

Notes on filtration of logics containing **K5**

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1 Preliminaries

Definition 1. A normal modal logic is a set of formulas that contains all Boolean tautologies, formulas $\Diamond p \vee \Diamond q \leftrightarrow \Diamond(p \vee q)$ and $\Diamond \perp \leftrightarrow \perp$, and is closed under Modus Ponens, Substitution, and Monotonicity: from $\varphi \rightarrow \psi$ infer $\Diamond \varphi \rightarrow \Diamond \psi$.

Definition 2. An Kripke model is a triple $\mathcal{M} = \langle W, R, \vartheta \rangle$, where $R \subseteq W \times W$, $\vartheta : PV \rightarrow 2^W$, and the connectives have the following semantics:

1. $\mathcal{M}, w \models p \Leftrightarrow w \in \vartheta(p)$
2. $\mathcal{M}, w \models \neg \varphi \Leftrightarrow \mathcal{M}, w \not\models \varphi$
3. $\mathcal{M}, w \models \varphi \vee \psi \Leftrightarrow \mathcal{M}, w \models \varphi$ or $\mathcal{M}, w \models \psi$
4. $\mathcal{M}, w \models \Diamond \varphi \Leftrightarrow \exists v \in R(w) \mathcal{M}, v \models \varphi$

Definition 3. Let $\mathcal{M} = \langle W, R, \vartheta \rangle$ be a Kripke model. A Kripke model $\mathcal{M}' = \langle W', R', \vartheta' \rangle$ is a generated submodel of \mathcal{M} , where:

1. $\emptyset \neq W' \subseteq W$ and W' is R -closed, i.e., for each $u \in W'$ and $v \in W$, uRv implies $v \in W'$
2. $R' = R \cap W' \times W'$
3. for every propositional variable p , $\vartheta'(p) = \vartheta(p) \cap W'$

Fact 1. Let $\mathcal{M} = \langle W, R, \vartheta \rangle$ be a Kripke model and $\mathcal{M}' = \langle W', R', \vartheta' \rangle$ its generated submodel, then for each $w \in W'$

$$\mathcal{M}, w \models \varphi \text{ iff } \mathcal{M}', w \models \varphi$$

By **K5** we mean the logic $\mathbf{K} \oplus A5$, where $A5 = \Diamond p \rightarrow \Box \Diamond p$. It is known that **K5** is the modal logic of all Euclidean frames. A frame is called Euclidean if for each x, y, z , xRy and xRz implies yRz .

Proposition 1. **K5** proves

1. $\Box^3 p \leftrightarrow \Box^2 p$
2. $\Box^2 \Diamond p \leftrightarrow \Box \Diamond p$
3. $\Box \Diamond \Box p \leftrightarrow \Box \Box p$
4. $\Box \Diamond^2 p \leftrightarrow \Box \Diamond p$

Proposition 2. Let \mathcal{M} be a **K5** model, xRy for $x, y \in W$ then one has

$$\mathcal{M}, x \models \Box \Diamond \varphi \text{ iff } \mathcal{M}, y \models \Box \Diamond \varphi.$$

Proof.

1. Suppose $\mathcal{M}, x \models \Box \Diamond \varphi$. Then $\mathcal{M}, y \models \Diamond \varphi$ and $\mathcal{M}, y \models \Box \Diamond \varphi$
2. Suppose $\mathcal{M}, y \models \Box \Diamond \varphi$, then $\mathcal{M}, x \models \Diamond \Box \Diamond \varphi$, so $\mathcal{M}, x \models \Box \Diamond \varphi$.

□

1.1 Filtrations: general definitions

Let $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$ be a Kripke model and Γ a set of formulas closed under subformulas. An equivalence relation \sim is set to have a finite index if the quotient set W / \sim is finite. The equivalence relation \sim_Γ induced by Γ is defined as

$$w \sim_\Gamma v \Leftrightarrow \forall \varphi \in \Gamma (\mathcal{M}, w \models \varphi \Leftrightarrow \mathcal{M}, v \models \varphi).$$

If Γ is finite, then \sim_Γ has a finite index. An equivalence relation \sim respects \sim_Γ , if $w \sim v$ implies $w \sim_\Gamma v$.

Definition 4. Let $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$ be a Kripke model and Γ be a Sub-closed set formulas. A Γ -filtration of \mathcal{M} is a model $\widehat{\mathcal{M}} = \langle \widehat{W}, \widehat{R}_1, \dots, \widehat{R}_n, \widehat{\vartheta} \rangle$ such that:

1. $\widehat{W} = W / \sim$, where \sim is an equivalence relation having a finite index that respects Γ
2. $\widehat{\vartheta}(p) = \{[x]_\sim \mid x \in W \ \& \ x \in \vartheta(p)\}$
3. For each $i \in I$ one has $\widehat{R}_i^{\min} \subseteq \widehat{R}_i \subseteq \widehat{R}_i^{\max}$. $\widehat{R}_{i,\sim}^{\min}$ is the i -th minimal filtered relation on \widehat{W} defined as

$$\widehat{x} \widehat{R}_{i,\sim}^{\min} \widehat{y} \Leftrightarrow \exists x' \sim x \exists y' \sim y x R_i y$$

$\widehat{R}_{\Gamma,i}^{\max}$ is the i -th maximal filtered relation on \widehat{W} induced by Γ defined as

$$\widehat{x} \widehat{R}_{\Gamma,i}^{\max} \widehat{y} \Leftrightarrow \forall \Box_i \varphi \in \Gamma (\mathcal{M}, x \models \Box_i \varphi \Rightarrow \mathcal{M}, y \models \varphi)$$

If Φ is finite subset of Γ and $\sim = \sim_\Phi$, then $\widehat{\mathcal{M}}$ is a definable Γ -filtration of \mathcal{M} through Φ . If $\sim = \sim_\Gamma$, then such a filtration by means of the definition above is called *strict*.

Lemma 1. Let Γ be a finite set of formulas closed under subformulas and $\widehat{\mathcal{M}}$ a filtration of \mathcal{M} through Γ , then for each $x \in W$ and for each $\varphi \in \Gamma$ one has

$$\mathcal{M}, x \models \varphi \Leftrightarrow \widehat{\mathcal{M}}, \widehat{x} \models \varphi$$

Definition 5. Let \mathbb{F} be a class of Kripke frames and Γ a finite set of formulas closed under subformulas. If for every model \mathcal{M} over $\mathcal{F} \in \mathbb{F}$ there exists a model that is a Γ -definable filtration of \mathcal{M} , then \mathbb{F} admits definable filtration. A class of models \mathbb{M} admits definable filtration if for every $\mathcal{M} \in \mathbb{M}$ there exists a model belonging to the same class that is a definable Γ -filtration of \mathcal{M} .

Lemma 2.

1. Let \mathcal{L} be a complete normal modal logic. If $\text{Frames}(\mathcal{L})$ admits filtration, then \mathcal{L} has the finite model property.
2. If the class of models $\text{Mod}(\mathcal{L})$ admits filtration, then \mathcal{L} has the finite model property and Kripke complete as well.

2 Filtration of Euclidean logics

First of all, let us ensure that a minimal filtration of an Euclidean frame is not necessary Euclidean. Let $[x] \sim_\Gamma [y]$ and $[x] \sim_\Gamma [z]$. Then for some $x' \in [x]$ $y' \in [y]$, one has $x'Ry'$ and $x''Rz'$ for some $x'' \in [x]$ and $z' \in [z]$. Clearly, we cannot claim that $x' = x''$ in general. Thus, minimal filtration does not preserve the required property.

Lemma 3. *K5 admit filtration.*

Proof. Let \mathcal{M} be a **K5**-model and Γ_0 a finite set of formulas closed under subformulas. Let us put $\Gamma = \Gamma_0 \cup \text{Sub}(\{\Diamond\Box\psi \mid \Box\psi \in \Gamma_0\}) \cup \Psi$, where $\Psi = \nabla_1\nabla_2\ldots\nabla_n\Box\psi$ for $\Box\psi \in \Gamma_0$ and $\nabla_i \in \{\Diamond, \Box\}$. By Proposition 1, any element of Φ has one of the four forms. Thus, $W \sim_{\equiv_\Gamma}$ has a finite index. We put $\hat{R} = R_\Gamma^{\max}$. \square

Definition 6. *A first-order formula is called Horn if it has the following form:*

$$\forall x_1, \dots, x_n (x_{i_1}Rx_{j_1} \wedge \dots \wedge x_{i_s}Rx_{j_s} \rightarrow x_kRx_l)$$

Definition 7. *Let H be a Horn property and $\langle W, R \rangle$ a Kripke frame. A Horn closure of a binary relation R is the minimal relation R^H containing R and satisfying H .*

Lemma 4. $R^H = \bigcup_{n < \omega} R_n$ where

1. $R_0 = R$.
2. $R_{n+1} = R_n \cup \{(a, b) \in W \mid \exists \vec{c} \in W \text{ } P(a, b, \vec{c})\}$, where P is a premise of H .

E -closure (an Euclidean Horn closure of a binary relation) has the following equivalent definitions:

Lemma 5. *Let $\mathcal{F} = \langle W, R \rangle$ be a Kripke frame. The following conditions are equivalent:*

1. R^E is the smallest Euclidean relation containing R .
2. $R^E = \bigcup_{i < \omega} R_i$, where
 - $R_0 = R$
 - $R_{n+1} = R_n \cup (R_n^{-1} \circ R_n)$
3. xR^Ey iff there exists $n < \omega$ such that either xRy or $\exists z_1, \dots, z_n$ with z_1Rx and $z_{n-1}Ry$ and for each $1 < i \leq n$ one has either $z_{i-1}Rz_i$ or z_iRz_{i-1} .
4. $R^E = R \cup \bigcup_{i < \omega} (R^{-1} \circ (R \circ R^{-1})^n \circ R)$.

Proof.

1. (1) \Rightarrow (2) Let us show that if R^E is the smallest Euclidean relation containing R , then $R^E = \bigcup_{i < \omega} R_i$. There are two inclusions:

- $R^E \subseteq \bigcup_{i < \omega} R_i$. Recall that R^E has the form (?):

$$R^E = \bigcap \{R' \mid R \subseteq R', \forall a, b \in W \text{ } R'(a, b) \Rightarrow \exists x \in W \text{ } R'(x, a) \ \& \ R'(x, b)\}$$

- $\bigcup_{i < \omega} R_i \subseteq R^E$. Let us show that $xR_n y$ for each $n < \omega$ implies $xR^E y$ by induction on n .
 If $n = 0$, then xRy , thus, $xR^E y$, since R is a subrelation of R^E . Suppose $n = m + 1$ and $xR_{m+1} y$. Let us show that $xR^E y$. From $xR_{m+1} y$, one has $(x, y) \in R^n \cup (R_n^{-1} \circ R_n)$. There are two cases:
 - $xR^n y$, one needs to merely apply the IH.
 - $xR_n^{-1} \circ R_n y$. Then $\exists z \in W$ $xR_n^{-1} z$ & $zR_n y$. That is, $zR_n x$ and $zR_n y$ for some z . R_n is already a subrelation of R^E . Thus, $zR^E x$ and $zR^E y$. That implies $xR^E y$.
- 2. (2) \Rightarrow (3) Let $(x, y) \in R_m$, let us the statement by induction on m .
 - (a) Suppose $m = 0$, then xRy , and the statement is shown putting $n = 0$.
 - (b) Suppose $m = p + 1$ and $xR_{p+1} y$. Assume that either xRy or $\exists z_1, \dots, z_p$ with $z_1 R x$ and $z_{p-1} R y$ and for each $1 < i \leq p$ one has either $z_{i-1} R z_i$ or $z_i R z_{i-1}$.
 $xR_{p+1} y$ implies $(x, y) \in R_p \cup (R_p^{-1} \circ R_p)$. If $(x, y) \in R_p$, then we merely apply the IH.
 Suppose $(x, y) \in R_p^{-1} \circ R_p$, then $(z, x) \in R_p$ and $(z, y) \in R_p$.
- 3. (3) \Rightarrow (4) Suppose either xRy or there exist $n \geq 1$ and z_1, \dots, z_n with $z_1 R x$ and $z_{n-1} R y$ and for each $1 < i \leq n$ one has either $z_{i-1} R z_i$ or $z_i R z_{i-1}$. If xRy , then we are done. Otherwise there exists $n \geq 1$ with the condition above. Then $(x, y) \in R_{n+1}$ that follows from the condition.
- 4. (4) \Rightarrow (1)

□

Lemma 6. Let $\mathcal{F} = \langle W, R \rangle$ be a Kripke frame. Let us define $R^E = \bigcup_{i < \omega} R_i$ where:

1. $R_0 = R$
2. $R_{n+1} = R_n \cup (R_n^{-1} \circ R_n)$

Then R^E is Euclidean.

Proof. Let $(x, y), (x, z) \in R^E$, one needs to show that $(y, z) \in R^E$. Clearly that $(x, y) \in R_i$ and $(x, z) \in R_j$ for some $i, j < \omega$. Thus, we need $(y, z) \in R_m$ for some m depending on i and j .

Let us consider the following cases:

1. $i = 0$ and $j = 0$
 Suppose $(x, y), (x, z) \in R_0 = R$, then $(y, z) \in R^{-1} \circ R$. Thus, $(y, z) \in R_1$
2. $i = 0$ and $j = k + 1$
 Suppose $(x, y) \in R$ and $(x, z) \in R_{k+1} = R_k \cup (R_k^{-1} \circ R_k)$. Clearly that $(x, y) \in R_{k+1}$ as well. It is obviously that $(y, z) \in R_{k+2}$ since $(y, x) \in R_{k+1}^{-1}$ and $(x, z) \in R_{k+1}$.
3. The case with $i = k + 1$ and $j = 0$ is similar to the previous one.
4. Suppose $i = m + 1$ and $j = k + 1$. That is, $(x, y) \in R_{m+1} = R_m \cup (R_m^{-1} \circ R_m)$ and $(x, z) \in R_{k+1} = R_k \cup (R_k^{-1} \circ R_k)$. Consider the following four subcases:
 - (a) Suppose $(x, y) \in R_m$ and $(x, z) \in R_k$ and $m \leq k$ without loss of generality. $m \leq k$ implies $R_m \subseteq R_k$ and $(x, y) \in R_k$ in particular. Thus, $(y, z) \in R_k^{-1} \circ R_k$, so $(y, z) \in R_{k+1}$.

(b) The rest of the cases are similar to the first one.

□

Theorem 1. *K45 admits strict filtrations.*

Proof. Let $\mathcal{M} = \langle W, R, \vartheta \rangle$ be a transitive Euclidean model and $\overline{\mathcal{M}} = \langle \overline{W}, \overline{R}, \overline{\vartheta} \rangle$ its minimal filtration through Γ , where Γ is finite and Sub-closed. Let us put $\hat{R} = \overline{R}^+ \cup \overline{R}^E$. Let us show that $\overline{R}^+ \cup \overline{R}^E \subseteq \overline{R}^{max}$.

That is, if $\mathcal{M}, y \models \varphi$ for $\diamond\varphi \in \Gamma$ and $\hat{x}\hat{R}\hat{y}$, then $\mathcal{M}, x \models \diamond\varphi$.

Let $\hat{x}\hat{R}\hat{y}$. Let us consider the case when $(\hat{x}, \hat{y}) \in \overline{R}^E$.

1. Suppose $(\hat{x}, \hat{y}) \in \overline{R}$, then $\mathcal{M}, x \models \diamond\varphi$ holds trivially by the definition of the minimal filtration.
2. Suppose the statement holds \overline{R}_n and $(\hat{x}, \hat{y}) \in \overline{R}_{n+1} = \overline{R}_n \cup (\overline{R}_n^{-1} \circ \overline{R}_n)$. We consider the case of $(\hat{x}, \hat{y}) \in (\overline{R}_n^{-1} \circ \overline{R}_n)$.

Then there exists \hat{z} such that $(\hat{z}, \hat{x}), (\hat{z}, \hat{y}) \in \overline{R}_n$.

By IH, $\mathcal{M}, z \models \diamond\varphi$.

$(\hat{z}, \hat{y}) \in \overline{R}_n$ iff there are $\hat{u}_1, \dots, \hat{u}_n$ such that

$$\hat{z} \xleftarrow{\hat{R}} \hat{u}_1 \xrightarrow{\hat{R}'} \hat{u}_2 \xrightarrow{\hat{R}'} \dots \xrightarrow{\hat{R}'} \hat{u}_{n-1} \xrightarrow{\hat{R}'} \hat{u}_n \xrightarrow{\hat{R}} \hat{y}$$

where \hat{R}' is either \hat{R} or \hat{R}^{-1} .

As it is known, $\diamond\varphi \rightarrow \square\varphi \in \mathbf{K45}$.

$\hat{u}_1\hat{z}$, that is, $u'_1 R z'$ for some $u'_1 \in \hat{u}_1$ and $z' \in \hat{z}$. That is, $\mathcal{M}, u'_1 \models \diamond\varphi$, so $\mathcal{M}, u'_1 \models \diamond\varphi$ and $\overline{\mathcal{M}}, \hat{u}_1 \models \diamond\varphi$.

We have $\hat{u}_1\hat{R}'\hat{u}_2$. Suppose $\mathcal{M}, u'_1 \models \diamond\varphi$ and $u''_1 R u'_2$. We also have $\mathcal{M}, u'_1 \models \square\varphi$, thus, $\mathcal{M}, u'_2 \models \diamond\varphi$.

Suppose $\hat{u}_2\hat{R}\hat{u}_1$ and $u'_2 R u''_1$, then $\mathcal{M}, u'_2 \models \diamond\varphi$.

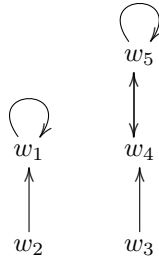
Similarly, we have $\mathcal{M}, u_i \models \diamond\varphi$ iff $\mathcal{M}, u_{i+1} \models \diamond\varphi$, whenever $\hat{u}_i\hat{R}'\hat{u}_{i+1}$.

Finally, we have $\hat{u}_n\hat{R}\hat{x}$. Thus, $u'_n R x'$ for some $u'_n \in \hat{u}_n$ and $x' \in \hat{x}$. $\mathcal{M}, u'_n \models \diamond\varphi$, so $\mathcal{M}, u'_n \models \square\varphi$. Then $\mathcal{M}, x' \models \diamond\varphi$. □

Theorem 2. *K5 does not admit strict filtrations.*

Proof. Let us consider a **K5** model whose Euclidean closure of the minimal filtration does not give us a filtration.

Let us consider a frame called \mathcal{F}_{bad} . We define this frame with the following graph:



□

Let us define a valuation ϑ such that $\vartheta(p) = \{w_5\}$ and $\vartheta(q) = \{w_1\}$. Let us consider a minimal filtration of \mathcal{M}_{bad} through the Sub-closure of $\Gamma = \{\neg p, \neg \Diamond p\}$.

Clearly that $w_2 \sim_\Gamma w_3$, since $\neg p$ and $\neg \Diamond p$ are true both at w_2 and w_3 .

Moreover, $R_{min} \cup (R_{min}^{-1} \circ R_{min})$ is not a subset of R_{max} since $(\hat{w}_1, \hat{w}_5) \in (R_{min}^{-1} \circ R_{min})$, but $\Diamond p$ is not true at w_5 .

Let us also note that strict filtrations of this model is not Euclidean. Suppose by contrary that $\hat{R}^\mathcal{E}$ is a strict filtration of that model. So $R_{min}^E \subseteq \hat{R}^\mathcal{E}$, since R_{min}^E is the minimal Euclidean relation containing R_{min} . On the other hand, $R_{min}^E \not\subseteq R_{max}$, so is not $\hat{R}^\mathcal{E}$.

3 Filtration for K4

Proposition 3. *Let R be a binary relation on $W \neq \emptyset$. Define $R^+ = \bigcup_{i < \omega} R_i$*

1. $R_0 = R$

2. $R_{n+1} = R_n \circ R$

Then R^+ is transitive

Lemma 7. *Let $\mathcal{M} = \langle W, R, \vartheta \rangle$ be a transitive model and $\overline{\mathcal{M}} = \langle \overline{W}, \overline{R}, \overline{\vartheta} \rangle$ its minimal filtration through a finite Sub-closed set of formulas Θ .*

Then $\overline{\mathcal{M}}^+ = \langle \overline{W}, (\overline{R})^+, \overline{\vartheta} \rangle$ is a Θ -filtration of \mathcal{M} .

Proof. $(\overline{R})^+$ obviously contains R . By the previous proposition, $(\overline{R})^+$ is transitive. Let us show that $(\overline{R})^+ \subseteq R_\Theta^{max}$.

Let $\hat{x}, \hat{y} \in \overline{W}$ with $\hat{x}(\overline{R})^+ \hat{y}$ and $\Box \varphi \in \Theta$ with $\mathcal{M}, x \models \Box \varphi$. Let us show that $\mathcal{M}, y \models \varphi$.

If $\hat{x}(\overline{R})^+ \hat{y}$, then there exist equivalence classes $\hat{x}_1, \dots, \hat{x}_n$ such that

$$\hat{x} \overline{R} \hat{x}_1 \overline{R} \dots \overline{R} \hat{x}_n \overline{R} \hat{y}$$

$\mathcal{M}, x \models \Box \varphi$ implies $\mathcal{M}, x \models \Box \Box \varphi$. Thus, $\overline{\mathcal{M}}, \hat{x} \models \Box \Box \varphi$.

$\hat{x} \overline{R} \hat{x}_1$, so there are $x_1 \in \hat{x}$ and $x_2 \in \hat{x}_1$ with $x_1 R x_2$. In particular, $\mathcal{M}, x_2 \models \Box \varphi$, so $\overline{\mathcal{M}}, \hat{x}_2 \models \Box \varphi$, and et cetera.

For each $i \in \{1, \dots, n\}$ we have $\mathcal{M}, x_i \models \Box \varphi$ which is shown inductively:

If $\mathcal{M}, x_i \models \Box \varphi$ for $x_i \in \hat{x}_i$, so $\mathcal{M}, x_i \models \Box \Box \varphi$, but there exist $x'_i \in \hat{x}_i$ and $x_{i+1} \in \hat{x}_{i+1}$, so $\mathcal{M}, x_{i+1} \models \Box \varphi$.

Finally, we have $\mathcal{M}, x_n \models \Box \varphi$ for $x_n \in \hat{x}_n$, but $\hat{x}_n \overline{R} \hat{y}$, so $\mathcal{M}, y' \models \varphi$ for each $y' \in \hat{y}$. Thus, φ is true at y as well. \square

Proof. Let $\hat{x}, \hat{y} \in \overline{W}$ with $\hat{x}(\overline{R})^+ \hat{y}$ and $\Box \varphi \in \Theta$ with $\mathcal{M}, x \models \Box \varphi$. Let us show that $\mathcal{M}, y \models \varphi$.

If $\hat{x}(\overline{R})^+ \hat{y}$, then there exist equivalence classes $\hat{x}_1, \dots, \hat{x}_n$ such that

$$\hat{x} \overline{R} \hat{x}_1 \overline{R} \dots \overline{R} \hat{x}_n \overline{R} \hat{y}$$

Let us show that $\mathcal{M}, \hat{x}_i \models \Box \varphi$ inductively:

1. $n = 1$ We have the following sequence:

$$\hat{x} \overline{R} \hat{x}_1 \overline{R} \hat{y}$$

$\hat{x} \overline{R} \hat{x}_1$, so there are $x' \in \hat{x}$ and $x'_1 \in \hat{x}_1$ such that $x' R x'_1$. $\Box \varphi$ is true at x' , so is $\Box \Box \varphi$. Then $\mathcal{M}, x'_1 \models \Box \varphi$ since $x'_1 \in R(x')$. So $\overline{\mathcal{M}}, \hat{x}_1 \models \Box \varphi$.

2. $n = i + 1$ The case is the following:

$$\hat{x}\bar{R}\hat{x}_1\bar{R}\dots\bar{R}\hat{x}_i\bar{R}\hat{x}_{i+1}\bar{R}\hat{y}$$

By IH, $\Box\varphi$ is true at \hat{x}_i , so is $\Box\Box\varphi$. Hence, we have $\bar{\mathcal{M}}, \hat{x}_{i+1} \models \Box\varphi$ since $\hat{x}_i\bar{R}\hat{x}_{i+1}$.

That is, for each $0 < n < \omega$, if we have a sequence of equivalence classes with $\hat{x}\bar{R}\hat{x}_1\bar{R}\dots\bar{R}\hat{x}_n\bar{R}\hat{y}$ where $\bar{\mathcal{M}}, \hat{x} \models \Box\varphi$, then $\bar{\mathcal{M}}, \hat{x}_n \models \Box\varphi$.

If $\hat{x}_n\bar{R}\hat{y}$, then there are $x'_n \in \hat{x}_n$ and $y' \in \hat{y}$ with $x'_n R y'$. $\mathcal{M}, x'_n \models \Box\varphi$ implies $\mathcal{M}, y' \models \varphi$, but y' and y are Γ -equivalent and $\varphi \in \Gamma$, so $\mathcal{M}, y \models \varphi$. \square

4 Finite “canonical” models

Let \mathcal{L} be a normal modal logic, $\mathcal{M}_{\mathcal{L}}$ its canonical model, and φ . Let us put $\Gamma = \text{Sub}(\varphi) \cup \{\neg\psi \mid \psi \in \text{Sub}(\varphi)\}$.

A subset $\Delta \subseteq \Gamma$ is a *finite \mathcal{L} -theory* if $\bigwedge \Delta \notin \mathcal{L}$. A subset Δ is maximal, if (the following are obviously equivalent):

1. Δ is maximal amongst finite \mathcal{L} -theories,
2. For each $\psi \in \text{Sub}(\varphi)$ either $\psi \in \Delta$ or $\neg\psi \in \Delta$.

Every finite \mathcal{L} -theory is clearly can be extended to some maximal one. It is the finite version of Lindenbaum’s lemma.

Definition 8. Let \mathcal{L} be a normal modal logic and $\varphi \notin \mathcal{L}$. A finite “canonical” model is a triple $\mathcal{M}_{\mathcal{L}}^{\varphi} = \langle W_{\mathcal{L}}^{\varphi}, R_{\mathcal{L}}^{\varphi}, \vartheta_{\mathcal{L}}^{\varphi} \rangle$, where

1. $W_{\mathcal{L}}^{\varphi}$ is the set all maximal theories that extend finite \mathcal{L} -theories
2. $R_{\mathcal{L}}^{\varphi}$ is a relation such that $\langle W_{\mathcal{L}}^{\varphi}, R_{\mathcal{L}}^{\varphi} \rangle$ is an \mathcal{L} -frame and

$$\forall \Box\psi \in \text{Sub}(\varphi) \quad \forall \Delta_1 \in W_{\mathcal{L}}^{\varphi} \quad (\Box\psi \in \Delta_1 \Leftrightarrow \forall \Delta_2 \in R_{\mathcal{L}}^{\varphi}(\Delta_1) \quad \psi \in \Delta_2)$$

3. $\vartheta_{\mathcal{L}}^{\varphi}(p) = \{\Delta \in W_{\mathcal{L}}^{\varphi} \mid p \in \Delta\}$.

Definition 9. Let φ be a modal formula, \mathcal{L} a logic that admits strict filtration, and \mathcal{M} an \mathcal{L} -model that refutes φ . A filtration diagram of a model $\widehat{\mathcal{M}}$ (a filtration of \mathcal{M} through φ) is a formula $\text{FDiagram}(\mathcal{M}_{\mathcal{L}}^{\text{Sub}(\varphi)}) = \bigwedge \Gamma \wedge \bigwedge \Delta$, where

- $\Gamma = \{\psi \in \text{Sub}(\varphi) \mid \exists w \in \widehat{\mathcal{M}} \widehat{M}, \hat{w} \models \psi\}$
- $\Delta = \{\neg\psi \in \text{Sub}(\varphi) \mid \psi \in \text{Sub}(\varphi) \ \& \ \mathcal{M} \not\models \psi\}$

Lemma 8. Let φ be a modal formula, \mathcal{L} a logic that admits strict filtration, and \mathcal{M} an \mathcal{L} -model that refutes φ . Then $\mathcal{M}_{\mathcal{L}}^{\text{Sub}(\varphi)} \models \text{FDiagram}(\mathcal{M}_{\mathcal{L}}^{\text{Sub}(\varphi)})$

Definition 10. Let \mathcal{M} be a model and Ψ a set of formulas, then a Ψ -theory of a model, denoted as Th_{Ψ} , is the set $\text{Th}_{\Psi} = \{\varphi \in \Psi \mid \mathcal{M} \models \varphi\}$.

Lemma 9. Let \mathcal{L} be a modal logic that admits strict filtrations and $\varphi \notin \mathcal{L}$. Let \mathcal{M} be a model that refutes φ and \mathcal{N} be a different model such that $\mathcal{N} \models \mathcal{L}$. If $\mathcal{N} \models \text{FDiagram}(\mathcal{M}_{\mathcal{L}}^{\text{Sub}(\varphi)})$, then $\text{Th}_{\text{Sub}(\varphi)}(\mathcal{M}_{\mathcal{L}}^{\text{Sub}(\varphi)}) = \text{Th}_{\text{Sub}(\varphi)}(\mathcal{N})$. In particular, \mathcal{N} refutes φ .

Theorem 3. *Let \mathcal{L} be a canonical logic, then the following are equivalent:*

1. \mathcal{L} admits strict filtrations
2. If $\varphi \notin \mathcal{L}$, there exists a finite “canonical” model $\mathcal{M}_{\mathcal{L}}^{\varphi}$ such that $\mathcal{M}_{\mathcal{L}}^{\varphi} \models \mathcal{L}$ and $\mathcal{M}_{\mathcal{L}}^{\varphi}$ refutes φ .

Proof.

1. (\Rightarrow)

Let $\varphi \notin \mathcal{L}$, there exists a model \mathcal{M} that refutes φ , so there exists $w \in \underline{\mathcal{M}}$ such that $\mathcal{M}, w \not\models \varphi$. But \mathcal{L} admits strict filtration, so there exists a model $\widehat{\mathcal{M}} = \langle \widehat{W}, \widehat{R}, \widehat{\vartheta} \rangle$, where $\widehat{W} = W / \sim_{\varphi}$ and $\widehat{R}^{min} \subseteq \widehat{R} \subseteq \widehat{R}^{max}$. In particular, $\widehat{\mathcal{M}}, \widehat{w} \not\models \varphi$.

By lemma above, $\widehat{\mathcal{M}} \models \text{FDiagram}(\widehat{\mathcal{M}})$

Consider a finite “canonical” model $\mathcal{M}_{\mathcal{L}}^{\text{Sub}(\varphi)} = \langle W_{\mathcal{L}}^{\text{Sub}(\varphi)}, R_{\mathcal{L}}^{\text{Sub}(\varphi)}, \vartheta_{\mathcal{L}}^{\text{Sub}(\varphi)} \rangle$. Let us show that $\mathcal{M}_{\mathcal{L}}^{\text{Sub}(\varphi)}$ refutes φ . For that, we show that $\mathcal{M}_{\mathcal{L}}^{\text{Sub}(\varphi)} \models \text{FDiagram}(\widehat{\mathcal{M}})$.

2. (\Leftarrow)

Let $\varphi \notin \mathcal{L}$ and $\mathcal{M}_{\mathcal{L}}^{\varphi} = \langle W_{\mathcal{L}}^{\varphi}, R_{\mathcal{L}}^{\varphi}, \vartheta_{\mathcal{L}}^{\varphi} \rangle$ be a finite “canonical” model that refutes φ . Let \mathcal{M} be an \mathcal{L} -model that also refutes φ . Let us show that there exists a model $\widehat{\mathcal{M}}$ such that $\widehat{\mathcal{M}}$ is a strict filtration of \mathcal{M} through $\text{Sub}(\varphi)$.

Let $\sim_{\text{Sub}(\varphi)}$ be an equivalence relation on $\underline{\mathcal{M}}$ defined as usual:

$$u \sim_{\text{Sub}(\varphi)} v \text{ iff } \forall \psi \in \text{Sub}(\varphi) \mathcal{M}, u \models \psi \Leftrightarrow \mathcal{M}, v \models \psi$$

Let $[u] \in W / \sim_{\text{Sub}(\varphi)}$, where $W = \underline{\mathcal{M}}$, consider its truth set $||[u]|| = \{\psi \in \text{Sub}(\varphi) \mid \mathcal{M}, u \models \psi\}$. $||[u]||$ is clearly a finite consistent theory and it consists in some $\Delta \in W_{\mathcal{L}}^{\varphi}$ by the finite analogue of Lindenbaum’s lemma. Such a Δ is clearly unique as well. Consider a map

$$\iota : W / \sim_{\text{Sub}(\varphi)} \rightarrow W_{\mathcal{L}}^{\varphi} \text{ with } \iota : [u] \mapsto \Delta \supseteq ||[u]||$$

Consider the set $\iota(W / \sim_{\text{Sub}(\varphi)})$ denoted as $\text{Im}(\iota)$. Let us show that $\text{Im}(\iota)$ is $R_{\mathcal{L}}^{\varphi}$ -closed.

Let $\iota([u]) R_{\mathcal{L}}^{\varphi} \Delta$, let us show that $\Delta \in \iota(W / \sim_{\text{Sub}(\varphi)})$, that is, $\Delta = \iota([v])$ for some $v \in W$.

Consider Δ as a finite ultrafilter. Thus, Δ is principal and it’s generated by $\psi_1 \wedge \dots \wedge \psi_n$, where each $\psi_i \in \text{Sub}(\varphi)$. So $\Diamond(\psi_1 \wedge \dots \wedge \psi_n) \in \iota([u])$. Then $\mathcal{M}, u \models \Diamond(\psi_1 \wedge \dots \wedge \psi_n)$. That implies $\mathcal{M}, v \models \psi_1 \wedge \dots \wedge \psi_n$ for some $v \in R(u)$. This is clearly a desired v since $||[v]|| \subseteq \Delta$ and such a Δ is unique.

Thus, $\langle \text{Im}(\iota), R_{\mathcal{L}}^{\varphi} \cap \text{Im}(\iota) \times \text{Im}(\iota), \vartheta'_{\mathcal{L}}^{\varphi} \rangle$, where $\vartheta'_{\mathcal{L}}^{\varphi}(p) = \vartheta_{\mathcal{L}}^{\varphi}(p) \cap \text{Im}(\iota)$ is a generated submodel of $\mathcal{M}_{\mathcal{L}}^{\varphi}$.

Consider a model $\widehat{\mathcal{M}} = \langle W / \sim_{\text{Sub}(\varphi)}, \widehat{R}, \widehat{\vartheta} \rangle$, where \widehat{R} is defined as

$$[u] \widehat{R} [v] \text{ iff } ||[u]|| R'_{\mathcal{L}}^{\varphi} ||[v]||, \text{ where } R'_{\mathcal{L}}^{\varphi} R_{\mathcal{L}}^{\varphi} \cap \text{Im}(\iota) \times \text{Im}(\iota).$$

Let us prove that $\widehat{\mathcal{M}}$ is a filtration of \mathcal{M} through $\text{Sub}(\varphi)$.

Let us show that $R_{min} \subseteq \widehat{R} \subseteq R_{max}$.

Suppose that there are $u' \in [u]$ and $v' \in [v]$ with $u'Rv'$. Let us show that $[u]\hat{R}[v]$. That is, let us show that $\Diamond\iota(v) \subseteq \iota(v)$. But if $\mathcal{M}, v' \models \psi$ for $\Diamond\psi \in \text{Sub}(\varphi)$, then $\mathcal{M}, u' \models \Diamond\psi$. Recall that $u' \sim_{\text{Sub}(\varphi)} u$ and $v' \sim_{\text{Sub}(\varphi)} v$, so we have $\Diamond\|[v]\| \subseteq \|[u]\|$. The same inclusion clearly holds for the maximal extensions of $\Diamond\|[v]\|$ and $\|[u]\|$.

The second inclusion follows from the definition of a relation in a finite “canonical” model.

□

In particular, **K5** fails to have the finite “canonical” model property that follows from the contraposition of the theorem above and Theorem 2.

References