Some Notes on Set Theory, Pt 1

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1 Cardinals

An ordinal number α is a *cardinal number* if no $\beta < \alpha$ such that $|\alpha| = |\beta|$. Further, we shall use κ, λ, μ to denote cardinal numbers.

Let W be a well-ordered set, then there exists an ordinal α such that $|W| = |\alpha|$, so we let:

$$|W| = \text{the least ordinal } \alpha \text{ such that } |W| = |\alpha|$$

An *aleph* is an infinite cardinal number.

Let α be an ordinal, then α^+ is the least cardinal bigger than α.

Lemma 1.1.

1. For every α there is a cardinal number κ such that $\kappa > \alpha$.
2. Let X be a set of cardinal, then $\sup X$ is a cardinal.

Proof.

1. Let X be a set, let

$$h(X) = \text{the least } \alpha \text{ such that no injection from } \alpha \text{ into } X$$

Consider $X \times X$, so $2^{X \times X}$ is the set of relations on X and there are well-orderings of subsets of X amongst all relations in $2^{X \times X}$, so consider the set

$$Y = \{R \subseteq Y \times Y \mid Y \subseteq X \;\&\; Y \text{ is a well-ordering } \}$$

So there is a set of ordinals:

$$Ord(Y) = \{\alpha \in \text{Ord} \mid \exists R \in Y \; \alpha \text{ is the order type of } Y\}$$

Note that $Ord(Y)$ is a set and take the least element ordinal β does not belong to $Ord(Y)$. So $h(X) = \beta$. To be more precise, we have:

$$\beta = \sup Ord(Y)$$

Then $|\alpha| < h(\alpha)$ for each ordinal α.

2. Let $\alpha = \sup X$. Let f be a one-to-one function from α onto some $\beta < \alpha$. Let κ be a cardinal such that $\beta < \kappa \leq \alpha$, then $|\kappa| = |\{f(\xi) \mid \xi < \kappa\}| \leq \beta$, so contradiction and α is a cardinal.

$\square$

The enumeration of all alephs is defined by transfinite induction:

- $\aleph_0 = \omega$
- $\aleph_{\alpha+1} = \aleph_\alpha^+ = \omega_{\alpha+1}$
- If β is a limit ordinal, then $\aleph_\beta = \omega_\beta = \sup\{\omega_\alpha \mid \alpha < \beta\}$.

A cardinal of the form $\aleph_{\alpha+1}$ is a *successor* cardinal, a cardinal $\aleph_\beta$ for limit β is a *limit cardinal*.

1.1 The ordering of $\alpha \times \alpha$

Define a well-ordering of the class $\mathrm{Ord} \times \mathrm{Ord}$ the following way:

$$\begin{gathered}(\alpha, \beta) < (\gamma, \delta) \text{ iff either } \max(\alpha, \beta) < \max(\gamma, \delta) \text{ or} \\ \max(\alpha, \beta) = \max(\gamma, \delta) \text{ and } \alpha < \gamma \text{ or} \\ \max(\alpha, \beta) = \max(\gamma, \delta) \text{ and } \alpha = \gamma \text{ and } \beta < \delta.\end{gathered}$$

Then $<$ is a well-ordering and linear relation on Ord. Moreover, $\alpha \times \alpha$ is the initial segment of $(\mathrm{Ord} \times \mathrm{Ord}, <)$ given by $(0, \alpha)$.

We let:

$$\Gamma(\alpha, \beta) = \text{the order type of } \{(\xi, \eta) \mid (\xi, \eta) < (\alpha, \beta)\}$$

Γ is also one-to-one:

$$(\alpha, \beta) < (\gamma, \delta) \text{ iff } \Gamma(\alpha, \beta) < \Gamma(\gamma, \delta)$$

Γ is increasing and continuous and $\Gamma(\alpha \times \alpha) = \alpha$ for arbitrarily large α.

Theorem 1.1. $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$

Proof. Let us show that $\Gamma(\omega_\alpha \times \omega_\alpha) = \omega_\alpha$.

1. If $\alpha = 0$, then $\Gamma(\omega \times \omega) = \omega$.
2. Let α be the least ordinal such that $\Gamma(\omega_\alpha \times \omega_\alpha) \neq \omega_\alpha$. Let β, γ be ordinals such that $\Gamma(\beta, \gamma) = \omega_\alpha$. Take $\delta < \omega_\alpha$ such that $\delta > \beta, \gamma$. $\delta \times \delta$ is the initial segment of Ord^2 and it contains (β, γ). So $\Gamma(\delta \times \delta) \supset \omega_\alpha = \Gamma(\beta, \gamma)$. Thus $|\delta \times \delta| \geq \aleph_\alpha$. But $|\delta \times \delta| = |\delta| \cdot |\delta| = |\delta|$. But $|\delta| < \aleph_\alpha$ by the assumption of minimality of α. Contradiction.

$\square$

As a corollary:

$$\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = \max(\aleph_\alpha, \aleph_\beta)$$

1.2 Cofinality

Let $\alpha, \beta > 0$ be limit ordinals. An increasing β-sequence $\langle \alpha_\xi : \xi < \beta \rangle$ is *cofinal* in α if $\lim_{\xi \to \beta} \alpha_\xi = \alpha$. A subset $X \subseteq \alpha$ is *cofinal* in α whenever $\sup X = \alpha$.

Let $\alpha > 0$ be a limit ordinal, the *cofinality* of α is:

$$\operatorname{cf} \alpha = \text{the least ordinal } \beta \text{ such that } \exists \ \langle \alpha_\xi : \xi < \beta \rangle \text{ such that } \lim_{\xi < \beta} \alpha_\xi = \alpha$$

2 The Baire Space

3 The Axiom of Choice

4 Cardinal Arithmetic via the Generalised Continuum Hypothesis