Representable cylindric algebras of dimension ω: the aspects of canonicity

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1 Intro

2 The problem itself

Suppose $\mathcal{C} \in \mathbf{RCA}_\omega$, whether $\mathcal{C}^+$ has a complete, ω-dimensional representation? [3]

3 Boolean algebras with operators and cylindric algebras

Definition 1.

1. *Let $\mathcal{B} = \langle B, +, -, 0, 1\rangle$ be a Boolean algebra. An operator is an n-ary function $\Omega : B^n \to B$ satisfying the following conditions:*
   - *Normality: for all $b_0, \ldots, b_{n-1} \in B$, if $b_1 = 0$ for some $i < n$, then*
   $$\Omega(b_0, \ldots, b_{i-1}, 0, b_{i+1}, \ldots, b_{n-1}) = 0$$
   - *Additivity: for all $b_0, \ldots, b_{n-1}, b, b' \in B$ we have*
   $$\Omega(b_0, \ldots, b_{i-1}, (b + b'), b_{i+1}, \ldots, b_{n-1}) = \\ \Omega(b_0, \ldots, b_{i-1}, b, b_{i+1}, \ldots, b_{n-1}) + \Omega(b_0, \ldots, b_{i-1}, b', b_{i+1}, \ldots, b_{n-1})$$
2. *Let I be an index set, a Boolean algebra with operators (BAO) is an algebra $\langle B, +, -, 0, 1, \{\Omega_i\}_{i \in I}\rangle$ such that $\langle B, +, -, 0, 1\rangle$ is a Boolean algebra and for each $i \in I$ Ω_i is an operator.*

Definition 2. *Let $\mathcal{B} = \langle B, +, -, 0, 1, \{\Omega_i\}_{i \in I}\rangle$ be a BAO, then*

1. *An opetator Ω is completely additive, if for each $b_0, \ldots, b_{n-1} \in B$ and $X \subseteq B$, one has*
$$\Omega(b_0, \ldots, b_{i-1}, \sum X, b_{i+1}, \ldots, b_{n-1}) = \sum_{x \in X} \Omega(b_0, \ldots, b_{i-1}, x, b_{i+1}, \ldots, b_{n-1})$$
2. *$\mathcal{B}$ is completely additive, if for each $i \in I$ Ω_i is additive,*
3. *A class $\mathcal{K}$ of BAOs is completely additive, if every $\mathcal{B} \in \mathcal{K}$ is completely additive.*

3.1 Atom structures and canonical extensions

Definition 3. *Let I be an index set and $\{\Omega_i\}_{i \in I}$ a set of function symbols*

1. *An atom structure is a relational structrure $\mathcal{F} = \langle W, \{R_i\}_{i \in I}\rangle$ such that R_i is a $n+1$-ary relation symbol, if $\Omega_{i \in I}$ is an n-ary function symbol,*
2. *Let $\mathcal{B}$ be an atomic BAO of the signature I, the atom structure of $\mathcal{B}$, written as $\mathbf{At}\mathcal{B}$, is an atom structure $\langle \mathrm{At}(\mathcal{B}), \{R_i\}_{i \in I}\rangle$ such that for each $a, b_0, \ldots, b_{n+1} \in \mathrm{At}(\mathcal{B})$ and for each $i \in I$*

$$\mathbf{At}\mathcal{B} \models R_i(a, b_0, \ldots, b_{n+1}) \textit{ iff } \mathcal{B} \models a \leqslant \Omega_i(b_0, \ldots, b_{n+1})$$

3. *Let $\mathcal{F} = \langle W, \{R_i\}_{i \in I}\rangle$ be an atom structure, the complex algebra of $\mathcal{F}$, written as $\mathbf{Cm}\mathcal{F}$, is a BAO $\langle \mathcal{P}(W), \cup, -, \varnothing, W, \{\Omega_{R_i}\}_{i \in I}\rangle$ such that for all $X_0, \ldots, X_{n-1} \subseteq W$ and for each $i \in I$*

$$\Omega_{R_i}(X_0, \ldots, X_{n-1}) = \{a \in W \mid \exists b_0 \in X_0 \ldots \exists b_{n-1} \in X_{n-1}\ \mathcal{F} \models R_i(a, b_0, \ldots, b_{n-1})\}$$

The following duality is due to Thomason [5].

Fact 1.

1. *Let $\mathcal{B}$ be a complete atomic BAO, then $\mathcal{B} \cong \mathbf{Cm}(\mathbf{At}(\mathcal{B}))$,*
2. *Let $\mathcal{F}$ be an atom structure, then $\mathcal{F} \cong \mathbf{At}(\mathbf{Cm}(\mathcal{B}))$.*

Let A be a non-empty subset of a Boolean algebra $\mathcal{B}$, A is a *filter*, if A is closed under finite infima and upwardly closed. A is an ultrafilter, if it has no non-trivial extensions. That is, if $A \subseteq A'$, then $A' = \mathcal{B}$.

Definition 4. *Let $\mathcal{B} = \langle B, +, -, 0, 1, \{\Omega_i\}_{i \in I}\rangle$ be a BAO and $\mathbf{Uf}(\mathcal{B})$ the set of its ultrafilters. The ultrafilter frame of $\mathcal{B}$ (or canonical frame) is a relational structure $\mathcal{F}_{\mathcal{B}} = \langle \mathbf{Uf}(\mathcal{B}), R_{\Omega_i}\rangle$ such that for each ultrafilters $\beta_0, \ldots, \beta_{n-1}, \gamma$ one has*

$$\mathbf{Uf}(\mathcal{B}) \models R_{\Omega_i}(\beta_0, \ldots, \beta_{n-1}, \gamma) \textit{ iff } \{\Omega(b_0, \ldots, b_{n-1}) \mid b_0 \in \beta_0, \ldots, b_{n-1} \in \beta_{n-1}\} \subseteq \gamma.$$

Definition 5. *Let $\mathcal{B}$ be a BAO, then*

1. *The canonical extension of $\mathcal{B}$ is a complex algebra of the canonical frame $\mathbf{Cm}(\mathcal{F}_{\mathcal{B}})$ denoted as $\mathcal{B}^+$,*
2. *The class of BAOs is canonical, if it is closed under canonical extensions.*

Theorem 1. *Let $\mathcal{A}$, $\mathcal{B}$ be BAOs,*

1. *There exists $\iota : \mathcal{A} \hookrightarrow \mathcal{A}^+$ such that $\iota : a \mapsto \{\gamma \in \mathbf{Uf}(\mathcal{A}) \mid a \in \gamma\}$.*
2. *If $i : \mathcal{A} \hookrightarrow \mathcal{B}$, then this embedding might be extented to the embedding $i^+ : \mathcal{A}^+ \hookrightarrow \mathcal{B}^+$*

Fact 2.

3.2 (Representable) cylindric algebras and cylindric set algebras

Definition 6.

A relation algebra is an algebra $\mathcal{R} = \langle R, 0, 1, +, -, ;, ^{\smile}, \mathbf{1'} \rangle$ *such that* $\langle R, 0, 1, +, - \rangle$ *is a Boolean algebra and the following equations hold, for each* $a, b, c \in R$*:*

1. $a; (b; c) = (a; b); c$
2. $(a + b); c = (a; c) + (b; c)$
3. $a; \mathbf{1'} = a$
4. $a^{\smile\smile} = a$
5. $(a + b)^{\smile} = a^{\smile} + b^{\smile}$
6. $(a; b)^{\smile} = b^{\smile}; a^{\smile}$
7. $a^{\smile}; (-(a; b)) \leqslant -b$

where $a \leqslant b$ *iff* $a + b = b$. **RA** *denotes the class of all relation algebras.*

Cylindric algebras provide a generalisation of relation algebras for relations of an arbitrary arity. Let α be an ordinal. Let ${}^{\alpha}U$ be the set of all functions mapping α to a non-empty set U. We denote $x(i) = x_i$ for $x \in {}^{\alpha}U$ and $i < \alpha$.

Definition 7.

1. *A subset of* ${}^{\alpha}U$ *is an* α*-ry relation on* U*. For* $i, j < \alpha$*, the* i, j*-diagonal* D_{ij} *is the set of all elements of* U *such that* $y_i = y_j$*.*

   If $i < \alpha$ *and* X *is an* α*-ry relation on* U*, then the* i*-th cylindrification* $C_i X$ *is the set of all elements of* U *that agree with some element of* X *on each coordinate except the* i*-th one. To be more precise,* $C_i X = \{y \in {}^{\alpha}U \mid \exists x \in X \forall i < \alpha \ (i \neq j \Rightarrow y_j = x_j)\}$*.*

2. *A cylindic set algebra of dimension* α *is an algebra consisting of a set* S *of* α*-ry relation on some base set* U *with the constants and operations* $0 = \varnothing$*,* $1 = {}^{\alpha}U$*,* $\cap$*,* $-$*, the diagonal elements* $\{D_{ij}\}_{i,j<\alpha}$*, the cylindrifications* $\{C\}_{i<\alpha}$*.*

   A generalised cylindric set algebra of dimension α *is a subdirect of cylindric algebras that have dimension* α

3. *A cylindric algebra of dimension* α *is an algebra* $\mathcal{C} = \langle \mathcal{B}, \{c_i\}_{i<\alpha}, \{d_{ij}\}_{i,j<\alpha} \rangle$ *such that*

   - $\mathcal{B}$ *is a Boolean algebra, for each* $i, j < \alpha$ c_i *is an operator and* $d_{ij} \in \mathcal{B}$
   - *For each* $i < \alpha$*,* $a \leqslant c_i a$*,* $c_i(a \wedge c_i b) = c_i a \wedge c_i b$ *and* $d_{ii} = 1$
   - *For every* $i, j < \alpha$*,* $c_i c_j a = c_j c_i a$
   - *If* $k \neq i, j < \alpha$*, then* $d_{ij} = c_k(d_{ij} \wedge d_{jk})$
   - *If* $i \neq j$*, then* $c_i(d_{ij} \wedge a) \wedge c_i(d_{ij} \wedge -a) = 0$

   $\mathbf{CA}_{\alpha}$ *is the class of all cylindric algebras of dimension* α

4. *An* α*-dimensional cylindric algebra* C *is representable, if it is isomorphic to a generalised cylindric set algebra of dimension* α*. Such is isomorphism is a representation of* C*.*

   $\mathbf{RCA}_{\alpha}$ *is the class of all representable cylindric algebras that have dimension* α*. In particular, we are interested in the case when* $\alpha = \omega$*.*

Definition 8. *Given a cylindric algebra of dimension α C, let x be a term of its signature, the substitution operator s^i_j have the following definition:*

$$s^i_j x = \begin{cases} x, \text{if } i = j \\ c_i(d_{ij} \wedge x), \text{otherwise} \end{cases}$$

It is well known that $\mathbf{RCA}_\alpha$ is a variety, $\mathbf{RCA}_\alpha$ ($\alpha \leqslant 2$) is finitely axiomatisable and $\mathbf{RCA}_\alpha$ ($2 < \alpha < \omega$) has no finite axiomatisation, see [2].

Let $\mathcal{A} \in \mathbf{C}_\omega$, then $\mathcal{A}$ has a *complete representation*, if this representation preserves all existing suprema.

Let us concretise the definition of a canonical extension for α-dimensional cylindric algebras:

Definition 9. *Let $\mathcal{C} = \langle C, +, -, 0, 1, \{d_{ij}\}_{i,j<\alpha}, \{c_i\}_{i<\alpha} \rangle$ $\mathcal{A}$ be a BAO of type $\mathbf{CA}_\alpha$ Let $\mathbf{Uf}(\mathcal{C})$ be the set of all ultrafilters of $\mathfrak{B}\mathcal{C}$, the Boolean part of $\mathcal{C}$.*

Let us define $\mathbf{C}_i : \mathbf{Uf}(\mathcal{C}) \to \mathbf{Uf}(\mathcal{C})$ for each $i, j < \alpha$ as

1. $\mathbf{C}_i \mathcal{X} = \{\mathcal{F} \in \mathbf{Uf}(\mathcal{C}) \mid \exists \mathcal{F}' \in \mathbf{Uf}(\mathcal{C}) \ (a \in \mathcal{F} \Rightarrow c_i a \in \mathcal{F}' R)\}$,
2. $D_{ij} = \{\mathcal{F} \in \mathbf{Uf}(\mathcal{C}) \mid d_{ij} \in \mathcal{F}\}$.

The structure $\mathcal{C}^+ = \langle \mathbf{Uf}(\mathcal{C}), \cup, -, \varnothing, C, \mathbf{C}_{i<\alpha}, \{D_{ij}\}_{i,j<\alpha} \rangle$ is called the canonical extension of $\mathcal{C}$.

Let us discuss the connection between representability and canonical extensions.

The following definitions and facts are due to Henkin, Monk, and Tarski [1].

Let $\mathcal{A} \in \mathbf{CA}_\alpha$ and $x \in \mathcal{A}$. Recall that *the dimension of* x is the set of all ordinals $\gamma < \alpha$ such that $c_\gamma x \neq x$. More formally,

$$\Delta x = \{\gamma \mid \gamma < \alpha \,\&\, c_\gamma x \neq x\}$$

Let us discuss some metamathematical intuitions standing behind the notion of a dimension. Let Θ be a first-order theory and $\mathcal{C}/\equiv_\Theta$ its Lindenbaum-Tarski algebra. Let φ be a formula in the signature of Θ. Then $\Delta(\varphi/\Theta)$ consists of all $\kappa < \alpha$ such that $\exists x_\kappa \varphi \leftrightarrow \varphi$ is not valid in Θ. That is, $\Delta(\varphi/\Theta)$ contains ordinals κ for which x_κ is free in φ. Moreover, $\Delta(\varphi/\Theta)$ consists only of those ordinals for which x_κ is free in every $\psi \in \varphi/\Theta$.

In particular, an element x is called *zero-dimensional* if $\Delta x = 0$. Zero-dimensional elements reflect equivalence classes of sentences in the Lindenbaum-Tarski algebra of a given first-order theory. Thus, the set of zero-dimensional elements form a Boolean algebras of sentences associated with Θ.

Definition 10. *Let $\mathcal{A}$ be an α-dimensional cylindric algebra. Let α be an ordinal and Γ a subset α, then an element $x \in \mathcal{A}$ is Γ-closed if $\Delta x \cdot \Gamma = \varnothing$. Alternatively, x is a Γ-cylinder.*

$\mathrm{Cl}_\Gamma\, \mathcal{A}$ *is the set of all Γ-closed elements.*

Metamathematically, Γ-closed elements reflect universal closures (is it correct?).

Let $\mathcal{C} = \langle C, +, -, 0, 1, \{d_{ij}\}_{i,j<\beta}, \{c\}_{c<\beta} \rangle$ be a β-dimensional cylindic algebra and $\alpha \leqslant \beta$ an ordinal. The *α-th reduct* of $\mathcal{C}$, denoted as $\mathfrak{Rd}_\alpha \mathcal{C}$, is an algebra having the form

$$\mathfrak{Rd}_\alpha \mathcal{C} = \langle C, +, -, 0, 1, \{d_{ij}\}_{i,j<\alpha}, \{c\}_{c<\alpha} \rangle$$

$\mathcal{B}$ is a subreduct of $\mathcal{C}$, denoted as $\mathcal{B} \subseteq^r \mathcal{C}$, if $\mathcal{B} \subseteq \mathfrak{Rd}_\gamma \mathcal{C}$ for some $\gamma \leqslant \beta$.

Definition 11. *Let $\mathcal{C}$ be a β-dimensional cylindic algebra and α an ordinal such that $\alpha \leqslant \beta$. The neat α-reduct of $\mathcal{C}$, denoted as $\mathfrak{Nr}_\alpha \mathcal{C}$, is the subalgbera $\mathcal{A}$ of $\mathfrak{Rd}_\alpha \mathcal{C}$ with $\mathcal{A} = \mathrm{Cl}_\kappa\, \mathcal{C}$ where $\alpha + \kappa = \beta$.*

Let $\mathbb{K}$ be a class of β-dimensional cylindic algebras, then we put

$$\mathbf{Nr}_\alpha\mathbb{K} = \{\mathfrak{Nr}_\alpha\mathcal{C} \mid \mathcal{C} \in \mathbb{K}\}$$

An algebra $\mathcal{B}$ is a neat subreduct of $\mathcal{C}$, or $\mathcal{B}$ is neatly embeddable to $\mathcal{C}$ if there exists an ordinal $\gamma \leqslant \alpha$ such that $\mathcal{C} \subseteq \mathfrak{Rd}_\gamma\mathcal{B}$.

One may define neat reducts alternatively as follows. Let $\mathcal{C}$ be a β-dimensional cylindic algebra and α an ordinal such that $\alpha \leqslant \beta$. The neat α-reduct of $\mathcal{C}$ is the α-dimensional cylindric algebra having the form

$$\mathfrak{Nr}_\alpha\mathcal{C} = \langle\{a \in \mathcal{C} \mid \forall j(\alpha \leqslant j \;\&\; j < \beta \Rightarrow c_j a = a)\}, +, -, 0, 1, \{d_{ij}\}_{i,j<\alpha}, \{c_\gamma\}_\gamma\rangle$$

We will adapting the following proof of the fact that **RRA** is canonical [1] to our case. This proof is due to Monk, but that was describe in McKenzie's thesis [4].

1. A relation algebra $\mathcal{A}$ is representable iff $\mathcal{A}$ neatly embeds to some omege-dimensional cylinric algebra,
2. If $\mathcal{A}$ neatly embeds in $\mathcal{A}$ then $\mathcal{A}^+$ neatly embeds in $\mathcal{B}^+$,
3. $\mathbf{CA}_\alpha$ is closed under canonical extensions,
4. Voilá.

Definition 12. *Let $\mathcal{C} \in \mathbf{CA}_\alpha$, where $\alpha \geqslant 3$. The relation algebra reduct of $\mathcal{C}$, written as $\mathfrak{Ra}(\mathcal{C})$, is the algebra having the form*

$$\langle \mathrm{dom}(\mathfrak{Nr}_2(\mathcal{C})), 0, 1, +, -, \mathbf{1'}, \smile, ; \rangle$$

where:

1. $+$, $-$, 0, *and* 1 *are defined as usual in* $\mathcal{C}$,
2. $\mathbf{1'} = d_{01} \in \mathfrak{Nr}_2(\mathcal{C})$,
3. $r^\smile = s_0^2 s_1^0 s_2^1 r$ *for* $r \in \mathfrak{Nr}_2(\mathcal{C})$,
4. *Let* $r, s \in \mathfrak{Nr}_2(\mathcal{C})$, *then* $r; s = c_2(s_2^1 r \cdot s_2^0 s)$

Moreover, $\mathfrak{Nr}_\beta(\mathcal{C})$ and $\mathfrak{Ra}(\mathcal{C})$ are closed under these operations. There is also the following fact by due to Henkin, Monk, and Tarski [2]:

Theorem 2. *Let $\mathcal{C} \in \mathbf{CA}_\alpha$ for $\alpha \geqslant 4$, then $\mathfrak{Ra}(\mathcal{C})$ is a relation algebra.*

The following characterisation results are by Henkin, Monk, and Tarski [2] as well:

Theorem 3.

1. $\mathbf{RA} = \mathbf{S}\mathfrak{Ra}\mathbf{CA}_4$,
2. $\mathbf{RRA} = \bigcap_{3 \leqslant n < \omega} \mathbf{S}\mathfrak{Ra}\mathbf{CA}_n = \mathbf{S}\mathfrak{Ra}\mathbf{CA}_\alpha$, *where α is an infinite ordinal.*

Let $\mathcal{C} \in \mathcal{CA}_\alpha$, then $\mathcal{R} \in \mathbf{RA}$ neatly embeds to $\mathcal{C}$, if $\mathcal{R}$ is isomorphic to some subalgebra of $\mathfrak{Ra}(\mathcal{C})$.

[1] This idea is by Ian Hodkinson

Theorem 4. **RRA** *is closed under canonical extensions.*

Proof. Let $\mathbf{R} \in \mathbf{RRA}$. By the second item of 3, every representable relation algebra is isomorphic to some subalgbera of the relation algebra reduct $\mathfrak{Ra}\mathcal{C}$ for some $\mathcal{C} \in \mathbf{CA}_\omega$. But neat embeddings repsect canonical extensions, so if $\mathbf{R} \hookrightarrow_n \mathcal{C}$, so is $\mathbf{R}^+ \hookrightarrow_n \mathcal{C}^+$. $\mathbf{CA}_\alpha$ is closed under canonical extensions, so is **RRA**. □

Lemma 1 (Henkin, Monk, Tarski)**.**
Let $\mathcal{A}$ be a BAO of type $\mathbf{CA}_\alpha$ and $\mathcal{B}$ be a β-dimensional cylindric algebra such that $\beta \leqslant \alpha$ and $\mathcal{A}$ neatly embeds to $\mathcal{B}$ by a complete embedding. Then $\mathcal{A}^+$ neatly embeds to $\mathcal{B}^+$ by a complete embedding.

Proof. □

Theorem 5 (This assumption is by Ian Hodkinson)**.**
Let $\mathcal{A}$ be a BAO of type $\mathbf{CA}_\omega$ such that $\mathcal{A}$ neatly embeds into $\mathbf{CA}_{\omega+\omega}$ by a complete embedding. Then $\mathcal{A}$ is completely representable as $\mathbf{CA}_\omega$.

Proof. Hmmmmm, I believe so. □

Lemma 1 and Theorem 5 imply the following theorem.

Theorem 6. *Let $\mathcal{C} \in \mathbf{RCA}_\omega$, then $\mathcal{C}^+ \in \mathbf{RCA}_\omega$. That is, $\mathbf{RCA}_\omega$ is closed under canonical extensions.*

Proof. :monkahmm: □

References

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