

Complete representability for canonical extensions of representable cylindric algebras

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1 The problem itself

- Given $\mathcal{C} \in \mathbf{RCA}_\omega$, whether \mathcal{C}^+ has a complete, ω -dimensional representation? The conjecture is yes. [6]
- Whether \mathbf{RCA}_ω is barely canonical. The conjecture is yes.

2 Atomic Representations

A representation of a Boolean algebra \mathcal{B} is an embedding h of \mathcal{B} to some field of sets.

Let $a \in \mathcal{B}$ be an element of a Boolean algebra \mathcal{B} , a is called an atom, if for every $b \in \mathcal{B}$ $b < a$ implies $b = 0$. That is, an atom is a minimal non-zero element. $\text{At}(\mathcal{B})$ is the set of all atoms of \mathcal{B} .

Let \mathcal{B} be a Boolean algebra and \mathcal{F} a field of sets such that $h : \mathcal{B} \rightarrow \mathcal{F}$ is a representation of \mathcal{B} , then \mathcal{B} is a complete representation of \mathcal{B} , if for every $A \subseteq \mathcal{B}$ we have the following whenever ΣA is defined:

$$h(\Sigma A) = \bigcup h[A]$$

A representation h is called atomic, if $x \in h(1)$ there exists $b \in \text{At}(\mathcal{B})$ such that $x \in h(b)$.

Theorem 1. *Let \mathcal{B} be a Boolean algebra, then \mathcal{B} is atomic iff \mathcal{B} is completely representable. See [5, Corollary 6].*

3 BAOs and Duality

By default, we assume that all operators are at most unary. Here is the rigorous definition:

Definition 1.

1. Let $\mathcal{B} = \langle B, +, -, 0, 1 \rangle$ be a Boolean algebra. An operator is a function $\Omega : B \rightarrow B$ satisfying the following conditions:
 - Normality: $\Omega(0) = 0$
 - Additivity: $\Omega(b + b') = \Omega(b) + \Omega(b')$
2. Let I be an index set, a Boolean algebra with operators (BAO) is an algebra $\langle B, +, -, 0, 1, (\Omega_i)_{i \in I} \rangle$ such that $\langle B, +, -, 0, 1 \rangle$ is a Boolean algebra and for each $i \in I$ Ω_i is an operator.

Definition 2. Let $\mathcal{B} = \langle B, +, -, 0, 1, (\Omega_i)_{i \in I} \rangle$ be a BAO, then

1. An operator Ω is completely additive, if for every $X \subseteq B$ such that ΣX is defined, one has

$$\Omega(\Sigma X) = \sum_{x \in X} \Omega(x)$$

2. \mathcal{B} is completely additive, if for each $i \in I$ Ω_i is additive,
3. A class \mathcal{K} of BAOs is completely additive, if every $\mathcal{B} \in \mathcal{K}$ is completely additive.

3.1 Atom structures and canonical extensions

Definition 3. Let I be an index set and $(\Omega_i)_{i \in I}$ a set of function symbols

1. A structure is a relational structure $\mathcal{F} = \langle W, (R_i)_{i \in I} \rangle$ such that R_i is a binary relation symbol for a function symbol $\Omega_{i \in I}$ with the corresponding index,
2. Let \mathcal{B} be an atomic BAO of the signature I , the atom structure of \mathcal{B} , written as $\mathfrak{At}\mathcal{B}$, is a structure $\langle \text{At}(\mathcal{B}), (R_i)_{i \in I} \rangle$ such that for all $a, b \in \text{At}(\mathcal{B})$ and for all $i \in I$

$$\mathfrak{At}\mathcal{B} \models R_i(a, b) \text{ iff } \mathcal{B} \models a \leq \Omega_i(b)$$

3. Let $\mathcal{F} = \langle W, (R_i)_{i \in I} \rangle$ be an atom structure, the complex algebra of \mathcal{F} , written as $\mathfrak{Cm}\mathcal{F}$, is a BAO $\langle \mathcal{P}(W), \cup, -, \emptyset, W, (\Omega_{R_i})_{i \in I} \rangle$ such that for all $X \subseteq W$ and for each $i \in I$:

$$\Omega_{R_i}(X) = \{a \in W \mid \exists b \in X \mathcal{F} \models R_i(a, b)\}$$

Definition 4. Let $\mathcal{F} = \langle W, (R_i)_{i \in I} \rangle$ and $\mathcal{F}' = \langle W', (R'_i)_{i \in I} \rangle$, then a function $f : \mathcal{F} \rightarrow \mathcal{F}'$ is a bounded morphism, if the following holds:

1. $x R_i y$ implies $f(x) R'_i f(y)$;
2. $f(x) R'_i z$, then there exists $y \in W$ such that $x R_i y$ and $f(y) = z$.

A bounded morphism $f : \mathcal{F} \rightarrow \mathcal{F}'$ is a p -morphism, if f is onto. $\mathcal{F} \twoheadrightarrow \mathcal{F}'$ iff there exists a p -morphism from \mathcal{F} onto \mathcal{F}' , or \mathcal{F}' is a p -morphic image of \mathcal{F} .

Definition 5. Let $\mathcal{F} = \langle W, (R_i)_{i \in I} \rangle$ is an inner substructure¹ of $\mathcal{F}' = \langle W', (R'_i)_{i \in I} \rangle$, if $W \subseteq W'$ and the embedding $\mathcal{F} \hookrightarrow \mathcal{F}'$ is a bounded morphism. Let \mathbb{F} be a class atom structures, then $\mathbb{S}(\mathbb{F})$ is the closure of \mathbb{F} under generated subframes.

Let \mathbb{F} be a class of structures, define:

1. $\mathfrak{Cm}(\mathbb{F}) = \{\mathcal{B} \mid \mathcal{B} \cong \mathfrak{Cm}(\mathcal{F}) \text{ for some } \mathcal{F} \in \mathbb{F}\}$.
2. $\mathbf{Up}(\mathbb{F})$ is the class of structures isomorphic to disjoint unions of elements of \mathbb{F} .
3. $\mathbf{S}(\mathbb{F})$ is the closure of \mathbb{F} under inner substructures.

Let A be a non-empty subset of a Boolean algebra \mathcal{B} , A is a *filter*, if A is closed under finite infima and it is upward closed. A is an *ultrafilter*, if it has no non-trivial extensions. That is, if $A \subseteq A'$, then $A' = \mathcal{B}$. This is a well-known fact that every filter can be extended to a maximal one using Zorn's lemma.

The following definition is due to, for example, [10, Definition 5.40].

¹Or alternatively, a generated subframe

Definition 6. Let $\mathcal{B} = \langle B, +, -, 0, 1, (\Omega_i)_{i \in I} \rangle$ be a BAO and $\mathbf{Spec}(\mathcal{B})$ the set of its ultrafilters. The ultrafilter frame of \mathcal{B} (or the canonical frame) is a relational structure $\mathcal{F}_{\mathcal{B}} = \langle \mathbf{Spec}(\mathcal{B}), R_{\Omega_i} \rangle$ such that for all ultrafilters U_1, U_2 one has

$$\mathbf{Spec}(\mathcal{B}) \models R_{\Omega_i}(U_1, U_2) \text{ iff } \{\Omega_i(b) \mid b \in U_1\} \subseteq U_2.$$

Given \mathcal{B} be a BAO, we denoted as \mathcal{B}^+ as the complex algebra of the canonical frame $\mathfrak{Cm}(\mathcal{F}_{\mathcal{B}})$, that is, the canonical extension of \mathcal{B} . A class of BAOs \mathbf{K} is canonical, if it is closed under canonical extensions. That is, $\mathcal{B}^+ \in \mathbf{K}$ whenever $\mathcal{B} \in \mathbf{K}$.

Theorem 2. Let \mathcal{A}, \mathcal{B} be BAOs,

1. There exists $\iota : \mathcal{A} \hookrightarrow \mathcal{A}^+$ such that $\iota : a \mapsto \{\gamma \in \mathbf{Spec}(\mathcal{A}) \mid a \in \gamma\}$.
2. $i : \mathcal{A} \hookrightarrow \mathcal{B}$ implies $i^+ : \mathcal{A}^+ \hookrightarrow \mathcal{B}^+$

4 Representable cylindric algebras

Let α be an ordinal. Denote $\{f \mid f\alpha \rightarrow U\}$ as ${}^\alpha U$. x_i stands for $x(i)$, where $x \in {}^\alpha U$ and $i < \alpha$.

A subset of ${}^\alpha U$ is an α -ry relation on U . For $i, j < \alpha$, the i, j -diagonal D_{ij} is the set of all elements of ${}^\alpha U$ such that $y_i = y_j$.

If $i < \alpha$ and X is an α -ry relation on U , then the i -th cylindrification $C_i X$ is the set of all elements of U that agree with some element of X on each coordinate except, perhaps, the i -th one. To be more precise,

$$C_i X = \{y \in {}^\alpha U \mid \exists x \in X \forall i < \alpha (i \neq j \Rightarrow y_j = x_j)\}.$$

We define the following equivalence relation for $i < \alpha$ and $x, y \in {}^\alpha U$:

$$x \equiv_i y \Leftrightarrow \forall j \in \alpha (i \neq j \Rightarrow x(j) = y(j))$$

Then one may reformulate the definition of the i -th cylindrification in the following way:

$$C_i X = \{y \in {}^\alpha U \mid \exists x \in X \ x \equiv_i y\}$$

According to this version of the definition, one may think of the cylindrification as an **S5** modal operator.

Definition 7. A cylindric set algebra of dimension α is an algebra consisting of a set S of α -ry relation on some base set U with the constants and operations $0 = \emptyset$, $1 = {}^\alpha U$, \cap , $-$, the diagonal elements $(D_{ij})_{i, j < \alpha}$, the cylindrifications $(C_i)_{i < \alpha}$. A generalised cylindric set algebra of dimension α is a subdirect of cylindric algebras that have dimension α . \mathbf{Cs}_α denotes the class of all cylindric set algebras of dimension α .

Definition 8. A cylindric algebra of dimension α is an algebra $\mathcal{C} = \langle \mathcal{B}, \{c_i\}_{i < \alpha}, \{d_{ij}\}_{i, j < \alpha} \rangle$ such that

- \mathcal{B} is a Boolean algebra, for each $i, j < \alpha$ c_i is an operator and $d_{ij} \in \mathcal{B}$
- For each $i < \alpha$, $a \leq c_i a$, $c_i(a \cdot c_i b) = c_i a \cdot c_i b$ and $d_{ii} = 1$
- For every $i, j < \alpha$, $c_i c_j a = c_j c_i a$
- If $k \neq i, j < \alpha$, then $d_{ij} = c_k(d_{ij} \cdot d_{jk})$

- If $i \neq j$, then $c_i(d_{ij} \cdot a) \cdot c_i(d_{ij} \cdot -a) = 0$

\mathbf{CA}_α is the class of all cylindric algebras of dimension α .

One may define a representation of a cylindric algebra explicitly in the following way:

Definition 9. Let \mathcal{A} be a cylindric algebra of dimension α . A representation of \mathcal{A} over the non-empty domain X is a map $f : \mathcal{A} \hookrightarrow 2^{\alpha U}$ such that:

1. $f(1) = \bigcup_{i \in I} {}^\alpha X_i$ for some disjoint family $\{X_i\}_{i \in I}$ where each $X_i \subseteq X$
2. $h : \mathcal{A} \rightarrow 2^{f(1)}$ is a representation of a Boolean reduct
3. for all $\lambda, \eta < \alpha$, $x \in h(d_{\lambda\eta})$ iff $x_\lambda = x_\eta$
4. for all $\lambda < \alpha$ and $a \in \mathcal{A}$, $x \in h(c_\lambda(a))$ iff there is $y \in X$ such that $x[\lambda \mapsto y] \in h(a)$

An α -dimensional cylindric algebra C is representable, if there exists a representation of h . \mathbf{RCA}_α is the class of all representable cylindric algebras that have dimension α . In particular, we are interested in the case $\alpha = \omega$.

It is well known that \mathbf{RCA}_α is a variety, \mathbf{RCA}_α ($\alpha \leq 2$) is finitely axiomatisable and \mathbf{RCA}_α ($2 < \alpha < \omega$) has no finite axiomatisation, see [4].

Let $\mathcal{A} \in \mathbf{CA}_\omega$, then \mathcal{A} has a *complete representation*, if its representation preserves all existing suprema. In other words, \mathcal{A} is *completely representable*.

5 \mathbf{RCA}_ω and canonicity

The following definition of an ω -frame is due to [9].

Definition 10. A cylindric ω -frame is a structure $\mathcal{F} = \langle W, (R_i)_{i < \omega}, (E_{ij})_{i, j < \omega} \rangle$ where $(R_i)_{i < \omega}$ are binary relations and $(E_{ij})_{i, j < \omega}$ are unary relations such that, for all $i, j, k < \omega$:

1. Every R_i is an equivalence relation on W ,
2. $R_i \circ R_j = R_j \circ R_i$, that is, the set $(R_i)_{i < \omega}$ forms a commutative semigroup under composition.
3. For all $x \in W$, $E_{ii}(x)$ holds.
4. For all $x, y, z \in W$, $xR_iy \ \& \ E_{ij}(y) \ \& \ xR_iz \ \& \ E_{ij}(y)$ implies $y = z$.
5. For all $x \in W$, $E_{ij}(x)$ iff there exists $y \in W$ such that xR_ky , $E_{ik}(y)$, and $E_{kj}(y)$.

\mathbf{Ca}_ω is the class of all ω -frames.

If $\mathcal{F} \in \mathbf{Ca}_\omega$ and $x \in \mathcal{F}$, then \mathcal{F}^x is a generated subframe generated by x , which is defined standardly. Generally, \mathcal{F}_1 is a generated subframe of \mathcal{F}_2 , if $\underline{\mathcal{F}_1} \subseteq \underline{\mathcal{F}_2}$ and $\underline{\mathcal{F}_1}$ is closed under R_{i2} equivalences for every $i < \omega$. That is:

For all $i < \omega$ and $x \in \mathcal{F}_1$, we have $R_{i2}(x) \subseteq \mathcal{F}_1$ and, thus, $R_{i1}(x) = R_{i2}(x)$.

We have the following connection between ω -frames and their generated subframes, which is standard for modal logic:

Proposition 1. Let $\mathcal{F} \in \mathbf{Ca}_\omega$, then

1. $\mathcal{F} = \coprod_{x \in \mathcal{F}} \mathcal{F}^x$,
2. $\mathfrak{Cm}(\mathcal{F}) \cong \prod_{x \in \mathcal{F}} \mathfrak{Cm}(\mathcal{F}^x)$,
3. $\mathfrak{Cm}(\mathcal{F}^x)$ is subdirectly irreducible.

It is known that \mathcal{Ca}_ω forms an elementary class, since one can express the conditions of an ω -frame with the first-order language.

The following fact is by Venema, see [9, Proposition 2.1.5]:

Proposition 2. *An ω -frame \mathcal{F} is cylindric iff $\mathfrak{Cm}(\mathcal{F})$ is a cylindric algebra of dimension ω .*

A cylindric ω -frame \mathcal{F} is completely representable, if $\mathfrak{Cm}(\mathcal{F})$ is completely representable as a cylindric algebra of dimension ω .

We are interested in the special case of cylindric ω -frames called Cartesian structure of dimension ω . To be more precise:

Definition 11. *Let U be a set and $V \subseteq {}^\omega U$ be a non-empty subset of the full Cartesian space of dimension ω , then an α -dimension Cartesian structure generated by V is an ω -frame $\mathfrak{S}(V) = \langle V, (R_i)_{i < \omega}, (E_{ij})_{i, j < \omega} \rangle$ such that:*

1. $R_i = \{(w, v) \mid w, v \in V, w_k = v_k, k < \omega, i \neq k\}$
2. $E_{ij} = \{w \in V \mid w_i = w_j\}$

$\mathfrak{S}({}^\omega U)$ is the full ω -dimensional Cartesian structure. \mathcal{Fct}_ω is the class of all full ω -dimensional Cartesian structures.

Clearly $\mathcal{Fct}_\omega \subseteq \mathcal{Ca}_\omega$.

We have the following connection between \mathbf{RCA}_ω , \mathbf{IGs}_ω , and complex algebras of full Cartesian structures:

$$\mathbf{RCA}_\omega = \mathbf{IGs}_\omega = \mathbf{S\mathfrak{Cm}Ud\mathcal{Fct}_\omega} = \mathbf{SP\mathfrak{Cm}\mathcal{Fct}_\omega}.$$

This follows from the fact that $\mathbf{Cs}_\omega = \mathfrak{Cm}\mathcal{Fct}_\omega$. Every generalised cylindric set algebra is a subdirect product of cylindric set algebras, thus, a generalised cylindric set algebra is a complex algebra of disjoint union of some full Cartesian spaces. But \mathbf{RCA}_ω is the closure of \mathbf{Cs}_ω under isomorphism.

Definition 12. *The weak Cartesian space with base U and dimension ω determined by $x \in {}^\omega U$ is the set:*

$${}^\omega U^{(x)} = \{y \in {}^\omega U \mid |\{k < \omega \mid x_k \neq y_k\}| < \aleph_0\}$$

$\mathfrak{S}({}^\omega U^{(x)})$ is a weak Cartesian structure of dimension ω . \mathcal{Wct}_ω is the class of all weak Cartesian structure of dimension ω up to isomorphism.

Note that we have $\mathcal{Wct}_\omega \subseteq \mathcal{Ca}_\omega$.

Every cylindric set algebra is a subalgebra of some complex algebra induced by an ω -dimensional Cartesian structure. In other words,

Lemma 1. $\mathbf{ICs}_\omega = \mathbf{S\mathfrak{Cm}\mathcal{Fct}_\omega}$.

In this section, we reproduce the results related to characterisation \mathbf{RCA}_ω . The following results are due to Goldblatt [3]. This denotes that a cylindric algebra of dimension algebra is representable iff it is isomorphic to a subalgebra of the complex algebra of disjoint sum of some full ω -dimensional Cartesian structure. Assuming the duality, this is equivalent to the standard definition of representability formulated in terms of subalgebras of subdirect products.

Lemma 2. $\mathbf{RCA}_\omega = \mathbf{S}\mathbf{CmS}\mathbf{Ud}\mathbf{Fct}_\omega = \mathbf{S}\mathbf{CmS}\mathbf{Ud}\mathbf{Wct}_\omega = \mathbf{IGws}_\omega$

Proof. □

Here we use the following fact related to canonical varieties generated by some class of complex algebras. Let \mathbf{K} be an elementary class of relational structures, then:

If \mathbf{K} is closed under p -morphic images, generated subframes, and disjoint unions,
then $\mathbf{S}\mathbf{Cm}\mathbf{K}$ is a canonical variety.

One may think of this fact a more abstract version of Fine's theorem which claims that every elementary modal logic is canonical [2]. This version denotes the same fact, but it is formulated in terms of varieties BAOs generated by complex algebras of some atom structures. We provide a more precise formulation of the fact above.

Let \mathbf{K} be a class of frames, denote the closure of \mathbf{K} under ultraproducts as \mathbf{PuK} .

Proposition 3. *Let \mathbf{K} be a class of frames, then $\mathbf{PuK} \subseteq \mathbf{HSUdK}$ implies that $\mathbf{S}\mathbf{CmS}\mathbf{UdK}$ is a canonical variety.*

This is a specialised version of [3, Theorem 4.4] formulated for dimension ω .

Theorem 3. \mathbf{RCA}_ω is a canonical variety.

Proof. We have $\mathbf{RCA}_\omega = \mathbf{S}\mathbf{CmS}\mathbf{Ud}\mathbf{Fct}_\omega$. That's enough to show that $\mathbf{PuFct}_\omega \subseteq \mathbf{HSUd}\mathbf{Fct}_\omega$. For that, we need the following claim:

Claim 1. $\mathbf{PuFct}_\omega \subseteq$

□

6 Canonicity of \mathbf{RCA}_n for finite n

In this section we consider classes \mathbf{RCA}_n , where $n < \omega$ is finite.

We provide the complete proof of the following theorem [7, Theorem 3.4.3].

Theorem 4. *Let $\mathcal{A} \in \mathbf{CA}_n$, then \mathcal{A} is representable iff \mathcal{A}^+ is completely representable.*

For that we need such model theoretic notions as saturation and types, see [8, Section 6.3].

Definition 13. *Let \mathcal{M} be a first-order structure of a signature L and $S \subseteq \mathcal{M}$. Let $L(S)$ be an extension of L with copies of elements from S as additional constants. We assume that $\text{Cnst}(L)$ and S are disjoint.*

1. *Let $n < \omega$, an n -type over S is a set \mathcal{T} of $L(S)$ formulas $A(\bar{x})$, where \bar{x} is a fixed n -tuple of elements from S . Notation: $\mathcal{T}(\bar{x})$. A type is an n -type for some $n < \omega$.*
2. *An n -type $\mathcal{T}(\bar{x})$ is realised in \mathcal{M} , if there exists $\bar{m} \in \mathcal{M}^n$ such that $\mathcal{M} \models A(\bar{m})$ for every $A \in \mathcal{T}(\bar{x})$. \mathcal{M} omits $\mathcal{T}(\bar{x})$, if $\mathcal{T}(\bar{x})$ is not realised in \mathcal{M} .*

3. $\mathcal{T}(\bar{x})$ is finitely satisfied in \mathcal{M} , if every finite subtype $\mathcal{T}_0(\bar{x}) \subseteq \mathcal{T}(\bar{x})$ is realised in \mathcal{M} . We can reformulate that as $\mathcal{M} \models \exists \bar{a} \bigwedge_{A \in \mathcal{T}_0} A(\bar{a})$.
4. Let T be a theory, then a type \mathcal{T} over the empty set of constants is T -consistent, if there exists a model $\mathcal{M} \models T$ such that \mathcal{T} is finitely satisfied in \mathcal{M} .
5. Let κ be a cardinal, then \mathcal{M} is κ -saturated, if for every $S \subseteq \mathcal{M}$ with $|S| < \kappa$ every finitely satisfied 1-type \mathcal{T} is realised in \mathcal{M} .

By default, a saturated model is an ω -saturated model for us.

The useful facts, they are from [1] and [8]:

Fact 1. Let \mathcal{M} be an FO-structure and κ a cardinal, then:

1. \mathcal{M} is κ -saturated, iff every finitely satisfiable α -type (an arbitrary $\alpha \leq \kappa$) with fewer than κ parameters is realised in \mathcal{M} .
2. If \mathcal{M} is κ -saturated, then \mathcal{M} is λ -saturated for every $\lambda < \kappa$.
3. Every consistent theory has a κ -saturated model and every model has an elementary κ -saturated extension.
4. Let $(\mathcal{M}_i)_{i < \omega}$ a family of structures of the (at most) countable signature and D a non-principal ultrafilter over ω , then $\Pi_D \mathcal{M}_i$ is ω_1 -saturated.

6.1 Proof of Theorem 4

Let $\mathcal{A} \in \mathbf{CA}_n$, then if \mathcal{A} is completely representable, then h , a complete representation of \mathcal{A} , is atomic. That is, $(a_1, \dots, a_n) \in h(1)$, then $(a_1, \dots, a_n) \in h(y)$ for some $y \in \text{At}(\mathcal{A})$.

Definition 14. Let \mathcal{A} be a cylindric algebra of dimension $n < \omega$. $L(\mathcal{A})$ is the first-order language that consists of equality plus n -ary predicate letters $(R_a^n)_{a \in \mathcal{A}}$. The $L(\mathcal{A})$ -theory $T_{\mathcal{A}}$ consists of the following sentences:

1. $A_+(a, b, c) := \forall x_1, \dots, x_n (R_a(x_1, \dots, x_n) \leftrightarrow R_b(x_1, \dots, x_n) \vee R_c(x_1, \dots, x_n))$. Informally, that means $\mathcal{A} \models a = b + c$.
2. $A_-(a, b) := \forall x_1, \dots, x_n (R_a(x_1, \dots, x_n) \leftrightarrow \neg R_b(x_1, \dots, x_n))$. That is, $\mathcal{A} \models a = -b$.
3. $A_{\neq 0}(a) := \exists x_1, \dots, x_n R_a(x_1, \dots, x_n)$. That is, $\mathcal{A} \models a \neq 0$.
4. $A_{c_i}(a) := \forall x_1, \dots, x_n (R_{c_i a}(x_1, \dots, x_n) \leftrightarrow \exists y_1, \dots, y_n (R_a(y_1, \dots, y_n) \wedge x_i = y_j))$, for $i < n$ and $j < n$ such that $i \neq j$. Informally, $\mathcal{A} \models c_i a = 1$.
5. $A_{d_{ij}} := \forall x_1, \dots, x_n (R_{d_{ij}}(x_1, \dots, x_n) \leftrightarrow x_i = x_j)$, for $i, j < n$.

In fact, we need to show the following implication:

If \mathcal{A} is representable, then A^+ is completely representable.

Assume that \mathcal{A} is representable, then the theory $T(\mathcal{A})$ is consistent, then it has an ω -saturated model \mathcal{M} by Fact 3. We have the following claim:

Claim 2. The set $U_{x_1, \dots, x_n} = \{a \in \mathcal{A} \mid \mathcal{M} \models R_a(x_1, \dots, x_n)\}$ is an ultrafilter of \mathcal{A} , for $x_1, \dots, x_n \in \mathcal{M}$ with $R_1(x_1, \dots, x_n)$.

Those U_{x_1, \dots, x_n} 's allow us to represent atoms of \mathcal{A}^+ .

We define a representation of \mathcal{A}^+ as a map $h : \mathcal{A}^+ \rightarrow 2^{\mathcal{M}^n}$ such that:

$$h : S \mapsto \{(x_1, \dots, x_n) \in 1^{\mathcal{M}} \mid U_{x_1, \dots, x_n} \in S\}, \text{ for } S \in \text{Spec}(\mathcal{A}).$$

Claim 3. Let $A_1, A_2 \in \text{Spec}(\mathcal{A})$

1. $h(0^{\mathcal{A}^+}) = \emptyset$
2. $h(-A_1) = -h(A_1)$
3. $h(1^{\mathcal{A}^+}) = 1^{\mathcal{M}}$
4. If $S \subseteq \text{Spec}(\mathcal{A})$, then $h(\bigcup S) = \bigcup_{U \in S} h(U)$

In particular, h is a Boolean homomorphism.

Proof.

1. $h(0^{\mathcal{A}^+}) = h(\emptyset) = \emptyset$.
2. From the definition of h .
3. $h(-A_1) = -h(A_1)$

Let $x_1, \dots, x_n \in 1^{\mathcal{M}}$, then we have:

$$(x_1, \dots, x_n) \in h(-A_1) \text{ iff } U_{x_1, \dots, x_n} \in -A_1 \text{ iff } U_{x_1, \dots, x_n} \notin A_1 \text{ iff } (x_1, \dots, x_n) \notin h(A_1)$$

4. Let $S = \bigcup_{i \in I} S_i$, where $S_i \in \text{Spec}(\mathcal{A})$ for every $i \in I$. Let $(x_1, \dots, x_n) \in 1^{\mathcal{M}}$, then we have:

$$\begin{aligned} (x_1, \dots, x_n) \in h\left(\bigcup_{i \in I} S_i\right) &\text{ iff } U_{x_1, \dots, x_n} \in \bigcup_{i \in I} S_i \text{ iff } \exists i \in I \text{ } U_{x_1, \dots, x_n} \in S_i \text{ iff} \\ &\exists i \in I \text{ } (x_1, \dots, x_n) \in h(S_i) \text{ iff } (x_1, \dots, x_n) \in \bigcup_{i \in I} h(S_i) \end{aligned}$$

□

Claim 4. h is injective.

Proof. Let $U \in \text{Spec}(\mathcal{A})$. The first is to show that $h(U)$ is non-empty. The following n -type:

$$T(x_1, \dots, x_n) = \{R_a(x_1, \dots, x_n) \mid a \in U\}$$

is finitely satisfied in \mathcal{M} .

Consider $T_0 = \{R_{a_1}(x_1, \dots, x_n), \dots, R_{a_k}(x_1, \dots, x_n)\} \subseteq T$. Then $a_1, \dots, a_k \in U$ and $a = a_1 \cdot \dots \cdot a_k \in U$. By the instance of the $A_{\neq 0}(a)$ -axiom, we have $\mathcal{M} \models \exists x_1, \dots, x_n R_a(x_1, \dots, x_n)$. $a \leq a_i$ for $i \leq k$, so we have $\mathcal{M} \models \exists x_1, \dots, x_n R_{a_i}(x_1, \dots, x_n)$ for every a_i with $i \leq k$ by the instance of the $A_+(a_i, a, a)$ -axiom. That makes every finite subtype of T satisfiable, thus the whole type is finitely satisfiable in \mathcal{M} . \mathcal{M} is ω -saturated, then T is realised in \mathcal{M} by some $(x_1, \dots, x_n) \in \mathcal{M}^n$ and, moreover, $\mathcal{M} \models 1(x_1, \dots, x_n)$. As we have already said, U_{x_1, \dots, x_n} is an ultrafilter, but $U_{x_1, \dots, x_n} \subseteq U$, thus $U = U_{x_1, \dots, x_n}$, so $(x_1, \dots, x_n) \in h(U)$.

That makes h one-to-one.

□

Claim 5.

1. $h(c_i^{\mathcal{A}^+} U) = C_i(h(U))$
2. $h(d_{ij}^{\mathcal{A}^+}) = D_{ij} \subseteq \text{Spec}(\mathcal{A})$

Proof.

1. Let $\bar{x} = (x_1, \dots, x_n) \in \mathcal{M}^n$ and $S \subseteq \text{Spec}(\mathcal{A})$. Assume $(x_1, \dots, x_n) \in h(c_i^{\mathcal{A}^+} S)$.

Let us show that $\bar{x} \in C_i(h(S))$, that is, there exists $\bar{y} = (y_1, \dots, y_n) \in h(S)$ such that $\bar{x} \equiv_i \bar{y}$.

Then $\mathcal{M} \models 1(x_1, \dots, x_n)$ and $U_{x_1, \dots, x_n} \in c_i^{\mathcal{A}^+} S$. But \mathcal{A}^+ is the complex algebra of the ultrafilter frame $\mathcal{F}_{\mathcal{A}}$. Then we have:

$$c_i^{\mathcal{A}^+} S = \{U_1 \in \text{Spec}(\mathcal{A}) \mid \exists U' \in S \ U_1 R_i U'\}$$

Then there must be an ultrafilter $U' \in S$ such that $U_{x_1, \dots, x_n} R_i U'$, that is, $c_i a \in U_{x_1, \dots, x_n}$ whenever $a \in U'$. Hence $\mathcal{M} \models R_{c_i}(x_1, \dots, x_n)$. By the $A_{c_i}(a)$ -axiom, we have

$$\mathcal{M} \models \exists z_1, \dots, z_n (R_a(z_1, \dots, z_n) \wedge x_i = z_j) \text{ for } i < n \text{ and } j < n \text{ such that } i \neq j.$$

Consider the following n -type with free variables z_1, \dots, z_n and parameters $x_1, \dots, x_n \in \mathcal{M}$:

$$T(z_1, \dots, z_n) = \{R_a(z_1, \dots, z_n) \wedge x_i = z_j \mid i < n, j < n, i \neq j, a \in U'\}.$$

Let us show that $T(z_1, \dots, z_n)$ is finitely satisfiable in \mathcal{M} . Consider a finite subset of T , say $T_0 = \{R_{b_k}(z_1, \dots, z_n) \wedge x_i = y_j \mid i < n, j < n, i \neq j, b_k \in U', k < \omega\}$. We put $p = p_1 \cdots p_k$ and $p \in U'$ since U' is a filter. Then we have:

$$\mathcal{M} \models \exists z_1, \dots, z_n (R_b(z_1, \dots, z_n) \wedge x_i = z_j) \text{ for } i < n \text{ and } j < n \text{ such that } i \neq j$$

Thus, we have, as required:

$$\mathcal{M} \models \exists z_1, \dots, z_n \bigwedge_{i=1}^k (R_{b_k}(z_1, \dots, z_n) \wedge x_i = z_j) \text{ for } i < n \text{ and } j < n \text{ such that } i \neq j.$$

As above, using ω -saturation, we conclude that T is realised in \mathcal{M} at an n -tuple $(y_1, \dots, y_n) = \bar{y}$. Then we have:

$$\mathcal{M} \models 1(\bar{y}), \bar{x} \equiv_i \bar{y}, U_{\bar{y}} \supseteq U'$$

Then $U_{\bar{y}} = U'$, then $\bar{y} \in h(S)$. Then $\bar{x} \in C_i(h(S))$.

Suppose for the converse, $\bar{x} = (x_1, \dots, x_n) \in C_i(h(S))$. We need $\bar{x} \in h(c_i^{\mathcal{A}^+}(S))$. Then there exists $\bar{y} = (y_1, \dots, y_n)$ such that $\bar{x} \equiv_i \bar{y}$ and $\bar{y} \in h(S)$. Then there exists an ultrafilter $U_{y_1, \dots, y_n} \in S$. Let us show that $\mathcal{M} \models 1(x_1, \dots, x_n)$ and $U_{x_1, \dots, x_n} \in c_i^{\mathcal{A}^+} U_{y_1, \dots, y_n}$. Let $a \in U_{y_1, \dots, y_n}$. Then we have $\mathcal{M} \models R_a(y_1, \dots, y_n)$. By the $A_{c_i}(a)$ axiom, we have $\mathcal{M} \models R_{c_i a}(x_1, \dots, x_n)$. Then $\mathcal{M} \models 1(x_1, \dots, x_n)$ and $c_i a \in U_{x_1, \dots, x_n}$, thus, $\bar{x} \in h(c_i^{\mathcal{A}^+}(S))$.

2. Let us show that h preserves cylindrifications.

Let $(x_1, \dots, x_n) \in \mathcal{M}^n$. Then $(x_1, \dots, x_n) \in D_{ij}$ iff $\mathcal{M} \models 1(x_1, \dots, x_n)$ and $x_i = x_j$ iff $U_{x_1, \dots, x_n} \in d_{ij}^{\mathcal{A}^+} = \{U \in \text{Spec}(\mathcal{A}) \mid d_{ij} \in U\}$ iff $\mathcal{M} \models d_{ij}^{\mathcal{M}}(x_1, \dots, x_n)$.

□

7 Representability games

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