Representable cylindric algebras of dimension ω: the aspects of canonicity and axiomatisability

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1 Intro

2 The problem itself

Suppose $\mathcal{C} \in \mathbf{RCA}_\omega$, whether $\mathcal{C}^+$ has a complete, ω-dimensional representation? [6]

3 Boolean algebras with operators and cylindric algebras, a bit of the backgroud

Let $a \in \mathcal{B}$ be an element of a Boolean algebra $\mathcal{B}$, a is called an atom, if for every $b \in \mathcal{B}$ $b < a$ implies $b = 0$. That is, an atom is a minimal non-zero element. $\mathrm{At}(\mathcal{B})$ is the set of all atoms of $\mathcal{B}$.

Let $\mathcal{B}$ be a Boolean algebra and $\mathcal{F}$ a field of sets such that $h : \mathcal{B} \to \mathcal{F}$ is a representation of $\mathcal{B}$, then $\mathcal{B}$ is a complete representation of $\mathcal{B}$, if for every $A \subseteq \mathcal{B}$ whenever ΣA we have the following:

$$h(\Sigma A) = \bigcup h[A]$$

Theorem 1. *Let $\mathcal{B}$ be a Boolean algebra, then $\mathcal{B}$ is atomic iff $\mathcal{B}$ is completely representable. See [4, Corollary 6].*

Definition 1.

1. *Let $\mathcal{B} = \langle B, +, -, 0, 1\rangle$ be a Boolean algebra. An operator is an n-ary function $\Omega : B^n \to B$ satisfying the following conditions:*
   - *Normality: for all $b_0, \dots, b_{n-1} \in B$, if $b_1 = 0$ for some $i < n$, then*
   $$\Omega(b_0, \dots, b_{i-1}, 0, b_{i+1}, \dots, b_{n-1}) = 0$$
   - *Additivity: for all $b_0, \dots, b_{n-1}, b, b' \in B$ we have*
   $$\Omega(b_0, \dots, b_{i-1}, (b + b'), b_{i+1}, \dots, b_{n-1}) = \\ \Omega(b_0, \dots, b_{i-1}, b, b_{i+1}, \dots, b_{n-1}) + \Omega(b_0, \dots, b_{i-1}, b', b_{i+1}, \dots, b_{n-1})$$
2. *Let I be an index set, a Boolean algebra with operators (BAO) is an algebra $\langle B, +, -, 0, 1, \{\Omega_i\}_{i\in I}\rangle$ such that $\langle B, +, -, 0, 1\rangle$ is a Boolean algebra and for each $i \in I$ Ω_i is an operator.*

Definition 2. *Let $\mathcal{B} = \langle B, +, -, 0, 1, \{\Omega_i\}_{i\in I}\rangle$ be a BAO, then*

1. *An opetator Ω is completely additive, if for each $b_0, \dots, b_{n-1} \in B$ and $X \subseteq B$, one has*

$$\Omega(b_0, \ldots, b_{i-1}, \sum X, b_{i+1}, \ldots, b_{n-1}) = \sum_{x \in X} \Omega(b_0, \ldots, b_{i-1}, x, b_{i+1}, \ldots, b_{n-1})$$

2. *$\mathcal{B}$ is completely additive, if for each $i \in I$ Ω_i is additive,*

3. *A class $\mathcal{K}$ of BAOs is completely additive, if every $\mathcal{B} \in \mathcal{K}$ is completely additive.*

3.1 Atom structures and canonical extensions

Definition 3. *Let I be an index set and $\{\Omega_i\}_{i \in I}$ a set of function symbols*

1. *An atom structure is a relational structrure $\mathcal{F} = \langle W, \{R_i\}_{i \in I} \rangle$ such that R_i is a $n+1$-ary relation symbol, where $\Omega_{i \in I}$ is an n-ary function symbol,*

2. *Let $\mathcal{B}$ be an atomic BAO of the signature I, the atom structure of $\mathcal{B}$, written as $\mathfrak{At}\mathcal{B}$, is an atom structure $\langle \mathrm{At}(\mathcal{B}), \{R_i\}_{i \in I} \rangle$ such that for each $a, b_0, \ldots, b_{n+1} \in \mathrm{At}(\mathcal{B})$ and for each $i \in I$*

$$\mathfrak{At}\mathcal{B} \models R_i(a, b_0, \ldots, b_{n+1}) \text{ iff } \mathcal{B} \models a \leqslant \Omega_i(b_0, \ldots, b_{n+1})$$

3. *Let $\mathcal{F} = \langle W, \{R_i\}_{i \in I} \rangle$ be an atom structure, the complex algebra of $\mathcal{F}$, written as $\mathbf{Cm}\mathcal{F}$, is a BAO $\langle \mathcal{P}(W), \cup, -, \varnothing, W, \{\Omega_{R_i}\}_{i \in I} \rangle$ such that for all $X_0, \ldots, X_{n-1} \subseteq W$ and for each $i \in I$*

$$\Omega_{R_i}(X_0, \ldots, X_{n-1}) = \{a \in W \mid \exists b_0 \in X_0 \ldots \exists b_{n-1} \in X_{n-1} \; \mathcal{F} \models R_i(a, b_0, \ldots, b_{n-1})\}$$

The following duality is due to Thomason [12].

Fact 1.

1. *Let $\mathcal{B}$ be a complete atomic BAO, then $\mathcal{B} \cong \mathfrak{Cm}(\mathbf{At}(\mathcal{B}))$,*

2. *Let $\mathcal{F}$ be an atom structure, then $\mathcal{F} \cong \mathfrak{At}(\mathfrak{Cm}(\mathcal{B}))$.*

Let A be a non-empty subset of a Boolean algebra $\mathcal{B}$, A is a *filter*, if A is closed under finite infima and upward closed. A is an ultrafilter, if it has no non-trivial extensions. That is, if $A \subseteq A'$, then $A' = \mathcal{B}$. This is a well-known fact that every filter can be extended to a maximal one using Zorn's lemma.

The following definition is due to, for example, [13, Definition 5.40].

Definition 4. *Let $\mathcal{B} = \langle B, +, -, 0, 1, \{\Omega_i\}_{i \in I} \rangle$ be a BAO and $\mathbf{Uf}(\mathcal{B})$ the set of its ultrafilters. The ultrafilter frame of $\mathcal{B}$ (or canonical frame) is a relational structure $\mathcal{F}_{\mathcal{B}} = \langle \mathbf{Uf}(\mathcal{B}), R_{\Omega_i} \rangle$ such that for each ultrafilters $\beta_0, \ldots, \beta_{n-1}, \gamma$ one has*

$$\mathbf{Uf}(\mathcal{B}) \models R_{\Omega_i}(\beta_0, \ldots, \beta_{n-1}, \gamma) \text{ iff } \{\Omega(b_0, \ldots, b_{n-1}) \mid b_0 \in \beta_0, \ldots, b_{n-1} \in \beta_{n-1}\} \subseteq \gamma.$$

Definition 5. *Let $\mathcal{B}$ be a BAO, then*

1. *The canonical extension of $\mathcal{B}$ is a complex algebra of the canonical frame $\mathfrak{Cm}(\mathcal{F}_{\mathcal{B}})$ denoted as $\mathcal{B}^+$,*

2. *The class of BAOs is canonical, if it is closed under canonical extensions.*

Theorem 2. *Let $\mathcal{A}$, $\mathcal{B}$ be BAOs,*

1. *There exists $\iota : \mathcal{A} \hookrightarrow \mathcal{A}^+$ such that $\iota : a \mapsto \{\gamma \in \mathbf{Uf}(\mathcal{A}) \mid a \in \gamma\}$.*

2. *$i : \mathcal{A} \hookrightarrow \mathcal{B}$ implies $i^+ : \mathcal{A}^+ \hookrightarrow \mathcal{B}^+$*

Fact 2.

4 Cylindric algebras and cylindric set algebras

Let α be an ordinal. Let ${}^{\alpha}U$ be the set of all functions mapping α to a non-empty set U. We denote $x(i) = x_i$ for $x \in {}^{\alpha}U$ and $i < \alpha$.

A subset of ${}^{\alpha}U$ is an α-ry relation on U. For $i, j < \alpha$, the i, j-*diagonal* D_{ij} is the set of all elements of ${}^{\alpha}U$ such that $y_i = y_j$.

If $i < \alpha$ and X is an α-ry relation on U, then the i-th cylindrification $C_i X$ is the set of all elements of U that agree with some element of X on each coordinate except, perhaps, the i-th one. To be more precise,

$$C_i X = \{y \in {}^{\alpha}U \mid \exists x \in X \forall i < \alpha \, (i \neq j \Rightarrow y_j = x_j)\}.$$

We define the following equivalence relation for $i < \alpha$ and $x, y \in {}^{\alpha}U$:

$$x \equiv_i y \Leftrightarrow \forall j \in \alpha \, (i \neq j \Rightarrow x(i) = y(j))$$

Then one may reformulate the definition of the i-th cylindrification in the following way:

$$C_i X = \{y \in {}^{\alpha}U \mid \exists x \in X \;\; x \equiv_i y\}$$

According to this version of the definiton, one may think of the cylindrification as an **S**5 modal operator.

The following definition is due to, e.g., [11]:

Definition 6. *Let $(\mathcal{A}_i)_{i \in I}$ be a family of algebras (of an abstract signature) and $\mathcal{A}$ is a subalgebra of $\prod\limits_{i \in I} \mathcal{A}_i$, then $\mathcal{A}$ is a subdirect product, if every projection is onto. That is, for every $i \in I$, $\pi_i[\mathcal{A}] = \mathcal{A}_i$.*

Definition 7. *A cylindic set algebra of dimension α is an algebra consisting of a set S of α-ry relation on some base set U with the constants and operations $0 = \varnothing$, $1 = {}^{\alpha}U$, $\cap$, $-$, the diagonal elements $\{D_{ij}\}_{i,j<\alpha}$, the cylindrifications $\{C_i\}_{i<\alpha}$. A generalised cylindric set algebra of dimension α is a subdirect of cylindric algebras that have dimension α.*

Definition 8. *A cylindric algebra of dimension α is an algebra $\mathcal{C} = \langle \mathcal{B}, \{c_i\}_{i<\alpha}, \{d_{ij}\}_{i,j<\alpha} \rangle$ such that*

- *$\mathcal{B}$ is a Boolean algebra, for each $i, j < \alpha$ c_i is an operator and $d_{ij} \in \mathcal{B}$*
- *For each $i < \alpha$, $a \leqslant c_i a$, $c_i(a \wedge c_i b) = c_i a \wedge c_i b$ and $d_{ii} = 1$*
- *For every $i, j < \alpha$, $c_i c_j a = c_j c_i a$*
- *If $k \neq i, j < \alpha$, then $d_{ij} = c_k(d_{ij} \wedge d_{jk})$*
- *If $i \neq j$, then $c_i(d_{ij} \wedge a) \wedge c_i(d_{ij} \wedge -a) = 0$*

$\mathbf{CA}_{\alpha}$ *is the class of all cylindric algebras of dimension α.*

Definition 9. *Let $\mathcal{A} = \langle A, +, -, 0, 1, (c_i)_{i<\alpha}, (d_i j)_{i,j<\alpha} \rangle \in \mathbf{CA}_{\alpha}$ and $\beta < \alpha$, then the β-reduct of $\mathcal{A}$ is an algebra $\mathcal{A}_{\beta} = \langle A, +, -, 0, 1, (c_i)_{i<\beta}, (d_i j)_{i,j<\beta} \rangle$,*

One may define a representation of a cylindric algebra explicitly in the following way:

Definition 10. *Let $\mathcal{A}$ be a cylindric algebra of dimension α. A representation of $\mathcal{A}$ over the non-empty domain X is a map $f : \mathcal{A} \hookrightarrow 2^{{}^{\alpha}U}$ such that:*

1. $f(1) = \bigcup_{i \in I} {}^{\alpha} X_i$ *for some disjoint family* $\{X_i\}_{i \in I}$ *where each* $X_i \subseteq X$

2. $h : \mathcal{A} \to 2^{f(1)}$ *is a representation of a Boolean reduct*

3. *for all* $\lambda, \eta < \alpha$, $x \in h(d_{\lambda\eta})$ *iff* $x_\lambda = x_\eta$

4. *for all* $\lambda < \alpha$ *and* $a \in \mathcal{A}$, $x \in h(c_\lambda(a))$ *iff there is* $y \in X$ *such that* $x[\lambda \mapsto y] \in h(a)$

An α-dimensional cylindric algebra C is representable, if it is isomorphic to a generalised cylindric set algebra of dimension α. Such is isomorphism is a representation of C. $\mathbf{RCA}_\alpha$ is the class of all representable cylindric algebras that have dimension α. In particular, we are interested in the case when $\alpha = \omega$.

Definition 11. *Given a cylindric algebra of dimension* α C*, let* x *be a term of its signature, the substitution operator* s^i_j *have the following definition:*

$$s^i_j x = \begin{cases} x, \text{ if } i = j \\ c_i(d_{ij} \wedge x), \text{ otherwise} \end{cases}$$

It is well known that $\mathbf{RCA}_\alpha$ is a variety, $\mathbf{RCA}_\alpha$ ($\alpha \leqslant 2$) is finitely axiomatisable and $\mathbf{RCA}_\alpha$ ($2 < \alpha < \omega$) has no finite axiomatisation, see [3].

Let $\mathcal{A} \in \mathbf{CA}_\omega$, then $\mathcal{A}$ has a *complete representation*, if this representation preserves all existing suprema. In other words, $\mathcal{A}$ is completely representable.

Let us concretise the definition of a canonical extension for $\mathbf{CA}_\alpha$-type BAOs.

Definition 12. *Let* $\mathcal{C} = \langle C, +, -, 0, 1, \{d_{ij}\}_{i,j<\alpha}, \{c_i\}_{i<\alpha} \rangle$ $\mathcal{A}$ *be a BAO of type* $\mathbf{CA}_\alpha$ *Let* $\mathbf{Uf}(\mathcal{C})$ *be the set of all ultrafilters of* $\mathfrak{B}\mathcal{C}$*, the Boolean part of* $\mathcal{C}$*.*

Let us define $\mathbf{C}_i : \mathbf{Uf}(\mathcal{C}) \to \mathbf{Uf}(\mathcal{C})$ *for each* $i, j < \alpha$ *as*

1. $\mathbf{C}_i \mathcal{X} = \{\mathcal{F} \in \mathbf{Uf}(\mathcal{C}) \mid \exists \mathcal{F}' \in \mathbf{Uf}(\mathcal{C}) \ (a \in \mathcal{F} \Rightarrow c_i a \in \mathcal{F}' R)\}$,

2. $D_{ij} = \{\mathcal{F} \in \mathbf{Uf}(\mathcal{C}) \mid d_{ij} \in \mathcal{F}\}$.

The structure $\mathcal{C}^+ = \langle \mathbf{Uf}(\mathcal{C}), \cup, -, \varnothing, C, \mathbf{C}_{i<\alpha}, \{D_{ij}\}_{i,j<\alpha} \rangle$ *is called the canonical extension of* $\mathcal{C}$*.*

4.1 Closed elements

The following definitions and facts are due to Henkin, Monk, and Tarski [2].

Let $\mathcal{A} \in \mathbf{CA}_\alpha$ and $x \in \mathcal{A}$. Recall that *the dimension of* x is the set of all ordinals $\gamma < \alpha$ such that $c_\gamma x \neq x$. More formally,

$$\Delta x = \{\gamma \mid \gamma < \alpha \ \& \ c_\gamma x \neq x\}$$

Let us discuss some metamathematical intuitions standing behind the notion of a dimension. Let Θ be a first-order theory and $\mathcal{C}/\equiv_\Theta$ its Lindenbaum-Tarski algebra. Let φ be a formula in the signature of Θ. Then $\Delta(\varphi/\Theta)$ consists of all $\kappa < \alpha$ such that $\exists x_\kappa \varphi \leftrightarrow \varphi$ is not valid in Θ. That is, $\Delta(\varphi/\Theta)$ contains ordinals κ for which x_κ is free in φ. Moreover, $\Delta(\varphi/\Theta)$ consists only of those ordinals for which x_κ is free in every $\psi \in \varphi/\Theta$.

In particular, an element x is called *zero-dimensional* if $\Delta x = 0$. Zero-dimensional elements correspond to equivalence classes of sentences in the Lindenbaum-Tarski algebra of a given first-order theory. Thus, the set of zero-dimensional elements form a Boolean algebras of sentences associated with Θ.

Definition 13. *Let $\mathcal{A}$ be an α-dimensional cylindric algebra. Let α be an ordinal and Γ a subset α, then an element $x \in \mathcal{A}$ is Γ-closed if $\Delta x \cdot \Gamma = \varnothing$. Alternatively, x is a Γ-cylinder.*

$\mathrm{Cl}_\Gamma \mathcal{A}$ *is the set of all* Γ*-closed elements.*

Metamathematically, Γ-closed elements reflect universal closures (is it correct?).

4.2 Neat reducts

Let $\mathcal{C} = \langle C, +, -, 0, 1, \{d_{ij}\}_{i,j<\beta}, \{c\}_{c<\beta}\rangle$ be a β-dimensional cylindic algebra and $\alpha \leqslant \beta$ an ordinal. The *α-th reduct* of $\mathcal{C}$, denoted as $\mathfrak{Rd}_\alpha \mathcal{C}$, is an algebra having the form

$$\mathfrak{Rd}_\alpha \mathcal{C} = \langle C, +, -, 0, 1, \{d_{ij}\}_{i,j<\alpha}, \{c\}_{c<\alpha}\rangle$$

$\mathcal{B}$ is a subreduct of $\mathcal{C}$, denoted as $\mathcal{B} \subseteq^r \mathcal{C}$, if $\mathcal{B} \subseteq \mathfrak{Rd}_\gamma \mathcal{C}$ for some $\gamma \leqslant \beta$.

Definition 14. *Let $\mathcal{C}$ be a β-dimensional cylindic algebra and α an ordinal such that $\alpha \leqslant \beta$. The neat α-reduct of $\mathcal{C}$, denoted as $\mathfrak{Nr}_\alpha \mathcal{C}$, is the subalgbera $\mathcal{A}$ of $\mathfrak{Rd}_\alpha \mathcal{C}$ with $\mathcal{A} = \mathrm{Cl}_\kappa \mathcal{C}$ where $\alpha + \kappa = \beta$.*

Let $\mathbb{K}$ be a class of β-dimensional cylindic algebras, then we put

$$\mathbf{Nr}_\alpha \mathbb{K} = \{\mathfrak{Nr}_\alpha \mathcal{C} \mid \mathcal{C} \in \mathbb{K}\}$$

An algebra $\mathcal{B}$ is a neat subreduct of $\mathcal{C}$, or $\mathcal{B}$ is neatly embeddable to $\mathcal{C}$ if there exists an ordinal $\gamma \leqslant \alpha$ such that $\mathcal{C} \subseteq \mathfrak{Rd}_\gamma \mathcal{B}$.

One may define neat reducts alternatively as follows. Let $\mathcal{C}$ be a β-dimensional cylindic algebra and α an ordinal such that $\alpha \leqslant \beta$. The neat α-reduct of $\mathcal{C}$ is the α-dimensional cylindric algebra having the form

$$\mathfrak{Nr}_\alpha \mathcal{C} = \langle \{a \in \mathcal{C} \mid \forall j (\alpha \leqslant j \;\&\; j < \beta \Rightarrow c_j a = a)\}, +, -, 0, 1, \{d_{ij}\}_{i,j<\alpha}, \{c_\gamma\}_\gamma\rangle$$

5 Completely representable cylindric algebras of dimension ω

Definition 15. *Let $\mathcal{A}$ be a BAO of type $\mathbf{CA}_\omega$, an $\mathcal{A}$-pre-network is a pair $\mathcal{N} = \langle N, l\rangle$, where N is a set of nodes and $l : {}^\omega N \to \mathrm{At}(\mathcal{A})$.*

$\mathcal{N}$ is a network, if the following conditions hold, for all $x, y \in {}^\omega N$ and $i, j < \omega$:

1. *$l(x) \leqslant d_{ij}$ iff $x_i = x_j$*
2. *$x \equiv_i y$ implies $l(x) \leqslant c_i l(y)$*

Let $\mathcal{N}_1 = \langle N_1, l_1\rangle$ and $\mathcal{N}_2 = \langle N_2, l_2\rangle$ be networks, then $\mathcal{N}_1 \subseteq \mathcal{N}_2$ if $N_1 \subseteq N_2$ and $l_1 = l_2 \restriction_{N_1}$.

Let $\Lambda \in \mathrm{Lim}$ and $\{\mathcal{N}_\lambda\}_{\lambda<\Lambda}$ a sequence of networks such that

$$\langle N_0, l_0\rangle \subseteq \langle N_1, l_1\rangle \subseteq \ldots \langle N_\lambda, l_\lambda\rangle \subseteq \ldots \text{ for } \lambda < \Lambda$$

then *the limit* of the sequence $\{\mathcal{N}_\lambda\}_{\lambda<\Lambda}$ is the network

$$\mathcal{N} = \langle N, l\rangle = \bigcup_{\lambda<\Lambda} \langle N_\lambda, l_\lambda\rangle$$

with nodes $N = \bigcup_{\lambda<\Lambda} N_\lambda$ and labelling $l = \bigcup_{\lambda<\Lambda} l_\lambda$, that is, for any $\lambda \in \Lambda$ and $x \in {}^\omega N$ one has $l(x) = l_\lambda(x)$.

The elements of ${}^\omega N$ are called *ω-dimensional hyperedges* of a network. One may identify a complete representation of an atomic cylindric-type algebra $\mathcal{A}$ with a set $\{\mathcal{N}_a \mid a \in \mathrm{At}(\mathcal{A})\}$ of $\mathcal{A}$-networks with the following additional condition:

- For each $a \in \mathrm{At}(\mathcal{A})$ there exists $x \in {}^\omega N_a$ such that $l_a(x) = a$ and for each $z \in {}^\omega N_a$ and $b \in \mathrm{At}(\mathcal{A})$, $i < \omega$ with $l_a(z) \leqslant c_i b$ there exists $y \in {}^\omega N_a$ such that $z \equiv_i y$ and $l_a(y) = b$.

We define a complete representation h of a cylindric-type algebra $\mathcal{A}$ as follows, for any $b \in \mathcal{A}$:

$$h(b) = \{x \mid \exists a \in \mathrm{At}(\mathcal{A}), x \in {}^\omega N_a, l_a(x) \leqslant b\}$$

Let us define an atomic game.

Definition 16. *Let $n \leqslant \omega$ an $\mathcal{A}$ be an atomic BAO of type $\mathbf{CA}_\omega$ that has countably atoms. The game $\mathcal{G}_n(\mathcal{A})$ is defined as follows. The game has two players: $\forall$ (Abelard, he/his) and $\exists$ (Héloïse, she/her). A play of the game $\mathcal{G}_n(\mathcal{A})$ is the sequence of networks*

$$\mathcal{N}_0 \subseteq \mathcal{N}_1 \subseteq \mathcal{N}_2 \subseteq \cdots \subseteq \mathcal{N}_k \subseteq \ldots \text{ for } k < n$$

The game consists of the following stages:

1. *(**Zero round**)*

   $\forall$ picks an atom $a \in \mathrm{At}(\mathcal{A})$ and $\exists$ plays a network $\mathcal{N}_0$. If there is no $x \in {}^\omega N_0$ such that $l_0(x) = a$, then $\forall$ wins the play.

2. *(**Successor round**)*

   Let $k < n$ such that $k + 1 < \kappa$ and a network $\mathcal{N}_\lambda = \langle N_k, l_k \rangle$ has been already played.

   $\forall$ picks $i \leqslant k$, $x \in {}^\omega N_k$, $a \in \mathrm{At}(\mathcal{A})$ such that $l_k(x) \leqslant c_i a$. We denote this move as (i, x, a). $\exists$ responds with a network $\mathcal{N}_{k+1} \supseteq \mathcal{N}_k$. $\forall$ wins, if there is no node $c \in N$ such that $l_{\lambda+1}(x[i/c]) = a$, then $\forall$ wins

3. *The limit of the play is defined as $\bigcup_{k<n} \mathcal{N}_k$. $\forall$ wins the play, if there exists $m < n$ such that $\exists$ does not win the m-th round. Otherwise, $\exists$ wins the play.*

Theorem 3. *Let $\mathcal{A}$ be an atomic ω-dimensional cylindric-type algebra and κ a cardinal such that $|\mathrm{At}(\mathcal{A})| = \kappa$, then the following are equivalent:*

1. *$\mathcal{A}$ is completely representable.*

2. *$\exists$ has a winning strategy in $\mathcal{G}_{\kappa+\omega}$.*

Proof.

1. $\Rightarrow$ If $\mathcal{A}$ is completely representable, then its Boolean reduct is completely representable as well by Theorem 1. $\exists$ maintains that embedding to win the play. TODO: write down this proof in more detail

2. $\Leftarrow$

   Suppose $\exists$ has a winning strategy in $\mathcal{G}_{\kappa+\omega}(\mathcal{A})$. In every round $\forall$ picks all possible $i < \omega$, $a \in \mathrm{At}(\mathcal{A})$, all possible hyperedges and all appropriate atoms and $\exists$ has a proper response for every $\forall$'s move.

   For each atom consider a play of the game with fewer than $\kappa+\omega$ nodes. For each $a \in \mathrm{At}(\mathcal{A})$ we associate a network $\mathcal{N}_a$, the resulting network of a corresponding game. Consider the set $\{\mathcal{N}_a \mid a \in \mathrm{At}(\mathcal{A})\}$.

   Let a be an atom, consider the network $\mathcal{N}_a = \langle V, l_a \rangle$. If there was not $x \in V$ such that $l_a(x) = a$, then $\forall$ would have a winning strategy, but that is not true, such an x does exist. The second item of this criterion follows from the presence of a winning strategy for $\exists$ as well.

   So we define a map *rep*:

$$\mathrm{rep}(a) = \{x \mid \exists b \in \mathrm{At}(\mathcal{A})\ x \in {}^{\omega}N_a, l_a(x) \leqslant b\}.$$

   We check that *rep* preserves cylindrifications and diagonal elements. Let $i, j < \omega$ and $a \in \mathcal{A}$:

   (a) Suppose $x \in \mathrm{rep}(c_i a)$, then there exists an atom b such that $x \in {}^{\omega}N_b$ with $l_b(x) \leqslant c_i a$. Then there exists $y \equiv_i x$ with $l_b(y) \leqslant a$, so $x \in \mathbf{C}_i(\mathrm{rep}(a))$.

   Let $x \in \mathbf{C}_i(\mathrm{rep}(a))$. We need $x \in (\mathrm{rep}(c_i a))$, that is, one needs to find an atom c such that $l_c(x) \leqslant c_i a$.

   We already know that there exists $y \equiv_i x$ such that $y \in \mathrm{rep}(a)$, that is, there exists an atom b such that $y \in {}^{\omega}N_b$ and $l_b(y) \leqslant a$.

   (b) If $x \in \mathrm{rep}(d_{ij})$, so there exists an atom b with $x \in {}^{\omega}N_a$ and $l_b(x) \leqslant d_{ij}$, then $x_i = x_j$, then $x \in D_{ij}$.

□

If $\mathcal{A} \in \mathbf{CA}_n$ is atomic, then we have the following criterion of complete representability formulated in terms of back-and-forth games, see [7, Theorem 3.3.3 and Corollary 3.3.5]. Here, the definition of a network for the case of $\mathbf{CA}_n$ is the same as Definition 15, but hyperedges have lenght n.

Theorem 4. *Let $\mathcal{A}$ be an atomic BAO of type $\mathbf{CA}_n$ such that $|\mathrm{At}(\mathcal{A})| = \kappa$, where κ is an arbitrary cardinal. Then the following are equivalent:*

1. *$\mathcal{A}$ is completely representable iff $\exists$ has a winning strategy in $\mathcal{G}_{\kappa+\omega}(\mathcal{A})$.*
2. *$\exists$ has a winning strategy in $\mathcal{G}_m(\mathcal{A})$ for every $m < \omega$ iff $\mathcal{A}$ is elementarily equivalent to a completely representable cylindric algebra.*

Now we formulate the Lyndon conditions for BAOs of type $\mathbf{CA}_\omega$, but we need their analogues for the finite case which are due to [5, Theorem 34]. Let $\mathcal{A}$ be an atomic BAO of type $\mathbf{CA}_n$. lc^n_k is a first-order sentence such that $\mathcal{A} \models lc^n_k$ iff $\exists$ has a winning strategy in the game $\mathcal{G}_k(\mathcal{A})$ of length $k < \omega$.

Consider the following set of formulas:

$$LC_\omega = \{lc^n_n \mid n < \omega\}$$

We shall think of LC_ω as the Lyndon conditions for atomic cylindric algebras of dimension ω.

Let us recall the definition of an ultraproduct, see [8, Section 9.5]. Let I be an index set, U an ultrafiler over I, an $(\mathcal{A}_i)_{i\in I}$ a family of structures of the same signature. The ultraproduct of $(\mathcal{A}_i)_{i\in I}$ modulo U, denoted as $\Pi_U\mathcal{A}_i$, is an algebra of the same signature whose elements are elements of the direct product $\Pi_{i\in I}\mathcal{A}_i$ factorised through an equivalence relation $\sim$ such that for every $a, b \in \Pi_{i\in I}\mathcal{A}_i$:

$$a \sim b \Leftrightarrow \{i \in I \mid a(i) = b(i)\} \in U$$

Lemma 1. *Let $\mathcal{A}$ be an atomic BAO of type $\mathbf{CA}_\omega$ such that for every $n < \omega$ $\mathcal{A}_n \models lc(n, n)$, then $\exists$ has a winning strategy in $\mathcal{G}_\omega(\Pi_U\mathcal{A})$, where $\Pi_U\mathcal{A}$ is a non-principal ultrapower of $\mathcal{A}$ and U is a non-principal ultrafilter over ω.*

Proof. HALP!!!!!!!! □

Theorem 5. *What's going on? :(*

Theorem 6. *Let $\mathcal{A}$ be a BAO of type $\mathbf{CA}_\omega$:*

1. *$\exists$ has a winning strategy in $\mathcal{G}_m(\mathcal{A})$ $(m < \omega)$, then $\exists$ has a winning strategy in $\mathcal{G}_\omega(\Pi_U\mathcal{A})$, where $\Pi_U\mathcal{A}$ is the non-principal ultrapower of $\mathcal{A}$ modulo U, an ultrafilter over ω.*

2. *$\exists$ has a winning strategy in $\mathcal{G}_m(\mathcal{A})$ (for every $m < \omega$) iff $\mathcal{A}$ is elementarily equivalent to a completely representable cylindric algebra of dimension ω.*

Proof.

The argument uses Łoś's Theorem, see [8, Theorem 9.5.1].

1.

2.

□

Theorem 7. *The elementary closure of the class of completely representable cylindric algebras of dimension ω is axiomatised with the axioms of $\mathbf{CA}_\omega$ and the Lyndon conditions.*

6 The result itself

Lemma 2. *Let $\mathcal{A}$ be a BAO of type $\mathbf{CA}_\alpha$ and $\mathcal{B}$ be a β-dimensional cylindric algebra such that $\beta \leqslant \alpha$ and $\mathcal{A}$ neatly embeds to $\mathcal{B}$ by a complete embedding.*

1. *$\mathcal{A}^+$ neatly embeds to $\mathcal{B}^+$ by a complete embedding.*

2. *$\mathcal{A}$ is atomic.*

Proof.

1. See [2, Remark 2.7.25].

2. Is it true?

□

Theorem 8 (This assumption is by Ian Hodkinson)**.**

Let $\mathcal{A}$ be a BAO of type $\mathbf{CA}_\omega$ such that $\mathcal{A}$ neatly embeds into $\mathbf{CA}_{\omega+\omega}$ by a complete embedding. Then $\mathcal{A}$ is completely representable as $\mathbf{CA}_\omega$.

Proof. Suppose $\mathcal{A} \subseteq \mathfrak{Nr}_\omega \mathcal{B}$, where $\mathcal{B} \in \mathbf{RCA}_{\omega+\omega}$ and the inclusion map $\rho : \mathcal{A} \hookrightarrow \mathfrak{Nr}_\omega \mathcal{B}$ is a complete embedding, that is:

$$\rho(\sum_{i \in I} a_i) = \sum_{i \in I} (\rho a_i), \text{ if } \sum_{i \in I} a_i \text{ exists.}$$

Let us show that $\mathcal{A}$ is atomic.

Consider $\rho(\mathcal{A})$. Let us show that $\exists$ has a winning strategy on $\mathcal{G}_{\kappa+\omega}(\rho(\mathcal{A}))$ $\square$

Lemma 2 and Theorem 8 imply the following theorem.

Theorem 9. *Let $\mathcal{C} \in \mathbf{RCA}_\omega$, then $\mathcal{C}^+ \in \mathbf{RCA}_\omega$. That is, $\mathbf{RCA}_\omega$ is closed under canonical extensions.*

Proof. $\square$

7 (Lack of) canonical axiomatisation of $\mathbf{CA}_\omega$

Here we are going to show that $\mathbf{CA}_\omega$ fails to have a canonical axiomatisation, the similar results for **RRA** and $\mathbf{RCA}_n$ for finite $n \geqslant 3$ have been shown by Hodkinson and Venema [9] and by Bulian and Hodkinson respectively [1].

8 Notes on the canonicity of RRA

Definition 17.

A relation algebra is an algebra $\mathcal{R} = \langle R, 0, 1, +, -, ;, \smile, \mathbf{1'} \rangle$ such that $\langle R, 0, 1, +, - \rangle$ is a Boolean algebra and the following equations hold, for each $a, b, c \in R$:

1. $a; (b; c) = (a; b); c$
2. $(a + b); c = (a; c) + (b; c)$
3. $a; \mathbf{1'} = a$
4. $a^{\smile\smile} = a$
5. $(a + b)^\smile = a^\smile + b^\smile$
6. $(a; b)^\smile = b^\smile; a^\smile$
7. $a^\smile; (-(a; b)) \leqslant -b$

where $a \leqslant b$ iff $a + b = b$. **RA** *denotes the class of all relation algebras.*

We will adapting the following proof of the fact that **RRA** is canonical [1] to our case. This proof is due to Monk, but that was describe in McKenzie's thesis [10].

1. A relation algebra $\mathcal{A}$ is representable iff $\mathcal{A}$ neatly embeds to some ω-dimensional cylinric algebra,

[1]This idea is by Ian Hodkinson

2. If $\mathcal{A}$ neatly embeds in $\mathcal{A}$ then $\mathcal{A}^+$ neatly embeds in $\mathcal{B}^+$,

3. $\mathbf{CA}_\alpha$ is closed under canonical extensions,

4. Voilá.

Definition 18. *Let $\mathcal{C} \in \mathbf{CA}_\alpha$, where $\alpha \geqslant 3$. The relation algebra reduct of $\mathcal{C}$, written as $\mathfrak{Ra}(\mathcal{C})$, is the algebra having the form*

$$\langle \mathrm{dom}(\mathfrak{Nr}_2(\mathcal{C})), 0, 1, +, -, \mathbf{1}', \smile, ; \rangle$$

where:

1. *$+$, $-$, 0, and 1 are defined as usual in $\mathcal{C}$,*

2. *$\mathbf{1}' = d_{01} \in \mathfrak{Nr}_2(\mathcal{C})$,*

3. *$r^\smile = s_0^2 s_1^0 s_2^1 r$ for $r \in \mathfrak{Nr}_2(\mathcal{C})$,*

4. *Let $r, s \in \mathfrak{Nr}_2(\mathcal{C})$, then $r; s = c_2(s_2^1 r \cdot s_2^0 s)$*

Moreover, $\mathfrak{Nr}_\beta(\mathcal{C})$ and $\mathfrak{Ra}(\mathcal{C})$ are closed under these operations. There is also the following fact by due to Henkin, Monk, and Tarski [3]:

Theorem 10. *Let $\mathcal{C} \in \mathbf{CA}_\alpha$ for $\alpha \geqslant 4$, then $\mathfrak{Ra}(\mathcal{C})$ is a relation algebra.*

The following characterisation results are by Henkin, Monk, and Tarski [3, 5.3.13, 5.3.16] as well:

Theorem 11.

1. *$\mathbf{RA} = \mathbf{S}\mathfrak{Ra}\mathbf{CA}_4$,*

2. *$\mathbf{RRA} = \bigcap_{3 \leqslant n < \omega} \mathbf{S}\mathfrak{Ra}\mathbf{CA}_n = \mathbf{S}\mathfrak{Ra}\mathbf{CA}_\alpha$, where α is an infinite ordinal.*

Let $\mathcal{C} \in \mathcal{CA}_\alpha$, then $\mathcal{R} \in \mathbf{RA}$ neatly embeds to $\mathcal{C}$, if $\mathcal{R}$ is isomorphic to some subalgebra of $\mathfrak{Ra}(\mathcal{C})$.

Theorem 12. **RRA** *is closed under canonical extensions.*

Proof. Let $\mathbf{R} \in \mathbf{RRA}$. By the second item of 11, every representable relation algebra is isomorphic to some subalgbera of the relation algebra reduct $\mathfrak{Ra}\mathcal{C}$ for some $\mathcal{C} \in \mathbf{CA}_\omega$. But neat embeddings repsect canonical extensions, so if $\mathbf{R} \hookrightarrow_n \mathcal{C}$, so is $\mathbf{R}^+ \hookrightarrow_n \mathcal{C}^+$. $\mathbf{CA}_\alpha$ is closed under canonical extensions, so is **RRA**. □

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