Stuff related to tensor products of modal algebras

Daniel Rogozin

1 Preliminaries

The basic definitions according to [1].

Definition 1. *An n-modal algebra is an algebra* $\mathcal{B} = \langle B, \vee, \neg, \bot, \Diamond_1, \dots, \Diamond_n \rangle$*, where* $\mathcal{B} = \langle B, \vee, \neg, \bot \rangle$ *is a Boolean algebra and for each* $i = 1, \dots, n$ $\Diamond_i$ *is an order-preserving unary function obeying:*

1. $\Diamond_i p \vee \Diamond_i q = \Diamond_i (p \vee q)$
2. $\Diamond_i \bot = \bot$

Definition 2. *An n-normal modal logic is a set of formulas containing Boolean tautologies, formulas* $\Diamond_i p \vee \Diamond_i q \leftrightarrow \Diamond_i (p \vee q)$ *and* $\Diamond_i \bot \leftrightarrow \bot$ *for* $i = 1, \dots, n$*, and is closed under* **MP**, **Sub**, *and the monotonicity rule: from* $\varphi \to \psi$ *infer* $\Diamond_i \varphi \to \Diamond_i \psi$.

Definition 3.

1. *A Kripke n-frame is a structure* $\mathcal{F} = \langle W, R_1, \dots, R_n \rangle$*, where* $R_i \subseteq W \times W$ *for each* $i = 1, \dots, n$.
2. *Let* $\mathcal{F}$ *be a Kripke frame. A Kripke model over* $\mathcal{F}$ *is a tuple* $\mathcal{M} = \langle \mathcal{F}, \vartheta \rangle$*, where* $\vartheta : \mathrm{PV} \to 2^W$ *is a valuation. Modalised formulas have the following semantics:*

$$\mathcal{M}, x \models \Diamond_i \varphi \Leftrightarrow \exists y \in R_i(x)\ \mathcal{M}, y \models \varphi.$$

   A formula φ *is true in* $\mathcal{M}$ *iff* $||\varphi||_{\mathcal{M}} = W$. φ *is valid in* $\mathcal{F} = \langle W, R_1, \dots, R_n \rangle$ *iff* $||\varphi||_{\mathcal{M}} = W$ *for every model* $\mathcal{M}$ *over* $\mathcal{F}$.

3. *The complex algebra of a Kripke frame* $\mathcal{F} = \langle W, R_1, \dots, R_n \rangle$ *is a complete modal algebra defined as* $\mathcal{F}^+ = \langle \mathcal{P}(W), \cup, \neg, \varnothing, R_1^{-1}, \dots, R_n^{-1} \rangle$.
4. *A general n-frame is a structure* $\mathcal{F} = \langle W, R_1, \dots, R_n, \mathcal{A} \rangle$*, where* $\mathcal{A}$ *is a subalgebra of* $\langle W, R_1, \dots, R_n \rangle^+$.

Definition 4. *Let* $\mathcal{F}$ *be an n-frame, then* $\mathrm{Log}(\mathcal{F}) = \{\varphi \mid \mathcal{F} \models \varphi\}$*. If* $\mathbb{F}$ *is a class of n-frames, then* $\mathrm{Log}(\mathbb{F}) = \bigcap\limits_{\mathcal{F} \in \mathbb{F}} \mathrm{Log}(\mathcal{F})$*. We use the same notation for logics of general frames and modal algebras.*

We discuss the background on products of modal logics and products of Kripke frames [8] [9].

Definition 5. *Let* $\mathcal{F}_1 = \langle W_1, R_1^1, \dots, R_1^n \rangle$ *be an n-frame and* $\mathcal{F}_2 = \langle W_2, R_2^1, \dots, R_2^m \rangle$*. The product frame of* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *is an* $n + m$*-frame of the form*

$$\mathcal{F}_1 \times \mathcal{F}_2 = \langle W_1 \times W_2, R_h^1, \ldots, R_h^n, R_v^1, \ldots, R_v^m \rangle$$

such that for all $u_1, u_2 \in W_1$ and for all $v_1, v_2 \in W_2$,

$$\langle u_1, v_1 \rangle R_h^i \langle u_2, v_2 \rangle \text{ iff } u_1 R_1^i u_2 \text{ and } v_1 = v_2 \text{ for } 1 \leqslant i \leqslant n.$$
$$\langle u_1, v_1 \rangle R_v^j \langle u_2, v_2 \rangle \text{ iff } u_1 = u_2 \text{ and } u_1 R_2^j u_2 \text{ for } 1 \leqslant j \leqslant m.$$

This operation on Kripke frames commutes with disjoint unions, p-morphic images, and generated subframes.

Let $\mathcal{L}_1$ be a normal n-modal logic and $\mathcal{L}_2$ a normal m-modal logic, the product of $\mathcal{L}_1$ and $\mathcal{L}_2$ is defined as

$$\mathcal{L}_1 \times \mathcal{L}_2 = \mathrm{Log}(\mathrm{Frames}(\mathcal{L}_1) \times \mathrm{Frames}(\mathcal{L}_2))$$

Proposition 1.

1. *Let $\mathcal{L}_1$, $\mathcal{L}_2$ be modal logics, then $\mathcal{L}_1 * \mathcal{L}_2 \subseteq \mathcal{L}_1 \times \mathcal{L}_2$.*
2. *Let $\mathcal{L}_1$, $\mathcal{L}_2$ be Kripke complete modal logics, then $\mathcal{L}_1 \times \mathcal{L}_2 = \mathrm{Log}(\mathrm{Frames}_r(\mathcal{L}_1) \times \mathrm{Frames}_r(\mathcal{L}_2))$, where $\mathrm{Frames}_r(\mathcal{L}_i) = \{\mathcal{F} \in \mathrm{Frames}(\mathcal{L}_i) \mid \mathcal{F} \text{ is rooted } \}$ for $i = 1, 2$.*

1.1 Axiomatising products

The following properties hold for a product frame having the form $\mathcal{F} = \langle W, R_1, \ldots, R_n \rangle$:

1. (left commutativity) $\forall x, y, z \in W \ (xR_j y \,\&\, yR_i z \Rightarrow \exists u \in W \ (xR_i u \,\&\, uR_j z))$
2. (right commutativity) $\forall x, y, z \in W \ (xR_i y \,\&\, yR_j z \Rightarrow \exists u \in W \ (xR_j u \,\&\, uR_i z))$
3. (Confluence) $\forall x, y, z \in W (xR_j y \,\&\, xR_i z \Rightarrow \exists u \in W (yR_i u \,\&\, zR_j u))$

The properties are expressed as modal formulas as well:

1. $\mathbf{comm}^{\mathbf{l}}_{ij} = \Diamond_j \Diamond_i p \to \Diamond_i \Diamond_j p$
2. $\mathbf{comm}^{\mathbf{r}}_{ij} = \Diamond_i \Diamond_j p \to \Diamond_j \Diamond_i p$
3. $\mathbf{cr}_{ij} = \Diamond_i \Box_j p \to \Box_i \Diamond_j p$

Definition 6. *Given unimodal modals logics $\mathcal{L}_1, \ldots, \mathcal{L}_n$, the commutator*

$$[\mathcal{L}_1, \ldots, \mathcal{L}_n]$$

is the smallest n-modal logic containing $\mathcal{L}_i$ and axioms $\mathbf{comm}^{\mathbf{l}}_{ij}$, $\mathbf{comm}^{\mathbf{r}}_{ij}$, and $\mathbf{cr}_{ij}$ for $i, j \in \{1, \ldots, n\}$ with $i \neq j$.

Since $\mathbf{comm}^{\mathbf{l}}_{ij}$, $\mathbf{comm}^{\mathbf{r}}_{ij}$, and $\mathbf{comm}^{\mathbf{r}}_{ij}$ are Salqvist formulas, one has

Proposition 2. *If $\mathcal{L}_1, \ldots, \mathcal{L}_n$ are canonical, then $[\mathcal{L}_1, \ldots, \mathcal{L}_n]$ is canonical, and thus, elementary and Kripke complete.*

Moreover, one has

Proposition 3. $[\mathcal{L}_1, \ldots, \mathcal{L}_n] \subseteq \mathcal{L}_1 \times \cdots \times \mathcal{L}_n$.

If the converse inclusion holds, then $\mathcal{L}_1, \ldots, \mathcal{L}_n$ are *product-mathcing*, see [3]. The examples of product-matching logics are Horn axiomatisable ones [9]. In particular, the following equality holds for Kripke complete and Horn axiomatisable logics $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$:

$$\mathcal{L}_1 \times \mathcal{L}_2 \times \mathcal{L}_3 = (\mathcal{L}_1 \times \mathcal{L}_2) \times \mathcal{L}_3 = \mathcal{L}_1 \times (\mathcal{L}_2 \times \mathcal{L}_3)$$

Generally, this is an open question whether the product of modal logics is associative.

2 Tensor products of modal algebras. The basic definitions and results

Definition 7. *A commutative associative algebra*

Definition 8. *Tensor product of them*

Definition 9. *Tensor product of modal algebras*

Definition 10. *Tensor product of general frames*

TODO: describe related underlying results

3 Note on incomplete modal logics

TODO: consider the system containing **GL**, McKinsey, seriality, and linearity.

4 Solution of Problem 1

5 Solution of Problem 4

Definition 11. *A finitely axiomatisable normal modal logic*

References

[1] A. Chagrov and M. Zakharyaschev. *Modal Logic.* Oxford logic guides. Clarendon Press, 1997.

[2] Dov Gabbay, Ilya Shapirovsky, and Valentin Shehtman. Products of modal logics and tensor products of modal algebras. *Journal of Applied Logic*, 12(4):570–583, 2014.

[3] Dov M Gabbay and Valentin B Shehtman. Products of modal logics, part 1. *Logic journal of IGPL*, 6(1):73–146, 1998.

[4] Robert Goldblatt. Logics of time and computation. csli, 1987.

[5] Robert Goldblatt. Varieties of complex algebras. *Annals of pure and applied logic*, 44(3):173–242, 1989.

[6] Christopher Hampson, Stanislav Kikot, Agi Kurucz, and Sérgio Marcelino. Non-finitely axiomatisable modal product logics with infinite canonical axiomatisations. *Annals of Pure and Applied Logic*, page 102786, 2020.

[7] Robin Hirsch and Ian Hodkinson. Step by step–building representations in algebraic logic. *The Journal of Symbolic Logic*, 62(1):225–279, 1997.

[8] Agi Kurucz. Combining modal logics. *Handbook of modal logic*, 3:869–924, 2007.

[9] Agi Kurucz, Frank Wolter, Michael Zakharyaschev, and Dov M Gabbay. *Many-dimensional modal logics: Theory and applications.* Elsevier, 2003.