

Note on filtration of logics containing **K5**

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1 Preliminaries

Definition 1. An n -normal modal logic is a set of formulas that contains all Boolean tautologies, formulas $\Diamond_i p \vee \Diamond_i q \leftrightarrow \Diamond_i(p \vee q)$ and $\Diamond_i \perp \leftrightarrow \perp$ for $i \leq n$, and is closed under modus ponens, substitution, and monotonicity: from $\varphi \rightarrow \psi$ infer $\Diamond_i \varphi \rightarrow \Diamond_i \psi$ for $i \leq n$.

Definition 2. An n -Kripke model is a triple $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$, where $R_i \subseteq W \times W$, $\vartheta : PV \rightarrow 2^W$, and the connectives have the following semantics:

1. $\mathcal{M}, w \models p \Leftrightarrow w \in \vartheta(p)$
2. $\mathcal{M}, w \models \varphi \Leftrightarrow \mathcal{M}, w \not\models \neg \varphi$
3. $\mathcal{M}, w \models \varphi \vee \psi \Leftrightarrow \mathcal{M}, w \models \varphi$ or $\mathcal{M}, w \models \psi$
4. $\mathcal{M}, w \models \Diamond_i \varphi \Leftrightarrow \exists v \in R_i(w) \mathcal{M}, v \models \varphi$

By **K5** we mean the logic $\mathbf{K} \oplus A5$, where $A5 = \Diamond p \rightarrow \Box \Diamond p$. It is known that **K5** is the modal logic of all Euclidean frames. A frame is called Euclidean if for each x, y, z , xRy and xRz implies yRz .

Proposition 1. Let $\mathcal{F} = \langle W, R \rangle$ be an Euclidean frame.

1. For each $x, y, z \in W$, xRy and xRz implies either yRz or zRy .
2. $R \subseteq R; R$, that is, R is dense.
3. For each $x \in W$, $R^*(x) = \{x\} \cup R(R(x))$.

Let $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$ be a Kripke model and Γ a set of formulas closed under subformulas. An equivalence relation \sim is set to have a finite index if the quotient set W / \sim is finite. The equivalence relation \sim_Γ induced by Γ is defined as

$$w \sim_\Gamma v \Leftrightarrow \forall \varphi \in \Gamma (\mathcal{M}, w \models \varphi \Leftrightarrow \mathcal{M}, v \models \varphi).$$

If Γ is finite, then \sim_Γ has a finite index. An equivalence relation \sim respects \sim_Γ , if $w \sim v$ implies $w \sim_\Gamma v$.

Definition 3. Let $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$ be a Kripke model and Γ be a Sub-closed set formulas. A Γ -filtration of \mathcal{M} is a model $\widehat{\mathcal{M}} = \langle \widehat{W}, \widehat{R}_1, \dots, \widehat{R}_n, \widehat{\vartheta} \rangle$ such that:

1. $\widehat{W} = W / \sim$, where \sim is an equivalence relation having a finite index that respects Γ
2. $\widehat{\vartheta}(p) = \{[x]_\sim \mid x \in W \text{ \& } x \in \vartheta(p)\}$

3. For each $i \in I$ one has $\hat{R}_i^{\min} \subseteq \hat{R}_i \subseteq \hat{R}_i^{\max}$. $\hat{R}_{i,\sim}^{\min}$ is the i -th minimal filtered relation on \widehat{W} defined as

$$\hat{x}\hat{R}_{i,\sim}^{\min}\hat{y} \Leftrightarrow \exists x' \sim x \exists y' \sim y xR_i y$$

$\hat{R}_{\Gamma,i}^{\max}$ is the i -th maximal filtered relation on \widehat{W} induced by Γ defined as

$$\hat{x}\hat{R}_{\Gamma,i}^{\max}\hat{y} \Leftrightarrow \forall \Box_i \varphi \in \Gamma (\mathcal{M}, x \models \Box_i \varphi \Rightarrow \mathcal{M}, y \models \varphi)$$

If Φ is finite subset of Γ and $\sim = \sim_\Phi$, then \widehat{M} is a definable Γ -filtration of \mathcal{M} through Φ . If $\sim = \sim_\Gamma$, then such a filtration by means of the definition above is called *strict*.

Lemma 1. *Let Γ be a finite set of formulas closed under subformulas and \widehat{M} a filtration of \mathcal{M} through Γ , then for each $x \in W$ and for each $\varphi \in \Gamma$ one has*

$$\mathcal{M}, x \models \varphi \Leftrightarrow \widehat{M}, \hat{x} \models \varphi$$

Definition 4. *Let \mathbb{F} be a class of Kripke frames and Γ a finite set of formulas closed under subformulas. If for every model \mathcal{M} over $\mathcal{F} \in \mathbb{F}$ there exists a model that is a Γ -definable filtration of \mathcal{M} , then \mathbb{F} admits definable filtration. A class of models \mathbb{M} admits definable filtration if for every $\mathcal{M} \in \mathbb{M}$ there exists a model belonging to the same class that is a definable Γ -filtration of \mathcal{M} .*

Lemma 2.

1. *Let \mathcal{L} be a complete normal modal logic. If $\text{Frames}(\mathcal{L})$ admits filtration, then \mathcal{L} has the finite model property.*
2. *If the class of models $\text{Mod}(\mathcal{L})$ admits filtration, then \mathcal{L} has the finite model property and Kripke complete as well.*

2 Filtration of Euclidean logics

First of all, let us ensure that a filtration of an Euclidean frame is not necessary finite. Let $[x] \sim_\Gamma [y]$ and $[x] \sim_\Gamma [z]$. Then for some $x' \in [x]$ $y' \in [y]$, one has $x'Ry'$ and $x''Rz'$ for some $x'' \in [x]$ and $z' \in [z]$. Clearly, we cannot claim that $x' = x''$ in general. Thus, minimal filtration does not preserve the required property.

- 2.1 Clusters for transitive frames
- 2.2 Clusters for transitive Euclidean frames
- 2.3 Clusters for Euclidean frames
- 3 Transitive closure stuff
- 4 PDLisation of Euclidean logics
- 5 Transitive closure and fusion

References

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