

Notes on filtration of logics containing **K5**

Daniel Rogozin

Let $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$ be a Kripke model and Γ a set of formulas closed under subformulas. An equivalence relation \sim is set to have a finite index if the quotient set W/\sim is finite. The equivalence relation \sim_Γ induced by Γ is defined as

$$w \sim_\Gamma v \Leftrightarrow \forall \varphi \in \Gamma (\mathcal{M}, w \models \varphi \Leftrightarrow \mathcal{M}, v \models \varphi).$$

If Γ is finite, then \sim_Γ has a finite index. An equivalence relation \sim respects \sim_Γ , if $w \sim v$ implies $w \sim_\Gamma v$.

Definition 1. Let $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$ be a Kripke model and Γ be a Sub-closed set formulas. A Γ -filtration of \mathcal{M} is a model $\widehat{\mathcal{M}} = \langle \widehat{W}, \widehat{R}_1, \dots, \widehat{R}_n, \widehat{\vartheta} \rangle$ such that:

1. $\widehat{W} = W/\sim$, where \sim is an equivalence relation having a finite index that respects Γ
2. $\widehat{\vartheta}(p) = \{[x]_\sim \mid x \in W \text{ \& } x \in \vartheta(p)\}$
3. For each $i \in I$ one has $\widehat{R}_i^{\min} \subseteq \widehat{R}_i \subseteq \widehat{R}_i^{\max}$. $\widehat{R}_{i,\sim}^{\min}$ is the i -th minimal filtered relation on \widehat{W} defined as

$$\widehat{x} \widehat{R}_{i,\sim}^{\min} \widehat{y} \Leftrightarrow \exists x' \sim x \exists y' \sim y x R_i y$$

$\widehat{R}_{\Gamma,i}^{\max}$ is the i -th maximal filtered relation on \widehat{W} induced by Γ defined as

$$\widehat{x} \widehat{R}_{\Gamma,i}^{\max} \widehat{y} \Leftrightarrow \forall \Box_i \varphi \in \Gamma (\mathcal{M}, x \models \Box_i \varphi \Rightarrow \mathcal{M}, y \models \varphi)$$

If Φ is finite subset of Γ and $\sim = \sim_\Phi$, then $\widehat{\mathcal{M}}$ is a definable Γ -filtration of \mathcal{M} through Φ . If $\sim = \sim_\Gamma$, then such a filtration by means of the definition above is called *strict*. A class of models \mathbb{M} admits strict filtrations for models (ASF), if for every Sub-closed set Γ and for every $\mathcal{M} \in \mathbb{M}$ there exists a Γ filtration of \mathcal{M} . A class of frames \mathbb{F} admits strict filtrations for frames, if for every Sub-closed set Γ and for every frame $\mathcal{F} \in \mathbb{F}$ and every model \mathcal{M} over \mathcal{F} there exists a Γ filtration of \mathcal{M} . If \mathcal{L} is canonical, then the ASF property for frames and ASF property for models are equivalent [1, Theorem 2.10].

Lemma 1. Let Γ be a finite set of formulas closed under subformulas and $\widehat{\mathcal{M}}$ a filtration of \mathcal{M} through Γ , then for each $x \in W$ and for each $\varphi \in \Gamma$ one has

$$\mathcal{M}, x \models \varphi \Leftrightarrow \widehat{\mathcal{M}}, \widehat{x} \models \varphi$$

Definition 2. Let \mathbb{F} be a class of Kripke frames and Γ a finite set of formulas closed under subformulas. If for every model \mathcal{M} over $\mathcal{F} \in \mathbb{F}$ there exists a model that is a Γ -definable filtration of \mathcal{M} , then \mathbb{F} admits definable filtration. A class of models \mathbb{M} admits definable filtration if for every $\mathcal{M} \in \mathbb{M}$ there exists a model belonging to the same class that is a definable Γ -filtration of \mathcal{M} .

Lemma 2.

1. Let \mathcal{L} be a complete normal modal logic. If $\text{Frames}(\mathcal{L})$ admits filtration, then \mathcal{L} has the finite model property.
2. If the class of models $\text{Mod}(\mathcal{L})$ admits filtration, then \mathcal{L} has the finite model property and it is Kripke complete as well.

Definition 3. A first-order formula is called *Horn* if it has the following form:

$$\forall x_1, \dots, x_n (x_{i_1} R x_{j_1} \wedge \dots \wedge x_{i_s} R x_{j_s} \rightarrow x_k R x_l)$$

Definition 4. Let H be a Horn property and $\langle W, R \rangle$ a Kripke frame. A Horn closure of a binary relation R is the minimal relation R^H containing R and satisfying H .

Lemma 3. $R^H = \bigcup_{n < \omega} R_n$ where

1. $R_0 = R$.
2. $R_{n+1} = R_n \cup \{(a, b) \in W \mid \exists \vec{c} \in W \text{ } P(a, b, \vec{c})\}$, where P is a premise of H .

E -closure (an Euclidean Horn closure of a binary relation) has the following equivalent definitions:

Lemma 4. Let $\mathcal{F} = \langle W, R \rangle$ be a Kripke frame. The following conditions are equivalent:

1. R^E is the smallest Euclidean relation containing R .
2. $R^E = \bigcup_{i < \omega} R_i$, where
 - $R_0 = R$
 - $R_{n+1} = R_n \cup (R_n^{-1} \circ R_n)$
3. $x R^E y$ iff there exists $n < \omega$ such that either $x R y$ or $\exists z_1, \dots, z_n$ with $z_1 R x$ and $z_{n-1} R y$ and for each $1 < i \leq n$ one has either $z_{i-1} R z_i$ or $z_i R z_{i-1}$.
4. $R^E = R \cup \bigcup_{i < \omega} (R^{-1} \circ (R \circ R^{-1})^n \circ R)$.

Proof.

1. (1) \Rightarrow (2) Let us show that if R^E is the smallest Euclidean relation containing R , then $R^E = \bigcup_{i < \omega} R_i$. There are two inclusions:
 - $R^E \subseteq \bigcup_{i < \omega} R_i$. Recall that R^E has the form (?):
$$R^E = \bigcap \{R' \mid R \subseteq R', \forall a, b \in W \text{ } R'(a, b) \Rightarrow \exists x \in W \text{ } R'(x, a) \& R'(x, b)\}$$
 - $\bigcup_{i < \omega} R_i \subseteq R^E$. Let us show that $x R_n y$ for each $n < \omega$ implies $x R^E y$ by induction on n . If $n = 0$, then $x R y$, thus, $x R^E y$, since R is a subrelation of R^E . Suppose $n = m + 1$ and $x R_{m+1} y$. Let us show that $x R^E y$. From $x R_{m+1} y$, one has $(x, y) \in R^n \cup (R_n^{-1} \circ R_n)$. There are two cases:
 - $x R^n y$, one needs to merely apply the IH.

– $xR_n^{-1} \circ R_n y$. Then $\exists z \in W$ $xR_n^{-1} z$ & $zR_n y$. That is, $zR_n x$ and $zR_n y$ for some z . R_n is already a subrelation of R^E . Thus, $zR^E x$ and $zR^E y$. That implies $xR^E y$.

2. (2) \Rightarrow (3) Let $(x, y) \in R_m$, let us the statement by induction on m .

(a) Suppose $m = 0$, then xRy , and the statement is shown putting $n = 0$.

(b) Suppose $m = p + 1$ and $xR_{p+1}y$. Assume that either xRy or $\exists z_1, \dots, z_p$ with z_1Rx and $z_{p-1}Ry$ and for each $1 < i \leq p$ one has either $z_{i-1}Rz_i$ or z_iRz_{i-1} .

$xR_{p+1}y$ implies $(x, y) \in R_p \cup (R_p^{-1} \circ R_p)$. If $(x, y) \in R_p$, then we merely apply the IH. Suppose $(x, y) \in R_p^{-1} \circ R_p$, then $(z, x) \in R_p$ and $(z, y) \in R_p$

3. (3) \Rightarrow (4) Suppose either xRy or there exist $n \geq 1$ and z_1, \dots, z_n with z_1Rx and $z_{n-1}Ry$ and for each $1 < i \leq n$ one has either $z_{i-1}Rz_i$ or z_iRz_{i-1} . If xRy , then we are done. Otherwise there exists $n \geq 1$ with the condition above. Then $(x, y) \in R_{n+1}$ that follows from the condition.

4. (4) \Rightarrow (1)

□

Lemma 5. Let $\mathcal{F} = \langle W, R \rangle$ be a Kripke frame. Let us define $R^E = \bigcup_{i < \omega} R_i$ where:

1. $R_0 = R$

2. $R_{n+1} = R_n \cup (R_n^{-1} \circ R_n)$

Then R^E is Euclidean.

Proof. Let $(x, y), (x, z) \in R^E$, one needs to show that $(y, z) \in R^E$. Clearly that $(x, y) \in R_i$ and $(x, z) \in R_j$ for some $i, j < \omega$. Thus, we need $(y, z) \in R_m$ for some m depending on i and j .

Let us consider the following cases:

1. $i = 0$ and $j = 0$

Suppose $(x, y), (x, z) \in R_0 = R$, then $(y, z) \in R^{-1} \circ R$. Thus, $(y, z) \in R_1$

2. $i = 0$ and $j = k + 1$

Suppose $(x, y) \in R$ and $(x, z) \in R_{k+1} = R_k \cup (R_k^{-1} \circ R_k)$. Clearly that $(x, y) \in R_{k+1}$ as well. It is obviously that $(y, z) \in R_{k+2}$ since $(y, x) \in R_{k+1}^{-1}$ and $(x, z) \in R_{k+1}$.

3. The case with $i = k + 1$ and $j = 0$ is similar to the previous one.

4. Suppose $i = m + 1$ and $j = k + 1$. That is, $(x, y) \in R_{m+1} = R_m \cup (R_m^{-1} \circ R_m)$ and $(x, z) \in R_{k+1} = R_k \cup (R_k^{-1} \circ R_k)$. Consider the following four subcases:

(a) Suppose $(x, y) \in R_m$ and $(x, z) \in R_k$ and $m \leq k$ without loss of generality. $m \leq k$ implies $R_m \subseteq R_k$ and $(x, y) \in R_k$ in particular. Thus, $(y, z) \in R_k^{-1} \circ R_k$, so $(y, z) \in R_{k+1}$.

(b) The rest of the cases are similar to the first one.

□

Theorem 1. K45 admits strict filtrations.

Proof. Let $\mathcal{M} = \langle W, R, \vartheta \rangle$ be a transitive Euclidean model and $\overline{\mathcal{M}} = \langle \overline{W}, \overline{R}, \overline{\vartheta} \rangle$ its minimal filtration through Γ , where Γ is finite and Sub-closed. Let us put $\hat{R} = \overline{R}^+ \cup \overline{R}^E$. Let us show that $\overline{R}^+ \cup \overline{R}^E \subseteq \overline{R}^{max}$.

That is, if $\mathcal{M}, y \models \varphi$ for $\diamond\varphi \in \Gamma$ and $\hat{x}\hat{R}\hat{y}$, then $\mathcal{M}, x \models \diamond\varphi$.

Let $\hat{x}\hat{R}\hat{y}$. Let us consider the case when $(\hat{x}, \hat{y}) \in \overline{R}^E$

1. Suppose $(\hat{x}, \hat{y}) \in \overline{R}$, then $\mathcal{M}, x \models \diamond\varphi$ holds trivially by the definition of the minimal filtration.
2. Suppose the statement holds \overline{R}_n and $(\hat{x}, \hat{y}) \in \overline{R}_{n+1} = \overline{R}_n \cup (\overline{R}_n^{-1} \circ \overline{R}_n)$. We consider the case of $(\hat{x}, \hat{y}) \in (\overline{R}_n^{-1} \circ \overline{R}_n)$.

Then there exists \hat{z} such that $(\hat{z}, \hat{x}), (\hat{z}, \hat{y}) \in \overline{R}_n$.

By IH, $\mathcal{M}, z \models \diamond\varphi$.

$(\hat{z}, \hat{y}) \in \overline{R}_n$ iff there are $\hat{u}_1, \dots, \hat{u}_n$ such that

$$\hat{z} \xleftarrow{\hat{R}} \hat{u}_1 \xrightarrow{\hat{R}'} \hat{u}_2 \xrightarrow{\hat{R}'} \dots \xrightarrow{\hat{R}'} \hat{u}_{n-1} \xrightarrow{\hat{R}'} \hat{u}_n \xrightarrow{\hat{R}} \hat{y}$$

where \hat{R}' is either \hat{R} or \hat{R}^{-1} .

As it is known, $\diamond\diamond\varphi \rightarrow \square\diamond\varphi \in \mathbf{K45}$.

$\hat{u}_1\hat{z}$, that is, $u'_1 R z'$ for some $u'_1 \in \hat{u}_1$ and $z' \in \hat{z}$. That is, $\mathcal{M}, u'_1 \models \diamond\diamond\varphi$, so $\mathcal{M}, u'_1 \models \diamond\varphi$ and $\overline{\mathcal{M}}, \hat{u}_1 \models \diamond\varphi$.

We have $\hat{u}_1\hat{R}'\hat{u}_2$. Suppose $\mathcal{M}, u''_1 \models \diamond\varphi$ and $u''_1 R u'_2$. We also have $\mathcal{M}, u''_1 \models \square\diamond\varphi$, thus, $\mathcal{M}, u'_2 \models \diamond\varphi$.

Suppose $\hat{u}_2\hat{R}\hat{u}_1$ and $u'_2 R u''_1$, then $\mathcal{M}, u'_2 \models \diamond\varphi$.

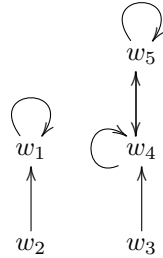
Similarly, we have $\mathcal{M}, u_i \models \diamond\varphi$ iff $\mathcal{M}, u_{i+1} \models \diamond\varphi$, whenever $\hat{u}_i\hat{R}'\hat{u}_{i+1}$.

Finally, we have $\hat{u}_n\hat{R}\hat{x}$. Thus, $u'_n R x'$ for some $u'_n \in \hat{u}_n$ and $x' \in \hat{x}$. $\mathcal{M}, u'_n \models \diamond\varphi$, so $\mathcal{M}, u'_n \models \square\diamond\varphi$. Then $\mathcal{M}, x' \models \diamond\varphi$. \square

Theorem 2. **K5** does not admit strict filtrations.

Proof. Let us consider a **K5** model whose Euclidean closure of the minimal filtration does not give us a filtration.

Let us consider a frame called \mathcal{F}_{bad} . We define this frame with the following graph:



\square

Let us define a valuation ϑ such that $\vartheta(p) = \{w_5\}$ and $\vartheta(q) = \{w_1\}$. Let us consider a minimal filtration of \mathcal{M}_{bad} through the Sub-closure of $\Gamma = \{\neg p, \neg\diamond p\}$.

Clearly that $w_2 \sim_\Gamma w_3$, since $\neg p$ and $\neg\diamond p$ are true both at w_2 and w_3 .

Moreover, $R_{min} \cup (R_{min}^{-1} \circ R_{min})$ is not a subset of R_{max} since $(\hat{w}_1, \hat{w}_5) \in (R_{min}^{-1} \circ R_{min})$, but $\Diamond p$ is not true at w_5 .

Let us also note that strict filtrations of this model is not Euclidean. Suppose by contrary that \hat{R}^E is a strict filtration of that model. So $R_{min}^E \subseteq \hat{R}^E$, since R_{min}^E is the minimal Euclidean relation containing R_{min} . On the other hand, $R_{min}^E \not\subseteq R_{max}$, so is not \hat{R}^E .

1 Finite “canonical” models

Let \mathcal{L} be a normal modal logic, $\mathcal{M}_{\mathcal{L}}$ its canonical model, and Γ a finite Sub-closed set of formulas. Let us put $\Gamma' = \text{Sub}(\varphi) \cup \{\neg\psi \mid \psi \in \text{Sub}(\varphi)\}$.

A subset $\Delta \subseteq \Gamma'$ is a *finite \mathcal{L} -consistent set* if $\neg \bigwedge \Delta \notin \mathcal{L}$. A subset Δ is maximal, if (the following are obviously equivalent):

1. Δ is maximal amongst finite \mathcal{L} -consistent sets,
2. For each $\psi \in \text{Sub}(\varphi)$ either $\psi \in \Delta$ or $\neg\psi \in \Delta$.

Every finite \mathcal{L} -theory is clearly can be extended to some maximal one. It is the finite version of Lindenbaum’s lemma.

Definition 5. Let \mathcal{L} be a modal logic and Γ be a finite Sub-closed set of formulas. A finite “canonical” model is a triple $\mathcal{M}_{\mathcal{L}}^{\Gamma} = \langle W_{\mathcal{L}}^{\Gamma}, R_{\mathcal{L}}^{\Gamma}, \vartheta_{\mathcal{L}}^{\Gamma} \rangle$, where

1. $W_{\mathcal{L}}^{\Gamma}$ is the set all maximal theories that extend finite \mathcal{L} -theories
2. $R_{\mathcal{L}}^{\Gamma}$ is a relation such that $\langle W_{\mathcal{L}}^{\Gamma}, R_{\mathcal{L}}^{\Gamma} \rangle$ is an \mathcal{L} -frame and

$$\forall \Box \psi \in \text{Sub}(\varphi) \quad \forall \Delta_1 \in W_{\mathcal{L}}^{\Gamma} \quad (\Box \psi \in \Delta_1 \Leftrightarrow \forall \Delta_2 \in R_{\mathcal{L}}^{\Gamma}(\Delta_1) \quad \psi \in \Delta_2)$$

3. $\vartheta_{\mathcal{L}}^{\Gamma}(p) = \{\Delta \in W_{\mathcal{L}}^{\Gamma} \mid p \in \Delta\}$ for every variable $p \in \Gamma$.

Lemma 6. Let \mathcal{L} be a modal logic and $\varphi \notin \mathcal{L}$, then $\mathcal{M}_{\mathcal{L}}^{\text{Sub}(\varphi)} \not\models \varphi$.

Lemma 7. Let \mathcal{L} be a modal logic and Γ a finite Sub-closed set of formulas, then if \mathcal{L} admits strict filtrations, then there exists a finite “canonical” model $\mathcal{M}_{\mathcal{L}}^{\Gamma}$ such that $\mathcal{M}_{\mathcal{L}}^{\Gamma} \models \mathcal{L}$.

Proof. (\Rightarrow) Let Γ be a finite Sub-closed of formulas. \mathcal{L} admits strict filtrations, so the filtration of the canonical model $\mathcal{M}_{\mathcal{L}}$ through Γ is also an \mathcal{L} -model. The underlying set of $\mathcal{M}_{\mathcal{L}} / \sim_{\Gamma}$ consists of maximal \mathcal{L} theories up to Γ -equivalence and this quotient set is finite.

It is readily checked that the quotient model $\mathcal{M}_{\mathcal{L}} / \sim_{\Gamma}$ satisfies Definition 5. \square

The converse implication does not have to true generally. We prove a bit more weak version of this implication. A logic admits \mathcal{L} admits \Diamond -filtrations, if for each model $\mathcal{M} \models \mathcal{L}$ and for each Γ , a Sub-closed set of formulas, there exists a model $\widehat{\mathcal{M}}$, a filtration of \mathcal{L} through $\text{Sub}(\Gamma \cup \Diamond \Gamma)$.

K5 is an example of such a logic since **K5** admits filtrations through sets having the form $\Gamma \cup \{\Diamond \Box \psi \mid \Box \psi \in \Gamma\}$.

Note that, if \mathcal{L} admits \Diamond -filtrations, so \mathcal{L} admits the “finite canonical” model property, the proof is the same as in Lemma 6.

Lemma 8. Let \mathcal{L} be a logic that admits \Diamond -filtrations and Γ a Sub-closed set of formulas, then there exists a “finite canonical” model $\mathcal{M}_{\mathcal{L}}^{\Gamma} \models \mathcal{L}$

Proof. Let Γ be a finite Sub-closed of formulas and $\mathcal{M}_{\mathcal{L}}^{\Gamma} = \langle W_{\mathcal{L}}^{\Gamma}, R_{\mathcal{L}}^{\Gamma}, \vartheta_{\mathcal{L}}^{\Gamma} \rangle$ a “finite canonical” model generated by Γ .

Let us show that there exists a model $\widehat{\mathcal{M}}$, a strict filtration of \mathcal{M} through $\text{Sub}(\Gamma \cup \Diamond \Gamma) := \Psi$.
Let \sim be an equivalence relation on \mathcal{M} defined as usual:

$$u \sim v \text{ iff } \forall \psi \in \Psi \ \mathcal{M}, u \models \psi \Leftrightarrow \mathcal{M}, v \models \psi$$

Let $[u] \in \mathcal{M}/\sim$, consider its truth set $||[u]|| = \{\psi \in \Psi \mid \mathcal{M}, u \models \psi\}$. $||[u]||$ is clearly a finite \mathcal{L} -consistent set and it is maximal. Consider a map:

$$\iota : W/\sim \rightarrow W_{\mathcal{L}}^{\Gamma} \text{ with } \iota : [u] \mapsto ||[u]||$$

Consider the set $\iota(W/\sim)$ that we denote as $\text{Im}(\iota)$. Let us show that $\text{Im}(\iota)$ is $R_{\mathcal{L}}^{\Gamma}$ -closed.

Let $\iota([u])R_{\mathcal{L}}^{\Gamma}\Delta$, let us show that $\Delta \in \iota(W/\sim)$, that is, $\Delta = \iota([v])$ for some $v \in W$.

Note that $\bigwedge_{\psi \in \Delta} \psi \in \Delta$ since Δ is finite. This conjunction has the form $\bigwedge_{\psi \in \Delta} \psi \in \Delta = \psi_1 \wedge \dots \wedge \psi_n$

for some $n < \omega$ and for some $\psi_i \in \Gamma \cup \neg\Gamma$, where $0 < i < \omega$.

Therefore $\mathcal{M}_{\mathcal{L}}^{\Gamma}, \iota([u]) \models \Diamond(\psi_1 \wedge \dots \wedge \psi_n)$. Then $\mathcal{M}, u \models \Diamond(\psi_1 \wedge \dots \wedge \psi_n)$. That implies $\mathcal{M}, v \models \psi_1 \wedge \dots \wedge \psi_n$ for some $v \in R(u)$. This is clearly a desired v since $||[v]|| \subseteq \Delta$ and such a Δ is unique.

Thus, $\langle \text{Im}(\iota), R_{\mathcal{L}}^{\Gamma}, \vartheta_{\mathcal{L}}^{\Gamma} \rangle$, where $\vartheta_{\mathcal{L}}^{\Gamma}(p) = \vartheta_{\mathcal{L}}^{\Gamma}(p) \cap \text{Im}(\iota)$ is a generated submodel of $\mathcal{M}_{\mathcal{L}}^{\Gamma}$.

Consider a model $\widehat{\mathcal{M}} = \langle W/\sim, \widehat{R}, \widehat{\vartheta} \rangle$, where \widehat{R} is defined as

$$[u]\widehat{R}[v] \text{ iff } ||[u]||R_{\mathcal{L}}^{\Gamma}||[v]||$$

Let us show that $R_{\min} \subseteq \widehat{R} \subseteq R_{\max}$.

Suppose that there are $u' \in [u]$ and $v' \in [v]$ with $u'Rv'$. Let us show that $[u]\widehat{R}[v]$. That is, let us show that $\Diamond \iota(v) \subseteq \iota(u)$.

Suppose $\psi \in \iota(v)$, that is, $\psi \in ||v||$, that denotes $\psi \in \Psi$ and $\mathcal{M}, v \models \psi$. Therefore, $\mathcal{M}, v' \models \psi$. Then $\mathcal{M}, u' \models \Diamond \psi$. $\Diamond \psi \in \Psi$ and $u \sim_{\Gamma} u'$, so $\mathcal{M}, u \models \Diamond \psi$.

The second inclusion follows from the definition of a relation in a finite “canonical” model. \square

In particular, **K5** has the finite “canonical” model property.

References

- [1] Stanislav Kikot, Ilya Shapirovsky, and Evgeny Zolin. Completeness of logics with the transitive closure modality and related logics. *arXiv preprint arXiv:2011.02205*, 2020.