

Model-theoretic aspects of relativised cylindric set algebras

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1 The problems themselves

1. Suppose $\mathcal{C} \in \mathbf{RCA}_\omega$, whether \mathcal{C} has a complete, ω -dimensional representation?
2. Is the class \mathbf{IG}_ω (the isomorphism-closure of the ω -dimensional cylindric relativised set algebras in which the unit is closed under substitutions and permutations) a variety, or even a pseudo-elementary class? Is it closed under ultraproducts?

2 Boolean algebras with operators and cylindric algebras

Definition 1.

1. Let $\mathcal{B} = \langle B, +, -, 0, 1 \rangle$ be a Boolean algebra. An operator is an n -ary function $\Omega : B^n \rightarrow B$ satisfying the following conditions:
 - Normality: for all $b_0, \dots, b_{n-1} \in B$, if $b_i = 0$ for some $i < n$, then
$$\Omega(b_0, \dots, b_{i-1}, 0, b_{i+1}, \dots, b_{n-1}) = 0$$
 - Additivity: for all $b_0, \dots, b_{n-1}, b, b' \in B$ we have
$$\Omega(b_0, \dots, b_{i-1}, (b + b'), b_{i+1}, \dots, b_{n-1}) = \Omega(b_0, \dots, b_{i-1}, b, b_{i+1}, \dots, b_{n-1}) + \Omega(b_0, \dots, b_{i-1}, b', b_{i+1}, \dots, b_{n-1})$$
2. Let I be an index set, a Boolean algebra with operators (BAO) is an algebra $\langle B, +, -, 0, 1, \{\Omega_i\}_{i \in I} \rangle$ such that $\langle B, +, -, 0, 1 \rangle$ is a Boolean algebra and for each $i \in I$ Ω_i is an operator.

Definition 2. Let $\mathcal{B} = \langle B, +, -, 0, 1, \{\Omega_i\}_{i \in I} \rangle$ be a BAO, then

1. An operator Ω is completely additive, if for each $b_0, \dots, b_{n-1} \in B$ and $X \subseteq B$, one has

$$\Omega(b_0, \dots, b_{i-1}, \sum X, b_{i+1}, \dots, b_{n-1}) = \sum_{x \in X} \Omega(b_0, \dots, b_{i-1}, x, b_{i+1}, \dots, b_{n-1})$$

2. \mathcal{B} is completely additive, if for each $i \in I$ Ω_i is additive,
3. A class \mathcal{K} of BAOs is completely additive, if every $\mathcal{B} \in \mathcal{K}$ is completely additive.

2.1 Atom structures and canonical extensions

Definition 3. Let I be an index set and $\{\Omega_i\}_{i \in I}$ a set of function symbols

1. An atom structure is a relational structure $\mathcal{F} = \langle W, \{R_i\}_{i \in I} \rangle$ such that R_i is a $n+1$ -ary relation symbol, if Ω_i is an n -ary function symbol,
2. Let \mathcal{B} be an atomic BAO of the signature I , the atom structure of \mathcal{B} , written as $\mathbf{At}\mathcal{B}$, is an atom structure $\langle \mathbf{At}(\mathcal{B}), \{R_i\}_{i \in I} \rangle$ such that for each $a, b_0, \dots, b_{n+1} \in \mathbf{At}(\mathcal{B})$ and for each $i \in I$

$$\mathbf{At}\mathcal{B} \models R_i(a, b_0, \dots, b_{n+1}) \text{ iff } \mathcal{B} \models a \leq \Omega_i(b_0, \dots, b_{n+1})$$

3. Let $\mathcal{F} = \langle W, \{R_i\}_{i \in I} \rangle$ be an atom structure, the complex algebra of \mathcal{F} , written as $\mathbf{Cm}\mathcal{F}$, is a BAO $\langle \mathcal{P}(W), \cup, -, \emptyset, W, \{\Omega_{R_i}\}_{i \in I} \rangle$ such that for all $X_0, \dots, X_{n-1} \subseteq W$ and for each $i \in I$

$$\Omega_{R_i}(X_0, \dots, X_{n-1}) = \{a \in W \mid \exists b_0 \in X_0 \dots \exists b_{n-1} \in X_{n-1} \mathcal{F} \models R_i(a, b_0, \dots, b_{n-1})\}$$

The following duality is due to Thomason [18].

Fact 1.

1. Let \mathcal{B} be a complete atomic BAO, then $\mathcal{B} \cong \mathbf{Cm}(\mathbf{At}(\mathcal{B}))$,
2. Let \mathcal{F} be an atom structure, then $\mathcal{F} \cong \mathbf{At}(\mathbf{Cm}(\mathcal{B}))$.

Let A be a non-empty subset of a Boolean algebra \mathcal{B} , A is a *filter*, if A is closed under finite infima and upwardly closed. A is an *ultrafilter*, if it has no non-trivial extensions. That is, if $A \subseteq A'$, then $A' = \mathcal{B}$.

Definition 4. Let $\mathcal{B} = \langle \mathcal{B}, +, -, 0, 1, \{\Omega_i\}_{i \in I} \rangle$ be a BAO and $\mathbf{Uf}(\mathcal{B})$ the set of its ultrafilters. The ultrafilter frame of \mathcal{B} (or canonical frame) is a relational structure $\mathcal{F}_{\mathcal{B}} = \langle \mathbf{Uf}(\mathcal{B}), R_{\Omega_i} \rangle$ such that for each ultrafilters $\beta_0, \dots, \beta_{n-1}, \gamma$ one has

$$\mathbf{Uf}(\mathcal{B}) \models R_{\Omega_i}(\beta_0, \dots, \beta_{n-1}, \gamma) \text{ iff } \{\Omega_i(b_0, \dots, b_{n-1}) \mid b_0 \in \beta_0, \dots, b_{n-1} \in \beta_{n-1}\} \subseteq \gamma.$$

Definition 5. Let \mathcal{B} be a BAO, then

1. The canonical extension of \mathcal{B} is a complex algebra of the canonical frame $\mathbf{Cm}(\mathcal{F}_{\mathcal{B}})$ denoted as \mathcal{B}^+ ,
2. The class of BAOs is canonical, if it is closed under canonical extensions.

Theorem 1. Let \mathcal{A}, \mathcal{B} be BAOs,

1. There exists $\iota : \mathcal{A} \hookrightarrow \mathcal{A}^+$ such that $\iota : a \mapsto \{\gamma \in \mathbf{Uf}(\mathcal{A}) \mid a \in \gamma\}$.
2. If $i : \mathcal{A} \hookrightarrow \mathcal{B}$, then this embedding might be extended to the embedding $i^+ : \mathcal{A}^+ \hookrightarrow \mathcal{B}^+$

Fact 2.

2.2 (Representable) cylindric algebras and cylindric set algebras

Let α be an ordinal. Let U^α be the set of all functions mapping α to a non-empty set U . We denote $x(i) = x_i$ for $x \in U^\alpha$ and $i < \alpha$.

Definition 6.

1. A subset of U^α is an α -ry relation on U . For $i, j < \alpha$, the i, j -diagonal D_{ij} is the set of all elements of U such that $y_i = y_j$. If $i < \alpha$ and X is an α -ry relation on U , then the i -th cylindrification $C_i X$ is the set of all elements of U that agree with some element of X on each coordinate except the i -th one. To be more precise, $C_i X = \{y \in U^\alpha \mid \exists x \in X \forall i < \alpha (i \neq j \Rightarrow y_j = x_j)\}$.
2. A cylindric set algebra of dimension α is an algebra consisting of a set S of α -ry relation on some base set U with the constants and operations $0 = \emptyset$, $1 = U^\alpha$, \cap , $-$, the diagonal elements $\{D_{ij}\}_{i,j < \alpha}$, the cylindrifications $\{C_i\}_{i < \alpha}$. A generalised cylindric set algebra of dimension α is a subdirect of cylindric algebras that have dimension α .
3. A cylindric algebra of dimension α is an algebra $\mathcal{C} = \langle \mathcal{B}, \{c_i\}_{i < \alpha}, \{d_{ij}\}_{i,j < \alpha} \rangle$ such that
 - \mathcal{B} is a Boolean algebra, for each $i, j < \alpha$ c_i is an operator and $d_{ij} \in \mathcal{B}$
 - For each $i < \alpha$, $a \leq c_i a$, $c_i(a \wedge c_i b) = c_i a \wedge c_i b$ and $d_{ii} = 1$
 - For every $i, j < \alpha$, $c_i c_j a = c_j c_i a$
 - If $k \neq i, j < \alpha$, then $d_{ij} = c_k(d_{ij} \wedge d_{jk})$
 - If $i \neq j$, then $c_i(d_{ij} \wedge a) \wedge c_i(d_{ij} \wedge -a) = 0$

\mathbf{CA}_α is the class of all cylindric algebras of dimension α

4. An α -dimensional cylindric algebra C is representable, if it is isomorphic to a generalised cylindric set algebra of dimension α . Such isomorphism is a representation of C . \mathbf{RCA}_α is the class of all representable cylindric algebras that have dimension α . In particular, we are interested in the case when $\alpha = \omega$.

2.3 Substitution in cylindric algebras

Definition 7. Given a cylindric algebra of dimension α C , let x be a term of its signature, the substitution operator s_j^i have the following definition:

$$s_j^i x = \begin{cases} x, & \text{if } i = j \\ c_i(d_{ij} \wedge x), & \text{otherwise} \end{cases}$$

Proposition 1. Let α be an ordinal and let $i, j, k, l < \alpha$. The following facts hold in \mathbf{CA}_α

1. $s_j^i x \leq c_i x$.
2. $s_j^i(x \wedge y) = s_j^i x \wedge s_j^i y$, $s_j^i(x \vee y) = s_j^i x \vee s_j^i y$, $-s_j^i x = s_j^i(-x)$. Moreover, s_j^i is completely additive.
3. $i \neq k, l$ implies $s_j^i d_{ik} = d_{jk}$ and $s_j^i d_{kl} = d_{kl}$.
4. $d_{jk} \wedge s_j^i = d_{jk} \wedge s_k^i$.

5. $s_j^i c_i x = c_i x$.
6. $k \neq i, j$ implies $s_j^i c_i x = c_i s_j^i x$.
7. $c_j s_j^i x = c_i s_i^j x$.
8. $i \neq j$ implies $c_i s_j^i x = s_j^i x$.
9. $i \neq k$ implies $s_j^i s_k^i = s_k^i x$.
10. If either $i \notin \{k, l\}$ and $k \notin \{i, j\}$, or $j = l$, then $s_j^i s_l^k x = s_l^k s_j^i x$.
11. $s_j^i s_i^j x = s_j^i x$.
12. $s_k^i s_i^j x = s_k^i s_k^j x = s_k^j s_j^i x$.

3 Model-theoretic and universal algebraic preliminaries

3.1 Ultraproducts

Here are the required notions and facts from model theory and universal algebra [9] [11] [17].

Let Λ be an index set and D an ultrafilter on the Boolean algebra $\langle \mathcal{P}(\Lambda), \cup, -, \Lambda, \emptyset \rangle$. Consider the product $M = \prod_{\lambda \in \Lambda} M_\lambda$ of the Ω -structures $\{M_\lambda\}_{\lambda \in \Lambda}$ and the equivalence relation on $\text{dom}(M)$ defined as

$$a_1 \sim a_2 \Leftrightarrow \{\lambda \in \Lambda \mid a_1(\lambda) = a_2(\lambda)\} \in D$$

Let us denote $\text{dom}(M)/\sim$ as U and $[a]_\sim$ as a/D , where $a \in \text{dom}(M)$. We also denote the *ultraproduct* of $\{M_\lambda\}_\lambda$ as $\prod_{\lambda \in \Lambda} M_\lambda/D$, or, for brevity, as $\prod_D M_\lambda$. The Ω -symbols have the following interpretation

1. If $c \in \text{Cnst}$, then $c^U = c^M/D$
2. If $f \in \text{Fn}$ is an n -ary function symbol and $\bar{a} \in M^n$, then $f^U(\bar{a}) = f^M(x) = f^M(\bar{a})/D$
3. If $R \in \text{Fn}$ is an n -ary relation symbol and $\bar{a} \in M^n$, then $U \models R(\bar{a}/D)$ iff $\{\lambda \in \Lambda \mid M_\lambda \models R(\bar{a}(\lambda))\} \in D$

The ultraproduct is principal if D is a principal filter.

Definition 8.

1. Let $\{M_\lambda\}_{\lambda \in \Lambda}$ be a set of Ω -structures such that every M_λ is isomorphic to the single structure M , then their ultraproduct over D is called the *ultrapower* over D . The denotation is $\prod_D M$ or M^Λ/D .
2. If $\prod_D M \cong N$ for some structure N , then M is an *ultraroot* of N .

Theorem 2 (Łos). Let $\{M_\lambda\}_{\lambda \in \Lambda}$ be Ω -structures and D an ultrafilter on Λ , and let $U = \prod_D M_\lambda$ be an ultraproduct of $\{M_\lambda\}_{\lambda \in \Lambda}$ over D . For each first-order formula $\varphi(x_1, \dots, x_n)$ and for each $a_1/D, \dots, a_n/D \in U$:

$$U \models \varphi(a_1/D, \dots, a_n/D) \text{ iff } \{\lambda \in \Lambda \mid \varphi(a_1(\lambda), \dots, a_n(\lambda))\} \in D$$

The Los has the following helpful corollary:

Corollary 1. *Let $\prod_D M$ be an ultrapower of M . For $a \in M$, let us define a function $\bar{a} : a \mapsto a/D$. Then such a map is an elementary embedding of M into $\prod_D M$.*

Moreover, any elementary equivalent structures have isomorphic ultrapowers.

Recall that a class of Ω -structures \mathbf{K} is called *elementary*, if $\mathbf{K} = \text{Mod}(\mathbf{T})$ for some first-order theory \mathbf{T} . In that case, \mathbf{T} is an axiomatisation of \mathbf{K} .

Theorem 3. *Let \mathbf{K} be a class Ω -structures, \mathbf{K} is elementary iff \mathbf{K} is closed under isomorphic copies, ultraroots, and ultrapowers.*

4 IG_ω and ultraproducts

5 IG_ω is (not) (pseudo-)elementary; is (not) a variety

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