Notes on filtrations for logics that contain **K5**

Daniel Rogozin

1 Preliminaries

Definition 1. *A normal modal logic is a set of formulas that contains all Boolean tautologies, formulas $\Diamond p \vee \Diamond q \leftrightarrow \Diamond(p \vee q)$ and $\Diamond\bot \leftrightarrow \bot$, and is closed under Modus Ponens, Substitution, and Monotonicity: from $\varphi \to \psi$ infer $\Diamond\varphi \to \Diamond\psi$.*

Definition 2. *An Kripke model is a triple $\mathcal{M} = \langle W, R, \vartheta \rangle$, where $R \subseteq W \times W$, $\vartheta : \mathrm{PV} \to 2^W$, and the connectives have the following semantics:*

1. $\mathcal{M}, w \models p \Leftrightarrow w \in \vartheta(p)$
2. $\mathcal{M}, w \models \neg\varphi \Leftrightarrow \mathcal{M}, w \not\models \varphi$
3. $\mathcal{M}, w \models \varphi \vee \psi \Leftrightarrow \mathcal{M}, w \models \varphi$ *or* $\mathcal{M}, w \models \psi$
4. $\mathcal{M}, w \models \Diamond\varphi \Leftrightarrow \exists v \in R(w)\, \mathcal{M}, v \models \varphi$

A modal logic $\mathcal{L}$ is called locally finite, if every fragment of $\mathcal{L}$ with a finite number of variables is finite. Equivalently, its weak canonical model generated by $k < \omega$ variables is finite. Moreover, if $\mathcal{L}$ is locally finite, then $\mathcal{L} = Log(\mathcal{F}_{\mathcal{L}} \upharpoonright k \mid k < \omega)$, where $\mathcal{F}_{\mathcal{L}} \upharpoonright k$ is a weak canonical frame generated by k variables.

Definition 3. *The modal depth of a formula is defined recursively by the following function:*

$$
\begin{gathered}
md(p_i) = 0 \\
md(\neg\varphi) = md(\varphi) \\
md(\varphi \vee \psi) = \max(md(\varphi), md(\psi)) \\
md(\Diamond\varphi) = 1 + md(\psi)
\end{gathered}
$$

Let $\mathcal{L}$ be a modal logic, then the modal depth of $\mathcal{L}$, denoted as $md(\mathcal{L})$, is the minimal n such that any $\varphi \in \mathcal{L}$ is equivalent to some ψ such that $md(\psi) = n$. If such an n does not exists, then $md(\mathcal{L}) = \infty$.

1.1 Filtrations

Let $\mathcal{M} = \langle W, R_1, \ldots, R_n, \vartheta \rangle$ be a Kripke model and Γ a set of formulas closed under subformulas. An equivalence relation $\sim$ is set to have a finite index if the quotient set $W/\sim$ is finite. The equivalence relation $\sim_\Gamma$ induced by Γ is defined as

$$w \sim_\Gamma v \Leftrightarrow \forall\varphi \in \Gamma\, (\mathcal{M}, w \models \varphi \Leftrightarrow \mathcal{M}, v \models \varphi).$$

If Γ is finite, then $\sim_\Gamma$ has a finite index. An equivalence relation $\sim$ respects $\sim_\Gamma$, if $w \sim v$ implies $w \sim_\Gamma v$.

The following definition of a filtration is due to, e.g., [9].

Definition 4. *Let $\mathcal{M} = \langle W, R_1, \ldots, R_n, \vartheta\rangle$ be a Kripke model and Γ be a Sub-closed set formulas. A Γ-filtration of $\mathcal{M}$ is a model $\widehat{\mathcal{M}} = \langle \widehat{W}, \widehat{R_1}, \ldots, \widehat{R_n}, \widehat{\vartheta}\rangle$ such that:*

1. *$\widehat{W} = W/\sim$, where $\sim$ is an equivalence relation that respects Γ*
2. *$\widehat{\vartheta}(p) = \{[x]_\sim \mid x \in W \;\&\; x \in \vartheta(p)\}$*
3. *For each $i \in I$ one has $\widehat{R}_i^{min} \subseteq \widehat{R}_i \subseteq \widehat{R}_i^{max}$. $\widehat{R}_{i,\sim}^{min}$ is the i-th minimal filtered relation on $\widehat{W}$ defined as*

$$\hat{x}\widehat{R}_{i,\sim}^{min}\hat{y} \Leftrightarrow \exists x' \sim x \, \exists y' \sim y \, xR_iy$$

$\widehat{R}_{\Gamma,i}^{max}$ is the i-th maximal filtered relation on $\widehat{W}$ induced by Γ defined as

$$\hat{x}\widehat{R}_{\Gamma,i}^{max}\hat{y} \Leftrightarrow \forall \Box_i\varphi \in \Gamma \, (\mathcal{M}, x \models \Box_i\varphi \Rightarrow \mathcal{M}, y \models \varphi)$$

Alternatively, one may reformulate the condition of the maximal filtered relations using $\Diamond$'s as follows. We will use this formulation occasionally:

$$\hat{x}\widehat{R}_{\Gamma,i}^{\max}\hat{y} \Leftrightarrow \forall \Diamond_i\varphi \in \Gamma \, (\mathcal{M}, y \models \varphi \Rightarrow \mathcal{M}, x \models \Diamond_i\varphi)$$

If Φ is an extension of Γ and $\sim = \sim_\Phi$, then $\widehat{\mathcal{M}}$ is a definable Γ-filtration of $\mathcal{M}$ through Φ. If $\sim = \sim_\Gamma$, then such a filtration by means of the definiton above is called *strict*.

A class of models $\mathbb{M}$ admits strict filtrations for models (ASF), if for every Sub-closed set Γ and for every $\mathcal{M} \in \mathbb{M}$ there exists a model $\widehat{\mathcal{M}}$ such that $\widehat{\mathcal{M}} \in \mathbb{M}$ and $\widehat{\mathcal{M}}$ is a filtration of $\mathcal{M}$ through Γ.

A class of frames $\mathbb{F}$ admits strict filtrations for frames, if for every Sub-closed set Γ and for every frame $\mathcal{F} \in \mathbb{F}$ and every model $\mathcal{M}$ over $\mathcal{F}$ there exists a Γ filtration of $\mathcal{M}$, and the underlying frame of this filtration belongs to $\mathbb{F}$.

If $\mathcal{L}$ is canonical, then the ASF property for frames and ASF property for models are equivalent, see [7, Theorem 2.10].

The key lemma about filtrations is the following, see [1, Theorem 2.39]:

Lemma 1. *Let Γ be a finite set of formulas closed under subformulas and $\widehat{\mathcal{M}}$ a filtration of $\mathcal{M}$ through Γ, then for each $x \in W$ and for each $\varphi \in \Gamma$ one has*

$$\mathcal{M}, x \models \varphi \Leftrightarrow \widehat{\mathcal{M}}, \hat{x} \models \varphi$$

Definition 5.

1. *Let $\mathbb{F}$ be a class of Kripke frames and Γ a finite set of formulas closed under subformulas. If for every model $\mathcal{M}$ over $\mathcal{F} \in \mathbb{F}$ there exists a model that is a Γ-definable filtration of $\mathcal{M}$, then $\mathbb{F}$ admits definable filtration.*
2. *A class of models $\mathbb{M}$ admits definable filtration if for every $\mathcal{M} \in \mathbb{M}$ there exists a model belonging to the same class that is a definable Γ-filtration of $\mathcal{M}$.*

Lemma 2.

1. *Let $\mathcal{L}$ be a complete normal modal logic. If* $\mathrm{Frames}(\mathcal{L})$ *admits filtration, then $\mathcal{L}$ has the finite model property.*
2. *If the class of models* $\mathrm{Mod}(\mathcal{L})$ *admits filtration, then $\mathcal{L}$ has the finite model property and it is Kripke complete as well.*

1.2 Horn closure

Definition 6. *A first-order formula is called Horn if it has the following form (see [3]):*

$$\forall x_1, \ldots, x_n (x_{i_1} R x_{j_1} \wedge \cdots \wedge x_{i_s} R x_{j_s} \to A), \text{ where } A \text{ is either } x_k R x_l \text{ or } \bot.$$

Definition 7. *Let H be a Horn property and $\langle W, R\rangle$ a Kripke frame. A Horn closure of a binary relation R is the minimal relation R^H containing R and satisfying H.*

Lemma 3. $R^H = \bigcup\limits_{n<\omega} R_n$ *where*

1. $R_0 = R$.
2. $R_{n+1} = R_n \cup \{(a,b) \in W \mid \exists \vec{c} \in W\ P(a,b,\vec{c})\}$*, where P is a premise of H.*

1.3 Bisimulations

A model $\mathcal{M}$ is called *weak* model, if its valuation is restriced to some finite subset of variables, namely $PV \upharpoonright k$, if the cardinality of this subset equals k.

Definition 8. *Let $\mathcal{M}_1 = \langle W_1, R_1, \vartheta_1\rangle$ and $\mathcal{M}_2 = \langle W_2, R_2, \vartheta_2\rangle$ be weak Kripke models (with the same variables), $x \in W_1$, and $y \in W_2$, then x and y are 0-equivalent, if the following holds:*

$$x \equiv_0 y \text{ iff for each } p \in PV \upharpoonright k \text{ (or for each } p \in PV\text{) one has } \mathcal{M}_1, x \models p \Leftrightarrow \mathcal{M}_2, y \models p$$

Definition 9. *Let $\mathcal{M}_1 = \langle W_1, R_1, \vartheta_1\rangle$ $\mathcal{M}_2 = \langle W_2, R_2, \vartheta_2\rangle$ be Kripke models, then a bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_2$ is a binary relation $E \subseteq W_1 \times W_2$ between two models such that:*

- *(0) $x_1 E x_2$ implies $x_1 \equiv_0 x_1$*
- *(**zig**) $x_1 E y_1$ and $x_1 R_1 z$ implies $y_1 R_2 y_2$ and $z E y_2$ for some $y_2 \in W_2$*
- *(**zag**) $x_1 E y_1$ and $y_1 R_2 y_2$ implies $x_1 R_1 x_2$ and $x_2 E y_2$ for some $x_2 \in W_1$*

Models with designated points $\langle \mathcal{M}_1, x\rangle$ and $\langle \mathcal{M}_2, y\rangle$ are bisimilar, if there exists a bisimulation E between them with xEy. We denote that as $\langle \mathcal{M}_1, x\rangle \underline{\leftrightarrow} \langle \mathcal{M}_2, y\rangle$. Moreover, we have

$$\langle \mathcal{M}_1, x\rangle \underline{\leftrightarrow} \langle \mathcal{M}_2, y\rangle \text{ implies } Th(\mathcal{M}_1, x) = Th(\mathcal{M}_2, y).$$

Definition 10. *Let $\langle \mathcal{M}_1, x\rangle$ and $\langle \mathcal{M}_2, y\rangle$ be models with designated points and $r < \omega$. The r-round bisimulation game $\mathcal{G}_r(\mathcal{M}_1, \mathcal{M}_2, x, y)$ is played by two players, $\forall$ (Abelard, man) and $\exists$ (Heloise, woman). The rules of the game are the following:*

1. *(**Round 0**) $\exists$ wins, if $x \equiv_0 y$. Otherwise, $\forall$ wins.*
2. *(**Round** $n+1$) Let $n+1 < r$. Suppose rounds $0, \ldots, n$ have been played. The n-th position is a pair (x_n, y_n). $\forall$ chooses x_{n+1} (or $y_n \in W_2$) with $x_n R_1 x_{n+1}$ (or $y_n R_1 y_{n+1}$). $\exists$ chooses y_{n+1} (x_{n+1}) with $y_n R_1 y_{n+1}$ (or $y_n R_1 y_{n+1}$) and $x_{n+1} \equiv_0 y_{n+1}$. $\forall$ ($\exists$) wins, if $\exists$ ($\forall$) cannot move.*

$\exists$ wins the game after r rounds. $\forall$ wins the game, if there exists $k < r$ such that $\forall$ wins in **Round** *k.*

There are two n-equivalences for formulas and games (the first one is due to Fine [4]).

Definition 11.

- $(\mathcal{M}_1, x) \equiv_n (\mathcal{M}_2, y)$ *iff for every* φ *such that* $md(\varphi) \leqslant n$ *we have* $\mathcal{M}_1, x \models \varphi$ *iff* $\mathcal{M}_2, y \models \varphi$.
- $(\mathcal{M}_1, x) \equiv_n (\mathcal{M}_2, y)$ *iff* $\exists$ *has a winning strategy in* $\mathcal{G}_n(\mathcal{M}_1, \mathcal{M}_2, x, y)$.

Lemma 4 (See [6])**.** *The definitions of* n*-equivalence are equivalent for weak Kripke models.*

Lemma 5. *(Stabilisation lemma)*
Let n *be a natural number and* $\mathcal{L}$ *a modal logic. If* $\equiv_n = \equiv_{n+1}$ *in a weak canonical model* $\mathcal{M}_{\mathcal{L}} \upharpoonright k$*, then* $md(\mathcal{L} \upharpoonright k) \leqslant n$.

Theorem 1 (See [5])**.** *Let* $\mathcal{L}$ *be a modal logic such that* $md(\mathcal{L}) = n$ *for some* $n < \omega$*, then* $\mathcal{L}$ *is locally finite.*

2 Filtrations for K5

The **K5**-closure (the Euclidean Horn closure of a binary relation) has the following equivalent definitions:

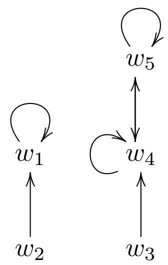
Lemma 6. *Let* $\mathcal{F} = \langle W, R \rangle$ *be a Kripke frame. The following are equivalent:*

1. $R^{\mathbf{K5}}$ *is the smallest Euclidean relation containing* R*, that is, the Horn closure of* R.
2. $R^{\mathbf{K5}} = \bigcup\limits_{i<\omega} R_i$*, where*
   - $R_0 = R$
   - $R_{n+1} = R_n \cup (R_n^{-1} \circ R_n)$
3. $xR^{\mathbf{K5}}y$ *iff there exists* $n < \omega$ *such that either* xRy *or* $\exists z_1, \ldots, z_n$ *with* z_1Rx *and* $z_{n-1}Ry$ *and for each* $1 < i \leqslant n$ *one has either* $z_{i-1}Rz_i$ *or* z_iRz_{i-1}.
4. $R^{\mathbf{K5}} = R \cup \bigcup\limits_{i<\omega} (R^{-1} \circ (R \cup R^{-1})^n \circ R)$.

Theorem 2. **K**5 *does not admit strict filtrations.*

Proof. Let us consider a **K**5 model whose Euclidean closure of the minimal filtration does not give us a filtration.

Let us consider a frame called $\mathcal{F}_{bad}$. We define this frame with the following graph:



Let us define a valuation ϑ such that $\vartheta(p) = \{w_5\}$ and $\vartheta(q) = \{w_1\}$. Let us consider a minimal filtration of $\mathcal{M}_{bad}$ through the Sub-closure of $\Gamma = \{\neg p, \neg\Diamond p\}$.

Clearly that $w_2 \sim_\Gamma w_3$, since $\neg p$ and $\neg\Diamond p$ are true both at w_2 and w_3.

Moreover, $R_{min} \cup (R_{min}^{-1} \circ R_{min})$ is not a subset of R_{max} since $(\hat{w_1}, \hat{w_5}) \in (R_{min}^{-1} \circ R_{min})$, but $\Diamond p$ is not true at w_5.

Let us also note that strict filtrations of this model is not Euclidean. Suppose by contrary that $\widehat{R}^{\mathcal{E}}$ is a strict filtraction of that model. So $R_{min}^{E} \subseteq \widehat{R}^{\mathcal{E}}$, since R_{min}^{E} is the minimal Euclidean relation containing R_{min}. On the other hand, $R_{min}^{E} \nsubseteq R_{max}$, so is not $\widehat{R}^{\mathcal{E}}$. $\square$

Theorem 3. **K5** *admits definable filtrations.*

Theorem 4. **K**45 *admits strict filtrations.*

Proof. Let $\mathcal{M} = \langle W, R, \vartheta \rangle$ be a transitive Euclidean model and $\overline{\mathcal{M}} = \langle \overline{W}, \overline{R}, \overline{\vartheta} \rangle$ its minimal filtration through Γ, where Γ is finite and Sub-closed. Let us put $\widehat{R} = \overline{R}^{+} \cup \overline{R}^{\mathbf{K5}}$. Let us show that $\overline{R}^{+} \cup \overline{R}^{\mathbf{K}45} \subseteq \overline{R}^{max}$.

That is, if $\mathcal{M}, y \models \varphi$ for $\Diamond\varphi \in \Gamma$ and $\hat{x}\widehat{R}\hat{y}$, then $\mathcal{M}, x \models \Diamond\varphi$.

Let $\hat{x}\widehat{R}\hat{y}$.

1. Suppose $(\hat{x}, \hat{y}) \in \overline{R}$, then $\mathcal{M}, x \models \Diamond\varphi$ holds trivially by the definition of the minimal filtration.
2. Let us consider the case when $(\hat{x}, \hat{y}) \in \overline{R}^{\mathbf{K}5}$. The second alternative is the same as for the **K**4-case, see [2, p. 141].

   Suppose the statement holds $\overline{R}_n$ and $(\hat{x}, \hat{y}) \in \overline{R}_{n+1} = \overline{R}_n \cup (\overline{R}_n^{-1} \circ \overline{R}_n)$. We consider the case of $(\hat{x}, \hat{y}) \in (\overline{R}_n^{-1} \circ \overline{R}_n)$.

   Then there exists $\hat{z}$ such that $(\hat{z}, \hat{x}), (\hat{z}, \hat{y}) \in \overline{R}_n$.

   By IH, $\mathcal{M}, z \models \Diamond\varphi$.

   $(\hat{z}, \hat{y}) \in \overline{R}_n$ iff there are $\hat{u}_1, \ldots, \hat{u}_n$ such that

   $$\hat{z} \xleftarrow[\widehat{R}]{} \hat{u}_1 \xrightarrow[\widehat{R}']{} \hat{u}_2 \xrightarrow[\widehat{R}']{} \ldots \xrightarrow[\widehat{R}']{} \hat{u}_{n-1} \xrightarrow[\widehat{R}']{} \hat{u}_n \xrightarrow[\widehat{R}]{} \hat{y}$$

   where $\widehat{R}'$ is either $\widehat{R}$ or $\widehat{R}^{-1}$.

As it is known, $\Diamond\Diamond\varphi \to \Box\Diamond\varphi \in \mathbf{K}45$.

$\hat{u}_1\hat{\hat{z}}$, that is, $u_1' R z'$ for some $u_1' \in \hat{u}_1$ and $z' \in \hat{z}$. That is, $\mathcal{M}, u_1' \models \Diamond\Diamond\varphi$, so $\mathcal{M}, u_1' \models \Diamond\varphi$ and $\overline{M}, \hat{u}_1 \models \Diamond\varphi$.

We have $\hat{u}_1 \widehat{R}' \hat{u}_2$. Suppose $\mathcal{M}, u_1'' \models \Diamond\varphi$ and $u_1'' R u_2'$. We also have $\mathcal{M}, u_1'' \models \Box\Diamond\varphi$, thus, $\mathcal{M}, u_2' \models \Diamond\varphi$.

Suppose $\hat{u}_2 \widehat{R} \hat{u}_1$ and $u_2' R u_1''$, then $\mathcal{M}, u_2' \models \Diamond\varphi$.

Similarly, we have $\mathcal{M}, u_i \models \Diamond\varphi$ iff $\mathcal{M}, u_{i+1} \models \Diamond\varphi$, whenever $\hat{u}_i \widehat{R}' \hat{u}_{i+1}$.

Finally, we have $\hat{u}_n \widehat{R} \hat{x}$. Thus, $u_n' R x'$ for some $u_n' \in \hat{u}_n$ and $x' \in \hat{x}$. $\mathcal{M}, u_n' \models \Diamond\varphi$, so $\mathcal{M}, u_n' \models \Box\Diamond\varphi$. Then $\mathcal{M}, x' \models \Diamond\varphi$. □

3 i,j-Euclideaness

Definition 12. *A binary relation $R \subseteq W \times W$ is called i,j-Euclidean for $i, j < \omega$, if for each x, y, z such that $xR^i y$ and $xR^j z$ implies xRz.*

Proposition 1. *Let $\mathcal{F} = \langle W, R \rangle$, then $\mathcal{F}$ is i,j-Euclidean iff $\mathcal{F} \models \Diamond^i p \to \Box^j \Diamond p$. As a corollary, the logic $\mathbf{K}5^{i,j} = \mathbf{K} \oplus \Diamond^i p \to \Box^j \Diamond p$ is Kripke-complete.*

Let R be a binary relation on $W \neq \varnothing$, then the i,j-Euclidean closure of R (where $i, j < \omega$), denoted as $R^{\mathbf{K}5_{i,j}}$, is a binary relation defined recursively as follows:

1. $R_0 = R$

2. $R_{n+1} = R_n \cup (((R_n)^{-1})^i \circ {R_n}^j)$

3. $R^{\mathbf{K5}_{i,j}} = \bigcup_{k<\omega} R_k$

Note that the following property holds not for every $\mathbf{K5}^{i,j}$, consider **KB** as an example (clearly, $\mathbf{KB} = \mathbf{K5}^{0,1}$), whose modal depth is ∞.

Exercise 1. $md(\mathbf{K5}) = 2$.

Proof. For the upper bound, let us show that $\equiv_2 = \equiv_3$ in every weak canonical model of **K5**, namely $\mathcal{L}_{\mathbf{K5}} \upharpoonright k$ for $k < \omega$. For that, we show that if $\exists$ can win $\mathcal{G}_2(\mathcal{L}_{\mathbf{K5}} \upharpoonright k, \Gamma, \Delta)$ (where Γ and Δ are maximal consistent $\mathbf{K5} \upharpoonright k$-theories), then she can the game $\mathcal{G}_2(\mathcal{L}_{\mathbf{K5}} \upharpoonright k, \Gamma, \Delta)$ as well.

If $\exists$ has a winning strategy in $\mathcal{G}_2(\mathcal{L}_{\mathbf{K5}} \upharpoonright k, \Gamma_0, \Delta_0)$, then, by Lemma 4, $\langle \mathcal{L}_{\mathbf{K5}} \upharpoonright k, \Gamma_0 \rangle \equiv_2 \langle \mathcal{L}_{\mathbf{K5}} \upharpoonright k, \Delta_0 \rangle$.

Therefore, Γ_0 and Δ_0 consist of the same formulas whose modal depth is less than or equal to 2.

Suppose (Γ_2, Δ_2) is the second position. Clearly $\Gamma_2 \equiv_0 \Delta_2$

Let us assume that $\forall$ picks Γ_3 such that $\Gamma_2 R_{\mathbf{K5}\upharpoonright k} \Gamma_3$. $\exists$ has to find a proper response, namely Δ_3, to win the game. $\square$

Exercise 2. *Let $i = j$, then $md(\mathbf{K5}^{i,i}) = i + 1$. So $\mathbf{K5}^{i,i}$ is locally finite.*

Proof. $\square$

Theorem 5. $md(\mathbf{K5}^{i,j}) = 1 + j + i$, *where $i > 0$.*

Theorem 6. $\mathbf{K5}^{i,j}$ *admits definable filtrations.*

Proof. Let $\mathcal{M} = \langle W, R, \vartheta \rangle$ and Γ a finite Sub-closed of formulas. We extend Γ as

$$\Delta = \Gamma \cup \mathrm{Sub}\{\Diamond^i \psi \mid \Box\psi \in \Gamma\} \cup \mathrm{Sub}\{\Box^j \Diamond\psi \mid \Diamond\psi \in \Gamma\}$$

Let $(\hat{x}, \hat{y}) \in R^{\mathbf{K5}_{i,j}}$, $\mathcal{M}, x \models \Box\psi$ for $\Box\psi$ in Δ. If $n = 0$, then the statement is obvious. Suppose $n = 1$. Then $(\hat{x}, \hat{y}) \in R_{min} \cup (((R_{min})^{-1})^i \circ {R_{min}}^j)$. Consider the second alternative. Then there exists $\hat{z}$ such that $(\hat{x}, \hat{z}) \in ((R_{min})^{-1})^i$ and $(\hat{z}, \hat{y}) \in {R_{min}}^j$.

That is, there are $\hat{x_1}, \hat{x_2}, \ldots, \hat{x_i}$ and $\hat{y_1}, \hat{y_2}, \ldots, \hat{y_j}$ such that

$$\hat{z} \xrightarrow{R_{min}} \hat{x_1} \xrightarrow{R_{min}} \hat{x_2} \xrightarrow{R_{min}} \ldots \xrightarrow{R_{min}} \hat{x_i} \xrightarrow{R_{min}} \hat{x}$$

$$\hat{z} \xrightarrow{R_{min}} \hat{y_1} \xrightarrow{R_{min}} \hat{y_2} \xrightarrow{R_{min}} \ldots \xrightarrow{R_{min}} \hat{y_j} \xrightarrow{R_{min}} \hat{y}$$

??? $\square$

4 The case of 2-transitivity

Let us define the logic $\mathcal{L}$ as $\mathbf{K} \oplus \Diamond\Diamond\Diamond p \to \Diamond p$. Let R be a binary relation, the $\mathcal{L}$-closure of R is defined (denoted as $R^{\text{✡}}$) as the following union:

$$R^{\text{✡}} = R \cup R^3 \cup R^5 \cup \cdots \cup R^{2k+1} \cup \ldots$$

Theorem 7. $\mathcal{L}$ *does not admit strict filtrations.*

Proof. Consider the following frame $\mathcal{F} = \langle W, R \rangle$:

$$\begin{tikzcd} x \arrow[loop above] \arrow[r] & y \end{tikzcd}$$

$$\begin{tikzcd} y_1 \arrow[r] \arrow[rr, bend right] & z_1 \arrow[r] & z \arrow[loop above] \end{tikzcd}$$

Clearly that $\mathcal{F}$ is an $\mathcal{L}$-frame. We define the valuation ϑ as follows:

$$\vartheta(p) = \{x\}$$
$$\vartheta(q) = \{y, y_1\}$$
$$\vartheta(r) = \{z\}$$

Let us put $\Gamma = \mathrm{Sub}\{p, q, \Diamond r\}$. We factorise W through $\sim_\Gamma$ and consider a model $\widehat{\mathcal{M}} = \langle W/ \sim_\Gamma, \widehat{R}, \widehat{\vartheta} \rangle$, where $\widehat{R} = (\widehat{R}_{min})^{\text{✡}}$. We have $(\hat{x}, \hat{z}) \in \widehat{R} \circ \widehat{R} \circ \widehat{R}$, but $\Diamond r$ is not true at x. $\square$

5 Fusion stuff

Definition 13. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be modal logics, then the fusion $\mathcal{L}_1 * \mathcal{L}_2$ is the minimal bimodal logic that contains $\mathcal{L}_1$ and $\mathcal{L}_2$ [8].*

Lemma 7. *Let Γ be a finite and* Sub*-closed set of formulas.*

1. *$\mathcal{M} = \langle W, R, \vartheta \rangle$ be a* **K**5*-model. Consider $\Gamma' = \Gamma \cup \{\Diamond\Box\psi \mid \Box\psi \in \Gamma\}$. Let Δ be any finite and* Sub*-closed extension of Γ' tThen a model $\widehat{M} = \langle W/ \sim_\Delta, (R^{min}_{\Gamma'})^{\mathbf{K5}}, \widehat{\vartheta} \rangle$ is a filtration of $\mathcal{M}$ through Γ'.*

2. *Let $\mathcal{L} = \mathbf{K} \oplus \Diamond\Diamond\Diamond p \to \Diamond p$, then we have the similar statement for $\mathcal{L}$, where Δ is a* Sub*-closed extension extension of Γ'*

   Proof.
   Recall that $(R^{min}_{\Delta})^{\mathbf{K5}}$ is defined inductively as:

   (a) $R^0_{\Gamma'} = R^{min}_{\Gamma'}$

   (b) $R^{n+1}_{\Gamma'} = R^n_{\Gamma'} \cup (R^{n}_{\Gamma'}{}^{-1} \circ R^n_{\Gamma'})$

   (c) $(R^{min}_{\Gamma'})^E = \bigcup_{k<\omega} R^k_{\Gamma'}$

   If $R^{min}_{\Gamma'}$ is already a subrelation of R^{max}, so the base case is self-evident.

   Suppose the statement holds for $R^n_{\Gamma'}$, $(\hat{x}, \hat{y}) \in R^{n+1}_{\Gamma'}$ such that $\mathcal{M}, y \models \psi$ for $\Diamond\psi \in \Gamma'$.

   According to the third item of Lemma 6, this is the same as there exist $\widehat{z_0}, \widehat{z_1}, \dots, \widehat{z_{n-1}}, \widehat{z_n}$ such that $\widehat{z_1} R^{min}_{\Gamma'} \hat{x}$, $\widehat{z_n} R^{min}_{\Gamma'} \hat{y}$, and for each $i \in n+1$ we have either $\widehat{z_i} R^{min}_{\Gamma'} \widehat{z_{i+1}}$ or $\widehat{z_{i+1}} R^{min}_{\Delta} \widehat{z_i}$.

   We visualise this with the following graph:

   $$\hat{x} \xleftarrow{R^{min}_{\Gamma'}} \widehat{z_0} \xleftrightarrow{R'} \widehat{z_1} \xleftrightarrow{R'} \dots \xleftrightarrow{R'} \widehat{z_{n-1}} \xrightarrow{R'} \widehat{z_n} \xrightarrow{R^{min}_{\Gamma'}} \hat{y}$$

   where R' is either $R^{min}_{\Gamma'}$ or its converse. We have $\mathcal{M}, z \models \Box\psi$, $\mathcal{M}, y \models \psi$, so $\mathcal{M}, z_n \models \Diamond\psi$. Since $\mathcal{M}$ is a **K**5-model, we have $\mathcal{M}, z_n \models \Box\Diamond\psi$.

   After that we apply the following property of **K**5-models:

Let $\mathcal{M} \models \mathbf{K}5$ and φ a formula, then for each $a, b \in \mathcal{M}$ such that aRb we have
$\mathcal{M}, a \models \Box\Diamond\varphi$ iff $\mathcal{M}, b \models \Box\Diamond\varphi$

So we have $\mathcal{M}, z_0 \models \Box\Diamond\varphi$. Note that we always stay within Γ'. Thus, $\mathcal{M}, x \models \Diamond\varphi$.

2. Let us prove the second item. Let $\mathcal{M}$ be an $\mathcal{L}$-model. The $\mathcal{L}$-closure of the minimal filtered relation modulo Δ, namely $R_{\Gamma'}^{min^{\mathcal{L}}}$ has the following form:

$$R_{\Gamma'}^{min^{\mathcal{L}}} = \bigcup_{k<\omega} R_{\Gamma'}^{min^{2k+1}}$$

We reformulate this closure equivalently as follows:

(a) $R_0 = R_{\Gamma'}^{min}$

(b) $R_{n+1} = R_n \cup ((R_{\Delta}^{min})^2 \circ R_n)$

(c) $R_{\Gamma'}^{min^{\mathcal{L}}} = \bigcup_{k<\omega}$

The base case is self-evident. Suppose the statement holds for R_n and $(\hat{x}, \hat{y}) \in (R_{\Gamma'}^{min})^2 \circ R_n$, that is, there exists $\hat{z}$ such that $(\hat{x}, \hat{z}) \in (R_{\Gamma'}^{min})^2$ and $(\hat{z}, \hat{y}) \in R_n$. By IH, we have $\mathcal{M}, z \models \Diamond\varphi$.

We have the following:

$$\hat{x} \xrightarrow{R_{\Delta}^{min}} \widehat{y_1} \xrightarrow{R_{\Gamma'}^{min}} \hat{z}$$

The sequence of implications if the following:

$$\widehat{\mathcal{M}}, \hat{z} \models \Diamond\varphi \Rightarrow \widehat{\mathcal{M}}, \widehat{y_1} \models \Diamond\Diamond\varphi \Rightarrow \mathcal{M}, x \models \Diamond\Diamond\Diamond\varphi \Rightarrow \mathcal{M}, x \models \Diamond\varphi$$

$\square$

Theorem 8.

1. $\mathbf{K}5 * \mathbf{K}5$ *admits definable filtrations.*

2. $\mathbf{K}5 * \cdots * \mathbf{K}5$ *admits definable filtrations.*

3. *If* $\mathcal{L}$ *admits strict filtrations, then* $\mathbf{K}5 * \mathcal{L}$ *admits definable filtrations*

4. *If* $\mathcal{L}_1, \ldots, \mathcal{L}_n$ *admit strict filtrations, then* $\mathbf{K}5 * \cdots * \cdots * \mathbf{K}5 * \mathcal{L}_1 * \cdots * \mathcal{L}_n$

5. *Let* $\mathcal{L} = \mathbf{K} \oplus \Diamond\Diamond\Diamond p \to \Diamond p$ *(here and below), then* $\mathcal{L} * \mathcal{L}$ *admits definable filtrations.*

6. *Let* $\mathcal{L} = \mathbf{K} \oplus \Diamond\Diamond\Diamond p \to \Diamond p$ *and* $\mathcal{L}_1$ *a logic that admits strict filtrations, then* $\mathcal{L} * \mathcal{L}_1$

Proof.

1. Let Γ be a finite Sub-closed set of bimodal formulas, $\mathcal{F} = \langle W, R_1, R_2 \rangle$ a $\mathbf{K}5 * \mathbf{K}5$-frame, and ϑ a valuation on $\mathcal{F}$. Denote $\langle \mathcal{F}, \vartheta \rangle$ as $\mathcal{M}$.

We introduce the set of fresh variables $V = \{p_\psi | \psi \in \Gamma\}$ and define a new model $\mathcal{M}' = \langle \mathcal{F}, \vartheta' \rangle$ as follows:

For all $\psi \in \Gamma$, $\mathcal{M}, x \models \psi \Leftrightarrow \mathcal{M}', x \models \psi \Leftrightarrow \mathcal{M}', x \models p_\psi$.

Consider these modifications of Γ and V:

$$\Gamma' = \Gamma \cup \{\Diamond_1\Box_1\psi \mid \Box_1\psi \in \Gamma\} \cup \{\Diamond_2\Box_2\psi \mid \Box_2\psi \in \Gamma\}$$
$$\Delta = V \cup \mathrm{Sub}(\{\Diamond\Box p_\psi \mid \Box_i\psi \in \Gamma, in = 1, 2\})$$

Let us define an equivalence relation $\sim_\Delta$ induced by Δ.

Consider $\mathcal{M}_i = \langle W, R_i, \vartheta' \rangle$, a reduct of $\mathcal{M}'$, we have:

(a) $\mathcal{M}_i, x \models \Box p_\psi$ iff $\mathcal{M}, x \models \Box_i\psi$

(b) $\mathcal{M}_i, x \models \Diamond\Box p_\psi$ iff $\mathcal{M}, x \models \Diamond_i\Box_i\psi$

So $\sim = \sim_{\Gamma'}$ by the construction. Let us put $\widehat{W} = W / \sim_{\Gamma'}$. Lemma 7 implies the following claim:

Claim 1. *Let $\widehat{R}_i = (R_\Delta^{min})^E$ and $\widehat{\vartheta(p)} = \{[x]_{\sim_i} \mid \mathcal{M}_i, x \models p\}$ for $p \in \Delta_1$, define $\widehat{\mathcal{M}}_i = \widehat{W}, \widehat{R}_i, \hat{\vartheta}$. Then $\widehat{\mathcal{M}}_i \models$ **K**5 and $\widehat{\mathcal{M}}_i$ is a filtration of $\mathcal{M}_i$ through Δ.*

Finally, we consider a model $\widehat{\mathcal{M}} = \langle \widehat{W}, \widehat{R}_1, \widehat{R}_2, \vartheta \rangle$, where $\widehat{R_{\Gamma'} i} = R_i{}_{\Gamma'}^{min\,E}$ and $\vartheta(p)$ is defined as usual for $p \in \Gamma$. $\widehat{\mathcal{M}}$ is a filtration of $\mathcal{M}$ through Γ'.

Let $\hat{x}\widehat{R_{\Gamma'} i}\hat{y}$ and $\mathcal{M}, x \models \Box_i\psi$ for $\Box_i\psi \in \Gamma$. Then $\mathcal{M}_i, x \models \Box p_\psi$, so $\widehat{\mathcal{M}_i}, \hat{x} \models \Box p_\psi$. By the claim above, $\widehat{\mathcal{M}_i}$ is a filtration of $\mathcal{M}_i$ through Δ, so $\mathcal{M}_i, y \models p_\psi$. Then $\mathcal{M}, y \models \psi$.

2. Likewise

3. The argument is the same as in the proof of the first item, except for **Claim** 1 that has the following formulation: Let $\widehat{R}_1 = (R_\Delta^{min})^E$ and $\widehat{R}_2 = (R_\Delta^{min})^{\mathcal{L}_1}$ Define a valuation as usual as $\widehat{\vartheta(p)} = \{[x]_{\sim_i} \mid \mathcal{M}_i, x \models p\}$ for $p \in \Delta_1$, define $\widehat{\mathcal{M}}_1 = \langle \widehat{W}, \widehat{R}_1, \hat{\vartheta} \rangle$ and $\widehat{\mathcal{M}}_2 = \langle \widehat{W}, \widehat{R}_2, \hat{\vartheta} \rangle$. Then $\widehat{\mathcal{M}}_1 \models$ **K**5 and $\widehat{\mathcal{M}}_1 \models$ **L** and $\widehat{\mathcal{M}}_i$ is a filtration of $\mathcal{M}_i$ through Δ.

4. Likewise

5. The argument is similar to the proof of first item, but filtrations are slightly different. Let $\mathcal{M} = \langle W, R_1, R_2, \vartheta \rangle$ be a $\mathcal{L} * \mathcal{L}$ model and Γ a Sub-closed set of formulas. As above, we define a set V and a model $\mathcal{M}'$. Define extensions of Γ and V:

$$\Gamma' = \Gamma \cup \{\Diamond_1\Diamond_1\psi \mid \Diamond_1\psi \in \Gamma\} \cup \{\Diamond_2\Diamond_2\psi \mid \Diamond_2\psi \in \Gamma\}$$
$$\Delta = V \cup \mathrm{Sub}(\{\Diamond\Diamond p_\psi \mid \Diamond\psi \in \Gamma, i = 1, 2\})$$

As above $\sim_\Delta = \Gamma'$ and $\widehat{\mathcal{M}'} = \langle W / \sim_\Delta, \widehat{R_i}, \hat{\vartheta} \rangle$ are filtrations of reducts of $\mathcal{M}'$ through Δ. Then $\widehat{\mathcal{M}} = \langle W / \sim_\Delta, \widehat{R_1}, \widehat{R_2}, \hat{\vartheta} \rangle$ is a required filtration of the original $\mathcal{M}$.

6. Extend Γ with $\{\Diamond_i\Diamond_i\psi \mid \Diamond_i\psi \in \Gamma, i = 1, 2\}$ and V with $\{\Diamond\Diamond p_\psi \mid \Diamond_i\psi, i = 1, 2\}$

$\Box$

Theorem 9. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be modal logics that admit definable filtrations. If* $\mathrm{Mod}(\mathcal{L}_1)$ *and* $\mathrm{Mod}(\mathcal{L}_2)$ *admit definable filtrations, so does* $\mathrm{Mod}(\mathcal{L}_1 * \mathcal{L}_2)$*.*

Proof. Let $\mathcal{M} = \langle W, R_1, R_2, \vartheta \rangle$ be an $\mathcal{L}_1 * \mathcal{L}_2$-model. We define a notation $\nabla = \{\neg\Diamond, \Diamond\neg, \Diamond\}$.

Both logics admit definable filtrations, so for every finite Sub-closed set Γ and for every $\mathfrak{M}$, an $\mathcal{L}_1$-model (or an $\mathcal{L}_2$ one) there exists there exists Δ, a extension of Γ having the form:

$$\Delta_1 = \Gamma \cup \mathrm{Sub}\{\nabla_1\nabla_2\ldots\nabla_n\Diamond\psi \mid \Diamond\psi \in \Gamma\} \text{ (for } \mathcal{L}_1)$$
$$\Delta_2 = \Gamma \cup \mathrm{Sub}\{\nabla_1\nabla_2\ldots\nabla_k\Diamond\psi \mid \Diamond\psi \in \Gamma\} \text{ (for } \mathcal{L}_2)$$

such that $\widehat{\mathfrak{M}} = \langle W/\sim_{\Delta_i}, \widehat{R}, \vartheta\rangle$ is a filtration of $\mathfrak{M}$ through the corresponding Δ_i.

Let V be a set of fresh variables indexed over Γ as in the proof for a fusion of **K5** with something else. Let $\mathcal{M}'$ be a model defined as previously. We extend V and Γ in the following way:

$$\Gamma' = \Gamma \cup \mathrm{Sub}\{\nabla_{11}\nabla_{21}\ldots\nabla_{n1}\Diamond_1\psi \mid \Diamond_1\psi \in \Gamma\} \cup \mathrm{Sub}\{\nabla_{12}\nabla_{22}\ldots\nabla_{n2}\Diamond_2\psi \mid \Diamond_2\psi \in \Gamma\}$$
$$\Delta = V \cup \mathrm{Sub}\{\nabla_1\nabla_2\ldots\nabla_n\Diamond p_\psi \mid \nabla_{n+1 1}\psi \in \Gamma'\} \cup \mathrm{Sub}\{\nabla_1\nabla_2\ldots\nabla_k\Diamond p_\psi \mid \Diamond_2\psi \in \Gamma\}.$$

By the construction, $\sim_{\Gamma'}=\sim_\Delta$. So we have filtrations for the corresponding reducts of $\mathcal{M}'$ through Δ as well as for the original $\mathcal{M}$. □

References

[1] Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal Logic*. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 2001.

[2] Alexander Chagrov and Michael Zakharyaschev. *Modal Logic*. Oxford Logic Guides, Volume 35. Oxford University Press, 1997.

[3] Chen Chung Chang and H Jerome Keisler. *Model theory*. Elsevier, 1990.

[4] Kit Fine. Logics containing k4. part i. *The Journal of Symbolic Logic*, 39(1):31–42, 1974.

[5] Dov M Gabbay and Valentin B Shehtman. Products of modal logics, part 1. *Logic journal of IGPL*, 6(1):73–146, 1998.

[6] Valentin Goranko and Martin Otto. Model theory of modal logic. In *Studies in Logic and Practical Reasoning*, volume 3, pages 249–329. Elsevier, 2007.

[7] Stanislav Kikot, Ilya Shapirovsky, and Evgeny Zolin. Completeness of logics with the transitive closure modality and related logics. *arXiv preprint arXiv:2011.02205*, 2020.

[8] Agi Kurucz. Combining modal logics. *Studies in Logic and Practical Reasoning*, 3:869–924, 2007.

[9] Valentin B Shehtman. Filtration via bisimulation. *Advances in modal logic*, 5:289–308, 2004.