Notes on filtration of logics containing **K5**

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1 Preliminaries

Definition 1. *An n-normal modal logic is a set of formulas that contains all Boolean tautologies, formulas $\Diamond_i p \vee \Diamond_i q \leftrightarrow \Diamond_i(p \vee q)$ and $\Diamond_i \bot \leftrightarrow \bot$ for $i \leqslant n$, and is closed under modus ponens, substitution, and monotonicity: from $\varphi \to \psi$ infer $\Diamond_i \varphi \to \Diamond_i \psi$ for $i \leqslant n$.*

Definition 2. *An n-Kripke model is a triple $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$, where $R_i \subseteq W \times W$, $\vartheta : \mathrm{PV} \to 2^W$, and the connectives have the following semantics:*

1. $\mathcal{M}, w \models p \Leftrightarrow w \in \vartheta(p)$
2. $\mathcal{M}, w \models \neg\varphi \Leftrightarrow \mathcal{M}, w \not\models \varphi$
3. $\mathcal{M}, w \models \varphi \vee \psi \Leftrightarrow \mathcal{M}, w \models \varphi$ *or* $\mathcal{M}, w \models \psi$
4. $\mathcal{M}, w \models \Diamond_i \varphi \Leftrightarrow \exists v \in R_i(w)\, \mathcal{M}, v \models \varphi$

By **K5** we mean the logic $\mathbf{K} \oplus A5$, where $A5 = \Diamond p \to \Box\Diamond p$. It is known that **K5** is the modal logic of all Euclidean frames. A frame is called Euclidean if for each x, y, z, xRy and xRz implies yRz.

Proposition 1. *Let $\mathcal{F} = \langle W, R \rangle$ be an Euclidean frame.*

1. *For each $x, y, z \in W$, xRy and xRz implies either yRz or zRy.*
2. *$R \subseteq R; R$, that is, R is dense.*
3. *For each $x \in W$, $R^*(x) = \{x\} \cup R(R(x))$.*
4. *$R^{-1}; R \subseteq R$.*

Let $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$ be a Kripke model and Γ a set of formulas closed under subformulas. An equivalence relation $\sim$ is set to have a finite index if the quotient set $W/\sim$ is finite. The equivalence relation $\sim_\Gamma$ induced by Γ is defined as

$$w \sim_\Gamma v \Leftrightarrow \forall \varphi \in \Gamma\, (\mathcal{M}, w \models \varphi \Leftrightarrow \mathcal{M}, v \models \varphi).$$

If Γ is finite, then $\sim_\Gamma$ has a finite index. An equivalence relation $\sim$ respects $\sim_\Gamma$, if $w \sim v$ implies $w \sim_\Gamma v$.

Definition 3. *Let $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$ be a Kripke model and Γ be a Sub-closed set formulas. A Γ-filtration of $\mathcal{M}$ is a model $\widehat{\mathcal{M}} = \langle \widehat{W}, \widehat{R_1}, \dots, \widehat{R_n}, \widehat{\vartheta} \rangle$ such that:*

1. *$\widehat{W} = W/\sim$, where $\sim$ is an equivalence relation having a finite index that respects Γ*

2. $\widehat{\vartheta}(p) = \{[x]_\sim \mid x \in W \,\&\, x \in \vartheta(p)\}$

3. *For each* $i \in I$ *one has* $\widehat{R}_i^{min} \subseteq \widehat{R}_i \subseteq \widehat{R}_i^{max}$. $\widehat{R}_{i,\sim}^{min}$ *is the* i*-th minimal filtered relation on* $\widehat{W}$ *defined as*

$$\hat{x}\widehat{R}_{i,\sim}^{min}\hat{y} \Leftrightarrow \exists x' \sim x\ \exists y' \sim y\ xR_iy$$

$\widehat{R}_{\Gamma,i}^{max}$ *is the* i*-th maximal filtered relation on* $\widehat{W}$ *induced by* Γ *defined as*

$$\hat{x}\widehat{R}_{\Gamma,i}^{max}\hat{y} \Leftrightarrow \forall \Box_i\varphi \in \Gamma\ (\mathcal{M}, x \models \Box_i\varphi \Rightarrow \mathcal{M}, y \models \varphi)$$

If Φ is finite subset of Γ and $\sim = \sim_\Phi$, then $\widehat{\mathcal{M}}$ is a definable Γ-filtration of $\mathcal{M}$ through Φ. If $\sim = \sim_\Gamma$, then such a filtration by means of the definiton above is called *strict*.

Lemma 1. *Let* Γ *be a finite set of formulas closed under subformulas and* $\widehat{\mathcal{M}}$ *a filtration of* $\mathcal{M}$ *through* Γ*, then for each* $x \in W$ *and for each* $\varphi \in \Gamma$ *one has*

$$\mathcal{M}, x \models \varphi \Leftrightarrow \widehat{\mathcal{M}}, \hat{x} \models \varphi$$

Definition 4. *Let* $\mathbb{F}$ *be a class of Kripke frames and* Γ *a finite set of formulas closed under subformulas. If for every model* $\mathcal{M}$ *over* $\mathcal{F} \in \mathbb{F}$ *there exists a model that is a* Γ*-definable filtration of* $\mathcal{M}$*, then* $\mathbb{F}$ *admits definable filtration. A class of models* $\mathbb{M}$ *admits definable filtration if for every* $\mathcal{M} \in \mathbb{M}$ *there exists a model belonging to the same class that is a definable* Γ*-filtration of* $\mathcal{M}$*.*

Lemma 2.

1. *Let* $\mathcal{L}$ *be a complete normal modal logic. If* Frames($\mathcal{L}$) *admits filtration, then* $\mathcal{L}$ *has the finite model property.*

2. *If the class of models* Mod($\mathcal{L}$) *admits filtration, then* $\mathcal{L}$ *has the finite model property and Kripke complete as well.*

2 Filtration of Euclidean logics

First of all, let us ensure that a filtration of an Euclidean frame is not necessary finite. Let $[x] \sim_\Gamma [y]$ and $[x] \sim_\Gamma [z]$. Then for some $x' \in [x]$ $y' \in [y]$, one has $x'Ry'$ and $x''Rz'$ for some $x'' \in [x]$ and $z' \in [z]$. Clearly, we cannot claim that $x' = x''$ in general. Thus, minimal filtration does not preserve the required property.

2.1 Strict filtration for logics containing K5

Definition 5. *Let* $\mathcal{F} = \langle W, R, \vartheta \rangle$ *be a Kripke model. Then its Euclidean Horn closure* (R^E*, E-closure) is defined as*

$$R^E = (\bigcup_{n<\omega} (R^{-1})^n); R$$

or, equivalently, $R^E = \bigcup\limits_{n<\omega} R_n$*, where* $R^0 = R$ *and* $R_{n+1} = R_n \cup (R^{-1}; R_n)$*.*

Lemma 3. *Let* $\mathcal{M} = \langle W, R, \vartheta \rangle$ *be an Euclidean model,* Γ *a set of* Sub*-closed formulas, and* $\sim$ *an equivalence relation having a finite index that respects* Γ*. Then its E-closure of the minimal filtration of* R *is a filtration itself.*

Proof. Let us show that $\widehat{R^E} = \bigcup_{n<\omega} (R_n)^{\min}_{\sim}$ is a filtration and $\langle W/\sim, \widehat{R}^E, \widehat{\vartheta}\rangle$ is an Euclidean model.

Let $\widehat{\mathcal{M}} = \langle \widehat{W}, \widehat{R}, \widehat{\vartheta}\rangle$ be a minimal filtration of an Euclidean model through $\sim$.

1. Suppose xRy, then $[x]\widehat{R}[y]$ by the definition of a minimal filtration. $\widehat{R}$ is clearly a subrelation of $\widehat{R}^E$, thus $[x]\widehat{R}^E[y]$.
2. Suppose $[x]\widehat{R^E}[y]$ and $\mathcal{M}, x \models \Box\varphi$ for $\Box\varphi \in \Gamma$. then $([x],[y]) \in \bigcup_{n<\omega} R_n$. Then $([x],[y]) \in R_n$ for some $n < \omega$. There are two cases:
   (a) $n = 0$, then $[x]\widehat{R}[y]$, so, obviously, one has $\mathcal{M}, y \models \varphi$
   (b) $n = m+1$. Suppose $[x]\widehat{R}_{m+1}[y]$. Thus, $[x]\widehat{R}_n \cup (\widehat{R}^{-1}; \widehat{R}_n)[y]$. There are the following two cases:
      i. $([x],[y]) \in \widehat{R}_n$, then the statement holds by IH.
      ii. $([x],[y]) \in \widehat{R}^{-1}; \widehat{R}_n$. Then there exists $[z]$ such that $[x]\widehat{R}^{-1}[z]$ and $[z]\widehat{R}_m[y]$, that is, $[z]\widehat{R}[x]$ and $[z]\widehat{R}_m[y]$. One has $\mathcal{M}, x \models \Box\varphi$, then $\widehat{\mathcal{M}}, [x] \models \Box\varphi$. So $\widehat{\mathcal{M}}, [z] \models \Diamond\Box\varphi$. On the other hand, $\mathcal{M}, z \models \Diamond\Box\varphi \to \Box\varphi$, so $\mathcal{M}, z \models \Box\varphi$ and, from that, $\widehat{\mathcal{M}}, [z] \models \Box\varphi$, and, thus, $\widehat{\mathcal{M}}, [y] \models \varphi$ and $\mathcal{M}, y \models \varphi$ by IH and the definition of a minimal filtration.

□

Corollary 1. **K**5 *admit strict filtrations.*

2.2 Clusters

Let $\mathcal{F} = \langle W, R\rangle$ be a transitive frame. Let us put $xR^{\bullet}y \Leftrightarrow xRy \,\&\, \neg(xRy)$. A point x is proper if xRx. Let us define the following equivalence relation:

$$x \equiv y \Leftrightarrow xRy \,\&\, yRx \vee x = y.$$

A cluster is an element of the quotient set $W/\equiv$. Given $x \in W$, C_x is a cluster containing x. Thus $C_x = \{x\} \cup \{y \,| xRyx\}$. The original relation lifts to the antisymmetric transitive relation on $W/\equiv$ defined as C_xRC_y iff xRy. A cluster C is called maximal if CRC' implies $C = C'$. A point is R-maximal if C_x is a maximal cluster, that is, $R^{\bullet}(x) = \varnothing$. A degenerated cluster is a singleton $\{x\}$ with $\neg(xRx)$. A cluster is called simple if it has the form $\{x\}$ with xRx. If $\langle W', R'\rangle$ is an inner substructure of $\langle W, R\rangle$, then every R'-cluster is an R-cluster and every R-cluster that intersects W' is a subset of W' and is an R'-cluster itself. Given a Kripke model $\mathcal{M}$, a set of formulas Γ is satisfied by a cluster C if every member of Γ is true at some point of C.

If clusters coincide then their poitns have the same theory in the original model:

Lemma 4. $C_x = C_y$ *implies* $\mathcal{M}, x \models \Box\varphi \Leftrightarrow \mathcal{M}, y \models \Box\varphi$

Let us describe the bulldozing technique allowing one to eliminate nondegenerated clusters [3]. Let $\mathcal{L}$ be a transitive logic and $\mathcal{F}$ its frame. We construct first a frame $\mathcal{F}^0 = \langle W^0, R^0\rangle$ replacing every nondegenerated frame C of W by $C^0 = \{\langle x, i\rangle \mid x \in C, i < \omega\}$. We also replace each degenerated cluster C by $\{\langle x, 0\rangle\}$. Elements of these subsets form W^0. The relation R^0 is defined as

$$\langle x, i\rangle R^0 \langle y, j\rangle \Leftrightarrow xR^{\bullet}y \text{ or } (x \equiv y \,\&\, i < j) \text{ or } i = j \,\&\, x <_C y$$

where $<_C$ is an arbitrary strict ordering on the proper cluster C containing x and y.

Each nondegenrated cluster C is replaced by an infinite set C_0 such that $\langle C_0, R_0\rangle$ is a strict linear order. Moreover, $\langle y, j\rangle$, a copy of y, occurs after $\langle y, j\rangle$, a copy of x.

Bulldozing might be extended to models $\mathcal{M} = \langle W, R, \vartheta\rangle$ defining ϑ^0 as follows

$$\vartheta^0(p_i) = \{\langle x, i\rangle \mid x \in \vartheta(p_i), i < \omega\}.$$

One may show inductively the following fact.

Lemma 5. $\mathcal{M}, x \models \varphi \Leftrightarrow \mathcal{M}^0, \langle x, i\rangle \models \varphi$

Let us concretise the case of transitive Euclidean frames. First of all, we consider clusters in **K**45 frames.

2.3

3 Transitive closure stuff

4 PDLisation of Euclidean logics

5 Transitive closure and fusion

References

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