The finite base property for some subreducts of representable relation algebras

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1 The Relation Algebras Background

We describe the basic definitions and results about relation algebras [10] [15].

Definition 1.

A relation algebra is an algebra $\mathcal{R} = \langle R, 0, 1, +, -, ;, ^{\smile}, \mathbf{1} \rangle$ such that $\langle R, 0, 1, +, - \rangle$ is a Boolean algebra and the following equations hold, for each $a, b, c \in R$:

1. $a;(b;c) = (a;b);c$
2. $(a+b);c = (a;c) + (b;c)$
3. $a;\mathbf{1} = a$
4. $a^{\smile\smile} = a$
5. $(a+b)^{\smile} = a^{\smile} + b^{\smile}$
6. $(a;b)^{\smile} = b^{\smile};a^{\smile}$
7. $a^{\smile};(-(a;b)) \leqslant -b$

*where $a \leqslant b$ iff $a + b = b$. **RA** denotes the class of all relation algebras.*

A relation algebra is called symmetric, if every element is self-converse. A relation algebra is called integral, if

$$a;b = 0 \Rightarrow a = 0 \text{ or } b = 0.$$

Definition 2. *A proper relation algebra is an algebra $\mathcal{R} = \langle R, 0, 1, \cup, -, |, ^{\smile}, \mathbf{1} \rangle$ such that $R \subseteq \mathcal{P}(W)$, where W is an equivalence relation; $0 = \varnothing$; $1 = W$; $\cap$, $\cup$, $-$ are set-theoretic intersection, union, and complement respectively; $|$ is relation composition, $^{\smile}$ is relation converse, $\mathbf{1}$ is a diagonal relation restricted to W, that is:*

1. $a|b = \{\langle x, z\rangle \mid \exists y \, \langle x, y\rangle \in a \,\&\, \langle y, z\rangle \in b\}$
2. $a^{\smile} = \{\langle x, y\rangle \mid \langle y, x\rangle \in a\}$
3. $\mathbf{1} = \{\langle x, y\rangle \mid x = y\}$

*The class of all proper relation algebras is denoted as **PRA**. **Rs** is the class of all relation set algebras, proper relation algebra with a diagonal subrelation as an identity. **RRA** is the class of all representable relation algebras, that is, the closure of **PRA** under isomorphic copies. That is, **RRA** = **IPRA**.*

Note that the (quasi)equational theories of those classes coincide, that is

$$\mathbf{IPRA} = \mathbf{RRA} = \mathbf{SPRs}$$

Moreover, **RRA** is a variety, but it cannot be defined by any set of first-order formulas [16].

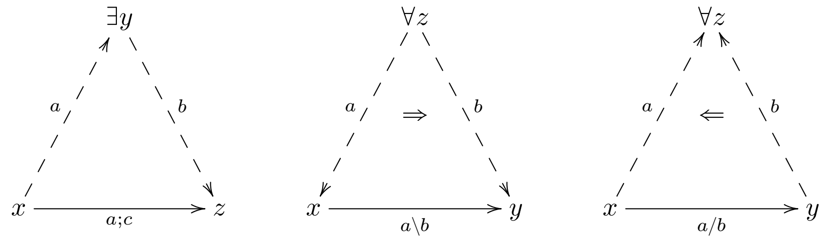
One may express residuals in every $\mathcal{R} \in \mathbf{RA}$ as follows, for every $a, b \in \mathcal{R}$:

1. $a \backslash b = -(a^{\smile}; -b)$
2. $a/b = -(-a; b^{\smile})$

Those residuals have the following interpretation in $\mathcal{R} \in \mathbf{PRA}$ (as well as in **RRA**), for every $a, b \in \mathcal{R}$:

1. $a \backslash b = \{\langle x, y \rangle \mid \forall z \, (z, x) \in a \Rightarrow (z, y) \in b\}$
2. $a/b = \{\langle x, y \rangle \mid \forall z \, (y, z) \in b \Rightarrow (x, z) \in a\}$

One may illustrate composition and residuals in **PRA** and **RRA** via the following triangles:



Given a subset of definable operations in **RA** τ, we denote the class of subalgebras of the τ-reducts by $\mathbf{R}(\tau)$. The algebras containing to this class are defined as restrictions of elements belonging to **Rs** to operations of τ. By $\mathbf{Q}(\tau)$ we mean a quasivariety generated by $\mathbf{R}(\tau)$. As in [12], we put $\mathbf{Q}(\tau)$ as the closure of $\mathbf{R}(\tau)$ under subalgebras and products assuming that $\mathbf{R}(\tau)$ is already closed under ultraproducts.

2 The Finite Base Property

We recall the underlying definitions according to [10, Section 19]

Definition 3. *Let* **K** *be a class of algebras of a signature* Ω, **K** *has the finite algebra property, if if any first-order* Ω*-sentence that is true in all finite algebras in* **K** *is true in every algebra in* **K**.

The finite base property is a version of the finite algebra property if **K** is a class of representable algebras:

Definition 4. *Let* **K** *be a class of representable algebras of a signature* Ω

1. **K** *has the finite base property if any first-order* Ω*-sentence that is true in every algebra in* **K** *having a representation over a finite base set is valid in* **K**.
2. **K** *has the finite algebra on finite base property if any finite algebra in* **K** *has a representation with finite base.*
3. **K** *has the finite algebra property for equations/quasi-identites if any equation/quasi-identity that is true in all finite algebras is true in every algebra in* **K**. *The finite base property for equations/quasi-identites is defined similarly.*

The following statements were shown in [3]. This lemma connects finite base property with finite algebra on finite base and finite algebra properties as follows:

Lemma 1. *Let* $\mathbf{K}$ *be a class of representable* Ω*-algebras:*

1. *If* $\mathbf{K}$ *has the finite algebra property, then it has the finite algebra and the finite base properties for equations/quasi-identites.*
2. *The finite algebra on finite base and the finite algebra properties implies the finite base property for* $\mathbf{K}$*. The same holds for equations/quasi-identites.*
3. *If any representation of an infinite algebra has an infinite base, then the finite base property implies the finite algebra one for* $\mathbf{K}$*.*
4. *Suppose* Ω *is finite and any subalgebra of a representable algebra is representable on the same base. Then the finite base property implies the finite algebra on finite base property.*

3 The Relation Residuated Semigroups Background

3.1 The underlying definitions and results

A *relation structure* (**RS**) is an arbitrary algebra of the signature $\Omega = \langle \circ, \backslash, /, \leqslant \rangle$, where $\circ, \backslash, /$ are binary function symbols and $\leqslant$ is a binary relation symbol.

Definition 5. *A residuated semigroup is an algebra* $\mathcal{S} = \langle S, \circ, \leqslant, \backslash, / \rangle$ *such that* $\langle S, \circ, \leqslant, \rangle$ *is an ordered residuated semigroup and the following equivalences hold for each* $a, b, c \in S$*:*

$$b \leqslant a \backslash c \Leftrightarrow a \circ b \leqslant c \Leftrightarrow a \leqslant c/b$$

ORS *is the class of all residuated semigroups.*

See this paper to have a proof of the following theorem [7]:

Theorem 1. *Every finite residuated semigroup is isomorphic to some residuated subsemigroup of some finite residuated lattice.*

Definition 6. *Let* A *be a set of binary relations on some base set* W *such that* $R = \cup A$ *is transitive and* $\{x, y \mid xRy\} = W$*. A relation residuated semigroup is an algebra* $\mathcal{A} = \langle A, ;, \backslash, /, \subseteq \rangle$ *where for each* $r, s \in A$

1. $r; s = \{\langle a, c \rangle \mid \exists b \in W \; (\langle a, b \rangle \in r \; \& \; \langle b, c \rangle \in s)\}$
2. $r \backslash s = \{\langle a, c \rangle \mid \forall b \in W \; (\langle b, a \rangle \in r \Rightarrow \langle b, c \rangle \in s)\}$
3. $r/s = \{\langle a, c \rangle \mid \forall b \in W \; (\langle c, b \rangle \in s \Rightarrow \langle a, b \rangle \in r)\}$
4. $r \leqslant s$ *iff* $r \subseteq s$*.*

Relation residuated semigroup are also called representable relativised relational structure (**RRS**).

See [6]

Theorem 2. *Every complete residuated semigroup* $\mathcal{A}$ *(quantale) is isomorphic to relational complete residuated semigroup on the underlying set of* $\mathcal{A}$*.*

Definition 7. *Let $\mathcal{A} = \langle A, \leqslant, ;, \backslash, / \rangle$ be a residuated semigroup. A representation of $\mathcal{A}$ is an inclusion map $h : \mathcal{A} \to \mathcal{A}'$, where $\mathcal{A}' \in$ **RRS** such that:*

1. *$dom(\mathcal{A}') = \{\hat{a} \mid a \in A\}$, where $\hat{a} = \{(b, c) \mid b \leqslant a; c\}$.*

Such a map preserves order, residuals, and composition. Andréka and Mikulás proved the following representation theorem for **ORS** in [4] that implies relational completeness of the Lambek calculus, the logic of **ORS**:

Theorem 3. **ORS** = **IRRS**, *where* **IRRS** *is a closure of* **RRS** *under isomorphic copies.*

Corollary 1. *Every finite representable residuated semigroup is isomorphic to representable residuated subsemigroup of some finite residuated lattice.*

Theorem 4. *Let $\mathcal{A} \in$ **RRS** and $|\mathcal{A}| < \omega$, then there exists a set W, a set A of binary relations on W, $R = \cup A$ with $dom(R) = A$ such that $\mathcal{A} \cong \langle A, |, \backslash, /, \subseteq \rangle$.*

Theorem 5. *The Lambek calculus is complete w.r.t finite relational models (has the fmp).*

4 Join-semilattice ordered semigroups

Definition 8. *A join-semilattice ordered semigroup ($\mathbf{OS}^{+}$) is an algebra $\mathcal{S} = \langle S, ;, + \rangle$ such that $\langle S, ; \rangle$ is a semigroup, $\langle S, + \rangle$ is a join-semilattice and the following equations hold for each $a, b.c \in S$:*

1. *$a; (b + c) = (a; b) + (a; c)$*
2. *$(a + b); c = (a; c) + (b; c)$*

This class is clearly a variety since $\mathbf{OS}^{+}$ has the equational definition so far as $+$ is defined as an associative, idempotent, and commutative operation.

Let A be a set of binary relations on some base set W such that $R = \cup A$ is transitive and $\{x, y \mid xRy\} = W$ as in Definition 6. A representable join semilattice-ordered semigroup is an algebra isomorphic to some join semilattice-ordered semigroup having the form $\mathcal{A} = \langle A, |, \cup \rangle$ such that ; is a relation composition as above and $\cup$ is the set-theoretic union. If $\mathcal{A}$ is representable, then $\mathcal{A} \in \mathbf{I}(\mathbf{R}(\cup, |))$. Let us recall some of underlying facts about representable join semilattice-ordered semigroups [2]:

Proposition 1.

1. *Let $\mathcal{A} = \langle A, +, ; \rangle$ be a join semilattice-ordered semigroup such that, for all $a, b \in A$:*
   (a) *If $a \nleqslant b$, then there exists an atom $c \leqslant a$ and $c \nleqslant b$.*
   (b) *If $c \leqslant a; b$ and c is an atom, then there exists an atom $a' \leqslant a$ such that $c \leqslant a' \cdot b$.*

   then $\mathcal{A}$ is representable.

2. *Let $\mathcal{A} = \langle A, ; \rangle$ be a posemigroup, then $\mathcal{A}$ is representable and such a representation preserve any existing finite suprema and infima, if*
   (a) *The set of atoms is closed under ;.*
   (b) *$\mathcal{A}$ has enough atoms, that is, if $x \in \mathrm{At}(A)$ and $z, w \in A$, then $x \leqslant z; w$ implies there exist atoms $z_1 \leqslant z$ and $w_1 \leqslant w$ such that $x \leqslant z_1; w_1$. If $z \nleqslant w$, then there exists an atom x such that $x \leqslant z$ and $x \nleqslant w$.*

Recall that a class of structures $\mathbf{K}$ is called finitely axiomatisable iff both $\mathbf{K}$ and its complement are closed ultraproducts and isomorphic copies.

It is known that the class of all representable join-semilattice ordered semigroups has no finite axiomatisation [1]. In other words,

Theorem 6. $\mathbf{R}(\cup, |)$ *is not finitely axiomatisable.*

4.1 The rainbow construction

Let us provide a proof of this fact using the rainbow technique [10] to show that the complement of $\mathbf{ROS}^+$ is not closed ultraproducts. This construction sometimes exploits the similar construction used by Andréka [2] and by Maddux [14]. We note that representability is not decidable for finite relation algebras [9] and this result has several generalisations [11]. Moreover, representability is undecidable for lattice-ordered semigroups and ordered complemented semigroups [17]. We use (more or less) a standard way of showing that the class of certain reducts of representable relation algebras has no finite axiomatisation, see [13] [8].

First of all, we recall several definitions such as colourings. We provide a sequence of symmetric, integral, finite relation algebras $\{\mathfrak{A}_n\}_{n<\omega}$ such that $\mathfrak{A}_n \notin \mathbf{RRA}$. The statement has been proved by Andreka [2] and reproduced here [5].

Given $n < \omega$, the set of atoms $\mathrm{At}(\mathfrak{A}_n)$ consists of the following elements:

- identity: $\mathbf{1}$, an atom with no colour,
- white: $\mathbf{w}$,
- greens: $\mathbf{g}_i$ for $1 \leqslant i \leqslant n$,
- yellows: $\mathbf{y}_i$ for $1 \leqslant i \leqslant n$,
- ivory: $\mathbf{i}$,
- reds: $\mathbf{r}_i$ for $1 \leqslant i \leqslant n$,
- blacks: $\mathbf{b}_i$ for $1 \leqslant i \leqslant n$.

We have the following steps:

Step 1. Let $\mathcal{A}_n$ be the upper semilattice presented with the set $\mathrm{At}(\mathfrak{A}_n) \cup \{0\}$ and the following relations for each $x \in \mathrm{At}(\mathfrak{A}_n)$ and for each $1 \leqslant i \leqslant n$.

1. $\mathbf{w} \leqslant \mathbf{g}_i + \mathbf{y}_i$,
2. $\mathbf{i} \leqslant \mathbf{y}_i + \mathbf{r}_i$,
3. $x + 0 = x$

Step 2. We define S, the set of two element subsets of $\mathcal{A}_n$:

$$S = \{\{\mathbf{w}, \mathbf{r}_1\}\} \cup \{\{\mathbf{g}_i, \mathbf{b}_i\} \mid 1 \leqslant i \leqslant n\} \cup \{\{\mathbf{y}_i, \mathbf{r}_i\} \mid 1 \leqslant i < n\} \cup \{\{\mathbf{y}_n, \mathbf{i}\}\}.$$

Step 3. The operations on $\mathfrak{A}_n$:

1. $1 = \sum \mathrm{At}(\mathcal{A}_n) \cup \{0\}$,
2. $0 = \emptyset$,

3. $x = x^{\smile}$,

4. $0; x = 0; x = 0$,

5. $\mathbf{1}; x = \mathbf{1}; x = x$,

6. $x; y = \begin{cases} \mathbf{i}, & \text{if } \{x, y\} \in S \\ 1, & \text{otherwise} \end{cases}$
   unless $x, y \in \{0, \mathbf{1}\}$.

Step 4. Define the following quasi-idenity:

$$q_n = \bigwedge_{1 \leqslant i \leqslant n} ((x \leqslant x_i^{'} + x_i^{''}) \wedge (y \leqslant y_i^{'} + y_i^{''})) \rightarrow$$
$$x; y \leqslant x; y_1^{'} + \sum_{1 \leqslant i < n} (x_i^{'}; y_i^{''} + x_i^{''}; y_{i+1}^{'}) + x_n^{'}; y_n^{''} + x_n^{''}; y$$

Lemma 2. *1. q_n is valid in* **RRA** *for each $n < \omega$.*

2. q_n fails in $\mathfrak{A}_n$.

Proof. The valuation ϑ defined as:

1. $\vartheta(x) = \mathbf{w}$
2. $\vartheta(x_i^{'}) = \mathbf{g}_i$
3. $\vartheta(x_i^{''}) = \mathbf{y}_i$
4. $\vartheta(y) = \mathbf{i}$
5. $\vartheta(y_i^{'}) = \mathbf{r}_i$
6. $\vartheta(y_i^{''}) = \mathbf{b}_i$

falsifies q_n in $\mathfrak{A}_n$.

TODO: visualise the reason for non-representability. $\square$

4.2 Networks and games

Definition 9. *Let $\mathcal{A}$ be a relation algebra. A network is a complete directed finite graph with edges labelled by elements of $\mathcal{A}$. Such a graph have the following form. $N = \langle E_N, l_N \rangle$, where $E_N = U_N \times U_N$ for some finite base set and $l_n : E_N \rightarrow \mathrm{At}(\mathcal{A})$ is function mapping each edge to some atom of A. This function obey the following requirements:*

1. *$l_N(x, y) \leqslant \mathbf{1}$ iff $x = y$*
2. *$l_N(x, y); l_N(y, z) \geqslant l_N(x, z)$*

Given two networks $N = \langle E_N, l_N \rangle$ and $N^{'} = \langle E_{N^{'}}, l_{N^{'}} \rangle$, N is a subnetwork of $N^{'}$ ($N \subseteq N^{'}$, or $N^{'}$ refines N) if $E_N \subseteq E_{N^{'}}$ and for each $x, y \in U_N$, $l_{N^{'}}(x, y) = l_N(x, y)$.

Definition 10. *Let $n < \omega$. We define a game $\mathcal{G}_n(\mathcal{A})$ for two players $\forall$ (Abelard) and $\exists$ (Héloïse). Abelard and Héloïse build a finite chain of networks $N_0 \subseteq N_1 \cdots \subseteq N_n$ as follows. In the first round $\forall$ picks an atom α and $\exists$ plays a network N_0 containing an edge (m_0, n_0) such that $l_n(m_0, n_0) = \alpha$. If $\alpha \leqslant \mathbf{1}$, then $m_0 = n_0$, otherwise $m_0 \neq n_0$. If $m_0 \neq n_0$, the edges (m_0, n_0) and (n_0, m_0) belong to Abelard. Suppose N_{i-1} for $i < n$ has been played, then*

- $\forall$ *chooses an edge* $(m,n) \in E_{N_{i-1}}$ *and atoms* $x, y \in \mathrm{At}(\mathcal{A})$ *such that* $l_{N_{i-1}}(m,n) \leqslant x;y$.
- $\exists$ *provides a network* $N_i = \langle E_{N_i}, l_{N_i} \rangle \supseteq N_{i-1}$ *such that there exists* $l \in U_{N_i}$ *such that* $l_{N_i}(m,l) = x$ *and* $l_{N_i}(l,n) = y$.

If $(m,n) \in E_i$ *such that* $m \neq n$ *and* $m,n \in U_{N_{i-1}}$*, then the owner of this edge is the same as in the previous round. The edges* (m,l) *and* (l,n) *and their converses belong to Abelard. The rest of the irreflexive edges belongs to Héloïse.* $\exists$ *wins a match of the game* $\mathcal{G}_n(\mathcal{A})$ *if she can provide a network* N_i *for each move of* $\forall$ *for each* $i \leqslant n$. $\exists$ *has a winning strategy if she can win all matches.*

This lemma has been proved by Hirsch and Hodkinson here [8]. This lemma provide a criterion of representability for relation algebras.

Lemma 3. *Let* $\mathcal{A}$ *be an atomic relation algebra. Then* $\exists$ *has a winning strategy in* $\mathcal{G}_n(\mathcal{A})$ *for each* $n < \omega$ *iff* $\mathcal{A}$ *is elementary equivalent to some completely representable relation algebra. If* $\exists$ *has a winning strategy, then* $\mathcal{A}$ *is representable since* **RRA** *is elementary.*

4.3 The ultraproduct

The second is to show that any non-trivial ultraproduct $\prod\limits_D \mathfrak{A}_n \in \mathbf{RRA}$, where D is an ultrafilter over $\mathcal{P}(\omega)$. We show that via the rainbow technique. Let us define networks and games according to [8].

Lemma 4. *Any non-trivial ultraproduct of* $\{\mathfrak{A}_n\}_{n<\omega}$ *is representable, that is, belongs to* **RRA**. *The same statement for non-trivial ultraproduct of reducts* $\{\mathfrak{S}_n\}_{n<\omega}$ *that belongs to* $\mathbf{R}(\cup, |)$.

According to the following claim, $\exists$ has a winning strategy on cofinitely many algebras that allows her to win a game on the ultraproduct. Thus, according to Lemma 3, the ultraproduct belongs to **RRA**.

Claim 1. *Let* $l < \omega$. $\exists$ *has a winning strategy for* $G_l(\mathfrak{A}_n)$ *for cofinitely many algebras* $\{\mathfrak{A}_n\}_{n<\omega}$.

4.4 The finite algebra on finite base for $\mathbf{R}(\cup, |)$ (or its failure)

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