Characterising representable positive relation algebras via Priestley duality

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1 Distributive lattice representation and priestley duality

Lemma 1. *Let $h : \mathcal{L} \to \mathcal{R}$ be a representation, then then*

$$h^{-1}[x] = \{a \in \mathcal{L} \mid x \in h(a)\}$$

is prime filter in $\mathcal{L}$.

Proof. Let $c \in h^{-1}[x]$ and $c \leqslant d$, then $x \in h(c)$, but h is order-preserving, so $x \in h(d)$. If $c, d \in h^{-1}[x]$, then $x \in h(c)$ and $x \in h(d)$, so $x \in h(c) \cap h(d)$, then $x \in h(c \cdot d)$, so $c \cdot d \in h^{-1}[x]$. Let $c + d \in h^{-1}[x]$, then $x \in h(c + d) = h(c) \cup h(d)$, so either $x \in h(c)$ or $x \in h(d)$, so either $c \in h^{-1}[x]$ or $d \in h^{-1}[x]$. □

2 Positive relation algebras and their representatition

3 Spectral spaces for positive relation algebras

4 Complete representability

An element a of a lattice $\mathcal{L}$ is *completely join-irreducible* if for every $A \subseteq \mathcal{L}$ such that $a \leqslant \bigvee A$ there exists $b \in A$ such that $a \leqslant b$.

A distributive lattice $\mathcal{L}$ is *completely representable* if there exists a representation $h : \mathcal{L} \to \mathcal{R}$ such that

$$\begin{aligned} f(\Sigma A) &= \bigcup\{f(a) \mid a \in A\} \\ f(\Pi A) &= \bigcap\{f(a) \mid a \in A\} \end{aligned}$$

for each $A \subseteq \mathcal{L}$ such that ΣA and ΠA exist.

A representation $h : \mathcal{L} \to \mathcal{R}$ is *refined* if for all $a \in \mathcal{L}$ there exists a completely join-irreducible j such that $a \in h(j)$.

Lemma 2. *Let $h : \mathcal{L} \to \mathcal{R}$ be a* refined *representation, then each $h^{-1}[x]$ is principal and is generated by completely join-irreducible element.*

Theorem 1. *Let $\mathcal{L}$ be a distributive lattice and let $h : \mathcal{L} \to \mathcal{R}$ be representation of $\mathcal{L}$ over the base set X, then h is refined iff it is complete.*

Proof. Suppose h is refined. Take any $b \in \mathcal{L}$, then

$$h(b) = \bigcup\{h(j) \mid j \in \mathcal{JR}(\mathcal{L}) \,\&\, j \leqslant a\}$$

Take $A \subseteq \mathcal{L}$ such that ΣA and ΠA exist.
TODO:
For converse, suppose h is complete $\square$

4.1 Completely representable distibutive lattices

4.2 Completely representable positive relation algebras

5 The main result

Theorem 2. *Let $\mathcal{A}$ be a positive relation algebra, then $\mathcal{R}$ is representable iff $(\mathcal{R}_+)^+$ is completely representable.*

Theorem 3. **RPRA** *is a canonical variety.*