Finitely axiomatisable varieties generated by representable relation algebra reducts

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1 Questions

1. Is the variety $\mathbf{V}(\mathbf{R}(\cdot, ;, \mathbf{1}))$ finitely axiomatisable?
2. Is the variety $\mathbf{V}(\mathbf{R}(\cdot, +, ;, \mathbf{1}))$ finitely axiomatisable?

2 Definitions

A structure $\mathcal{M} = (M, \cdot, ;, \mathbf{1})$ is called a *lower-semilattice ordered monoid* if the following axioms hold:

- $(M, \cdot)$ is a meet-semilattice,
- $(M, ;, \mathbf{1})$ is a monoid,
- $(\mathbf{1} \cdot x);(\mathbf{1} \cdot y) = \mathbf{1} \cdot x \cdot y$,
- $(\mathbf{1} \cdot x);(y \cdot z) = (\mathbf{1} \cdot x); y \cdot z$,
- $(x \cdot y);(\mathbf{1} \cdot z) = x \cdot y;(\mathbf{1} \cdot z)$.

LSMod is the class (variety) of all lower-semilattice ordered monoids.

A lower-semilattice ordered monoid $\mathcal{M}$ is *representable* if there exists a map $f : \mathcal{M} \to 2^X$ for some $X \neq \varnothing$ such that:

- f is one-to-one
- $f(a;b) = f(a)|f(b)$
- $f(\mathbf{1}) = \mathbf{Id}$
- $f(a \cdot b) = f(a) \cap f(b)$

$\mathbf{R}(\cdot, ;, \mathbf{1})$ is the class of all representable lower-semilattice ordered monoids

2.1 Stone-style representation for lower-semilattices

Let $\mathcal{L}$ be a lower semilattice.

A subset $F \subseteq \mathcal{F}(\omega)$ is a filter if F is a downward closed meet-subsemilattice. F is principal if there is some $a \in F$ such that $F = \uparrow a$. Let $A \subseteq \mathcal{F}(\omega)$, then $\langle A \rangle = \cup_{a \in A} \uparrow a$, the filter generated by F.

A subset $I \subseteq \mathcal{L}$ is an *ideal* if it is upward closed and updirected, that is, $a, b \in I$ implies there exists some c such that $a, b \leqslant c$. It is known that F is a prime filter iff $L \backslash F$ is a prime ideal.

Theorem 1. *Let $\mathcal{L}$ be a lower semilattice. Then $\mathcal{L} \hookrightarrow (2^{\mathrm{Spec}(\mathcal{L})})$ whenever $\mathcal{L}$ is distributive.*

2.2 Finite axiomatisability

Theorem 2. $\mathbf{LSMod} = \mathbf{V}(\mathbf{R}(\cdot, ;, \mathbf{1}))$.

The right-to-left inclusion is obvious. To show the left-to-right inclusion, one needs to show that the free lower-semilattice ordered monoid with ω generators $\mathcal{F}(\omega)$ is representable.

A network is a structure $\mathcal{N} = (V, E, l)$, where V is a set of vertices, E is a set of edges and $l : E \to 2^{\mathcal{F}(\omega)}$ is a labelling function with the following data:

- Each $l(x, y)$ is a filter,
- $l(x, y); l(y, z) \subseteq l(y, z)$,
- $\mathbf{1}' \in l(x, y)$ implies $x = y$,
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We define a back-and-forth representability game $\mathcal{G}_\omega(\mathcal{F}(\omega))$ with two players $\forall$ and $\exists$.

3 The distributive lattice case

Theorem 3. *Let $\mathcal{L}$ be a distributive lattice, then $\mathcal{L} \hookrightarrow 2^{\mathrm{Spec}(\mathcal{L})}$.*

Given a distributive-lattice ordered monoid $\mathcal{M}$, its canonical extension is a structure $\mathcal{M}_+ = (\mathrm{Spec}(\mathcal{M}), \subseteq, R, E)$, where

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Definition 1. *Join-irreducibles...*

TODO: complete representation vs atomic representation vs representations via join-irreducibles
TODO: Birkhoff representation
TODO: Raney representation

Theorem 4. *$(\mathcal{F}_\omega)_+^+$ is completely representable.*

References

[BJ11] Guram Bezhanishvili and Ramon Jansana. Priestley style duality for distributive meet-semilattices. *Studia Logica*, 98:83–122, 2011.

[CG20] Sergio A Celani and Luciano J González. A categorical duality for semilattices and lattices. *Applied Categorical Structures*, 28:853–875, 2020.

[Mik17] Szabolcs Mikulas. Ordered monoids: Languages and relations. *arXiv preprint arXiv:1704.01391*, 2017.