

Representable cylindric algebras of dimension ω : the aspects of canonicity

Daniel Rogozin

1 Intro

2 The problem itself

Suppose $\mathcal{C} \in \mathbf{RCA}_\omega$, whether \mathcal{C}^+ has a complete, ω -dimensional representation? [4]

3 Boolean algebras with operators and cylindric algebras

Definition 1.

1. Let $\mathcal{B} = \langle B, +, -, 0, 1 \rangle$ be a Boolean algebra. An operator is an n -ary function $\Omega : B^n \rightarrow B$ satisfying the following conditions:

- Normality: for all $b_0, \dots, b_{n-1} \in B$, if $b_i = 0$ for some $i < n$, then

$$\Omega(b_0, \dots, b_{i-1}, 0, b_{i+1}, \dots, b_{n-1}) = 0$$

- Additivity: for all $b_0, \dots, b_{n-1}, b, b' \in B$ we have

$$\begin{aligned} \Omega(b_0, \dots, b_{i-1}, (b + b'), b_{i+1}, \dots, b_{n-1}) = \\ \Omega(b_0, \dots, b_{i-1}, b, b_{i+1}, \dots, b_{n-1}) + \Omega(b_0, \dots, b_{i-1}, b', b_{i+1}, \dots, b_{n-1}) \end{aligned}$$

2. Let I be an index set, a Boolean algebra with operators (BAO) is an algebra $\langle B, +, -, 0, 1, \{\Omega_i\}_{i \in I} \rangle$ such that $\langle B, +, -, 0, 1 \rangle$ is a Boolean algebra and for each $i \in I$ Ω_i is an operator.

Definition 2. Let $\mathcal{B} = \langle B, +, -, 0, 1, \{\Omega_i\}_{i \in I} \rangle$ be a BAO, then

1. An operator Ω is completely additive, if for each $b_0, \dots, b_{n-1} \in B$ and $X \subseteq B$, one has

$$\Omega(b_0, \dots, b_{i-1}, \sum X, b_{i+1}, \dots, b_{n-1}) = \sum_{x \in X} \Omega(b_0, \dots, b_{i-1}, x, b_{i+1}, \dots, b_{n-1})$$

2. \mathcal{B} is completely additive, if for each $i \in I$ Ω_i is additive,
3. A class \mathcal{K} of BAOs is completely additive, if every $\mathcal{B} \in \mathcal{K}$ is completely additive.

3.1 Atom structures and canonical extensions

Definition 3. Let I be an index set and $\{\Omega_i\}_{i \in I}$ a set of function symbols

1. An atom structure is a relational structure $\mathcal{F} = \langle W, \{R_i\}_{i \in I} \rangle$ such that R_i is a $n+1$ -ary relation symbol, if Ω_i is an n -ary function symbol,
2. Let \mathcal{B} be an atomic BAO of the signature I , the atom structure of \mathcal{B} , written as $\mathbf{At}\mathcal{B}$, is an atom structure $\langle \mathbf{At}(\mathcal{B}), \{R_i\}_{i \in I} \rangle$ such that for each $a, b_0, \dots, b_{n+1} \in \mathbf{At}(\mathcal{B})$ and for each $i \in I$

$$\mathbf{At}\mathcal{B} \models R_i(a, b_0, \dots, b_{n+1}) \text{ iff } \mathcal{B} \models a \leq \Omega_i(b_0, \dots, b_{n+1})$$

3. Let $\mathcal{F} = \langle W, \{R_i\}_{i \in I} \rangle$ be an atom structure, the complex algebra of \mathcal{F} , written as $\mathbf{Cm}\mathcal{F}$, is a BAO $\langle \mathcal{P}(W), \cup, -, \emptyset, W, \{\Omega_{R_i}\}_{i \in I} \rangle$ such that for all $X_0, \dots, X_{n-1} \subseteq W$ and for each $i \in I$

$$\Omega_{R_i}(X_0, \dots, X_{n-1}) = \{a \in W \mid \exists b_0 \in X_0 \dots \exists b_{n-1} \in X_{n-1} \mathcal{F} \models R_i(a, b_0, \dots, b_{n-1})\}$$

The following duality is due to Thomason [6].

Fact 1.

1. Let \mathcal{B} be a complete atomic BAO, then $\mathcal{B} \cong \mathbf{Cm}(\mathbf{At}(\mathcal{B}))$,
2. Let \mathcal{F} be an atom structure, then $\mathcal{F} \cong \mathbf{At}(\mathbf{Cm}(\mathcal{B}))$.

Let A be a non-empty subset of a Boolean algebra \mathcal{B} , A is a *filter*, if A is closed under finite infima and upwardly closed. A is an *ultrafilter*, if it has no non-trivial extensions. That is, if $A \subseteq A'$, then $A' = \mathcal{B}$.

Definition 4. Let $\mathcal{B} = \langle B, +, -, 0, 1, \{\Omega_i\}_{i \in I} \rangle$ be a BAO and $\mathbf{Uf}(\mathcal{B})$ the set of its ultrafilters. The ultrafilter frame of \mathcal{B} (or canonical frame) is a relational structure $\mathcal{F}_{\mathcal{B}} = \langle \mathbf{Uf}(\mathcal{B}), R_{\Omega_i} \rangle$ such that for each ultrafilters $\beta_0, \dots, \beta_{n-1}, \gamma$ one has

$$\mathbf{Uf}(\mathcal{B}) \models R_{\Omega_i}(\beta_0, \dots, \beta_{n-1}, \gamma) \text{ iff } \{\Omega_i(b_0, \dots, b_{n-1}) \mid b_0 \in \beta_0, \dots, b_{n-1} \in \beta_{n-1}\} \subseteq \gamma.$$

Definition 5. Let \mathcal{B} be a BAO, then

1. The canonical extension of \mathcal{B} is a complex algebra of the canonical frame $\mathbf{Cm}(\mathcal{F}_{\mathcal{B}})$ denoted as \mathcal{B}^+ ,
2. The class of BAOs is canonical, if it is closed under canonical extensions.

Theorem 1. Let \mathcal{A}, \mathcal{B} be BAOs,

1. There exists $\iota : \mathcal{A} \hookrightarrow \mathcal{A}^+$ such that $\iota : a \mapsto \{\gamma \in \mathbf{Uf}(\mathcal{A}) \mid a \in \gamma\}$.
2. If $i : \mathcal{A} \hookrightarrow \mathcal{B}$, then this embedding might be extended to the embedding $i^+ : \mathcal{A}^+ \hookrightarrow \mathcal{B}^+$

Fact 2.

3.2 (Representable) cylindric algebras and cylindric set algebras

Cylindric algebras provide a generalisation of relation algebras for relations of an arbitrary arity. Let α be an ordinal. Let ${}^\alpha U$ be the set of all functions mapping α to a non-empty set U . We denote $x(i) = x_i$ for $x \in {}^\alpha U$ and $i < \alpha$.

Definition 6.

1. A subset of ${}^\alpha U$ is an α -ry relation on U . For $i, j < \alpha$, the i, j -diagonal D_{ij} is the set of all elements of U such that $y_i = y_j$.

If $i < \alpha$ and X is an α -ry relation on U , then the i -th cylindrification $C_i X$ is the set of all elements of U that agree with some element of X on each coordinate except the i -th one. To be more precise, $C_i X = \{y \in {}^\alpha U \mid \exists x \in X \forall i < \alpha (i \neq j \Rightarrow y_j = x_j)\}$.

2. A cylindric set algebra of dimension α is an algebra consisting of a set S of α -ry relation on some base set U with the constants and operations $0 = \emptyset$, $1 = {}^\alpha U$, \cap , $-$, the diagonal elements $\{D_{ij}\}_{i,j < \alpha}$, the cylindrifications $\{C_i\}_{i < \alpha}$.

A generalised cylindric set algebra of dimension α is a subdirect of cylindric algebras that have dimension α

3. A cylindric algebra of dimension α is an algebra $\mathcal{C} = \langle \mathcal{B}, \{c_i\}_{i < \alpha}, \{d_{ij}\}_{i,j < \alpha} \rangle$ such that

- \mathcal{B} is a Boolean algebra, for each $i, j < \alpha$ c_i is an operator and $d_{ij} \in \mathcal{B}$
- For each $i < \alpha$, $a \leq c_i a$, $c_i(a \wedge c_i b) = c_i a \wedge c_i b$ and $d_{ii} = 1$
- For every $i, j < \alpha$, $c_i c_j a = c_j c_i a$
- If $k \neq i, j < \alpha$, then $d_{ij} = c_k(d_{ij} \wedge d_{jk})$
- If $i \neq j$, then $c_i(d_{ij} \wedge a) \wedge c_i(d_{ij} \wedge -a) = 0$

\mathbf{CA}_α is the class of all cylindric algebras of dimension α

4. An α -dimensional cylindric algebra C is representable, if it is isomorphic to a generalised cylindric set algebra of dimension α . Such an isomorphism is a representation of C .

\mathbf{RCA}_α is the class of all representable cylindric algebras that have dimension α . In particular, we are interested in the case when $\alpha = \omega$.

One may define a representation explicitly in the following way:

Definition 7. Let \mathcal{A} be a cylindric algebra of dimension α . A representation of \mathcal{A} over the non-empty domain X is a map $f : \mathcal{A} \hookrightarrow 2^{{}^\alpha X}$ such that:

1. $f(1) = \bigcup_{i \in I} {}^\alpha X_i$ for some disjoint family $\{X_i\}_{i \in I}$ where each $X_i \subseteq X$
2. $h : \mathcal{A} \rightarrow 2^{f(1)}$ is a representation of a Boolean reduct
3. for all $\lambda, \eta < \alpha$, $x \in h(d_{\lambda\eta})$ iff $x_\lambda = x_\eta$
4. for all $\lambda < \alpha$ and $a \in \mathcal{A}$, $x \in h(c_\lambda(a))$ iff there is $y \in X$ such that $x[\lambda \mapsto y] \in h(a)$

Definition 8. Given a cylindric algebra of dimension α C , let x be a term of its signature, the substitution operator s_j^i have the following definition:

$$s_j^i x = \begin{cases} x, & \text{if } i = j \\ c_i(d_{ij} \wedge x), & \text{otherwise} \end{cases}$$

It is well known that \mathbf{RCA}_α is a variety, \mathbf{RCA}_α ($\alpha \leq 2$) is finitely axiomatisable and \mathbf{RCA}_α ($2 < \alpha < \omega$) has no finite axiomatisation, see [2].

Let $\mathcal{A} \in \mathbf{C}_\omega$, then \mathcal{A} has a *complete representation*, if this representation preserves all existing suprema. In other words, \mathcal{A} is completely representable.

Definition 9. *Atom*

Let us concretise the definition of a canonical extension for α -dimensional cylindric algebras:

Definition 10. Let $\mathcal{C} = \langle C, +, -, 0, 1, \{d_{ij}\}_{i,j < \alpha}, \{c_i\}_{i < \alpha} \rangle$ be a BAO of type \mathbf{CA}_α . Let $\mathbf{Uf}(\mathcal{C})$ be the set of all ultrafilters of \mathfrak{BC} , the Boolean part of \mathcal{C} .

Let us define $\mathbf{C}_i : \mathbf{Uf}(\mathcal{C}) \rightarrow \mathbf{Uf}(\mathcal{C})$ for each $i, j < \alpha$ as

1. $\mathbf{C}_i \mathcal{X} = \{\mathcal{F} \in \mathbf{Uf}(\mathcal{C}) \mid \exists \mathcal{F}' \in \mathbf{Uf}(\mathcal{C}) (a \in \mathcal{F} \Rightarrow c_i a \in \mathcal{F}' R)\},$
2. $D_{ij} = \{\mathcal{F} \in \mathbf{Uf}(\mathcal{C}) \mid d_{ij} \in \mathcal{F}\}.$

The structure $\mathcal{C}^+ = \langle \mathbf{Uf}(\mathcal{C}), \cup, -, \emptyset, C, \mathbf{C}_{i < \alpha}, \{D_{ij}\}_{i,j < \alpha} \rangle$ is called the canonical extension of \mathcal{C} .

Let us discuss the connection between representability and canonical extensions.

The following definitions and facts are due to Henkin, Monk, and Tarski [1].

Let $\mathcal{A} \in \mathbf{CA}_\alpha$ and $x \in \mathcal{A}$. Recall that the *dimension of x* is the set of all ordinals $\gamma < \alpha$ such that $c_\gamma x \neq x$. More formally,

$$\Delta x = \{\gamma \mid \gamma < \alpha \text{ \& } c_\gamma x \neq x\}$$

Let us discuss some metamathematical intuitions standing behind the notion of a dimension. Let Θ be a first-order theory and $\mathcal{C}/\equiv_\Theta$ its Lindenbaum-Tarski algebra. Let φ be a formula in the signature of Θ . Then $\Delta(\varphi/\Theta)$ consists of all $\kappa < \alpha$ such that $\exists x_\kappa \varphi \leftrightarrow \varphi$ is not valid in Θ . That is, $\Delta(\varphi/\Theta)$ contains ordinals κ for which x_κ is free in φ . Moreover, $\Delta(\varphi/\Theta)$ consists only of those ordinals for which x_κ is free in every $\psi \in \varphi/\Theta$.

In particular, an element x is called *zero-dimensional* if $\Delta x = 0$. Zero-dimensional elements reflect equivalence classes of sentences in the Lindenbaum-Tarski algebra of a given first-order theory. Thus, the set of zero-dimensional elements form a Boolean algebras of sentences associated with Θ .

Definition 11. Let \mathcal{A} be an α -dimensional cylindric algebra. Let α be an ordinal and Γ a subset α , then an element $x \in \mathcal{A}$ is Γ -closed if $\Delta x \cdot \Gamma = \emptyset$. Alternatively, x is a Γ -cylinder.

$\text{Cl}_\Gamma \mathcal{A}$ is the set of all Γ -closed elements.

Metamathematically, Γ -closed elements reflect universal closures (is it correct?).

Let $\mathcal{C} = \langle C, +, -, 0, 1, \{d_{ij}\}_{i,j < \beta}, \{c_i\}_{i < \beta} \rangle$ be a β -dimensional cylindric algebra and $\alpha \leq \beta$ an ordinal. The α -th *reduct* of \mathcal{C} , denoted as $\mathfrak{Rd}_\alpha \mathcal{C}$, is an algebra having the form

$$\mathfrak{Rd}_\alpha \mathcal{C} = \langle C, +, -, 0, 1, \{d_{ij}\}_{i,j < \alpha}, \{c_i\}_{i < \alpha} \rangle$$

\mathcal{B} is a subreduct of \mathcal{C} , denoted as $\mathcal{B} \subseteq^r \mathcal{C}$, if $\mathcal{B} \subseteq \mathfrak{Rd}_\gamma \mathcal{C}$ for some $\gamma \leq \beta$.

Definition 12. Let \mathcal{C} be a β -dimensional cylindric algebra and α an ordinal such that $\alpha \leq \beta$. The *neat α -reduct* of \mathcal{C} , denoted as $\mathfrak{Nr}_\alpha \mathcal{C}$, is the subalgebra \mathcal{A} of $\mathfrak{Rd}_\alpha \mathcal{C}$ with $\mathcal{A} = \text{Cl}_\kappa \mathcal{C}$ where $\alpha + \kappa = \beta$.

Let \mathbb{K} be a class of β -dimensional cylindric algebras, then we put

$$\mathbf{Nr}_\alpha \mathbb{K} = \{\mathfrak{Nr}_\alpha \mathcal{C} \mid \mathcal{C} \in \mathbb{K}\}$$

An algebra \mathcal{B} is a neat subreduct of \mathcal{C} , or \mathcal{B} is neatly embeddable to \mathcal{C} if there exists an ordinal $\gamma \leq \alpha$ such that $\mathcal{C} \subseteq \mathfrak{Rd}_\gamma \mathcal{B}$.

One may define neat reducts alternatively as follows. Let \mathcal{C} be a β -dimensional cylindric algebra and α an ordinal such that $\alpha \leq \beta$. The neat α -reduct of \mathcal{C} is the α -dimensional cylindric algebra having the form

$$\mathfrak{Nr}_\alpha \mathcal{C} = \langle \{a \in \mathcal{C} \mid \forall j (\alpha \leq j \ \& \ j < \beta \Rightarrow c_j a = a)\}, +, -, 0, 1, \{d_{ij}\}_{i,j < \alpha}, \{c_\gamma\}_\gamma \rangle$$

4 The result

Lemma 1. *Let \mathcal{A} be a BAO of type \mathbf{CA}_α and \mathcal{B} be a β -dimensional cylindric algebra such that $\beta \leq \alpha$ and \mathcal{A} neatly embeds to \mathcal{B} by a complete embedding.*

1. \mathcal{A}^+ neatly embeds to \mathcal{B}^+ by a complete embedding.
2. \mathcal{A} is atomic.

Proof.

1. See [1, Remark 2.7.25].
2. Is it true?

□

Lemma 2 (See [3]). *A Boolean algebra is completely representable iff it has an atomic representation.*

Definition 13. *Let \mathcal{A} be a BAO of type \mathbf{CA}_ω , an \mathcal{A} -pre-network is a pair $\mathcal{N} = \langle N, l \rangle$, where N is a set of nodes and $l : {}^\omega N \rightarrow \text{At}(\mathcal{A})$.*

\mathcal{N} is a network, if the following conditions hold, for all $x, y \in {}^\omega N$ and $i, j < \omega$:

1. $l(x) \leq d_{ij}$ iff $x(i) = x(j)$
2. $x \equiv_i y$ implies $l(x) \leq c_i l(y)$

Let $\mathcal{N}_1 = \langle N_1, l_1 \rangle$ and $\mathcal{N}_2 = \langle N_2, l_2 \rangle$ be networks, then $\mathcal{N}_1 \subseteq \mathcal{N}_2$ if $N_1 \subseteq N_2$ and $l_1 = l_2 \upharpoonright_{N_1}$.

Let $\Lambda \in \text{Lim}$ and $\{\mathcal{N}_\lambda\}_{\lambda < \Lambda}$ a sequence of networks such that

$$\langle N_0, l_0 \rangle \subseteq \langle N_1, l_1 \rangle \subseteq \dots \subseteq \langle N_\lambda, l_\lambda \rangle \subseteq \dots \text{ for } \lambda < \Lambda$$

then the limit of the sequence $\{\mathcal{N}_\lambda\}_{\lambda < \Lambda}$ is the network

$$\mathcal{N} = \langle N, l \rangle = \bigcup_{\lambda < \Lambda} \langle N_\lambda, l_\lambda \rangle$$

with nodes $N = \bigcup_{\lambda < \Lambda} N_\lambda$ and labelling $l = \bigcup_{\lambda < \Lambda} l_\lambda$, that is, for any $\lambda \in \Lambda$ and $x \in {}^\omega N$ one has $l(x) = l_\lambda(x)$.

The elements of ${}^\omega N$ are called ω -dimensional hyperedges of a network. One may identify a complete representation of an atomic cylindric-type algebra \mathcal{A} with a set $\{\mathcal{N}_a \mid a \in \text{At}(\mathcal{A})\}$ of \mathcal{A} -networks with the following additional condition:

- For each $a \in \text{At}(\mathcal{A})$ there exists $x \in {}^\omega N_a$ such that $l_a(x) = a$ and for each $z \in {}^\omega N_a$ and $b \in \text{At}(\mathcal{A})$, $i < \omega$ with $l_a(z) \leq c_i b$ there exists $y \in {}^\omega N_a$ such that $z \equiv_i y$ and $l_a(y) = b$.

We define a complete representation h of a cylindric-type algebra \mathcal{A} as follows, for any $b \in \mathcal{A}$:

$$h(b) = \{x \mid \exists a \in \text{At}(\mathcal{A}), x \in {}^\omega N_a, l_a(x) \leq b\}$$

Let us define an atomic game.

Definition 14. Let \mathcal{A} be an atomic BAO of type \mathbf{CA}_ω and $\kappa > 0$ a cardinal. The game $\mathcal{G}^\kappa(\mathcal{A})$ is defined as follows. The game has two players: \forall (Abelard) and \exists (Héloïse). A play of the game is the sequence

$$\mathcal{N}_0 \subseteq \mathcal{N}_1 \subseteq \mathcal{N}_2 \subseteq \dots \subseteq \mathcal{N}_\lambda \subseteq \dots \text{ for } \lambda < \kappa$$

The game consists of the following stages:

1. **(Zero round)**

\forall picks an atom $a \in \text{At}(\mathcal{A})$ and \exists plays a network \mathcal{N}_0 . If there is no $x \in {}^\omega N_0$ such that $l_0(x) = a$, then \forall wins the play.

2. **(Successor round)**

Let $0 < \lambda$ be a cardinal such that $\lambda + 1 < \kappa$ and $\mathcal{N}_\lambda = \langle N_\lambda, l_\lambda \rangle$ is a network that has been played.

\forall picks $i < \omega$, $x \in {}^\omega N_\lambda$, $a \in \text{At}$ such that $l_\lambda(x) \leq c_i a$. We denote this move as (i, x, a) . \exists responds with a network $\mathcal{N}_{\lambda+1} \supseteq \mathcal{N}_\lambda$. \forall wins, if there is no node $c \in N$ such that $l_{\lambda+1}(x[i/c]) = a$, then \forall wins

3. The limit of the play is defined as $\bigcup_{\lambda < \kappa} \mathcal{N}_\lambda$. \forall wins the play, if there exists $\kappa_1 < \kappa$ such that \exists does not win the κ_1 th-round. Otherwise, \exists wins the play.

Theorem 2. Let \mathcal{A} be an atomic ω -dimensional cylindric-type algebra and κ a cardinal such that $|\text{At}(\mathcal{A})| = \kappa$, then the following are equivalent:

1. \mathcal{A} is completely representable.
2. \exists has a winning strategy in $\mathcal{G}^{\kappa+\omega}$.

Proof.

1. \Rightarrow If \mathcal{A} is completely representable, then its Boolean reduct is representable as well. \exists maintains that embedding to win the play.

2. \Leftarrow

Suppose \exists has a winning strategy in $\mathcal{G}^{\kappa+\omega}(\mathcal{A})$. In every round \forall picks all possible $i < \omega$, $a \in \text{At}(\mathcal{A})$, all possible hyperedges and all appropriate atoms. ... Question: Is this claim correct?

□

Theorem 3 (This assumption is by Ian Hodkinson).

Let \mathcal{A} be a BAO of type \mathbf{CA}_ω such that \mathcal{A} neatly embeds into $\mathbf{CA}_{\omega+\omega}$ by a complete embedding. Then \mathcal{A} is completely representable as \mathbf{CA}_ω .

Proof. Suppose $\mathcal{A} \subseteq \mathfrak{Rt}_\omega \mathcal{B}$, where $\mathcal{B} \in \mathbf{RCA}_{\omega+\omega}$ and the inclusion map $\rho : \mathcal{A} \hookrightarrow \mathfrak{Rt}_\omega \mathcal{B}$ is a complete embedding, that is:

$$\rho(\sum_{i \in I} a_i) = \sum_{i \in I} (\rho a_i), \text{ if } \sum_{i \in I} a_i \text{ exists.}$$

Let us show that \mathcal{A} is atomic.

Consider $\rho(\mathcal{A})$. Let us show that \exists has a winning strategy on $\mathcal{G}^{\kappa+\omega}(\rho(\mathcal{A}))$ □

Lemma 1 and Theorem 3 imply the following theorem.

Theorem 4. *Let $\mathcal{C} \in \mathbf{RCA}_\omega$, then $\mathcal{C}^+ \in \mathbf{RCA}_\omega$. That is, \mathbf{RCA}_ω is closed under canonical extensions.*

Proof. □

5 Sahlqvist and canonical axiomatisation of \mathbf{CA}_ω

6 Notes on the canonicity of \mathbf{RRA}

Definition 15.

A relation algebra is an algebra $\mathcal{R} = \langle R, 0, 1, +, -, ;, \smile, \mathbf{1}' \rangle$ such that $\langle R, 0, 1, +, - \rangle$ is a Boolean algebra and the following equations hold, for each $a, b, c \in R$:

1. $a; (b; c) = (a; b); c$
2. $(a + b); c = (a; c) + (b; c)$
3. $a; \mathbf{1}' = a$
4. $a^{\smile\smile} = a$
5. $(a + b)^{\smile} = a^{\smile} + b^{\smile}$
6. $(a; b)^{\smile} = b^{\smile}; a^{\smile}$
7. $a^{\smile}; -(a; b) \leq -b$

where $a \leq b$ iff $a + b = b$. \mathbf{RA} denotes the class of all relation algebras.

We will adapting the following proof of the fact that \mathbf{RRA} is canonical ¹ to our case. This proof is due to Monk, but that was describe in McKenzie's thesis [5].

1. A relation algebra \mathcal{A} is representable iff \mathcal{A} neatly embeds to some ω -dimensional cylindric algebra,
2. If \mathcal{A} neatly embeds in \mathcal{A} then \mathcal{A}^+ neatly embeds in \mathcal{B}^+ ,
3. \mathbf{CA}_α is closed under canonical extensions,
4. Voilà.

Definition 16. Let $\mathcal{C} \in \mathbf{CA}_\alpha$, where $\alpha \geq 3$. The relation algebra reduct of \mathcal{C} , written as $\mathfrak{Ra}(\mathcal{C})$, is the algebra having the form

¹This idea is by Ian Hodkinson

$$\langle \text{dom}(\mathfrak{Nr}_2(\mathcal{C})), 0, 1, +, -, \mathbf{1}', \smile, ; \rangle$$

where:

1. $+$, $-$, 0 , and 1 are defined as usual in \mathcal{C} ,
2. $\mathbf{1}' = d_{01} \in \mathfrak{Nr}_2(\mathcal{C})$,
3. $r \smile = s_0^2 s_1^0 s_2^1 r$ for $r \in \mathfrak{Nr}_2(\mathcal{C})$,
4. Let $r, s \in \mathfrak{Nr}_2(\mathcal{C})$, then $r; s = c_2(s_2^1 r \cdot s_2^0 s)$

Moreover, $\mathfrak{Nr}_\beta(\mathcal{C})$ and $\mathfrak{Ra}(\mathcal{C})$ are closed under these operations. There is also the following fact by due to Henkin, Monk, and Tarski [2]:

Theorem 5. *Let $\mathcal{C} \in \mathbf{CA}_\alpha$ for $\alpha \geq 4$, then $\mathfrak{Ra}(\mathcal{C})$ is a relation algebra.*

The following characterisation results are by Henkin, Monk, and Tarski [2, 5.3.13, 5.3.16] as well:

Theorem 6.

1. $\mathbf{RA} = \mathbf{SRA} \mathbf{CA}_4$,
2. $\mathbf{RRA} = \bigcap_{3 \leq n < \omega} \mathbf{SRA} \mathbf{CA}_n = \mathbf{SRA} \mathbf{CA}_\alpha$, where α is an infinite ordinal.

Let $\mathcal{C} \in \mathcal{CA}_\alpha$, then $\mathcal{R} \in \mathbf{RA}$ neatly embeds to \mathcal{C} , if \mathcal{R} is isomorphic to some subalgebra of $\mathfrak{Ra}(\mathcal{C})$.

Theorem 7. *\mathbf{RRA} is closed under canonical extensions.*

Proof. Let $\mathbf{R} \in \mathbf{RRA}$. By the second item of 6, every representable relation algebra is isomorphic to some subalgebra of the relation algebra reduct $\mathfrak{Ra}\mathcal{C}$ for some $\mathcal{C} \in \mathbf{CA}_\omega$. But neat embeddings respect canonical extensions, so if $\mathbf{R} \hookrightarrow_n \mathcal{C}$, so is $\mathbf{R}^+ \hookrightarrow_n \mathcal{C}^+$. \mathbf{CA}_α is closed under canonical extensions, so is \mathbf{RRA} . \square

References

- [1] L Henkin, J D Monk, and A Tarski. *Cylindric Algebras Part I*. North-Holland, 1971.
- [2] Leon Henkin, J. Donald Monk, and Alfred Tarski. Cylindric algebras. part ii. *Journal of Symbolic Logic*, 53(2):651–653, 1988.
- [3] Robin Hirsch and Ian Hodkinson. Complete representations in algebraic logic. *Journal of Symbolic Logic*, pages 816–847, 1997.
- [4] Robin Hirsch and Ian Hodkinson. *Relation algebras by games*. Elsevier, 2002.
- [5] Ralph McKenzie. The representation of relation algebras. *PhD Thesis, University of Colorado*, 1968.
- [6] Steven K Thomason. Categories of frames for modal logic. *The journal of symbolic logic*, 40(3):439–442, 1975.