Notes on filtrations for logics that contain **K5**

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1 Preliminaries

Let $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$ be a Kripke model and Γ a set of formulas closed under subformulas. An equivalence relation $\sim$ is set to have a finite index if the quotient set $W/\sim$ is finite. The equivalence relation $\sim_\Gamma$ induced by Γ is defined as

$$w \sim_\Gamma v \Leftrightarrow \forall \varphi \in \Gamma \, (\mathcal{M}, w \models \varphi \Leftrightarrow \mathcal{M}, v \models \varphi).$$

If Γ is finite, then $\sim_\Gamma$ has a finite index. An equivalence relation $\sim$ respects $\sim_\Gamma$, if $w \sim v$ implies $w \sim_\Gamma v$.

Definition 1. *Let $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$ be a Kripke model and Γ be a Sub-closed set formulas. A Γ-filtration of $\mathcal{M}$ is a model $\widehat{\mathcal{M}} = \langle \widehat{W}, \widehat{R_1}, \dots, \widehat{R_n}, \hat{\vartheta} \rangle$ such that:*

1. *$\widehat{W} = W/\sim$, where $\sim$ is an equivalence relation having a finite index that respects Γ*
2. *$\hat{\vartheta}(p) = \{[x]_\sim \mid x \in W \ \& \ x \in \vartheta(p)\}$*
3. *For each $i \in I$ one has $\widehat{R}_i^{min} \subseteq \widehat{R}_i \subseteq \widehat{R}_i^{max}$. $\widehat{R}_{i,\sim}^{min}$ is the i-th minimal filtered relation on $\widehat{W}$ defined as*

$$\hat{x} \widehat{R}_{i,\sim}^{min} \hat{y} \Leftrightarrow \exists x' \sim x \, \exists y' \sim y \, x R_i y$$

$\widehat{R}_{\Gamma,i}^{max}$ is the i-th maximal filtered relation on $\widehat{W}$ induced by Γ defined as

$$\hat{x} \widehat{R}_{\Gamma,i}^{max} \hat{y} \Leftrightarrow \forall \Box_i \varphi \in \Gamma \, (\mathcal{M}, x \models \Box_i \varphi \Rightarrow \mathcal{M}, y \models \varphi)$$

If Φ is finite subset of Γ and $\sim = \sim_\Phi$, then $\widehat{\mathcal{M}}$ is a definable Γ-filtration of $\mathcal{M}$ through Φ. If $\sim = \sim_\Gamma$, then such a filtration by means of the definiton above is called *strict.* A class of models $\mathbb{M}$ admits strict filtrations for models (ASF), if for every Sub-closed set Γ and for every $\mathcal{M} \in \mathbb{M}$ there exists a Γ filtration of $\mathcal{M}$. A class of frames $\mathbb{F}$ admits strict filtrations for frames, if for every Sub-closed set Γ and for every frame $\mathcal{F} \in \mathbb{F}$ and every model $\mathcal{M}$ over $\mathcal{F}$ there exists a Γ filtration of $\mathcal{M}$. If $\mathcal{L}$ is canonical, then the ASF property for frames and ASF property for models are equivalent [1, Theorem 2.10].

Lemma 1. *Let Γ be a finite set of formulas closed under subformulas and $\widehat{\mathcal{M}}$ a filtration of $\mathcal{M}$ through Γ, then for each $x \in W$ and for each $\varphi \in \Gamma$ one has*

$$\mathcal{M}, x \models \varphi \Leftrightarrow \widehat{\mathcal{M}}, \hat{x} \models \varphi$$

Definition 2. *Let $\mathbb{F}$ be a class of Kripke frames and Γ a finite set of formulas closed under subformulas. If for every model $\mathcal{M}$ over $\mathcal{F} \in \mathbb{F}$ there exists a model that is a Γ-definable filtration of $\mathcal{M}$, then $\mathbb{F}$ admits definable filtration. A class of models $\mathbb{M}$ admits definable filtration if for every $\mathcal{M} \in \mathbb{M}$ there exists a model belonging to the same class that is a definable Γ-filtration of $\mathcal{M}$.*

Lemma 2.

1. *Let $\mathcal{L}$ be a complete normal modal logic. If $\mathrm{Frames}(\mathcal{L})$ admits filtration, then $\mathcal{L}$ has the finite model property.*

2. *If the class of models $\mathrm{Mod}(\mathcal{L})$ admits filtration, then $\mathcal{L}$ has the finite model property and it is Kripke complete as well.*

Definition 3. *A first-order formula is called Horn if it has the following form:*

$$\forall x_1, \ldots, x_n (x_{i_1} R x_{j_1} \wedge \cdots \wedge x_{i_s} R x_{j_s} \to x_k R x_l)$$

Definition 4. *Let H be a Horn property and $\langle W, R \rangle$ a Kripke frame. A Horn closure of a binary relation R is the minimal relation R^H containing R and satisfying H.*

Lemma 3. $R^H = \bigcup\limits_{n<\omega} R_n$ *where*

1. $R_0 = R$.

2. $R_{n+1} = R_n \cup \{(a, b) \in W \mid \exists \vec{c} \in W\ P(a, b, \vec{c})\}$, *where P is a premise of H.*

2 Filtrations for K5

E-closure (an Euclidean Horn closure of a binary relation) has the following equivalent definitions:

Lemma 4. *Let $\mathcal{F} = \langle W, R \rangle$ be a Kripke frame. The following conditions are equivalent:*

1. *R^E is the smallest Euclidean relation containing R.*

2. $R^E = \bigcup\limits_{i<\omega} R_i$, *where*
    - $R_0 = R$
    - $R_{n+1} = R_n \cup (R_n^{-1} \circ R_n)$

3. *$xR^E y$ iff there exists $n < \omega$ such that either xRy or $\exists z_1, \ldots, z_n$ with $z_1 R x$ and $z_{n-1} R y$ and for each $1 < i \leqslant n$ one has either $z_{i-1} R z_i$ or $z_i R z_{i-1}$.*

4. $R^E = R \cup \bigcup\limits_{i<\omega} (R^{-1} \circ (R \circ R^{-1})^n \circ R)$.

Proof.

1. (1) $\Rightarrow$ (2) Let us show that if R^E is the smallest Euclidean relation containing R, then $R^E = \bigcup\limits_{i<\omega} R_i$. There are two inclusions:
    - $R^E \subseteq \bigcup\limits_{i<\omega} R_i$. Recall that R^E has the form (?):

$$R^E = \bigcap\{R' \mid R \subseteq R', \forall a, b \in W\ R'(a,b) \Rightarrow \exists x \in W\ R'(x,a)\ \&\ R'(x,b)\}$$

- $\bigcup\limits_{i<\omega} R_i \subseteq R^E$. Let us show that xR_ny for each $n < \omega$ implies xR^Ey by induction on n. If $n = 0$, then xRy, thus, xR^Ey, since R is a subrelation of R^E. Suppose $n = m+1$ and $xR_{m+1}y$. Let us show that xR^Ey. From $xR_{m+1}y$, one has $(x,y) \in R^n \cup (R_n^{-1} \circ R_n)$. There are two cases:
  - xR^ny, one needs to merely apply the IH.
  - $xR_n^{-1} \circ R_ny$. Then $\exists z \in W\ xR_n^{-1}z\ \&\ zR_n$. That is, zR_nx and zR_ny for some z. R_n is already a subrelation of R^E. Thus, zR^Ex and zR^Ey. That implies xR^Ey.

2. (2) $\Rightarrow$ (3) Let $(x,y) \in R_m$, let us the statement by induction on m.
   (a) Suppose $m = 0$, then xRy, and the statement is shown putting $n = 0$.
   (b) Suppose $m = p+1$ and $xR_{p+1}y$. Assume that either xRy or $\exists z_1, \dots, z_p$ with z_1Rx and $z_{p-1}Ry$ and for each $1 < i \leqslant p$ one has either $z_{i-1}Rz_i$ or z_iRz_{i-1}.
   $xR_{p+1}y$ implies $(x,y) \in R_p \cup (R_p^{-1} \circ R_p)$. If $(x,y) \in R_p$, then we merely apply the IH. Suppose $(x,y) \in R_p^{-1} \circ R_p$, then $(z,x) \in R_p$ and $(z,y) \in R_p$
3. (3) $\Rightarrow$ (4) Suppose either xRy or there exist $n \geqslant 1$ and $z_1, \dots, z_n$ with z_1Rx and $z_{n-1}Ry$ and for each $1 < i \leqslant n$ one has either $z_{i-1}Rz_i$ or z_iRz_{i-1}. If xRy, then we are done. Otherwise there exists $n \geqslant 1$ with the condition above. Then $(x,y) \in R_{n+1}$ that follows from the condition.
4. (4) $\Rightarrow$ (1)

$\square$

Lemma 5. *Let $\mathcal{F} = \langle W, R \rangle$ be a Kripke frame. Let us define $R^E = \bigcup\limits_{i<\omega} R_i$ where:*

1. $R_0 = R$
2. $R_{n+1} = R_n \cup ({R_n}^{-1} \circ R_n)$

Then R^E is Euclidean.

Proof. Let $(x,y), (x,z) \in R^E$, one needs to show that $(y,z) \in R^E$. Clearly that $(x,y) \in R_i$ and $(x,z) \in R_j$ for some $i, j < \omega$. Thus, we need $(y,z) \in R_m$ for some m depending on i and j.

Let us consider the following cases:

1. $i = 0$ and $j = 0$
   Suppose $(x,y), (x,z) \in R_0 = R$, then $(y,z) \in R^{-1} \circ R$. Thus, $(y,z) \in R_1$
2. $i = 0$ and $j = k+1$
   Suppose $(x,y) \in R$ and $(x,z) \in R_{k+1} = R_k \cup ({R_k}^{-1} \circ R_k)$. Clearly that $(x,y) \in R_{k+1}$ as well. It is obviously that $(y,z) \in R_{k+2}$ since $(y,x) \in R_{k+1}^{-1}$ and $(x,z) \in R_{k+1}$.
3. The case with $i = k+1$ and $j = 0$ is similar to the previous one.
4. Suppose $i = m+1$ and $j = k+1$. That is, $(x,y) \in R_{m+1} = R_m \cup ({R_m}^{-1} \circ R_m)$ and $(x,z) \in R_{k+1} = R_k \cup ({R_k}^{-1} \circ R_k)$. Consider the following four subcases:

(a) Suppose $(x, y) \in R_m$ and $(x, z) \in R_k$ and $m \leqslant k$ without loss of generality. $m \leqslant k$ implies $R_m \subseteq R_k$ and $(x, y) \in R_k$ in particular. Thus, $(y, z) \in {R_k}^{-1} \circ R_k$, so $(y, z) \in R_{k+1}$.

(b) The rest of the cases are similar to the first one.

$\square$

Theorem 1. **K**45 *admits strict filtrations.*

Proof. Let $\mathcal{M} = \langle W, R, \vartheta \rangle$ be a transitive Euclidean model and $\overline{\mathcal{M}} = \langle \overline{W}, \overline{R}, \overline{\vartheta} \rangle$ its minimal filtration through Γ, where Γ is finite and Sub-closed. Let us put $\widehat{R} = \overline{R}^{+} \cup \overline{R}^{E}$. Let us show that $\overline{R}^{+} \cup \overline{R}^{E} \subseteq \overline{R}^{max}$.

That is, if $\mathcal{M}, y \models \varphi$ for $\Diamond\varphi \in \Gamma$ and $\hat{x}\widehat{R}\hat{y}$, then $\mathcal{M}, x \models \Diamond\varphi$.

Let $\hat{x}\widehat{R}\hat{y}$. Let us consider the case when $(\hat{x}, \hat{y}) \in \overline{R}^{E}$

1. Suppose $(\hat{x}, \hat{y}) \in \overline{R}$, then $\mathcal{M}, x \models \Diamond\varphi$ holds trivially by the definition of the minimal filtration.
2. Suppose the statement holds $\overline{R}_n$ and $(\hat{x}, \hat{y}) \in \overline{R}_{n+1} = \overline{R}_n \cup (\overline{R}_n^{-1} \circ \overline{R}_n)$. We consider the case of $(\hat{x}, \hat{y}) \in (\overline{R}_n^{-1} \circ \overline{R}_n)$.

   Then there exists $\hat{z}$ such that $(\hat{z}, \hat{x}), (\hat{z}, \hat{y}) \in \overline{R}_n$.

   By IH, $\mathcal{M}, z \models \Diamond\varphi$.

   $(\hat{z}, \hat{y}) \in \overline{R}_n$ iff there are $\hat{u}_1, \ldots, \hat{u}_n$ such that

   $$\hat{z} \xleftarrow[\widehat{R}]{} \hat{u}_1 \xrightarrow[\widehat{R}']{} \hat{u}_2 \xrightarrow[\widehat{R}']{} \ldots \xrightarrow[\widehat{R}']{} \hat{u}_{n-1} \xrightarrow[\widehat{R}']{} \hat{u}_n \xrightarrow[\widehat{R}]{} \hat{y}$$

   where $\widehat{R}'$ is either $\widehat{R}$ or $\widehat{R}^{-1}$.

As it is known, $\Diamond\Diamond\varphi \to \Box\Diamond\varphi \in \mathbf{K}45$.

$\hat{u}_1\widehat{\hat{z}}$, that is, $u_1' R z'$ for some $u_1' \in \hat{u}_1$ and $z' \in \hat{z}$. That is, $\mathcal{M}, u_1' \models \Diamond\Diamond\varphi$, so $\mathcal{M}, u_1' \models \Diamond\varphi$ and $\overline{M}, \hat{u}_1 \models \Diamond\varphi$.

We have $\hat{u}_1\widehat{R}'\hat{u}_2$. Suppose $\mathcal{M}, u_1'' \models \Diamond\varphi$ and $u_1'' R u_2'$. We also have $\mathcal{M}, u_1'' \models \Box\Diamond\varphi$, thus, $\mathcal{M}, u_2' \models \Diamond\varphi$.

Suppose $\hat{u}_2\widehat{R}\hat{u}_1$ and $u_2' R u_1''$, then $\mathcal{M}, u_2' \models \Diamond\varphi$.

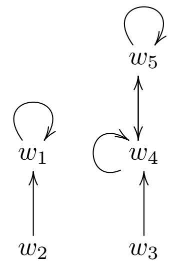
Similarly, we have $\mathcal{M}, u_i \models \Diamond\varphi$ iff $\mathcal{M}, u_{i+1} \models \Diamond\varphi$, whenever $\hat{u}_i\widehat{R}'\hat{u}_{i+1}$.

Finally, we have $\hat{u}_n\widehat{R}\hat{x}$. Thus, $u_n' R x'$ for some $u_n' \in \hat{u}_n$ and $x' \in \hat{x}$. $\mathcal{M}, u_n' \models \Diamond\varphi$, so $\mathcal{M}, u_n' \models \Box\Diamond\varphi$. Then $\mathcal{M}, x' \models \Diamond\varphi$. $\square$

Theorem 2. **K**5 *does not admit strict filtrations.*

Proof. Let us consider a **K**5 model whose Euclidean closure of the minimal filtration does not give us a filtration.

Let us consider a frame called $\mathcal{F}_{bad}$. We define this frame with the following graph:



Let us define a valuation ϑ such that $\vartheta(p) = \{w_5\}$ and $\vartheta(q) = \{w_1\}$. Let us consider a minimal filtration of $\mathcal{M}_{bad}$ through the Sub-closure of $\Gamma = \{\neg p, \neg\Diamond p\}$.

Clearly that $w_2 \sim_\Gamma w_3$, since $\neg p$ and $\neg\Diamond p$ are true both at w_2 and w_3.

Moreover, $R_{min} \cup (R_{min}^{-1} \circ R_{min})$ is not a subset of R_{max} since $(\hat{w_1}, \hat{w_5}) \in (R_{min}^{-1} \circ R_{min})$, but $\Diamond p$ is not true at w_5.

Let us also note that strict filtrations of this model is not Euclidean. Suppose by contrary that $\widehat{R}^{\mathcal{E}}$ is a strict filtraction of that model. So $R_{min}^{E} \subseteq \widehat{R}^{\mathcal{E}}$, since R_{min}^{E} is the minimal Euclidean relation containing R_{min}. On the other hand, $R_{min}^{E} \nsubseteq R_{max}$, so is not $\widehat{R}^{\mathcal{E}}$.

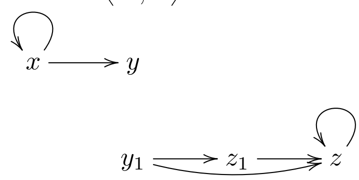
$\square$

Let us define the logic $\mathcal{L}$ as $\mathbf{K} \oplus \Diamond\Diamond\Diamond p \to \Diamond p$. Let R be a binary relation, the $\mathcal{L}$-closure of R is defined (denoted as $R^{\text{✡}}$) as the following union:

$$R^{\text{✡}} = R \cup R^3 \cup R^5 \cup \dots \cup R^{2k+1} \cup \dots$$

Theorem 3. *$\mathcal{L}$ does not admit strict filtrations.*

Proof. Consider the following frame $\mathcal{F} = \langle W, R\rangle$:



Clearly that $\mathcal{F}$ is an $\mathcal{L}$-frame. We define the valuation ϑ as follows:

$$\vartheta(p) = \{x\}$$
$$\vartheta(q) = \{y, y_1\}$$
$$\vartheta(r) = \{z\}$$

Let us put $\Gamma = \mathrm{Sub}\{p, q, \Diamond r\}$. We factorise W through $\sim_\Gamma$ and consider a model $\widehat{\mathcal{M}} = \langle W/\sim_\Gamma, \widehat{R}, \widehat{\vartheta}\rangle$, where $\widehat{R} = (\widehat{R}_{min})^{\text{✡}}$. We have $(\hat{x}, \hat{z}) \in \widehat{R} \circ \widehat{R} \circ \widehat{R}$, but $\Diamond r$ is not true at x. $\square$

3 Finite "canonical" models

Let $\mathcal{L}$ be a normal modal logic, $\mathcal{M}_{\mathcal{L}}$ its canonical model, and Γ a finite Sub-closed set of formulas. Let us put $\Gamma' = \mathrm{Sub}(\varphi) \cup \{\neg\psi \mid \psi \in \mathrm{Sub}(\varphi)\}$.

A subset $\Delta \subseteq' \Gamma$ is a *finite $\mathcal{L}$-consistent set* if $\neg \bigwedge \Delta \notin \mathcal{L}$. A subset Δ is maximal, if (the following are obviously equivalent):

1. Δ is maximal amongst finite $\mathcal{L}$-consistent sets,

2. For each $\psi \in \mathrm{Sub}(\varphi)$ either $\psi \in \Delta$ or $\neg\psi \in \Delta$.

Every finite $\mathcal{L}$-theory is clearly can be extended to some maximal one. It is the finite version of Lindenbaum's lemma.

Definition 5. *Let $\mathcal{L}$ be a modal logic and Γ be a finite* Sub*-closed set of formulas. A finite "canonical" model is a triple $\mathcal{M}_{\mathcal{L}}^{\Gamma} = \langle W_{\mathcal{L}}^{\Gamma}, R_{\mathcal{L}}^{\Gamma}, \vartheta_{\mathcal{L}}^{\Gamma}\rangle$, where*

1. *$W_{\mathcal{L}}^{\Gamma}$ is the set all maximal theories that extend finite $\mathcal{L}$-theories*
2. *$R_{\mathcal{L}}^{\Gamma}$ is a relation such that $\langle W_{\mathcal{L}}^{\Gamma}, R_{\mathcal{L}}^{\Gamma}\rangle$ is an $\mathcal{L}$-frame and*
$$\forall \Box\psi \in \mathrm{Sub}(\varphi)\ \forall \Delta_1 \in W_{\mathcal{L}}^{\Gamma}\ (\Box\psi \in \Delta_1 \Leftrightarrow \forall \Delta_2 \in R_{\mathcal{L}}^{\Gamma}(\Delta_1)\ \psi \in \Delta_2)$$
3. *$\vartheta_{\mathcal{L}}^{\Gamma}(p) = \{\Delta \in W_{\mathcal{L}}^{\varphi} \mid p \in \Delta\}$ for every variable $p \in \Gamma$.*

Lemma 6. *Let $\mathcal{L}$ be a modal logic and $\varphi \notin \mathcal{L}$, then $\mathcal{M}_{\mathcal{L}}^{\mathrm{Sub}(\varphi)} \not\models \varphi$.*

Lemma 7. *Let $\mathcal{L}$ be a modal logic and Γ a finite* Sub*-closed set of formulas, then if $\mathcal{L}$ admits strict filtrations, then there exists a finite "canonical" model $\mathcal{M}_{\mathcal{L}}^{\Gamma}$ such that $\mathcal{M}_{\mathcal{L}}^{\Gamma} \models \mathcal{L}$.*

Proof. ($\Rightarrow$) Let Γ be a finite Sub-closed of formulas. $\mathcal{L}$ admits strict filtrations, so the filtration of the canonical model $\mathcal{M}_{\mathcal{L}}$ through Γ is also an $\mathcal{L}$-model. The underlying set of $\mathcal{M}_{\mathcal{L}}/\sim_{\Gamma}$ consists of maximal $\mathcal{L}$ theories up to Γ-equivalence and this quotient set is finite.

It is readily checked that the quotient model $\mathcal{M}_{\mathcal{L}}/\sim_{\Gamma}$ satifsfies Definition 5. □

The converse implication does not have to true generally. **GL** might be an example of a logic that has the "finite canonical" model property with no filtrations.

4 Fusion stuff

Definition 6. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be modal logics, then the fusion $\mathcal{L}_1 * \mathcal{L}_2$ is the minimal bimodal logic that contains $\mathcal{L}_1$ and $\mathcal{L}_2$ [2].*

Lemma 8.

- *Let Γ be a finite and* Sub*-closed set of formulas and $\mathcal{M} = \langle W, R, \vartheta\rangle$. Consider $\Gamma' = \Gamma \cup \{\Diamond\Box\psi \mid \Box\psi \in \Gamma\}$. Let Δ be any finite and* Sub*-closed extension of Γ'. Then a model $\widehat{M} = \langle W/\sim_{\Delta}, (R_{\Delta}^{min})^E, \widehat{\vartheta}\rangle$ is a filtration of $\mathcal{M}$ through Δ.*
- *The same for $\mathbf{K} \oplus \Diamond\Diamond\Diamond p \to \Diamond p$*

Proof. Recall that $(R_{\Delta}^{min})^E$ is defined inductively as:

1. $R_{\Delta}^0 = R_{\Delta}^{min}$
2. $R_{\Delta}^{n+1} = R_{\Delta}^n \cup (R_{\Delta}^{n\,-1} \circ R_{\Delta}^n)$
3. $(R_{\Delta}^{min})^E = \bigcup_{k<\omega} R_{\Delta}^k$

If R_{Δ}^{min} is already a subrelation of R^{max}, so the base case is self-evident.

Suppose the statement holds for R_{Δ}^n, $(\hat{x}, \hat{y}) \in R_{\Delta}^{n+1}$, that is, there exists $\hat{z}$ such that $(\hat{z}, \hat{x}), (\hat{z}, \hat{y}) \in R_{\Delta}^n$. Let $\mathcal{M}, x \models \Box\psi$ (for $\Box\psi \in \Gamma$).

We rewrite $(\hat{z}, \hat{x}) \in R_{\Delta}^n$ as the following sausage (for some $\widehat{u_1}, \widehat{u_2}, \ldots, \widehat{u_{n-1}}, \widehat{u_n}$):

$$\hat{z} \xleftarrow{R_{\Delta}^{min}} \widehat{u_1} \xrightarrow{R'} \widehat{u_2} \xrightarrow{R'} \ldots \xrightarrow{R'} \widehat{u_{n-1}} \xrightarrow{R'} \widehat{u_n} \xrightarrow{R_{\Delta}^{min}} \hat{x}$$

where R' is either R_{Δ}^{min} or its converse. We have $\mathcal{M}, z \models \Box\psi$, since $\mathcal{M}, x \models \Box\psi$ implies $\mathcal{M}, u_n \models \Diamond\Box\psi$. After that we apply the following property of $\mathbf{K}5$-models:

For each $a, b \in M_i$ such that aR_ib we have $\mathcal{M}_i, a \models \Diamond\Box p_\psi$ iff $\mathcal{M}_i, b \models \Diamond\Box p_\psi$

Note that we always stay within Δ since $\Gamma' \subseteq \Delta$. Finally, we conclude $\mathcal{M}, x \models \psi$ from IH and $\mathcal{M}, z \models \Box\psi$.

The second item is shown similarly. $\Box$

Theorem 4.

1. *$\mathbf{K}5 * \mathbf{K}5$ admits definable filtrations.*
2. *$\mathbf{K}5 * \cdots * \mathbf{K}5$ admits definable filtrations.*
3. *If $\mathcal{L}$ admits strict filtrations, then $\mathbf{K}5 * \mathcal{L}$ admits definable filtrations*
4. *If $\mathcal{L}_1, \ldots, \mathcal{L}_n$ admit strict filtrations, then $\mathbf{K}5 * \cdots * \cdots * \mathbf{K}5 * \mathcal{L}_1 * \cdots * \mathcal{L}_n$*
5. *Let $\mathcal{L} = \mathbf{K} \oplus \Diamond\Diamond\Diamond p \to \Diamond p$ (here and below), then $\mathcal{L} * \mathcal{L}$ admits definable filtrations.*
6. *Let $\mathcal{L} = \mathbf{K} \oplus \Diamond\Diamond\Diamond p \to \Diamond p$ and $\mathcal{L}_1$ a logic that admits strict filtrations, then $\mathcal{L} * \mathcal{L}_1$*

Proof.

1. Let Γ be a finite Sub-closed set of bimodal formulas, $\mathcal{F} = \langle W, R_1, R_2 \rangle$ a $\mathbf{K}5 * \mathbf{K}5$-frame, and ϑ a valuation on $\mathcal{F}$. Denote $\langle \mathcal{F}, \vartheta \rangle$ as $\mathcal{M}$.

   We introduce the set of fresh variables $V = \{p_\psi | \psi \in \Gamma\}$ and define a new model $\mathcal{M}' = \langle \mathcal{F}, \vartheta' \rangle$ as follows:

   $$\text{For all } \psi \in \Gamma,\ \mathcal{M}, x \models \psi \Leftrightarrow \mathcal{M}', x \models \psi \Leftrightarrow \mathcal{M}', x \models p_\psi.$$

   Consider these modifications of Γ and V:

   $$\Gamma' = \Gamma \cup \{\Diamond_1\Box_1\psi \mid \Box_1\psi \in \Gamma\} \cup \{\Diamond_2\Box_2\psi \mid \Box_2\psi \in \Gamma\}$$
   $$\Delta = V \cup \mathrm{Sub}(\{\Diamond\Box p_\psi \mid \Box_i\psi \in \Gamma, in = 1, 2\})$$

   Let us define an equivalence relation $\sim_\Delta$ induced by Δ.

   Consider $\mathcal{M}_i = \langle W, R_i, \vartheta' \rangle$, a reduct of $\mathcal{M}'$, we have:

   (a) $\mathcal{M}_i, x \models \Box p_\psi$ iff $\mathcal{M}, x \models \Box_i\psi$
   (b) $\mathcal{M}_i, x \models \Diamond\Box p_\psi$ iff $\mathcal{M}, x \models \Diamond_i\Box_i\psi$

   So $\sim = \sim_{\Gamma'}$ by the construction. Let us put $\widehat{W} = W/\sim_{\Gamma'}$. Lemma 8 implies the following claim:

   Claim 1. *Let $\hat{R}_i = (R^{min}_\Delta)^E$ and $\widehat{\vartheta(p)} = \{[x]_{\sim_i} \mid \mathcal{M}_i, x \models p\}$ for $p \in \Delta_1$, define $\widehat{\mathcal{M}}_i = \widehat{W}, \hat{R}_i, \hat{\vartheta}$. Then $\widehat{\mathcal{M}}_i \models \mathbf{K}5$ and $\widehat{\mathcal{M}_i}$ is a filtration of $\mathcal{M}_i$ through Δ.*

   Finally, we consider a model $\widehat{\mathcal{M}} = \langle \widehat{W}, \hat{R}_1, \hat{R}_2, \vartheta \rangle$, where $\widehat{R_{\Gamma'}}_i = {R_i}_{\Gamma'}^{min\,E}$ and $\vartheta(p)$ is defined as usual for $p \in \Gamma$. $\widehat{\mathcal{M}}$ is a filtration of $\mathcal{M}$ through Γ'.

   Let $\hat{x}\widehat{R_{\Gamma'}}_i\hat{y}$ and $\mathcal{M}, x \models \Box_i\psi$ for $\Box_i\psi \in \Gamma$. Then $\mathcal{M}_i, x \models \Box p_\psi$, so $\widehat{\mathcal{M}_i}, \hat{x} \models \Box p_\psi$. By the claim above, $\widehat{\mathcal{M}_i}$ is a filtration of $\mathcal{M}_i$ through Δ, so $\mathcal{M}_i, y \models p_\psi$. Then $\mathcal{M}, y \models \psi$.

2. Likewise

3. The argument is the same as in the proof of the first item, except for **Claim** 1 that has the following formulation: Let $\widehat{R}_1 = (R^{min}_\Delta)^E$ and $\widehat{R}_2 = (R^{min}_\Delta)^{\mathcal{L}_1}$ Define a valuation as usual as $\widehat{\vartheta(p)} = \{[x]_{\sim_i} \mid \mathcal{M}_i, x \models p\}$ for $p \in \Delta_1$, define $\widehat{\mathcal{M}}_1 = \langle \widehat{W}, \widehat{R}_1, \widehat{\vartheta}\rangle$ and $\widehat{\mathcal{M}}_2 = \langle \widehat{W}, \widehat{R}_2, \widehat{\vartheta}\rangle$. Then $\widehat{\mathcal{M}}_1 \models \mathbf{K}5$ and $\widehat{\mathcal{M}}_1 \models \mathbf{L}$ and $\widehat{\mathcal{M}}_i$ is a filtration of $\mathcal{M}_i$ through Δ.

4. Likewise

5. The argument is similar to the proof of first item, but filtrations are slightly different. Let $\mathcal{M} = \langle W, R_1, R_2, \vartheta\rangle$ be a $\mathcal{L} * \mathcal{L}$ model and Γ a Sub-closed set of formulas. As above, we define a set V and a model $\mathcal{M}'$. Define extensions of Γ and V:

$$\Gamma' = \Gamma \cup \{\Diamond_1\Diamond_1\psi \mid \Diamond_1\psi \in \Gamma\} \cup \{\Diamond_2\Diamond_2\psi \mid \Diamond_2\psi \in \Gamma\}$$
$$\Delta = V \cup \mathrm{Sub}(\{\Diamond\Diamond p_\psi \mid \Diamond\psi \in \Gamma, i = 1, 2\})$$

As above $\sim_\Delta = \Gamma'$ and $\widehat{\mathcal{M}'} = \langle W/\sim_\Delta, \widehat{R_i}, \widehat{\vartheta}\rangle$ are filtrations of reducts of $\mathcal{M}'$ through Δ. Then $\widehat{\mathcal{M}} = \langle W/\sim_\Delta, \widehat{R_1}, \widehat{R_2}, \widehat{\vartheta}\rangle$ is a required filtration of the original $\mathcal{M}$.

6. Extend Γ with $\{\Diamond_i\Diamond_i\psi \mid \Diamond_i\psi \in \Gamma, i = 1, 2\}$ and V with $\{\Diamond\Diamond p_\psi \mid \Diamond_i\psi, i = 1, 2\}$

□

Theorem 5. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be modal logics that admit definable filtrations. If* $\mathrm{Mod}(\mathcal{L}_1)$ *and* $\mathrm{Mod}(\mathcal{L}_2)$ *admit definable filtrations, so does* $\mathrm{Mod}(\mathcal{L}_1 * \mathcal{L}_2)$*.*

Proof. Let $\mathcal{M} = \langle W, R_1, R_2, \vartheta\rangle$ be an $\mathcal{L}_1 * \mathcal{L}_2$. We define a notation $\nabla = \{\neg\Diamond, \Diamond\neg, \Diamond\}$.

Both logics admit definable filtrations, so for every finite Sub-closed set Γ and for every $\mathfrak{M}$, an $\mathcal{L}_1$-model (or an $\mathcal{L}_2$ one) there exists there exists Δ, a extension of Γ having the form:

$$\Delta_1 = \Gamma \cup \mathrm{Sub}\{\nabla_1\nabla_2 \ldots \nabla_n\Diamond\psi \mid \Diamond\psi \in \Gamma\} \text{ (for } \mathcal{L}_1)$$
$$\Delta_2 = \Gamma \cup \mathrm{Sub}\{\nabla_1\nabla_2 \ldots \nabla_k\Diamond\psi \mid \Diamond\psi \in \Gamma\} \text{ (for } \mathcal{L}_2)$$

such that $\widehat{\mathfrak{M}} = \langle W/\sim_{\Delta_i}, \widehat{R}, \vartheta\rangle$ is a filtration of $\mathfrak{M}$ through the corresponding Δ_i.

Let V be a set of fresh variables indexed over Γ as in the proof for a fusion of $\mathbf{K5}$ with something else. Let $\mathcal{M}'$ be a model defined as previously. We extend V and Γ in the following way:

$$\Gamma' = \Gamma \cup \mathrm{Sub}\{\nabla_{11}\nabla_{21} \ldots \nabla_{n1}\Diamond_1\psi \mid \Diamond_1\psi \in \Gamma\} \cup \mathrm{Sub}\{\nabla_{12}\nabla_{22} \ldots \nabla_{n2}\Diamond_2\psi \mid \Diamond_2\psi \in \Gamma\}$$
$$\Delta = V \cup \mathrm{Sub}\{\nabla_1\nabla_2 \ldots \nabla_n\Diamond p_\psi \mid \nabla_{n+1\,1}\psi \in \Gamma'\} \cup \mathrm{Sub}\{\nabla_1\nabla_2 \ldots \nabla_k\Diamond p_\psi \mid \Diamond_2\psi \in \Gamma\}.$$

By the construction, $\sim_{\Gamma'} = \sim_\Delta$. So we have filtrations for the corresponding reducts of $\mathcal{M}'$ through Δ as well as for the original $\mathcal{M}$. □

References

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