Model-theoretic aspects of relativised cylindric set algebras

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1 Model-theoretic and universal algebraic preliminaries

1.1 Ultraproducts

Here are the required notions and facts from model theory and universal algebra [**?**] [**?**] [**?**].

Let A be a non-empty set, an *ultrafilter* on A is a set of subsets $U \subseteq \mathcal{P}(\mathcal{P})(A)$ such that A is closed under intersections, $\subseteq$-upwardly closed, and either $X \in U$ or $-X \in U$, where $X \subseteq A$. An ultrafilter is called principal if it has the form $\uparrow X = \{Y \in \mathcal{P}(A) \mid X \subseteq Y\}$.

Let $\Omega = \langle \mathrm{Cnst}, \mathrm{Fn}, \mathrm{Pred} \rangle$ be a signature and Λ an index set, and let $\{M_\lambda\}_{\lambda \in \Lambda}$ be an indexed set of Ω structures. The Ω-structure

$$M = \prod_{\lambda \in \Lambda} M_\lambda$$

is called a *product* that defined as follows. Its domain is the Cartesian product of the domains of M_λ. $a \in M$ is a function $\Lambda \to \bigcup_{\lambda \in \Lambda} \mathrm{dom}(M_\lambda)$ such that $a(\lambda) \in M_\lambda$ for each $\lambda \in \Lambda$. Given $\lambda \in \Lambda$ and $a_\lambda \in M_\lambda$, we denote the function mapping λ to a_λ as $\langle a_\lambda \mid \lambda \in \Lambda \rangle$. We define the interpretation of Ω-symbols as

1. If $c \in \mathrm{Cnst}$, then $c^M = \langle c^{M_\lambda} \mid \lambda \in \Lambda \rangle$

2. If $f \in \mathrm{Fn}$ is an n-ary function symbol, then $f^M(\overline{a}) = \langle f^{M_\lambda}(\overline{a}) \mid \lambda \in \Lambda \rangle$, where $\overline{n} \in M^n$

3. If $R \in \mathrm{Pred}$ is an n-ary relation symbol and $\overline{n} \in M^a$, then $R^M(\overline{a}) = \langle R^{M_\lambda}(\overline{a}) \mid \lambda \in \Lambda \rangle$

Given $\lambda \in \Lambda$, we define the λth projection as $\pi_\lambda : M \to M_\lambda$ such that $\pi_\lambda(a) = a(\lambda)$.

Let Λ be an index set and D an ultrafilter on the Boolean algebra $\langle \mathcal{P}(\Lambda), \cup, -, \Lambda, \varnothing \rangle$. Consider the product $M = \prod_{\lambda \in \Lambda} M_\lambda$ of the Ω-structures $\{M_\lambda\}_{\lambda \in \Lambda}$ and the equivalence relation on $\mathrm{dom}(M)$ defined as

$$a_1 \sim a_2 \Leftrightarrow \{\lambda \in \Lambda \mid a_1(\lambda) = a_2(\lambda)\} \in D$$

Let us denote $\mathrm{dom}(M)/\sim$ as U and $[a]_\sim$ as a/D, where $a \in \mathrm{dom}(M)$. We also denote the *ultraproduct* of $\{M_\lambda\}_\lambda$ as $\prod_{\lambda \in \Lambda} M_\lambda/D$, or, for brevity, as $\prod_D M_\lambda$. The Ω-symbols have the following interpretation

1. If $c \in \mathrm{Cnst}$, then $c^U = c^M/D$

2. If $f \in \mathrm{Fn}$ is an n-ary function symbol and $\overline{a} \in M^n$, then $f^U(\overline{a}) = f^M(x) = f^M(\overline{a})/D$

3. If $R \in \mathrm{Fn}$ is an n-ary relation symbol and $\overline{a} \in M^n$, then $U \models R(\overline{a}/D)$ iff $\{\lambda \in \Lambda \mid M_\lambda \models R(\overline{a}(\lambda))\} \in D$

The ultraproduct is principal if D is a principal filter.

Definition 1.

1. Let $\{M_\lambda\}_{\lambda \in \Lambda}$ be a set of Ω-structures such that every M_λ is isomorphic to the single structure M, then their ultraproduct over D is called the ultrapower over D. The denotation is $\prod_D M$ or M^Λ/D.

2. If $\prod_D M \cong N$ for some structure N, then M is an ultraroot of N.

Theorem 1 (Łos)**.** Let $\{M_\lambda\}_{\lambda \in \Lambda}$ be Ω-structures and D an ultrafilter on Λ, and let $U = \prod_D M_\lambda$ be an ultraproduct of $\{M_\lambda\}_{\lambda \in \Lambda}$ over D. For each first-order formula $\varphi(x_1, \ldots, x_n)$ and for each $a_1/D, \ldots, a_n/D \in U$:

$$U \models \varphi(a_1/D, \ldots, a_n/D) \text{ iff } \{\lambda \in \Lambda \mid \varphi(a_1(\lambda), \ldots, a_n(\lambda))\} \in D$$

The Łos has the following helpful corollary:

Corollary 1. Let $\prod_D M$ be an ultrapower of M. For $a \in M$, let us define a function $\overline{a} : a \mapsto a/D$. Then such a map is an elementary embedding of M into $\prod_D M$.

Moreover, any elementary equivalent structures have isomorphic ultrapowers.

Recall that a class of Ω-structures **K** is called *elementary*, if **K** = Mod(T) for some first-order theory **T**. In that case, T is an axiomatisation of **K**.

Theorem 2. Let **K** be a class Ω-structures, **K** is elementary iff **K** is closed under isomorphic copies, ultraroots, and ultrapowers.

1.2   Preliminaries from universal algebra

Definition 2. Let **K** be a class of Ω-structures, then **K** is a variety, if it is defined by some set of equations. The variety generated by **K** is the smallest variety contating **K**. **K** is a quasi-variety, if it is defined by some set of quasi-identities.

Given a class **K** of Ω-structures, then **I(K)**, **S(K)**, **H(K)**, and **P(K)** are the classes of isomorphic copies, algebras isomorphic to subalgebras belonging to **K**, algebras isomorphic to homomorphic images belonging to **K**, and algebras isomorphic to direct products belonging to **K**. We claim that **I(K)** $\subseteq$ **S(K)**. **Up(K)** is the class of algebras isomorphic to ultraproducts belonging to **K**.

Theorem 3. Let **K** be a class of Ω-structures

1. **K** is a variety iff **H(K)**, **S(K)**, **P(K)** $\subseteq$ **K**

2. **HSP(K)** = **H(S(P(K)))** is the smallest variety containg **K**

3. **K** is a quasi-variety iff it is closed under subalgebras, products, and ultraproducts, iff **SPUp(K)** = **K**.

1.3 Subdirect products

Definition 3.

1. Let $\{A\}_{i \in I}$ be Ω-structures, a subdirect product of $\langle A_i \mid i \in I \rangle$ is a subalgebra B of $\prod_{i \in I} A_i$ such that for each $i \in I$, a projection map $\pi_i : B \to A_i$ is a surjection.

2. A subdirect representation of an Ω-structure is an embedding $f : A \to \prod_{i \in I} A_i$ for some I and $\{A_i\}_{i \in I}$ such that $f \circ \pi_i : A \to A_i$ is a surjection.

3. An Ω-structure A is subdirectly irreducible if for every subdirect representation $f : A \to \prod_{i \in I} A_i$ there exists a projection π_i such that $f \circ \pi_i$ is an isomorphism.

4. **Sir(K)** is the class of subdirectly irreducible structures belonging to **K**.

5. A subdirect decomposition of A if there exists a subdirect representation $f : A \to \prod_{i \in I} A_i$ such that every A_i is subdirectly irreducible.

It is known that every Boolean algebra with operators has a subdirect demcomposition. Moreover, that implies:

Theorem 4.

1. If **K** is a variety, then every element of **K** has a subdirect decomposition with some subdirect irreducible elements of **K**.

2. If **K** is a variety and ε is an equation, $\textbf{Sir}(\textbf{K}) \models \varepsilon \Leftrightarrow \textbf{K} \models \varepsilon$.

1.4 Pseudo-elementary classes

2 Cylindric algebras

2.1 (Representable) cylindric algebras and cylindric set algebras

Let α be an ordinal. Let U^α be the set of all functions mapping α to a non-empty set U. We denote $x(i) = x_i$ for $x \in U^\alpha$ and $i < \alpha$.

Definition 4.

1. A subset of U^α is an α-ry relation on U. For $i, j < \alpha$, the i, j-diagonal D_{ij} is the set of all elements of U such that $y_i = y_j$. If $i < \alpha$ and X is an α-ry relation on U, then the i-th cylindrification $C_i X$ is the set of all elements of U that agree with some element of X on each coordinate except the i-th one. To be more precise, $C_i X = \{y \in U^\alpha \mid \exists x \in X \forall i < \alpha \, (i \neq j \Rightarrow y_j = x_j)\}$.

2. A cylindic set algebra of dimension α is an algebra consisting of a set S of α-ry relation on some base set U with the constants and operations $0 = \emptyset$, $1 = U^\alpha$, $\cap$, $-$, the diagonal elements $\{D_{ij}\}_{i,j<\alpha}$, the cylindrifications $\{C\}_{i<\alpha}$. A generalised cylindric set algebra of dimension α is a subdirect of cylindric algebras that have dimension α

3. A cylindric algebra of dimension α is an algebra $\mathcal{C} = \langle \mathcal{B}, \{c_i\}_{i<\alpha}, \{d_{ij}\}_{i,j<\alpha} \rangle$ such that

- $\mathcal{B}$ is a Boolean algebra, for each $i, j < \alpha$ c_i is an operator and $d_{ij} \in \mathcal{B}$
- For each $i < \alpha$, $a \leqslant c_i a$, $c_i(a \wedge c_i b) = c_i a \wedge c_i b$ and $d_{ii} = 1$
- For every $i, j < \alpha$, $c_i c_j a = c_j c_i a$
- If $k \neq i, j < \alpha$, then $d_{ij} = c_k(d_{ij} \wedge d_{jk})$
- If $i \neq j$, then $c_i(d_{ij} \wedge a) \wedge c_i(d_{ij} \wedge -a) = 0$

$\mathbf{CA}_\alpha$ is the class of all cylindric algebras of dimension α

4. An α-dimensional cylindric algebra C is representable, if it is isomorphic to a generalised cylindric set algebra of dimension α. Such is isomorphism is a representation of C. $\mathbf{RCA}_\alpha$ is the class of all representable cylindric algebras that have dimension α.

2.2 Substitution in cylindric algebras

Definition 5. *Given a cylindric algebra of dimension α C, let x be a term of its signature, the substitution operator s_j^i have the following definition:*

$$s_j^i x = \begin{cases} x, \text{if } i = j \\ c_i(d_{ij} \wedge x), \text{otherwise} \end{cases}$$

Proposition 1. *Let α be an ordinal and let $i, j, k, l < \alpha$. The following facts hold in $\mathbf{CA}_\alpha$*

1. $s_j^i x \leqslant c_i x$.

2. $s_j^i(x \wedge y) = s_j^i x \wedge s_j^i y$, $s_j^i(x \vee y) = s_j^i x \vee s_j^i y$, $-s_j^i x = s_j^i(-x)$. *Moreover, s_j^i is completely additive.*

3. $i \neq k, l$ *implies* $s_j^i d_{ik} = d_{jk}$ *and* $s_j^i d_{kl} = d_{kl}$.

4. $d_{jk} \wedge s_j^i = d_{jk} \wedge s_k^i$.

5. $s_j^i c_i x = c_i x$.

6. $k \neq i, j$ *implies* $s_j^i c_i x = c_i s_j^i x$.

7. $c_j s_j^i x = c_i s_i^j x$.

8. $i \neq j$ *implies* $c_i s_j^i x = s_j^i x$.

9. $i \neq k$ *implies* $s_j^i s_k^i = s_k^i x$.

10. *If either* $i \notin \{k, l\}$ *and* $k \notin \{i, j\}$, *or* $j = l$, *then* $s_j^i s_l^k x = s_l^k s_j^i x$.

11. $s_j^i s_i^j x = s_j^i x$.

12. $s_k^i s_i^j x = s_k^i s_k^j x = s_k^j s_j^i x$

3 IG$_\omega$ and ultraproducts

4 IG$_\omega$ is (not) a variety; is (not) (pseudo-)elementary

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