

# Notes on filtration of logics containing **K5**

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## 1 Preliminaries

**Definition 1.** An  $n$ -normal modal logic is a set of formulas that contains all Boolean tautologies, formulas  $\Diamond_i p \vee \Diamond_i q \leftrightarrow \Diamond_i(p \vee q)$  and  $\Diamond_i \perp \leftrightarrow \perp$  for  $i \leq n$ , and is closed under modus ponens, substitution, and monotonicity: from  $\varphi \rightarrow \psi$  infer  $\Diamond_i \varphi \rightarrow \Diamond_i \psi$  for  $i \leq n$ .

**Definition 2.** An  $n$ -Kripke model is a triple  $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$ , where  $R_i \subseteq W \times W$ ,  $\vartheta : PV \rightarrow 2^W$ , and the connectives have the following semantics:

1.  $\mathcal{M}, w \models p \Leftrightarrow w \in \vartheta(p)$
2.  $\mathcal{M}, w \models \neg \varphi \Leftrightarrow \mathcal{M}, w \not\models \varphi$
3.  $\mathcal{M}, w \models \varphi \vee \psi \Leftrightarrow \mathcal{M}, w \models \varphi$  or  $\mathcal{M}, w \models \psi$
4.  $\mathcal{M}, w \models \Diamond_i \varphi \Leftrightarrow \exists v \in R_i(w) \mathcal{M}, v \models \varphi$

By **K5** we mean the logic  $\mathbf{K} \oplus A5$ , where  $A5 = \Diamond p \rightarrow \Box \Diamond p$ . It is known that **K5** is the modal logic of all Euclidean frames. A frame is called Euclidean if for each  $x, y, z$ ,  $xRy$  and  $xRz$  implies  $yRz$ .

**Proposition 1.** **K5** proves

1.  $\Box^3 p \leftrightarrow \Box^2 p$
2.  $\Box^2 \Diamond p \leftrightarrow \Box \Diamond p$
3.  $\Box \Diamond \Box p \leftrightarrow \Box \Box p$
4.  $\Box \Diamond^2 p \leftrightarrow \Box \Diamond p$

**Proposition 2.** Let  $\mathcal{M}$  be a **K5** model,  $xRy$  for  $x, y \in W$  then one has

$$\mathcal{M}, x \models \Box \Diamond \varphi \text{ iff } \mathcal{M}, y \models \Box \Diamond \varphi.$$

*Proof.*

1. Suppose  $\mathcal{M}, x \models \Box \Diamond \varphi$ . Then  $\mathcal{M}, y \models \Diamond \varphi$  and  $\mathcal{M}, y \models \Box \Diamond \varphi$
2. Suppose  $\mathcal{M}, y \models \Box \Diamond \varphi$ , then  $\mathcal{M}, x \models \Diamond \Box \Diamond \varphi$ , so  $\mathcal{M}, x \models \Box \Diamond \varphi$ .

□

## 1.1 Filtrations: general definitions

Let  $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$  be a Kripke model and  $\Gamma$  a set of formulas closed under subformulas. An equivalence relation  $\sim$  is set to have a finite index if the quotient set  $W / \sim$  is finite. The equivalence relation  $\sim_\Gamma$  induced by  $\Gamma$  is defined as

$$w \sim_\Gamma v \Leftrightarrow \forall \varphi \in \Gamma (\mathcal{M}, w \models \varphi \Leftrightarrow \mathcal{M}, v \models \varphi).$$

If  $\Gamma$  is finite, then  $\sim_\Gamma$  has a finite index. An equivalence relation  $\sim$  respects  $\sim_\Gamma$ , if  $w \sim v$  implies  $w \sim_\Gamma v$ .

**Definition 3.** Let  $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$  be a Kripke model and  $\Gamma$  be a Sub-closed set formulas. A  $\Gamma$ -filtration of  $\mathcal{M}$  is a model  $\widehat{\mathcal{M}} = \langle \widehat{W}, \widehat{R}_1, \dots, \widehat{R}_n, \widehat{\vartheta} \rangle$  such that:

1.  $\widehat{W} = W / \sim$ , where  $\sim$  is an equivalence relation having a finite index that respects  $\Gamma$
2.  $\widehat{\vartheta}(p) = \{[x]_\sim \mid x \in W \ \& \ x \in \vartheta(p)\}$
3. For each  $i \in I$  one has  $\widehat{R}_i^{\min} \subseteq \widehat{R}_i \subseteq \widehat{R}_i^{\max}$ .  $\widehat{R}_{i,\sim}^{\min}$  is the  $i$ -th minimal filtered relation on  $\widehat{W}$  defined as

$$\widehat{R}_{i,\sim}^{\min} \hat{=} \exists x' \sim x \exists y' \sim y \ x R_i y$$

$\widehat{R}_{\Gamma,i}^{\max}$  is the  $i$ -th maximal filtered relation on  $\widehat{W}$  induced by  $\Gamma$  defined as

$$\widehat{R}_{\Gamma,i}^{\max} \hat{=} \forall \Box_i \varphi \in \Gamma (\mathcal{M}, x \models \Box_i \varphi \Rightarrow \mathcal{M}, y \models \varphi)$$

If  $\Phi$  is finite subset of  $\Gamma$  and  $\sim = \sim_\Phi$ , then  $\widehat{\mathcal{M}}$  is a definable  $\Gamma$ -filtration of  $\mathcal{M}$  through  $\Phi$ . If  $\sim = \sim_\Gamma$ , then such a filtration by means of the definition above is called *strict*.

**Lemma 1.** Let  $\Gamma$  be a finite set of formulas closed under subformulas and  $\widehat{\mathcal{M}}$  a filtration of  $\mathcal{M}$  through  $\Gamma$ , then for each  $x \in W$  and for each  $\varphi \in \Gamma$  one has

$$\mathcal{M}, x \models \varphi \Leftrightarrow \widehat{\mathcal{M}}, \hat{x} \models \varphi$$

**Definition 4.** Let  $\mathbb{F}$  be a class of Kripke frames and  $\Gamma$  a finite set of formulas closed under subformulas. If for every model  $\mathcal{M}$  over  $\mathcal{F} \in \mathbb{F}$  there exists a model that is a  $\Gamma$ -definable filtration of  $\mathcal{M}$ , then  $\mathbb{F}$  admits definable filtration. A class of models  $\mathbb{M}$  admits definable filtration if for every  $\mathcal{M} \in \mathbb{M}$  there exists a model belonging to the same class that is a definable  $\Gamma$ -filtration of  $\mathcal{M}$ .

**Lemma 2.**

1. Let  $\mathcal{L}$  be a complete normal modal logic. If  $\text{Frames}(\mathcal{L})$  admits filtration, then  $\mathcal{L}$  has the finite model property.
2. If the class of models  $\text{Mod}(\mathcal{L})$  admits filtration, then  $\mathcal{L}$  has the finite model property and Kripke complete as well.

## 2 Filtration of Euclidean logics

First of all, let us ensure that a minimal filtration of an Euclidean frame is not necessary Euclidean. Let  $[x] \sim_\Gamma [y]$  and  $[x] \sim_\Gamma [z]$ . Then for some  $x' \in [x]$   $y' \in [y]$ , one has  $x'Ry'$  and  $x''Rz'$  for some  $x'' \in [x]$  and  $z' \in [z]$ . Clearly, we cannot claim that  $x' = x''$  in general. Thus, minimal filtration does not preserve the required property.

**Lemma 3.** *K5 admit filtration.*

*Proof.* Let  $\mathcal{M}$  be a **K5**-model and  $\Gamma_0$  a finite set of formulas closed under subformulas. Let us put  $\Gamma = \Gamma_0 \cup \text{Sub}(\{\Diamond\Box\psi \mid \Box\psi \in \Gamma_0\}) \cup \Psi$ , where  $\Psi = \nabla_1\nabla_2\ldots\nabla_n\Box\psi$  for  $\Box\psi \in \Gamma_0$  and  $\nabla_i \in \{\Diamond, \Box\}$ . By Proposition 1, any element of  $\Phi$  has one of the four forms. Thus,  $W \sim_{\equiv_\Gamma}$  has a finite index. We put  $\hat{R} = R_\Gamma^{\max}$ .  $\square$

**Definition 5.** *A first-order formula is called Horn if it has the following form:*

$$\forall x_1, \dots, x_n (x_{i_1}Rx_{j_1} \wedge \dots \wedge x_{i_s}Rx_{j_s} \rightarrow x_kRx_l)$$

**Definition 6.** *Let  $H$  be a Horn property and  $\langle W, R \rangle$  a Kripke frame. A Horn closure of a binary relation  $R$  is the minimal relation  $R^H$  containing  $R$  and satisfying  $H$ .*

**Lemma 4.**  $R^H = \bigcup_{n < \omega} R_n$  where

1.  $R_0 = R$ .
2.  $R_{n+1} = R_n \cup \{(a, b) \in W \mid \exists \vec{c} \in W \text{ } P(a, b, \vec{c})\}$ , where  $P$  is a premise of  $H$ .

$E$ -closure (an Euclidean Horn closure of a binary relation) has the following equivalent definitions:

**Lemma 5.** *Let  $\mathcal{F} = \langle W, R \rangle$  be a Kripke frame. The following conditions are equivalent:*

1.  $R^E$  is the smallest Euclidean relation containing  $R$ .
2.  $R^E = \bigcup_{i < \omega} R_i$ , where
  - $R_0 = R$
  - $R_{n+1} = R_n \cup (R_n^{-1} \circ R_n)$
3.  $xR^E y$  iff there exists  $n < \omega$  such that either  $xRy$  or  $\exists z_1, \dots, z_n$  with  $z_1Rx$  and  $z_{n-1}Ry$  and for each  $1 < i \leq n$  one has either  $z_{i-1}Rz_i$  or  $z_iRz_{i-1}$ .
4.  $R^E = R \cup \bigcup_{i < \omega} (R^{-1} \circ (R \circ R^{-1})^n \circ R)$ .

*Proof.*

1. (1)  $\Rightarrow$  (2) Let us show that if  $R^E$  is the smallest Euclidean relation containing  $R$ , then  $R^E = \bigcup_{i < \omega} R_i$ . There are two inclusions:

- $R^E \subseteq \bigcup_{i < \omega} R_i$ . Recall that  $R^E$  has the form (?):

$$R^E = \bigcap \{R' \mid R \subseteq R', \forall a, b \in W \text{ } R'(a, b) \Rightarrow \exists x \in W \text{ } R'(x, a) \ \& \ R'(x, b)\}$$

- $\bigcup_{i < \omega} R_i \subseteq R^E$ . Let us show that  $xR_n y$  for each  $n < \omega$  implies  $xR^E y$  by induction on  $n$ .  
 If  $n = 0$ , then  $xRy$ , thus,  $xR^E y$ , since  $R$  is a subrelation of  $R^E$ . Suppose  $n = m + 1$  and  $xR_{m+1}y$ . Let us show that  $xR^E y$ . From  $xR_{m+1}y$ , one has  $(x, y) \in R^n \cup (R_n^{-1} \circ R_n)$ . There are two cases:
  - $xR^n y$ , one needs to merely apply the IH.
  - $xR_n^{-1} \circ R_n y$ . Then  $\exists z \in W$   $xR_n^{-1}z$  &  $zR_n y$ . That is,  $zR_n x$  and  $zR_n y$  for some  $z$ .  $R_n$  is already a subrelation of  $R^E$ . Thus,  $zR^E x$  and  $zR^E y$ . That implies  $xR^E y$ .
- 2. (2)  $\Rightarrow$  (3) Let  $(x, y) \in R_m$ , let us the statement by induction on  $m$ .
  - (a) Suppose  $m = 0$ , then  $xRy$ , and the statement is shown putting  $n = 0$ .
  - (b) Suppose  $m = p + 1$  and  $xR_{p+1}y$ . Assume that either  $xRy$  or  $\exists z_1, \dots, z_p$  with  $z_1Rx$  and  $z_{p-1}Ry$  and for each  $1 < i \leq p$  one has either  $z_{i-1}Rz_i$  or  $z_iRz_{i-1}$ .  
 $xR_{p+1}y$  implies  $(x, y) \in R_p \cup (R_p^{-1} \circ R_p)$ . If  $(x, y) \in R_p$ , then we merely apply the IH.  
 Suppose  $(x, y) \in R_p^{-1} \circ R_p$ , then  $(z, x) \in R_p$  and  $(z, y) \in R_p$ .
- 3. (3)  $\Rightarrow$  (4) Suppose either  $xRy$  or there exist  $n \geq 1$  and  $z_1, \dots, z_n$  with  $z_1Rx$  and  $z_{n-1}Ry$  and for each  $1 < i \leq n$  one has either  $z_{i-1}Rz_i$  or  $z_iRz_{i-1}$ . If  $xRy$ , then we are done. Otherwise there exists  $n \geq 1$  with the condition above. Then  $(x, y) \in R_{n+1}$  that follows from the condition.
- 4. (4)  $\Rightarrow$  (1)

□

**Lemma 6.** Let  $\mathcal{F} = \langle W, R \rangle$  be a Kripke frame. Let us define  $R^E = \bigcup_{i < \omega} R_i$  where:

1.  $R_0 = R$
2.  $R_{n+1} = R_n \cup (R_n^{-1} \circ R_n)$

Then  $R^E$  is Euclidean.

*Proof.* Let  $(x, y), (x, z) \in R^E$ , one needs to show that  $(y, z) \in R^E$ . Clearly that  $(x, y) \in R_i$  and  $(x, z) \in R_j$  for some  $i, j < \omega$ . Thus, we need  $(y, z) \in R_m$  for some  $m$  depending on  $i$  and  $j$ .

Let us consider the following cases:

1.  $i = 0$  and  $j = 0$   
 Suppose  $(x, y), (x, z) \in R_0 = R$ , then  $(y, z) \in R^{-1} \circ R$ . Thus,  $(y, z) \in R_1$
2.  $i = 0$  and  $j = k + 1$   
 Suppose  $(x, y) \in R$  and  $(x, z) \in R_{k+1} = R_k \cup (R_k^{-1} \circ R_k)$ . Clearly that  $(x, y) \in R_{k+1}$  as well. It is obviously that  $(y, z) \in R_{k+2}$  since  $(y, x) \in R_{k+1}^{-1}$  and  $(x, z) \in R_{k+1}$ .
3. The case with  $i = k + 1$  and  $j = 0$  is similar to the previous one.
4. Suppose  $i = m + 1$  and  $j = k + 1$ . That is,  $(x, y) \in R_{m+1} = R_m \cup (R_m^{-1} \circ R_m)$  and  $(x, z) \in R_{k+1} = R_k \cup (R_k^{-1} \circ R_k)$ . Consider the following four subcases:
  - (a) Suppose  $(x, y) \in R_m$  and  $(x, z) \in R_k$  and  $m \leq k$  without loss of generality.  $m \leq k$  implies  $R_m \subseteq R_k$  and  $(x, y) \in R_k$  in particular. Thus,  $(y, z) \in R_k^{-1} \circ R_k$ , so  $(y, z) \in R_{k+1}$ .

(b) The rest of the cases are similar to the first one.

□

**Theorem 1.** *K45 admits strict filtrations.*

*Proof.* Let  $\mathcal{M} = \langle W, R, \vartheta \rangle$  be a transitive Euclidean model and  $\overline{\mathcal{M}} = \langle \overline{W}, \overline{R}, \overline{\vartheta} \rangle$  its minimal filtration through  $\Gamma$ , where  $\Gamma$  is finite and Sub-closed. Let us put  $\hat{R} = \overline{R}^+ \cup \overline{R}^E$ . Let us show that  $\overline{R}^+ \cup \overline{R}^E \subseteq \overline{R}^{max}$ .

That is, if  $\mathcal{M}, y \models \varphi$  for  $\diamond\varphi \in \Gamma$  and  $\hat{x}\hat{R}\hat{y}$ , then  $\mathcal{M}, x \models \diamond\varphi$ .

Let  $\hat{x}\hat{R}\hat{y}$ . Let us consider the case when  $(\hat{x}, \hat{y}) \in \overline{R}^E$ .

1. Suppose  $(\hat{x}, \hat{y}) \in \overline{R}$ , then  $\mathcal{M}, x \models \diamond\varphi$  holds trivially by the definition of the minimal filtration.
2. Suppose the statement holds  $\overline{R}_n$  and  $(\hat{x}, \hat{y}) \in \overline{R}_{n+1} = \overline{R}_n \cup (\overline{R}_n^{-1} \circ \overline{R}_n)$ . We consider the case of  $(\hat{x}, \hat{y}) \in (\overline{R}_n^{-1} \circ \overline{R}_n)$ .

Then there exists  $\hat{z}$  such that  $(\hat{z}, \hat{x}), (\hat{z}, \hat{y}) \in \overline{R}_n$ .

By IH,  $\mathcal{M}, z \models \diamond\varphi$ .

$(\hat{z}, \hat{y}) \in \overline{R}_n$  iff there are  $\hat{u}_1, \dots, \hat{u}_n$  such that

$$\hat{z} \xleftarrow{\hat{R}} \hat{u}_1 \xrightarrow{\hat{R}'} \hat{u}_2 \xrightarrow{\hat{R}'} \dots \xrightarrow{\hat{R}'} \hat{u}_{n-1} \xrightarrow{\hat{R}'} \hat{u}_n \xrightarrow{\hat{R}} \hat{y}$$

where  $\hat{R}'$  is either  $\hat{R}$  or  $\hat{R}^{-1}$ .

As it is known,  $\diamond\varphi \rightarrow \square\varphi \in \mathbf{K45}$ .

$\hat{u}_1\hat{z}$ , that is,  $u'_1 R z'$  for some  $u'_1 \in \hat{u}_1$  and  $z' \in \hat{z}$ . That is,  $\mathcal{M}, u'_1 \models \diamond\varphi$ , so  $\mathcal{M}, u'_1 \models \diamond\varphi$  and  $\overline{\mathcal{M}}, \hat{u}_1 \models \diamond\varphi$ .

We have  $\hat{u}_1\hat{R}'\hat{u}_2$ . Suppose  $\mathcal{M}, u'_1 \models \diamond\varphi$  and  $u''_1 R u'_2$ . We also have  $\mathcal{M}, u'_1 \models \square\varphi$ , thus,  $\mathcal{M}, u'_2 \models \diamond\varphi$ .

Suppose  $\hat{u}_2\hat{R}\hat{u}_1$  and  $u'_2 R u''_1$ , then  $\mathcal{M}, u'_2 \models \diamond\varphi$ .

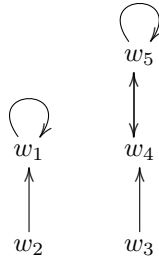
Similarly, we have  $\mathcal{M}, u_i \models \diamond\varphi$  iff  $\mathcal{M}, u_{i+1} \models \diamond\varphi$ , whenever  $\hat{u}_i\hat{R}'\hat{u}_{i+1}$ .

Finally, we have  $\hat{u}_n\hat{R}\hat{x}$ . Thus,  $u'_n R x'$  for some  $u'_n \in \hat{u}_n$  and  $x' \in \hat{x}$ .  $\mathcal{M}, u'_n \models \diamond\varphi$ , so  $\mathcal{M}, u'_n \models \square\varphi$ . Then  $\mathcal{M}, x' \models \diamond\varphi$ . □

**Theorem 2.** *K5 does not admit strict filtrations.*

*Proof.* Let us consider a **K5** model whose Euclidean closure of the minimal filtration does not give us a filtration.

Let us consider a frame called  $\mathcal{F}_{bad}$ . We define this frame with the following graph:



□

Let us define a valuation  $\vartheta$  such that  $\vartheta(p) = \{w_5\}$  and  $\vartheta(q) = \{w_1\}$ . Let us consider a minimal filtration of  $\mathcal{M}_{bad}$  through the Sub-closure of  $\Gamma = \{\neg p, \neg \Diamond p\}$ .

Clearly that  $w_2 \sim_\Gamma w_3$ , since  $\neg p$  and  $\neg \Diamond p$  are true both at  $w_2$  and  $w_3$ .

Moreover,  $R_{min} \cup (R_{min}^{-1} \circ R_{min})$  is not a subset of  $R_{max}$  since  $(\hat{w}_1, \hat{w}_5) \in (R_{min}^{-1} \circ R_{min})$ , but  $\Diamond p$  is not true at  $w_5$ .

Let us also note that strict filtrations of this model is not Euclidean. Suppose by contrary that  $\hat{R}^\mathcal{E}$  is a strict filtration of that model. So  $R_{min}^E \subseteq \hat{R}^\mathcal{E}$ , since  $R_{min}^E$  is the minimal Euclidean relation containing  $R_{min}$ . On the other hand,  $R_{min}^E \not\subseteq R_{max}$ , so is not  $\hat{R}^\mathcal{E}$ .

### 3 Filtration for K4

**Proposition 3.** *Let  $R$  be a binary relation on  $W \neq \emptyset$ . Define  $R^+ = \bigcup_{i < \omega} R_i$*

1.  $R_0 = R$
2.  $R_{n+1} = R_n \circ R$

*Then  $R^+$  is transitive*

**Lemma 7.** *Let  $\mathcal{M} = \langle W, R, \vartheta \rangle$  be a transitive model and  $\overline{\mathcal{M}} = \langle \overline{W}, \overline{R}, \overline{\vartheta} \rangle$  its minimal filtration through a finite Sub-closed set of formulas  $\Theta$ .*

*Then  $\overline{\mathcal{M}}^+ = \langle \overline{W}, (\overline{R})^+, \overline{\vartheta} \rangle$  is a  $\Theta$ -filtration of  $\mathcal{M}$ .*

*Proof.*  $(\overline{R})^+$  obviously contains  $R$ . By the previous proposition,  $(\overline{R})^+$  is transitive. Let us show that  $(\overline{R})^+ \subseteq R_\Theta^{max}$ .

Let  $\hat{x}, \hat{y} \in \overline{W}$  with  $\hat{x}(\overline{R})^+ \hat{y}$  and  $\Box \varphi \in \Theta$  with  $\mathcal{M}, x \models \Box \varphi$ . Let us show that  $\mathcal{M}, y \models \varphi$ .

If  $\hat{x}(\overline{R})^+ \hat{y}$ , then there exist equivalence classes  $\hat{x}_1, \dots, \hat{x}_n$  such that

$$\hat{x} \overline{R} \hat{x}_1 \overline{R} \dots \overline{R} \hat{x}_n \overline{R} \hat{y}$$

$\mathcal{M}, x \models \Box \varphi$  implies  $\mathcal{M}, x \models \Box \Box \varphi$ . Thus,  $\overline{\mathcal{M}}, \hat{x} \models \Box \Box \varphi$ .

$\hat{x} \overline{R} \hat{x}_1$ , so there are  $x_1 \in \hat{x}$  and  $x_2 \in \hat{x}_1$  with  $x_1 R x_2$ . In particular,  $\mathcal{M}, x_2 \models \Box \varphi$ , so  $\overline{\mathcal{M}}, \hat{x}_2 \models \Box \varphi$ , and et cetera.

For each  $i \in \{1, \dots, n\}$  we have  $\mathcal{M}, x_i \models \Box \varphi$  which is shown inductively:

If  $\mathcal{M}, x_i \models \Box \varphi$  for  $x_i \in \hat{x}_i$ , so  $\mathcal{M}, x_i \models \Box \Box \varphi$ , but there exist  $x'_i \in \hat{x}_i$  and  $x_{i+1} \in \hat{x}_{i+1}$ , so  $\mathcal{M}, x_{i+1} \models \Box \varphi$ .

Finally, we have  $\mathcal{M}, x_n \models \Box \varphi$  for  $x_n \in \hat{x}_n$ , but  $\hat{x}_n \overline{R} \hat{y}$ , so  $\mathcal{M}, y' \models \varphi$  for each  $y' \in \hat{y}$ . Thus,  $\varphi$  is true at  $y$  as well.  $\square$

*Proof.* Let  $\hat{x}, \hat{y} \in \overline{W}$  with  $\hat{x}(\overline{R})^+ \hat{y}$  and  $\Box \varphi \in \Theta$  with  $\mathcal{M}, x \models \Box \varphi$ . Let us show that  $\mathcal{M}, y \models \varphi$ .

If  $\hat{x}(\overline{R})^+ \hat{y}$ , then there exist equivalence classes  $\hat{x}_1, \dots, \hat{x}_n$  such that

$$\hat{x} \overline{R} \hat{x}_1 \overline{R} \dots \overline{R} \hat{x}_n \overline{R} \hat{y}$$

Let us show that  $\mathcal{M}, \hat{x}_i \models \Box \varphi$  inductively:

1.  $n = 1$  We have the following sequence:

$$\hat{x} \overline{R} \hat{x}_1 \overline{R} \hat{y}$$

$\hat{x} \overline{R} \hat{x}_1$ , so there are  $x' \in \hat{x}$  and  $x'_1 \in \hat{x}_1$  such that  $x' R x'_1$ .  $\Box \varphi$  is true at  $x'$ , so is  $\Box \Box \varphi$ . Then  $\mathcal{M}, x'_1 \models \Box \varphi$  since  $x'_1 \in R(x')$ . So  $\overline{\mathcal{M}}, \hat{x}_1 \models \Box \varphi$ .

2.  $n = i + 1$  The case is the following:

$$\hat{x}\bar{R}\hat{x}_1\bar{R}\dots\bar{R}\hat{x}_i\bar{R}\hat{x}_{i+1}\bar{R}\hat{y}$$

By IH,  $\Box\varphi$  is true at  $\hat{x}_i$ , so is  $\Box\Box\varphi$ . Hence, we have  $\bar{\mathcal{M}}, \hat{x}_{i+1} \models \Box\varphi$  since  $\hat{x}_i\bar{R}\hat{x}_{i+1}$ .

That is, for each  $0 < n < \omega$ , if we have a sequence of equivalence classes with  $\hat{x}\bar{R}\hat{x}_1\bar{R}\dots\bar{R}\hat{x}_n\bar{R}\hat{y}$  where  $\bar{\mathcal{M}}, \hat{x} \models \Box\varphi$ , then  $\bar{\mathcal{M}}, \hat{x}_n \models \Box\varphi$ .

If  $\hat{x}_n\bar{R}\hat{y}$ , then there are  $x'_n \in \hat{x}_n$  and  $y' \in \hat{y}$  with  $x'_n R y'$ .  $\mathcal{M}, x'_n \models \Box\varphi$  implies  $\mathcal{M}, y' \models \varphi$ , but  $y'$  and  $y$  are  $\Gamma$ -equivalent and  $\varphi \in \Gamma$ , so  $\mathcal{M}, y \models \varphi$ .  $\square$

## 4 Finite “canonical” models

Let  $\mathcal{L}$  be a normal modal logic,  $\mathcal{M}_{\mathcal{L}}$  its canonical model, and  $\varphi$ . Let us put  $\Gamma = \text{Sub}(\varphi) \cup \{\neg\psi \mid \psi \in \text{Sub}(\varphi)\}$ .

A subset  $\Delta \subseteq \Gamma$  is a *finite  $\mathcal{L}$ -theory* if  $\bigwedge \Delta \notin \mathcal{L}$ . A subset  $\Delta$  is maximal, if (the following are obviously equivalent):

1.  $\Delta$  is maximal amongst finite  $\mathcal{L}$ -theories,
2. For each  $\psi \in \text{Sub}(\varphi)$  either  $\psi \in \Delta$  or  $\neg\psi \in \Delta$ .

Every finite  $\mathcal{L}$ -theory is clearly can be extended to some maximal one. It is the finite version of Lindenbaum’s lemma.

**Definition 7.** Let  $\mathcal{L}$  be a normal modal logic and  $\varphi \notin \mathcal{L}$ . A finite “canonical” model is a triple  $\mathcal{M}_{\mathcal{L}}^{\varphi} = \langle W_{\mathcal{L}}^{\varphi}, R_{\mathcal{L}}^{\varphi}, \vartheta_{\mathcal{L}}^{\varphi} \rangle$ , where

1.  $W_{\mathcal{L}}^{\varphi}$  is the set all maximal theories that extend finite  $\mathcal{L}$ -theories
2.  $R_{\mathcal{L}}^{\varphi}$  is a relation such that  $\langle W_{\mathcal{L}}^{\varphi}, R_{\mathcal{L}}^{\varphi} \rangle$  is an  $\mathcal{L}$ -frame and

$$\forall \Box\psi \in \text{Sub}(\varphi) \quad \forall \Delta_1 \in W_{\mathcal{L}}^{\varphi} \quad (\Box\psi \in \Delta_1 \Leftrightarrow \forall \Delta_2 \in R_{\mathcal{L}}^{\varphi}(\Delta_1) \quad \psi \in \Delta_2)$$

3.  $\vartheta_{\mathcal{L}}^{\varphi}(p) = \{\Delta \in W_{\mathcal{L}}^{\varphi} \mid p \in \Delta\}$ .

**Definition 8.** Let  $\varphi$  be a modal formula,  $\mathcal{L}$  a logic that admits strict filtration, and  $\mathcal{M}$  an  $\mathcal{L}$ -model that refutes  $\varphi$ . A filtration diagram of a model  $\widehat{\mathcal{M}}$  (a filtration of  $\mathcal{M}$  through  $\varphi$ ) is a formula  $\text{FDiagram}(\mathcal{M}_{\mathcal{L}}^{\text{Sub}(\varphi)}) = \bigwedge \Gamma \wedge \bigwedge \Delta$ , where

- $\Gamma = \{\psi \in \text{Sub}(\varphi) \mid \exists w \in \widehat{\mathcal{M}} \widehat{M}, \hat{w} \models \psi\}$
- $\Delta = \{\neg\psi \in \text{Sub}(\varphi) \mid \psi \in \text{Sub}(\varphi) \ \& \ \mathcal{M} \models \psi\}$

**Lemma 8.** Let  $\varphi$  be a modal formula,  $\mathcal{L}$  a logic that admits strict filtration, and  $\mathcal{M}$  an  $\mathcal{L}$ -model that refutes  $\varphi$ . Then  $\mathcal{M}_{\mathcal{L}}^{\text{Sub}(\varphi)} \models \text{FDiagram}(\mathcal{M}_{\mathcal{L}}^{\text{Sub}(\varphi)})$

**Definition 9.** Let  $\mathcal{M}$  be a model and  $\Psi$  a set of formulas, then a  $\Psi$ -theory of a model, denoted as  $\text{Th}_{\Psi}$ , is the set  $\text{Th}_{\Psi} = \{\varphi \in \Psi \mid \mathcal{M} \models \varphi\}$ .

**Lemma 9.** Let  $\mathcal{L}$  be a modal logic that admits strict filtrations and  $\varphi \notin \mathcal{L}$ . Let  $\mathcal{M}$  be a model that refutes  $\varphi$  and  $\mathcal{N}$  be a different model such that  $\mathcal{N} \models \mathcal{L}$ . If  $\mathcal{N} \models \text{FDiagram}(\mathcal{M}_{\mathcal{L}}^{\text{Sub}(\varphi)})$ , then  $\text{Th}_{\text{Sub}(\varphi)}(\mathcal{M}_{\mathcal{L}}^{\text{Sub}(\varphi)}) = \text{Th}_{\text{Sub}(\varphi)}(\mathcal{N})$ . In particular,  $\mathcal{N}$  refutes  $\varphi$ .

**Theorem 3.** *Let  $\mathcal{L}$  be a complete normal modal logic, then the following are equivalent:*

1.  $\mathcal{L}$  admits strict filtrations
2. If  $\varphi \notin \mathcal{L}$ , there exists a finite “canonical” model  $\mathcal{M}_{\mathcal{L}}^{\varphi}$  such that  $\mathcal{M}_{\mathcal{L}}^{\varphi} \models \mathcal{L}$  and  $\mathcal{M}_{\mathcal{L}}^{\varphi}$  refutes  $\varphi$ .

*Proof.*

1. ( $\Rightarrow$ )

Let  $\varphi \notin \mathcal{L}$ , there exists a model  $\mathcal{M}$  that refutes  $\varphi$ , so there exists  $w \in \underline{\mathcal{M}}$  such that  $\mathcal{M}, w \models \varphi$ . But  $\mathcal{L}$  admits strict filtration, so there exists a model  $\widehat{\mathcal{M}} = \langle \widehat{W}, \widehat{R}, \widehat{\vartheta} \rangle$ , where  $\widehat{W} = W / \sim_{\varphi}$  and  $\widehat{R}^{min} \subseteq \widehat{R} \subseteq \widehat{R}^{max}$ . In particular,  $\widehat{\mathcal{M}}, \widehat{w} \models \varphi$ .

By lemma above,  $\widehat{\mathcal{M}} \models \text{FDiagram}(\widehat{\mathcal{M}})$

Consider a finite “canonical” model  $\mathcal{M}_{\mathcal{L}}^{\text{Sub}(\varphi)} = \langle W_{\mathcal{L}}^{\text{Sub}(\varphi)}, R_{\mathcal{L}}^{\text{Sub}(\varphi)}, \vartheta_{\mathcal{L}}^{\text{Sub}(\varphi)} \rangle$ . Let us show that  $\mathcal{M}_{\mathcal{L}}^{\text{Sub}(\varphi)}$  refutes  $\varphi$ . For that, we show that  $\mathcal{M}_{\mathcal{L}}^{\text{Sub}(\varphi)} \models \text{FDiagram}(\widehat{\mathcal{M}})$ .

2. ( $\Leftarrow$ )

Let  $\mathcal{M}_{\mathcal{L}}^{\varphi}$  be a finite “canonical” model that refutes  $\varphi$ .

Consider  $W_{\mathcal{L}}^{\varphi} / \sim_{\text{Sub}(\varphi)}$ . Clearly  $W_{\mathcal{L}}^{\varphi} = W_{\mathcal{L}}^{\varphi} / \sim_{\text{Sub}(\varphi)}$  since

$$\Delta_1 \sim_{\text{Sub}(\varphi)} \Delta_2 \text{ iff } \forall \psi \in \text{Sub}(\varphi) \ W_{\mathcal{L}}^{\varphi}, \Delta_1 \models \psi \Leftrightarrow W_{\mathcal{L}}^{\varphi}, \Delta_2 \models \psi \text{ iff } \Delta_1 = \Delta_2.$$

$R_{\mathcal{L}}^{\varphi}$  also satisfies the required definition. Suppose  $\Box \psi \in \varphi$  and  $\mathcal{M}_{\mathcal{L}}^{\varphi}, \Delta_1 \models \Box \psi$  and  $\Delta_1 R_{\mathcal{L}}^{\varphi} \Delta_2$ . Clearly that  $\psi \in \Delta_2$  by the definition of a finite “canonical” model.

Therefore,  $\mathcal{L}$  admits strict filtrations since  $\mathcal{M}_{\mathcal{L}}^{\varphi}$  is already finite. In particular, **K5** fails to have the finite “canonical” model property that follows from the contraposition of this statement and Theorem 2.

□

## References