

Model-theoretic aspects of relativised cylindric set algebras

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1 Model-theoretic and universal algebraic preliminaries

1.1 Ultraproducts

Here are the required notions and facts from model theory and universal algebra [5] [6] [11].

Let A be a non-empty set, an *ultrafilter* on A is a set of subsets $U \subseteq \mathcal{P}(A)$ such that A is closed under intersections, \subseteq -upwardly closed, and either $X \in U$ or $-X \in U$, where $X \subseteq A$. An ultrafilter is called *principal* if it has the form $\uparrow X = \{Y \in \mathcal{P}(A) \mid X \subseteq Y\}$.

Let $\Omega = \langle \text{Cnst}, \text{Fn}, \text{Pred} \rangle$ be a signature and Λ an index set, and let $\{M_\lambda\}_{\lambda \in \Lambda}$ be an indexed set of Ω structures. The Ω -structure

$$M = \prod_{\lambda \in \Lambda} M_\lambda$$

is called a *product* that defined as follows. Its domain is the Cartesian product of the domains of M_λ . $a \in M$ is a function $\Lambda \rightarrow \bigcup_{\lambda \in \Lambda} \text{dom}(M_\lambda)$ such that $a(\lambda) \in M_\lambda$ for each $\lambda \in \Lambda$. Given $\lambda \in \Lambda$ and $a_\lambda \in M_\lambda$, we denote the function mapping λ to a_λ as $\langle a_\lambda \mid \lambda \in \Lambda \rangle$. We define the interpretation of Ω -symbols as

1. If $c \in \text{Cnst}$, then $c^M = \langle c^{M_\lambda} \mid \lambda \in \Lambda \rangle$
2. If $f \in \text{Fn}$ is an n -ary function symbol, then $f^M(\bar{a}) = \langle f^{M_\lambda}(\bar{a}) \mid \lambda \in \Lambda \rangle$, where $\bar{n} \in M^n$
3. If $R \in \text{Pred}$ is an n -ary relation symbol and $\bar{n} \in M^a$, then $R^M(\bar{a}) = \langle R^{M_\lambda}(\bar{a}) \mid \lambda \in \Lambda \rangle$

Given $\lambda \in \Lambda$, we define the λ th projection as $\pi_\lambda : M \rightarrow M_\lambda$ such that $\pi_\lambda(a) = a(\lambda)$.

Let Λ be an index set and D an ultrafilter on the Boolean algebra $\langle \mathcal{P}(\Lambda), \cup, -, \Lambda, \emptyset \rangle$. Consider the product $M = \prod_{\lambda \in \Lambda} M_\lambda$ of the Ω -structures $\{M_\lambda\}_{\lambda \in \Lambda}$ and the equivalence relation on $\text{dom}(M)$ defined as

$$a_1 \sim a_2 \Leftrightarrow \{\lambda \in \Lambda \mid a_1(\lambda) = a_2(\lambda)\} \in D$$

Let us denote $\text{dom}(M)/\sim$ as U and $[a]_\sim$ as a/D , where $a \in \text{dom}(M)$. We also denote the *ultraproduct* of $\{M_\lambda\}_\lambda$ as $\prod_{\lambda \in \Lambda} M_\lambda/D$, or, for brevity, as $\prod_D M_\lambda$. The Ω -symbols have the following interpretation

1. If $c \in \text{Cnst}$, then $c^U = c^M/D$
2. If $f \in \text{Fn}$ is an n -ary function symbol and $\bar{a} \in M^n$, then $f^U(\bar{a}) = f^M(x) = f^M(\bar{a})/D$
3. If $R \in \text{Fn}$ is an n -ary relation symbol and $\bar{a} \in M^n$, then $U \models R(\bar{a}/D)$ iff $\{\lambda \in \Lambda \mid M_\lambda \models R(\bar{a}(\lambda))\} \in D$

The ultraproduct is principal if D is a principal filter.

Definition 1.

1. Let $\{M_\lambda\}_{\lambda \in \Lambda}$ be a set of Ω -structures such that every M_λ is isomorphic to the single structure M , then their ultraproduct over D is called the ultrapower over D . The denotation is $\prod_D M$ or M^Λ/D .
2. If $\prod_D M \cong N$ for some structure N , then M is an ultraroot of N .

Theorem 1 (Los). Let $\{M_\lambda\}_{\lambda \in \Lambda}$ be Ω -structures and D an ultrafilter on Λ , and let $U = \prod_D M_\lambda$ be an ultraproduct of $\{M_\lambda\}_{\lambda \in \Lambda}$ over D . For each first-order formula $\varphi(x_1, \dots, x_n)$ and for each $a_1/D, \dots, a_n/D \in U$:

$$U \models \varphi(a_1/D, \dots, a_n/D) \text{ iff } \{\lambda \in \Lambda \mid \varphi(a_1(\lambda), \dots, a_n(\lambda))\} \in D$$

The Los has the following helpful corollary:

Corollary 1. Let $\prod_D M$ be an ultrapower of M . For $a \in M$, let us define a function $\bar{a} : a \mapsto a/D$. Then such a map is an elementary embedding of M into $\prod_D M$.

Moreover, any elementary equivalent structures have isomorphic ultrapowers.

Recall that a class of Ω -structures \mathbf{K} is called *elementary*, if $\mathbf{K} = \text{Mod}(\mathbf{T})$ for some first-order theory \mathbf{T} . In that case, \mathbf{T} is an axiomatisation of \mathbf{K} .

Theorem 2. Let \mathbf{K} be a class Ω -structures, \mathbf{K} is elementary iff \mathbf{K} is closed under isomorphic copies, ultraroots, and ultrapowers.

1.2 Preliminaries from universal algebra

Definition 2. Let \mathbf{K} be a class of Ω -structures, then \mathbf{K} is a *variety*, if it is defined by some set of equations. The variety generated by \mathbf{K} is the smallest variety containing \mathbf{K} . \mathbf{K} is a *quasi-variety*, if it is defined by some set of quasi-identities.

Given a class \mathbf{K} of Ω -structures, then $\mathbf{I}(\mathbf{K})$, $\mathbf{S}(\mathbf{K})$, $\mathbf{H}(\mathbf{K})$, and $\mathbf{P}(\mathbf{K})$ are the classes of isomorphic copies, algebras isomorphic to subalgebras belonging to \mathbf{K} , algebras isomorphic to homomorphic images belonging to \mathbf{K} , and algebras isomorphic to direct products belonging to \mathbf{K} . We claim that $\mathbf{I}(\mathbf{K}) \subseteq \mathbf{S}(\mathbf{K})$. $\mathbf{Up}(\mathbf{K})$ is the class of algebras isomorphic to ultraproducts belonging to \mathbf{K} .

Theorem 3. Let \mathbf{K} be a class of Ω -structures

1. \mathbf{K} is a variety iff $\mathbf{H}(\mathbf{K}), \mathbf{S}(\mathbf{K}), \mathbf{P}(\mathbf{K}) \subseteq \mathbf{K}$
2. $\mathbf{HSP}(\mathbf{K}) = \mathbf{H}(\mathbf{S}(\mathbf{P}(\mathbf{K})))$ is the smallest variety containing \mathbf{K}
3. \mathbf{K} is a quasi-variety iff it is closed under subalgebras, products, and ultraproducts, iff $\mathbf{SPUp}(\mathbf{K}) = \mathbf{K}$.

1.3 Subdirect products

Definition 3.

1. Let $\{A_i\}_{i \in I}$ be Ω -structures, a subdirect product of $\langle A_i \mid i \in I \rangle$ is a subalgebra B of $\prod_{i \in I} A_i$ such that for each $i \in I$, a projection map $\pi_i : B \rightarrow A_i$ is a surjection.
2. A subdirect representation of an Ω -structure is an embedding $f : A \rightarrow \prod_{i \in I} A_i$ for some I and $\{A_i\}_{i \in I}$ such that $f \circ \pi_i : A \rightarrow A_i$ is a surjection.
3. An Ω -structure A is subdirectly irreducible if for every subdirect representation $f : A \rightarrow \prod_{i \in I} A_i$ there exists a projection π_i such that $f \circ \pi_i$ is an isomorphism.
4. $\mathbf{Sir}(\mathbf{K})$ is the class of subdirectly irreducible structures belonging to \mathbf{K} .
5. A subdirect decomposition of A if there exists a subdirect representation $f : A \rightarrow \prod_{i \in I} A_i$ such that every A_i is subdirectly irreducible.

It is known that every Boolean algebra with operators has a subdirect decomposition. Moreover, that implies:

Theorem 4.

1. If \mathbf{K} is a variety, then every element of \mathbf{K} has a subdirect decomposition with some subdirect irreducible elements of \mathbf{K} .
2. If \mathbf{K} is a variety and ε is an equation, $\mathbf{Sir}(\mathbf{K}) \models \varepsilon \Leftrightarrow \mathbf{K} \models \varepsilon$.

1.4 Pseudo-elementary classes

2 Cylindric algebras

2.1 (Representable) cylindric algebras and cylindric set algebras

Let α be an ordinal. Let U^α be the set of all functions mapping α to a non-empty set U . We denote $x(i) = x_i$ for $x \in U^\alpha$ and $i < \alpha$.

Definition 4.

1. A subset of U^α is an α -ry relation on U . For $i, j < \alpha$, the i, j -diagonal D_{ij} is the set of all elements of U such that $y_i = y_j$. If $i < \alpha$ and X is an α -ry relation on U , then the i -th cylindrification $C_i X$ is the set of all elements of U that agree with some element of X on each coordinate except the i -th one. To be more precise, $C_i X = \{y \in U^\alpha \mid \exists x \in X \forall i < \alpha (i \neq j \Rightarrow y_j = x_j)\}$.
2. A cylindric set algebra of dimension α is an algebra consisting of a set S of α -ry relation on some base set U with the constants and operations $0 = \emptyset$, $1 = U^\alpha$, \cap , $-$, the diagonal elements $\{D_{ij}\}_{i, j < \alpha}$, the cylindrifications $\{C_i\}_{i < \alpha}$. A generalised cylindric set algebra of dimension α is a subdirect of cylindric algebras that have dimension α .
3. A cylindric algebra of dimension α is an algebra $\mathcal{C} = \langle \mathcal{B}, \{c_i\}_{i < \alpha}, \{d_{ij}\}_{i, j < \alpha} \rangle$ such that

- \mathcal{B} is a Boolean algebra, for each $i, j < \alpha$ c_i is an operator and $d_{ij} \in \mathcal{B}$
- For each $i < \alpha$, $a \leq c_i a$, $c_i(a \wedge c_i b) = c_i a \wedge c_i b$ and $d_{ii} = 1$
- For every $i, j < \alpha$, $c_i c_j a = c_j c_i a$
- If $k \neq i, j < \alpha$, then $d_{ij} = c_k(d_{ij} \wedge d_{jk})$
- If $i \neq j$, then $c_i(d_{ij} \wedge a) \wedge c_i(d_{ij} \wedge -a) = 0$

\mathbf{CA}_α is the class of all cylindric algebras of dimension α

4. An α -dimensional cylindric algebra C is representable, if it is isomorphic to a generalised cylindric set algebra of dimension α . Such isomorphism is a representation of C . \mathbf{RCA}_α is the class of all representable cylindric algebras that have dimension α .

2.2 Substitution in cylindric algebras

Definition 5. Given a cylindric algebra of dimension α C , let x be a term of its signature, the substitution operator s_j^i have the following definition:

$$s_j^i x = \begin{cases} x, & \text{if } i = j \\ c_i(d_{ij} \wedge x), & \text{otherwise} \end{cases}$$

Proposition 1. Let α be an ordinal and let $i, j, k, l < \alpha$. The following facts hold in \mathbf{CA}_α

1. $s_j^i x \leq c_i x$.
2. $s_j^i(x \wedge y) = s_j^i x \wedge s_j^i y$, $s_j^i(x \vee y) = s_j^i x \vee s_j^i y$, $-s_j^i x = s_j^i(-x)$. Moreover, s_j^i is completely additive.
3. $i \neq k, l$ implies $s_j^i d_{ik} = d_{jk}$ and $s_j^i d_{kl} = d_{kl}$.
4. $d_{jk} \wedge s_j^i = d_{jk} \wedge s_k^i$.
5. $s_j^i c_i x = c_i x$.
6. $k \neq i, j$ implies $s_j^i c_i x = c_i s_j^i x$.
7. $c_j s_j^i x = c_i s_i^j x$.
8. $i \neq j$ implies $c_i s_j^i x = s_j^i x$.
9. $i \neq k$ implies $s_j^i s_k^i = s_k^i$.
10. If either $i \notin \{k, l\}$ and $k \notin \{i, j\}$, or $j = l$, then $s_j^i s_l^k x = s_l^k s_j^i x$.
11. $s_j^i s_i^j x = s_j^i x$.
12. $s_k^i s_i^j x = s_k^i s_k^j x = s_k^j s_j^i x$

3 IG_ω class

4 Cylindric modal logic

5 IG_ω and ultraproducts

6 IG_ω is (not) a variety; is (not) (pseudo-)elementary

References

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