The finite base property for some relation algebras subreducts

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1 The Relation Algebras Background

We describe the basic definitions and results about relation algebras [8] [16].

Definition 1.

1. *A relation algebra is an algebra* $\mathcal{R} = \langle R, 0, 1, \wedge, \vee, \neg, ;, ^{\smile}, \mathbf{1} \rangle$ *such that* $\langle R, 0, 1, \wedge, \vee, \neg \rangle$ *is a Boolean algebra and the following equations hold, for each* $a, b, c \in R$*:*

   (a) $a; (b; c) = (a; b); c$

   (b) $(a \vee b); c = (a; c) \vee (b; c)$

   (c) $a; \mathbf{1} = a$

   (d) $a^{\smile\smile} = a$

   (e) $(a \vee b)^{\smile} = a^{\smile} \vee b^{\smile}$

   (f) $(a; b)^{\smile} = b^{\smile}; a^{\smile}$

   (g) $a^{\smile}; (\neg(a; b)) \leqslant \neg b$

   where $a \leqslant b$ *iff* $a \wedge b = a$ *iff* $a \vee b = b$. **RA** *denotes the class of all relation algebras.*

2. *A proper relation algebra is an algebra* $\mathcal{R} = \langle R, 0, 1, \wedge, \vee, \neg, ;, ^{\smile}, \mathbf{1} \rangle$ *such that* $R \subseteq \mathcal{P}(W)$*, where* W *is an equivalence relation;* $0 = \varnothing$*;* $1 = W$*;* $\wedge$*,* $\vee$*,* $\neg$ *are set-theoretic intersection, union, and complement respectively;* ; *is relation composition,* $^{\smile}$ *is relation converse,* $\mathbf{1}$ *is a diagonal relation restricted to* W*, that is:*

   (a) $a; b = \{\langle x, z \rangle \mid \exists y \, \langle x, y \rangle \in a \;\&\; \langle y, z \rangle \in b\}$

   (b) $a^{\smile} = \{\langle x, y \rangle \mid \langle y, x \rangle \in a\}$

   (c) $\mathbf{1} = \{\langle x, y \rangle \mid x = y\}$

   The class of all proper relation algebras is denoted as **PRA**. **Rs** *is the class of all relation set algebras, proper relation algebra with a diagonal subrelation as an identity.* **RRA** *is the class of all representable relation algebras, that is, the closure of* **PRA** *under isomorphic copies. That is,* **RRA** = **IPRA**.

Note that the (quasi)equational theories of those classes coincide, that is

$$\mathbf{IPRA} = \mathbf{RRA} = \mathbf{SPRs}$$

Moreover, **RRA** is a variety, but it cannot be defined by any set of first-order formulas [19] [].

One may express residuals in every $\mathcal{R} \in \mathbf{RA}$ as follows, for every $a, b \in \mathcal{R}$:

1. $a\backslash b = \neg(a^{\smile}; \neg b)$
2. $a/b = \neg(\neg a; b^{\smile})$

Those residuals have the following interpretation in $\mathcal{R} \in \mathbf{PRA}$ (as well as in $\mathbf{RRA}$), for every $a, b \in \mathcal{R}$:

1. $a\backslash b = \{\langle x, y\rangle \mid \forall z\,(z, x) \in a \Rightarrow (z, y) \in b\}$
2. $a/b = \{\langle x, y\rangle \mid \forall z\,(y, z) \in b \Rightarrow (x, z) \in a\}$

One may illustrate composition and residuals in **PRA** and **RRA** via the following triangles:

$$
\begin{tikzcd}
 & \exists y \arrow[dr, dashed, "b"] & \\
x \arrow[ur, dashed, "a"] \arrow[rr, "a;c"'] & & z
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \forall z \arrow[dl, dashed, "a"'] \arrow[dr, dashed, "b"] \arrow[d, phantom, "\Rightarrow"] & \\
x \arrow[rr, "a\backslash b"'] & {} & y
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \forall z \arrow[d, phantom, "\Leftarrow"] & \\
x \arrow[ur, dashed, "a"] \arrow[rr, "a/b"'] & {} & y \arrow[ul, dashed, "b"']
\end{tikzcd}
$$

Given a subset of definable operations in **RA** τ, we denote the class of subalgebras of the τ-reducts by $\mathbf{R}(\tau)$. The algebras containing to this class are defined as restrictions of elements belonging to **Rs** to operations of τ. By $\mathbf{Q}(\tau)$ we mean a quasivariety generated by $R(\tau)$. As in [11], we put $\mathbf{Q}(\tau)$ as the closure of $\mathbf{R}(\tau)$ under subalgebras and products assuming that $\mathbf{R}(\tau)$ is already closed under ultraproducts.

2 The Finite Base Property

We recall the underlying definitions according to [8, Section 19]

Definition 2. *Let* **K** *be a class of algebras of a signature* Ω*,* **K** *has the finite algebra property, if if any first-order* Ω*-sentence that is true in all finite algebras in* **K** *is true in every algebra in* **K***.*

The finite base property is a version of the finite algebra property if **K** is a class of representable algebras:

Definition 3. *Let* **K** *be a class of representable algebras of a signature* Ω

1. **K** *has the finite base property if any first-order* Ω*-sentence that is true in every algebra in* **K** *having a representation over a finite base set is valid in* **K***.*
2. **K** *has the finite algebra on finite base property if any finite algebra in* **K** *has a representation with finite base.*
3. **K** *has the finite algebra property for equations/quasi-identites if any equation/quasi-identity that is true in all finite algebras is true in every algebra in* **K***. The finite base property for equations/quasi-identites is defined similarly.*

The following statements were shown in [3]. This lemma connects finite base property with finite algebra on finite base and finite algebra properties as follows:

Lemma 1. *Let* **K** *be a class of representable* Ω*-algebras:*

1. *If* $\mathbf{K}$ *has the finite algebra property, then it has the finite algebra and the finite base properties for equations/quasi-identites.*

2. *The finite algebra on finite base and the finite algebra properties implies the finite base property for* $\mathbf{K}$*. The same holds for equations/quasi-identites.*

3. *If any representation of an infinite algebra has an infinite base, then the finite base property implies the finite algebra one for* $\mathbf{K}$*.*

4. *Suppose* Ω *is finite and any subalgebra of a representable algebra is representable on the same base. Then the finite base property implies the finite algebra on finite base property.*

3 The Relation Residuated Semigroups Background

3.1 The underlying definitions and results

A *relation structure* (**RS**) is an arbitrary algebra of the signature $\Omega = \langle \cdot, \backslash, /, \leqslant \rangle$, where $\cdot, \backslash, /$ are binary function symbols and $\leqslant$ is a binary relation symbol.

Definition 4. *A residuated semigroup is an algebra* $\mathcal{S} = \langle S, \cdot, \leqslant, \backslash, / \rangle$ *such that* $\langle S, \cdot, \leqslant, \rangle$ *is an ordered residuated semigroup and the following equivalences hold for each* $a, b, c \in S$*:*

$$b \leqslant a \backslash c \Leftrightarrow a \cdot b \leqslant c \Leftrightarrow a \leqslant c/b$$

ORS *is the class of all residuated semigroups.*

Definition 5. *Let* A *be a set of binary relations on some base set* W *such that* $R = \cup A$ *is transitive and* $\{x, y \mid xRy\} = W$*. A relation residuated semigroup is an algebra* $\mathcal{A} = \langle A, ;, \backslash, /, \subseteq \rangle$ *where for each* $r, s \in A$

1. $r; s = \{\langle a, c \rangle \mid \exists b \in W\ (\langle a, b \rangle \in r \ \&\ \langle b, c \rangle \in s)\}$

2. $r \backslash s = \{\langle a, c \rangle \mid \forall b \in W\ (\langle b, a \rangle \in r \Rightarrow \langle b, c \rangle \in s)\}$

3. $r/s = \{\langle a, c \rangle \mid \forall b \in W\ (\langle c, b \rangle \in s \Rightarrow \langle a, b \rangle \in r)\}$

Relation residuated semigroup are also called representable relativised relational structure (**RRS**).

Andréka and Mikulás proved the following representation theorem for **ORS** in [4] that implies relational completeness of the Lambek calculus, the logic of **ORS**:

Theorem 1. **ORS** = **IRRS***, where* **IRRS** *is a closure of* **RRS** *under isomorphic copies.*

3.2 The finite base property for RRS

Definition 6. *A relativised representation*

Definition 7. *The standard translation*

TODO: take a look at relativised representations and loosely guarded fragments in general TODO: realise whether it makes sense to use the technique similar to [8, Theorem 19.13] used for weakly associative algebras.

Theorem 2. *Let* $\mathcal{A}$ *be a finite residuated semigroup and* $|\mathcal{A}| < \omega$*, then* $\mathcal{A}$ *has a finite relativised representation.*

Theorem 3. *Let $\mathcal{A}$ be a finite representable residuated semigroup, then $\mathcal{A}$ is isomorphic to representable residuated semigroup a domain of which is finite.*

Proof. That might follow from the previous theorem, Theorem 1, and something else. $\square$

Corollary 1. *The Lambek calculus has the fmp and the universal theory of* **IRRS** *is NP-complete.*

The hypothetical plan is the following one:

1. Define properly relativised representation for residuated semigroups, that should look like ternary Kripke frames for the basic Lambek calculus or arrow logic.
2. Define the standard translation to such first-order relation structures. TODO: take a look at loosely guarded fragment stuff.
3. Every finite residuated semigroup has a finite relativised representation.
4. If every Π_1-statement φ of the language of residuated semigroups that is valid in every residuated semingroup is valid in algebra having a finite relativised representation (one may use here Theorem 1 somehow), then φ is valid in **ORS** as well as in **IRRS**.
5. Every finite residuated semigroup should have a finite relativised representation.
6. Construct a finitely based relation residuated semigroup from that (an analogue of complex algebra or smth like that). This item is the most non-trivial one.
7. As a corollary, the first-order universal first-order theory of **IRRS** should be decidable and (it seems so) NP-complete (that should follow from the results in [23]). The Lambek calculus is decidable that was shown syntactically via cut elimination and subformula property. Here we would have an alternative way of showing decidability for some substructural logics.

4 Join-semilattice ordered semigroups

Definition 8. *A join-semilattice ordered semigroup ($\mathbf{OS}^{\vee}$) is an algebra $\mathcal{S} = \langle S, \cdot, \vee \rangle$ such that $\langle S, \cdot \rangle$ is a semigroup, $\langle S, \vee \rangle$ is a join-semilattice and the following equations hold for each $a, b.c \in S$:*

1. $a \cdot (b \vee c) = (a \cdot b) \vee (a \cdot c)$
2. $(a \vee b) \cdot c = (a \cdot c) \vee (b \cdot c)$

This class is clearly a variety since $\mathbf{OS}^{\vee}$ has the equational definition so far as $\vee$ is defined as an associative, idempotent, and commutative operation.

Let A be a set of binary relations on some base set W such that $R = \cup A$ is transitive and $\{x, y \mid xRy\} = W$ as in Definition 5. A relation join-semilattice ordered semigroup ($\mathbf{ROS}^{\vee}$) is an algebra of binary relations $\mathcal{A} = \langle A, |, \cup \rangle$ such that ; is a relation composition as above and $\cup$ is the set-theoretic union. Let us define a representation of

Definition 9. *Let $\mathcal{S} = \langle S, \cdot, \vee \rangle$ a join-semilattice ordered semigroup. A representation of $\mathcal{S}$ is a map $h : S \to 2^{D \times D}$ for some base set D satisfying the following conditions*

- $h(a) = h(b) \Rightarrow a = b$

- $h(a \vee b) = h(a) \vee h(b)$
- $h(a; b) = h(a)|h(b) = \{(e, d) \in D \times D \mid \exists f \in D\ (e, f) \in h(a)\ \&\ (f, d) \in h(b)\}$

It is known that the class of all representable join-semilattice ordered semigroups has no finite axiomatisation [1]. In other words,

Theorem 4. *The equational and quasiequational theories of $R(;, \vee)$ is not finitely based.*

Let us provide a proof of this fact using the rainbow technique [8] to show that the complement of $\mathbf{ROS}^{\vee}$ is not closed ultraproducts. This is (more or less) a standard way, see [14]. We note that representability is not decidable for finite relation algebras [7]. Moreover, representability is undecidable for lattice-ordered semigroups and ordered complemented semigroups [21].

First of all, we recall several definitions such as colourings. We define a sequence of relation algebras $\{\mathfrak{A}_n\}_{n<\omega}$ each of which belongs to $\mathbf{RA}$. We need these algebras to show that their $\{;, \vee\}$-reducts are not representable. That is, we are seeking to show that

Given $n < \omega$, the set of atoms $\mathrm{At}(\mathfrak{A}_n)$ consists of the following elements:

- identity: $\mathbf{1}$, an atom with no colour
- greens: $\mathbf{g}_i$ for $0 \leqslant i \leqslant 2^n$
- yellow: $\mathbf{y}$
- black: $\mathbf{b}$
- whites: $\mathbf{b}_{ij}$ for $0 \leqslant i \leqslant j \leqslant 2^n$
- reds: $\mathbf{r}_i$ for $0 < i \leqslant 2^n$

We claim that every atom is self-converse ($a^{\smile} = a$). Given $x, y, z \in \mathfrak{A}_n$, a triple (x, y, z) is an inconsistent triangle if

$$x \wedge (y; z) = y \wedge (z; x) = z \wedge (x; y) = 0$$

We define the set of inconsistent triangles explicitly as follows.

- $(\mathbf{g}_i, \mathbf{g}_i, \mathbf{g}_i)$ for $0 \leqslant i \leqslant 2^n$
- $(\mathbf{y}, \mathbf{y}, \mathbf{y})$ for $0 \leqslant i \leqslant 2^n$
- $(\mathbf{g}_i, \mathbf{g}_i, \mathbf{w}_{kj})$ for $0 \leqslant i \leqslant 2^n$ and $0 \leqslant k \leqslant j \leqslant 2^n$
- $(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_k)$ unless $i + k = j$ or $i + k = j$ or $j + k = i$
- $(\mathbf{g}_i, \mathbf{g}_{i+1}, \mathbf{r})$ unless $j = 1$
- $(\mathbf{g}_i, \mathbf{y}, \mathbf{w}_{jk})$ unless $i \in \{j, k\}$

$(\mathbf{g}_i, \mathbf{g}_i, \mathbf{w}_{kj})$ stands for $\mathbf{g}_i \wedge (\mathbf{g}_i; \mathbf{w}_{kj}) = \mathbf{g}_i \wedge (z; \mathbf{g}_i) = \mathbf{w}_{kj} \wedge (\mathbf{g}_i; \mathbf{g}_i) = 0$, and so on.

Lemma 2. *For each $n < \omega$, $\mathfrak{A}_n$ does not belong* $\mathbf{RRA}$. *The $(\vee, ;)$-reduct $\mathfrak{S}_n$ of $\mathfrak{A}_n$ is not representable as well.*

Proof. See [14] to have a proof that $\mathfrak{A}_n \notin \mathbf{RRA}$.

We prove that □

4.1 The finite algebra on finite base for RJSOS (or its failure)

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