Note on filtration of logics containing **K5**

Daniel Rogozin

1 Preliminaries

Definition 1. *An n-normal modal logic is a set of formulas that contains all Boolean tautologies, formulas $\Diamond_i p \vee \Diamond_i q \leftrightarrow \Diamond_i (p \vee q)$ and $\Diamond_i \bot \leftrightarrow \bot$ for $i \leqslant n$, and is closed under modus ponens, substitution, and monotonicity: from $\varphi \to \psi$ infer $\Diamond_i \varphi \to \Diamond_i \psi$ for $i \leqslant n$.*

Definition 2. *An n-Kripke model is a triple $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$, where $R_i \subseteq W \times W$, $\vartheta : \mathrm{PV} \to 2^W$, and the connectives have the following semantics:*

1. $\mathcal{M}, w \models p \Leftrightarrow w \in \vartheta(p)$
2. $\mathcal{M}, w \models \neg\varphi \Leftrightarrow \mathcal{M}, w \not\models \varphi$
3. $\mathcal{M}, w \models \varphi \vee \psi \Leftrightarrow \mathcal{M}, w \models \varphi$ *or* $\mathcal{M}, w \models \psi$
4. $\mathcal{M}, w \models \Diamond_i \varphi \Leftrightarrow \exists v \in R_i(w)\, \mathcal{M}, v \models \varphi$

By **K5** we mean the logic $\mathbf{K} \oplus A5$, where $A5 = \Diamond p \to \Box \Diamond p$. It is known that **K5** is the modal logic of all Euclidean frames. A frame is called Euclidean if for each x, y, z, xRy and xRz implies yRz.

Proposition 1. *Let $\mathcal{F} = \langle W, R \rangle$ be an Euclidean frame.*

1. *For each $x, y, z \in W$, xRy and xRz implies either yRz or zRy.*
2. *$R \subseteq R;R$, that is, R is dense.*
3. *For each $x \in W$, $R^*(x) = \{x\} \cup R(R(x))$.*

Let $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$ be a Kripke model and Γ a set of formulas closed under subformulas. An equivalence relation $\sim$ is set to have a finite index if the quotient set $W/\sim$ is finite. The equivalence relation $\sim_\Gamma$ induced by Γ is defined as

$$w \sim_\Gamma v \Leftrightarrow \forall \varphi \in \Gamma\, (\mathcal{M}, w \models \varphi \Leftrightarrow \mathcal{M}, v \models \varphi).$$

If Γ is finite, then $\sim_\Gamma$ has a finite index. An equivalence relation $\sim$ respects $\sim_\Gamma$, if $w \sim v$ implies $w \sim_\Gamma v$.

Definition 3. *Let $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$ be a Kripke model and Γ be a Sub-closed set formulas. A Γ-filtration of $\mathcal{M}$ is a model $\widehat{\mathcal{M}} = \langle \widehat{W}, \widehat{R_1}, \dots, \widehat{R_n}, \widehat{\vartheta} \rangle$ such that:*

1. *$\widehat{W} = W/\sim$, where $\sim$ is an equivalence relation having a finite index that respects Γ*
2. *$\widehat{\vartheta}(p) = \{[x]_\sim \mid x \in W \;\&\; x \in \vartheta(p)\}$*

3. *For each $i \in I$ one has $\widehat{R}_i^{min} \subseteq \widehat{R}_i \subseteq \widehat{R}_i^{max}$. $\widehat{R}_{i,\sim}^{min}$ is the i-th minimal filtered relation on $\widehat{W}$ defined as*

$$\hat{x}\widehat{R}_{i,\sim}^{min}\hat{y} \Leftrightarrow \exists x' \sim x\, \exists y' \sim y\, xR_iy$$

$\widehat{R}_{\Gamma,i}^{max}$ is the i-th maximal filtered relation on $\widehat{W}$ induced by Γ defined as

$$\hat{x}\widehat{R}_{\Gamma,i}^{max}\hat{y} \Leftrightarrow \forall \Box_i\varphi \in \Gamma\, (\mathcal{M}, x \models \Box_i\varphi \Rightarrow \mathcal{M}, y \models \varphi)$$

If Φ is finite subset of Γ and $\sim = \sim_\Phi$, then $\widehat{M}$ is a definable Γ-filtration of $\mathcal{M}$ through Φ. If $\sim = \sim_\Gamma$, then such a filtration by means of the definiton above is called *strict*.

Lemma 1. *Let Γ be a finite set of formulas closed under subformulas and $\widehat{M}$ a filtration of $\mathcal{M}$ through Γ, then for each $x \in W$ and for each $\varphi \in \Gamma$ one has*

$$\mathcal{M}, x \models \varphi \Leftrightarrow \widehat{M}, \hat{x} \models \varphi$$

Definition 4. *Let $\mathbb{F}$ be a class of Kripke frames and Γ a finite set of formulas closed under subformulas. If for every model $\mathcal{M}$ over $\mathcal{F} \in \mathbb{F}$ there exists a model that is a Γ-definable filtration of $\mathcal{M}$, then $\mathbb{F}$ admits definable filtration. A class of models $\mathbb{M}$ admits definable filtration if for every $\mathcal{M} \in \mathbb{M}$ there exists a model belonging to the same class that is a definable Γ-filtration of $\mathcal{M}$.*

Lemma 2.

1. *Let $\mathcal{L}$ be a complete normal modal logic. If* $\mathrm{Frames}(\mathcal{L})$ *admits filtration, then $\mathcal{L}$ has the finite model property.*
2. *If the class of models* $\mathrm{Mod}(\mathcal{L})$ *admits filtration, then $\mathcal{L}$ has the finite model property and Kripke complete as well.*

2 Filtration of Euclidean logics

First of all, let us ensure that a filtration of an Euclidean frame is not necessary finite. Let $[x] \sim_\Gamma [y]$ and $[x] \sim_\Gamma [z]$. Then for some $x' \in [x]$ $y' \in [y]$, one has $x'Ry'$ and $x''Rz'$ for some $x'' \in [x]$ and $z' \in [z]$. Clearly, we cannot claim that $x' = x''$ in general. Thus, minimal filtration does not preserve the required property.

2.1 Clusters

Let $\mathcal{F} = \langle W, R\rangle$ be a transitive frame. Let us put $xR^\bullet y \Leftrightarrow xRy \,\&\, \neg(xRy)$. A point x is proper if xRx. Let us define the following equivalence relation:

$$x \equiv y \Leftrightarrow xRy \,\&\, yRx \vee x = y.$$

A cluster is an element of the quotient set $W/\equiv$. Given $x \in W$, C_x is a cluster containing x. Thus $C_x = \{x\} \cup \{y \,|xRyx\}$. The original relation lifts to the antisymmetric transitive relation on $W/\equiv$ defined as C_xRC_y iff xRy. A cluster C is called maximal if CRC' implies $C = C'$. A point is R-maximal if C_x is a maximal cluster, that is, $R^\bullet(x) = \varnothing$. A degenerated cluster is a singleton $\{x\}$ with $\neg(xRx)$. A cluster is called simple if it has the form $\{x\}$ with xRx. If $\langle W', R'\rangle$ is an inner substructure of $\langle W, R\rangle$, then every R'-cluster is an R-cluster and every R-cluster that intersects W' is a subset of W' and is an R'-cluster itself. Given a Kripke model $\mathcal{M}$, a set of formulas Γ is satisfied by a cluster C if every member of Γ is true at some point of C.

If clusters coincide then their poitns have the same theory in the original model:

Lemma 3. $C_x = C_y$ *implies* $\mathcal{M}, x \models \Box\varphi \Leftrightarrow \mathcal{M}, y \models \Box\varphi$

Let us describe the bulldozing technique allowing one to eliminate nondegenerated clusters [3]. Let $\mathcal{L}$ be a transitive logic and $\mathcal{F}$ its frame. We construct first a frame $\mathcal{F}^0 = \langle W^0, R^0 \rangle$ replacing every nondegenerated frame C of W by $C^0 = \{\langle x, i\rangle \mid x \in C, i < \omega\}$. We also replace each degenerated cluster C by $\{\langle x, 0\rangle\}$. Elements of these subsets form W^0. The relation R^0 is defined as

$$\langle x, i\rangle R^0 \langle y, j\rangle \Leftrightarrow xR^\bullet y \text{ or } (x \equiv y \;\&\; i < j) \text{ or } i = j \;\&\; x <_C y$$

where $<_C$ is an arbitrary strict ordering on the proper cluster C containing x and y.

Each nondegenrated cluster C is replaced by an infinite set C_0 such that $\langle C_0, R_0\rangle$ is a strict linear order. Moreover, $\langle y, j\rangle$, a copy of y, occurs after $\langle y, j\rangle$, a copy of x.

Bulldozing might be extended to models $\mathcal{M} = \langle W, R, \vartheta\rangle$ defining ϑ^0 as follows

$$\vartheta^0(p_i) = \{\langle x, i\rangle \mid x \in \vartheta(p_i), i < \omega\}.$$

One may show inductively the following fact.

Lemma 4. $\mathcal{M}, x \models \varphi \Leftrightarrow \mathcal{M}^0, \langle x, i\rangle \models \varphi$

Let us concretise the case of transitive Euclidean frames. First of all, we consider clusters in **K**45 frames.

3 Transitive closure stuff

4 PDLisation of Euclidean logics

5 Transitive closure and fusion

References

[1] Philippe Balbiani, Dimiter Georgiev, and Tinko Tinchev. Modal correspondence theory in the class of all euclidean frames. *Journal of Logic and Computation*, 28(1):119–131, 2018.

[2] Patrick Blackburn, Maarten De Rijke, and Yde Venema. *Modal logic*, volume 53. Cambridge University Press, 2002.

[3] Robert Bull and Krister Segerberg. Basic modal logic. In *Handbook of philosophical logic*, pages 1–88. Springer, 1984.

[4] Michael J Fischer and Richard E Ladner. Propositional dynamic logic of regular programs. *Journal of computer and system sciences*, 18(2):194–211, 1979.

[5] Olivier Gasquet, Andreas Herzig, Bilal Said, and François Schwarzentruber. Modal logics with transitive closure. In *Kripke's Worlds*, pages 157–189. Springer, 2014.

[6] Robert Goldblatt. Logics of time and computation. csli, 1987.

[7] Robert Goldblatt and Ian Hodkinson. The finite model property for logics with the tangle modality. *Studia Logica*, 106(1):131–166, 2018.

[8] Stanislav Kikot, Ilya Shapirovsky, and Evgeny Zolin. Filtration safe operations on frames. *Advances in modal logic*, 10:333–352, 2014.

[9] Grigori Schwarz. In search of a "true" logic of knowledge: the nonmonotonic perspective. *Artificial Intelligence*, 79(1):39–63, 1995.

[10] Krister Segerberg. An essay in classical modal logic (3 vols.). *Filoso ska studier*, (13), 1971.

[11] Ilya Shapirovsky and Evgeny Zolin. On completeness of logics enriched with transitive closure modality. *Topology, Algebra and Categories in Logic 2015*, page 255, 2015.

[12] Ilya B Shapirovsky. Glivenko's theorem, finite height, and local tabularity. *arXiv preprint arXiv:1806.06899*, 2018.