Characterising representable positive relation algebras via Priestley duality

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1 Distributive lattice representation and Priestley duality

Lemma 1. *Let $h : \mathcal{L} \to \mathcal{R}$ be a representation, then then*

$$h^{-1}[x] = \{a \in \mathcal{L} \mid x \in h(a)\}$$

is prime filter in $\mathcal{L}$.

Proof. Let $c \in h^{-1}[x]$ and $c \leqslant d$, then $x \in h(c)$, but h is order-preserving, so $x \in h(d)$. If $c, d \in h^{-1}[x]$, then $x \in h(c)$ and $x \in h(d)$, so $x \in h(c) \cap h(d)$, then $x \in h(c \cdot d)$, so $c \cdot d \in h^{-1}[x]$. Let $c + d \in h^{-1}[x]$, then $x \in h(c + d) = h(c) \cup h(d)$, so either $x \in h(c)$ or $x \in h(d)$, so either $c \in h^{-1}[x]$ or $d \in h^{-1}[x]$. $\square$

1.1 Priestley duality

2 Representatiting positive relation algebras

3 Spectral spaces for positive relation algebras

4 Complete representability

4.1 Completely representable distibutive lattices

TODO: read [EH12]

4.2 Completely representable positive relation algebras

5 The main result

For that we need such model theoretic notions as saturation and types, see [Hod93, Section 6.3].

Definition 1. *Let $\mathcal{M}$ be a first-order structure of a signature L and $S \subseteq \mathcal{M}$. Let $L(S)$ be an extension of L with copies of elements from S as additional constants. We assume that $Cnst(L)$ and S are disjoint.*

1. *Let $n < \omega$, an n-type over S is a set $\mathcal{T}$ of $L(S)$ formulas $A(\overline{x})$, where $\overline{x}$ is a fixed n-tuple of elements from S. Notation: $\mathcal{T}(\overline{x})$. A type is an n-type for some $n < \omega$.*

2. *An n-type $\mathcal{T}(\overline{x})$ is realised in $\mathcal{M}$, if there exists $\overline{m} \in \mathcal{M}^n$ such that $\mathcal{M} \models A(\overline{m})$ for every $A \in \mathcal{T}(\overline{x})$. $\mathcal{M}$ omits $\mathcal{T}(\overline{x})$, if $\mathcal{T}(\overline{x})$ is not realised in $\mathcal{M}$.*

3. *$\mathcal{T}(\overline{x})$ is finitely satisfied in $\mathcal{M}$, if every finite subtype $\mathcal{T}_0(\overline{x}) \subseteq \mathcal{T}(\overline{x})$ is realised in $\mathcal{M}$. We can reformulate that as $\mathcal{M} \models \exists \overline{a} \bigwedge\limits_{A \in \mathcal{T}_0} A(\overline{a})$.*

4. *Let T be a theory, then a type $\mathcal{T}$ over the empty set of constants is T-consistent, if there exists a model $\mathcal{M} \models T$ such that $\mathcal{T}$ is finitely satisfied in $\mathcal{M}$.*

5. *Let κ be a cardinal, then $\mathcal{M}$ is κ-saturated, if for every $S \subseteq \mathcal{M}$ with $|S| < \kappa$ every finitely satisfied 1-type $\mathcal{T}$ is realised in $\mathcal{M}$.*

By default, a saturated model is an ω-saturated model for us.
The useful facts, they are from [CK90] and [Hod93]:

Fact 1. *Let $\mathcal{M}$ be an FO-structue and κ a cardinal, then:*

1. *$\mathcal{M}$ is κ-saturated iff every finitely satisfiable α-type (an arbitrary $\alpha \leqslant \kappa$) with fewer than κ parameters is realised in $\mathcal{M}$.*

2. *If $\mathcal{M}$ is κ-saturated, then $\mathcal{M}$ is λ-saturated for every $\lambda < \kappa$.*

3. *Every consistent theory has a κ-saturated model and every model has an elementary κ-saturated extension.*

4. *Let $(\mathcal{M}_i)_{i<\omega}$ a family of structures of the (at most) countable signature and D a non-principal ultrafilter over ω, then $\Pi_D \mathcal{M}_i$ is ω_1-saturated.*

Let $\mathcal{A}$ be a positive relation algebra, define the first-order relational language of the form

$$\mathcal{L}(\mathcal{A}) = (=, \{R_a^2\}_{a \in \mathcal{A}})$$

The $\mathcal{L}(\mathcal{A})$-theory $T_{\mathcal{A}}$ consists of the following statements:

- $\sigma_1 = \forall x \forall y (\mathbf{1'}(x, y) \leftrightarrow (x = y))$
- $\sigma_+(R, S, T) = \forall x \forall y (R(x, y) \leftrightarrow S(x, y) \vee T(x, y))$
- $\sigma_.(R, S, T) = \forall x \forall y (R(x, y) \leftrightarrow S(x, y) \wedge T(x, y))$
- $\sigma_;(R, S, T) = \forall x \forall y (R(x, y) \leftrightarrow \exists z (S(x, z) \wedge T(z, y)))$
- $\sigma_\smile(R, S) = \forall x \forall y (R(x, y) \leftrightarrow S(y, x))$
- $\sigma_{\neq 0} = \exists x \exists y R(x, y)$ for any R_a such that $a \neq 0$
- $\sigma_0 = \neg \exists x \exists y 0(x, y)$
- $\sigma_{\mathbf{1}} = \forall x \forall y (R(x, y) \to \mathbf{1}(x, y))$

Proposition 1. *$T_{\mathcal{A}}$ is satisfiable whenever $\mathcal{A}$ is representable.*

Theorem 1. *Let $\mathcal{A}$ be a positive relation algebra, then $\mathcal{R}$ is representable iff $(\mathcal{R}_+)^+$ is completely representable.*

Theorem 2. **RPRA** *is a canonical variety.*

References

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