# Characterising representable positive relation algebras with Priestley duality

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## 1 Distributive lattice representation and Priestley duality

Given a bounded distributive lattice $\mathcal{L}$, a proper subset $F \subset \mathcal{L}$ is said to be a *filter* if it is upward closed and closer under finite infima. A filter $F$ is *prime* if $a + b \in F$ implies either $a \in F$ or $b \in F$. The spectrum of $\mathcal{L}$, denoted as $\mathrm{Spec}(\mathcal{L})$, is the set of all prime filters.

A filter is *complete* if whenever $\Pi T$ exists for $T \subseteq F$, then $\Pi T \in F$. A filter is *completely prime* if whenever $\Sigma T$ exists for $T \subseteq F$, then there exists $t \in T$ such that $t \in F$. The dual definitions are for ideals.

**Proposition 1.** *Let $h : \mathcal{L} \to \mathcal{R}$ be a representation, then then*

$$h^{-1}[x] = \{a \in \mathcal{L} \mid x \in h(a)\} \in \mathrm{Spec}(\mathcal{L})$$

Recall that a *Priestley space* is a triple $\mathcal{X} = (X, \tau, \leqslant)$ such that $(X, \tau)$ is a compact topological space, $(X, \leqslant)$ is a bounded poset such that if $x \not\leqslant y$, then there exists a clopen $U$ such that $x \in U$ and $y \notin U$. Given a bounded distributive lattice $\mathcal{L}$, define the map $\phi : \mathcal{L} \to 2^{\mathrm{Spec}(\mathcal{L})}$ such that

$$\phi : a \mapsto \{F \in \mathrm{Spec}(\mathcal{L}) \mid a \in F\}$$

**Fact 1.**

1. *The sets $\phi(a)$ and $-\phi(a)$ form the subbasis of the topology $\tau$ on* $\mathrm{Spec}(\mathcal{L})$*.*
2. $(\mathrm{Spec}(\mathcal{L}), \tau, \subseteq)$ *is a Priestley space.*

Given a Priestley space $\mathcal{X} = (X, \tau, \leqslant)$, the set $\mathrm{ClOp}(\mathcal{X})$ consists of all clopens of $\mathcal{X}$. The structure $(\mathrm{ClOp}(\mathcal{X}), \cap, \cup, \varnothing, X)$ is a distributive lattice.

**Fact 2.** *Let $\mathcal{L}$ be a distributive lattice and let $\mathcal{X}$ be a Priestley space:*

1. $\mathcal{L} \hookrightarrow \mathcal{L}^{+} = (2^{\mathrm{Spec}(\mathcal{L})}, \cap, \cup, \varnothing, \mathrm{Spec}(\mathcal{L}))$,
2. $\mathcal{L} \cong \mathrm{ClOp}(\mathrm{Spec}(\mathcal{L}))$,
3. $\mathcal{X} \cong \mathrm{Spec}(\mathrm{ClOp}(\mathcal{X}))$,
4. *The categories of Priestley spaces and bounded distributive lattices are dually equivalent.*

### 1.1 Completely representable distibutive lattices

Let $\mathcal{L}$ be a bounded distributive lattice, then a set $S \subseteq 2^{\mathcal{L}}$ is said to be distinguishing if for every $a, b \in \mathcal{L}$ such that $a \neq b$ there exists $s \in S$ such that either $a \in s$ and $b \notin b$ or vice versa.

**Theorem 1.** *Let $\mathcal{L}$ be a bounded distributive lattice, then*

1. *$\mathcal{L}$ is completely representable iff $\mathcal{L}$ has a distinguishing set of complete, completely prime filters,*
2. *$(\mathcal{L}_+)^+$ is completely representable.*

TODO: read [EH12]

# 2 Representatiting positive relation algebras

**Definition 1.** *A positive relation algebra is a algebra $\mathcal{R} = (R, \cdot, +, ;, ^\smile, 0, 1, \mathbf{1'})$ such that*

1. *$(R, \cdot, +, 0, 1)$ is a bounded distributive lattice,*
2. *$(R, ;, \mathbf{1'})$ is a monoid,*
3. *for all $a, b, c \in R$*
   (a) $a; (b + c) = a; b + a; c$,
   (b) $a^{\smile\smile} = a$,
   (c) $(a + b)^\smile = a^\smile + b^\smile$,
   (d) $(a; b)^\smile = b^\smile; a^\smile$,
   (e) $(a; b) \cdot c^\smile = 0 \leftrightarrow (b; c) \cdot c^\smile = 0$.

A positive relation algebra $\mathcal{R}$ is *representable* if there exists a one-to-one function $h : \mathcal{R} \to 2^{X \times X}$ over the base set $X \neq \varnothing$ such that:

- $f(a \cdot b) = f(a) \cap f(b)$,
- $f(a + b) = f(a) + f(b)$,
- $f(0) = \varnothing$,
- $f(1) = \bigcup_{a \in \mathcal{R}} f(a)$,
- $f(\mathbf{1'}) = \Delta_X$,
- $f(a; b) = \{(x, z) \mid \exists y \in X\ ((x, y) \in f(a)\ \&\ (y, z) \in f(b))\} = f(a)|f(b)$,
- $f(a^\smile) = \{(y, x) \mid (x, y) \in f(a)\}$.

A positive relation algebra is *completely representable* if it is representable and its bounded distributive lattice reduct is completely representable.

# 3 Priestley spaces for positive relation algebras

A PRA-space is a structure $(X, \tau, \leqslant, R, I, E)$ where $X = (X, \tau, \leqslant)$ is a Priestley space and $R \subseteq X^3$, $I \subseteq X^2$ and $E \subseteq X$ such that:

- For all $x, y, z, w \in X$ there exists $u \in X$ such that $R(x, y, u)$ and $R(u, z, w)$ iff there exists $v \in X$ such that $R(y, z, v)$ and $R(x, v, w)$,
- If $A, B \subseteq X$ are upward closed, so is $R[A, B, \_]$,
- $I(A)$ is upward closed clopen whenever $A$ is upward closed clopen,
- $I(x)$ is closed for each $x \in X$,
- For all $x, y, z \in X$, $x \leqslant y$ and $I(x, z)$ imply $I(y, z)$,
- For all $x, y \in X$ there exists $z \in X$ such that $x = y$ iff $I(z, y)$ and $I(x, z)$,
- For all $x, y, z \in X$ there exists $u \in X$ such that $I(u, z)$ and $R(x, y, u)$ iff there exist $u, w \in X$ such that $R(v, w, z)$, $I(y, v)$ and $I(x, w)$.
- For all $x, y, u, v \in X$, $R(u, v, y)$ and $I(x, u)$ implies $R(x, y, v)$,
- $E$ is upward closed clopen such that for each clopen $A \subseteq X$ one has

$$R[A, E, \_] = R[E, A, \_] = A$$

**Fact 3.** *Let $\mathcal{R}$ be a positive relation algebra, then*

1. *If $F \subseteq \mathcal{R}$ is a filter, then for each $X, Y \subseteq \mathcal{R}$ $X; Y \subseteq F$ iff $\uparrow (X; Y) \subseteq F$*
2. *If $F_1, F_2 \subseteq \mathcal{R}$ are filters, so is $\uparrow (F_1; F_2)$,*
3. *Let $F_1, F_2$ be filters and $F_3 \in \mathrm{Spec}(\mathcal{R})$ such that $\uparrow (F_1; F_2) \subseteq F_3$, then there are $F_1', F_2' \in \mathrm{Spec}(\mathcal{R})$ such that $F_1 \subseteq F_1'$, $F_2 \subseteq F_2'$ and $\uparrow (F_1'; F_2') \subseteq F_3$.*

**Theorem 2.** *Let $\mathcal{R}$ be a positive relation algebra and $\mathcal{X}$ a PRA-space, then*

1. $\mathcal{R} \hookrightarrow (2^{\mathrm{Spec}(R)}, \cup, \cap, \varnothing, \mathrm{Spec}(R), \bullet, \iota, \epsilon)$,
2. $\mathcal{R} \cong \mathrm{ClOp}(\mathrm{Spec}(\mathcal{R}))$,
3. $\mathcal{X} \cong \mathrm{Spec}(\mathrm{ClOp}(\mathcal{X}))$,
4. *The categories of positive relation algebras and PRA-spaces are dually equivalent.*

# 4 The main result

For that we need such model theoretic notions as saturation and types, see [Hod93, Section 6.3].

**Definition 2.** *Let $\mathcal{M}$ be a first-order structure of a signature $L$ and $S \subseteq \mathcal{M}$. Let $L(S)$ be an extension of $L$ with copies of elements from $S$ as additional constants. We assume that $Cnst(L)$ and $S$ are disjoint.*

1. *Let $n < \omega$, an $n$-type over $S$ is a set $\mathcal{T}$ of $L(S)$ formulas $A(\overline{x})$, where $\overline{x}$ is a fixed $n$-tuple of elements from $S$. Notation: $\mathcal{T}(\overline{x})$. A type is an $n$-type for some $n < \omega$.*

2. *An $n$-type $\mathcal{T}(\overline{x})$ is realised in $\mathcal{M}$, if there exists $\overline{m} \in \mathcal{M}^n$ such that $\mathcal{M} \models A(\overline{m})$ for every $A \in \mathcal{T}(\overline{x})$. $\mathcal{M}$ omits $\mathcal{T}(\overline{x})$, if $\mathcal{T}(\overline{x})$ is not realised in $\mathcal{M}$.*

3. *$\mathcal{T}(\overline{x})$ is finitely satisfied in $\mathcal{M}$, if every finite subtype $\mathcal{T}_0(\overline{x}) \subseteq \mathcal{T}(\overline{x})$ is realised in $\mathcal{M}$. We can reformulate that as $\mathcal{M} \models \exists \overline{a} \bigwedge\limits_{A \in \mathcal{T}_0} A(\overline{a})$.*

4. *Let $T$ be a theory, then a type $\mathcal{T}$ over the empty set of constants is $T$-consistent, if there exists a model $\mathcal{M} \models T$ such that $\mathcal{T}$ is finitely satisfied in $\mathcal{M}$.*

5. *Let $\kappa$ be a cardinal, then $\mathcal{M}$ is $\kappa$-saturated, if for every $S \subseteq \mathcal{M}$ with $|S| < \kappa$ every finitely satisfied 1-type $\mathcal{T}$ is realised in $\mathcal{M}$.*

By default, a saturated model is an $\omega$-saturated model for us.
The useful facts, they are from [CK90] and [Hod93]:

**Fact 4.** *Let $\mathcal{M}$ be an FO-structue and $\kappa$ a cardinal, then:*

1. *$\mathcal{M}$ is $\kappa$-saturated iff every finitely satisfiable $\alpha$-type (an arbitrary $\alpha \leqslant \kappa$) with fewer than $\kappa$ parameters is realised in $\mathcal{M}$.*

2. *If $\mathcal{M}$ is $\kappa$-saturated, then $\mathcal{M}$ is $\lambda$-saturated for every $\lambda < \kappa$.*

3. *Every consistent theory has a $\kappa$-saturated model and every model has an elementary $\kappa$-saturated extension.*

4. *Let $(\mathcal{M}_i)_{i<\omega}$ a family of structures of the (at most) countable signature and $D$ a non-principal ultrafilter over $\omega$, then $\Pi_D \mathcal{M}_i$ is $\omega_1$-saturated.*

Let $\mathcal{A}$ be a positive relation algebra, define the first-order relational language of the form

$$\mathcal{L}(\mathcal{A}) = (=, \{R^2_a\}_{a \in \mathcal{A}})$$

The $\mathcal{L}(\mathcal{A})$-theory $T_{\mathcal{A}}$ consists of the following statements:

- $\sigma_1 = \forall x \forall y (\mathbf{1'}(x, y) \leftrightarrow (x = y))$
- $\sigma_+(R, S, T) = \forall x \forall y (R(x, y) \leftrightarrow S(x, y) \vee T(x, y))$
- $\sigma_\cdot(R, S, T) = \forall x \forall y (R(x, y) \leftrightarrow S(x, y) \wedge T(x, y))$
- $\sigma_;(R, S, T) = \forall x \forall y (R(x, y) \leftrightarrow \exists z (S(x, z) \wedge T(z, y)))$
- $\sigma_{\smile}(R, S) = \forall x \forall y (R(x, y) \leftrightarrow S(y, x))$
- $\sigma_{\neq 0} = \exists x \exists y R(x, y)$ for any $R_a$ such that $a \neq 0$
- $\sigma_0 = \neg \exists x \exists y 0(x, y)$
- $\sigma_{\mathbf{1}} = \forall x \forall y (R(x, y) \to \mathbf{1}(x, y))$

**Proposition 2.** *$T_{\mathcal{A}}$ is satisfiable whenever $\mathcal{A}$ is representable.*

**Theorem 3.** *Let $\mathcal{A}$ be a positive relation algebra, then $\mathcal{R}$ is representable iff $\mathcal{R}^+$ is completely representable.*

*Proof.* The right-to-left implication is easy. If $\mathcal{R}^+$ is representable (no completeness needed here), so is $\mathrm{ClOp}(\mathrm{Spec}(\mathcal{L}))$ as a subalgebra of $\mathcal{R}^+$. But, by Priestley duality for positive relation algebras, $\mathrm{ClOp}(\mathrm{Spec}(\mathcal{L})) \cong \mathcal{L}$, so $\mathcal{L}$ is representable.

Assume that $\mathcal{R}$ is representable, then $T_{\mathcal{R}}$ is satisfiable, let $M \models T_{\mathcal{R}}$ and $M$ is $\omega$-saturated. Define a map $h : \mathcal{R}^+ \to 2^{M\times M}$ as

$$h : F \mapsto \{(x, y) \in 1^M \mid f_{x,y} \in F\}$$

where

$$f_{x,y} = \{a \in \mathcal{R} \mid M \models R_a(x, y)\}$$

**Claim 1.** *$f_{x,y} \in \mathrm{Spec}(\mathcal{R})$ whenever $M \models 1(x, y)$.*

*Proof.* Let $a \in f_{x,y}$ and $a \leqslant b$.

Then $M \models R_a(x, y)$. But $a \leqslant b$ iff $a \cdot b = a$, so, from the axiom $R.(R_a, R_a, R_b)$ we have $M \models R_b(x, y)$.

Let $a, b \in f_{x,y}$, so $M \models R_a(x, y) \wedge R_b(x, y)$, so $R_{a\cdot b}(x, y)$ by the axiom $R.(R_a, R_a, R_b)$ again.

Let $a + b \in f_{x,y}$, so $M \models R_{a+b}(x, y)$, then we have either $M \models R_a(x, y)$ or $M \models R_b(x, y)$, so either $a \in f_{x,y}$ or $b \in f_{x,y}$. □

**Claim 2.** *$h$ is one-to-one.*

**Claim 3.** *$h$ is a complete representation of the bounded lattice reduct.*

*Proof.* Follows from the fact that $\mathrm{Spec}(\mathcal{L})$ is a Hausdorff space. (????) □

**Claim 4.** *$h$ preserves the structure of an involutive monoid.*

□

**Theorem 4. RPRA** *is a canonical variety. (is it a variety at all?)*

# 5 Union-free reducts of positive relation algebras

TODO: [BJ11].

# References

[AM11] Hajnal Andréka and Szabolcs Mikulás. Axiomatizability of positive algebras of binary relations. *Algebra universalis*, 66(1-2):7, 2011.

[BJ11] Guram Bezhanishvili and Ramon Jansana. Priestley style duality for distributive meet-semilattices. *Studia Logica*, 98:83–122, 2011.

[CK90] Chen Chung Chang and H Jerome Keisler. *Model theory.* Elsevier, 1990.

[EH12] Robert Egrot and Robin Hirsch. Completely representable lattices. *Algebra universalis*, 67(3):205–217, 2012.

[Gal03] Nikolaos Galatos. *Varieties of residuated lattices.* Vanderbilt University, 2003.

[Geh14] Mai Gehrke. Canonical extensions, esakia spaces, and universal models. *Leo Esakia on duality in modal and intuitionistic logics*, pages 9–41, 2014.

[GJ04] Mai Gehrke and Bjarni Jónsson. Bounded distributive lattice expansions. *Mathematica Scandinavica*, pages 13–45, 2004.

[Gol89] Robert Goldblatt. Varieties of complex algebras. *Annals of pure and applied logic*, 44(3):173–242, 1989.

[HH97] Robin Hirsch and Ian Hodkinson. Complete representations in algebraic logic. *The Journal of Symbolic Logic*, 62(3):816–847, 1997.

[HH02] Robin Hirsch and Ian Hodkinson. *Relation algebras by games*. Elsevier, 2002.

[HM00] Ian Hodkinson and Szabolcs Mikulás. Axiomatizability of reducts of algebras of relations. *Algebra universalis*, 43:127–156, 2000.

[Hod93] Wilfrid Hodges. *Model theory*. Cambridge University Press, 1993.

[Jón82] Bjarni Jónsson. Varieties of relation algebras. *Algebra universalis*, 15(3):273–298, 1982.