# Notes on filtration of logics containing **K5**

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## 1 Preliminaries

**Definition 1.** *A normal modal logic is a set of formulas that contains all Boolean tautologies, formulas* $\Diamond p \vee \Diamond q \leftrightarrow \Diamond(p \vee q)$ *and* $\Diamond\bot \leftrightarrow \bot$*, and is closed under Modus Ponens, Substitution, and Monotonicity: from* $\varphi \to \psi$ *infer* $\Diamond\varphi \to \Diamond\psi$*.*

**Definition 2.** *An Kripke model is a triple* $\mathcal{M} = \langle W, R, \vartheta \rangle$*, where* $R \subseteq W \times W$*,* $\vartheta : \mathrm{PV} \to 2^W$*, and the connectives have the following semantics:*

1. $\mathcal{M}, w \models p \Leftrightarrow w \in \vartheta(p)$
2. $\mathcal{M}, w \models \neg\varphi \Leftrightarrow \mathcal{M}, w \not\models \varphi$
3. $\mathcal{M}, w \models \varphi \vee \psi \Leftrightarrow \mathcal{M}, w \models \varphi$ *or* $\mathcal{M}, w \models \psi$
4. $\mathcal{M}, w \models \Diamond\varphi \Leftrightarrow \exists v \in R(w)\, \mathcal{M}, v \models \varphi$

**Definition 3.** *Let* $\mathcal{M} = \langle W, R, \vartheta \rangle$ *be a Kripke model. A Kripke model* $\mathcal{M}' = \langle W', R', \vartheta' \rangle$ *is a generated submodel of* $\mathcal{M}$*, where:*

1. $\varnothing \neq W' \subseteq W$ *and* $W'$ *is* $R$*-closed, i.e., for each* $u \in W'$ *and* $v \in W$*,* $uRv$ *implies* $v \in W'$
2. $R' = R \cap W' \times W'$
3. *for every propositional variable* $p$*,* $\vartheta(p)' = \vartheta(p) \cap W'$

**Fact 1.** *Let* $\mathcal{M} = \langle W, R, \vartheta \rangle$ *be a Kripke model and* $\mathcal{M}' = \langle W', R', \vartheta' \rangle$ *its generated submodel, then for each* $w \in W'$

$$\mathcal{M}, w \models \varphi \text{ iff } \mathcal{M}', w \models \varphi$$

By **K5** we mean the logic $\mathbf{K} \oplus A5$, where $A5 = \Diamond p \to \Box\Diamond p$. It is known that **K5** is the modal logic of all Euclidean frames. A frame is called Euclidean if for each $x, y, z$, $xRy$ and $xRz$ implies $yRz$.

**Proposition 1.** **K5** *proves*

1. $\Box^3 p \leftrightarrow \Box^2 p$
2. $\Box^2\Diamond p \leftrightarrow \Box\Diamond p$
3. $\Box\Diamond\Box p \leftrightarrow \Box\Box p$
4. $\Box\Diamond^2 p \leftrightarrow \Box\Diamond p$

**Proposition 2.** *Let $\mathcal{M}$ be a* $\mathbf{K}5$ *model, $xRy$ for $x, y \in W$ then one has*

$$\mathcal{M}, x \models \Box\Diamond\varphi \textit{ iff } \mathcal{M}, y \models \Box\Diamond\varphi.$$

*Proof.*

1. Suppose $\mathcal{M}, x \models \Box\Diamond\varphi$. Then $\mathcal{M}, y \models \Diamond\varphi$ and $\mathcal{M}, y \models \Box\Diamond\varphi$
2. Suppose $\mathcal{M}, y \models \Box\Diamond\varphi$, then $\mathcal{M}, x \models \Diamond\Box\Diamond\varphi$, so $\mathcal{M}, x \models \Box\Diamond\varphi$.

□

## 1.1 Filtrations: general definitions

Let $\mathcal{M} = \langle W, R_1, \ldots, R_n, \vartheta \rangle$ be a Kripke model and $\Gamma$ a set of formulas closed under subformulas. An equivalence relation $\sim$ is set to have a finite index if the quotient set $W/\sim$ is finite. The equivalence relation $\sim_\Gamma$ induced by $\Gamma$ is defined as

$$w \sim_\Gamma v \Leftrightarrow \forall \varphi \in \Gamma\, (\mathcal{M}, w \models \varphi \Leftrightarrow \mathcal{M}, v \models \varphi).$$

If $\Gamma$ is finite, then $\sim_\Gamma$ has a finite index. An equivalence relation $\sim$ respects $\sim_\Gamma$, if $w \sim v$ implies $w \sim_\Gamma v$.

**Definition 4.** *Let $\mathcal{M} = \langle W, R_1, \ldots, R_n, \vartheta \rangle$ be a Kripke model and $\Gamma$ be a Sub-closed set formulas. A $\Gamma$-filtration of $\mathcal{M}$ is a model $\widehat{\mathcal{M}} = \langle \widehat{W}, \widehat{R_1}, \ldots, \widehat{R_n}, \widehat{\vartheta} \rangle$ such that:*

1. *$\widehat{W} = W/\sim$, where $\sim$ is an equivalence relation having a finite index that respects $\Gamma$*
2. *$\widehat{\vartheta}(p) = \{[x]_\sim \mid x \in W \;\&\; x \in \vartheta(p)\}$*
3. *For each $i \in I$ one has $\widehat{R}_i^{min} \subseteq \widehat{R}_i \subseteq \widehat{R}_i^{max}$. $\widehat{R}_{i,\sim}^{min}$ is the $i$-th minimal filtered relation on $\widehat{W}$ defined as*

   $$\hat{x}\widehat{R}_{i,\sim}^{min}\hat{y} \Leftrightarrow \exists x' \sim x\, \exists y' \sim y\; xR_i y$$

   *$\widehat{R}_{\Gamma,i}^{max}$ is the $i$-th maximal filtered relation on $\widehat{W}$ induced by $\Gamma$ defined as*

   $$\hat{x}\widehat{R}_{\Gamma,i}^{max}\hat{y} \Leftrightarrow \forall \Box_i\varphi \in \Gamma\, (\mathcal{M}, x \models \Box_i\varphi \Rightarrow \mathcal{M}, y \models \varphi)$$

If $\Phi$ is finite subset of $\Gamma$ and $\sim = \sim_\Phi$, then $\widehat{\mathcal{M}}$ is a definable $\Gamma$-filtration of $\mathcal{M}$ through $\Phi$. If $\sim = \sim_\Gamma$, then such a filtration by means of the definiton above is called *strict*. A class of models $\mathbb{M}$ admits strict filtrations for models (ASF), if for every Sub-closed set $\Gamma$ and for every $\mathcal{M} \in \mathbb{M}$ there exists a $\Gamma$ filtration of $\mathcal{M}$. A class of frames $\mathbb{F}$ admits strict filtrations for frames, if for every Sub-closed set $\Gamma$ and for every frame $\mathcal{F} \in \mathbb{F}$ and every model $\mathcal{M}$ over $\mathcal{F}$ there exists a $\Gamma$ filtration of $\mathcal{M}$. If $\mathcal{L}$ is canonical, then the ASF property for frames and ASF property for models are equivalent [1, Theorem 2.10].

**Lemma 1.** *Let $\Gamma$ be a finite set of formulas closed under subformulas and $\widehat{\mathcal{M}}$ a filtration of $\mathcal{M}$ through $\Gamma$, then for each $x \in W$ and for each $\varphi \in \Gamma$ one has*

$$\mathcal{M}, x \models \varphi \Leftrightarrow \widehat{\mathcal{M}}, \hat{x} \models \varphi$$

**Definition 5.** *Let $\mathbb{F}$ be a class of Kripke frames and $\Gamma$ a finite set of formulas closed under subformulas. If for every model $\mathcal{M}$ over $\mathcal{F} \in \mathbb{F}$ there exists a model that is a $\Gamma$-definable filtration of $\mathcal{M}$, then $\mathbb{F}$ admits definable filtration. A class of models $\mathbb{M}$ admits definable filtration if for every $\mathcal{M} \in \mathbb{M}$ there exists a model belonging to the same class that is a definable $\Gamma$-filtration of $\mathcal{M}$.*

**Lemma 2.**

1. *Let $\mathcal{L}$ be a complete normal modal logic. If* $\mathrm{Frames}(\mathcal{L})$ *admits filtration, then $\mathcal{L}$ has the finite model property.*
2. *If the class of models* $\mathrm{Mod}(\mathcal{L})$ *admits filtration, then $\mathcal{L}$ has the finite model property and Kripke complete as well.*

## 2 Filtration of Euclidean logics

First of all, let us ensure that a minimal filtration of an Euclidean frame is not necessary Euclidean. Let $[x] \sim_\Gamma [y]$ and $[x] \sim_\Gamma [z]$. Then for some $x' \in [x]$ $y' \in [y]$, one has $x'Ry'$ and $x''Rz'$ for some $x'' \in [x]$ and $z' \in [z]$. Clearly, we cannot claim that $x' = x''$ in general. Thus, minimal filtration does not preserve the required property.

**Lemma 3.** **K5** *admit filtration.*

*Proof.* Let $\mathcal{M}$ be a **K5**-model and $\Gamma_0$ a finite set of formulas closed under subformulas. Let us put $\Gamma = \Gamma_0 \cup \mathrm{Sub}(\{\Diamond\Box\psi \mid \Box\psi \in \Gamma_0\}) \cup \Psi$, where $\Psi = \nabla_1\nabla_2 \dots \nabla_n\Box\psi$ for $\Box\psi \in \Gamma_0$ and $\nabla_i \in \{\Diamond, \Box\}$. By Proposition 1, any element of $\Phi$ has one of the four forms. Thus, $W \sim_{\equiv_\Gamma}$ has a finite index. We put $\widehat{R} = R_\Gamma^{\max}$. ☐

**Definition 6.** *A first-order formula is called Horn if it has the following form:*

$$\forall x_1, \dots, x_n (x_{i_1} R x_{j_1} \wedge \dots \wedge x_{i_s} R x_{j_s} \to x_k R x_l)$$

**Definition 7.** *Let $H$ be a Horn property and $\langle W, R\rangle$ a Kripke frame. A Horn closure of a binary relation $R$ is the minimal relation $R^H$ containing $R$ and satisfying $H$.*

**Lemma 4.** $R^H = \bigcup\limits_{n<\omega} R_n$ *where*

1. $R_0 = R$.
2. $R_{n+1} = R_n \cup \{(a, b) \in W \mid \exists \vec{c} \in W\ P(a, b, \vec{c})\}$, *where $P$ is a premise of $H$.*

$E$-closure (an Euclidean Horn closure of a binary relation) has the following equivalent definitions:

**Lemma 5.** *Let $\mathcal{F} = \langle W, R\rangle$ be a Kripke frame. The following conditions are equivalent:*

1. *$R^E$ is the smallest Euclidean relation containing $R$.*
2. $R^E = \bigcup\limits_{i<\omega} R_i$, *where*
   - $R_0 = R$
   - $R_{n+1} = R_n \cup (R_n^{-1} \circ R_n)$

3. $xR^E y$ *iff there exists* $n < \omega$ *such that either* $xRy$ *or* $\exists z_1, \ldots, z_n$ *with* $z_1 Rx$ *and* $z_{n-1}Ry$ *and for each* $1 < i \leqslant n$ *one has either* $z_{i-1}Rz_i$ *or* $z_i R z_{i-1}$.

4. $R^E = R \cup \bigcup\limits_{i<\omega} (R^{-1} \circ (R \circ R^{-1})^n \circ R)$.

*Proof.*

1. (1) $\Rightarrow$ (2) Let us show that if $R^E$ is the smallest Euclidean relation containing $R$, then $R^E = \bigcup\limits_{i<\omega} R_i$. There are two inclusions:

   - $R^E \subseteq \bigcup\limits_{i<\omega} R_i$. Recall that $R^E$ has the form (?):
   $$R^E = \bigcap\{R' \mid R \subseteq R', \forall a, b \in W\ R'(a, b) \Rightarrow \exists x \in W\ R'(x, a)\ \&\ R'(x, b)\}$$
   - $\bigcup\limits_{i<\omega} R_i \subseteq R^E$. Let us show that $xR_n y$ for each $n < \omega$ implies $xR^E y$ by induction on $n$. If $n = 0$, then $xRy$, thus, $xR^E y$, since $R$ is a subrelation of $R^E$. Suppose $n = m+1$ and $xR_{m+1}y$. Let us show that $xR^E y$. From $xR_{m+1}y$, one has $(x, y) \in R^n \cup (R_n^{-1} \circ R_n)$. There are two cases:
     - $xR^n y$, one needs to merely apply the IH.
     - $xR_n^{-1} \circ R_n y$. Then $\exists z \in W\ xR_n^{-1} z\ \&\ zR_n$. That is, $zR_n x$ and $zR_n y$ for some $z$. $R_n$ is already a subrelation of $R^E$. Thus, $zR^E x$ and $zR^E y$. That implies $xR^E y$.

2. (2) $\Rightarrow$ (3) Let $(x, y) \in R_m$, let us the statement by induction on $m$.

   (a) Suppose $m = 0$, then $xRy$, and the statement is shown putting $n = 0$.

   (b) Suppose $m = p + 1$ and $xR_{p+1}y$. Assume that either $xRy$ or $\exists z_1, \ldots, z_p$ with $z_1 Rx$ and $z_{p-1}Ry$ and for each $1 < i \leqslant p$ one has either $z_{i-1}Rz_i$ or $z_i R z_{i-1}$.
   $xR_{p+1}y$ implies $(x, y) \in R_p \cup (R_p^{-1} \circ R_p)$. If $(x, y) \in R_p$, then we merely apply the IH. Suppose $(x, y) \in R_p^{-1} \circ R_p$, then $(z, x) \in R_p$ and $(z, y) \in R_p$

3. (3) $\Rightarrow$ (4) Suppose either $xRy$ or there exist $n \geqslant 1$ and $z_1, \ldots, z_n$ with $z_1 Rx$ and $z_{n-1}Ry$ and for each $1 < i \leqslant n$ one has either $z_{i-1}Rz_i$ or $z_i R z_{i-1}$. If $xRy$, then we are done. Otherwise there exists $n \geqslant 1$ with the condition above. Then $(x, y) \in R_{n+1}$ that follows from the condition.

4. (4) $\Rightarrow$ (1)

$\square$

**Lemma 6.** *Let* $\mathcal{F} = \langle W, R \rangle$ *be a Kripke frame. Let us define* $R^E = \bigcup\limits_{i<\omega} R_i$ *where:*

1. $R_0 = R$
2. $R_{n+1} = R_n \cup ({R_n}^{-1} \circ R_n)$

*Then* $R^E$ *is Euclidean.*

*Proof.* Let $(x, y), (x, z) \in R^E$, one needs to show that $(y, z) \in R^E$. Clearly that $(x, y) \in R_i$ and $(x, z) \in R_j$ for some $i, j < \omega$. Thus, we need $(y, z) \in R_m$ for some $m$ depending on $i$ and $j$.

Let us consider the following cases:

1. $i = 0$ and $j = 0$

   Suppose $(x, y), (x, z) \in R_0 = R$, then $(y, z) \in R^{-1} \circ R$. Thus, $(y, z) \in R_1$

2. $i = 0$ and $j = k + 1$

   Suppose $(x, y) \in R$ and $(x, z) \in R_{k+1} = R_k \cup ({R_k}^{-1} \circ R_k)$. Clearly that $(x, y) \in R_{k+1}$ as well. It is obviously that $(y, z) \in R_{k+2}$ since $(y, x) \in R^{-1}_{k+1}$ and $(x, z) \in R_{k+1}$.

3. The case with $i = k + 1$ and $j = 0$ is similar to the previous one.

4. Suppose $i = m + 1$ and $j = k + 1$. That is, $(x, y) \in R_{m+1} = R_m \cup ({R_m}^{-1} \circ R_m)$ and $(x, z) \in R_{k+1} = R_k \cup ({R_k}^{-1} \circ R_k)$. Consider the following four subcases:

   (a) Suppose $(x, y) \in R_m$ and $(x, z) \in R_k$ and $m \leqslant k$ without loss of generality. $m \leqslant k$ implies $R_m \subseteq R_k$ and $(x, y) \in R_k$ in particular. Thus, $(y, z) \in {R_k}^{-1} \circ R_k$, so $(y, z) \in R_{k+1}$.

   (b) The rest of the cases are similar to the first one.

$\square$

**Theorem 1.** **K**45 *admits strict filtrations.*

*Proof.* Let $\mathcal{M} = \langle W, R, \vartheta \rangle$ be a transitive Euclidean model and $\overline{\mathcal{M}} = \langle \overline{W}, \overline{R}, \overline{\vartheta} \rangle$ its minimal filtration through $\Gamma$, where $\Gamma$ is finite and Sub-closed. Let us put $\widehat{R} = \overline{R}^{+} \cup \overline{R}^{E}$. Let us show that $\overline{R}^{+} \cup \overline{R}^{E} \subseteq \overline{R}^{max}$.

That is, if $\mathcal{M}, y \models \varphi$ for $\Diamond\varphi \in \Gamma$ and $\hat{x}\widehat{R}\hat{y}$, then $\mathcal{M}, x \models \Diamond\varphi$.

Let $\hat{x}\widehat{R}\hat{y}$. Let us consider the case when $(\hat{x}, \hat{y}) \in \overline{R}^{E}$

1. Suppose $(\hat{x}, \hat{y}) \in \overline{R}$, then $\mathcal{M}, x \models \Diamond\varphi$ holds trivially by the definition of the minimal filtration.

2. Suppose the statement holds $\overline{R}_n$ and $(\hat{x}, \hat{y}) \in \overline{R}_{n+1} = \overline{R}_n \cup (\overline{R}_n^{-1} \circ \overline{R}_n)$. We consider the case of $(\hat{x}, \hat{y}) \in (\overline{R}_n^{-1} \circ \overline{R}_n)$.

   Then there exists $\hat{z}$ such that $(\hat{z}, \hat{x}), (\hat{z}, \hat{y}) \in \overline{R}_n$.

   By IH, $\mathcal{M}, z \models \Diamond\varphi$.

   $(\hat{z}, \hat{y}) \in \overline{R}_n$ iff there are $\hat{u}_1, \ldots, \hat{u}_n$ such that

$$\hat{z} \xleftarrow[\widehat{R}]{} \hat{u}_1 \xrightarrow[\widehat{R}']{} \hat{u}_2 \xrightarrow[\widehat{R}']{} \ldots \xrightarrow[\widehat{R}']{} \hat{u}_{n-1} \xrightarrow[\widehat{R}']{} \hat{u}_n \xrightarrow[\widehat{R}]{} \hat{y}$$

   where $\widehat{R}'$ is either $\widehat{R}$ or $\widehat{R}^{-1}$.

As it is known, $\Diamond\Diamond\varphi \to \Box\Diamond\varphi \in \mathbf{K}45$.

$\hat{u}_1\hat{\hat{z}}$, that is, $u'_1 R z'$ for some $u'_1 \in \hat{u}_1$ and $z' \in \hat{z}$. That is, $\mathcal{M}, u'_1 \models \Diamond\Diamond\varphi$, so $\mathcal{M}, u'_1 \models \Diamond\varphi$ and $\overline{M}, \hat{u}_1 \models \Diamond\varphi$.

We have $\hat{u}_1\widehat{R}'\hat{u}_2$. Suppose $\mathcal{M}, u''_1 \models \Diamond\varphi$ and $u''_1 R u'_2$. We also have $\mathcal{M}, u''_1 \models \Box\Diamond\varphi$, thus, $\mathcal{M}, u'_2 \models \Diamond\varphi$.

Suppose $\hat{u}_2\widehat{R}\hat{u}_1$ and $u'_2 R u''_1$, then $\mathcal{M}, u'_2 \models \Diamond\varphi$.

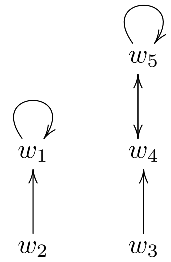
Similarly, we have $\mathcal{M}, u_i \models \Diamond\varphi$ iff $\mathcal{M}, u_{i+1} \models \Diamond\varphi$, whenever $\hat{u}_i\widehat{R}'\hat{u}_{i+1}$.

Finally, we have $\hat{u}_n\widehat{R}\hat{x}$. Thus, $u'_n R x'$ for some $u'_n \in \hat{u}_n$ and $x' \in \hat{x}$. $\mathcal{M}, u'_n \models \Diamond\varphi$, so $\mathcal{M}, u'_n \models \Box\Diamond\varphi$. Then $\mathcal{M}, x' \models \Diamond\varphi$. $\square$

**Theorem 2.** **K**5 *does not admit strict filtrations.*

*Proof.* Let us consider a **K**5 model whose Euclidean closure of the minimal filtration does not give us a filtration.

Let us consider a frame called $\mathcal{F}_{bad}$. We define this frame with the following graph:



$\square$

Let us define a valuation $\vartheta$ such that $\vartheta(p) = \{w_5\}$ and $\vartheta(q) = \{w_1\}$. Let us consider a minimal filtration of $\mathcal{M}_{bad}$ through the Sub-closure of $\Gamma = \{\neg p, \neg\Diamond p\}$.

Clearly that $w_2 \sim_\Gamma w_3$, since $\neg p$ and $\neg\Diamond p$ are true both at $w_2$ and $w_3$.

Moreover, $R_{min} \cup (R_{min}^{-1} \circ R_{min})$ is not a subset of $R_{max}$ since $(\hat{w_1}, \hat{w_5}) \in (R_{min}^{-1} \circ R_{min})$, but $\Diamond p$ is not true at $w_5$.

Let us also note that strict filtrations of this model is not Euclidean. Suppose by contrary that $\widehat{R}^{\mathcal{E}}$ is a strict filtraction of that model. So $R_{min}^{E} \subseteq \widehat{R}^{\mathcal{E}}$, since $R_{min}^{E}$ is the minimal Euclidean relation containing $R_{min}$. On the other hand, $R_{min}^{E} \nsubseteq R_{max}$, so is not $\widehat{R}^{\mathcal{E}}$.

# 3 Filtration for K4

**Proposition 3.** *Let $R$ be a binary relation on $W \neq \varnothing$. Define $R^+ = \bigcup\limits_{i<\omega} R_i$*

1. $R_0 = R$
2. $R_{n+1} = R_n \circ R$

*Then $R^+$ is transitive*

**Lemma 7.** *Let $\mathcal{M} = \langle W, R, \vartheta\rangle$ be a transitive model and $\overline{\mathcal{M}} = \langle \overline{W}, \overline{R}, \overline{\vartheta}\rangle$ its minimal filtration through a finite* Sub-*closed set of formulas $\Theta$.*

*Then $\overline{\mathcal{M}}^+ = \langle \overline{W}, (\overline{R})^+, \overline{\vartheta}\rangle$ is a $\Theta$-filtration of $\mathcal{M}$.*

*Proof.* $(\overline{R})^+$ obviously contains $R$. By the previous proposition, $(\overline{R})^+$ is transitive. Let us show that $(\overline{R})^+ \subseteq R_{\Theta}^{max}$.

Let $\hat{x}, \hat{y} \in \overline{W}$ with $\hat{x}(\overline{R})^+\hat{y}$ and $\Box\varphi \in \Theta$ with $\mathcal{M}, x \models \Box\varphi$. Let us show that $\mathcal{M}, y \models \varphi$.

If $\hat{x}(\overline{R})^+\hat{y}$, then there exist equivalence classes $\hat{x}_1, \ldots, \hat{x}_n$ such that

$$\hat{x}\overline{R}\hat{x}_1\overline{R}\ldots\overline{R}\hat{x}_n\overline{R}\hat{y}$$

$\mathcal{M}, x \models \Box\varphi$ implies $\mathcal{M}, x \models \Box\Box\varphi$. Thus, $\overline{M}, \hat{x} \models \Box\Box\varphi$.

$\hat{x}\overline{R}\hat{x}_1$, so there are $x_1 \in \hat{x}$ and $x_2 \in \hat{x}_1$ with $x_1 R x_2$. In particular, $\mathcal{M}, x_2 \models \Box\varphi$, so $\overline{\mathcal{M}}, \hat{x}_2 \models \Box\varphi$, and et cetera.

For each $i \in \{1, \ldots, n\}$ we have $\mathcal{M}, x_i \models \Box\varphi$ which is shown inductively:

If $\mathcal{M}, x_i \models \Box\varphi$ for $x_i \in \hat{x}_i$, so $\mathcal{M}, x_i \models \Box\Box\varphi$, but there exist $x_i{}' \in \hat{x}_i$ and $x_{i+1} \in \hat{x}_{i+1}$, so $\mathcal{M}, x_{i+1} \models \Box\varphi$.

Finally, we have $\mathcal{M}, x_n \models \Box\varphi$ for $x_n \in \hat{x}_n$, but $\hat{x}_n \overline{R} \hat{y}$, so $\mathcal{M}, y' \models \varphi$ for each $y' \in \hat{y}$. Thus, $\varphi$ is true at $y$ as well. $\Box$

*Proof.* Let $\hat{x}, \hat{y} \in \overline{W}$ with $\hat{x}(\overline{R})^+\hat{y}$ and $\Box\varphi \in \Theta$ with $\mathcal{M}, x \models \Box\varphi$. Let us show that $\mathcal{M}, y \models \varphi$.

If $\hat{x}(\overline{R})^+\hat{y}$, then there exist equivalence classes $\hat{x}_1, \dots, \hat{x}_n$ such that

$$\hat{x}\overline{R}\hat{x}_1\overline{R}\dots\overline{R}\hat{x}_n\overline{R}\hat{y}$$

Let us show that $\mathcal{M}, \hat{x}_i \models \Box\varphi$ inductively:

1. $n = 1$ We have the following sequence:

$$\hat{x}\overline{R}\hat{x}_1\overline{R}\hat{y}$$

   $\hat{x}\overline{R}\hat{x}_1$, so there are $x' \in \hat{x}$ and $x'_1 \in \hat{x}_1$ such that $x'Rx'_1$. $\Box\varphi$ is true at $x'$, so is $\Box\Box\varphi$. Then $\mathcal{M}, x'_1 \models \Box\varphi$ since $x'_1 \in R(x')$. So $\overline{\mathcal{M}}, \hat{x}_1 \models \Box\varphi$.

2. $n = i + 1$ The case is the following:

$$\hat{x}\overline{R}\hat{x}_1\overline{R}\dots\overline{R}\hat{x}_i\overline{R}\hat{x}_{i+1}\overline{R}\hat{y}$$

   By IH, $\Box\varphi$ is true at $\hat{x}_i$, so is $\Box\Box\varphi$. Hence, we have $\overline{\mathcal{M}}, \hat{x}_{i+1} \models \Box\varphi$ since $\hat{x}_i\overline{R}\hat{x}_{i+1}$.

That is, for each $0 < n < \omega$, if we have a sequence of equivalence classes with $\hat{x}\overline{R}\hat{x}_1\overline{R}\dots\overline{R}\hat{x}_n\overline{R}\hat{y}$ where $\overline{\mathcal{M}}, \hat{x} \models \Box\varphi$, then $\overline{\mathcal{M}}, \hat{x}_n \models \Box\varphi$.

If $\hat{x}_n\overline{R}\hat{y}$, then there are $x'_n \in \hat{x}_n$ and $y' \in \hat{y}$ with $x'_nRy'$. $\mathcal{M}, x'_n \models \Box\varphi$ implies $\mathcal{M}, y' \models \varphi$, but $y'$ and $y$ are $\Gamma$-equivalent and $\varphi \in \Gamma$, so $\mathcal{M}, y \models \varphi$. $\Box$

# 4 Finite "canonical" models

Let $\mathcal{L}$ be a normal modal logic, $\mathcal{M}_{\mathcal{L}}$ its canonical model, and $\varphi$. Let us put $\Gamma = \mathrm{Sub}(\varphi) \cup \{\neg\psi | \psi \in \mathrm{Sub}(\varphi)\}$.

A subset $\Delta \subseteq \Gamma$ is a *finite $\mathcal{L}$-consistent set* if $\neg \bigwedge \Delta \notin \mathcal{L}$. A subset $\Delta$ is maximal, if (the following are obviously equivalent):

1. $\Delta$ is maximal amongst finite $\mathcal{L}$-consistent sets,
2. For each $\psi \in \mathrm{Sub}(\varphi)$ either $\psi \in \Delta$ or $\neg\psi \in \Delta$.

Every finite $\mathcal{L}$-theory is clearly can be extended to some maximal one. It is the finite version of Lindenbaum's lemma.

**Definition 8.** *Let $\mathcal{L}$ be a normal modal logic and $\varphi \notin \mathcal{L}$. A finite "canonical" model is a triple $\mathcal{M}_{\mathcal{L}}^{\varphi} = \langle W_{\mathcal{L}}^{\varphi}, R_{\mathcal{L}}^{\varphi}, \vartheta_{\mathcal{L}}^{\varphi}\rangle$, where*

1. *$W_{\mathcal{L}}^{\varphi}$ is the set all maximal theories that extend finite $\mathcal{L}$-theories*
2. *$R_{\mathcal{L}}^{\varphi}$ is a relation such that $\langle W_{\mathcal{L}}^{\varphi}, R_{\mathcal{L}}^{\varphi}\rangle$ is an $\mathcal{L}$-frame and*

$$\forall \Box\psi \in \mathrm{Sub}(\varphi)\ \forall \Delta_1 \in W_{\mathcal{L}}^{\varphi}\ (\Box\psi \in \Delta_1 \Leftrightarrow \forall \Delta_2 \in R_{\mathcal{L}}^{\varphi}(\Delta_1)\ \psi \in \Delta_2)$$

3. *$\vartheta_{\mathcal{L}}^{\varphi}(p) = \{\Delta \in W_{\mathcal{L}}^{\varphi} \mid p \in \Delta\}$.*

**Theorem 3.** *$\mathcal{L}$ be a modal logic, then the following are equivalent:*

1. *$\mathcal{L}$ admits strict filtrations*

2. *If $\varphi \notin \mathcal{L}$, there exists a finite "canonical" model $\mathcal{M}^{\varphi}_{\mathcal{L}}$ such that $\mathcal{M}^{\varphi}_{\mathcal{L}} \models \mathcal{L}$ and $\mathcal{M}^{\varphi}_{\mathcal{L}}$ refutes $\varphi$.*

*Proof.*

1. ($\Rightarrow$) Let $\varphi \notin \mathcal{L}$, let us put $\Gamma = \mathrm{Sub}(\varphi)$. $\mathcal{L}$. In particular, $\mathcal{M}_{\mathcal{L}} \not\models \varphi$, where $\mathcal{M}_{\mathcal{L}}$ is the canonical model of $\mathcal{L}$. $\mathcal{L}$ admits strict filtrations, so the filtration of $\mathcal{M}_{\mathcal{L}}$ through $\Gamma$ is also an $\mathcal{L}$-model. The underlying set of $\mathcal{M}_{\mathcal{L}}/\sim_{\Gamma}$ consists of maximal $\mathcal{L}$ theories up to $\Gamma$-equivalence and this quotient set is finite.

   It is readily checked that the quotient model $\mathcal{M}_{\mathcal{L}}/\sim_{\Gamma}$ satifsfies Definition 8.

2. ($\Leftarrow$)

   Let $\varphi \notin \mathcal{L}$ and $\mathcal{M}^{\varphi}_{\mathcal{L}} = \langle W^{\varphi}_{\mathcal{L}}, R^{\varphi}_{\mathcal{L}}, \vartheta^{\varphi}_{\mathcal{L}} \rangle$ be a finite "canonical" model that refutes $\varphi$. Let $\mathcal{M}$ be an $\mathcal{L}$-model that also refutes $\varphi$. Let us show that there exists a model $\widehat{\mathcal{M}}$ such that $\widehat{\mathcal{M}}$ is a strict filtration of $\mathcal{M}$ through $\mathrm{Sub}(\varphi)$.

   Let $\sim_{\mathrm{Sub}(\varphi)}$ be an equivalence relation on $\underline{\mathcal{M}}$ defined as usual:

   $$u \sim_{\mathrm{Sub}(\varphi)} v \text{ iff } \forall \psi \in \mathrm{Sub}(\varphi)\ \mathcal{M}, u \models \psi \Leftrightarrow \mathcal{M}, v \models \psi$$

   Let $[u] \in W/\sim_{\mathrm{Sub}(\varphi)}$, where $W = \underline{\mathcal{M}}$, consider its truth set $||[u]|| = \{\psi \in \mathrm{Sub}(\varphi) \mid \mathcal{M}, u \models \psi\}$. $||[u]||$ is clearly a finite $\mathcal{L}$-consistent set and it is maximal. Consider a map

   $$\iota : W/\sim_{\mathrm{Sub}(\varphi)} \to W^{\varphi}_{\mathcal{L}} \text{ with } \iota : [u] \mapsto ||[u]||$$

   Consider the set $\iota(W/\sim_{\mathrm{Sub}(\varphi)})$ that we denote as $\mathrm{Im}(\iota)$. Let us show that $\mathrm{Im}(\iota)$ is $R^{\varphi}_{\mathcal{L}}$-closed.

   Let $\iota([u]) R^{\varphi}_{\mathcal{L}} \Delta$, let us show that $\Delta \in \iota(W/\sim_{\mathrm{Sub}(\varphi)})$, that is, $\Delta = \iota([v])$ for some $v \in W$.

   Note that $\bigwedge\limits_{\psi \in \Delta} \psi \in \Delta$ since $\Delta$ is finite. This conjuction has the form $\bigwedge\limits_{\psi \in \Delta} \psi \in \Delta = \psi_1 \wedge \dots \wedge \psi_n$ for some $n < \omega$ and for some $\psi_i \in \mathrm{Sub}(\varphi) \cup \neg\,\mathrm{Sub}(\varphi)$, where $0 < i < \omega$.

   Therefore $\mathcal{M}^{\varphi}_{\mathcal{L}}, \iota([u]) \models \Diamond(\psi_1 \wedge \dots \wedge \psi_n)$. Then $\mathcal{M}, u \models \Diamond(\psi_1 \wedge \dots \wedge \psi_n)$. That implies $\mathcal{M}, v \models \psi_1 \wedge \dots \wedge \psi_n$ for some $v \in R(u)$. This is clearly a desired $v$ since $||[v]|| \subseteq \Delta$ and such a $\Delta$ is unique.

   Thus, $\langle \mathrm{Im}(\iota), R^{\varphi}_{\mathcal{L}}, \vartheta'^{\varphi}_{\mathcal{L}} \rangle$, where $\vartheta'^{\varphi}_{\mathcal{L}}(p) = \vartheta^{\varphi}_{\mathcal{L}}(p) \cap \mathrm{Im}(\iota)$ is a generated submodel of $\mathcal{M}^{\varphi}_{\mathcal{L}}$.

   Consider a model $\widehat{\mathcal{M}} = \langle W/\sim_{\mathrm{Sub}(\varphi)}, \widehat{R}, \widehat{\vartheta} \rangle$, where $\widehat{R}$ is defined as

   $$[u] \widehat{R} [v] \text{ iff } ||[u]|| R^{\varphi}_{\mathcal{L}} ||[v]||$$

   Let us show that $R_{min} \subseteq \widehat{R} \subseteq R_{max}$.

   Suppose that there are $u' \in [u]$ and $v' \in [v]$ with $u' R v'$. Let us show that $[u]\widehat{R}[v]$. That is, let us show that $\Diamond\iota(v) \subseteq \iota(v)$. But if $\mathcal{M}, v' \models \psi$ for $\Diamond\psi \in \mathrm{Sub}(\varphi)$, then $\mathcal{M}, u' \models \Diamond\psi$. Recall that $u' \sim_{\mathrm{Sub}(\varphi)} u$ and $v' \sim_{\mathrm{Sub}(\varphi)} v$, so we have $\Diamond||[v]|| \subseteq ||[u]||$.

   The second inclusion follows from the definition of a relation in a finite "canonical" model.

□

In particular, **K**5 fails to have the finite "canonical" model property that follows from the contraposition of the theorem above and Theorem 2.

## References

[1] Stanislav Kikot, Ilya Shapirovsky, and Evgeny Zolin. Completeness of logics with the transitive closure modality and related logics. *arXiv preprint arXiv:2011.02205*, 2020.