Varieties of representable relation algebra reducts

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1 Question

Let τ be a subsignature of operations expressible in the signature of RA. $\mathbf{R}(\tau)$ is a variety iff ??.

Investigate discriminators for representable relation algebra reducts.

2 Varieties and discriminators

Standardly, a class of algebras is called *variety*, if it can be determined by some equational theory, or, equivalently, it is closed under homomorphic images, subalgebras and direct products. Given a class of algebras $\mathcal{K}$, $\mathbf{V}(\mathcal{K})$ is a variety generated by $\mathcal{K}$ or, equivalently, $\mathbf{HSP}(\mathcal{K})$, the closure of $\mathcal{K}$ under homomorphic images, subalgebras and direct products.

Let $\{A_i \mid i \in I\}$ be an indexed family of algebras, then a subalgebra $A \subseteq \Pi_{i \in I} A_i$ is a subdirect product if $\pi_i(A) = A_i$. An embedding $\alpha : A \to \Pi_{i \in I} A_i$ is subdirect if $\alpha(A)$ is subdirect product. An algebra is subdirectly irreducible if for every subdirect embedding $\alpha : A \to \Pi_{i \in I} A_i$ there exists $i \in I$ such that $\pi_i \circ \alpha : A \to A_i$ is an isomorphism.

An equivalence relation θ on an algebra A is called *congruence*, if θ respects any operation. $\mathbf{Con}(A)$ is the set of all congruences on A. An algebra is called *simple*, if $\mathbf{Con}(A/\theta) = \{\Delta, \nabla\}$, where Δ and ∇ are trivial congruences. One can obtain a simple algebra by factorising it through the maximal congruence [SB81, Theorem 8].

One can equivalently define subdirectly irreducible algebras using congruences as follows. an algebra is subdirectly irreducible iff either A is trivial or there exists a minimal congruence in $\mathbf{Con}(A) \backslash \{\Delta\}$.

Recall that a Stone space is a compact Hausdorff zero-dimensional topological space. A subdirect product $A \subseteq \Pi_{x \in X} A_x$ over a Stone space X if

1. for all $a, b \in A$ $\{x \in X \mid a(x) = b(x)\}$ is clopen.
2. for all $a, b \in A$ and for all clopen $Y \subseteq X$ $a \restriction_Y \cup a \restriction_{X \backslash Y} \in A$.

A variety $\mathcal{V}$ is *arithmetical*, if it is congruence-permutable and congruence-distributive, or, equivalently, there exists a ternary term p such that:

$$\mathcal{V} \models p(x, y, x) \approx p(x, y, y) \approx p(y, y, x) \approx x \tag{1}$$

A ternary term $t(x, y, z)$ for an algebra A if, for all $a, b, c \in A$:

$$t(a, b, c) = \begin{cases} c & \text{if } a = b \\ a & \text{otherwise} \end{cases} \tag{2}$$

A variety $\mathcal{V}$ is called discriminator if there exists a class $\mathcal{K}$ such that $\mathcal{V} = \mathbf{V}(\mathcal{K})$ and there exists a term $t(x, y, z)$, which is a discriminator term for every member of $\mathcal{K}$. It is known that if an algebra A has a discriminator term, then A is simple [SB81, Lemma 9.2]. Moreover, we have the following property of discriminator terms, see [SB81, Theorem 9.4].

Theorem 1. *Let $t(x, y, z)$ be a discriminator term for every member of a class $\mathcal{K}$:*

1. *$\mathbf{V}(\mathcal{K})$ is an arithmetical variety.*
2. *Every indecomposable member of $\mathbf{V}(\mathcal{K})$ is simple.*
3. *Simple algebras are precisely members of $ISP_U(\mathcal{K}_+)$.*
4. *Every member of $\mathbf{V}(\mathcal{K})$ is isomorphic to a Boolean product of simple algebras.*

3 BAOs, relation algebras and their reducts

3.1 Discriminator varieties of BAOs

Let B be a Boolean algebra, an operator is an n-ary function $f : A^n \to A$ such that, for all $x_1, \dots, x_n, x, y \in B$:

- $f(x_1, \dots, x + y, \dots, x_n) = f(x_1, \dots, x, \dots, x_n) + f(x_1, \dots, y, \dots, x_n)$
- $f(x_1, \dots, 0, \dots, x_n) = 0$

A Boolean algebra with operators is an algebra $M = (B, (f_i)_{i \in I})$, where each f_i is an operator.

In the case of BAOs, one can define discriminator simpler, as an unary term $d(x)$ such that, for all $a \in M$, where M is a BAO:

$$d(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{otherwise} \end{cases} \tag{3}$$

One can characterise discriminator varieties as follows, see [AGM$^+$98, Lemma 2.1]:

Theorem 2. *Let $\mathcal{V}$ be a variety of BAOs and $d(x)$ a unary term, then the following are equivalent:*

1. *d is a discriminator variety.*
2. *The following equations are valid in $\mathcal{V}$:*
   (a) $x \leqslant d(x)$
   (b) $d(d(x)) \leqslant d(x)$
   (c) $d(-d(x)) \leqslant -d(x)$
   (d) $f(x_0, \dots, x_{n-1}) \leqslant d(x_i)$ *for all $n > 0$ and for every operator f of M*

3.2 Relation algebras and their reducts

In this subsection, we consider relation algebras, a kind of BAOs.

Definition 1.
A relation algebra is an algebra $\mathcal{R} = (R, 0, 1, +, -, ;, ^\smile, \mathbf{1}')$ such that $(R, 0, 1, +, -)$ is a Boolean algebra and the following hold:

1. *$(R, ;, \mathbf{1}')$ is a monoid*

2. $(a+b);c = (a;c) + (b;c)$

3. $a^{\smile\smile} = a$

4. $(a+b)^{\smile} = a^{\smile} + b^{\smile}$

5. $(a;b)^{\smile} = b^{\smile};a^{\smile}$

6. $a^{\smile};(-(a;b)) \leqslant -b$

where $a \leqslant b$ *iff* $a + b = b$. **RA** *denotes the class of all relation algebras.*

Definition 2. *A proper relation algebra is an algebra* $\mathcal{R} = (R, \varnothing, W, \cup, -, |, ^{\smile}, \mathbf{1})$ *such that* $R \subseteq \mathcal{P}(W)$*, where* $W \subseteq X \times X$ *is an equivalence relation;* $|$ *is relation composition,* $^{\smile}$ *is relation converse,* **Id** *is a diagonal subset of* W*, that is:*

1. $a|b = \{(x,z) \mid \exists y\ (x,y) \in a\ \&\ (y,z) \in b\}$

2. $a^{\smile} = \{(x,y) \mid (y,x) \in a\}$

3. $\mathbf{Id} = \{(x,y) \mid x = y\}$

The class of all proper relation algebras is denoted as **PRA**. **Rs** *is the class of all relation set algebras, proper relation algebra with a diagonal subrelation as an identity.* **RRA** *is the class of all representable relation algebras, that is, the closure of* **PRA** *under isomorphic copies.*

3.3 Varieties

3.3.1 Boolean algebras with residuated operators

Let $\mathcal{A}_0$ be a Boolean algebra, a unary operation on f is called *residuated* if there exists a *residual* operation g such that for all $a, b \in \mathcal{A}_0$:

$$f(a) \leqslant b \text{ iff } a \leqslant f(b)$$

Equivalently, f is residuated if there exists a conjugate operation h such that for all $a, b \in \mathcal{A}_0$:

$$f(a) \cdot b = 0 \text{ iff } a \wedge h(b) = 0$$

f is *self-conjugate* whenever f is equal to its conjugate operation.

Theorem 3. *Let* f *be a residuated operator on* $\mathcal{A}_0$*, then*

1. f *is normal and completely additive,*

2. f *and* h *are conjugate operations on* $\mathcal{A}_0$ *iff they are normal and the following holds for all* $a, b \in \mathcal{A}_0$

$$f(a) \cdot b \leqslant f(a \cdot h(b)) \text{ and } a \cdot h(b) \leqslant h(f(a) \cdot b)$$

Two n-ary operators f and h are conjugate in the i-th argument if $f_{\vec{a},i}$ is conjugate to $g_{\vec{a},i}$. Let $\mathcal{A}$ be a BAO, $\mathcal{A}$ is called a *Boolean algebra with residuated operators* if for each operator f of $\mathcal{A}$ and for each $i < ar(f)$ there exists an $ar(f)$-ry term t, which is conjugate to f in the i-th argument. **RA** is an example, where $\smile$ is self-conjugate and composition has conjugate terms $x;y^{\smile}$ and $x^{\smile};y$.

An r-algebra is a Boolean algebra with three residuated binary operations $\bullet$, $\rhd$ and $\lhd$, where $\rhd$ and $\lhd$ are right and left conjugates to $\bullet$, that is, the following conditions are equivalent:

1. $(x \bullet y) \cdot z = 0$
2. $(x \rhd z) \cdot y = 0$
3. $(z \lhd y) \cdot x = 0$

3.3.2 Residuated monoids

3.3.3 Positive relation algebras

3.3.4 Domain algebras

3.4 Non-varieties

3.5 Unknown

4 Decidability aspects

4.1 Current results

4.2 Problems

References

[AGM+98] Hajnal Andréka, Steven Givant, Szabolcs Mikulás, István Németi, and András Simon. Notions of density that imply representability in algebraic logic. *Annals of Pure and Applied logic*, 91(2-3):93–190, 1998.

[BP94] WJ Blok and Don Pigozzi. On the structure of varieties with equationally definable principal congruences iii. *Algebra Universalis*, 32(4):545–608, 1994.

[Bur92] Stanley Burris. Discriminator varieties and symbolic computation. *Journal of Symbolic Computation*, 13(2):175–207, 1992.

[HH02] Robin Hirsch and Ian Hodkinson. *Relation algebras by games*. Elsevier, 2002.

[Hir18] Robin Hirsch. Decidability of equational theories for subsignatures of relation algebra. In *International Conference on Relational and Algebraic Methods in Computer Science*, pages 87–96. Springer, 2018.

[HM11] Robin Hirsch and Szabolcs Mikulás. Positive fragments of relevance logic and algebras of binary relations. *The Review of Symbolic Logic*, 4(1):81–105, 2011.

[Jip93] Peter Jipsen. Discriminator varieties of boolean algebras with residuated operators. *Banach Center Publications*, 28(1):239–252, 1993.

[Mik11] Szabolcs Mikulás. On representable ordered residuated semigroups. *Logic Journal of the IGPL-Interest Group in Pure and Applied Logics*, 19(1):233, 2011.

[Pra91] Vaughan Pratt. Action logic and pure induction. In *European Workshop on Logics in Artificial Intelligence*, pages 97–120. Springer, 1991.

[Pra05] Vaughan Pratt. Dynamic algebras as a well-behaved fragment of relation algebras. In *Algebraic Logic and Universal Algebra in Computer Science: Conference, Ames, Iowa, USA June 1–4, 1988 Proceedings*, pages 77–110. Springer, 2005.

[SB81] Hanamantagouda P Sankappanavar and Stanley Burris. A course in universal algebra. *Graduate Texts Math*, 78:56, 1981.

[Vag01] Diego Vaggione. Characterization of discriminator varieties. *Proceedings of the American Mathematical Society*, 129(3):663–666, 2001.