Notes on filtration of logics containing **K5**

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Let $\mathcal{M} = \langle W, R_1, \ldots, R_n, \vartheta \rangle$ be a Kripke model and Γ a set of formulas closed under subformulas. An equivalence relation $\sim$ is set to have a finite index if the quotient set $W/\sim$ is finite. The equivalence relation $\sim_\Gamma$ induced by Γ is defined as

$$w \sim_\Gamma v \Leftrightarrow \forall \varphi \in \Gamma \, (\mathcal{M}, w \models \varphi \Leftrightarrow \mathcal{M}, v \models \varphi).$$

If Γ is finite, then $\sim_\Gamma$ has a finite index. An equivalence relation $\sim$ respects $\sim_\Gamma$, if $w \sim v$ implies $w \sim_\Gamma v$.

Definition 1. *Let $\mathcal{M} = \langle W, R_1, \ldots, R_n, \vartheta \rangle$ be a Kripke model and Γ be a Sub-closed set formulas. A Γ-filtration of $\mathcal{M}$ is a model $\widehat{\mathcal{M}} = \langle \widehat{W}, \widehat{R_1}, \ldots, \widehat{R_n}, \widehat{\vartheta} \rangle$ such that:*

1. *$\widehat{W} = W/\sim$, where $\sim$ is an equivalence relation having a finite index that respects Γ*
2. *$\widehat{\vartheta}(p) = \{[x]_\sim \mid x \in W \,\&\, x \in \vartheta(p)\}$*
3. *For each $i \in I$ one has $\widehat{R}_i^{min} \subseteq \widehat{R}_i \subseteq \widehat{R}_i^{max}$. $\widehat{R}_{i,\sim}^{min}$ is the i-th minimal filtered relation on $\widehat{W}$ defined as*

$$\hat{x} \widehat{R}_{i,\sim}^{min} \hat{y} \Leftrightarrow \exists x' \sim x \, \exists y' \sim y \; x R_i y$$

$\widehat{R}_{\Gamma,i}^{max}$ is the i-th maximal filtered relation on $\widehat{W}$ induced by Γ defined as

$$\hat{x} \widehat{R}_{\Gamma,i}^{max} \hat{y} \Leftrightarrow \forall \Box_i \varphi \in \Gamma \, (\mathcal{M}, x \models \Box_i \varphi \Rightarrow \mathcal{M}, y \models \varphi)$$

If Φ is finite subset of Γ and $\sim = \sim_\Phi$, then $\widehat{\mathcal{M}}$ is a definable Γ-filtration of $\mathcal{M}$ through Φ. If $\sim = \sim_\Gamma$, then such a filtration by means of the definiton above is called *strict*. A class of models $\mathbb{M}$ admits strict filtrations for models (ASF), if for every Sub-closed set Γ and for every $\mathcal{M} \in \mathbb{M}$ there exists a Γ filtration of $\mathcal{M}$. A class of frames $\mathbb{F}$ admits strict filtrations for frames, if for every Sub-closed set Γ and for every frame $\mathcal{F} \in \mathbb{F}$ and every model $\mathcal{M}$ over $\mathcal{F}$ there exists a Γ filtration of $\mathcal{M}$. If $\mathcal{L}$ is canonical, then the ASF property for frames and ASF property for models are equivalent [1, Theorem 2.10].

Lemma 1. *Let Γ be a finite set of formulas closed under subformulas and $\widehat{\mathcal{M}}$ a filtration of $\mathcal{M}$ through Γ, then for each $x \in W$ and for each $\varphi \in \Gamma$ one has*

$$\mathcal{M}, x \models \varphi \Leftrightarrow \widehat{\mathcal{M}}, \hat{x} \models \varphi$$

Definition 2. *Let $\mathbb{F}$ be a class of Kripke frames and Γ a finite set of formulas closed under subformulas. If for every model $\mathcal{M}$ over $\mathcal{F} \in \mathbb{F}$ there exists a model that is a Γ-definable filtration of $\mathcal{M}$, then $\mathbb{F}$ admits definable filtration. A class of models $\mathbb{M}$ admits definable filtration if for every $\mathcal{M} \in \mathbb{M}$ there exists a model belonging to the same class that is a definable Γ-filtration of $\mathcal{M}$.*

Lemma 2.

1. *Let $\mathcal{L}$ be a complete normal modal logic. If* $\mathrm{Frames}(\mathcal{L})$ *admits filtration, then $\mathcal{L}$ has the finite model property.*

2. *If the class of models* $\mathrm{Mod}(\mathcal{L})$ *admits filtration, then $\mathcal{L}$ has the finite model property and it is Kripke complete as well.*

Definition 3. *A first-order formula is called Horn if it has the following form:*

$$\forall x_1, \ldots, x_n (x_{i_1} R x_{j_1} \wedge \cdots \wedge x_{i_s} R x_{j_s} \to x_k R x_l)$$

Definition 4. *Let H be a Horn property and $\langle W, R \rangle$ a Kripke frame. A Horn closure of a binary relation R is the minimal relation R^H containing R and satisfying H.*

Lemma 3. $R^H = \bigcup\limits_{n<\omega} R_n$ *where*

1. $R_0 = R$.

2. $R_{n+1} = R_n \cup \{(a, b) \in W \mid \exists \vec{c} \in W \ P(a, b, \vec{c})\}$, *where P is a premise of H.*

E-closure (an Euclidean Horn closure of a binary relation) has the following equivalent definitions:

Lemma 4. *Let $\mathcal{F} = \langle W, R \rangle$ be a Kripke frame. The following conditions are equivalent:*

1. *R^E is the smallest Euclidean relation containing R.*

2. $R^E = \bigcup\limits_{i<\omega} R_i$, *where*
   - $R_0 = R$
   - $R_{n+1} = R_n \cup (R_n^{-1} \circ R_n)$

3. *$xR^E y$ iff there exists $n < \omega$ such that either xRy or $\exists z_1, \ldots, z_n$ with $z_1 R x$ and $z_{n-1} R y$ and for each $1 < i \leqslant n$ one has either $z_{i-1} R z_i$ or $z_i R z_{i-1}$.*

4. $R^E = R \cup \bigcup\limits_{i<\omega} (R^{-1} \circ (R \circ R^{-1})^n \circ R)$.

Proof.

1. (1) $\Rightarrow$ (2) Let us show that if R^E is the smallest Euclidean relation containing R, then $R^E = \bigcup\limits_{i<\omega} R_i$. There are two inclusions:
   - $R^E \subseteq \bigcup\limits_{i<\omega} R_i$. Recall that R^E has the form (?):
   $$R^E = \bigcap \{R' \mid R \subseteq R', \forall a, b \in W \ R'(a, b) \Rightarrow \exists x \in W \ R'(x, a) \ \& \ R'(x, b)\}$$
   - $\bigcup\limits_{i<\omega} R_i \subseteq R^E$. Let us show that $xR_n y$ for each $n < \omega$ implies $xR^E y$ by induction on n. If $n = 0$, then xRy, thus, $xR^E y$, since R is a subrelation of R^E. Suppose $n = m+1$ and $xR_{m+1} y$. Let us show that $xR^E y$. From $xR_{m+1} y$, one has $(x, y) \in R^n \cup (R_n^{-1} \circ R_n)$. There are two cases:
     - $xR^n y$, one needs to merely apply the IH.

– $xR_n^{-1} \circ R_n y$. Then $\exists z \in W \; xR_n^{-1}z \;\&\; zR_n$. That is, $zR_n x$ and $zR_n y$ for some z. R_n is already a subrelation of R^E. Thus, $zR^E x$ and $zR^E y$. That implies $xR^E y$.

2. (2) $\Rightarrow$ (3) Let $(x, y) \in R_m$, let us the statement by induction on m.

   (a) Suppose $m = 0$, then xRy, and the statement is shown putting $n = 0$.

   (b) Suppose $m = p + 1$ and $xR_{p+1}y$. Assume that either xRy or $\exists z_1, \dots, z_p$ with z_1Rx and $z_{p-1}Ry$ and for each $1 < i \leqslant p$ one has either $z_{i-1}Rz_i$ or z_iRz_{i-1}.

   $xR_{p+1}y$ implies $(x, y) \in R_p \cup (R_p^{-1} \circ R_p)$. If $(x, y) \in R_p$, then we merely apply the IH. Suppose $(x, y) \in R_p^{-1} \circ R_p$, then $(z, x) \in R_p$ and $(z, y) \in R_p$

3. (3) $\Rightarrow$ (4) Suppose either xRy or there exist $n \geqslant 1$ and $z_1, \dots, z_n$ with z_1Rx and $z_{n-1}Ry$ and for each $1 < i \leqslant n$ one has either $z_{i-1}Rz_i$ or z_iRz_{i-1}. If xRy, then we are done. Otherwise there exists $n \geqslant 1$ with the condition above. Then $(x, y) \in R_{n+1}$ that follows from the condition.

4. (4) $\Rightarrow$ (1)

$\square$

Lemma 5. *Let $\mathcal{F} = \langle W, R\rangle$ be a Kripke frame. Let us define $R^E = \bigcup\limits_{i<\omega} R_i$ where:*

1. $R_0 = R$

2. $R_{n+1} = R_n \cup ({R_n}^{-1} \circ R_n)$

Then R^E is Euclidean.

Proof. Let $(x, y), (x, z) \in R^E$, one needs to show that $(y, z) \in R^E$. Clearly that $(x, y) \in R_i$ and $(x, z) \in R_j$ for some $i, j < \omega$. Thus, we need $(y, z) \in R_m$ for some m depending on i and j.

Let us consider the following cases:

1. $i = 0$ and $j = 0$

   Suppose $(x, y), (x, z) \in R_0 = R$, then $(y, z) \in R^{-1} \circ R$. Thus, $(y, z) \in R_1$

2. $i = 0$ and $j = k + 1$

   Suppose $(x, y) \in R$ and $(x, z) \in R_{k+1} = R_k \cup ({R_k}^{-1} \circ R_k)$. Clearly that $(x, y) \in R_{k+1}$ as well. It is obviously that $(y, z) \in R_{k+2}$ since $(y, x) \in R_{k+1}^{-1}$ and $(x, z) \in R_{k+1}$.

3. The case with $i = k + 1$ and $j = 0$ is similar to the previous one.

4. Suppose $i = m + 1$ and $j = k + 1$. That is, $(x, y) \in R_{m+1} = R_m \cup ({R_m}^{-1} \circ R_m)$ and $(x, z) \in R_{k+1} = R_k \cup ({R_k}^{-1} \circ R_k)$. Consider the following four subcases:

   (a) Suppose $(x, y) \in R_m$ and $(x, z) \in R_k$ and $m \leqslant k$ without loss of generality. $m \leqslant k$ implies $R_m \subseteq R_k$ and $(x, y) \in R_k$ in particular. Thus, $(y, z) \in {R_k}^{-1} \circ R_k$, so $(y, z) \in R_{k+1}$.

   (b) The rest of the cases are similar to the first one.

$\square$

Theorem 1. **K**45 *admits strict filtrations.*

Proof. Let $\mathcal{M} = \langle W, R, \vartheta \rangle$ be a transitive Euclidean model and $\overline{\mathcal{M}} = \langle \overline{W}, \overline{R}, \overline{\vartheta} \rangle$ its minimal filtration through Γ, where Γ is finite and Sub-closed. Let us put $\widehat{R} = \overline{R}^{+} \cup \overline{R}^{E}$. Let us show that $\overline{R}^{+} \cup \overline{R}^{E} \subseteq \overline{R}^{max}$.

That is, if $\mathcal{M}, y \models \varphi$ for $\Diamond\varphi \in \Gamma$ and $\hat{x}\widehat{R}\hat{y}$, then $\mathcal{M}, x \models \Diamond\varphi$.

Let $\hat{x}\widehat{R}\hat{y}$. Let us consider the case when $(\hat{x}, \hat{y}) \in \overline{R}^{E}$

1. Suppose $(\hat{x}, \hat{y}) \in \overline{R}$, then $\mathcal{M}, x \models \Diamond\varphi$ holds trivially by the definition of the minimal filtration.
2. Suppose the statement holds $\overline{R}_n$ and $(\hat{x}, \hat{y}) \in \overline{R}_{n+1} = \overline{R}_n \cup (\overline{R}_n^{-1} \circ \overline{R}_n)$. We consider the case of $(\hat{x}, \hat{y}) \in (\overline{R}_n^{-1} \circ \overline{R}_n)$.

   Then there exists $\hat{z}$ such that $(\hat{z}, \hat{x}), (\hat{z}, \hat{y}) \in \overline{R}_n$.

   By IH, $\mathcal{M}, z \models \Diamond\varphi$.

   $(\hat{z}, \hat{y}) \in \overline{R}_n$ iff there are $\hat{u}_1, \ldots, \hat{u}_n$ such that

$$\hat{z} \xleftarrow[\widehat{R}]{} \hat{u}_1 \xrightarrow[\widehat{R}']{} \hat{u}_2 \xrightarrow[\widehat{R}']{} \ldots \xrightarrow[\widehat{R}']{} \hat{u}_{n-1} \xrightarrow[\widehat{R}']{} \hat{u}_n \xrightarrow[\widehat{R}]{} \hat{y}$$

   where $\widehat{R}'$ is either $\widehat{R}$ or $\widehat{R}^{-1}$.

As it is known, $\Diamond\Diamond\varphi \to \Box\Diamond\varphi \in \mathbf{K}45$.

$\hat{u}_1\hat{\hat{z}}$, that is, $u_1' R z'$ for some $u_1' \in \hat{u}_1$ and $z' \in \hat{z}$. That is, $\mathcal{M}, u_1' \models \Diamond\Diamond\varphi$, so $\mathcal{M}, u_1' \models \Diamond\varphi$ and $\overline{M}, \hat{u}_1 \models \Diamond\varphi$.

We have $\hat{u}_1\widehat{R}'\hat{u}_2$. Suppose $\mathcal{M}, u_1'' \models \Diamond\varphi$ and $u_1'' R u_2'$. We also have $\mathcal{M}, u_1'' \models \Box\Diamond\varphi$, thus, $\mathcal{M}, u_2' \models \Diamond\varphi$.

Suppose $\hat{u}_2\widehat{R}\hat{u}_1$ and $u_2' R u_1''$, then $\mathcal{M}, u_2' \models \Diamond\varphi$.

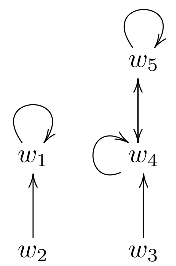
Similarly, we have $\mathcal{M}, u_i \models \Diamond\varphi$ iff $\mathcal{M}, u_{i+1} \models \Diamond\varphi$, whenever $\hat{u}_i\widehat{R}'\hat{u}_{i+1}$.

Finally, we have $\hat{u}_n\widehat{R}\hat{x}$. Thus, $u_n' R x'$ for some $u_n' \in \hat{u}_n$ and $x' \in \hat{x}$. $\mathcal{M}, u_n' \models \Diamond\varphi$, so $\mathcal{M}, u_n' \models \Box\Diamond\varphi$. Then $\mathcal{M}, x' \models \Diamond\varphi$. □

Theorem 2. **K**5 *does not admit strict filtrations.*

Proof. Let us consider a **K**5 model whose Euclidean closure of the minimal filtration does not give us a filtration.

Let us consider a frame called $\mathcal{F}_{bad}$. We define this frame with the following graph:



□

Let us define a valuation ϑ such that $\vartheta(p) = \{w_5\}$ and $\vartheta(q) = \{w_1\}$. Let us consider a minimal filtration of $\mathcal{M}_{bad}$ through the Sub-closure of $\Gamma = \{\neg p, \neg\Diamond p\}$.

Clearly that $w_2 \sim_\Gamma w_3$, since $\neg p$ and $\neg\Diamond p$ are true both at w_2 and w_3.

Moreover, $R_{min} \cup (R_{min}^{-1} \circ R_{min})$ is not a subset of R_{max} since $(\hat{w_1}, \hat{w_5}) \in (R_{min}^{-1} \circ R_{min})$, but $\Diamond p$ is not true at w_5.

Let us also note that strict filtrations of this model is not Euclidean. Suppose by contrary that $\widehat{R}^{\mathcal{E}}$ is a strict filtraction of that model. So $R_{min}^{E} \subseteq \widehat{R}^{\mathcal{E}}$, since R_{min}^{E} is the minimal Euclidean relation containing R_{min}. On the other hand, $R_{min}^{E} \nsubseteq R_{max}$, so is not $\widehat{R}^{\mathcal{E}}$.

1 Finite "canonical" models

Let $\mathcal{L}$ be a normal modal logic, $\mathcal{M}_{\mathcal{L}}$ its canonical model, and Γ a finite Sub-closed set of formulas. Let us put $\Gamma' = \mathrm{Sub}(\varphi) \cup \{\neg\psi \mid \psi \in \mathrm{Sub}(\varphi)\}$.

A subset $\Delta \subseteq' \Gamma$ is a *finite $\mathcal{L}$-consistent set* if $\neg \bigwedge \Delta \notin \mathcal{L}$. A subset Δ is maximal, if (the following are obviously equivalent):

1. Δ is maximal amongst finite $\mathcal{L}$-consistent sets,
2. For each $\psi \in \mathrm{Sub}(\varphi)$ either $\psi \in \Delta$ or $\neg\psi \in \Delta$.

Every finite $\mathcal{L}$-theory is clearly can be extended to some maximal one. It is the finite version of Lindenbaum's lemma.

Definition 5. *Let $\mathcal{L}$ be a modal logic and Γ be a finite* Sub*-closed set of formulas. A finite "canonical" model is a triple $\mathcal{M}_{\mathcal{L}}^{\Gamma} = \langle W_{\mathcal{L}}^{\Gamma}, R_{\mathcal{L}}^{\Gamma}, \vartheta_{\mathcal{L}}^{\Gamma} \rangle$, where*

1. *$W_{\mathcal{L}}^{\Gamma}$ is the set all maximal theories that extend finite $\mathcal{L}$-theories*
2. *$R_{\mathcal{L}}^{\Gamma}$ is a relation such that $\langle W_{\mathcal{L}}^{\Gamma}, R_{\mathcal{L}}^{\Gamma} \rangle$ is an $\mathcal{L}$-frame and*

$$\forall \Box\psi \in \mathrm{Sub}(\varphi) \ \ \forall \Delta_1 \in W_{\mathcal{L}}^{\Gamma} \ \ (\Box\psi \in \Delta_1 \Leftrightarrow \forall \Delta_2 \in R_{\mathcal{L}}^{\Gamma}(\Delta_1) \ \ \psi \in \Delta_2)$$

3. *$\vartheta_{\mathcal{L}}^{\Gamma}(p) = \{\Delta \in W_{\mathcal{L}}^{\varphi} \mid p \in \Delta\}$ for every variable $p \in \Gamma$.*

Lemma 6. *Let $\mathcal{L}$ be a modal logic and $\varphi \notin \mathcal{L}$, then $\mathcal{M}_{\mathcal{L}}^{\mathrm{Sub}(\varphi)} \not\models \varphi$.*

Lemma 7. *Let $\mathcal{L}$ be a modal logic and Γ a finite* Sub*-closed set of formulas, then if $\mathcal{L}$ admits strict filtrations, then there exists a finite "canonical" model $\mathcal{M}_{\mathcal{L}}^{\Gamma}$ such that $\mathcal{M}_{\mathcal{L}}^{\Gamma} \models \mathcal{L}$.*

Proof. ($\Rightarrow$) Let Γ be a finite Sub-closed of formulas. $\mathcal{L}$ admits strict filtrations, so the filtration of the canonical model $\mathcal{M}_{\mathcal{L}}$ through Γ is also an $\mathcal{L}$-model. The underlying set of $\mathcal{M}_{\mathcal{L}} / \sim_{\Gamma}$ consists of maximal $\mathcal{L}$ theories up to Γ-equivalence and this quotient set is finite.

It is readily checked that the quotient model $\mathcal{M}_{\mathcal{L}} / \sim_{\Gamma}$ satifsfies Definition 5. $\Box$

The converse implication does not have to true generally.

Recall that $\mathbf{GL} = \mathbf{K} \oplus \Diamond(p \wedge \neg\Diamond p) \to \Diamond p$. $\mathbf{GL}$ does not admit filtrations, but one may prove that $\mathbf{GL}$ has the FMP using the selective filtation technique.

Definition 6. *Let $\mathcal{M} = \langle W, R, \vartheta \rangle$ be a model and Γ a set closed under subformulas, then a weak submodel $\mathcal{M} = \langle W', R', \eta \rangle$ (that is, $W' \subseteq W$, $R' \subseteq R$, $\eta = \vartheta \upharpoonright_{W'}$) is called a selective filtration of $\mathcal{M}$ through Γ is the following holds, for $\Diamond\psi \in \Gamma$, $x \in W'$:*

$$\mathcal{M}, x \models \Diamond\psi \text{ iff } \mathcal{M}, y \models \psi \text{ for some } y \in R'(x)$$

Theorem 3. $\mathbf{GL}$ *admits the "finite canonical" model property.*

Proof. Let Γ be a finite **Sub**-closed set. So, $\Gamma = \mathbf{Sub}(\gamma_1) \cup \ldots \mathbf{Sub}(\gamma_n)$, for some $\gamma_1, \ldots, \gamma_n \in \Gamma$ and every γ_i is **GL**-consistent.

γ is true in $\mathcal{M}_\Gamma = \langle V, R^{\mathbf{GL}}, \vartheta \rangle$, where

$$V_\Gamma = \bigcup_{\psi \in \Gamma} \{x \in W_{\mathbf{GL}} \mid \mathcal{M}_{\mathbf{GL}}, x \models \psi \wedge \neg \Diamond \psi\}$$

since $\mathcal{M}_\Gamma$ is a selective filtration of $\mathcal{M}_{\mathbf{M}}$ through Γ.

Let $x \in V$, consider $||x|| = \{\psi \in \Gamma \mid \mathcal{M}_{\mathbf{GL}}, x \models \psi\}$ and define $xR^{\mathbf{GL}}y$ implies $||x||R_\Gamma||y||$. Every $||x||$ is clearly finite. The valuation map is usual, $\vartheta_\Gamma(p) = \{||x|| \mid p \in ||x||\}$ for $p \in \mathrm{PV} \cap \Gamma$. We put W_Γ

Let us check that $\langle W_\Gamma, R_\Gamma, \vartheta_\Gamma \rangle$ is a "finite canonical" **GL**-model. Let us check that R_Γ is transitive and Noetherian and it has no relflexive points. If W_Γ had a relfexive point, say $||y||$, then we would have $\mathcal{M}, y \models \psi \to \Diamond \psi$, but that is not impossible. Transitivity is obvious.

Clearly that $xR^{\mathbf{GL}}y$ implies $||x||R_\Gamma||y||$ and vice versa. So, if we had an infinite decreasing R_Γ-chain, then we would have an infinite decreasing $R^{\mathbf{GL}}$-chain, but that is not possible. So $\langle W_\Gamma, R_\Gamma, \vartheta_\Gamma \rangle$ is a "finite canonical" **GL**-model. $\square$

So we have an example of a logic that has the "finite canonical" model property with no filtrations.

References

[1] Stanislav Kikot, Ilya Shapirovsky, and Evgeny Zolin. Completeness of logics with the transitive closure modality and related logics. *arXiv preprint arXiv:2011.02205*, 2020.