

# Notes on filtration of logics containing **K5**

Daniel Rogozin

Let  $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$  be a Kripke model and  $\Gamma$  a set of formulas closed under subformulas. An equivalence relation  $\sim$  is set to have a finite index if the quotient set  $W/\sim$  is finite. The equivalence relation  $\sim_\Gamma$  induced by  $\Gamma$  is defined as

$$w \sim_\Gamma v \Leftrightarrow \forall \varphi \in \Gamma (\mathcal{M}, w \models \varphi \Leftrightarrow \mathcal{M}, v \models \varphi).$$

If  $\Gamma$  is finite, then  $\sim_\Gamma$  has a finite index. An equivalence relation  $\sim$  respects  $\sim_\Gamma$ , if  $w \sim v$  implies  $w \sim_\Gamma v$ .

**Definition 1.** Let  $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$  be a Kripke model and  $\Gamma$  be a Sub-closed set formulas. A  $\Gamma$ -filtration of  $\mathcal{M}$  is a model  $\widehat{\mathcal{M}} = \langle \widehat{W}, \widehat{R}_1, \dots, \widehat{R}_n, \widehat{\vartheta} \rangle$  such that:

1.  $\widehat{W} = W/\sim$ , where  $\sim$  is an equivalence relation having a finite index that respects  $\Gamma$
2.  $\widehat{\vartheta}(p) = \{[x]_\sim \mid x \in W \text{ \& } x \in \vartheta(p)\}$
3. For each  $i \in I$  one has  $\widehat{R}_i^{\min} \subseteq \widehat{R}_i \subseteq \widehat{R}_i^{\max}$ .  $\widehat{R}_{i,\sim}^{\min}$  is the  $i$ -th minimal filtered relation on  $\widehat{W}$  defined as

$$\hat{x} \widehat{R}_{i,\sim}^{\min} \hat{y} \Leftrightarrow \exists x' \sim x \exists y' \sim y x R_i y$$

$\widehat{R}_{\Gamma,i}^{\max}$  is the  $i$ -th maximal filtered relation on  $\widehat{W}$  induced by  $\Gamma$  defined as

$$\hat{x} \widehat{R}_{\Gamma,i}^{\max} \hat{y} \Leftrightarrow \forall \Box_i \varphi \in \Gamma (\mathcal{M}, x \models \Box_i \varphi \Rightarrow \mathcal{M}, y \models \varphi)$$

If  $\Phi$  is finite subset of  $\Gamma$  and  $\sim = \sim_\Phi$ , then  $\widehat{\mathcal{M}}$  is a definable  $\Gamma$ -filtration of  $\mathcal{M}$  through  $\Phi$ . If  $\sim = \sim_\Gamma$ , then such a filtration by means of the definition above is called *strict*. A class of models  $\mathbb{M}$  admits strict filtrations for models (ASF), if for every Sub-closed set  $\Gamma$  and for every  $\mathcal{M} \in \mathbb{M}$  there exists a  $\Gamma$  filtration of  $\mathcal{M}$ . A class of frames  $\mathbb{F}$  admits strict filtrations for frames, if for every Sub-closed set  $\Gamma$  and for every frame  $\mathcal{F} \in \mathbb{F}$  and every model  $\mathcal{M}$  over  $\mathcal{F}$  there exists a  $\Gamma$  filtration of  $\mathcal{M}$ . If  $\mathcal{L}$  is canonical, then the ASF property for frames and ASF property for models are equivalent [1, Theorem 2.10].

**Lemma 1.** Let  $\Gamma$  be a finite set of formulas closed under subformulas and  $\widehat{\mathcal{M}}$  a filtration of  $\mathcal{M}$  through  $\Gamma$ , then for each  $x \in W$  and for each  $\varphi \in \Gamma$  one has

$$\mathcal{M}, x \models \varphi \Leftrightarrow \widehat{\mathcal{M}}, \hat{x} \models \varphi$$

**Definition 2.** Let  $\mathbb{F}$  be a class of Kripke frames and  $\Gamma$  a finite set of formulas closed under subformulas. If for every model  $\mathcal{M}$  over  $\mathcal{F} \in \mathbb{F}$  there exists a model that is a  $\Gamma$ -definable filtration of  $\mathcal{M}$ , then  $\mathbb{F}$  admits definable filtration. A class of models  $\mathbb{M}$  admits definable filtration if for every  $\mathcal{M} \in \mathbb{M}$  there exists a model belonging to the same class that is a definable  $\Gamma$ -filtration of  $\mathcal{M}$ .

**Lemma 2.**

1. Let  $\mathcal{L}$  be a complete normal modal logic. If  $\text{Frames}(\mathcal{L})$  admits filtration, then  $\mathcal{L}$  has the finite model property.
2. If the class of models  $\text{Mod}(\mathcal{L})$  admits filtration, then  $\mathcal{L}$  has the finite model property and it is Kripke complete as well.

**Definition 3.** A first-order formula is called *Horn* if it has the following form:

$$\forall x_1, \dots, x_n (x_{i_1} R x_{j_1} \wedge \dots \wedge x_{i_s} R x_{j_s} \rightarrow x_k R x_l)$$

**Definition 4.** Let  $H$  be a Horn property and  $\langle W, R \rangle$  a Kripke frame. A Horn closure of a binary relation  $R$  is the minimal relation  $R^H$  containing  $R$  and satisfying  $H$ .

**Lemma 3.**  $R^H = \bigcup_{n < \omega} R_n$  where

1.  $R_0 = R$ .
2.  $R_{n+1} = R_n \cup \{(a, b) \in W \mid \exists \vec{c} \in W \text{ } P(a, b, \vec{c})\}$ , where  $P$  is a premise of  $H$ .

$E$ -closure (an Euclidean Horn closure of a binary relation) has the following equivalent definitions:

**Lemma 4.** Let  $\mathcal{F} = \langle W, R \rangle$  be a Kripke frame. The following conditions are equivalent:

1.  $R^E$  is the smallest Euclidean relation containing  $R$ .
2.  $R^E = \bigcup_{i < \omega} R_i$ , where
  - $R_0 = R$
  - $R_{n+1} = R_n \cup (R_n^{-1} \circ R_n)$
3.  $xR^E y$  iff there exists  $n < \omega$  such that either  $xRy$  or  $\exists z_1, \dots, z_n$  with  $z_1 Rx$  and  $z_{n-1} Ry$  and for each  $1 < i \leq n$  one has either  $z_{i-1} R z_i$  or  $z_i R z_{i-1}$ .
4.  $R^E = R \cup \bigcup_{i < \omega} (R^{-1} \circ (R \circ R^{-1})^n \circ R)$ .

*Proof.*

1. (1)  $\Rightarrow$  (2) Let us show that if  $R^E$  is the smallest Euclidean relation containing  $R$ , then  $R^E = \bigcup_{i < \omega} R_i$ . There are two inclusions:
  - $R^E \subseteq \bigcup_{i < \omega} R_i$ . Recall that  $R^E$  has the form (?):
$$R^E = \bigcap \{R' \mid R \subseteq R', \forall a, b \in W \text{ } R'(a, b) \Rightarrow \exists x \in W \text{ } R'(x, a) \& R'(x, b)\}$$
  - $\bigcup_{i < \omega} R_i \subseteq R^E$ . Let us show that  $xR_n y$  for each  $n < \omega$  implies  $xR^E y$  by induction on  $n$ . If  $n = 0$ , then  $xRy$ , thus,  $xR^E y$ , since  $R$  is a subrelation of  $R^E$ . Suppose  $n = m+1$  and  $xR_{m+1} y$ . Let us show that  $xR^E y$ . From  $xR_{m+1} y$ , one has  $(x, y) \in R^n \cup (R_n^{-1} \circ R_n)$ . There are two cases:
    - $xR^n y$ , one needs to merely apply the IH.

–  $xR_n^{-1} \circ R_n y$ . Then  $\exists z \in W$   $xR_n^{-1} z$  &  $zR_n y$ . That is,  $zR_n x$  and  $zR_n y$  for some  $z$ .  $R_n$  is already a subrelation of  $R^E$ . Thus,  $zR^E x$  and  $zR^E y$ . That implies  $xR^E y$ .

2. (2)  $\Rightarrow$  (3) Let  $(x, y) \in R_m$ , let us the statement by induction on  $m$ .

(a) Suppose  $m = 0$ , then  $xRy$ , and the statement is shown putting  $n = 0$ .

(b) Suppose  $m = p + 1$  and  $xR_{p+1}y$ . Assume that either  $xRy$  or  $\exists z_1, \dots, z_p$  with  $z_1Rx$  and  $z_{p-1}Ry$  and for each  $1 < i \leq p$  one has either  $z_{i-1}Rz_i$  or  $z_iRz_{i-1}$ .

$xR_{p+1}y$  implies  $(x, y) \in R_p \cup (R_p^{-1} \circ R_p)$ . If  $(x, y) \in R_p$ , then we merely apply the IH. Suppose  $(x, y) \in R_p^{-1} \circ R_p$ , then  $(z, x) \in R_p$  and  $(z, y) \in R_p$

3. (3)  $\Rightarrow$  (4) Suppose either  $xRy$  or there exist  $n \geq 1$  and  $z_1, \dots, z_n$  with  $z_1Rx$  and  $z_{n-1}Ry$  and for each  $1 < i \leq n$  one has either  $z_{i-1}Rz_i$  or  $z_iRz_{i-1}$ . If  $xRy$ , then we are done. Otherwise there exists  $n \geq 1$  with the condition above. Then  $(x, y) \in R_{n+1}$  that follows from the condition.

4. (4)  $\Rightarrow$  (1)

□

**Lemma 5.** Let  $\mathcal{F} = \langle W, R \rangle$  be a Kripke frame. Let us define  $R^E = \bigcup_{i < \omega} R_i$  where:

1.  $R_0 = R$

2.  $R_{n+1} = R_n \cup (R_n^{-1} \circ R_n)$

Then  $R^E$  is Euclidean.

*Proof.* Let  $(x, y), (x, z) \in R^E$ , one needs to show that  $(y, z) \in R^E$ . Clearly that  $(x, y) \in R_i$  and  $(x, z) \in R_j$  for some  $i, j < \omega$ . Thus, we need  $(y, z) \in R_m$  for some  $m$  depending on  $i$  and  $j$ .

Let us consider the following cases:

1.  $i = 0$  and  $j = 0$

Suppose  $(x, y), (x, z) \in R_0 = R$ , then  $(y, z) \in R^{-1} \circ R$ . Thus,  $(y, z) \in R_1$

2.  $i = 0$  and  $j = k + 1$

Suppose  $(x, y) \in R$  and  $(x, z) \in R_{k+1} = R_k \cup (R_k^{-1} \circ R_k)$ . Clearly that  $(x, y) \in R_{k+1}$  as well. It is obviously that  $(y, z) \in R_{k+2}$  since  $(y, x) \in R_{k+1}^{-1}$  and  $(x, z) \in R_{k+1}$ .

3. The case with  $i = k + 1$  and  $j = 0$  is similar to the previous one.

4. Suppose  $i = m + 1$  and  $j = k + 1$ . That is,  $(x, y) \in R_{m+1} = R_m \cup (R_m^{-1} \circ R_m)$  and  $(x, z) \in R_{k+1} = R_k \cup (R_k^{-1} \circ R_k)$ . Consider the following four subcases:

(a) Suppose  $(x, y) \in R_m$  and  $(x, z) \in R_k$  and  $m \leq k$  without loss of generality.  $m \leq k$  implies  $R_m \subseteq R_k$  and  $(x, y) \in R_k$  in particular. Thus,  $(y, z) \in R_k^{-1} \circ R_k$ , so  $(y, z) \in R_{k+1}$ .

(b) The rest of the cases are similar to the first one.

□

**Theorem 1.** K45 admits strict filtrations.

*Proof.* Let  $\mathcal{M} = \langle W, R, \vartheta \rangle$  be a transitive Euclidean model and  $\overline{\mathcal{M}} = \langle \overline{W}, \overline{R}, \overline{\vartheta} \rangle$  its minimal filtration through  $\Gamma$ , where  $\Gamma$  is finite and Sub-closed. Let us put  $\hat{R} = \overline{R}^+ \cup \overline{R}^E$ . Let us show that  $\overline{R}^+ \cup \overline{R}^E \subseteq \overline{R}^{max}$ .

That is, if  $\mathcal{M}, y \models \varphi$  for  $\diamond\varphi \in \Gamma$  and  $\hat{x}\hat{R}\hat{y}$ , then  $\mathcal{M}, x \models \diamond\varphi$ .

Let  $\hat{x}\hat{R}\hat{y}$ . Let us consider the case when  $(\hat{x}, \hat{y}) \in \overline{R}^E$

1. Suppose  $(\hat{x}, \hat{y}) \in \overline{R}$ , then  $\mathcal{M}, x \models \diamond\varphi$  holds trivially by the definition of the minimal filtration.
2. Suppose the statement holds  $\overline{R}_n$  and  $(\hat{x}, \hat{y}) \in \overline{R}_{n+1} = \overline{R}_n \cup (\overline{R}_n^{-1} \circ \overline{R}_n)$ . We consider the case of  $(\hat{x}, \hat{y}) \in (\overline{R}_n^{-1} \circ \overline{R}_n)$ .

Then there exists  $\hat{z}$  such that  $(\hat{z}, \hat{x}), (\hat{z}, \hat{y}) \in \overline{R}_n$ .

By IH,  $\mathcal{M}, z \models \diamond\varphi$ .

$(\hat{z}, \hat{y}) \in \overline{R}_n$  iff there are  $\hat{u}_1, \dots, \hat{u}_n$  such that

$$\hat{z} \xleftarrow{\hat{R}} \hat{u}_1 \xrightarrow{\hat{R}'} \hat{u}_2 \xrightarrow{\hat{R}'} \dots \xrightarrow{\hat{R}'} \hat{u}_{n-1} \xrightarrow{\hat{R}'} \hat{u}_n \xrightarrow{\hat{R}} \hat{y}$$

where  $\hat{R}'$  is either  $\hat{R}$  or  $\hat{R}^{-1}$ .

As it is known,  $\diamond\diamond\varphi \rightarrow \square\diamond\varphi \in \mathbf{K45}$ .

$\hat{u}_1\hat{z}$ , that is,  $u'_1 R z'$  for some  $u'_1 \in \hat{u}_1$  and  $z' \in \hat{z}$ . That is,  $\mathcal{M}, u'_1 \models \diamond\diamond\varphi$ , so  $\mathcal{M}, u'_1 \models \diamond\varphi$  and  $\overline{\mathcal{M}}, \hat{u}_1 \models \diamond\varphi$ .

We have  $\hat{u}_1\hat{R}'\hat{u}_2$ . Suppose  $\mathcal{M}, u''_1 \models \diamond\varphi$  and  $u''_1 R u'_2$ . We also have  $\mathcal{M}, u''_1 \models \square\diamond\varphi$ , thus,  $\mathcal{M}, u'_2 \models \diamond\varphi$ .

Suppose  $\hat{u}_2\hat{R}\hat{u}_1$  and  $u'_2 R u''_1$ , then  $\mathcal{M}, u'_2 \models \diamond\varphi$ .

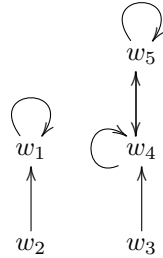
Similarly, we have  $\mathcal{M}, u_i \models \diamond\varphi$  iff  $\mathcal{M}, u_{i+1} \models \diamond\varphi$ , whenever  $\hat{u}_i\hat{R}'\hat{u}_{i+1}$ .

Finally, we have  $\hat{u}_n\hat{R}\hat{x}$ . Thus,  $u'_n R x'$  for some  $u'_n \in \hat{u}_n$  and  $x' \in \hat{x}$ .  $\mathcal{M}, u'_n \models \diamond\varphi$ , so  $\mathcal{M}, u'_n \models \square\diamond\varphi$ . Then  $\mathcal{M}, x' \models \diamond\varphi$ .  $\square$

**Theorem 2.** **K5** does not admit strict filtrations.

*Proof.* Let us consider a **K5** model whose Euclidean closure of the minimal filtration does not give us a filtration.

Let us consider a frame called  $\mathcal{F}_{bad}$ . We define this frame with the following graph:



Let us define a valuation  $\vartheta$  such that  $\vartheta(p) = \{w_5\}$  and  $\vartheta(q) = \{w_1\}$ . Let us consider a minimal filtration of  $\mathcal{M}_{bad}$  through the Sub-closure of  $\Gamma = \{\neg p, \neg\diamond p\}$ .

Clearly that  $w_2 \sim_\Gamma w_3$ , since  $\neg p$  and  $\neg\diamond p$  are true both at  $w_2$  and  $w_3$ .

Moreover,  $R_{min} \cup (R_{min}^{-1} \circ R_{min})$  is not a subset of  $R_{max}$  since  $(\hat{w}_1, \hat{w}_5) \in (R_{min}^{-1} \circ R_{min})$ , but  $\diamond p$  is not true at  $w_5$ .

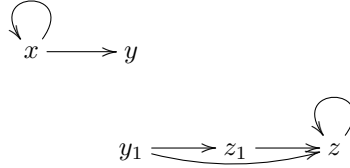
Let us also note that strict filtrations of this model is not Euclidean. Suppose by contrary that  $\hat{R}^{\mathcal{E}}$  is a strict filtration of that model. So  $R_{min}^E \subseteq \hat{R}^{\mathcal{E}}$ , since  $R_{min}^E$  is the minimal Euclidean relation containing  $R_{min}$ . On the other hand,  $R_{min}^E \not\subseteq R_{max}$ , so is not  $\hat{R}^{\mathcal{E}}$ .  $\square$

Let us define the logic  $\mathcal{L}$  as  $\mathbf{K} \oplus \Diamond \Diamond p \rightarrow \Diamond p$ . Let  $R$  be a binary relation, the  $\mathcal{L}$ -closure of  $R$  is defined (denoted as  $R^{\hat{\Diamond}}$ ) as the following union:

$$R^{\hat{\Diamond}} = R \cup R^3 \cup R^5 \cup \dots \cup R^{2k+1} \cup \dots$$

**Theorem 3.**  $\mathcal{L}$  does not admit strict filtrations.

*Proof.* Consider the following frame  $\mathcal{F} = \langle W, R \rangle$ :



Clearly that  $\mathcal{F}$  is an  $\mathcal{L}$ -frame. We define the valuation  $\vartheta$  as follows:

$$\begin{aligned} \vartheta(p) &= \{x\} \\ \vartheta(q) &= \{y, y_1\} \\ \vartheta(r) &= \{z\} \end{aligned}$$

Let us put  $\Gamma = \text{Sub}\{p, q, \Diamond r\}$ . We factorise  $W$  through  $\sim_\Gamma$  and consider a model  $\hat{\mathcal{M}} = \langle W / \sim_\Gamma, \hat{R}, \hat{\vartheta} \rangle$ , where  $\hat{R} = (R_{min})^{\hat{\Diamond}}$ . We have  $(\hat{x}, \hat{z}) \in \hat{R} \circ \hat{R} \circ \hat{R}$ , but  $\Diamond r$  is not true at  $x$ .  $\square$

## 1 Finite “canonical” models

Let  $\mathcal{L}$  be a normal modal logic,  $\mathcal{M}_{\mathcal{L}}$  its canonical model, and  $\Gamma$  a finite Sub-closed set of formulas. Let us put  $\Gamma' = \text{Sub}(\varphi) \cup \{\neg\psi \mid \psi \in \text{Sub}(\varphi)\}$ .

A subset  $\Delta \subseteq \Gamma'$  is a *finite  $\mathcal{L}$ -consistent set* if  $\neg \bigwedge \Delta \notin \mathcal{L}$ . A subset  $\Delta$  is maximal, if (the following are obviously equivalent):

1.  $\Delta$  is maximal amongst finite  $\mathcal{L}$ -consistent sets,
2. For each  $\psi \in \text{Sub}(\varphi)$  either  $\psi \in \Delta$  or  $\neg\psi \in \Delta$ .

Every finite  $\mathcal{L}$ -theory is clearly can be extended to some maximal one. It is the finite version of Lindenbaum’s lemma.

**Definition 5.** Let  $\mathcal{L}$  be a modal logic and  $\Gamma$  be a finite Sub-closed set of formulas. A finite “canonical” model is a triple  $\mathcal{M}_{\mathcal{L}}^\Gamma = \langle W_{\mathcal{L}}^\Gamma, R_{\mathcal{L}}^\Gamma, \vartheta_{\mathcal{L}}^\Gamma \rangle$ , where

1.  $W_{\mathcal{L}}^\Gamma$  is the set all maximal theories that extend finite  $\mathcal{L}$ -theories
2.  $R_{\mathcal{L}}^\Gamma$  is a relation such that  $\langle W_{\mathcal{L}}^\Gamma, R_{\mathcal{L}}^\Gamma \rangle$  is an  $\mathcal{L}$ -frame and

$$\forall \Box \psi \in \text{Sub}(\varphi) \quad \forall \Delta_1 \in W_{\mathcal{L}}^\Gamma \quad (\Box \psi \in \Delta_1 \Leftrightarrow \forall \Delta_2 \in R_{\mathcal{L}}^\Gamma(\Delta_1) \quad \psi \in \Delta_2)$$

3.  $\vartheta_{\mathcal{L}}^\Gamma(p) = \{\Delta \in W_{\mathcal{L}}^\Gamma \mid p \in \Delta\}$  for every variable  $p \in \Gamma$ .

**Lemma 6.** Let  $\mathcal{L}$  be a modal logic and  $\varphi \notin \mathcal{L}$ , then  $\mathcal{M}_{\mathcal{L}}^{\text{Sub}(\varphi)} \not\models \varphi$ .

**Lemma 7.** Let  $\mathcal{L}$  be a modal logic and  $\Gamma$  a finite Sub-closed set of formulas, then if  $\mathcal{L}$  admits strict filtrations, then there exists a finite “canonical” model  $\mathcal{M}_{\mathcal{L}}^{\Gamma}$  such that  $\mathcal{M}_{\mathcal{L}}^{\Gamma} \models \mathcal{L}$ .

*Proof.* ( $\Rightarrow$ ) Let  $\Gamma$  be a finite Sub-closed of formulas.  $\mathcal{L}$  admits strict filtrations, so the filtration of the canonical model  $\mathcal{M}_{\mathcal{L}}$  through  $\Gamma$  is also an  $\mathcal{L}$ -model. The underlying set of  $\mathcal{M}_{\mathcal{L}}/\sim_{\Gamma}$  consists of maximal  $\mathcal{L}$  theories up to  $\Gamma$ -equivalence and this quotient set is finite.

It is readily checked that the quotient model  $\mathcal{M}_{\mathcal{L}}/\sim_{\Gamma}$  satisfies Definition 5.  $\square$

The converse implication does not have to true generally. **GL** might be an example of a logic that has the “finite canonical” model property with no filtrations.

## 2 Fusion stuff

**Definition 6.** Let  $\mathcal{L}_1$  and  $\mathcal{L}_2$  be modal logics, then the fusion  $\mathcal{L}_1 * \mathcal{L}_2$  is the minimal bimodal logic that contains  $\mathcal{L}_1$  and  $\mathcal{L}_2$  [2].

**Lemma 8.** Let  $\Gamma$  be a finite and Sub-closed set of formulas and  $\mathcal{M} = \langle W, R, \vartheta \rangle$ . Consider  $\Gamma' = \Gamma \cup \{\Diamond \Box \psi \mid \Box \psi \in \Gamma\}$ . Let  $\Delta$  be any finite and Sub-closed extension of  $\Gamma'$ . Then a model  $\widehat{\mathcal{M}} = \langle W/\sim_{\Delta}, (R_{\Delta}^{\min})^E, \widehat{\vartheta} \rangle$  is a filtration of  $\mathcal{M}$  through  $\Delta$ .

*Proof.* Recall that  $(R_{\Delta}^{\min})^E$  is defined inductively as:

1.  $R_{\Delta}^0 = R_{\Delta}^{\min}$
2.  $R_{\Delta}^{n+1} = R_{\Delta}^n \cup (R_{\Delta}^{n-1} \circ R_{\Delta}^n)$
3.  $(R_{\Delta}^{\min})^E = \bigcup_{k < \omega} R_{\Delta}^k$

If  $R_{\Delta}^{\min}$  is already a subrelation of  $R^{\max}$ , so the base case is self-evident.

Suppose the statement holds for  $R_{\Delta}^n$ ,  $(\hat{x}, \hat{y}) \in R_{\Delta}^{n+1}$ , that is, there exists  $\hat{z}$  such that  $(\hat{z}, \hat{x}), (\hat{z}, \hat{y}) \in R_{\Delta}^n$ . Let  $\mathcal{M}, x \models \Box \psi$  (for  $\Box \psi \in \Gamma$ ).

We rewrite  $(\hat{z}, \hat{x}) \in R_{\Delta}^n$  as the following sausage (for some  $\widehat{u}_1, \widehat{u}_2, \dots, \widehat{u}_{n-1}, \widehat{u}_n$ ):

$$\hat{z} \xleftarrow{R_{\Delta}^{\min}} \widehat{u}_1 \xrightarrow{R'} \widehat{u}_2 \xrightarrow{R'} \dots \xrightarrow{R'} \widehat{u}_{n-1} \xrightarrow{R'} \widehat{u}_n \xrightarrow{R_{\Delta}^{\min}} \hat{x}$$

where  $R'$  is either  $R_{\Delta}^{\min}$  or its converse. We have  $\mathcal{M}, z \models \Box \psi$ , since  $\mathcal{M}, x \models \Box \psi$  implies  $\mathcal{M}, u_n \models \Diamond \Box \psi$ . After that we apply the following property of **K5**-models:

For each  $a, b \in M_i$  such that  $aR_ib$  we have  $\mathcal{M}_i, a \models \Diamond \Box p_{\psi}$  iff  $\mathcal{M}_i, b \models \Diamond \Box p_{\psi}$

Note that we always stay within  $\Delta$  since  $\Gamma' \subseteq \Delta$ . Finally, we conclude  $\mathcal{M}, x \models \psi$  from IH and  $\mathcal{M}, z \models \Box \psi$ .  $\square$

**Theorem 4.**

1. **K5** \* **K5** admits definable filtrations.
2. **K5** \*  $\dots$  \* **K5** admits definable filtrations.
3. If  $\mathcal{L}_1, \dots, \mathcal{L}_n$  admit strict filtrations, then **K5** \*  $\dots$  \* **K5** \*  $\mathcal{L}_1$  \*  $\dots$  \*  $\mathcal{L}_n$

*Proof.*

1. Let  $\Gamma$  be a finite Sub-closed set of bimodal formulas,  $\mathcal{F} = \langle W, R_1, R_2 \rangle$  a  $\mathbf{K5} * \mathbf{K5}$ -frame, and  $\vartheta$  a valuation on  $\mathcal{F}$ . Denote  $\langle \mathcal{F}, \vartheta \rangle$  as  $\mathcal{M}$ .

We introduce the set of fresh variables  $V = \{p_\psi \mid \psi \in \Gamma\}$  and define a new model  $\mathcal{M}' = \langle \mathcal{F}, \vartheta' \rangle$  as follows:

$$\text{For all } \psi \in \Gamma, \mathcal{M}, x \models \psi \Leftrightarrow \mathcal{M}', x \models \psi \Leftrightarrow \mathcal{M}', x \models p_\psi.$$

Consider these modifications of  $\Gamma$  and  $V$ :

$$\begin{aligned} \Gamma' &= \Gamma \cup \{\Diamond_1 \Box_1 \psi \mid \Box_1 \psi \in \Gamma\} \cup \{\Diamond_2 \Box_2 \psi \mid \Box_2 \psi \in \Gamma\} \\ \Delta &= V \cup \text{Sub}(\{\Diamond \Box p_\psi \mid \Box_i \psi \in \Gamma, in = 1, 2\}) \end{aligned}$$

Let us define an equivalence relation  $\sim_\Delta$  induced by  $\Delta$ .

Consider  $\mathcal{M}_i = \langle W, R_i, \vartheta' \rangle$ , a reduct of  $\mathcal{M}'$ , we have:

- (a)  $\mathcal{M}_i, x \models \Box p_\psi$  iff  $\mathcal{M}, x \models \Box_i \psi$
- (b)  $\mathcal{M}_i, x \models \Diamond \Box p_\psi$  iff  $\mathcal{M}, x \models \Diamond_i \Box_i \psi$

So  $\sim = \sim_{\Gamma'}$  by the construction. Let us put  $\widehat{W} = W / \sim_{\Gamma'}$ . Lemma 8 implies the following claim:

**Claim 1.** Let  $\widehat{R}_i = (R_{\Delta}^{min})^E$  and  $\widehat{\vartheta}(p) = \{[x]_{\sim_i} \mid \mathcal{M}_i, x \models p\}$  for  $p \in \Delta_1$ , define  $\widehat{\mathcal{M}}_i = \langle \widehat{W}, \widehat{R}_i, \widehat{\vartheta} \rangle$ . Then  $\widehat{\mathcal{M}}_i \models \mathbf{K5}$  and  $\widehat{\mathcal{M}}_i$  is a filtration of  $\mathcal{M}_i$  through  $\Delta$ .

Finally, we consider a model  $\widehat{\mathcal{M}} = \langle \widehat{W}, \widehat{R}_1, \widehat{R}_2, \widehat{\vartheta} \rangle$ , where  $\widehat{R}_{\Gamma' i} = R_{i\Gamma'}^{minE}$  and  $\widehat{\vartheta}(p)$  is defined as usual for  $p \in \Gamma$ .  $\widehat{\mathcal{M}}$  is a filtration of  $\mathcal{M}$  through  $\Gamma'$ .

Let  $\widehat{x} \widehat{R}_{\Gamma' i} \widehat{y}$  and  $\mathcal{M}, x \models \Box_i \psi$  for  $\Box_i \psi \in \Gamma$ . Then  $\mathcal{M}_i, x \models \Box p_\psi$ , so  $\widehat{\mathcal{M}}_i, \widehat{x} \models \Box p_\psi$ . By the claim above,  $\widehat{\mathcal{M}}_i$  is a filtration of  $\mathcal{M}_i$  through  $\Delta$ , so  $\mathcal{M}_i, y \models p_\psi$ . Then  $\mathcal{M}, y \models \psi$ .

2. Likewise

□

**Theorem 5.** Let  $\mathcal{L}_1$  and  $\mathcal{L}_2$  be modal logics such that  $\text{Fr}(\mathcal{L}_1)$  and  $\text{Fr}(\mathcal{L}_2)$  admit filtrations, so does  $\text{Fr}(\mathcal{L}_1 * \mathcal{L}_2)$ .

*Proof.*

□

## References

- [1] Stanislav Kikot, Ilya Shapirovsky, and Evgeny Zolin. Completeness of logics with the transitive closure modality and related logics. *arXiv preprint arXiv:2011.02205*, 2020.
- [2] Agi Kurucz. Combining modal logics. *Studies in Logic and Practical Reasoning*, 3:869–924, 2007.