The finite base property for subreducts of representable relation algebras

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1 The Relation Algebras Background

We recall the basic definitions and results about relation algebras [10] [15].

Definition 1.
A relation algebra is an algebra $\mathcal{R} = \langle R, 0, 1, +, -, ;, ^{\smile}, \mathbf{1} \rangle$ *such that* $\langle R, 0, 1, +, - \rangle$ *is a Boolean algebra and the following equations hold, for each* $a, b, c \in R$*:*

1. $a; (b; c) = (a; b); c$
2. $(a + b); c = (a; c) + (b; c)$
3. $a; \mathbf{1} = a$
4. $a^{\smile\smile} = a$
5. $(a + b)^{\smile} = a^{\smile} + b^{\smile}$
6. $(a; b)^{\smile} = b^{\smile}; a^{\smile}$
7. $a^{\smile}; (-(a; b)) \leqslant -b$

where $a \leqslant b$ *iff* $a + b = b$. **RA** *denotes the class of all relation algebras.*

A relation algebra is called symmetric, if every element is self-converse. A relation algebra is called integral, if

$$a; b = 0 \Rightarrow a = 0 \text{ or } b = 0.$$

Definition 2. *A proper relation algebra is an algebra* $\mathcal{R} = \langle R, 0, 1, \cup, -, |, ^{\smile}, \mathbf{1} \rangle$ *such that* $R \subseteq \mathcal{P}(W)$*, where* W *is an equivalence relation;* $0 = \varnothing$*;* $1 = W$*;* $\cap$*,* $\cup$*,* $-$ *are set-theoretic intersection, union, and complement respectively;* $|$ *is relation composition,* $^{\smile}$ *is relation converse,* $\mathbf{1}$ *is a diagonal relation restricted to* W*, that is:*

1. $a|b = \{\langle x, z \rangle \mid \exists y \, \langle x, y \rangle \in a \;\&\; \langle y, z \rangle \in b\}$
2. $a^{\smile} = \{\langle x, y \rangle \mid \langle y, x \rangle \in a\}$
3. $\mathbf{1} = \{\langle x, y \rangle \mid x = y\}$

The class of all proper relation algebras is denoted as **PRA**. **Rs** *is the class of all relation set algebras, proper relation algebra with a diagonal subrelation as an identity.* **RRA** *is the class of all representable relation algebras, that is, the closure of* **PRA** *under isomorphic copies. That is,* **RRA** = **IPRA**.

Note that the (quasi)equational theories of those classes coincide, that is

$$\mathbf{IPRA} = \mathbf{RRA} = \mathbf{SPRs}$$

RRA is a variety, but it cannot be defined by any set of first-order formulas. It is also known that **RA** and **RRA** are equational classes, but **RRA** is not finitely axiomatisable.

2 Join-semilattice ordered semigroups

Definition 3. *A join-semilattice ordered semigroup ($\mathbf{OS}^+$) is an algebra $\mathcal{S} = \langle S, ;, + \rangle$ such that $\langle S, ; \rangle$ is a semigroup, $\langle S, + \rangle$ is a join-semilattice and the following equations hold for each $a, b.c \in S$:*

1. $a; (b + c) = (a; b) + (a; c)$
2. $(a + b); c = (a; c) + (b; c)$

This class is clearly a variety since $\mathbf{OS}^+$ has the equational definition so far as $+$ is defined as an associative, idempotent, and commutative operation.

Let A be a set of binary relations on some base set W such that $R = \cup A$ is transitive and $\{x, y \mid xRy\} = W$ as in Definition **??**. A representable join semilattice-ordered semigroup is an algebra isomorphic to some join semilattice-ordered semigroup having the form $\mathcal{A} = \langle A, |, \cup \rangle$ such that ; is a relation composition as above and $\cup$ is the set-theoretic union. If $\mathcal{A}$ is representable, then $\mathcal{A} \in \mathbf{I}(\mathbf{R}(\cup, |))$. Let us recall some of underlying facts about representable join semilattice-ordered semigroups [2]:

Proposition 1.

1. *Let $\mathcal{A} = \langle A, +, ; \rangle$ be a join semilattice-ordered semigroup such that, for all $a, b \in A$:*
   (a) *If $a \nleqslant b$, then there exists an atom $c \leqslant a$ and $c \nleqslant b$.*
   (b) *If $c \leqslant a; b$ and c is an atom, then there exists an atom $a' \leqslant a$ such that $c \leqslant a' \cdot b$.*

   then $\mathcal{A}$ is representable.
2. *Let $\mathcal{A} = \langle A, ; \rangle$ be a posemigroup, then $\mathcal{A}$ is representable and such a representation preserve any existing finite suprema and infima, if*
   (a) *The set of atoms is closed under ;.*
   (b) *$\mathcal{A}$ has enough atoms, that is, if $x \in \mathrm{At}(A)$ and $z, w \in A$, then $x \leqslant z; w$ implies there exist atoms $z_1 \leqslant z$ and $w_1 \leqslant w$ such that $x \leqslant z_1; w_1$. If $z \nleqslant w$, then there exists an atom x such that $x \leqslant z$ and $x \nleqslant w$.*

Recall that a class of structures **K** is called finitely axiomatisable iff both **K** and its complement are closed ultraproducts and isomorphic copies.

It is known that the class of all representable join-semilattice ordered semigroups has no finite axiomatisation [1]. In other words,

Theorem 1. $\mathbf{R}(\cup, |)$ *is not finitely axiomatisable.*

2.1 The rainbow construction

Let us provide a proof of this fact using the rainbow technique [10] to show that the complement of $\mathbf{ROS}^{+}$ is not closed ultraproducts. This construction sometimes exploits the similar construction used by Andréka [2] and by Maddux [14]. We note that representability is not decidable for finite relation algebras [9] and this result has several generalisations [11]. Moreover, representability is undecidable for lattice-ordered semigroups and ordered complemented semigroups [16]. We use (more or less) a standard way of showing that the class of certain reducts of representable relation algebras has no finite axiomatisation, see [13] [8].

First of all, we recall several definitions such as colourings. We provide a sequence of symmetric, integral, finite relation algebras $\{\mathfrak{A}_n\}_{n<\omega}$ such that $\mathfrak{A}_n \notin \mathbf{RRA}$. The statement has been proved by Andreka [2] and reproduced here [5].

Given $n < \omega$, the set of atoms $\mathrm{At}(\mathfrak{A}_n)$ consists of the following elements:

- identity: $\mathbf{1}$, an atom with no colour,
- white: $\mathbf{w}$,
- greens: $\mathbf{g}_i$ for $1 \leqslant i \leqslant n$,
- yellows: $\mathbf{y}_i$ for $1 \leqslant i \leqslant n$,
- reds: $\mathbf{r}_i$ for $1 \leqslant i \leqslant n$,
- blacks: $\mathbf{b}_i$ for $1 \leqslant i \leqslant n$,
- ivory: $\mathbf{i}$.
- orange: $\mathbf{o}$

We have the following steps:

Step 1. Let $\mathcal{A}_n$ be the Boolean algebra presented with the set of generators $\mathrm{At}(\mathfrak{A}_n)$ and the following relations for each $x \in \mathrm{At}(\mathfrak{A}_n)$ and for each $1 \leqslant i \leqslant n$.

1. $\mathbf{w} + \mathbf{g}_i + \mathbf{y}_i = \mathbf{g}_i + \mathbf{y}_i$,
2. $\mathbf{i} + \mathbf{y}_i + \mathbf{r}_i = \mathbf{y}_i + \mathbf{r}_i$.

Step 2. We define S, the set of two element subsets of $\mathcal{A}_n$:

$$S = \{\{\mathbf{w}, \mathbf{r}_1\}\} \cup \{\{\mathbf{g}_i, \mathbf{b}_i\} \mid 1 \leqslant i \leqslant n\} \cup \{\{\mathbf{y}_i, \mathbf{r}_i\} \mid 1 \leqslant i < n\} \cup \{\{\mathbf{y}_n, \mathbf{i}\}\}.$$

Step 3. The operations on $\mathfrak{A}_n$:

1. $1 = \sum \mathrm{At}(\mathcal{A}_n)$,
2. $x = x^{\smile}$,
3. $0; x = 0; x = 0$,
4. $\mathbf{1}; x = \mathbf{1}; x = x$,
5. $x; y = \begin{cases} \mathbf{o}, & \text{if } \{x, y\} \in S \\ 1, & \text{otherwise} \end{cases}$
   unless $x, y \in \{0, \mathbf{1}\}$.

Step 4. Define the following quasi-idenity:

$$q_n = \bigwedge_{1 \leqslant i \leqslant n} ((x \leqslant x_i^{'} + x_i^{''}) \wedge (y \leqslant y_i^{'} + y_i^{''})) \to$$
$$x; y \leqslant x; y_1^{'} + \sum_{1 \leqslant i < n} (x_i^{'}; y_i^{''} + x_i^{''}; y_{i+1}^{'}) + x_n^{'}; y_n^{''} + x_n^{''}; y$$

Lemma 1. *1. q_n is valid in* **RRA** *for each $n < \omega$.*

2. q_n fails in $\mathfrak{A}_n$.

Proof. The valuation ϑ defined as:

1. $\vartheta(x) = \mathbf{w}$
2. $\vartheta(x_i^{'}) = \mathbf{g}_i$
3. $\vartheta(x_i^{''}) = \mathbf{y}_i$
4. $\vartheta(y) = \mathbf{i}$
5. $\vartheta(y_i^{'}) = \mathbf{r}_i$
6. $\vartheta(y_i^{''}) = \mathbf{b}_i$

falsifies q_n in $\mathfrak{A}_n$.

TODO: visualise the reason for non-representability. □

2.2 Networks and games

Definition 4. *Let $\mathcal{A}$ be a relation algebra. A network is a complete directed finite graph with edges labelled by elements of $\mathcal{A}$. Such a graph have the following form. $N = \langle U_N, E_N, l_N \rangle$, where $E_N = U_N \times U_N$ for some finite base set and $l_n : E_N \to \mathcal{A}$ is function mapping each edge to some atom of A. This function obey the following requirements:*

1. *$l_N(x, y) \leqslant \mathbf{1}$ iff $x = y$*
2. *$l_N(x, y); l_N(y, z) \geqslant l_N(x, z)$*

A network is called atomic if all the edges are labelled by atoms.

Given two networks $N = \langle E_N, l_N \rangle$ and $N^{'} = \langle E_{N^{'}}, l_{N^{'}} \rangle$, N is a subnetwork of $N^{'}$ ($N \subseteq N^{'}$, or $N^{'}$ refines N) if $E_N \subseteq E_{N^{'}}$ and for each $x, y \in U_N$, $l_{N^{'}}(x, y) = l_N(x, y)$.

Definition 5. *Let $n < \omega$. We define a game $\mathcal{G}_n(\mathcal{A})$ for two players $\forall$ (Abelard) and $\exists$ (Héloïse). Abelard and Héloïse build a finite chain of networks $N_0 \subseteq N_1 \cdots \subseteq N_n$ as follows. In the first round $\forall$ picks an atom α and $\exists$ plays a network N_0 containing an edge (m_0, n_0) such that $l_n(m_0, n_0) = \alpha$. If $\alpha \leqslant \mathbf{1}$, then $m_0 = n_0$, otherwise $m_0 \neq n_0$. If $m_0 \neq n_0$, the edges (m_0, n_0) and (n_0, m_0) belong to Abelard. Suppose N_{i-1} for $i < n$ has been played, then*

- *$\forall$ chooses an edge $(m, n) \in E_{N_{i-1}}$ and atoms $x, y \in \mathrm{At}(\mathcal{A})$ such that $l_{N_{i-1}}(m, n) \leqslant x; y$.*
- *$\exists$ provides a network $N_i = \langle E_{N_i}, l_{N_i} \rangle \supseteq N_{i-1}$ such that there exists $l \in U_{N_i}$ such that $l_{N_i}(m, l) = x$ and $l_{N_i}(l, n) = y$.*

If $(m,n) \in E_i$ such that $m \neq n$ and $m, n \in U_{N_{i-1}}$, then the owner of this edge is the same as in the previous round. The edges (m,l) and (l,n) and their converses belong to Abelard. The rest of the irreflexive edges belongs to Héloïse. $\exists$ wins a match of the game $\mathcal{G}_n(\mathcal{A})$ if she can provide a network N_i for each move of $\forall$ for each $i \leqslant n$. $\exists$ has a winning strategy if she can win all matches.

This lemma has been proved by Hirsch and Hodkinson here [8]. This lemma provide a criterion of representability for relation algebras.

Lemma 2. *Let $\mathcal{A}$ be an atomic relation algebra. Then $\exists$ has a winning strategy in $\mathcal{G}_n(\mathcal{A})$ for each $n < \omega$ iff $\mathcal{A}$ is elementary equivalent to some completely representable relation algebra. If $\exists$ has a winning strategy, then $\mathcal{A}$ is representable since* **RRA** *is elementary.*

2.3 The ultraproduct

The second is to show that any non-trivial ultraproduct $\prod\limits_{D} \mathfrak{A}_n \in \mathbf{RRA}$, where D is an ultrafilter over $\mathcal{P}(\omega)$. We show that via the rainbow technique. Let us define networks and games according to [8].

Lemma 3. *Any non-trivial ultraproduct of $\{\mathfrak{A}_n\}_{n<\omega}$ is representable, that is, belongs to* **RRA**. *The same statement for non-trivial ultraproduct of reducts $\{\mathfrak{S}_n\}_{n<\omega}$ that belongs to $\mathbf{R}(\cup, |)$.*

According to the following claim, $\exists$ has a winning strategy on cofinitely many algebras that allows her to win a game on the ultraproduct. Thus, according to Lemma 2, the ultraproduct belongs to **RRA**.

Claim 1. *Let $l < \omega$. $\exists$ has a winning strategy for $G_l(\mathfrak{A}_n)$ for cofinitely many algebras beloging to the sequence $\{\mathfrak{A}_n\}_{n<\omega}$.*

Let us consider the following sequence of networks $\{\mathcal{N}_n\}_{n<\omega}$ labelled by atoms of $\mathfrak{A}_n$.

1. $n = 0$, then the $N_0 = \langle U_0, E_0, l_0 \rangle$ has the form:

Claim 2.

2.4 The finite algebra on finite base for $\mathbf{R}(\cup, |)$ (or its failure)

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