Notes on filtration of logics containing **K5**

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Let $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$ be a Kripke model and Γ a set of formulas closed under subformulas. An equivalence relation $\sim$ is set to have a finite index if the quotient set $W/\sim$ is finite. The equivalence relation $\sim_\Gamma$ induced by Γ is defined as

$$w \sim_\Gamma v \Leftrightarrow \forall \varphi \in \Gamma\, (\mathcal{M}, w \models \varphi \Leftrightarrow \mathcal{M}, v \models \varphi).$$

If Γ is finite, then $\sim_\Gamma$ has a finite index. An equivalence relation $\sim$ respects $\sim_\Gamma$, if $w \sim v$ implies $w \sim_\Gamma v$.

Definition 1. *Let $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$ be a Kripke model and Γ be a Sub-closed set formulas. A Γ-filtration of $\mathcal{M}$ is a model $\widehat{\mathcal{M}} = \langle \widehat{W}, \widehat{R_1}, \dots, \widehat{R_n}, \hat{\vartheta} \rangle$ such that:*

1. *$\widehat{W} = W/\sim$, where $\sim$ is an equivalence relation having a finite index that respects Γ*
2. *$\hat{\vartheta}(p) = \{[x]_\sim \mid x \in W \ \&\ x \in \vartheta(p)\}$*
3. *For each $i \in I$ one has $\hat{R}_i^{min} \subseteq \hat{R}_i \subseteq \hat{R}_i^{max}$. $\hat{R}_{i,\sim}^{min}$ is the i-th minimal filtered relation on $\widehat{W}$ defined as*

$$\hat{x} \hat{R}_{i,\sim}^{min} \hat{y} \Leftrightarrow \exists x' \sim x\ \exists y' \sim y\ x R_i y$$

$\hat{R}_{\Gamma,i}^{max}$ is the i-th maximal filtered relation on $\widehat{W}$ induced by Γ defined as

$$\hat{x} \hat{R}_{\Gamma,i}^{max} \hat{y} \Leftrightarrow \forall \Box_i \varphi \in \Gamma\, (\mathcal{M}, x \models \Box_i \varphi \Rightarrow \mathcal{M}, y \models \varphi)$$

If Φ is finite subset of Γ and $\sim = \sim_\Phi$, then $\widehat{\mathcal{M}}$ is a definable Γ-filtration of $\mathcal{M}$ through Φ. If $\sim = \sim_\Gamma$, then such a filtration by means of the definiton above is called *strict*. A class of models $\mathbb{M}$ admits strict filtrations for models (ASF), if for every Sub-closed set Γ and for every $\mathcal{M} \in \mathbb{M}$ there exists a Γ filtration of $\mathcal{M}$. A class of frames $\mathbb{F}$ admits strict filtrations for frames, if for every Sub-closed set Γ and for every frame $\mathcal{F} \in \mathbb{F}$ and every model $\mathcal{M}$ over $\mathcal{F}$ there exists a Γ filtration of $\mathcal{M}$. If $\mathcal{L}$ is canonical, then the ASF property for frames and ASF property for models are equivalent [1, Theorem 2.10].

Lemma 1. *Let Γ be a finite set of formulas closed under subformulas and $\widehat{\mathcal{M}}$ a filtration of $\mathcal{M}$ through Γ, then for each $x \in W$ and for each $\varphi \in \Gamma$ one has*

$$\mathcal{M}, x \models \varphi \Leftrightarrow \widehat{\mathcal{M}}, \hat{x} \models \varphi$$

Definition 2. *Let $\mathbb{F}$ be a class of Kripke frames and Γ a finite set of formulas closed under subformulas. If for every model $\mathcal{M}$ over $\mathcal{F} \in \mathbb{F}$ there exists a model that is a Γ-definable filtration of $\mathcal{M}$, then $\mathbb{F}$ admits definable filtration. A class of models $\mathbb{M}$ admits definable filtration if for every $\mathcal{M} \in \mathbb{M}$ there exists a model belonging to the same class that is a definable Γ-filtration of $\mathcal{M}$.*

Lemma 2.

1. *Let $\mathcal{L}$ be a complete normal modal logic. If* $\mathrm{Frames}(\mathcal{L})$ *admits filtration, then $\mathcal{L}$ has the finite model property.*

2. *If the class of models* $\mathrm{Mod}(\mathcal{L})$ *admits filtration, then $\mathcal{L}$ has the finite model property and it is Kripke complete as well.*

Definition 3. *A first-order formula is called Horn if it has the following form:*

$$\forall x_1, \dots, x_n (x_{i_1} R x_{j_1} \wedge \dots \wedge x_{i_s} R x_{j_s} \to x_k R x_l)$$

Definition 4. *Let H be a Horn property and $\langle W, R \rangle$ a Kripke frame. A Horn closure of a binary relation R is the minimal relation R^H containing R and satisfying H.*

Lemma 3. $R^H = \bigcup\limits_{n<\omega} R_n$ *where*

1. $R_0 = R$.

2. $R_{n+1} = R_n \cup \{(a, b) \in W \mid \exists \vec{c} \in W \; P(a, b, \vec{c})\}$, *where P is a premise of H.*

E-closure (an Euclidean Horn closure of a binary relation) has the following equivalent definitions:

Lemma 4. *Let $\mathcal{F} = \langle W, R \rangle$ be a Kripke frame. The following conditions are equivalent:*

1. *R^E is the smallest Euclidean relation containing R.*

2. $R^E = \bigcup\limits_{i<\omega} R_i$, *where*
   - $R_0 = R$
   - $R_{n+1} = R_n \cup (R_n^{-1} \circ R_n)$

3. *$xR^E y$ iff there exists $n < \omega$ such that either xRy or $\exists z_1, \dots, z_n$ with $z_1 R x$ and $z_{n-1} R y$ and for each $1 < i \leqslant n$ one has either $z_{i-1} R z_i$ or $z_i R z_{i-1}$.*

4. $R^E = R \cup \bigcup\limits_{i<\omega} (R^{-1} \circ (R \circ R^{-1})^n \circ R)$.

Proof.

1. (1) $\Rightarrow$ (2) Let us show that if R^E is the smallest Euclidean relation containing R, then $R^E = \bigcup\limits_{i<\omega} R_i$. There are two inclusions:
   - $R^E \subseteq \bigcup\limits_{i<\omega} R_i$. Recall that R^E has the form (?):

     $$R^E = \bigcap \{R' \mid R \subseteq R', \forall a, b \in W \; R'(a, b) \Rightarrow \exists x \in W \; R'(x, a) \; \& \; R'(x, b)\}$$

   - $\bigcup\limits_{i<\omega} R_i \subseteq R^E$. Let us show that $xR_n y$ for each $n < \omega$ implies $xR^E y$ by induction on n. If $n = 0$, then xRy, thus, $xR^E y$, since R is a subrelation of R^E. Suppose $n = m+1$ and $xR_{m+1} y$. Let us show that $xR^E y$. From $xR_{m+1} y$, one has $(x, y) \in R^n \cup (R_n^{-1} \circ R_n)$. There are two cases:
     - $xR^n y$, one needs to merely apply the IH.

- $xR_n^{-1} \circ R_n y$. Then $\exists z \in W\ xR_n^{-1}z \ \&\ zR_n$. That is, zR_nx and zR_ny for some z. R_n is already a subrelation of R^E. Thus, zR^Ex and zR^Ey. That implies xR^Ey.

2. (2) $\Rightarrow$ (3) Let $(x, y) \in R_m$, let us the statement by induction on m.
   (a) Suppose $m = 0$, then xRy, and the statement is shown putting $n = 0$.
   (b) Suppose $m = p + 1$ and $xR_{p+1}y$. Assume that either xRy or $\exists z_1, \dots, z_p$ with z_1Rx and $z_{p-1}Ry$ and for each $1 < i \leqslant p$ one has either $z_{i-1}Rz_i$ or z_iRz_{i-1}.
   $xR_{p+1}y$ implies $(x, y) \in R_p \cup (R_p^{-1} \circ R_p)$. If $(x, y) \in R_p$, then we merely apply the IH. Suppose $(x, y) \in R_p^{-1} \circ R_p$, then $(z, x) \in R_p$ and $(z, y) \in R_p$
3. (3) $\Rightarrow$ (4) Suppose either xRy or there exist $n \geqslant 1$ and $z_1, \dots, z_n$ with z_1Rx and $z_{n-1}Ry$ and for each $1 < i \leqslant n$ one has either $z_{i-1}Rz_i$ or z_iRz_{i-1}. If xRy, then we are done. Otherwise there exists $n \geqslant 1$ with the condition above. Then $(x, y) \in R_{n+1}$ that follows from the condition.
4. (4) $\Rightarrow$ (1)

□

Lemma 5. *Let $\mathcal{F} = \langle W, R\rangle$ be a Kripke frame. Let us define $R^E = \bigcup\limits_{i<\omega} R_i$ where:*

1. $R_0 = R$
2. $R_{n+1} = R_n \cup ({R_n}^{-1} \circ R_n)$

Then R^E is Euclidean.

Proof. Let $(x, y), (x, z) \in R^E$, one needs to show that $(y, z) \in R^E$. Clearly that $(x, y) \in R_i$ and $(x, z) \in R_j$ for some $i, j < \omega$. Thus, we need $(y, z) \in R_m$ for some m depending on i and j.

Let us consider the following cases:

1. $i = 0$ and $j = 0$
   Suppose $(x, y), (x, z) \in R_0 = R$, then $(y, z) \in R^{-1} \circ R$. Thus, $(y, z) \in R_1$
2. $i = 0$ and $j = k + 1$
   Suppose $(x, y) \in R$ and $(x, z) \in R_{k+1} = R_k \cup ({R_k}^{-1} \circ R_k)$. Clearly that $(x, y) \in R_{k+1}$ as well. It is obviously that $(y, z) \in R_{k+2}$ since $(y, x) \in R_{k+1}^{-1}$ and $(x, z) \in R_{k+1}$.
3. The case with $i = k + 1$ and $j = 0$ is similar to the previous one.
4. Suppose $i = m + 1$ and $j = k + 1$. That is, $(x, y) \in R_{m+1} = R_m \cup ({R_m}^{-1} \circ R_m)$ and $(x, z) \in R_{k+1} = R_k \cup ({R_k}^{-1} \circ R_k)$. Consider the following four subcases:
   (a) Suppose $(x, y) \in R_m$ and $(x, z) \in R_k$ and $m \leqslant k$ without loss of generality. $m \leqslant k$ implies $R_m \subseteq R_k$ and $(x, y) \in R_k$ in particular. Thus, $(y, z) \in {R_k}^{-1} \circ R_k$, so $(y, z) \in R_{k+1}$.
   (b) The rest of the cases are similar to the first one.

□

Theorem 1. **K**45 *admits strict filtrations.*

Proof. Let $\mathcal{M} = \langle W, R, \vartheta \rangle$ be a transitive Euclidean model and $\overline{\mathcal{M}} = \langle \overline{W}, \overline{R}, \overline{\vartheta} \rangle$ its minimal filtration through Γ, where Γ is finite and Sub-closed. Let us put $\widehat{R} = \overline{R}^{+} \cup \overline{R}^{E}$. Let us show that $\overline{R}^{+} \cup \overline{R}^{E} \subseteq \overline{R}^{max}$.

That is, if $\mathcal{M}, y \models \varphi$ for $\Diamond\varphi \in \Gamma$ and $\hat{x}\widehat{R}\hat{y}$, then $\mathcal{M}, x \models \Diamond\varphi$.

Let $\hat{x}\widehat{R}\hat{y}$. Let us consider the case when $(\hat{x}, \hat{y}) \in \overline{R}^{E}$

1. Suppose $(\hat{x}, \hat{y}) \in \overline{R}$, then $\mathcal{M}, x \models \Diamond\varphi$ holds trivially by the definition of the minimal filtration.
2. Suppose the statement holds $\overline{R}_n$ and $(\hat{x}, \hat{y}) \in \overline{R}_{n+1} = \overline{R}_n \cup (\overline{R}_n^{-1} \circ \overline{R}_n)$. We consider the case of $(\hat{x}, \hat{y}) \in (\overline{R}_n^{-1} \circ \overline{R}_n)$.

   Then there exists $\hat{z}$ such that $(\hat{z}, \hat{x}), (\hat{z}, \hat{y}) \in \overline{R}_n$.

   By IH, $\mathcal{M}, z \models \Diamond\varphi$.

   $(\hat{z}, \hat{y}) \in \overline{R}_n$ iff there are $\hat{u}_1, \ldots, \hat{u}_n$ such that

   $$\hat{z} \xleftarrow[\widehat{R}]{} \hat{u}_1 \xrightarrow[\widehat{R}']{} \hat{u}_2 \xrightarrow[\widehat{R}']{} \ldots \xrightarrow[\widehat{R}']{} \hat{u}_{n-1} \xrightarrow[\widehat{R}']{} \hat{u}_n \xrightarrow[\widehat{R}]{} \hat{y}$$

   where $\widehat{R}'$ is either $\widehat{R}$ or $\widehat{R}^{-1}$.

As it is known, $\Diamond\Diamond\varphi \to \Box\Diamond\varphi \in \mathbf{K}45$.

$\hat{u}_1\hat{\hat{z}}$, that is, $u_1' R z'$ for some $u_1' \in \hat{u}_1$ and $z' \in \hat{z}$. That is, $\mathcal{M}, u_1' \models \Diamond\Diamond\varphi$, so $\mathcal{M}, u_1' \models \Diamond\varphi$ and $\overline{M}, \hat{u}_1 \models \Diamond\varphi$.

We have $\hat{u}_1\widehat{R}'\hat{u}_2$. Suppose $\mathcal{M}, u_1'' \models \Diamond\varphi$ and $u_1'' R u_2'$. We also have $\mathcal{M}, u_1'' \models \Box\Diamond\varphi$, thus, $\mathcal{M}, u_2' \models \Diamond\varphi$.

Suppose $\hat{u}_2\widehat{R}\hat{u}_1$ and $u_2' R u_1''$, then $\mathcal{M}, u_2' \models \Diamond\varphi$.

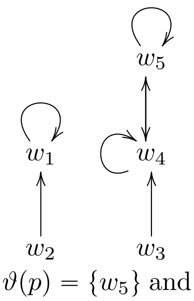
Similarly, we have $\mathcal{M}, u_i \models \Diamond\varphi$ iff $\mathcal{M}, u_{i+1} \models \Diamond\varphi$, whenever $\hat{u}_i\widehat{R}'\hat{u}_{i+1}$.

Finally, we have $\hat{u}_n\widehat{R}\hat{x}$. Thus, $u_n' R x'$ for some $u_n' \in \hat{u}_n$ and $x' \in \hat{x}$. $\mathcal{M}, u_n' \models \Diamond\varphi$, so $\mathcal{M}, u_n' \models \Box\Diamond\varphi$. Then $\mathcal{M}, x' \models \Diamond\varphi$. $\Box$

Theorem 2. **K**5 *does not admit strict filtrations.*

Proof. Let us consider a **K**5 model whose Euclidean closure of the minimal filtration does not give us a filtration.

Let us consider a frame called $\mathcal{F}_{bad}$. We define this frame with the following graph:



Let us define a valuation ϑ such that $\vartheta(p) = \{w_5\}$ and $\vartheta(q) = \{w_1\}$. Let us consider a minimal filtration of $\mathcal{M}_{bad}$ through the Sub-closure of $\Gamma = \{\neg p, \neg\Diamond p\}$.

Clearly that $w_2 \sim_\Gamma w_3$, since $\neg p$ and $\neg\Diamond p$ are true both at w_2 and w_3.

Moreover, $R_{min} \cup (R_{min}^{-1} \circ R_{min})$ is not a subset of R_{max} since $(\hat{w_1}, \hat{w_5}) \in (R_{min}^{-1} \circ R_{min})$, but $\Diamond p$ is not true at w_5.

Let us also note that strict filtrations of this model is not Euclidean. Suppose by contrary that $\widehat{R}^{\mathcal{E}}$ is a strict filtraction of that model. So $R^{E}_{min} \subseteq \widehat{R}^{\mathcal{E}}$, since R^{E}_{min} is the minimal Euclidean relation containing R_{min}. On the other hand, $R^{E}_{min} \nsubseteq R_{max}$, so is not $\widehat{R}^{\mathcal{E}}$.

$\square$

Let us define the logic $\mathcal{L}$ as $\mathbf{K} \oplus \Diamond\Diamond\Diamond p \to \Diamond p$. Let R be a binary relation, the $\mathcal{L}$-closure of R is defined (denoted as $R^{\davidsstar}$) as the following union:

$$R^{\davidsstar} = R \cup R^3 \cup R^5 \cup \dots \cup R^{2k+1} \cup \dots$$

Theorem 3. *$\mathcal{L}$ does not admit strict filtrations.*

Proof. Consider the following frame $\mathcal{F} = \langle W, R \rangle$:

$$\begin{tikzcd} x \arrow[loop above] \arrow[r] & y \end{tikzcd}$$

$$\begin{tikzcd} y_1 \arrow[r] \arrow[rr, bend right] & z_1 \arrow[r] & z \arrow[loop above] \end{tikzcd}$$

Clearly that $\mathcal{F}$ is an $\mathcal{L}$-frame. We define the valuation ϑ as follows:

$$\vartheta(p) = \{x\}$$
$$\vartheta(q) = \{y, y_1\}$$
$$\vartheta(r) = \{z\}$$

Let us put $\Gamma = \mathrm{Sub}\{p, q, \Diamond r\}$. We factorise W through $\sim_\Gamma$ and consider a model $\widehat{\mathcal{M}} = \langle W/\sim_\Gamma, \widehat{R}, \widehat{\vartheta} \rangle$, where $\widehat{R} = (\widehat{R}_{min})^{\davidsstar}$. We have $(\hat{x}, \hat{z}) \in \widehat{R} \circ \widehat{R} \circ \widehat{R}$, but $\Diamond r$ is not true at x. $\square$

1 Finite "canonical" models

Let $\mathcal{L}$ be a normal modal logic, $\mathcal{M}_{\mathcal{L}}$ its canonical model, and Γ a finite Sub-closed set of formulas. Let us put $\Gamma' = \mathrm{Sub}(\varphi) \cup \{\neg\psi \mid \psi \in \mathrm{Sub}(\varphi)\}$.

A subset $\Delta \subseteq' \Gamma$ is a *finite $\mathcal{L}$-consistent set* if $\neg \bigwedge \Delta \notin \mathcal{L}$. A subset Δ is maximal, if (the following are obviously equivalent):

1. Δ is maximal amongst finite $\mathcal{L}$-consistent sets,
2. For each $\psi \in \mathrm{Sub}(\varphi)$ either $\psi \in \Delta$ or $\neg\psi \in \Delta$.

Every finite $\mathcal{L}$-theory is clearly can be extended to some maximal one. It is the finite version of Lindenbaum's lemma.

Definition 5. *Let $\mathcal{L}$ be a modal logic and Γ be a finite* Sub-*closed set of formulas. A finite "canonical" model is a triple $\mathcal{M}^{\Gamma}_{\mathcal{L}} = \langle W^{\Gamma}_{\mathcal{L}}, R^{\Gamma}_{\mathcal{L}}, \vartheta^{\Gamma}_{\mathcal{L}} \rangle$, where*

1. *$W^{\Gamma}_{\mathcal{L}}$ is the set all maximal theories that extend finite $\mathcal{L}$-theories*
2. *$R^{\Gamma}_{\mathcal{L}}$ is a relation such that $\langle W^{\Gamma}_{\mathcal{L}}, R^{\Gamma}_{\mathcal{L}} \rangle$ is an $\mathcal{L}$-frame and*

$$\forall \Box\psi \in \mathrm{Sub}(\varphi)\ \forall \Delta_1 \in W^{\Gamma}_{\mathcal{L}}\ (\Box\psi \in \Delta_1 \Leftrightarrow \forall \Delta_2 \in R^{\Gamma}_{\mathcal{L}}(\Delta_1)\ \psi \in \Delta_2)$$

3. *$\vartheta^{\Gamma}_{\mathcal{L}}(p) = \{\Delta \in W^{\varphi}_{\mathcal{L}} \mid p \in \Delta\}$ for every variable $p \in \Gamma$.*

Lemma 6. *Let $\mathcal{L}$ be a modal logic and $\varphi \notin \mathcal{L}$, then $\mathcal{M}_{\mathcal{L}}^{\mathrm{Sub}(\varphi)} \not\models \varphi$.*

Lemma 7. *Let $\mathcal{L}$ be a modal logic and Γ a finite* Sub*-closed set of formulas, then if $\mathcal{L}$ admits strict filtrations, then there exists a finite "canonical" model $\mathcal{M}_{\mathcal{L}}^{\Gamma}$ such that $\mathcal{M}_{\mathcal{L}}^{\Gamma} \models \mathcal{L}$.*

Proof. ($\Rightarrow$) Let Γ be a finite Sub-closed of formulas. $\mathcal{L}$ admits strict filtrations, so the filtration of the canonical model $\mathcal{M}_{\mathcal{L}}$ through Γ is also an $\mathcal{L}$-model. The underlying set of $\mathcal{M}_{\mathcal{L}}/\sim_{\Gamma}$ consists of maximal $\mathcal{L}$ theories up to Γ-equivalence and this quotient set is finite.

It is readily checked that the quotient model $\mathcal{M}_{\mathcal{L}}/\sim_{\Gamma}$ satifsfies Definition 5. $\square$

The converse implication does not have to true generally. **GL** might be an example of a logic that has the "finite canonical" model property with no filtrations.

2 Fusion stuff

Definition 6. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be modal logics, then the fusion $\mathcal{L}_1 * \mathcal{L}_2$ is the minimal bimodal logic that contains $\mathcal{L}_1$ and $\mathcal{L}_2$ [2].*

Theorem 4.

1. $\mathbf{K}5 * \mathbf{K}5$ *admits definable filtrations.*

2. $\mathbf{K}5 * \cdots * \mathbf{K}5$ *admits definable filtrations.*

3. *If $\mathcal{L}_1, \ldots, \mathcal{L}_n$ admit strict filtrations, then $\mathbf{K}5 * \cdots * \cdots * \mathbf{K}5 * \mathcal{L}_1 * \cdots * \mathcal{L}_n$*

Proof.

1. Let Γ be a finite Sub-closed set of bimodal formulas, $\mathcal{F} = \langle W, R_1, R_2 \rangle$ a $\mathbf{K}5 * \mathbf{K}5$-frame, and ϑ a valuation on $\mathcal{F}$. Denote $\langle \mathcal{F}, \vartheta \rangle$ as $\mathcal{M}$.

   We introduce the set of fresh variables $V = \{p_\psi | \psi \in \Gamma\}$ and define a new model $\mathcal{M}' = \langle \mathcal{F}, \vartheta' \rangle$ as follows:

   $$\text{For all } \psi \in \Gamma,\ \mathcal{M}, x \models \psi \Leftrightarrow \mathcal{M}', x \models \psi \Leftrightarrow \mathcal{M}', x \models p_\psi.$$

   Consider these modifications of Γ and V:

   $$\begin{aligned} \Gamma' &= \Gamma \cup \{\Diamond_1 \Box_1 \psi \mid \Box_1 \psi \in \Gamma\} \cup \{\Diamond_2 \Box_2 \psi \mid \Box_2 \psi \in \Gamma\} \\ \Delta_1 &= V \cup \mathrm{Sub}(\{\Diamond \Box p_\psi \mid \Box_1 \psi \in \Gamma\}) \\ \Delta_2 &= V \cup \mathrm{Sub}(\{\Diamond \Box p_\psi \mid \Box_2 \psi \in \Gamma\}) \end{aligned}$$

   Let us define equivalence relations $\sim_1$ and $\sim_2$ induced by Δ_1 and Δ_2 respectively.

   Consider $\mathcal{M}_i = \langle W, R_i, \vartheta' \rangle$, a reduct of $\mathcal{M}'$, we have:

   (a) $\mathcal{M}_i, x \models \Box p_\psi$ iff $\mathcal{M}, x \models \Box_i \psi$

   (b) $\mathcal{M}_i, x \models \Diamond \Box p_\psi$ iff $\mathcal{M}, x \models \Diamond_i \Box_i \psi$

   So $\sim_i = \sim_{\Gamma'}$ by the construction. Let us put $\widehat{W} = W/\sim_{\Gamma'}$.

   Claim 1. *Let $\widehat{R}_i = (R_{\Delta_i}^{min})^E$ and $\widehat{\vartheta(p)} = \{[x]_{\sim_i} \mid \mathcal{M}_i, x \models p\}$ for $p \in \Delta_1$, define $\widehat{\mathcal{M}}_i = \widehat{W}, \widehat{R}_i, \hat{\vartheta}$. Then $\widehat{\mathcal{M}}_i \models \mathbf{K}5$ and $\widehat{\mathcal{M}}_i$ is a filtration of $\mathcal{M}_i$ through Δ_i.*

Proof. Let us show that $\widehat{R}_i \subseteq R^{max}_{\Delta_i}$ by induction. Let $\hat{x}\widehat{R}_i\hat{y}$ such that $\mathcal{M}_i, x \models \Box p_\psi$. We need $\mathcal{M}_i, y \models p_\psi$. Recall that $\widehat{R}_i$ is defined inductively as:

(a) $R^0_{\Delta_i} = R^{min}_{\Delta_i}$

(b) $R^{n+1}_{\Delta_i} = R^n_{\Delta_i} \cup ({R^n_{\Delta_i}}^{-1} \circ R^n_{\Delta_i})$

(c) $(R^{min}_{\Delta_i})^E = \bigcup_{k<\omega} R^k_{\Delta_i}$

(a) $n = 0$. This is obvious.

(b) $m = n+1$. Suppose the statement holds for $R^n_{\Delta_i}$, we also have $(\hat{x}, \hat{y}) \in R^n_{\Delta_i} \cup ({R^n_{\Delta_i}}^{-1} \circ R^n_{\Delta_i})$. If $(\hat{x}, \hat{y}) \in R^n_{\Delta_i}$, then we just apply IH.

$(\hat{x}, \hat{y}) \in {R^n_{\Delta_i}}^{-1} \circ R^n_{\Delta_i} \in R^n_{\Delta_i}$, then there exists $\hat{z}$ such that $(\hat{z}, \hat{x}) \in R^n_{\Delta_i}$ and $(\hat{z}, \hat{y}) \in R^n_{\Delta_i}$. We rewrite $(\hat{z}, \hat{x}) \in R^n_{\Delta_i}$ as the following sausage (for some $\widehat{u_1}, \widehat{u_2}, \ldots, \widehat{u_{n-1}}, \widehat{u_n}$):

$$\hat{z} \xleftarrow{R^{min}_{\Delta_i}} \widehat{u_1} \xrightarrow{R'} \widehat{u_2} \xrightarrow{R'} \ldots \qquad \widehat{u_{n-1}} \xrightarrow{R'} \widehat{u_n} \xrightarrow{R^{min}_{\Delta_i}} \hat{x}$$

where R' is either $R^{min}_{\Delta_i}$ or its converse. We clearly have $\mathcal{M}, z \models \Box p_\psi$, since $\mathcal{M}, x \models \Box p_\psi$ implies $\mathcal{M}, u_n \models \Diamond\Box p_\psi$. After that we apply the following property of **K**5-models:

For each $a, b \in M_i$ such that aR_ib we have $\mathcal{M}_i, a \models \Diamond\Box p_\psi$ iff $\mathcal{M}_i, b \models \Diamond\Box p_\psi$

Then $\mathcal{M}_i, y \models p_\psi$ by IH.

$\square$

Finally, we consider a model $\widehat{\mathcal{M}} = \langle \widehat{W}, \widehat{R}_1, \widehat{R}_2, \vartheta \rangle$, where $\widehat{R_{\Gamma'}}_i = {R_i}^{min^E}_{\Gamma'}$ and $\vartheta(p)$ is defined as usual for $p \in \Gamma$. $\widehat{\mathcal{M}}$ is a filtration of $\mathcal{M}$ through Γ'.

Let $\hat{x}\widehat{R_{\Gamma'}}_i\hat{y}$ and $\mathcal{M}, x \models \Box_i\psi$ for $\Box_i\psi \in \Gamma$. Then $\mathcal{M}_i, x \models \Box p_\psi$, so $\widehat{\mathcal{M}_i}, \hat{x} \models \Box p_\psi$. By the claim above, $\widehat{\mathcal{M}_i}$ is a filtration of $\mathcal{M}_i$ through Δ_i, so $\mathcal{M}_i, y \models p_\psi$. Then $\mathcal{M}, y \models \psi$.

2. Likewise

3.

$\square$

Theorem 5. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be modal logics such that $\mathrm{Fr}(\mathcal{L}_1)$ and $\mathrm{Fr}(\mathcal{L}_2)$ admit filtrations, so does $\mathrm{Fr}(\mathcal{L}_1 * \mathcal{L}_2)$.*

Proof. $\square$

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