

Notes on filtrations for logics that contain **K5**

Daniel Rogozin

1 Preliminaries

Let $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$ be a Kripke model and Γ a set of formulas closed under subformulas. An equivalence relation \sim is set to have a finite index if the quotient set W / \sim is finite. The equivalence relation \sim_Γ induced by Γ is defined as

$$w \sim_\Gamma v \Leftrightarrow \forall \varphi \in \Gamma (\mathcal{M}, w \models \varphi \Leftrightarrow \mathcal{M}, v \models \varphi).$$

If Γ is finite, then \sim_Γ has a finite index. An equivalence relation \sim respects \sim_Γ , if $w \sim v$ implies $w \sim_\Gamma v$.

The following definition of a filtration is due to, e.g., [6].

Definition 1. Let $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$ be a Kripke model and Γ be a Sub-closed set formulas. A Γ -filtration of \mathcal{M} is a model $\widehat{\mathcal{M}} = \langle \widehat{W}, \widehat{R}_1, \dots, \widehat{R}_n, \widehat{\vartheta} \rangle$ such that:

1. $\widehat{W} = W / \sim$, where \sim is an equivalence relation that respects Γ
2. $\widehat{\vartheta}(p) = \{[x]_\sim \mid x \in W \text{ \& } x \in \vartheta(p)\}$
3. For each $i \in I$ one has $\widehat{R}_i^{\min} \subseteq \widehat{R}_i \subseteq \widehat{R}_i^{\max}$. $\widehat{R}_{i,\sim}^{\min}$ is the i -th minimal filtered relation on \widehat{W} defined as

$$\widehat{x} \widehat{R}_{i,\sim}^{\min} \widehat{y} \Leftrightarrow \exists x' \sim x \exists y' \sim y x R_i y$$

$\widehat{R}_{\Gamma,i}^{\max}$ is the i -th maximal filtered relation on \widehat{W} induced by Γ defined as

$$\widehat{x} \widehat{R}_{\Gamma,i}^{\max} \widehat{y} \Leftrightarrow \forall \Box_i \varphi \in \Gamma (\mathcal{M}, x \models \Box_i \varphi \Rightarrow \mathcal{M}, y \models \varphi)$$

Alternatively, one may reformulate the condition of the maximal filtered relations using \Diamond 's as follows. We will use this formulation occasionally:

$$\widehat{x} \widehat{R}_{\Gamma,i}^{\max} \widehat{y} \Leftrightarrow \forall \Diamond_i \varphi \in \Gamma (\mathcal{M}, y \models \varphi \Rightarrow \mathcal{M}, x \models \Diamond_i \varphi)$$

If Φ is an extension of Γ and $\sim = \sim_\Phi$, then $\widehat{\mathcal{M}}$ is a definable Γ -filtration of \mathcal{M} through Φ . If $\sim = \sim_\Gamma$, then such a filtration by means of the definition above is called *strict*.

A class of models \mathbb{M} admits strict filtrations for models (ASF), if for every Sub-closed set Γ and for every $\mathcal{M} \in \mathbb{M}$ there exists a model $\widehat{\mathcal{M}}$ such that $\widehat{\mathcal{M}} \in \mathbb{M}$ and $\widehat{\mathcal{M}}$ is a filtration of \mathcal{M} through Γ .

A class of frames \mathbb{F} admits strict filtrations for frames, if for every Sub-closed set Γ and for every frame $\mathcal{F} \in \mathbb{F}$ and every model \mathcal{M} over \mathcal{F} there exists a Γ filtration of \mathcal{M} , and the underlying frame of this filtration belongs to \mathbb{F} .

If \mathcal{L} is canonical, then the ASF property for frames and ASF property for models are equivalent, see [4, Theorem 2.10].

The key lemma about filtrations is the following, see [1, Theorem 2.39]:

Lemma 1. *Let Γ be a finite set of formulas closed under subformulas and $\widehat{\mathcal{M}}$ a filtration of \mathcal{M} through Γ , then for each $x \in W$ and for each $\varphi \in \Gamma$ one has*

$$\mathcal{M}, x \models \varphi \Leftrightarrow \widehat{\mathcal{M}}, \hat{x} \models \varphi$$

Definition 2.

1. *Let \mathbb{F} be a class of Kripke frames and Γ a finite set of formulas closed under subformulas. If for every model \mathcal{M} over $\mathcal{F} \in \mathbb{F}$ there exists a model that is a Γ -definable filtration of \mathcal{M} , then \mathbb{F} admits definable filtration.*
2. *A class of models \mathbb{M} admits definable filtration if for every $\mathcal{M} \in \mathbb{M}$ there exists a model belonging to the same class that is a definable Γ -filtration of \mathcal{M} .*

Lemma 2.

1. *Let \mathcal{L} be a complete normal modal logic. If $\text{Frames}(\mathcal{L})$ admits filtration, then \mathcal{L} has the finite model property.*
2. *If the class of models $\text{Mod}(\mathcal{L})$ admits filtration, then \mathcal{L} has the finite model property and it is Kripke complete as well.*

Definition 3. *A first-order formula is called Horn if it has the following form (see [3]):*

$$\forall x_1, \dots, x_n (x_{i_1} R x_{j_1} \wedge \dots \wedge x_{i_s} R x_{j_s} \rightarrow A), \text{ where } A \text{ is either } x_k R x_l \text{ or } \perp.$$

Definition 4. *Let H be a Horn property and $\langle W, R \rangle$ a Kripke frame. A Horn closure of a binary relation R is the minimal relation R^H containing R and satisfying H .*

Lemma 3. $R^H = \bigcup_{n < \omega} R_n$ where

1. $R_0 = R$.
2. $R_{n+1} = R_n \cup \{(a, b) \in W \mid \exists \vec{c} \in W \text{ } P(a, b, \vec{c})\}$, where P is a premise of H .

2 Filtrations for K5

The **K5**-closure (an Euclidean Horn closure of a binary relation) has the following equivalent definitions:

Lemma 4. *Let $\mathcal{F} = \langle W, R \rangle$ be a Kripke frame and $R \subseteq R^{\mathbf{K5}}$. The following conditions are equivalent:*

1. $R^{\mathbf{K5}}$ is the smallest Euclidean relation containing R , that is, the Horn closure of R .
2. $R^{\mathbf{K5}} = \bigcup_{i < \omega} R_i$, where
 - $R_0 = R$
 - $R_{n+1} = R_n \cup (R_n^{-1} \circ R_n)$

3. $xR^{\mathbf{K5}}y$ iff there exists $n < \omega$ such that either xRy or $\exists z_1, \dots, z_n$ with z_1Rx and $z_{n-1}Ry$ and for each $1 < i \leq n$ one has either $z_{i-1}Rz_i$ or z_iRz_{i-1} .
4. $R^{\mathbf{K5}} = R \cup \bigcup_{i < \omega} (R^{-1} \circ (R \cup R^{-1})^n \circ R)$.

Proof.

1. (1) \Rightarrow (2) Let us show that if R^E is the smallest Euclidean relation containing R , then $R^E = \bigcup_{i < \omega} R_i$. There are two inclusions:
- $R^E \subseteq \bigcup_{i < \omega} R_i$. Recall that R^E has the form (?):

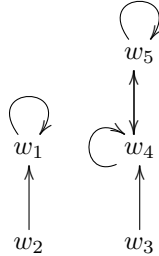
$$R^E = \bigcap \{R' \mid R \subseteq R', \forall a, b \in W \ R'(a, b) \Rightarrow \exists x \in W \ R'(x, a) \ \& \ R'(x, b)\}$$
 - $\bigcup_{i < \omega} R_i \subseteq R^E$. Let us show that xR_ny for each $n < \omega$ implies xR^Ey by induction on n .
 If $n = 0$, then xRy , thus, xR^Ey , since R is a subrelation of R^E . Suppose $n = m + 1$ and $xR_{m+1}y$. Let us show that xR^Ey . From $xR_{m+1}y$, one has $(x, y) \in R^n \cup (R_n^{-1} \circ R_n)$. There are two cases:
 - xR^ny , one needs to merely apply the IH.
 - $xR_n^{-1} \circ R_ny$. Then $\exists z \in W \ xR_n^{-1}z \ \& \ zR_ny$. That is, zR_nx and zR_ny for some z . R_n is already a subrelation of R^E . Thus, zR^Ex and zR^Ey . That implies xR^Ey .
2. (2) \Rightarrow (3) Let $(x, y) \in R_m$, let us the statement by induction on m .
- (a) Suppose $m = 0$, then xRy , and the statement is shown putting $n = 0$.
- (b) Suppose $m = p + 1$ and $xR_{p+1}y$. Assume that either xRy or $\exists z_1, \dots, z_p$ with z_1Rx and $z_{p-1}Ry$ and for each $1 < i \leq p$ one has either $z_{i-1}Rz_i$ or z_iRz_{i-1} .
 $xR_{p+1}y$ implies $(x, y) \in R_p \cup (R_p^{-1} \circ R_p)$. If $(x, y) \in R_p$, then we merely apply the IH. Suppose $(x, y) \in R_p^{-1} \circ R_p$, then $(z, x) \in R_p$ and $(z, y) \in R_p$.
3. (3) \Rightarrow (4) Suppose either xRy or there exist $n \geq 1$ and z_1, \dots, z_n with z_1Rx and $z_{n-1}Ry$ and for each $1 < i \leq n$ one has either $z_{i-1}Rz_i$ or z_iRz_{i-1} . If xRy , then we are done. Otherwise there exists $n \geq 1$ with the condition above. Then $(x, y) \in R_{n+1}$ that follows from the condition.
4. (4) \Rightarrow (1)

□

Theorem 1. $\mathbf{K5}$ does not admit strict filtrations.

Proof. Let us consider a $\mathbf{K5}$ model whose Euclidean closure of the minimal filtration does not give us a filtration.

Let us consider a frame called \mathcal{F}_{bad} . We define this frame with the following graph:



Let us define a valuation ϑ such that $\vartheta(p) = \{w_5\}$ and $\vartheta(q) = \{w_1\}$. Let us consider a minimal filtration of \mathcal{M}_{bad} through the Sub-closure of $\Gamma = \{\neg p, \neg \Diamond p\}$.

Clearly that $w_2 \sim_\Gamma w_3$, since $\neg p$ and $\neg \Diamond p$ are true both at w_2 and w_3 .

Moreover, $R_{min} \cup (R_{min}^{-1} \circ R_{min})$ is not a subset of R_{max} since $(\hat{w}_1, \hat{w}_5) \in (R_{min}^{-1} \circ R_{min})$, but $\Diamond p$ is not true at w_5 .

Let us also note that strict filtrations of this model is not Euclidean. Suppose by contrary that $\hat{R}^\mathcal{E}$ is a strict filtration of that model. So $R_{min}^E \subseteq \hat{R}^\mathcal{E}$, since R_{min}^E is the minimal Euclidean relation containing R_{min} . On the other hand, $R_{min}^E \not\subseteq R_{max}$, so is not $\hat{R}^\mathcal{E}$. \square

Theorem 2. **K5** admits definable filtrations.

Theorem 3. **K45** admits strict filtrations.

Proof. Let $\mathcal{M} = \langle W, R, \vartheta \rangle$ be a transitive Euclidean model and $\overline{\mathcal{M}} = \langle \overline{W}, \overline{R}, \overline{\vartheta} \rangle$ its minimal filtration through Γ , where Γ is finite and Sub-closed. Let us put $\hat{R} = \overline{R}^+ \cup \overline{R}^{\mathbf{K5}}$. Let us show that $\overline{R}^+ \cup \overline{R}^{\mathbf{K45}} \subseteq \overline{R}^{max}$.

That is, if $\mathcal{M}, y \models \varphi$ for $\Diamond \varphi \in \Gamma$ and $\hat{x}\hat{R}\hat{y}$, then $\mathcal{M}, x \models \Diamond \varphi$.

Let $\hat{x}\hat{R}\hat{y}$.

1. Suppose $(\hat{x}, \hat{y}) \in \overline{R}$, then $\mathcal{M}, x \models \Diamond \varphi$ holds trivially by the definition of the minimal filtration.
2. Let us consider the case when $(\hat{x}, \hat{y}) \in \overline{R}^{\mathbf{K5}}$. The second alternative is the same as for the **K4**-case, see [2, p. 141].

Suppose the statement holds \overline{R}_n and $(\hat{x}, \hat{y}) \in \overline{R}_{n+1} = \overline{R}_n \cup (\overline{R}_n^{-1} \circ \overline{R}_n)$. We consider the case of $(\hat{x}, \hat{y}) \in (\overline{R}_n^{-1} \circ \overline{R}_n)$.

Then there exists \hat{z} such that $(\hat{z}, \hat{x}), (\hat{z}, \hat{y}) \in \overline{R}_n$.

By IH, $\mathcal{M}, z \models \Diamond \varphi$.

$(\hat{z}, \hat{y}) \in \overline{R}_n$ iff there are $\hat{u}_1, \dots, \hat{u}_n$ such that

$$\hat{z} \xleftarrow{\hat{R}} \hat{u}_1 \xrightarrow{\hat{R}'} \hat{u}_2 \xrightarrow{\hat{R}'} \dots \xrightarrow{\hat{R}'} \hat{u}_{n-1} \xrightarrow{\hat{R}'} \hat{u}_n \xrightarrow{\hat{R}} \hat{y}$$

where \hat{R}' is either \hat{R} or \hat{R}^{-1} .

As it is known, $\Diamond \Diamond \varphi \rightarrow \Box \Diamond \varphi \in \mathbf{K45}$.

$\hat{u}_1 \hat{z}$, that is, $u'_1 R z'$ for some $u'_1 \in \hat{u}_1$ and $z' \in \hat{z}$. That is, $\mathcal{M}, u'_1 \models \Diamond \Diamond \varphi$, so $\mathcal{M}, u'_1 \models \Diamond \varphi$ and $\overline{\mathcal{M}}, \hat{u}_1 \models \Diamond \varphi$.

We have $\hat{u}_1 \hat{R}' \hat{u}_2$. Suppose $\mathcal{M}, u''_1 \models \Diamond \varphi$ and $u''_1 R u'_2$. We also have $\mathcal{M}, u''_1 \models \Box \Diamond \varphi$, thus, $\mathcal{M}, u'_2 \models \Diamond \varphi$.

Suppose $\hat{u}_2 \hat{R} \hat{u}_1$ and $u'_2 R u''_1$, then $\mathcal{M}, u'_2 \models \Diamond \varphi$.

Similarly, we have $\mathcal{M}, u_i \models \Diamond \varphi$ iff $\mathcal{M}, u_{i+1} \models \Diamond \varphi$, whenever $\hat{u}_i \hat{R}' \hat{u}_{i+1}$.

Finally, we have $\hat{u}_n \hat{R} \hat{x}$. Thus, $u'_n R x'$ for some $u'_n \in \hat{u}_n$ and $x' \in \hat{x}$. $\mathcal{M}, u'_n \models \Diamond \varphi$, so $\mathcal{M}, u'_n \models \Box \Diamond \varphi$. Then $\mathcal{M}, x' \models \Diamond \varphi$. \square

3 i,j-Euclideaness

A binary relation $R \subseteq W \times W$ is called i, j -Euclidean for $i, j < \omega$, if for each x, y, z such that $xR^i y$ and $xR^j z$ implies xRz .

Proposition 1. *Let $\mathcal{F} = \langle W, R \rangle$, then \mathcal{F} is i, j -Euclidean iff $\mathcal{F} \models \Diamond^i p \rightarrow \Box^j \Diamond p$. As a corollary, the logic $\mathbf{K5}^{i,j} = \mathbf{K} \oplus \Diamond^i p \rightarrow \Box^j \Diamond p$ is Kripke-complete.*

Let R be a binary relation on $W \neq \emptyset$, then the i, j -Euclidean closure of R (where $i, j < \omega$), denoted as $R^{\mathbf{K5}^{i,j}}$, is a binary relation defined recursively as follows:

1. $R_0 = R$
2. $R_{n+1} = R_n \cup (((R_n)^{-1})^i \circ R_n^j)$
3. $R^{E_{i,j}} = \bigcup_{k < \omega} R_k$

Note that the following property holds not for every $\mathbf{K5}^{i,j}$, consider \mathbf{KB} as an example (clearly, $\mathbf{KB} = \mathbf{K5}^{0,1}$).

Lemma 5. $\mathbf{K5}^{i,j}$ has n non-equivalent sequences of modalities, if . *TODO: define n .*

Theorem 4. $\mathbf{K5}^{i,j}$ admits definable filtrations.

Proof. Let $\mathcal{M} = \langle W, R, \vartheta \rangle$ and Γ a finite Sub-closed of formulas. We extend Γ as

$$\Delta = \Gamma \cup \text{Sub}\{\Diamond^i \psi \mid \Box \psi \in \Gamma\} \cup \text{Sub}\{\Box^j \Diamond \psi \mid \Diamond \psi \in \Gamma\}$$

Let $(\hat{x}, \hat{y}) \in R^{E_{i,j}}$, $\mathcal{M}, x \models \Box \psi$ for $\Box \psi$ in Δ . If $n = 0$, then the statement is obvious. Suppose $n = 1$. Then $(\hat{x}, \hat{y}) \in R_{\min} \cup (((R_{\min})^{-1})^i \circ R_{\min}^j)$. Consider the second alternative. Then there exists \hat{z} such that $(\hat{x}, \hat{z}) \in ((R_{\min})^{-1})^i$ and $(\hat{z}, \hat{y}) \in R_{\min}^j$.

That is, there are $\hat{x}_1, \hat{x}_2, \dots, \hat{x}_i$ and $\hat{y}_1, \hat{y}_2, \dots, \hat{y}_j$ such that

$$\hat{z} \xrightarrow{R_{\min}} \hat{x}_1 \xrightarrow{R_{\min}} \hat{x}_2 \xrightarrow{R_{\min}} \dots \xrightarrow{R_{\min}} \hat{x}_i \xrightarrow{R_{\min}} \hat{x}$$

$$\hat{z} \xrightarrow{R_{\min}} \hat{y}_1 \xrightarrow{R_{\min}} \hat{y}_2 \xrightarrow{R_{\min}} \dots \xrightarrow{R_{\min}} \hat{y}_j \xrightarrow{R_{\min}} \hat{y}$$

???

□

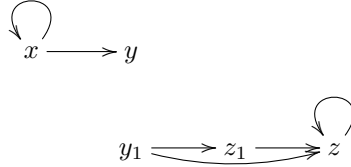
4 The case of 2-transitivity

Let us define the logic \mathcal{L} as $\mathbf{K} \oplus \Diamond \Diamond p \rightarrow \Diamond p$. Let R be a binary relation, the \mathcal{L} -closure of R is defined (denoted as R^{\star}) as the following union:

$$R^{\star} = R \cup R^3 \cup R^5 \cup \dots \cup R^{2k+1} \cup \dots$$

Theorem 5. \mathcal{L} does not admit strict filtrations.

Proof. Consider the following frame $\mathcal{F} = \langle W, R \rangle$:



Clearly that \mathcal{F} is an \mathcal{L} -frame. We define the valuation ϑ as follows:

$$\begin{aligned}\vartheta(p) &= \{x\} \\ \vartheta(q) &= \{y, y_1\} \\ \vartheta(r) &= \{z\}\end{aligned}$$

Let us put $\Gamma = \text{Sub}\{p, q, \Diamond r\}$. We factorise W through \sim_Γ and consider a model $\widehat{\mathcal{M}} = \langle W / \sim_\Gamma, \widehat{R}, \widehat{\vartheta} \rangle$, where $\widehat{R} = (\widehat{R}_{\min})^\Delta$. We have $(\hat{x}, \hat{z}) \in \widehat{R} \circ \widehat{R} \circ \widehat{R}$, but $\Diamond r$ is not true at x . \square

5 Fusion stuff

Definition 5. Let \mathcal{L}_1 and \mathcal{L}_2 be modal logics, then the fusion $\mathcal{L}_1 * \mathcal{L}_2$ is the minimal bimodal logic that contains \mathcal{L}_1 and \mathcal{L}_2 [5].

Lemma 6. Let Γ be a finite and Sub-closed set of formulas.

1. $\mathcal{M} = \langle W, R, \vartheta \rangle$ be a **K5**-model. Consider $\Gamma' = \Gamma \cup \{\Diamond \Box \psi \mid \Box \psi \in \Gamma\}$. Let Δ be any finite and Sub-closed extension of Γ' that contains all $\Diamond \Box \psi$ for each $\Box \psi \in \Delta$. Then a model $\widehat{\mathcal{M}} = \langle W / \sim_\Delta, (R_\Delta^{\min})^E, \widehat{\vartheta} \rangle$ is a filtration of \mathcal{M} through Δ .
2. Let $\mathcal{L} = \mathbf{K} \oplus \Diamond \Diamond \Diamond p \rightarrow \Diamond p$, then we have the similar statement for \mathcal{L} , where Δ is an extension of $\Gamma' = \{\Diamond \Diamond \psi \mid \Diamond \psi \in \Delta\}$ that contains all $\Diamond \Diamond \varphi$ for $\Diamond \varphi \in \Delta$.

Proof.

Recall that $(R_\Delta^{\min})^E$ is defined inductively as:

- (a) $R_\Delta^0 = R_\Delta^{\min}$
- (b) $R_\Delta^{n+1} = R_\Delta^n \cup (R_\Delta^n)^{-1} \circ R_\Delta^n$
- (c) $(R_\Delta^{\min})^E = \bigcup_{k < \omega} R_\Delta^k$

If R_Δ^{\min} is already a subrelation of R^{\max} , so the base case is self-evident.

Suppose the statement holds for R_Δ^n , $(\hat{x}, \hat{y}) \in R_\Delta^{n+1}$ such that $\mathcal{M}, y \models \psi$ for $\Diamond \psi \in \Delta$.

According to the third item of Lemma 4, this is the same as there exist $\widehat{z}_0, \widehat{z}_1, \dots, \widehat{z}_{n-1}, \widehat{z}_n$ such that $\widehat{z}_1 R_\Delta^{\min} \hat{x}$, $\widehat{z}_n R_\Delta^{\min} \hat{y}$, and for each $i \in n+1$ we have either $\widehat{z}_i R_\Delta^{\min} \widehat{z}_{i+1}$ or $\widehat{z}_{i+1} R_\Delta^{\min} \widehat{z}_i$.

We visualise this with the following graph:

$$\hat{x} \xleftarrow{R_\Delta^{\min}} \widehat{z}_0 \xleftrightarrow{R'} \widehat{z}_1 \xleftrightarrow{R'} \dots \xleftrightarrow{R'} \widehat{z}_{n-1} \xrightarrow{R'} \widehat{z}_n \xrightarrow{R_\Delta^{\min}} \hat{y}$$

where R' is either R_Δ^{\min} or its converse. We have $\mathcal{M}, z \models \Box \psi$, $\mathcal{M}, y \models \psi$, so $\mathcal{M}, z_n \models \Diamond \psi$. Since \mathcal{M} is a **K5**-model, we have $\mathcal{M}, z_n \models \Box \Diamond \psi$.

After that we apply the following property of **K5**-models:

$$\begin{aligned}\text{Let } \mathcal{M} \models \mathbf{K5} \text{ and } \varphi \text{ a formula, then for each } a, b \in \mathcal{M} \text{ such that } aRb \text{ we have} \\ \mathcal{M}, a \models \Box \Diamond \varphi \text{ iff } \mathcal{M}, b \models \Box \Diamond \varphi\end{aligned}$$

So we have $\mathcal{M}, z_0 \models \Box \Diamond \varphi$. Note that we always stay within Δ . Thus, $\mathcal{M}, x \models \Diamond \varphi$.

2. Let us prove the second item. Let \mathcal{M} be an \mathcal{L} -model. The \mathcal{L} -closure of the minimal filtered relation modulo Δ , namely $R_\Delta^{\min \mathcal{L}}$ has the following form:

$$R_{\Delta}^{min\mathcal{L}} = \bigcup_{k < \omega} R_{\Delta}^{min 2^{k+1}}$$

We reformulate this closure equivalently as follows:

- (a) $R_0 = R_{\Delta}^{min}$
- (b) $R_{n+1} = R_n \cup ((R_{\Delta}^{min})^2 \circ R_n)$
- (c) $R_{\Delta}^{min\mathcal{L}} = \bigcup_{k < \omega} R_k$

The base case is self-evident. Suppose the statement holds for R_n and $(\hat{x}, \hat{y}) \in (R_{\Delta}^{min})^2 \circ R_n$, that is, there exists \hat{z} such that $(\hat{x}, \hat{z}) \in (R_{\Delta}^{min})^2$ and $(\hat{z}, \hat{y}) \in R_n$. By IH, we have $\mathcal{M}, z \models \Diamond\varphi$.

We have the following:

$$\hat{x} \xrightarrow{R_{\Delta}^{min}} \hat{y}_1 \xrightarrow{R_{\Delta}^{min}} \hat{z}$$

The sequence of implications is the following:

$$\widehat{\mathcal{M}}, \hat{z} \models \Diamond\varphi \Rightarrow \widehat{\mathcal{M}}, \hat{y}_1 \models \Diamond\Diamond\varphi \Rightarrow \mathcal{M}, x \models \Diamond\Diamond\Diamond\varphi \Rightarrow \mathcal{M}, x \models \Diamond\varphi$$

□

Theorem 6.

1. **K5 * K5** admits definable filtrations.
2. **K5 * ... * K5** admits definable filtrations.
3. If \mathcal{L} admits strict filtrations, then **K5 * L** admits definable filtrations
4. If $\mathcal{L}_1, \dots, \mathcal{L}_n$ admit strict filtrations, then **K5 * ... * K5 * L₁ * ... * L_n**
5. Let $\mathcal{L} = \mathbf{K} \oplus \Diamond\Diamond\Diamond p \rightarrow \Diamond p$ (here and below), then $\mathcal{L} * \mathcal{L}$ admits definable filtrations.
6. Let $\mathcal{L} = \mathbf{K} \oplus \Diamond\Diamond\Diamond p \rightarrow \Diamond p$ and \mathcal{L}_1 a logic that admits strict filtrations, then $\mathcal{L} * \mathcal{L}_1$

Proof.

1. Let Γ be a finite Sub-closed set of bimodal formulas, $\mathcal{F} = \langle W, R_1, R_2 \rangle$ a **K5 * K5**-frame, and ϑ a valuation on \mathcal{F} . Denote $\langle \mathcal{F}, \vartheta \rangle$ as \mathcal{M} .

We introduce the set of fresh variables $V = \{p_{\psi} \mid \psi \in \Gamma\}$ and define a new model $\mathcal{M}' = \langle \mathcal{F}, \vartheta' \rangle$ as follows:

$$\text{For all } \psi \in \Gamma, \mathcal{M}, x \models \psi \Leftrightarrow \mathcal{M}', x \models \psi \Leftrightarrow \mathcal{M}', x \models p_{\psi}.$$

Consider these modifications of Γ and V :

$$\begin{aligned} \Gamma' &= \Gamma \cup \{\Diamond_1 \Box_1 \psi \mid \Box_1 \psi \in \Gamma\} \cup \{\Diamond_2 \Box_2 \psi \mid \Box_2 \psi \in \Gamma\} \\ \Delta &= V \cup \text{Sub}(\{\Diamond \Box p_{\psi} \mid \Box_i \psi \in \Gamma, in = 1, 2\}) \end{aligned}$$

Let us define an equivalence relation \sim_{Δ} induced by Δ .

Consider $\mathcal{M}_i = \langle W, R_i, \vartheta' \rangle$, a reduct of \mathcal{M}' , we have:

- (a) $\mathcal{M}_i, x \models \Box p_\psi$ iff $\mathcal{M}, x \models \Box_i \psi$
- (b) $\mathcal{M}_i, x \models \Diamond \Box p_\psi$ iff $\mathcal{M}, x \models \Diamond_i \Box_i \psi$

So $\sim = \sim_{\Gamma'}$ by the construction. Let us put $\widehat{W} = W / \sim_{\Gamma'}$. Lemma 6 implies the following claim:

Claim 1. Let $\widehat{R}_i = (R_{\Delta}^{min})^E$ and $\widehat{\vartheta}(p) = \{[x]_{\sim_i} \mid \mathcal{M}_i, x \models p\}$ for $p \in \Delta_1$, define $\widehat{\mathcal{M}}_i = \langle \widehat{W}, \widehat{R}_i, \widehat{\vartheta} \rangle$. Then $\widehat{\mathcal{M}}_i \models \mathbf{K5}$ and $\widehat{\mathcal{M}}_i$ is a filtration of \mathcal{M}_i through Δ .

Finally, we consider a model $\widehat{\mathcal{M}} = \langle \widehat{W}, \widehat{R}_1, \widehat{R}_2, \vartheta \rangle$, where $\widehat{R}_{\Gamma'_i} = R_{\Gamma'_i}^{min^E}$ and $\vartheta(p)$ is defined as usual for $p \in \Gamma$. $\widehat{\mathcal{M}}$ is a filtration of \mathcal{M} through Γ' .

Let $\widehat{x} \widehat{R}_{\Gamma'_i} \widehat{y}$ and $\mathcal{M}, x \models \Box_i \psi$ for $\Box_i \psi \in \Gamma$. Then $\mathcal{M}_i, x \models \Box p_\psi$, so $\widehat{\mathcal{M}}_i, \widehat{x} \models \Box p_\psi$. By the claim above, $\widehat{\mathcal{M}}_i$ is a filtration of \mathcal{M}_i through Δ , so $\mathcal{M}_i, y \models p_\psi$. Then $\mathcal{M}, y \models \psi$.

2. Likewise
3. The argument is the same as in the proof of the first item, except for **Claim 1** that has the following formulation: Let $\widehat{R}_1 = (R_{\Delta}^{min})^E$ and $\widehat{R}_2 = (R_{\Delta}^{min})^{\mathcal{L}_1}$. Define a valuation as usual as $\widehat{\vartheta}(p) = \{[x]_{\sim_i} \mid \mathcal{M}_i, x \models p\}$ for $p \in \Delta_1$, define $\widehat{\mathcal{M}}_1 = \langle \widehat{W}, \widehat{R}_1, \widehat{\vartheta} \rangle$ and $\widehat{\mathcal{M}}_2 = \langle \widehat{W}, \widehat{R}_2, \widehat{\vartheta} \rangle$. Then $\widehat{\mathcal{M}}_1 \models \mathbf{K5}$ and $\widehat{\mathcal{M}}_1 \models \mathbf{L}$ and $\widehat{\mathcal{M}}_i$ is a filtration of \mathcal{M}_i through Δ .
4. Likewise
5. The argument is similar to the proof of first item, but filtrations are slightly different. Let $\mathcal{M} = \langle W, R_1, R_2, \vartheta \rangle$ be a $\mathcal{L} * \mathcal{L}$ model and Γ a Sub-closed set of formulas. As above, we define a set V and a model \mathcal{M}' . Define extensions of Γ and V :

$$\begin{aligned} \Gamma' &= \Gamma \cup \{\Diamond_1 \Diamond_1 \psi \mid \Diamond_1 \psi \in \Gamma\} \cup \{\Diamond_2 \Diamond_2 \psi \mid \Diamond_2 \psi \in \Gamma\} \\ \Delta &= V \cup \text{Sub}(\{\Diamond \Diamond p_\psi \mid \Diamond \psi \in \Gamma, i = 1, 2\}) \end{aligned}$$

As above $\sim_{\Delta} = \Gamma'$ and $\widehat{\mathcal{M}}' = \langle W / \sim_{\Delta}, \widehat{R}_i, \widehat{\vartheta} \rangle$ are filtrations of reducts of \mathcal{M}' through Δ . Then $\widehat{\mathcal{M}} = \langle W / \sim_{\Delta}, \widehat{R}_1, \widehat{R}_2, \widehat{\vartheta} \rangle$ is a required filtration of the original \mathcal{M} .

6. Extend Γ with $\{\Diamond_i \Diamond_i \psi \mid \Diamond_i \psi \in \Gamma, i = 1, 2\}$ and V with $\{\Diamond \Diamond p_\psi \mid \Diamond_i \psi, i = 1, 2\}$

□

Theorem 7. Let \mathcal{L}_1 and \mathcal{L}_2 be modal logics that admit definable filtrations. If $\text{Mod}(\mathcal{L}_1)$ and $\text{Mod}(\mathcal{L}_2)$ admit definable filtrations, so does $\text{Mod}(\mathcal{L}_1 * \mathcal{L}_2)$.

Proof. Let $\mathcal{M} = \langle W, R_1, R_2, \vartheta \rangle$ be an $\mathcal{L}_1 * \mathcal{L}_2$. We define a notation $\nabla = \{\neg \Diamond, \Diamond \neg, \Diamond\}$.

Both logics admit definable filtrations, so for every finite Sub-closed set Γ and for every \mathfrak{M} , an \mathcal{L}_1 -model (or an \mathcal{L}_2 one) there exists there exists Δ , a extension of Γ having the form:

$$\begin{aligned} \Delta_1 &= \Gamma \cup \text{Sub}\{\nabla_1 \nabla_2 \dots \nabla_n \Diamond \psi \mid \Diamond \psi \in \Gamma\} \text{ (for } \mathcal{L}_1) \\ \Delta_2 &= \Gamma \cup \text{Sub}\{\nabla_1 \nabla_2 \dots \nabla_k \Diamond \psi \mid \Diamond \psi \in \Gamma\} \text{ (for } \mathcal{L}_2) \end{aligned}$$

such that $\widehat{\mathfrak{M}} = \langle W / \sim_{\Delta_i}, \widehat{R}, \vartheta \rangle$ is a filtration of \mathfrak{M} through the corresponding Δ_i .

Let V be a set of fresh variables indexed over Γ as in the proof for a fusion of **K5** with something else. Let \mathcal{M}' be a model defined as previously. We extend V and Γ in the following way:

$$\begin{aligned}\Gamma' &= \Gamma \cup \text{Sub}\{\nabla_{11}\nabla_{21}\dots\nabla_{n1}\Diamond_1\psi \mid \Diamond_1\psi \in \Gamma\} \cup \text{Sub}\{\nabla_{12}\nabla_{22}\dots\nabla_{n2}\Diamond_2\psi \mid \Diamond_2\psi \in \Gamma\} \\ \Delta &= V \cup \text{Sub}\{\nabla_1\nabla_2\dots\nabla_n\Diamond p_\psi \mid \nabla_{n+1}\psi \in \Gamma'\} \cup \text{Sub}\{\nabla_1\nabla_2\dots\nabla_k\Diamond p_\psi \mid \Diamond_2\psi \in \Gamma\}.\end{aligned}$$

By the construction, $\sim_{\Gamma'} = \sim_{\Delta}$. So we have filtrations for the corresponding reducts of \mathcal{M}' through Δ as well as for the original \mathcal{M} . \square

References

- [1] Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal Logic*. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 2001.
- [2] Alexander Chagrov and Michael Zakharyashev. *Modal Logic*. Oxford Logic Guides, Volume 35. Oxford University Press, 1997.
- [3] Chen Chung Chang and H Jerome Keisler. *Model theory*. Elsevier, 1990.
- [4] Stanislav Kikot, Ilya Shapirovsky, and Evgeny Zolin. Completeness of logics with the transitive closure modality and related logics. *arXiv preprint arXiv:2011.02205*, 2020.
- [5] Agi Kurucz. Combining modal logics. *Studies in Logic and Practical Reasoning*, 3:869–924, 2007.
- [6] Valentin B Shehtman. Filtration via bisimulation. *Advances in modal logic*, 5:289–308, 2004.