Stuff related to tensor products of modal algebras

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1 Preliminaries

The basic definitions according to [1].

Definition 1. *An n-modal algebra is an algebra $\mathcal{B} = \langle B, \vee, \neg, \bot, \Diamond_1, \dots, \Diamond_n \rangle$, where $\mathcal{B} = \langle B, \vee, \neg, \bot \rangle$ is a Boolean algebra and for each $i = 1, \dots, n$ $\Diamond_i$ is an order-preserving unary function obeying:*

1. $\Diamond_i p \vee \Diamond_i q = \Diamond_i (p \vee q)$
2. $\Diamond_i \bot = \bot$

Definition 2. *An n-normal modal logic is a set of formulas containing Boolean tautologies, formulas $\Diamond_i p \vee \Diamond_i q \leftrightarrow \Diamond_i (p \vee q)$ and $\Diamond_i \bot \leftrightarrow \bot$ for $i = 1, \dots, n$, and is closed under* **MP**, **Sub**, *and the monotonicity rule: from $\varphi \to \psi$ infer $\Diamond_i \varphi \to \Diamond_i \psi$.*

Definition 3.

1. *A Kripke n-frame is a structure $\mathcal{F} = \langle W, R_1, \dots, R_n \rangle$, where $R_i \subseteq W \times W$ for each $i = 1, \dots, n$.*
2. *Let $\mathcal{F}$ be a Kripke frame. A Kripke model over $\mathcal{F}$ is a tuple $\mathcal{M} = \langle \mathcal{F}, \vartheta \rangle$, where $\vartheta : \mathrm{PV} \to 2^W$ is a valuation. Modalised formulas have the following semantics:*

$$\mathcal{M}, x \models \Diamond_i \varphi \Leftrightarrow \exists y \in R_i(x) \; \mathcal{M}, y \models \varphi.$$

   A formula φ is true in $\mathcal{M}$ iff $||\varphi||_{\mathcal{M}} = W$. φ is valid in $\mathcal{F} = \langle W, R_1, \dots, R_n \rangle$ iff $||\varphi||_{\mathcal{M}} = W$ for every model $\mathcal{M}$ over $\mathcal{F}$.

3. *The complex algebra of a Kripke frame $\mathcal{F} = \langle W, R_1, \dots, R_n \rangle$ is a complete modal algebra defined as $\mathcal{F}^+ = \langle \mathcal{P}(W), \cup, \neg, \varnothing, R_1^{-1}, \dots, R_n^{-1} \rangle$.*
4. *A general n-frame is a structure $\mathcal{F} = \langle W, R_1, \dots, R_n, \mathcal{A} \rangle$, where $\mathcal{A}$ is a subalgebra of $\langle W, R_1, \dots, R_n \rangle^+$.*

Definition 4. *Let $\mathcal{F}$ be an n-frame, then $\mathrm{Log}(\mathcal{F}) = \{\varphi \mid \mathcal{F} \models \varphi\}$. If $\mathbb{F}$ is a class of n-frames, then $\mathrm{Log}(\mathbb{F}) = \bigcap\limits_{\mathcal{F} \in \mathbb{F}} \mathrm{Log}(\mathcal{F})$. We use the same notation for logics of general frames and modal algebras.*

We discuss the background on products of modal logics and products of Kripke frames [8] [9].

Definition 5. *Let $\mathcal{F}_1 = \langle W_1, R_1^1, \dots, R_1^n \rangle$ be an n-frame and $\mathcal{F}_2 = \langle W_2, R_2^1, \dots, R_2^m \rangle$. The product frame of $\mathcal{F}_1$ and $\mathcal{F}_2$ is an $n + m$-frame of the form*

$$\mathcal{F}_1 \times \mathcal{F}_2 = \langle W_1 \times W_2, R_h^1, \ldots, R_h^n, R_v^1, \ldots, R_v^m \rangle$$

such that for all $u_1, u_2 \in W_1$ *and for all* $v_1, v_2 \in W_2$,

$$\langle u_1, v_1 \rangle R_h^i \langle u_2, v_2 \rangle \text{ iff } u_1 R_1^i u_2 \text{ and } v_1 = v_2 \text{ for } 1 \leqslant i \leqslant n.$$
$$\langle u_1, v_1 \rangle R_v^j \langle u_2, v_2 \rangle \text{ iff } u_1 = u_2 \text{ and } u_1 R_2^j u_2 \text{ for } 1 \leqslant j \leqslant m.$$

Let $\mathcal{L}_1$ be a normal n-modal logic and $\mathcal{L}_2$ a normal m-modal logic, the product of $\mathcal{L}_1$ and $\mathcal{L}_2$ is defined as

$$\mathcal{L}_1 \times \mathcal{L}_2 = \mathrm{Log}(\mathrm{Frames}(\mathcal{L}_1) \times \mathrm{Frames}(\mathcal{L}_2))$$

Proposition 1.

1. *Let* $\mathcal{L}_1$, $\mathcal{L}_2$ *be modal logics, then* $\mathcal{L}_1 * \mathcal{L}_2 \subseteq \mathcal{L}_1 \times \mathcal{L}_2$.
2. *Let* $\mathcal{L}_1$, $\mathcal{L}_2$ *be Kripke complete modal logics, then* $\mathcal{L}_1 \times \mathcal{L}_2 = \mathrm{Log}(\mathrm{Frames}_r(\mathcal{L}_1) \times \mathrm{Frames}_r(\mathcal{L}_2))$, *where* $\mathrm{Frames}_r(\mathcal{L}_i) = \{\mathcal{F} \in \mathrm{Frames}(\mathcal{L}_i) \mid \mathcal{F} \text{ is rooted }\}$ *for* $i = 1, 2$.

1.1 Axiomatising products

The following properties hold for a product frame having the form $\mathcal{F} = \langle W, R_1, \ldots, R_n \rangle$:

1. commutativity 1
2. commutativity 2
3. confluence

2 Tensor products of modal algebras. The basic definitions and results

Definition 6. *A commutative associative algebra*

Definition 7. *Tensor product of them*

Definition 8. *Tensor product of modal algebras*

Definition 9. *Tensor product of general frames*

TODO: describe related underlying results

3 Note on incomplete modal logics

TODO: consider the system containing **GL**, McKinsey, seriality, and linearity.

4 Solution of Problem 1

5 Solution of Problem 4

Definition 10. *A finitely axiomatisable normal modal logic*

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