

Notes on filtration of logics containing **K5**

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1 Preliminaries

Definition 1. An n -normal modal logic is a set of formulas that contains all Boolean tautologies, formulas $\Diamond_i p \vee \Diamond_i q \leftrightarrow \Diamond_i(p \vee q)$ and $\Diamond_i \perp \leftrightarrow \perp$ for $i \leq n$, and is closed under modus ponens, substitution, and monotonicity: from $\varphi \rightarrow \psi$ infer $\Diamond_i \varphi \rightarrow \Diamond_i \psi$ for $i \leq n$.

Definition 2. An n -Kripke model is a triple $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$, where $R_i \subseteq W \times W$, $\vartheta : PV \rightarrow 2^W$, and the connectives have the following semantics:

1. $\mathcal{M}, w \models p \Leftrightarrow w \in \vartheta(p)$
2. $\mathcal{M}, w \models \neg \varphi \Leftrightarrow \mathcal{M}, w \not\models \varphi$
3. $\mathcal{M}, w \models \varphi \vee \psi \Leftrightarrow \mathcal{M}, w \models \varphi$ or $\mathcal{M}, w \models \psi$
4. $\mathcal{M}, w \models \Diamond_i \varphi \Leftrightarrow \exists v \in R_i(w) \mathcal{M}, v \models \varphi$

By **K5** we mean the logic $\mathbf{K} \oplus A5$, where $A5 = \Diamond p \rightarrow \Box \Diamond p$. It is known that **K5** is the modal logic of all Euclidean frames. A frame is called Euclidean if for each x, y, z , xRy and xRz implies yRz .

Proposition 1. Let $\mathcal{F} = \langle W, R \rangle$ be an Euclidean frame.

1. For each $x, y, z \in W$, xRy and xRz implies either yRz or zRy .
2. For each $x \in W$, $R^*(x) = \{x\} \cup R(R(x))$.
3. $R^{-1} \circ R \subseteq R$.

Proposition 2. **K5** proves

1. $\Box^3 p \leftrightarrow \Box^2 p$
2. $\Box^2 \Diamond p \leftrightarrow \Box \Diamond p$
3. $\Box \Diamond \Box p \leftrightarrow \Box \Box p$
4. $\Box \Diamond^2 p \leftrightarrow \Box \Diamond p$

Proposition 3. Let \mathcal{M} be a **K5** model, xRy for $x, y \in W$ then one has

$$\mathcal{M}, x \models \Diamond \Box \varphi \text{ iff } \mathcal{M}, y \models \Diamond \Box \varphi.$$

Proof.

1. Suppose $\mathcal{M}, x \models \Diamond \Box \varphi$. One also has $\mathcal{M}, x \models \Diamond \Box \varphi \rightarrow \Box \Diamond \Box \varphi$, so $\mathcal{M}, x \models \Box \Diamond \Box \varphi$. Thus, $\mathcal{M}, y \models \Diamond \Box \varphi$ since $y \in R(x)$.
2. Suppose $\mathcal{M}, y \models \Diamond \Box \varphi$, then $\mathcal{M}, y \models \Box \varphi$, so $\mathcal{M}, x \models \Diamond \Box \varphi$.

□

1.1 Filtrations: general definitions

Let $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$ be a Kripke model and Γ a set of formulas closed under subformulas. An equivalence relation \sim is set to have a finite index if the quotient set W / \sim is finite. The equivalence relation \sim_Γ induced by Γ is defined as

$$w \sim_\Gamma v \Leftrightarrow \forall \varphi \in \Gamma (\mathcal{M}, w \models \varphi \Leftrightarrow \mathcal{M}, v \models \varphi).$$

If Γ is finite, then \sim_Γ has a finite index. An equivalence relation \sim respects \sim_Γ , if $w \sim v$ implies $w \sim_\Gamma v$.

Definition 3. Let $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$ be a Kripke model and Γ be a Sub-closed set formulas. A Γ -filtration of \mathcal{M} is a model $\widehat{\mathcal{M}} = \langle \widehat{W}, \widehat{R}_1, \dots, \widehat{R}_n, \widehat{\vartheta} \rangle$ such that:

1. $\widehat{W} = W / \sim$, where \sim is an equivalence relation having a finite index that respects Γ
2. $\widehat{\vartheta}(p) = \{[x]_\sim \mid x \in W \ \& \ x \in \vartheta(p)\}$
3. For each $i \in I$ one has $\widehat{R}_i^{\min} \subseteq \widehat{R}_i \subseteq \widehat{R}_i^{\max}$. $\widehat{R}_{i,\sim}^{\min}$ is the i -th minimal filtered relation on \widehat{W} defined as

$$\widehat{R}_{i,\sim}^{\min} \Leftrightarrow \exists x' \sim x \exists y' \sim y \ x R_i y$$

$\widehat{R}_{\Gamma,i}^{\max}$ is the i -th maximal filtered relation on \widehat{W} induced by Γ defined as

$$\widehat{R}_{\Gamma,i}^{\max} \Leftrightarrow \forall \Box_i \varphi \in \Gamma (\mathcal{M}, x \models \Box_i \varphi \Rightarrow \mathcal{M}, y \models \varphi)$$

If Φ is finite subset of Γ and $\sim = \sim_\Phi$, then $\widehat{\mathcal{M}}$ is a definable Γ -filtration of \mathcal{M} through Φ . If $\sim = \sim_\Gamma$, then such a filtration by means of the definition above is called *strict*.

Lemma 1. Let Γ be a finite set of formulas closed under subformulas and $\widehat{\mathcal{M}}$ a filtration of \mathcal{M} through Γ , then for each $x \in W$ and for each $\varphi \in \Gamma$ one has

$$\mathcal{M}, x \models \varphi \Leftrightarrow \widehat{\mathcal{M}}, \widehat{x} \models \varphi$$

Definition 4. Let \mathbb{F} be a class of Kripke frames and Γ a finite set of formulas closed under subformulas. If for every model \mathcal{M} over $\mathcal{F} \in \mathbb{F}$ there exists a model that is a Γ -definable filtration of \mathcal{M} , then \mathbb{F} admits definable filtration. A class of models \mathbb{M} admits definable filtration if for every $\mathcal{M} \in \mathbb{M}$ there exists a model belonging to the same class that is a definable Γ -filtration of \mathcal{M} .

Lemma 2.

1. Let \mathcal{L} be a complete normal modal logic. If $\text{Frames}(\mathcal{L})$ admits filtration, then \mathcal{L} has the finite model property.
2. If the class of models $\text{Mod}(\mathcal{L})$ admits filtration, then \mathcal{L} has the finite model property and Kripke complete as well.

2 Filtration of Euclidean logics

First of all, let us ensure that a minimal filtration of an Euclidean frame is not necessary Euclidean. Let $[x] \sim_\Gamma [y]$ and $[x] \sim_\Gamma [z]$. Then for some $x' \in [x]$ $y' \in [y]$, one has $x'Ry'$ and $x''Rz'$ for some $x'' \in [x]$ and $z' \in [z]$. Clearly, we cannot claim that $x' = x''$ in general. Thus, minimal filtration does not preserve the required property.

Lemma 3. *K5 admit filtration.*

Proof. Let \mathcal{M} be a **K5**-model and Γ_0 a finite set of formulas closed under subformulas. Let us put $\Gamma = \Gamma_0 \cup \text{Sub}(\{\Diamond\Box\psi \mid \Box\psi \in \Gamma_0\}) \cup \Psi$, where $\Psi = \nabla_1\nabla_2\ldots\nabla_n\Box\psi$ for $\Box\psi \in \Gamma_0$ and $\nabla_i \in \{\Diamond, \Box\}$. By Proposition 2, any element of Φ has one of the four forms. Thus, $W \sim_{\equiv_\Gamma}$ has a finite index. We put $\hat{R} = R_\Gamma^{\max}$. \square

Definition 5. *A first-order formula is called Horn if it has the following form:*

$$\forall x_1, \dots, x_n (x_{i_1}Rx_{j_1} \wedge \dots \wedge x_{i_s}Rx_{j_s} \rightarrow x_kRx_l)$$

Definition 6. *Let H be a Horn property and $\langle W, R \rangle$ a Kripke frame. A Horn closure of a binary relation R is the minimal relation R^H containing R and satisfying H .*

Lemma 4. $R^H = \bigcup_{n < \omega} R_n$ where

1. $R_0 = R$.
2. $R_{n+1} = R_n \cup \{(a, b) \in W \mid \exists \vec{c} \in W \text{ } P(a, b, \vec{c})\}$, where P is a premise of H .

E -closure (an Euclidean Horn closure of a binary relation) has the following equivalent definitions:

Lemma 5. *Let $\mathcal{F} = \langle W, R \rangle$ be a Kripke frame. The following conditions are equivalent:*

1. R^E is the smallest Euclidean relation containing R .
2. $R^E = \bigcup_{i < \omega} R_i$, where
 - $R_0 = R$
 - $R_{n+1} = R_n \cup (R_n^{-1} \circ R_n)$
3. xR^Ey iff there exists $n < \omega$ such that either xRy or $\exists z_1, \dots, z_n$ with z_1Rx and $z_{n-1}Ry$ and for each $1 < i \leq n$ one has either $z_{i-1}Rz_i$ or z_iRz_{i-1} .
4. $R^E = R \cup \bigcup_{i < \omega} (R^{-1} \circ (R \circ R^{-1})^n \circ R)$.

Proof.

1. (1) \Rightarrow (2) Let us show that if R^E is the smallest Euclidean relation containing R , then $R^E = \bigcup_{i < \omega} R_i$. There are two inclusions:

- $R^E \subseteq \bigcup_{i < \omega} R_i$. Recall that R^E has the form (?):

$$R^E = \bigcap \{R' \mid R \subseteq R', \forall a, b \in W \text{ } R'(a, b) \Rightarrow \exists x \in W \text{ } R'(x, a) \ \& \ R'(x, b)\}$$

- $\bigcup_{i < \omega} R_i \subseteq R^E$. Let us show that $xR_n y$ for each $n < \omega$ implies $xR^E y$ by induction on n .
 If $n = 0$, then xRy , thus, $xR^E y$, since R is a subrelation of R^E . Suppose $n = m+1$ and $xR_{m+1}y$. Let us show that $xR^E y$. From $xR_{m+1}y$, one has $(x, y) \in R^n \cup (R_n^{-1} \circ R_n)$. There are two cases:
 - $xR^n y$, one needs to merely apply the IH.
 - $xR_n^{-1} \circ R_n y$. Then $\exists z \in W$ $xR_n^{-1}z$ & $zR_n y$. That is, $zR_n x$ and $zR_n y$ for some z . R_n is already a subrelation of R^E . Thus, $zR^E x$ and $zR^E y$. That implies $xR^E y$.
- 2. (2) \Rightarrow (3) Let $(x, y) \in R_m$, let us the statement by induction on m .
 - (a) Suppose $m = 0$, then xRy , and the statement is shown putting $n = 0$.
 - (b) Suppose $m = p+1$ and $xR_{p+1}y$. Assume that either xRy or $\exists z_1, \dots, z_p$ with $z_1 R x$ and $z_{p-1} R y$ and for each $1 < i \leq p$ one has either $z_{i-1} R z_i$ or $z_i R z_{i-1}$.
 $xR_{p+1}y$ implies $(x, y) \in R_p \cup (R_p^{-1} \circ R_p)$. If $(x, y) \in R_p$, then we merely apply the IH.
 Suppose $(x, y) \in R_p^{-1} \circ R_p$, then $(z, x) \in R_p$ and $(z, y) \in R_p$.
- 3. (3) \Rightarrow (4) Suppose either xRy or there exist $n \geq 1$ and z_1, \dots, z_n with $z_1 R x$ and $z_{n-1} R y$ and for each $1 < i \leq n$ one has either $z_{i-1} R z_i$ or $z_i R z_{i-1}$. If xRy , then we are done.
 Otherwise there exists $n \geq 1$ with the condition above. Then $(x, y) \in R_{n+1}$ that follows from the condition.
- 4. (4) \Rightarrow (1)

□

Lemma 6. Let $\mathcal{F} = \langle W, R \rangle$ be a Kripke frame. Let us define $R^E = \bigcup_{i < \omega} R_i$ where:

1. $R_0 = R$
2. $R_{n+1} = R_n \cup (R_n^{-1} \circ R_n)$

Then R^E is Euclidean.

Proof. Let $(x, y), (x, z) \in R^E$, one needs to show that $(y, z) \in R^E$. Clearly that $(x, y) \in R_i$ and $(x, z) \in R_j$ for some $i, j < \omega$. Thus, we need $(y, z) \in R_m$ for some m depending on i and j .

Let us consider the following cases:

1. $i = 0$ and $j = 0$
 Suppose $(x, y), (x, z) \in R_0 = R$, then $(y, z) \in R^{-1} \circ R$. Thus, $(y, z) \in R_1$
2. $i = 0$ and $j = k+1$
 Suppose $(x, y) \in R$ and $(x, z) \in R_{k+1} = R_k \cup (R_k^{-1} \circ R_k)$. Clearly that $(x, y) \in R_{k+1}$ as well. It is obviously that $(y, z) \in R_{k+2}$ since $(y, x) \in R_{k+1}^{-1}$ and $(x, z) \in R_{k+1}$.
3. The case with $i = k+1$ and $j = 0$ is similar to the previous one.
4. Suppose $i = m+1$ and $j = k+1$. That is, $(x, y) \in R_{m+1} = R_m \cup (R_m^{-1} \circ R_m)$ and $(x, z) \in R_{k+1} = R_k \cup (R_k^{-1} \circ R_k)$. Consider the following four subcases:
 - (a) Suppose $(x, y) \in R_m$ and $(x, z) \in R_k$ and $m \leq k$ without loss of generality. $m \leq k$ implies $R_m \subseteq R_k$ and $(x, y) \in R_k$ in particular. Thus, $(y, z) \in R_k^{-1} \circ R_k$, so $(y, z) \in R_{k+1}$.

(b) The rest of the cases are similar to the first one.

□

Lemma 7. *Let $\mathcal{M} = \langle W, R, \vartheta \rangle$ be an Euclidean model, Γ a set of Sub-closed formulas, and \sim an equivalence relation having a finite index that respects Γ , then $\hat{R} = (R_{\Phi}^{min})^E \subseteq R_{\Gamma}^{max}$, where $\Phi = \Gamma \cup \{\Diamond \Box \varphi \mid \Box \varphi \in \Gamma\}$.*

*Thus, **K5** admits strict filtrations.*

Proof. Recall that $(R_{\Phi}^{min})^E$ has the form $(R_{\Phi}^{min})^E = \bigcup_{n < \omega} (R_{\Phi}^{min})_n$, where

1. $(R_{\Phi}^{min})_0 = R_{\Phi}^{min}$
2. $(R_{\Phi}^{min})_{m+1} = (R_{\Phi}^{min})_n \cup (((R_{\Phi}^{min})_n)^{-1} \circ (R_{\Phi}^{min})_n)$

One needs to show that for each $n < \omega$ $(R_{\Phi}^{min})_n \subseteq R_{\Gamma}^{max}$. We prove this by induction. Suppose $\mathcal{M}, x \models \Box \varphi$ for $\Box \varphi \in \Phi$ and $[x](R_{\Phi}^{min})^E[y]$. We need $\mathcal{M}, y \models \varphi$.

1. $([x], [y]) \in (R_{\Phi}^{min})_0$, then $([x], [y]) \in R_{\Phi}^{min}$. Then there exist $x' \in [x]$ and $y' \in [y]$ such that $x'Ry'$. So $\mathcal{M}, x' \models \Box \varphi$ and, thus, $\mathcal{M}, y' \models \varphi$. Then $\mathcal{M}, y' \models \varphi$ as well since $y' \in [y]$.
2. $([x], [y]) \in (R_{\Phi}^{min})_{m+1}$, then $([x], [y]) \in (R_{\Phi}^{min})_m \cup (((R_{\Phi}^{min})_m)^{-1} \circ R_{\Phi}^{min})_m$.

If $([x], [y]) \in (R_{\Phi}^{min})_m$, then we apply the IH.

Suppose $([x], [y]) \in (R_{\Phi}^{min})_m^{-1} \circ (R_{\Phi}^{min})_m$, then there exists $[z] \in W / \sim_{\Phi}$ such that $([z], [x]) \in (R_{\Phi}^{min})_m$ and $([z], [y]) \in (R_{\Phi}^{min})_m$.

Then one has the following picture (using Lemma 5):

$$[z] \xleftarrow{R_{\Phi}^{min}} [z_1] \xrightarrow{R'} [z_2] \xrightarrow{R'} \dots \xrightarrow{R'} [z_{m-1}] \xrightarrow{R'} [z_m] \xrightarrow{R_{\Phi}^{min}} [x]$$

$$[z] \xleftarrow{R_{\Phi}^{min}} [z'_1] \xrightarrow{R'} [z'_2] \xrightarrow{R'} \dots \xrightarrow{R'} [z'_{m-1}] \xrightarrow{R'} [z'_m] \xrightarrow{R_{\Phi}^{min}} [y]$$

Where R' is either R_{Φ}^{min} or its converse. One has $\mathcal{M}, x \models \Box \varphi$ for $\Box \varphi \in \Phi$, where \widehat{M} is the minimal filtration of \mathcal{M} through Φ . One has $[z_m]R_{\Phi}^{min}[x]$, then $a_m Ra$ for some $a_m \in [z_m]$ and $a \in [x]$. Thus, $\mathcal{M}, a_m \models \Diamond \Box \varphi$ and, thus, $\widehat{\mathcal{M}}, [z_m] \models \Diamond \Box \varphi$.

Applying Proposition 3 several times, one may show that $\widehat{\mathcal{M}}, [z_1] \models \Diamond \Box \varphi$. One has $[z_1]R_{\Phi}^{min}[z]$, then for some $a \in [z]$ and $a_1 \in [z_1]$ we have $a_1 Ra$.

Then $\mathcal{M}, a \models \Box \varphi$ and $\widehat{\mathcal{M}}, [z] \models \Box \varphi$.

We have $[z'_1]R_{\Phi}^{min}[z]$, thus, $a'_1 Ra'$ for some $a'_1 \in [z'_1]$ and $a' \in [z]$. Then $\mathcal{M}, a'_1 \models \Diamond \Box \varphi$, and, thus, $\widehat{\mathcal{M}}, [z'_1] \models \Diamond \Box \varphi$.

One may show that $\widehat{\mathcal{M}}, [z'_m] \models \Diamond \Box \varphi$ in the same way via Lemma 3. Thus, $\mathcal{M}, z'_m \models \Diamond \Box \varphi$. We also have $\mathcal{M}, z'_m \models \Diamond \Box \varphi \rightarrow \Box \varphi$, and, thus, $\mathcal{M}, z'_m \models \Box \varphi$. Then $\widehat{\mathcal{M}}, [z'_m] \models \Box \varphi$.

One has $[z'_m]R_{\Phi}^{min}[y]$, then $a'_m Ry'$ for some $a'_m \in [z'_m]$ and $y' \in [y]$. Then $\mathcal{M}, y' \models \varphi$. But $y' \sim_{\Phi} y$, so $\mathcal{M}, y \models \varphi$.

□

2.1 Clusters

Let $\mathcal{F} = \langle W, R \rangle$ be a transitive frame. Let us put $xR^\bullet y \Leftrightarrow xRy \ \& \ \neg(xRy)$. A point x is proper if xRx . Let us define the following equivalence relation:

$$x \equiv y \Leftrightarrow xRy \ \& \ yRx \vee x = y.$$

A cluster is an element of the quotient set W/\equiv . Given $x \in W$, C_x is a cluster containing x . Thus $C_x = \{x\} \cup \{y \mid xRyx\}$. The original relation lifts to the antisymmetric transitive relation on W/\equiv defined as $C_x RC_y$ iff xRy . A cluster C is called maximal if CRC' implies $C = C'$. A point is R -maximal if C_x is a maximal cluster, that is, $R^\bullet(x) = \emptyset$. A degenerated cluster is a singleton $\{x\}$ with $\neg(xRx)$. A cluster is called simple if it has the form $\{x\}$ with xRx . If $\langle W', R' \rangle$ is an inner substructure of $\langle W, R \rangle$, then every R' -cluster is an R -cluster and every R -cluster that intersects W' is a subset of W' and is an R' -cluster itself. Given a Kripke model \mathcal{M} , a set of formulas Γ is satisfied by a cluster C if every member of Γ is true at some point of C .

If clusters coincide then their points have the same theory in the original model:

Lemma 8. $C_x = C_y$ implies $\mathcal{M}, x \models \Box\varphi \Leftrightarrow \mathcal{M}, y \models \Box\varphi$

Let us describe the bulldozing technique allowing one to eliminate nondegenerated clusters [3]. Let \mathcal{L} be a transitive logic and \mathcal{F} its frame. We construct first a frame $\mathcal{F}^0 = \langle W^0, R^0 \rangle$ replacing every nondegenerated cluster C of W by $C^0 = \{\langle x, i \rangle \mid x \in C, i < \omega\}$. We also replace each degenerated cluster C by $\{\langle x, 0 \rangle\}$. Elements of these subsets form W^0 . The relation R^0 is defined as

$$\langle x, i \rangle R^0 \langle y, j \rangle \Leftrightarrow xR^\bullet y \text{ or } (x \equiv y \ \& \ i < j) \text{ or } i = j \ \& \ x <_C y$$

where $<_C$ is an arbitrary strict ordering on the proper cluster C containing x and y .

Each nondegenerated cluster C is replaced by an infinite set C_0 such that $\langle C_0, R_0 \rangle$ is a strict linear order. Moreover, $\langle y, j \rangle$, a copy of y , occurs after $\langle y, j \rangle$, a copy of x .

Bulldozing might be extended to models $\mathcal{M} = \langle W, R, \vartheta \rangle$ defining ϑ^0 as follows

$$\vartheta^0(p_i) = \{\langle x, i \rangle \mid x \in \vartheta(p_i), i < \omega\}.$$

One may show inductively the following fact.

Lemma 9. $\mathcal{M}, x \models \varphi \Leftrightarrow \mathcal{M}^0, \langle x, i \rangle \models \varphi$

Let us concretise the case of transitive Euclidean frames. First of all, we consider clusters in K45 frames.

2.2

3 Transitive closure stuff

4 PDLisation of Euclidean logics

5 Transitive closure and fusion

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