Complete representability for canonical extensions of representable cylindric algebras

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1 The problem itself

- Given $\mathcal{C} \in \mathbf{RCA}_\omega$, whether $\mathcal{C}^+$ has a complete, ω-dimensional representation? The conjecture is yes. [5]
- Whether $\mathbf{RCA}_\omega$ is barely canonical. The conjecture is yes.

2 Atomic Representations

A representation of a Boolean algebra $\mathcal{B}$ is an embedding h of $\mathcal{B}$ to some field of sets.

Let $a \in \mathcal{B}$ be an element of a Boolean algebra $\mathcal{B}$, a is called an atom, if for every $b \in \mathcal{B}$ $b < a$ implies $b = 0$. That is, an atom is a minimal non-zero element. $\mathrm{At}(\mathcal{B})$ is the set of all atoms of $\mathcal{B}$.

Let $\mathcal{B}$ be a Boolean algebra and $\mathcal{F}$ a field of sets such that $h : \mathcal{B} \to \mathcal{F}$ is a representation of $\mathcal{B}$, then $\mathcal{B}$ is a complete representation of $\mathcal{B}$, if for every $A \subseteq \mathcal{B}$ we have the following whenever ΣA is defined:

$$h(\Sigma A) = \bigcup h[A]$$

A representation h is called atomic, if $x \in h(1)$ there exists $b \in \mathrm{At}(\mathcal{B})$ such that $x \in h(b)$.

Theorem 1. *Let $\mathcal{B}$ be a Boolean algebra, then $\mathcal{B}$ is atomic iff $\mathcal{B}$ is completely representable. See [4, Corollary 6].*

3 BAOs and Duality

By default, we assume that all operators are at most unary. Here is the rigorous definition:

Definition 1.

1. *Let $\mathcal{B} = \langle B, +, -, 0, 1 \rangle$ be a Boolean algebra. An operator is a function $\Omega : B \to B$ satisfying the following conditions:*
   - *Normality:* $\Omega(0) = 0$
   - *Additivity:* $\Omega(b + b') = \Omega(b) + \Omega(b')$
2. *Let I be an index set, a Boolean algebra with operators (BAO) is an algebra $\langle B, +, -, 0, 1, (\Omega_i)_{i \in I} \rangle$ such that $\langle B, +, -, 0, 1 \rangle$ is a Boolean algebra and for each $i \in I$ Ω_i is an operator.*

Definition 2. *Let $\mathcal{B} = \langle B, +, -, 0, 1, (\Omega_i)_{i\in I}\rangle$ be a BAO, then*

1. *An opetator Ω is completely additive, if for every $X \subseteq B$ such that ΣX is defined, one has*

$$\Omega(\sum X) = \sum_{x\in X} \Omega(x)$$

2. *$\mathcal{B}$ is completely additive, if for each $i \in I$ Ω_i is additive,*

3. *A class $\mathcal{K}$ of BAOs is completely additive, if every $\mathcal{B} \in \mathcal{K}$ is completely additive.*

3.1 Atom structures and canonical extensions

Definition 3. *Let I be an index set and $(\Omega_i)_{i\in I}$ a set of function symbols*

1. *A structure is a relational structrure $\mathcal{F} = \langle W, (R_i)_{i\in I}\rangle$ such that R_i is a binary relation symbol for a function symbol $\Omega_{i\in I}$ with the corresponding index,*

2. *Let $\mathcal{B}$ be an atomic BAO of the signature I, the atom structure of $\mathcal{B}$, written as $\mathfrak{At}\mathcal{B}$, is a structure $\langle \mathrm{At}(\mathcal{B}), (R_i)_{i\in I}\rangle$ such that for all $a, b \in \mathrm{At}(\mathcal{B})$ and for all $i \in I$*

$$\mathfrak{At}\mathcal{B} \models R_i(a, b) \text{ iff } \mathcal{B} \models a \leqslant \Omega_i(b)$$

3. *Let $\mathcal{F} = \langle W, (R_i)_{i\in I}\rangle$ be an atom structure, the complex algebra of $\mathcal{F}$, written as $\mathfrak{Cm}\mathcal{F}$, is a BAO $\langle \mathcal{P}(W), \cup, -, \varnothing, W, (\Omega_{R_i})_{i\in I}\rangle$ such that for all $X \subseteq W$ and for each $i \in I$:*

$$\Omega_{R_i}(X) = \{a \in W \mid \exists b \in X\ \mathcal{F} \models R_i(a, b)\}$$

Definition 4. *Let $\mathcal{F} = \langle W, (R_i)_{i\in I}\rangle$ and $\mathcal{F}' = \langle W', ({R'}_i)_{i\in I}\rangle$, then a function $f : \mathcal{F} \to \mathcal{F}'$ is a bounded morphism, if the following holds:*

1. *xR_iy implies $f(x){R'}_if(y)$;*

2. *$f(x){R'}_iz$, then there exists $y \in W$ such that xR_iy and $f(y) = z$.*

A bounded morphism $f : \mathcal{F} \to \mathcal{F}'$ is a p-morphism, if f is onto. $\mathcal{F} \twoheadrightarrow \mathcal{F}'$ iff there exists a p-morphism from $\mathcal{F}$ onto $\mathcal{F}'$, or $\mathcal{F}'$ is a p-morphic image of $\mathcal{F}$.

Definition 5. *Let $\mathcal{F} = \langle W, (R_i)_{i\in I}\rangle$ is an inner substructure [1] of $\mathcal{F}' = \langle W', ({R'}_i)_{i\in I}\rangle$, if $W \subseteq W'$ and the embedding $\mathcal{F} \hookrightarrow \mathcal{F}'$ is a bounded morphism. Let $\mathbb{F}$ be a class atom structures, then $\mathbb{S}(\mathbb{F})$ is the closure of $\mathbb{F}$ under generated subframes.*

Let $\mathbb{F}$ be a class of structures, define:

1. $\mathfrak{Cm}(\mathbb{F}) = \{\mathcal{B} \mid \mathcal{B} \cong \mathfrak{Cm}(\mathcal{F})$ for some $\mathcal{F} \in \mathbf{F}\}$.

2. $\mathbf{Up}(\mathbb{F})$ is the class of structures isomorphic to disjoint unions of elements of $\mathbb{F}$.

3. $\mathbf{S}(\mathbb{F})$ is the closure of $\mathbb{F}$ under inner substructures.

Let A be a non-empty subset of a Boolean algebra $\mathcal{B}$, A is a *filter*, if A is closed under finite infima and it is upward closed. A is an ultrafilter, if it has no non-trivial extensions. That is, if $A \subseteq A'$, then $A' = \mathcal{B}$. This is a well-known fact that every filter can be extended to a maximal one using Zorn's lemma.

The following definition is due to, for example, [7, Definition 5.40].

[1] Or alternatively, a generated subframe

Definition 6. *Let $\mathcal{B} = \langle B, +, -, 0, 1, (\Omega_i)_{i \in I} \rangle$ be a BAO and* $\mathbf{Spec}(\mathcal{B})$ *the set of its ultrafilters. The ultrafilter frame of $\mathcal{B}$ (or the canonical frame) is a relational structure $\mathcal{F}_{\mathcal{B}} = \langle \mathbf{Spec}(\mathcal{B}), R_{\Omega_i} \rangle$ such that for all ultrafilters U_1, U_2 one has*

$$\mathbf{Spec}(\mathcal{B}) \models R_{\Omega_i}(U_1, U_2) \text{ iff } \{\Omega_i(b) \mid b \in U_1\} \subseteq U_2.$$

Given $\mathcal{B}$ be a BAO, we denoted as $\mathcal{B}^+$ as the complex algebra of the canonical frame $\mathfrak{Cm}(\mathcal{F}_{\mathcal{B}})$, that is, *the canonical extension* of $\mathcal{B}$. A class of BAOs $\mathbf{K}$ is canonical, if it is closed under canonical extensions. That is, $\mathcal{B}^+ \in \mathbf{K}$ whenever $\mathcal{B} \in \mathbf{K}$.

Theorem 2. *Let $\mathcal{A}$, $\mathcal{B}$ be BAOs,*

1. *There exists $\iota : \mathcal{A} \hookrightarrow \mathcal{A}^+$ such that $\iota : a \mapsto \{\gamma \in \mathbf{Spec}(\mathcal{A}) \mid a \in \gamma\}$.*
2. *$i : \mathcal{A} \hookrightarrow \mathcal{B}$ implies $i^+ : \mathcal{A}^+ \hookrightarrow \mathcal{B}^+$*

4 Representable cylindric algebras

Let α be an ordinal. Denote $\{f \mid f\alpha \to U\}$ as ${}^{\alpha}U$. x_i stands for $x(i)$, where $x \in {}^{\alpha}U$ and $i < \alpha$.

A subset of ${}^{\alpha}U$ is an α-ry relation on U. For $i, j < \alpha$, the *i, j-diagonal* D_{ij} is the set of all elements of ${}^{\alpha}U$ such that $y_i = y_j$.

If $i < \alpha$ and X is an α-ry relation on U, then the i-th cylindrification $C_i X$ is the set of all elements of U that agree with some element of X on each coordinate except, perhaps, the i-th one. To be more precise,

$$C_i X = \{y \in {}^{\alpha}U \mid \exists x \in X \forall i < \alpha \, (i \neq j \Rightarrow y_j = x_j)\}.$$

We define the following equivalence relation for $i < \alpha$ and $x, y \in {}^{\alpha}U$:

$$x \equiv_i y \Leftrightarrow \forall j \in \alpha \, (i \neq j \Rightarrow x(i) = y(j))$$

Then one may reformulate the definition of the i-th cylindrification in the following way:

$$C_i X = \{y \in {}^{\alpha}U \mid \exists x \in X \;\; x \equiv_i y\}$$

According to this version of the definiton, one may think of the cylindrification as an $\mathbf{S}5$ modal operator.

Definition 7. *A cylindic set algebra of dimension α is an algebra consisting of a set S of α-ry relation on some base set U with the constants and operations $0 = \varnothing$, $1 = {}^{\alpha}U$, $\cap$, $-$, the diagonal elements $(D_{ij})_{i,j<\alpha}$, the cylindrifications $(C_i)_{i<\alpha}$. A generalised cylindric set algebra of dimension α is a subdirect of cylindric algebras that have dimension α. $\mathbf{Cs}_{\alpha}$ denotes the class of all cylindric set algebras of dimension α.*

Definition 8. *A cylindric algebra of dimension α is an algebra $\mathcal{C} = \langle \mathcal{B}, \{c_i\}_{i<\alpha}, \{d_{ij}\}_{i,j<\alpha} \rangle$ such that*

- *$\mathcal{B}$ is a Boolean algebra, for each $i, j < \alpha$ c_i is an operator and $d_{ij} \in \mathcal{B}$*
- *For each $i < \alpha$, $a \leqslant c_i a$, $c_i(a \cdot c_i b) = c_i a \cdot c_i b$ and $d_{ii} = 1$*
- *For every $i, j < \alpha$, $c_i c_j a = c_j c_i a$*
- *If $k \neq i, j < \alpha$, then $d_{ij} = c_k(d_{ij} \cdot d_{jk})$*

- *If $i \neq j$, then $c_i(d_{ij} \cdot a) \cdot c_i(d_{ij} \cdot -a) = 0$*

$\mathbf{CA}_\alpha$ *is the class of all cylindric algebras of dimension α.*

One may define a representation of a cylindric algebra explicitly in the following way:

Definition 9. *Let $\mathcal{A}$ be a cylindric algebra of dimension α. A representation of $\mathcal{A}$ over the non-empty domain X is a map $f : \mathcal{A} \hookrightarrow 2^{^\alpha U}$ such that:*

1. *$f(1) = \bigcup\limits_{i \in I} {}^\alpha X_i$ for some disjoint family $\{X_i\}_{i \in I}$ where each $X_i \subseteq X$*
2. *$h : \mathcal{A} \to 2^{f(1)}$ is a representation of a Boolean reduct*
3. *for all $\lambda, \eta < \alpha$, $x \in h(d_{\lambda\eta})$ iff $x_\lambda = x_\eta$*
4. *for all $\lambda < \alpha$ and $a \in \mathcal{A}$, $x \in h(c_\lambda(a))$ iff there is $y \in X$ such that $x[\lambda \mapsto y] \in h(a)$*

An α-dimensional cylindric algebra C is representable, if there exists a representation of h. $\mathbf{RCA}_\alpha$ is the class of all representable cylindric algebras that have dimension α. In particular, we are interested in the case $\alpha = \omega$.

It is well known that $\mathbf{RCA}_\alpha$ is a variety, $\mathbf{RCA}_\alpha$ ($\alpha \leqslant 2$) is finitely axiomatisable and $\mathbf{RCA}_\alpha$ ($2 < \alpha < \omega$) has no finite axiomatisation, see [3].

Let $\mathcal{A} \in \mathbf{CA}_\omega$, then $\mathcal{A}$ has a *complete representation*, if its representation preserves all existing suprema. In other words, $\mathcal{A}$ is *completely representable.*

5 $\mathbf{RCA}_\omega$ and canonicity

The following definition of an ω-frame is due to [6].

Definition 10. *A cylindric ω-frame is a structure $\mathcal{F} = \langle W, (R_i)_{i<\omega}, (E_{ij})_{i,j<\omega} \rangle$ where $(R_i)_{i<\omega}$ are binary relations and $(E_ij)_{i,j<\omega}$ are unary relations such that, for all $i, j, k < \omega$:*

1. *Every R_i is an equivalence relation on W,*
2. *$R_i \circ R_j = R_j \circ R_i$, that is, the set $(R_i)_{i<\omega}$ forms a commutative semigroup under composition.*
3. *For all $x \in W$, $E_{ii}(x)$ holds.*
4. *For all $x, y, z \in W$, xR_iy & $E_{ij}(y)$ & xR_iz & $E_{ij}(y)$ implies $y = z$.*
5. *For all $x \in W$, $E_{ij}(x)$ iff there exists $y \in W$ such that xR_ky, $E_{ik}(y)$, and $E_{kj}(y)$.*

$\mathcal{C}\mathfrak{a}_\omega$ is the class of all ω-frames.

If $\mathcal{F} \in \mathcal{C}\mathfrak{a}_\omega$ and $x \in \mathcal{F}$, then $\mathcal{F}^x$ is a generated subframe generated by x, which is defined standardly. Generally, $\mathcal{F}_1$ is a generated subframe of $\mathcal{F}_2$, if $\underline{\mathcal{F}_1} \subseteq \underline{\mathcal{F}_2}$ and $\underline{\mathcal{F}_1}$ is closed under under R_{i2} equivalences for every $i < \omega$. That is:

For all $i < \omega$ and $x \in \mathcal{F}_1$, we have $R_{i2}(x) \subseteq \mathcal{F}_1$ and, thus, $R_{i1}(x) = R_{i2}(x)$.

We have the following connection betweens ω-frames and their generated subframes, which is standard for modal logic:

Proposition 1. *Let $\mathcal{F} \in \mathcal{C}\mathfrak{a}_\omega$, then*

1. $\mathcal{F} = \coprod_{x \in \mathcal{F}} \mathcal{F}^x$,

2. $\mathfrak{Cm}(\mathcal{F}) \cong \prod_{x \in \mathcal{F}} \mathfrak{Cm}(\mathcal{F}^x)$,

3. $\mathfrak{Cm}(\mathcal{F}^x)$ *is subdirectly irreducible.*

It is known that $\mathcal{Ca}_\omega$ forms an elementary class, since one can express the conditions of an ω-frame with the first-order language.

The following fact is by Venema, see [6, Proposition 2.1.5]:

Proposition 2. *An ω-frame $\mathcal{F}$ is cylindric iff $\mathfrak{Cm}(\mathcal{F})$ is a cylindric algebra of dimension ω.*

A cylindric ω-frame $\mathcal{F}$ is completely representable, if $\mathfrak{Cm}(\mathcal{F})$ is completely representable as a cylindric algebra of dimension ω.

We are interested in the special case of cylindric ω-frames called Cartesian structure of dimension ω. To be more precise:

Definition 11. *Let U be a set and $V \subseteq {}^\omega U$ be a non-empty subset of the full Cartesian space of dimension ω, then an α-dimension Cartesian structure generated by V is an ω-frame $\mathfrak{S}(V) = \langle V, (R_i)_{i<\omega}, (E_{ij})_{i,j<\omega} \rangle$ such that:*

1. $R_i = \{(w, v) \mid w, v \in V, w_k = w_k, k < \omega, i \neq k\}$

2. $E_{ij} = \{w \in V \mid w_i = w_j\}$

$\mathfrak{S}({}^\omega U)$ is the full ω-dimensional Cartesian structure. $\mathcal{F}\mathfrak{ct}_\omega$ is the class of all full ω-dimensional Cartesian structures.

Clearly $\mathcal{F}\mathfrak{ct}_\omega \subseteq \mathcal{Ca}_\omega$.

We have the following connection between $\mathbf{RCA}_\omega$, $\mathbf{IGs}_\omega$, and complex algebras of full Cartesian structures:

$$\mathbf{RCA}_\omega = \mathbf{IGs}_\omega = \mathbf{S}\mathfrak{Cm}\mathbf{Ud}\mathcal{F}\mathfrak{ct}_\omega = \mathbf{SP}\mathfrak{Cm}\mathcal{F}\mathfrak{ct}_\omega.$$

This follows from the fact that $\mathbf{Cs}_\omega = \mathfrak{Cm}\mathcal{F}\mathfrak{ct}_\omega$. Every generalised cylindric set algebra is a subdirect product of cylindric set algebras, thus, a generalised cylindric set algebra is a complex algebra of disjoint union of some full Cartesian spaces. But $\mathbf{RCA}_\omega$ is the closure of $\mathbf{Cs}_\omega$ under isomorphism.

Definition 12. *The weak Cartesian space with base U and dimension ω determined by $x \in {}^\omega U$ is the set:*

$${}^\omega U^{(x)} = \{y \in {}^\omega U \mid |\{k < \omega \mid x_k \neq y_k\}| < \aleph_0\}$$

$\mathfrak{S}({}^\omega U^{(x)})$ is a weak Cartesian structure of dimension ω. $\mathcal{W}\mathfrak{ct}_\omega$ is the class of all weak Cartesian structure of dimension ω up to isomorphism.

Note that we have $\mathcal{W}\mathfrak{ct}_\omega \subseteq \mathcal{Ca}_\omega$.

Every cylindric set algebra is a subalgebra of some complex algebra induced by an ω-dimensional Cartesian structure. In other words,

Lemma 1. $\mathbf{ICs}_\omega = \mathbf{S}\mathfrak{Cm}\mathcal{F}\mathfrak{ct}_\omega$.

In this section, we reproduce the results related to characterisation $\mathbf{RCA}_\omega$. The following results are due to Goldblatt [2]. This denotes that a cylindric algebra of dimension algebra is representable iff it is isomorphic to a subalgebra of the complex algebra of disjoint sum of some full ω-dimensional Cartesian structure. Assuming the duality, this is equivalent to the standard definition of representability formulated in terms of sublagebras of subdirect products.

Lemma 2. $\mathbf{RCA}_\omega = \mathbf{S}\,\mathfrak{Cm}\mathbb{S}\mathbf{Ud}\mathcal{F}\mathfrak{ct}_\omega = \mathbf{S}\,\mathfrak{Cm}\mathbb{S}\mathbf{Ud}\mathcal{W}\mathfrak{ct}_\omega = \mathbf{IGws}_\omega$

Proof. □

Here we use the following fact related to canonical varieties generated by some class of complex algebras. Let $\mathbf{K}$ be an elementary class of relational structures, then:

If $\mathbf{K}$ is closed under p-morphic images, generated subframes, and disjoint unious, then $\mathbf{S}\mathfrak{Cm}\mathbf{K}$ is a canonical variety.

One may think of this fact a more abstract version of Fine's theorem which claims that every elementary modal logic is canonical [1]. This version denotes the same fact, but it is formulated in terms of varieties BAOs generated by complex algebras of some atom structures.

This is a specialised version of [2, Theorem 4.4] formulated for dimension ω.

Theorem 3. $\mathbf{RCA}_\omega$ *is a canonical variety.*

Proof. By Lemma 2, □

6 Representability games

References

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