Notes on filtrations for logics that contain **K5**

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1 Preliminaries

Let $\mathcal{M} = \langle W, R_1, \ldots, R_n, \vartheta \rangle$ be a Kripke model and Γ a set of formulas closed under subformulas. An equivalence relation $\sim$ is set to have a finite index if the quotient set $W/\sim$ is finite. The equivalence relation $\sim_\Gamma$ induced by Γ is defined as

$$w \sim_\Gamma v \Leftrightarrow \forall \varphi \in \Gamma \, (\mathcal{M}, w \models \varphi \Leftrightarrow \mathcal{M}, v \models \varphi).$$

If Γ is finite, then $\sim_\Gamma$ has a finite index. An equivalence relation $\sim$ respects $\sim_\Gamma$, if $w \sim v$ implies $w \sim_\Gamma v$.

Definition 1. *Let $\mathcal{M} = \langle W, R_1, \ldots, R_n, \vartheta \rangle$ be a Kripke model and Γ be a Sub-closed set formulas. A Γ-filtration of $\mathcal{M}$ is a model $\widehat{\mathcal{M}} = \langle \widehat{W}, \widehat{R_1}, \ldots, \widehat{R_n}, \widehat{\vartheta} \rangle$ such that:*

1. *$\widehat{W} = W/\sim$, where $\sim$ is an equivalence relation having a finite index that respects Γ*

2. *$\widehat{\vartheta}(p) = \{[x]_\sim \mid x \in W \,\&\, x \in \vartheta(p)\}$*

3. *For each $i \in I$ one has $\widehat{R}_i^{min} \subseteq \widehat{R}_i \subseteq \widehat{R}_i^{max}$. $\widehat{R}_{i,\sim}^{min}$ is the i-th minimal filtered relation on $\widehat{W}$ defined as*

$$\hat{x} \widehat{R}_{i,\sim}^{min} \hat{y} \Leftrightarrow \exists x' \sim x \, \exists y' \sim y \; x R_i y$$

$\widehat{R}_{\Gamma,i}^{max}$ is the i-th maximal filtered relation on $\widehat{W}$ induced by Γ defined as

$$\hat{x} \widehat{R}_{\Gamma,i}^{max} \hat{y} \Leftrightarrow \forall \square_i \varphi \in \Gamma \, (\mathcal{M}, x \models \square_i \varphi \Rightarrow \mathcal{M}, y \models \varphi)$$

If Φ is finite subset of Γ and $\sim = \sim_\Phi$, then $\widehat{\mathcal{M}}$ is a definable Γ-filtration of $\mathcal{M}$ through Φ. If $\sim = \sim_\Gamma$, then such a filtration by means of the definiton above is called *strict*. A class of models $\mathbb{M}$ admits strict filtrations for models (ASF), if for every Sub-closed set Γ and for every $\mathcal{M} \in \mathbb{M}$ there exists a Γ filtration of $\mathcal{M}$. A class of frames $\mathbb{F}$ admits strict filtrations for frames, if for every Sub-closed set Γ and for every frame $\mathcal{F} \in \mathbb{F}$ and every model $\mathcal{M}$ over $\mathcal{F}$ there exists a Γ filtration of $\mathcal{M}$. If $\mathcal{L}$ is canonical, then the ASF property for frames and ASF property for models are equivalent [1, Theorem 2.10].

Lemma 1. *Let Γ be a finite set of formulas closed under subformulas and $\widehat{\mathcal{M}}$ a filtration of $\mathcal{M}$ through Γ, then for each $x \in W$ and for each $\varphi \in \Gamma$ one has*

$$\mathcal{M}, x \models \varphi \Leftrightarrow \widehat{\mathcal{M}}, \hat{x} \models \varphi$$

Definition 2. *Let $\mathbb{F}$ be a class of Kripke frames and Γ a finite set of formulas closed under subformulas. If for every model $\mathcal{M}$ over $\mathcal{F} \in \mathbb{F}$ there exists a model that is a Γ-definable filtration of $\mathcal{M}$, then $\mathbb{F}$ admits definable filtration. A class of models $\mathbb{M}$ admits definable filtration if for every $\mathcal{M} \in \mathbb{M}$ there exists a model belonging to the same class that is a definable Γ-filtration of $\mathcal{M}$.*

Lemma 2.

1. *Let $\mathcal{L}$ be a complete normal modal logic. If $\mathrm{Frames}(\mathcal{L})$ admits filtration, then $\mathcal{L}$ has the finite model property.*

2. *If the class of models $\mathrm{Mod}(\mathcal{L})$ admits filtration, then $\mathcal{L}$ has the finite model property and it is Kripke complete as well.*

Definition 3. *A first-order formula is called Horn if it has the following form:*

$$\forall x_1, \ldots, x_n (x_{i_1} R x_{j_1} \wedge \cdots \wedge x_{i_s} R x_{j_s} \rightarrow x_k R x_l)$$

Definition 4. *Let H be a Horn property and $\langle W, R \rangle$ a Kripke frame. A Horn closure of a binary relation R is the minimal relation R^H containing R and satisfying H.*

Lemma 3. $R^H = \bigcup\limits_{n < \omega} R_n$ *where*

1. $R_0 = R$.

2. $R_{n+1} = R_n \cup \{(a,b) \in W \mid \exists \vec{c} \in W \, P(a,b,\vec{c})\}$, *where P is a premise of H.*

2 Filtrations for K5

E-closure (an Euclidean Horn closure of a binary relation) has the following equivalent definitions:

Lemma 4. *Let $\mathcal{F} = \langle W, R \rangle$ be a Kripke frame. The following conditions are equivalent:*

1. R^E *is the smallest Euclidean relation containing R.*

2. $R^E = \bigcup\limits_{i < \omega} R_i$, *where*

   - $R_0 = R$
   - $R_{n+1} = R_n \cup (R_n^{-1} \circ R_n)$

3. $x R^E y$ *iff there exists $n < \omega$ such that either xRy or $\exists z_1, \ldots, z_n$ with $z_1 R x$ and $z_{n-1} R y$ and for each $1 < i \leqslant n$ one has either $z_{i-1} R z_i$ or $z_i R z_{i-1}$.*

4. $R^E = R \cup \bigcup\limits_{i < \omega} (R^{-1} \circ (R \circ R^{-1})^n \circ R)$.

Proof.

1. $(1) \Rightarrow (2)$ Let us show that if R^E is the smallest Euclidean relation containing R, then $R^E = \bigcup\limits_{i < \omega} R_i$. There are two inclusions:

   - $R^E \subseteq \bigcup\limits_{i < \omega} R_i$. Recall that R^E has the form (?):

$$R^E = \bigcap \{R' \mid R \subseteq R', \forall a, b \in W\ R'(a,b) \Rightarrow \exists x \in W\ R'(x,a)\ \&\ R'(x,b)\}$$

- $\bigcup_{i<\omega} R_i \subseteq R^E$. Let us show that xR_ny for each $n < \omega$ implies xR^Ey by induction on n.

  If $n = 0$, then xRy, thus, xR^Ey, since R is a subrelation of R^E. Suppose $n = m+1$ and $xR_{m+1}y$. Let us show that xR^Ey. From $xR_{m+1}y$, one has $(x,y) \in R^n \cup (R_n^{-1} \circ R_n)$. There are two cases:

  - xR^ny, one needs to merely apply the IH.
  - $xR_n^{-1} \circ R_n y$. Then $\exists z \in W\ xR_n^{-1}z\ \&\ zR_n$. That is, zR_nx and zR_ny for some z. R_n is already a subrelation of R^E. Thus, zR^Ex and zR^Ey. That implies xR^Ey.

2. $(2) \Rightarrow (3)$ Let $(x,y) \in R_m$, let us the statement by induction on m.

   (a) Suppose $m = 0$, then xRy, and the statement is shown putting $n = 0$.

   (b) Suppose $m = p + 1$ and $xR_{p+1}y$. Assume that either xRy or $\exists z_1, \ldots, z_p$ with z_1Rx and $z_{p-1}Ry$ and for each $1 < i \leqslant p$ one has either $z_{i-1}Rz_i$ or z_iRz_{i-1}.

   $xR_{p+1}y$ implies $(x,y) \in R_p \cup (R_p^{-1} \circ R_p)$. If $(x,y) \in R_p$, then we merely apply the IH. Suppose $(x,y) \in R_p^{-1} \circ R_p$, then $(z,x) \in R_p$ and $(z,y) \in R_p$

3. $(3) \Rightarrow (4)$ Suppose either xRy or there exist $n \geqslant 1$ and $z_1, \ldots, z_n$ with z_1Rx and $z_{n-1}Ry$ and for each $1 < i \leqslant n$ one has either $z_{i-1}Rz_i$ or z_iRz_{i-1}. If xRy, then we are done. Otherwise there exists $n \geqslant 1$ with the condition above. Then $(x,y) \in R_{n+1}$ that follows from the condition.

4. $(4) \Rightarrow (1)$

$\square$

Lemma 5. *Let $\mathcal{F} = \langle W, R \rangle$ be a Kripke frame. Let us define $R^E = \bigcup_{i<\omega} R_i$ where:*

1. $R_0 = R$

2. $R_{n+1} = R_n \cup (R_n^{-1} \circ R_n)$

Then R^E is Euclidean.

Proof. Let $(x,y), (x,z) \in R^E$, one needs to show that $(y,z) \in R^E$. Clearly that $(x,y) \in R_i$ and $(x,z) \in R_j$ for some $i, j < \omega$. Thus, we need $(y,z) \in R_m$ for some m depending on i and j.

Let us consider the following cases:

1. $i = 0$ and $j = 0$

   Suppose $(x,y), (x,z) \in R_0 = R$, then $(y,z) \in R^{-1} \circ R$. Thus, $(y,z) \in R_1$

2. $i = 0$ and $j = k + 1$

   Suppose $(x,y) \in R$ and $(x,z) \in R_{k+1} = R_k \cup (R_k^{-1} \circ R_k)$. Clearly that $(x,y) \in R_{k+1}$ as well. It is obviously that $(y,z) \in R_{k+2}$ since $(y,x) \in R_{k+1}^{-1}$ and $(x,z) \in R_{k+1}$.

3. The case with $i = k + 1$ and $j = 0$ is similar to the previous one.

4. Suppose $i = m + 1$ and $j = k + 1$. That is, $(x,y) \in R_{m+1} = R_m \cup (R_m^{-1} \circ R_m)$ and $(x,z) \in R_{k+1} = R_k \cup (R_k^{-1} \circ R_k)$. Consider the following four subcases:

(a) Suppose $(x, y) \in R_m$ and $(x, z) \in R_k$ and $m \leqslant k$ without loss of generality. $m \leqslant k$ implies $R_m \subseteq R_k$ and $(x, y) \in R_k$ in particular. Thus, $(y, z) \in R_k^{-1} \circ R_k$, so $(y, z) \in R_{k+1}$.

(b) The rest of the cases are similar to the first one.

$\square$

Theorem 1. K45 *admits strict filtrations.*

Proof. Let $\mathcal{M} = \langle W, R, \vartheta \rangle$ be a transitive Euclidean model and $\overline{\mathcal{M}} = \langle \overline{W}, \overline{R}, \overline{\vartheta} \rangle$ its minimal filtration through Γ, where Γ is finite and Sub-closed. Let us put $\widehat{R} = \overline{R}^+ \cup \overline{R}^E$. Let us show that $\overline{R}^+ \cup \overline{R}^E \subseteq \overline{R}^{max}$.

That is, if $\mathcal{M}, y \models \varphi$ for $\Diamond \varphi \in \Gamma$ and $\hat{x} \widehat{R} \hat{y}$, then $\mathcal{M}, x \models \Diamond \varphi$.

Let $\hat{x} \widehat{R} \hat{y}$. Let us consider the case when $(\hat{x}, \hat{y}) \in \overline{R}^E$

1. Suppose $(\hat{x}, \hat{y}) \in \overline{R}$, then $\mathcal{M}, x \models \Diamond \varphi$ holds trivially by the definition of the minimal filtration.

2. Suppose the statement holds $\overline{R}_n$ and $(\hat{x}, \hat{y}) \in \overline{R}_{n+1} = \overline{R}_n \cup (\overline{R}_n^{-1} \circ \overline{R}_n)$. We consider the case of $(\hat{x}, \hat{y}) \in (\overline{R}_n^{-1} \circ \overline{R}_n)$.

   Then there exists $\hat{z}$ such that $(\hat{z}, \hat{x}), (\hat{z}, \hat{y}) \in \overline{R}_n$.

   By IH, $\mathcal{M}, z \models \Diamond \varphi$.

   $(\hat{z}, \hat{y}) \in \overline{R}_n$ iff there are $\hat{u}_1, \ldots, \hat{u}_n$ such that

$$\hat{z} \xleftarrow{\widehat{R}} \hat{u}_1 \xrightarrow{\widehat{R}'} \hat{u}_2 \xrightarrow{\widehat{R}'} \ldots \xrightarrow{\widehat{R}'} \hat{u}_{n-1} \xrightarrow{\widehat{R}'} \hat{u}_n \xrightarrow{\widehat{R}} \hat{y}$$

   where $\widehat{R}'$ is either $\widehat{R}$ or $\widehat{R}^{-1}$.

As it is known, $\Diamond \Diamond \varphi \to \Box \Diamond \varphi \in \mathbf{K}45$.

$\hat{u}_1 \widehat{\hat{z}}$, that is, $u_1' R z'$ for some $u_1' \in \hat{u}_1$ and $z' \in \hat{z}$. That is, $\mathcal{M}, u_1' \models \Diamond \Diamond \varphi$, so $\mathcal{M}, u_1' \models \Diamond \varphi$ and $\overline{\mathcal{M}}, \hat{u}_1 \models \Diamond \varphi$.

We have $\hat{u}_1 \widehat{R}' \hat{u}_2$. Suppose $\mathcal{M}, u_1'' \models \Diamond \varphi$ and $u_1'' R u_2'$. We also have $\mathcal{M}, u_1'' \models \Box \Diamond \varphi$, thus, $\mathcal{M}, u_2' \models \Diamond \varphi$.

Suppose $\hat{u}_2 \widehat{R} \hat{u}_1$ and $u_2' R u_1''$, then $\mathcal{M}, u_2' \models \Diamond \varphi$.
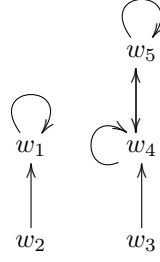
Similarly, we have $\mathcal{M}, u_i \models \Diamond \varphi$ iff $\mathcal{M}, u_{i+1} \models \Diamond \varphi$, whenever $\hat{u}_i \widehat{R}' \hat{u}_{i+1}$.

Finally, we have $\hat{u}_n \widehat{R} \hat{x}$. Thus, $u_n' R x'$ for some $u_n' \in \hat{u}_n$ and $x' \in \hat{x}$. $\mathcal{M}, u_n' \models \Diamond \varphi$, so $\mathcal{M}, u_n' \models \Box \Diamond \varphi$. Then $\mathcal{M}, x' \models \Diamond \varphi$. $\square$

Theorem 2. K5 *does not admit strict filtrations.*

Proof. Let us consider a **K**5 model whose Euclidean closure of the minimal filtration does not give us a filtration.

Let us consider a frame called $\mathcal{F}_{bad}$. We define this frame with the following graph:

Let us define a valuation ϑ such that $\vartheta(p) = \{w_5\}$ and $\vartheta(q) = \{w_1\}$. Let us consider a minimal filtration of $\mathcal{M}_{bad}$ through the Sub-closure of $\Gamma = \{\neg p, \neg \Diamond p\}$.

Clearly that $w_2 \sim_\Gamma w_3$, since $\neg p$ and $\neg \Diamond p$ are true both at w_2 and w_3.

Moreover, $R_{min} \cup (R_{min}^{-1} \circ R_{min})$ is not a subset of R_{max} since $(\hat{w_1}, \hat{w_5}) \in (R_{min}^{-1} \circ R_{min})$, but $\Diamond p$ is not true at w_5.

Let us also note that strict filtrations of this model is not Euclidean. Suppose by contrary that $\hat{R}^\mathcal{E}$ is a strict filtraction of that model. So $R_{min}^E \subseteq \hat{R}^\mathcal{E}$, since R_{min}^E is the minimal Euclidean relation containing R_{min}. On the other hand, $R_{min}^E \nsubseteq R_{max}$, so is not $\hat{R}^\mathcal{E}$.

$\square$

3   i,j-Euclideaness

Let R be a binary relation on $W \neq \varnothing$, then the i,j-Euclidean closure of R (where $i,j < \omega$), denoted as $R^{E_{i,j}}$, is a binary relation defined recursively as follows:

1. $R_0 = R$

2. $R_{n+1} = R_n \cup (((R_n)^{-1})^i \circ R_n{}^j)$

3. $R^{E_{i,j}} = \bigcup_{k<\omega} R_k$

Proposition 1. *If R be a binary relation on $W \neq \varnothing$, $R^{E_{i,j}}$ is i,j-Euclidean.*

Proof. By induction. Let $(x, y_1), (y_1, y_2), (y_2, y_3), \ldots, (y_{i-1}, y_i), (x, z_1), (z_1, z_2), (z_2, z_3), \ldots, (z_{j-1}, z_j) \in R^{E_{i,j}}$. That is, each of these pairs belong to some R_k and every $k < \omega$ for every pair is its own. If $k = 0$ for each pair, then $(y_i, z_j) \in R_1$. If some those k is non-zero, then we take the greatest amongst them, say e and all given pairs will be elements of R_{e+1}. $\square$

Theorem 3. $\mathbf{K5}^{i,j} = \mathbf{K} \oplus \Diamond^i p \to \Box^j \Diamond p$ *admits definable filtrations.*

Proof. Let $\mathcal{M} = \langle W, R, \vartheta \rangle$ and Γ a finite Sub-closed of formulas. We extend Γ as

$$\Delta = \Gamma \cup \text{Sub}\{\Diamond^i \psi \mid \Box \psi \in \Gamma\} \cup \text{Sub}\{\Box^j \Diamond \psi \mid \Diamond \psi \in \Gamma\}$$

Let $(\hat{x}, \hat{y}) \in R^{E_{i,j}}$, $\mathcal{M}, x \models \Box \psi$ for $\Box \psi$ in Δ. If $n = 0$, then the statement is obvious. Suppose $n = 1$. Then $(\hat{x}, \hat{y}) \in R_{min} \cup (((R_{min})^{-1})^i \circ R_{min}{}^j)$. Consider the second alternative. Then there exists $\hat{z}$ such that $(\hat{x}, \hat{z}) \in ((R_{min})^{-1})^i$ and $(\hat{z}, \hat{y}) \in R_{min}{}^j$.

That is, there are $\hat{x_1}, \hat{x_2}, \ldots, \hat{x_i}$ and $\hat{y_1}, \hat{y_2}, \ldots, \hat{y_j}$ such that

$$\hat{z} \xrightarrow{R_{min}} \hat{x_1} \xrightarrow{\boldsymbol{R_{min}}} \hat{x_2} \xrightarrow{R_{min}} \ldots \xrightarrow{R_{min}} \hat{x_i} \xrightarrow{\boldsymbol{R_{min}}} \hat{x}$$

$$\hat{z} \xrightarrow{R_{min}} \hat{y_1} \xrightarrow{\boldsymbol{R_{min}}} \hat{y_2} \xrightarrow{R_{min}} \ldots \xrightarrow{R_{min}} \hat{y_j} \xrightarrow{\boldsymbol{R_{min}}} \hat{y}$$
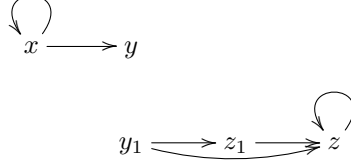
???

$\square$

4 The case of 2-transitivity

Let us define the logic $\mathcal{L}$ as $\mathbf{K} \oplus \Diamond\Diamond\Diamond p \to \Diamond p$. Let R be a binary relation, the $\mathcal{L}$-closure of R is defined (denoted as $R^{\Diamond}$) as the following union:

$$R^{\Diamond} = R \cup R^3 \cup R^5 \cup \cdots \cup R^{2k+1} \cup \ldots$$

Theorem 4. $\mathcal{L}$ *does not admit strict filtrations.*

Proof. Consider the following frame $\mathcal{F} = \langle W, R \rangle$:

$$x \longrightarrow y$$

$$y_1 \longrightarrow z_1 \longrightarrow z$$

Clearly that $\mathcal{F}$ is an $\mathcal{L}$-frame. We define the valuation ϑ as follows:

$$\vartheta(p) = \{x\}$$
$$\vartheta(q) = \{y, y_1\}$$
$$\vartheta(r) = \{z\}$$

Let us put $\Gamma = \mathrm{Sub}\{p, q, \Diamond r\}$. We factorise W through $\sim_\Gamma$ and consider a model $\widehat{\mathcal{M}} = \langle W/\sim_\Gamma, \widehat{R}, \widehat{\vartheta}\rangle$, where $\widehat{R} = (\widehat{R}_{min})^{\Diamond}$. We have $(\hat{x}, \hat{z}) \in \widehat{R} \circ \widehat{R} \circ \widehat{R}$, but $\Diamond r$ is not true at x. $\qquad\square$

5 Finite "canonical" models

Let $\mathcal{L}$ be a normal modal logic, $\mathcal{M}_\mathcal{L}$ its canonical model, and Γ a finite Sub-closed set of formulas. Let us put $\Gamma' = \mathrm{Sub}(\varphi) \cup \{\neg\psi \mid \psi \in \mathrm{Sub}(\varphi)\}$.

A subset $\Delta \subseteq' \Gamma$ is a *finite $\mathcal{L}$-consistent set* if $\neg\bigwedge\Delta \notin \mathcal{L}$. A subset Δ is maximal, if (the following are obviously equivalent):

1. Δ is maximal amongst finite $\mathcal{L}$-consistent sets,

2. For each $\psi \in \mathrm{Sub}(\varphi)$ either $\psi \in \Delta$ or $\neg\psi \in \Delta$.

Every finite $\mathcal{L}$-theory is clearly can be extended to some maximal one. It is the finite version of Lindenbaum's lemma.

Definition 5. *Let $\mathcal{L}$ be a modal logic and Γ be a finite Sub-closed set of formulas. A finite "canonical" model is a triple $\mathcal{M}_\mathcal{L}^\Gamma = \langle W_\mathcal{L}^\Gamma, R_\mathcal{L}^\Gamma, \vartheta_\mathcal{L}^\Gamma \rangle$, where*

1. $W_\mathcal{L}^\Gamma$ *is the set all maximal theories that extend finite $\mathcal{L}$-theories*

2. $R_\mathcal{L}^\Gamma$ *is a relation such that $\langle W_\mathcal{L}^\Gamma, R_\mathcal{L}^\Gamma \rangle$ is an $\mathcal{L}$-frame and*

$$\forall\Box\psi \in \mathrm{Sub}(\varphi) \ \forall\Delta_1 \in W_\mathcal{L}^\Gamma \ (\Box\psi \in \Delta_1 \Leftrightarrow \forall\Delta_2 \in R_\mathcal{L}^\Gamma(\Delta_1) \ \psi \in \Delta_2)$$

3. $\vartheta_\mathcal{L}^\Gamma(p) = \{\Delta \in W_\mathcal{L}^\varphi \mid p \in \Delta\}$ *for every variable $p \in \Gamma$.*

Lemma 6. *Let $\mathcal{L}$ be a modal logic and $\varphi \notin \mathcal{L}$, then $\mathcal{M}_\mathcal{L}^{\mathrm{Sub}(\varphi)} \not\models \varphi$.*

Lemma 7. *Let $\mathcal{L}$ be a modal logic and Γ a finite Sub-closed set of formulas, then if $\mathcal{L}$ admits strict filtrations, then there exists a finite "canonical" model $\mathcal{M}_{\mathcal{L}}^{\Gamma}$ such that $\mathcal{M}_{\mathcal{L}}^{\Gamma} \models \mathcal{L}$.*

Proof. ($\Rightarrow$) Let Γ be a finite Sub-closed of formulas. $\mathcal{L}$ admits strict filtrations, so the filtration of the canonical model $\mathcal{M}_{\mathcal{L}}$ through Γ is also an $\mathcal{L}$-model. The underlying set of $\mathcal{M}_{\mathcal{L}}/\sim_{\Gamma}$ consists of maximal $\mathcal{L}$ theories up to Γ-equivalence and this quotient set is finite.

It is readily checked that the quotient model $\mathcal{M}_{\mathcal{L}}/\sim_{\Gamma}$ satifsfies Definition 5. $\qquad\square$

The converse implication does not have to true generally. **GL** might be an example of a logic that has the "finite canonical" model property with no filtrations.

6 Fusion stuff

Definition 6. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be modal logics, then the fusion $\mathcal{L}_1 * \mathcal{L}_2$ is the minimal bimodal logic that contains $\mathcal{L}_1$ and $\mathcal{L}_2$ [2].*

Lemma 8.

- *Let Γ be a finite and Sub-closed set of formulas and $\mathcal{M} = \langle W, R, \vartheta \rangle$. Consider $\Gamma' = \Gamma \cup \{\Diamond\Box\psi \mid \Box\psi \in \Gamma\}$. Let Δ be any finite and Sub-closed extension of Γ'. Then a model $\widehat{M} = \langle W/\sim_{\Delta}, (R_{\Delta}^{min})^E, \widehat{\vartheta} \rangle$ is a filtration of $\mathcal{M}$ through Δ.*

- *The same for $\mathbf{K} \oplus \Diamond\Diamond\Diamond p \to \Diamond p$*

Proof. Recall that $(R_{\Delta}^{min})^E$ is defined inductively as:

1. $R_{\Delta}^0 = R_{\Delta}^{min}$

2. $R_{\Delta}^{n+1} = R_{\Delta}^n \cup (R_{\Delta}^n{}^{-1} \circ R_{\Delta}^n)$

3. $(R_{\Delta}^{min})^E = \bigcup_{k<\omega} R_{\Delta}^k$

If R_{Δ}^{min} is already a subrelation of R^{max}, so the base case is self-evident.

Suppose the statement holds for R_{Δ}^n, $(\hat{x}, \hat{y}) \in R_{\Delta}^{n+1}$, that is, there exists $\hat{z}$ such that $(\hat{z}, \hat{x}), (\hat{z}, \hat{y}) \in R_{\Delta}^n$. Let $\mathcal{M}, x \models \Box\psi$ (for $\Box\psi \in \Gamma$).

We rewrite $(\hat{z}, \hat{x}) \in R_{\Delta}^n$ as the following sausage (for some $\widehat{u_1}, \widehat{u_2}, \ldots, \widehat{u_{n-1}}, \widehat{u_n}$):

$$\hat{z} \xleftarrow{R_{\Delta}^{min}} \widehat{u_1} \xrightarrow{R'} \widehat{u_2} \xrightarrow{R'} \ldots \xrightarrow{R'} \widehat{u_{n-1}} \xrightarrow{R'} \widehat{u_n} \xrightarrow{R_{\Delta}^{min}} \hat{x}$$

where R' is either R_{Δ}^{min} or its converse. We have $\mathcal{M}, z \models \Box\psi$, since $\mathcal{M}, x \models \Box\psi$ implies $\mathcal{M}, u_n \models \Diamond\Box\psi$. After that we apply the following property of **K**5-models:

For each $a, b \in M_i$ such that $aR_i b$ we have $\mathcal{M}_i, a \models \Diamond\Box p_\psi$ iff $\mathcal{M}_i, b \models \Diamond\Box p_\psi$

Note that we always stay within Δ since $\Gamma' \subseteq \Delta$. Finally, we conclude $\mathcal{M}, x \models \psi$ from IH and $\mathcal{M}, z \models \Box\psi$.

The second item is shown similarly. $\qquad\square$

Theorem 5.

1. $\mathbf{K}5 * \mathbf{K}5$ *admits definable filtrations.*

2. $\mathbf{K}5 * \cdots * \mathbf{K}5$ *admits definable filtrations.*

3. *If $\mathcal{L}$ admits strict filtrations, then $\mathbf{K}5 * \mathcal{L}$ admits definable filtrations*

4. *If $\mathcal{L}_1, \ldots, \mathcal{L}_n$ admit strict filtrations, then $\mathbf{K}5 * \cdots * \cdots * \mathbf{K}5 * \mathcal{L}_1 * \cdots * \mathcal{L}_n$*

5. *Let $\mathcal{L} = \mathbf{K} \oplus \Diamond\Diamond\Diamond p \to \Diamond p$ (here and below), then $\mathcal{L} * \mathcal{L}$ admits definable filtrations.*

6. *Let $\mathcal{L} = \mathbf{K} \oplus \Diamond\Diamond\Diamond p \to \Diamond p$ and $\mathcal{L}_1$ a logic that admits strict filtrations, then $\mathcal{L} * \mathcal{L}_1$*

Proof.

1. Let Γ be a finite Sub-closed set of bimodal formulas, $\mathcal{F} = \langle W, R_1, R_2 \rangle$ a $\mathbf{K}5 * \mathbf{K}5$-frame, and ϑ a valuation on $\mathcal{F}$. Denote $\langle \mathcal{F}, \vartheta \rangle$ as $\mathcal{M}$.

   We introduce the set of fresh variables $V = \{p_\psi | \psi \in \Gamma\}$ and define a new model $\mathcal{M}' = \langle \mathcal{F}, \vartheta' \rangle$ as follows:

   $$\text{For all } \psi \in \Gamma, \ \mathcal{M}, x \models \psi \Leftrightarrow \mathcal{M}', x \models \psi \Leftrightarrow \mathcal{M}', x \models p_\psi.$$

   Consider these modifications of Γ and V:

   $$\Gamma' = \Gamma \cup \{\Diamond_1\Box_1\psi \mid \Box_1\psi \in \Gamma\} \cup \{\Diamond_2\Box_2\psi \mid \Box_2\psi \in \Gamma\}$$
   $$\Delta = V \cup \mathrm{Sub}(\{\Diamond\Box p_\psi \mid \Box_i\psi \in \Gamma, in = 1, 2\})$$

   Let us define an equivalence relation $\sim_\Delta$ induced by Δ.

   Consider $\mathcal{M}_i = \langle W, R_i, \vartheta' \rangle$, a reduct of $\mathcal{M}'$, we have:

   (a) $\mathcal{M}_i, x \models \Box p_\psi$ iff $\mathcal{M}, x \models \Box_i\psi$

   (b) $\mathcal{M}_i, x \models \Diamond\Box p_\psi$ iff $\mathcal{M}, x \models \Diamond_i\Box_i\psi$

   So $\sim = \sim_{\Gamma'}$ by the construction. Let us put $\widehat{W} = W / \sim_{\Gamma'}$. Lemma 8 implies the following claim:

   Claim 1. *Let $\widehat{R}_i = (R_\Delta^{min})^E$ and $\widehat{\vartheta(p)} = \{[x]_{\sim_i} \mid \mathcal{M}_i, x \models p\}$ for $p \in \Delta_1$, define $\widehat{\mathcal{M}}_i = \widehat{W}, \widehat{R}_i, \widehat{\vartheta}$. Then $\widehat{\mathcal{M}}_i \models \mathbf{K}5$ and $\widehat{\mathcal{M}}_i$ is a filtration of $\mathcal{M}_i$ through Δ.*

   Finally, we consider a model $\widehat{\mathcal{M}} = \langle \widehat{W}, \widehat{R}_1, \widehat{R}_2, \vartheta \rangle$, where $\widehat{R_{\Gamma'}}_i = R_{i\Gamma'}^{min}{}^E$ and $\vartheta(p)$ is defined as usual for $p \in \Gamma$. $\widehat{\mathcal{M}}$ is a filtration of $\mathcal{M}$ through Γ'.

   Let $\hat{x}\widehat{R_{\Gamma'}}_i\hat{y}$ and $\mathcal{M}, x \models \Box_i\psi$ for $\Box_i\psi \in \Gamma$. Then $\mathcal{M}_i, x \models \Box p_\psi$, so $\widehat{\mathcal{M}}_i, \hat{x} \models \Box p_\psi$. By the claim above, $\widehat{\mathcal{M}}_i$ is a filtration of $\mathcal{M}_i$ through Δ, so $\mathcal{M}_i, y \models p_\psi$. Then $\mathcal{M}, y \models \psi$.

2. Likewise

3. The argument is the same as in the proof of the first item, except for **Claim** 1 that has the following formulation: Let $\widehat{R}_1 = (R_\Delta^{min})^E$ and $\widehat{R}_2 = (R_\Delta^{min})^{\mathcal{L}_1}$ Define a valuation as usual as $\widehat{\vartheta(p)} = \{[x]_{\sim_i} \mid \mathcal{M}_i, x \models p\}$ for $p \in \Delta_1$, define $\widehat{\mathcal{M}}_1 = \langle \widehat{W}, \widehat{R}_1, \widehat{\vartheta} \rangle$ and $\widehat{\mathcal{M}}_2 = \langle \widehat{W}, \widehat{R}_2, \widehat{\vartheta} \rangle$.

   Then $\widehat{\mathcal{M}}_1 \models \mathbf{K}5$ and $\widehat{\mathcal{M}}_1 \models \mathbf{L}$ and $\widehat{\mathcal{M}}_i$ is a filtration of $\mathcal{M}_i$ through Δ.

4. Likewise

5. The argument is similar to the proof of first item, but filtrations are slightly different. Let $\mathcal{M} = \langle W, R_1, R_2, \vartheta \rangle$ be a $\mathcal{L} * \mathcal{L}$ model and Γ a Sub-closed set of formulas. As above, we define a set V and a model $\mathcal{M}'$. Define extensions of Γ and V:

$$\Gamma' = \Gamma \cup \{\Diamond_1 \Diamond_1 \psi \mid \Diamond_1 \psi \in \Gamma\} \cup \{\Diamond_2 \Diamond_2 \psi \mid \Diamond_2 \psi \in \Gamma\}$$
$$\Delta = V \cup \mathrm{Sub}(\{\Diamond \Diamond p_\psi \mid \Diamond \psi \in \Gamma, i = 1, 2\})$$

As above $\sim_\Delta = \Gamma'$ and $\widehat{\mathcal{M}'} = \langle W/\sim_\Delta, \widehat{R_i}, \widehat{\vartheta} \rangle$ are filtrations of reducts of $\mathcal{M}'$ through Δ. Then $\widehat{\mathcal{M}} = \langle W/\sim_\Delta, \widehat{R_1}, \widehat{R_2}, \widehat{\vartheta} \rangle$ is a required filtration of the original $\mathcal{M}$.

6. Extend Γ with $\{\Diamond_i \Diamond_i \psi \mid \Diamond_i \psi \in \Gamma, i = 1, 2\}$ and V with $\{\Diamond \Diamond p_\psi \mid \Diamond_i \psi, i = 1, 2\}$

$\square$

Theorem 6. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be modal logics that admit definable filtrations. If $\mathrm{Mod}(\mathcal{L}_1)$ and $\mathrm{Mod}(\mathcal{L}_2)$ admit definable filtrations, so does $\mathrm{Mod}(\mathcal{L}_1 * \mathcal{L}_2)$.*

Proof. Let $\mathcal{M} = \langle W, R_1, R_2, \vartheta \rangle$ be an $\mathcal{L}_1 * \mathcal{L}_2$. We define a notation $\nabla = \{\neg \Diamond, \Diamond \neg, \Diamond\}$.

Both logics admit definable filtrations, so for every finite Sub-closed set Γ and for every $\mathfrak{M}$, an $\mathcal{L}_1$-model (or an $\mathcal{L}_2$ one) there exists there exists Δ, a extension of Γ having the form:

$$\Delta_1 = \Gamma \cup \mathrm{Sub}\{\nabla_1 \nabla_2 \ldots \nabla_n \Diamond \psi \mid \Diamond \psi \in \Gamma\} \text{ (for } \mathcal{L}_1\text{)}$$
$$\Delta_2 = \Gamma \cup \mathrm{Sub}\{\nabla_1 \nabla_2 \ldots \nabla_k \Diamond \psi \mid \Diamond \psi \in \Gamma\} \text{ (for } \mathcal{L}_2\text{)}$$

such that $\widehat{\mathfrak{M}} = \langle W/\sim_{\Delta_i}, \widehat{R}, \vartheta \rangle$ is a filtration of $\mathfrak{M}$ through the corresponding Δ_i.

Let V be a set of fresh variables indexed over Γ as in the proof for a fusion of **K5** with something else. Let $\mathcal{M}'$ be a model defined as previously. We extend V and Γ in the following way:

$$\Gamma' = \Gamma \cup \mathrm{Sub}\{\nabla_{11} \nabla_{21} \ldots \nabla_{n1} \Diamond_1 \psi \mid \Diamond_1 \psi \in \Gamma\} \cup \mathrm{Sub}\{\nabla_{12} \nabla_{22} \ldots \nabla_{n2} \Diamond_2 \psi \mid \Diamond_2 \psi \in \Gamma\}$$
$$\Delta = V \cup \mathrm{Sub}\{\nabla_1 \nabla_2 \ldots \nabla_n \Diamond p_\psi \mid \nabla_{n+1 1} \psi \in \Gamma'\} \cup \mathrm{Sub}\{\nabla_1 \nabla_2 \ldots \nabla_k \Diamond p_\psi \mid \Diamond_2 \psi \in \Gamma\}.$$

By the construction, $\sim_{\Gamma'} = \sim_\Delta$. So we have filtrations for the corresponding reducts of $\mathcal{M}'$ through Δ as well as for the original $\mathcal{M}$. $\square$

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