Stuff related to tensor products of modal algebras

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1 Preliminaries

The basic definitions according to [1].

Definition 1. *An n-modal algebra is an algebra $\mathcal{B} = \langle B, \vee, \neg, \bot, \Diamond_1, \dots, \Diamond_n \rangle$, where $\mathcal{B} = \langle B, \vee, \neg, \bot \rangle$ is a Boolean algebra and for each $i = 1, \dots, n$ $\Diamond_i$ is an order-preserving unary function obeying:*

1. $\Diamond_i p \vee \Diamond_i q = \Diamond_i (p \vee q)$
2. $\Diamond_i \bot = \bot$

Definition 2. *An n-normal modal logic is a set of formulas containing Boolean tautologies, formulas $\Diamond_i p \vee \Diamond_i q \leftrightarrow \Diamond_i (p \vee q)$ and $\Diamond_i \bot \leftrightarrow \bot$ for $i = 1, \dots, n$, and is closed under* **MP**, **Sub**, *and the monotonicity rule: from $\varphi \to \psi$ infer $\Diamond_i \varphi \to \Diamond_i \psi$.*

Definition 3.

1. *A Kripke n-frame is a structure $\mathcal{F} = \langle W, R_1, \dots, R_n \rangle$, where $R_i \subseteq W \times W$ for each $i = 1, \dots, n$.*
2. *Let $\mathcal{F}$ be a Kripke frame. A Kripke model over $\mathcal{F}$ is a tuple $\mathcal{M} = \langle \mathcal{F}, \vartheta \rangle$, where $\vartheta : \mathrm{PV} \to 2^W$ is a valuation. Modalised formulas have the following semantics:*

$$\mathcal{M}, x \models \Diamond_i \varphi \Leftrightarrow \exists y \in R_i(x)\; \mathcal{M}, y \models \varphi.$$

A formula φ is true in $\mathcal{M}$ iff $||\varphi||_{\mathcal{M}} = W$. φ is valid in $\mathcal{F} = \langle W, R_1, \dots, R_n \rangle$ iff $||\varphi||_{\mathcal{M}} = W$ for every model $\mathcal{M}$ over $\mathcal{F}$.

3. *The complex algebra of a Kripke frame $\mathcal{F} = \langle W, R_1, \dots, R_n \rangle$ is a complete modal algebra defined as $\mathcal{F}^+ = \langle \mathcal{P}(W), \cup, \neg, \varnothing, R_1^{-1}, \dots, R_n^{-1} \rangle$.*
4. *A general n-frame is a structure $\mathcal{F} = \langle W, R_1, \dots, R_n, \mathcal{A} \rangle$, where $\mathcal{A}$ is a subalgebra of $\langle W, R_1, \dots, R_n \rangle^+$.*

Definition 4. *Let $\mathcal{F}$ be an n-frame, then $\mathrm{Log}(\mathcal{F}) = \{\varphi \mid \mathcal{F} \models \varphi\}$. If $\mathbb{F}$ is a class of n-frames, then $\mathrm{Log}(\mathbb{F}) = \bigcap\limits_{\mathcal{F} \in \mathbb{F}} \mathrm{Log}(\mathcal{F})$. We use the same notation for logics of general frames and modal algebras.*

We discuss the background on products of modal logics and products of Kripke frames [8] [9].

Definition 5. *Let $\mathcal{F}_1 = \langle W_1, R_1^1, \dots, R_1^n \rangle$ be an n-frame and $\mathcal{F}_2 = \langle W_2, R_2^1, \dots, R_2^m \rangle$. The product frame of $\mathcal{F}_1$ and $\mathcal{F}_2$ is an $n + m$-frame of the form*

$$\mathcal{F}_1 \times \mathcal{F}_2 = \langle W_1 \times W_2, R_h^1, \ldots, R_h^n, R_v^1, \ldots, R_v^m \rangle$$

such that for all $u_1, u_2 \in W_1$ *and for all* $v_1, v_2 \in W_2$*,*

$$\langle u_1, v_1 \rangle R_h^i \langle u_2, v_2 \rangle \text{ iff } u_1 R_1^i u_2 \text{ and } v_1 = v_2 \text{ for } 1 \leqslant i \leqslant n.$$
$$\langle u_1, v_1 \rangle R_v^j \langle u_2, v_2 \rangle \text{ iff } u_1 = u_2 \text{ and } u_1 R_2^j u_2 \text{ for } 1 \leqslant j \leqslant m.$$

This operation on Kripke frames commutes with disjoint unions, p-morphic images, and generated subframes.

Let $\mathcal{L}_1$ be a normal n-modal logic and $\mathcal{L}_2$ a normal m-modal logic, the product of $\mathcal{L}_1$ and $\mathcal{L}_2$ is defined as

$$\mathcal{L}_1 \times \mathcal{L}_2 = \mathrm{Log}(\mathrm{Frames}(\mathcal{L}_1) \times \mathrm{Frames}(\mathcal{L}_2))$$

Proposition 1.

1. *Let* $\mathcal{L}_1$*,* $\mathcal{L}_2$ *be modal logics, then* $\mathcal{L}_1 * \mathcal{L}_2 \subseteq \mathcal{L}_1 \times \mathcal{L}_2$*.*
2. *Let* $\mathcal{L}_1$*,* $\mathcal{L}_2$ *be Kripke complete modal logics, then* $\mathcal{L}_1 \times \mathcal{L}_2 = \mathrm{Log}(\mathrm{Frames}_r(\mathcal{L}_1) \times \mathrm{Frames}_r(\mathcal{L}_2))$*, where* $\mathrm{Frames}_r(\mathcal{L}_i) = \{\mathcal{F} \in \mathrm{Frames}(\mathcal{L}_i) \mid \mathcal{F} \textit{ is rooted } \}$ *for* $i = 1, 2$*.*

1.1 Axiomatising products

The following properties hold for a product frame having the form $\mathcal{F} = \langle W, R_1, \ldots, R_n \rangle$:

1. (left commutativity) $\forall x, y, z \in W \ (x R_j y \ \& \ y R_i z \Rightarrow \exists u \in W \ (x R_i u \ \& \ u R_j z))$
2. (right commutativity) $\forall x, y, z \in W \ (x R_i y \ \& \ y R_j z \Rightarrow \exists u \in W \ (x R_j u \ \& \ u R_i z))$
3. (Confluence) $\forall x, y, z \in W (x R_j y \ \& \ x R_i z \Rightarrow \exists u \in W (y R_i u \ \& \ z R_j u))$

The properties are expressed as modal formulas as well:

1. $\mathbf{comm}^{\mathbf{l}}_{ij} = \Diamond_j \Diamond_i p \to \Diamond_i \Diamond_j p$
2. $\mathbf{comm}^{\mathbf{r}}_{ij} = \Diamond_i \Diamond_j p \to \Diamond_j \Diamond_i p$
3. $\mathbf{cr}_{ij} = \Diamond_i \Box_j p \to \Box_i \Diamond_j p$

Definition 6. *Given unimodal modals logics* $\mathcal{L}_1, \ldots, \mathcal{L}_n$*, the commutator*

$$[\mathcal{L}_1, \ldots, \mathcal{L}_n]$$

is the smallest n*-modal logic containing* $\mathcal{L}_i$ *and axioms* $\mathbf{comm}^{\mathbf{l}}_{ij}$*,* $\mathbf{comm}^{\mathbf{r}}_{ij}$*, and* $\mathbf{cr}_{ij}$ *for* $i, j \in \{1, \ldots, n\}$ *with* $i \neq j$*.*

Since $\mathbf{comm}^{\mathbf{l}}_{ij}$, $\mathbf{comm}^{\mathbf{r}}_{ij}$, and $\mathbf{comm}^{\mathbf{r}}_{ij}$ are Salqvist formulas, one has

Proposition 2. *If* $\mathcal{L}_1, \ldots, \mathcal{L}_n$ *are canonical, then* $[\mathcal{L}_1, \ldots, \mathcal{L}_n]$ *is canonical, and thus, elementary and Kripke complete.*

Moreover, one has

Proposition 3. $[\mathcal{L}_1, \ldots, \mathcal{L}_n] \subseteq \mathcal{L}_1 \times \cdots \times \mathcal{L}_n$*.*

If the converse inclusion holds, then $\mathcal{L}_1, \ldots, \mathcal{L}_n$ are *product-mathcing*, see [3]. The examples of product-matching logics are Horn axiomatisable ones [9]. In particular, the following equality holds for Kripke complete and Horn axiomatisable logics $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$:

$$\mathcal{L}_1 \times \mathcal{L}_2 \times \mathcal{L}_3 = (\mathcal{L}_1 \times \mathcal{L}_2) \times \mathcal{L}_3 = \mathcal{L}_1 \times (\mathcal{L}_2 \times \mathcal{L}_3)$$

Generally, this is an open question whether the product of modal logics is associative.

2 Tensor products of modal algebras. The basic definitions and results

Definition 7. *A commutative associative algebra*

Definition 8. *Tensor product of them*

Definition 9. *Tensor product of modal algebras*

Definition 10. *Tensor product of general frames*

TODO: describe related underlying results

3 Note on incomplete modal logics

TODO: consider the system containing **GL**, McKinsey, seriality, and linearity.

4 Solution of Problem 1

5 Solution of Problem 4

Definition 11. *A finitely axiomatisable normal modal logic*

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