/(a_1 + 1/(a_2 + \dots)))$ , so we have a continuous bijection between infinite sequences and irrational points of the open interval  $(0, 1)$ . Moreover, the Baire space is homeomorphic to the space of irrational numbers.

Now we describe the characterisation of perfect sets in the Baire space.

Let  $\text{Seq}$  be the set of all finite sequences in  $\mathcal{N}$ . A *tree* is a set  $T \subset \text{Seq}$  satisfying:

If  $t \in T$  and there exists  $n < \omega$  such that  $s = t \upharpoonright n$ , then  $s \in T$ .

Let  $T$  be a tree, let  $[T]$  be the set of all infinite paths through  $T$ :

$$[T] = \{f \in \mathcal{N} \mid \forall n < \omega \ f \upharpoonright n \in T\}$$

For each  $T$ , the set  $[T]$  is closed in the Baire space. Let  $f \in \mathcal{N}$  such that  $f \notin [T]$ . Then there exists  $n < \omega$  such that  $s = f \upharpoonright n \notin T$ , so the open neighbourhood of  $f$   $O(s) = \{g \in \mathcal{N} \mid g \supset s\}$ . Thus  $[T]$  is closed.

Conversely, let  $F$  be closed in  $\mathcal{N}$ , then the set

$$T_F = \{s \in \text{Seq} \mid \exists f \in F \ s \subset f\}$$

is a tree and one can verify that  $[T_F] = F$ . If  $f \in \mathcal{N}$  such that  $f \upharpoonright n \in T$  for each  $n < \omega$ , then for each  $n$  there is some  $g \in F$  such that  $g \upharpoonright n = f \upharpoonright n$ , so  $f \in F$  since  $F$  is closed.

If  $f$  is an isolated point of a closed set  $F$  in  $\mathcal{N}$ , then there is  $n \in \mathbb{N}$  such that no  $g \in F$  such that  $g \neq f$  and  $g \upharpoonright n = f \upharpoonright n$ , so we have no branching starting from the  $n$ -th position.

So we have the notion of a perfect set  $P$  in the Baire space. A tree  $T$  is *perfect* if  $t \in T$ , then there exist incomparable  $t_1, t_2 \supset t$  such that both of them are in  $T$  and neither  $t_1 \subset t_2$  nor  $t_2 \subset t_1$ .

**Theorem 2.7.** A closed set  $F \subset \mathcal{N}$  is perfect iff the tree  $T_F$  is perfect.

Let us discuss the Cantor-Bendixon analysis of closed subsets of the Baire space. Let  $T$  be a tree, define:

$$T' = \{t \in T \mid \exists t_1, t_2 \supset t \ (t_1, t_2 \in T \ \& \ \neg(t_1 \subset t_2 \vee t_2 \subset t_1))\}$$

Then a set  $T$  is perfect iff  $T = T' \neq \emptyset$ .

$[T] - [T']$  is at most countable: take  $f \in [T]$  such that  $f \notin [T']$ . Take  $s_f = f \upharpoonright n$  where  $n < \omega$  is the smallest index such that  $f \upharpoonright n \notin T'$ . If  $f, g \in [T] - [T']$ , then  $s_f \neq s_g$  by the definition of  $T'$ , so the mapping  $f \mapsto s_f$  is one-to-one.

Now let:

$$\begin{aligned} T_0 &= T \\ T_{\alpha+1} &= T'_\alpha \\ T_\alpha &= \bigcap_{\beta < \alpha} T_\beta \text{ if } \alpha > 0 \text{ is limit.} \end{aligned}$$

We have  $T_0 \supset T_1 \supset \dots \supset T_\alpha \supset \dots$ .  $T_0$  is at most countable, so there is  $\theta < \omega_1$  at which the sequence stabilises. If  $T_\theta \neq \emptyset$ , then  $T_\theta$  is perfect.

One can verify that:

$$[\bigcap_{\beta < \alpha} T_\beta] = \bigcap_{\beta < \alpha} [T_\beta]$$

so we have

$$[T] - [T_\theta] = \bigcup_{\beta < \alpha} ([T_\alpha - T'_\alpha])$$

and the set  $[T] - [T_\theta]$  is at most countable. So we have a version of Theorem 2.5 for the Baire space.

### 3 The Axiom of Choice

Recall that the axiom of choice (AC) says that if we have a family of sets  $S$  such that  $\emptyset \notin S$ , then we have a *choice function* on  $S$  such that  $f(X) \in X$ .

In some cases we can show the existence of a choice function without using the axiom of choice. For example, for families of a complete lattice, the choice function can return the supremum or infimum of each set belonging to a family.

Using the axiom of choice one can also show that every infinite set has cardinality equal to  $\aleph_\alpha$  for some  $\alpha$ .

#### Theorem 3.1. (Zermelo)

Every set can be well-ordered.

*Proof.* Let  $A$  be a set. It is sufficient to construct a transfinite sequence  $\langle a_\alpha : \alpha < \theta \rangle$  that enumerates  $A$ . We do that by induction and by using the choice function  $f$  on non-empty subsets of  $A$ . For  $\alpha$  we let:

$$a_\alpha = f(A - \{a_\xi \mid \xi < \alpha\})$$

whenever  $A - \{a_\xi \mid \xi < \alpha\}$  is non-empty. Let  $\theta$  be the smallest ordinal such that  $A = \{a_\alpha \mid \alpha < \theta\}$ . Thus  $\langle a_\alpha : \alpha < \theta \rangle$  enumerates  $A$ .  $\square$

As it is well-known, Zermelo's theorem implies the axiom of choice. Let  $S$  be a family of sets such that  $\emptyset \notin S$ . By Zermelo's theorem, we can well-order  $\cup S$ , so let  $f(X)$  be the smallest element of  $X$ .

Note that Zermelo's theorem also implies that  $\mathbb{R}$  can be well ordered and also that  $2^{\aleph_0}$  is an aleph and  $2^{\aleph_0} \geq \aleph_1$ .

Another important consequence of the axiom of choice:

**Theorem 3.2.** The union of a countable family of countable sets is countable.

*Proof.* Let  $A_n$  be a countable set for each  $n < \omega$ . For each  $n$  let us choose an enumeration  $\langle a_{n,k} : k < \omega \rangle$  of  $A_n$ . So we have a projection of  $\mathbb{N} \times \mathbb{N}$  onto  $\bigcup_{n < \omega} A_n$  by mapping  $(n, k) \mapsto a_{n,k}$ .  $\square$

In fact, the theorem above can be generalised the following way:

**Theorem 3.3.**  $|S| \leq S \cdot \sup\{|X| \mid X \in S\}$ .

*Proof.* Let  $\kappa = |S|$  and  $\lambda = \sup\{|X| \mid X \in S\}$ . We have  $S = \{X_\alpha \mid \alpha < \kappa\}$  and for each  $\alpha < \kappa$  we choose an enumeration  $X_\alpha = \{a_{\alpha,\beta} \mid \beta < \lambda_\alpha\}$  where  $\lambda_\alpha = |X_\alpha|$ . Clearly that  $\lambda_\alpha \leq \lambda$  for each  $\alpha < \kappa$ . So we have a projection of  $\kappa \times \lambda$  onto  $\cup S$  by mapping  $(\alpha, \beta) \mapsto a_{\alpha,\beta}$ .  $\square$

**Corollary 3.1.** For every  $\alpha$   $\aleph_{\alpha+1}$  is a regular cardinal.

*Proof.* If  $\aleph_{\alpha+1}$  were singular for some  $\alpha$ , then  $\omega_{\alpha+1}$  would be the union of at most  $\aleph_\alpha$  sets of cardinality  $\aleph_\alpha$  by Lemma 1.6, which would mean that  $\aleph_{\alpha+1} = \aleph_\alpha$  by Theorem 3.3. Contradiction.  $\square$

Let  $(P, <)$  be a poset, an element  $a \in P$  is *maximal* if no  $b \in P$  such that  $b > a$ . Let  $X$  be a non-empty subset of  $P$ , then  $c$  is the *upper bound* of  $X$  if  $c \geq X$ .  $X$  is a *chain* in  $P$  if any two elements of  $X$  are comparable.

**Theorem 3.4. (Zorn)**

Let  $(P, <)$  be a poset such that every chain  $C$  has an upper bound, then  $P$  has a maximal element.

*Proof.* Let  $f$  be a choice function on non-empty subsets of  $P$ . We construct a chain  $C$  leading to a maximal element.

Construct the following elements by induction:

$a_\alpha$  = an element of  $P$  such that  $a_\alpha > a_\xi$  for every  $\xi > \alpha$  if it exists

If  $\alpha > 0$  is a limit ordinal, then  $C_\alpha$  is a chain in  $P$  and  $a_\alpha$  does exist. Eventually, there is  $\theta$  such that no  $a_{\theta+1} > a_\theta$ . Thus  $a_\theta$  is maximal.  $\square$

As it is known, Zorn's lemma implies the axiom of choice. Let  $S$  be a family of non-empty sets, then we check that the set  $\{f \mid f \text{ is a choice function on some } S' \subset S\}$  ordered by inclusion satisfies the condition of Zorn's lemma, so a maximal element of that poset is a choice function on  $S$ .

There is a weaker version of the axiom of choice for countable families of non-empty sets. The countable AC implies Theorem 3 and regularity of  $\aleph_1$ , but the countable AC is too weak to show that  $\mathbb{R}$  can be well-ordered.

There is a stronger version of the countable AC.

**Definition 3.1. (The Principle of Dependent Choice (DC))**

Let  $R$  be a binary relation on  $A$  such that for all  $x \in A$  there exists  $y \in A$  such that  $yRx$ , then there is a sequence  $a_0, a_1, \dots, a_n, \dots$  for  $n < \omega$  such that:

$$\forall n < \omega (a_{n+1}Ra_n)$$

The Principle of Dependent Choices allows characterising well orderings and (as well as well-founded relations) the following way:

**Lemma 3.1.** Let  $(A, <)$  be a poset, then the following are equivalent:

1.  $(A, <)$  is a well-ordering.
2. No infinite sequences  $a_0, a_1, \dots, a_n, \dots$  for  $n < \omega$  such that:

$$a_0 > a_1 > \dots > a_n > \dots$$

### 3.1 Cardinal Arithmetic the Generalised Continuum Hypothesis

We shall be discussing the cardinal exponentiation operator for the rest of the section.

**Lemma 3.2.** Let  $\lambda$  be infinite and  $2 \leq \kappa \leq \lambda$ , then  $\kappa^\lambda = 2^\lambda$ .

*Proof.*  $2^\lambda \leq \kappa^\lambda \leq (2^\kappa)^\lambda = 2^{\kappa \cdot \lambda} = 2^\lambda$ . □

The evaluation of  $\kappa^\lambda$  is more complicated when  $\lambda < \kappa$ . If  $2^\lambda \geq \kappa$ , then we have  $\kappa^\lambda = 2^\lambda$  since  $\kappa \leq (2^\lambda)^\lambda = 2^\lambda$ . But if  $2^\lambda < \kappa$ , the only thing we can conclude:

$$\kappa \leq \kappa^\lambda \leq 2^\kappa$$

which is already known by Cantor's theorem.

Let  $\lambda$  be a cardinal and let  $A$  be a set such that  $|A| \geq \lambda$ , we let:

$$[A]^\lambda = \{X \in 2^A \mid |X| = \lambda\}$$

**Lemma 3.3.** If  $|A| = \kappa \geq \lambda$ , then the set  $[A]^\lambda$  has cardinality  $\kappa^\lambda$ .

*Proof.* On the one hand every function  $f : \lambda \rightarrow A$  is a subset of  $\lambda \times A$  and  $|f| = \lambda$ . Thus:

$$\kappa^\lambda \leq |[\lambda \times A]^\lambda| = |[A]^\lambda|$$

On the other hand, there is a one-to-one function  $F : [A]^\lambda \rightarrow A^\lambda$ . If  $X \in [A]^\lambda$ , let  $F(X)$  be some function  $f$  on  $\lambda$  whose range is  $X$ . □

Let  $\lambda$  be a limit cardinal, let:

$$\kappa^{<\lambda} = \sup\{\kappa^\mu \mid \mu \text{ is a cardinal such that } \mu < \lambda\}$$

We also define  $\kappa^{<\lambda^+}$  for successors  $\lambda^+$ .

Let  $\kappa$  be an infinite cardinal and  $|A| \geq \kappa$ , let:

$$[A]^{<\kappa} = \{X \in 2^A \mid |X| < \kappa\}$$

Clearly, the cardinality of  $[A]^{<\kappa}$  is  $|A|^{<\kappa}$ .

### 3.2 Infinite Sums and Products

Let  $\{\kappa_i \mid i \in I\}$  be an indexed family of cardinals, define:

$$\sum_{i \in I} \kappa_i = \left| \bigcup_{i \in I} X_i \right|$$

where each for  $i \in I$   $|X_i| = \kappa_i$ . Note that, by the Axiom of Choice, the definition of sum does not depend on the choice of  $\{X_i \mid i \in I\}$ .

Let  $\lambda, \kappa$  be cardinals and let  $\kappa_i = \kappa$ , then:

$$\sum_{i < \lambda} \kappa_i = \lambda \cdot \kappa$$

More generally, we have:

**Lemma 3.4.** Let  $\lambda$  be an infinite cardinal and  $\kappa_i > 0$  for each  $i < \lambda$ , then:

$$\sum_{i < \lambda} \kappa_i = \lambda \cdot \sup_{i < \lambda} \kappa_i$$

*Proof.* Let  $\kappa = \sup_{i < \lambda} \kappa_i$  and  $\sigma = \sum_{i < \lambda} \kappa_i$ . On the one hand, we have  $\forall i < \lambda \ \kappa_i \leq \kappa$ , so

$$\sum_{i < \lambda} \kappa_i \leq \lambda \cdot \kappa$$

On the other hand, since  $\kappa_i \geq 1$  for each  $i$ , we have

$$\lambda = \sum_{i < \lambda} 1 \leq \sigma$$

$\sigma \geq \kappa_i$  for each  $i$ , so we have

$$\sigma \geq \sup_{i < \lambda} \kappa_i = \kappa$$

So  $\sigma \geq \lambda \cdot \kappa$ . □

Let  $\{X_i \mid i \in I\}$  be an indexed family of sets, we let:

$$\prod_{i \in I} X_i = \{f \mid f \text{ is a function on } I \text{ such that } \forall i \in I \ f(i) \in X_i\}$$

If each of  $X_i$ 's is non-empty, then the whole product is non-empty and this is equivalent to the axiom of choice.

Let  $\{\kappa_i \mid i \in I\}$  be a family of cardinals, define:

$$\prod_{i \in I} \kappa_i = \left| \prod_{i \in I} X_i \right|$$

where for each  $i$   $X_i$  is a set of cardinality of  $\kappa_i$ . As in the case of sum, assuming the axiom of choice, one can show that the definition of product does not depend on the choice of  $X_i$ 's.

If  $\kappa_i = \kappa$  for each  $i \in I$  and  $I$  has cardinality  $\lambda$ , then:

$$\prod_{i \in I} \kappa_i = \lambda$$

If  $I$  is a disjoint union  $I = \bigcup_{j \in J} A_j$ , then:

$$\prod_{i \in I} \kappa_i = \prod_{j \in J} \left( \prod_{i \in A_j} \kappa_i \right)$$

If  $\kappa_i \geq 2$  for each  $i \in I$ , then:

$$\sum_{i \in I} \kappa_i \leq \prod_{i \in I} \kappa_i$$

If  $I$  is finite, then the inequality is self-evident. Assume  $I$  is infinite. We have:

$$\prod_{i \in I} \kappa_i \geq \prod_{i \in I} 2 = 2^{|I|} > |I|$$

We show that  $\sum_i \kappa_i \leq |I| \cdot \prod_i \kappa_i$ .

Let  $\{X_i \mid i \in I\}$  be a disjoint family such that for each  $i \in I$   $|X_i| = \kappa_i$ . Assign each  $x \in \bigcup_i X_i$  to a pair  $(i, f)$  such that  $x \in X_i$  and  $f \in \prod_i X_i$  such that  $f(i) = x$ .

**Lemma 3.5.** Let  $\lambda$  be an infinite cardinal and let  $\langle \kappa_i : i < \lambda \rangle$  be a non-decreasing sequence of ordinals, then

$$\prod_{i \in I} \kappa_i = \left( \sup_{i \in I} \kappa_i \right)^\lambda$$

*Proof.* Let  $\kappa = \sup_i \kappa_i$ . Since  $\kappa_i \leq \kappa$  for each  $i < \lambda$ , we have:

$$\prod_{i \in I} \kappa_i \leq \prod_{i \in I} \kappa = \kappa^\lambda$$

Let us show  $\kappa^\lambda \leq \prod_{i \in I} \kappa_i$ .

Consider a partition of  $\lambda$  into  $\lambda$  disjoint sets  $A_j$ , each of which has cardinality  $\lambda$ :

$$\lambda = \bigcup_{j < \lambda} A_j$$

For each  $j < \lambda$  we have:

$$\kappa = \sup_{i \in A_j} \kappa_i \leq \prod_{i \in A_j} \kappa_i$$

And thus:

$$\prod_{i \in I} \kappa_i = \prod_{j < \lambda} \left( \prod_{i \in A_j} \kappa_i \right) \geq \prod_{j < \lambda} \kappa = \kappa^\lambda$$

□

**Theorem 3.5. (König)**

Assume  $\kappa_i < \lambda_i$  for each  $i \in I$ , then:

$$\sum_{i \in I} \kappa_i < \prod_{i \in I} \lambda_i$$

*Proof.* Let us show  $\sum_i \kappa_i \not\geq \prod_i \lambda_i$ . Let  $\{T_i \mid i \in I\}$  be an indexed family such that  $|T_i| = \lambda_i$ . It suffices to show that if we have a family  $\{Z_i \mid i \in I\}$  of subsets of  $T = \prod_i T_i$  such that  $|Z_i| < \kappa_i$  for each  $i$ , then  $\cup_i Z_i \neq T$ .

For every  $i \in I$ , let  $S_i$  be the projection of  $Z_i$  into the  $i$ -th coordinate:

$$S_i = \{f(i) \mid f \in Z_i\}$$

As far as  $|Z_i| < |T_i|$ , we have  $S_i \subset T_i$  and  $S_i \neq T_i$  for each  $i \in I$ . Let  $f \in T$  be a function such that  $f(i) \notin S_i$ .  $f$  does not belong to any  $Z_i$ , so  $\cup_i Z_i \neq T$ . □

**Corollary 3.2.**  $\kappa < 2^\kappa$

*Proof.*  $\sum_{i < \kappa} 1 < \prod_{i < \kappa} 2$ . □

**Corollary 3.3.** For each  $\alpha$   $\aleph_\alpha < \text{cf}(2^{\aleph_\alpha})$ .

*Proof.* Let us show that if for each  $i < \omega_\alpha$   $\kappa_i < 2^{\aleph_\alpha}$ , then  $\sum_{i < \omega_\alpha} \kappa_i < 2^{\aleph_\alpha}$ . Let  $\lambda_i = 2^{\aleph_\alpha}$ .

$$\sum_{i < \omega_\alpha} \kappa_i < \prod_{i < \omega_\alpha} \lambda_i = (2^{\aleph_\alpha})^{\aleph_\alpha} = 2^{\aleph_\alpha}$$

□

**Corollary 3.4.** For all  $\alpha, \beta$   $\aleph_\beta < \text{cf}(\aleph_\alpha^{\aleph_\beta})$ .

*Proof.* We show that if  $\kappa_i < \aleph_\alpha^{\aleph_\beta}$  for each  $i < \omega_\beta$ , then  $\sum_{i < \omega_\beta} \kappa_i < \aleph_\alpha^{\aleph_\beta}$ . Let  $\lambda_i = \aleph_\alpha^{\aleph_\beta}$ , then

$$\sum_{i < \omega_\beta} \kappa_i < \prod_{i < \omega_\beta} \lambda_i = \aleph_\alpha^{\aleph_\beta}$$

□

**Corollary 3.5.** Let  $\kappa$  be an infinite cardinal, then  $\kappa < \kappa^{\text{cf } \kappa}$

*Proof.* Let  $i < \text{cf } \kappa$  and  $\kappa_i < \kappa$  be such that  $\kappa = \sum_{i < \text{cf } \kappa} \kappa_i$ .

$$\kappa = \sum_{i < \text{cf } \kappa} \kappa_i < \prod_{i < \text{cf } \kappa} \kappa = \kappa^{\text{cf } \kappa}.$$

□

### 3.3 The Continuum Function

Cantor's theorem claims that  $\aleph_\alpha < 2^{\aleph_\alpha}$ , so  $\aleph_{\alpha+1} \leq 2^{\aleph_\alpha}$  for each  $\alpha$ . The *Generalised Continuum Hypothesis* (GCH) is the statement

$$2^{\aleph_\alpha} = \aleph_{\alpha+1}$$

for each  $\alpha$ . GCH is independent of ZFC, but ZFC + GCH proves the following properties of cardinal exponentiation:

**Theorem 3.6.** Assume GCH. Let  $\kappa$  and  $\lambda$  be infinite cardinals, then:

1. If  $\kappa \leq \lambda$ , then  $\kappa^\lambda = \lambda^+$ .
2. If  $\text{cf } \kappa \leq \lambda < \kappa$ , then  $\kappa^\lambda = \kappa^+$ .
3. If  $\lambda < \text{cf } \kappa$ , then  $\kappa^\lambda = \kappa$ .

*Proof.*

1. By Lemma 3.2 we have  $\kappa^\lambda = 2^\lambda$ , but  $2^\lambda = \lambda^+$ .
2. Combine Lemma 3.3 and Lemma 3.4.
3. By Lemma 1.5 we have:

$$\kappa^\lambda = \{\alpha^\lambda \mid \alpha < \kappa\}$$

so:

$$|\alpha^\lambda| \leq 2^{|\alpha| \cdot \lambda} = (|\alpha| \cdot \lambda)^+ \leq \kappa$$

□

The *beth function* is defined by induction:

1.  $\beth_0 = \aleph_0$
2.  $\beta_{\alpha+1} = 2^{\beta_\alpha}$
3.  $\beta_\alpha = \sup\{\beta_\beta \mid \beta < \alpha\}$  if  $\alpha$  is limit ordinal.

So we can reword GCH as  $\beta_\alpha = \aleph_\alpha$  for all  $\alpha$ .

Now we study the behaviour of the continuum function  $\kappa \mapsto 2^\kappa$  assuming no GCH.

**Theorem 3.7.** Let  $\kappa, \lambda$  be cardinals, then

1. If  $\kappa < \lambda$ , then  $2^\kappa \leq 2^\lambda$ .
2.  $\kappa < \text{cf } 2^\kappa$
3. If  $\kappa$  is a limit cardinal, then  $2^\kappa = (2^{<\kappa})^{\text{cf } \kappa}$



*Proof.*

1. Fairly obvious.
2. Corollary 3.3.
3. Let  $\kappa = \Sigma_{i < \text{cf } \kappa} \kappa_i$  where each  $\kappa_i < \kappa$  for each  $i$ . We have

$$2^\kappa = 2^{\Sigma_{i < \text{cf } \kappa} \kappa_i} = \prod_{i < \text{cf } \kappa} 2^{\kappa_i} \leq \prod_{i < \text{cf } \kappa} 2^{< \kappa} = (2^{< \kappa})^{\text{cf } \kappa} \leq (2^\kappa)^{\text{cf } \kappa} \leq 2^\kappa$$

□

## 4 The Axiom of Regularity