

Some Notes on Set Theory, Pt 1

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1 Cardinals

An ordinal number α is a *cardinal number* if no $\beta < \alpha$ such that $|\alpha| = |\beta|$. Further, we shall use κ, λ, μ to denote cardinal numbers.

Let W be a well-ordered set, then there exists an ordinal α such that $|W| = |\alpha|$, so we let:

$$|W| = \text{the least ordinal } \alpha \text{ such that } |W| = |\alpha|$$

An *aleph* is an infinite cardinal number.

Let α be an ordinal, then α^+ is the least cardinal bigger than α .

Lemma 1.1.

1. For every α there is a cardinal number κ such that $\kappa > \alpha$.
2. Let X be a set of cardinal, then $\sup X$ is a cardinal.

Proof.

1. Let X be a set, let

$$h(X) = \text{the least } \alpha \text{ such that no injection from } \alpha \text{ into } X$$

Consider $X \times X$, so $2^{X \times X}$ is the set of relations on X and there are well-orderings of subsets of X amongst all relations in $2^{X \times X}$, so consider the set

$$Y = \{R \subseteq X \times X \mid R \text{ is a well-ordering}\}$$

So there is a set of ordinals:

$$\text{Ord}(Y) = \{\alpha \in \text{Ord} \mid \exists R \in Y \text{ } \alpha \text{ is the order type of } R\}$$

Note that $\text{Ord}(Y)$ is a set and take the least element ordinal β does not belong to $\text{Ord}(Y)$. So $h(X) = \beta$. To be more precise, we have:

$$\beta = \sup \text{Ord}(Y)$$

Then $|\alpha| < h(\alpha)$ for each ordinal α .

2. Let $\alpha = \sup X$. Let f be a one-to-one function from α onto some $\beta < \alpha$. Let κ be a cardinal such that $\beta < \kappa \leq \alpha$, then $|\kappa| = |\{f(\xi) \mid \xi < \kappa\}| \leq \beta$, so contradiction and α is a cardinal.

□

The enumeration of all alephs is defined by transfinite induction:

- $\aleph_0 = \omega$
- $\aleph_{\alpha+1} = \aleph_\alpha^+ = \omega_{\alpha+1}$
- If β is a limit ordinal, then $\aleph_\beta = \omega_\beta = \sup\{\omega_\alpha \mid \alpha < \beta\}$.

A cardinal of the form $\aleph_{\alpha+1}$ is a *successor* cardinal, a cardinal \aleph_β for limit β is a *limit cardinal*.

1.1 The ordering of $\alpha \times \alpha$

Define a well-ordering of the class $\text{Ord} \times \text{Ord}$ the following way:

$$\begin{aligned} (\alpha, \beta) < (\gamma, \delta) \text{ iff either } \max(\alpha, \beta) < \max(\gamma, \delta) \text{ or} \\ \max(\alpha, \beta) = \max(\gamma, \delta) \text{ and } \alpha < \gamma \text{ or} \\ \max(\alpha, \beta) = \max(\gamma, \delta) \text{ and } \alpha = \gamma \text{ and } \beta < \delta. \end{aligned}$$

Then $<$ is a well-ordering and linear relation on Ord . Moreover, $\alpha \times \alpha$ is the initial segment of $(\text{Ord} \times \text{Ord}, <)$ given by $(0, \alpha)$.

We let:

$$\Gamma(\alpha, \beta) = \text{the order type of } \{(\xi, \eta) \mid (\xi, \eta) < (\alpha, \beta)\}$$

Γ is also one-to-one:

$$(\alpha, \beta) < (\gamma, \delta) \text{ iff } \Gamma(\alpha, \beta) < \Gamma(\gamma, \delta)$$

Γ is increasing and continuous and $\Gamma(\alpha \times \alpha) = \alpha$ for arbitrarily large α .

Theorem 1.1. $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$

Proof. Let us show that $\Gamma(\omega_\alpha \times \omega_\alpha) = \omega_\alpha$.

1. If $\alpha = 0$, then $\Gamma(\omega \times \omega) = \omega$.
2. Let α be the least ordinal such that $\Gamma(\omega_\alpha \times \omega_\alpha) \neq \omega_\alpha$. Let β, γ be ordinals such that $\Gamma(\beta, \gamma) = \omega_\alpha$. Take $\delta < \omega_\alpha$ such that $\delta > \beta, \gamma$. $\delta \times \delta$ is the initial segment of Ord^2 and it contains (β, γ) . So $\Gamma(\delta \times \delta) \supset \omega_\alpha = \Gamma(\beta, \gamma)$. Thus $|\delta \times \delta| \geq \aleph_\alpha$. But $|\delta \times \delta| = |\delta| \cdot |\delta| = |\delta|$. But $|\delta| < \aleph_\alpha$ by the assumption of minimality of α . Contradiction.

□

As a corollary:

$$\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = \max(\aleph_\alpha, \aleph_\beta)$$

1.2 Cofinality

Let $\alpha, \beta > 0$ be limit ordinals. An increasing β -sequence $\langle \alpha_\xi : \xi < \beta \rangle$ is *cofinal* in α if $\lim_{\xi \rightarrow \beta} \alpha_\xi = \alpha$. A subset $X \subseteq \alpha$ is *cofinal* in α whenever $\sup X = \alpha$.

Let $\alpha > 0$ be a limit ordinal, the *cofinality* of α is:

$\text{cf } \alpha =$ the least ordinal β such that $\exists \langle \alpha_\xi : \xi < \beta \rangle$ such that $\lim_{\xi \rightarrow \beta} \alpha_\xi = \alpha$

Note that for each α $\text{cf } \alpha$ is a limit ordinal and $\text{cf } \alpha \leq \alpha$.

Lemma 1.2. For each α $\text{cf}(\text{cf } \alpha) \leq \text{cf } \alpha$.

Proof. Let $\langle \alpha_\xi : \xi < \beta \rangle$ be cofinal in α and let $\langle \xi_\nu : \nu < \gamma \rangle$ be cofinal in β .

Consider $\langle \alpha_{\xi_\nu} : \nu < \gamma \rangle$, then

$$\lim_{\nu < \gamma} \alpha_{\xi_\nu} = \alpha$$

since the limit of a subsequence equals the limit of a sequence as in usual real analysis or topology. \square

Lemma 1.3. Let α be a non-zero limit ordinal, then

1. If $A \subseteq \alpha$ and $\sup A = \alpha$, the order-type of A is at least $\text{cf } \alpha$.
2. Let $\beta_0 \leq \beta_1 \leq \dots \leq \beta_\xi \leq \dots$ for $\xi < \gamma$ be a non-decreasing sequence of ordinals such that $\lim_{\xi \rightarrow \gamma} \beta_\xi = \alpha$, then $\text{cf } \gamma = \alpha$.

Proof. 1. The order-type of A is the length of the increasing enumeration of A , the limit of which (as an increasing sequence) is α .

2. If $\gamma = \lim_{\nu \rightarrow \text{cf } \gamma} \xi_\nu$, then $\alpha = \lim_{\nu \rightarrow \text{cf } \gamma} \beta_{\xi_\nu}$, and the non-decreasing sequence $\langle \beta_{\xi_\nu} : \nu < \text{cf } \gamma \rangle$ has an increasing sequence of the length at most $\text{cf } \gamma$ and it has the same limit, so $\text{cf } \alpha \leq \text{cf } \gamma$.

To show $\text{cf } \gamma \leq \text{cf } \alpha$, assume $\alpha = \lim_{\nu \rightarrow \text{cf } \alpha} \alpha_\nu$. Take $\nu < \text{cf } \alpha$, let ξ_ν be the least ξ greater than all ξ_ι for $\iota < \nu$ such that $\beta_{\xi_\iota} > \alpha_\nu$. We have $\alpha = \lim_{\nu \rightarrow \text{cf } \alpha} \beta_{\xi_\nu}$, so $\gamma = \lim_{\nu \rightarrow \text{cf } \alpha} \xi_\nu$, so the inequation is proved. \square

An infinite cardinal \aleph_α is *regular* if $\text{cf } \aleph_\alpha = \aleph_\alpha$. \aleph_α is *singular* if $\text{cf } \aleph_\alpha < \aleph_\alpha$.

Lemma 1.4. Let α be a limit ordinal, then $\text{cf } \alpha$ is a regular cardinal.

Proof. If α is not a cardinal, then there exists an ordinal $\beta < \alpha$ such that $|\beta| = |\alpha|$, then we construct a cofinal sequence in α of length $|\beta|$, then $\text{cf } \alpha = |\beta|$ and $\text{cf } \alpha < \alpha$. \square

Let κ be a limit ordinal, a subset $X \subset \kappa$ is *bounded* if $\sup X < \kappa$ and *unbounded* if $\sup X = \kappa$.

Lemma 1.5. Let κ be an aleph, then:

1. If $X \subset \kappa$ and $|X| < \text{cf } \kappa$, then X is bounded.
2. If $\lambda \nmid \text{cf } \kappa$ and $f : \lambda \rightarrow \kappa$, then $\text{Im } f$ is bounded in κ .

Proof. 1. Let X be such subset of κ and assume X is unbounded, so $\sup X = \kappa$. By 1 of Lemma 1.3, the order-type of X is at least $\text{cf } \kappa$, which contradicts to $|X| < \text{cf } \kappa$, so X is bounded.

2. Follows from the first item.

□

Lemma 1.6. (Hausdorff)

Let κ be a cardinal, then the following are equivalent:

1. κ is singular.
2. There is a cardinal $\lambda < \kappa$ and a family $\{S_\xi \mid \xi < \lambda\}$ such that each $S_\xi \subset \kappa$, $|S_\xi| < \kappa$ and $\kappa = \bigcup_{\xi < \lambda} S_\xi$.

Proof.

1. (1) \Rightarrow (2).

If κ is singular, then there is an increasing sequence $\langle \alpha_\xi : \xi < \text{cf } \kappa \rangle$, so a family of required subsets is actually a family of those α_ξ 's and $\lambda = \text{cf } \kappa$ which is strictly less than κ since κ is singular.

2. (2) \Rightarrow (1).

Let λ be the least cardinal such that $\lambda < \kappa$ and there exists a family $\{S_\xi \mid \xi < \lambda\}$ where each $S_\xi \subset \kappa$, $|S_\xi| < \kappa$ and

$$\kappa = \bigcup_{\xi < \lambda} S_\xi$$

For each $\xi < \lambda$, let β_ξ be the order-type of $\bigcup_{\nu < \xi} S_\nu$. The sequence $\langle \beta_\xi : \xi < \lambda \rangle$ is non-decreasing and each $\beta_\xi < \kappa$ for all $\xi < \lambda$ since λ is minimal.

Let us show that $\lim_{\xi \rightarrow \kappa} \beta_\xi = \kappa$ to show that $\text{cf } \kappa \leq \lambda$.

Assume $\beta = \lim_{\xi \rightarrow \kappa} \beta_\xi$. There is a one-to-one mapping $f : \bigcup_{\xi < \beta} S_\xi \rightarrow \lambda \times \beta$ such that:

$$f : \alpha \mapsto (\xi, \gamma)$$

where ξ is the least ordinal such that $\alpha \in S_\xi$ and γ is the order-type of $S_\xi \cap \gamma$.

We have $\lambda < \kappa$ and $|\lambda \times \beta| = \lambda \cdot |\beta|$, then $\kappa = \beta$.

□

Theorem 1.2. Let κ be an infinite cardinal, then $\kappa < \kappa^{\text{cf } \kappa}$.

Proof. Let F be a collection of κ functions from $\text{cf } \kappa$ to κ :

$$F = \{f_\alpha : \text{cf } \kappa \rightarrow \kappa \mid \alpha < \kappa\}$$

Let us construct f that does not belong to F .

We have $\kappa = \lim_{\xi < \text{cf } \kappa} \alpha_\xi$, for $\xi < \text{cf } \kappa$ we let:

$$f(\xi) = \text{least } \gamma \text{ such that } \gamma \neq \forall \alpha < \alpha_\xi f_\alpha \neq \gamma$$

Such γ does exist and f is different from all the f_α . □

An uncountable cardinal κ is *weakly inaccessible* if it is limit and regular, but we cannot prove the existence of weakly inaccessible cardinals in ZFC.

2 Real Numbers and The Baire Space

3 The Axiom of Choice

4 Cardinal Arithmetic via the Generalised Continuum Hypothesis