

# Note on filtration of logics containing **K5**

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## 1 Preliminaries

**Definition 1.** An  $n$ -normal modal logic is a set of formulas that contains all Boolean tautologies, formulas  $\Diamond_i p \vee \Diamond_i q \leftrightarrow \Diamond_i(p \vee q)$  and  $\Diamond_i \perp \leftrightarrow \perp$  for  $i \leq n$ , and is closed under modus ponens, substitution, and monotonicity: from  $\varphi \rightarrow \psi$  infer  $\Diamond_i \varphi \rightarrow \Diamond_i \psi$  for  $i \leq n$ .

**Definition 2.** An  $n$ -Kripke model is a triple  $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$ , where  $R_i \subseteq W \times W$ ,  $\vartheta : PV \rightarrow 2^W$ , and the connectives have the following semantics:

1.  $\mathcal{M}, w \models p \Leftrightarrow w \in \vartheta(p)$
2.  $\mathcal{M}, w \models \varphi \Leftrightarrow \mathcal{M}, w \not\models \neg \varphi$
3.  $\mathcal{M}, w \models \varphi \vee \psi \Leftrightarrow \mathcal{M}, w \models \varphi$  or  $\mathcal{M}, w \models \psi$
4.  $\mathcal{M}, w \models \Diamond_i \varphi \Leftrightarrow \exists v \in R_i(w) \mathcal{M}, v \models \varphi$

By **K5** we mean the logic  $\mathbf{K} \oplus A5$ , where  $A5 = \Diamond p \rightarrow \Box \Diamond p$ . It is known that **K5** is the modal logic of all Euclidean frames. A frame is called Euclidean if for each  $x, y, z$ ,  $xRy$  and  $xRz$  implies  $yRz$ .

**Proposition 1.** Let  $\mathcal{F} = \langle W, R \rangle$  be an Euclidean frame.

1. For each  $x, y, z \in W$ ,  $xRy$  and  $xRz$  implies either  $yRz$  or  $zRy$ .
2.  $R \subseteq R; R$ , that is,  $R$  is dense.
3. For each  $x \in W$ ,  $R^*(x) = \{x\} \cup R(R(x))$ .

Let  $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$  be a Kripke model and  $\Gamma$  a set of formulas closed under subformulas. An equivalence relation  $\sim$  is set to have a finite index if the quotient set  $W / \sim$  is finite. The equivalence relation  $\sim_\Gamma$  induced by  $\Gamma$  is defined as

$$w \sim_\Gamma v \Leftrightarrow \forall \varphi \in \Gamma (\mathcal{M}, w \models \varphi \Leftrightarrow \mathcal{M}, v \models \varphi).$$

If  $\Gamma$  is finite, then  $\sim_\Gamma$  has a finite index. An equivalence relation  $\sim$  respects  $\sim_\Gamma$ , if  $w \sim v$  implies  $w \sim_\Gamma v$ .

**Definition 3.** Let  $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$  be a Kripke model and  $\Gamma$  be a Sub-closed set formulas. A  $\Gamma$ -filtration of  $\mathcal{M}$  is a model  $\widehat{\mathcal{M}} = \langle \widehat{W}, \widehat{R}_1, \dots, \widehat{R}_n, \widehat{\vartheta} \rangle$  such that:

1.  $\widehat{W} = W / \sim$ , where  $\sim$  is an equivalence relation having a finite index that respects  $\Gamma$
2.  $\widehat{\vartheta}(p) = \{[x]_\sim \mid x \in W \text{ \& } x \in \vartheta(p)\}$

3. For each  $i \in I$  one has  $\hat{R}_i^{\min} \subseteq \hat{R}_i \subseteq \hat{R}_i^{\max}$ .  $\hat{R}_{i,\sim}^{\min}$  is the  $i$ -th minimal filtered relation on  $\widehat{W}$  defined as

$$\hat{x}\hat{R}_{i,\sim}^{\min}\hat{y} \Leftrightarrow \exists x' \sim x \exists y' \sim y xR_i y$$

$\hat{R}_{\Gamma,i}^{\max}$  is the  $i$ -th maximal filtered relation on  $\widehat{W}$  induced by  $\Gamma$  defined as

$$\hat{x}\hat{R}_{\Gamma,i}^{\max}\hat{y} \Leftrightarrow \forall \Box_i \varphi \in \Gamma (\mathcal{M}, x \models \Box_i \varphi \Rightarrow \mathcal{M}, y \models \varphi)$$

If  $\Phi$  is finite subset of  $\Gamma$  and  $\sim = \sim_\Phi$ , then  $\widehat{M}$  is a definable  $\Gamma$ -filtration of  $\mathcal{M}$  through  $\Phi$ . If  $\sim = \sim_\Gamma$ , then such a filtration by means of the definition above is called *strict*.

**Lemma 1.** Let  $\Gamma$  be a finite set of formulas closed under subformulas and  $\widehat{M}$  a filtration of  $\mathcal{M}$  through  $\Gamma$ , then for each  $x \in W$  and for each  $\varphi \in \Gamma$  one has

$$\mathcal{M}, x \models \varphi \Leftrightarrow \widehat{M}, \hat{x} \models \varphi$$

**Definition 4.** Let  $\mathbb{F}$  be a class of Kripke frames and  $\Gamma$  a finite set of formulas closed under subformulas. If for every model  $\mathcal{M}$  over  $\mathcal{F} \in \mathbb{F}$  there exists a model that is a  $\Gamma$ -definable filtration of  $\mathcal{M}$ , then  $\mathbb{F}$  admits definable filtration. A class of models  $\mathbb{M}$  admits definable filtration if for every  $\mathcal{M} \in \mathbb{M}$  there exists a model belonging to the same class that is a definable  $\Gamma$ -filtration of  $\mathcal{M}$ .

**Lemma 2.**

1. Let  $\mathcal{L}$  be a complete normal modal logic. If  $\text{Frames}(\mathcal{L})$  admits filtration, then  $\mathcal{L}$  has the finite model property.
2. If the class of models  $\text{Mod}(\mathcal{L})$  admits filtration, then  $\mathcal{L}$  has the finite model property and Kripke complete as well.

## 2 Filtration of Euclidean logics

First of all, let us ensure that a filtration of an Euclidean frame is not necessary finite. Let  $[x] \sim_\Gamma [y]$  and  $[x] \sim_\Gamma [z]$ . Then for some  $x' \in [x]$   $y' \in [y]$ , one has  $x'Ry'$  and  $x''Rz'$  for some  $x'' \in [x]$  and  $z' \in [z]$ . Clearly, we cannot claim that  $x' = x''$  in general. Thus, minimal filtration does not preserve the required property.

## References

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