Notes on filtration of logics containing **K5**

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1 Preliminaries

Definition 1. *An n-normal modal logic is a set of formulas that contains all Boolean tautologies, formulas $\Diamond_i p \vee \Diamond_i q \leftrightarrow \Diamond_i (p \vee q)$ and $\Diamond_i \bot \leftrightarrow \bot$ for $i \leqslant n$, and is closed under modus ponens, substitution, and monotonicity: from $\varphi \to \psi$ infer $\Diamond_i \varphi \to \Diamond_i \psi$ for $i \leqslant n$.*

Definition 2. *An n-Kripke model is a triple $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$, where $R_i \subseteq W \times W$, $\vartheta : \mathrm{PV} \to 2^W$, and the connectives have the following semantics:*

1. $\mathcal{M}, w \models p \Leftrightarrow w \in \vartheta(p)$
2. $\mathcal{M}, w \models \neg\varphi \Leftrightarrow \mathcal{M}, w \not\models \varphi$
3. $\mathcal{M}, w \models \varphi \vee \psi \Leftrightarrow \mathcal{M}, w \models \varphi$ *or* $\mathcal{M}, w \models \psi$
4. $\mathcal{M}, w \models \Diamond_i \varphi \Leftrightarrow \exists v \in R_i(w)\ \mathcal{M}, v \models \varphi$

By **K5** we mean the logic $\mathbf{K} \oplus A5$, where $A5 = \Diamond p \to \Box\Diamond p$. It is known that **K5** is the modal logic of all Euclidean frames. A frame is called Euclidean if for each x, y, z, xRy and xRz implies yRz.

Proposition 1. K5 *proves*

1. $\Box^3 p \leftrightarrow \Box^2 p$
2. $\Box^2 \Diamond p \leftrightarrow \Box\Diamond p$
3. $\Box\Diamond\Box p \leftrightarrow \Box\Box p$
4. $\Box\Diamond^2 p \leftrightarrow \Box\Diamond p$

Proposition 2. *Let $\mathcal{M}$ be a $\mathbf{K}5$ model, xRy for $x, y \in W$ then one has*

$$\mathcal{M}, x \models \Diamond\Box\varphi \text{ iff } \mathcal{M}, y \models \Diamond\Box\varphi.$$

Proof.

1. Suppose $\mathcal{M}, x \models \Diamond\Box\varphi$. One also has $\mathcal{M}, x \models \Diamond\Box\varphi \to \Box\Diamond\Box\varphi$, so $\mathcal{M}, x \models \Box\Diamond\Box\varphi$. Thus, $\mathcal{M}, y \models \Diamond\Box\varphi$ since $y \in R(x)$.
2. Suppose $\mathcal{M}, y \models \Diamond\Box\varphi$, then $\mathcal{M}, y \models \Box\varphi$, so $\mathcal{M}, x \models \Diamond\Box\varphi$.

$\Box$

1.1 Filtrations: general definitions

Let $\mathcal{M} = \langle W, R_1, \ldots, R_n, \vartheta \rangle$ be a Kripke model and Γ a set of formulas closed under subformulas. An equivalence relation $\sim$ is set to have a finite index if the quotient set $W/\sim$ is finite. The equivalence relation $\sim_\Gamma$ induced by Γ is defined as

$$w \sim_\Gamma v \Leftrightarrow \forall \varphi \in \Gamma \, (\mathcal{M}, w \models \varphi \Leftrightarrow \mathcal{M}, v \models \varphi).$$

If Γ is finite, then $\sim_\Gamma$ has a finite index. An equivalence relation $\sim$ respects $\sim_\Gamma$, if $w \sim v$ implies $w \sim_\Gamma v$.

Definition 3. *Let $\mathcal{M} = \langle W, R_1, \ldots, R_n, \vartheta \rangle$ be a Kripke model and Γ be a Sub-closed set formulas. A Γ-filtration of $\mathcal{M}$ is a model $\widehat{\mathcal{M}} = \langle \widehat{W}, \widehat{R_1}, \ldots, \widehat{R_n}, \widehat{\vartheta} \rangle$ such that:*

1. *$\widehat{W} = W/\sim$, where $\sim$ is an equivalence relation having a finite index that respects Γ*
2. *$\widehat{\vartheta}(p) = \{[x]_\sim \mid x \in W \;\&\; x \in \vartheta(p)\}$*
3. *For each $i \in I$ one has $\widehat{R}_i^{min} \subseteq \widehat{R}_i \subseteq \widehat{R}_i^{max}$. $\widehat{R}_{i,\sim}^{min}$ is the i-th minimal filtered relation on $\widehat{W}$ defined as*

$$\hat{x} \widehat{R}_{i,\sim}^{min} \hat{y} \Leftrightarrow \exists x' \sim x \, \exists y' \sim y \, x R_i y$$

$\widehat{R}_{\Gamma,i}^{max}$ is the i-th maximal filtered relation on $\widehat{W}$ induced by Γ defined as

$$\hat{x} \widehat{R}_{\Gamma,i}^{max} \hat{y} \Leftrightarrow \forall \Box_i \varphi \in \Gamma \, (\mathcal{M}, x \models \Box_i \varphi \Rightarrow \mathcal{M}, y \models \varphi)$$

If Φ is finite subset of Γ and $\sim = \sim_\Phi$, then $\widehat{\mathcal{M}}$ is a definable Γ-filtration of $\mathcal{M}$ through Φ. If $\sim = \sim_\Gamma$, then such a filtration by means of the definiton above is called *strict*.

Lemma 1. *Let Γ be a finite set of formulas closed under subformulas and $\widehat{\mathcal{M}}$ a filtration of $\mathcal{M}$ through Γ, then for each $x \in W$ and for each $\varphi \in \Gamma$ one has*

$$\mathcal{M}, x \models \varphi \Leftrightarrow \widehat{\mathcal{M}}, \hat{x} \models \varphi$$

Definition 4. *Let $\mathbb{F}$ be a class of Kripke frames and Γ a finite set of formulas closed under subformulas. If for every model $\mathcal{M}$ over $\mathcal{F} \in \mathbb{F}$ there exists a model that is a Γ-definable filtration of $\mathcal{M}$, then $\mathbb{F}$ admits definable filtration. A class of models $\mathbb{M}$ admits definable filtration if for every $\mathcal{M} \in \mathbb{M}$ there exists a model belonging to the same class that is a definable Γ-filtration of $\mathcal{M}$.*

Lemma 2.

1. *Let $\mathcal{L}$ be a complete normal modal logic. If* $\mathrm{Frames}(\mathcal{L})$ *admits filtration, then $\mathcal{L}$ has the finite model property.*
2. *If the class of models* $\mathrm{Mod}(\mathcal{L})$ *admits filtration, then $\mathcal{L}$ has the finite model property and Kripke complete as well.*

2 Filtration of Euclidean logics

First of all, let us ensure that a minimal filtration of an Euclidean frame is not necessary Euclidean. Let $[x] \sim_\Gamma [y]$ and $[x] \sim_\Gamma [z]$. Then for some $x' \in [x]$ $y' \in [y]$, one has $x'Ry'$ and $x''Rz'$ for some $x'' \in [x]$ and $z' \in [z]$. Clearly, we cannot claim that $x' = x''$ in general. Thus, minimal filtration does not preserve the required property.

Lemma 3. **K5** *admit filtration.*

Proof. Let $\mathcal{M}$ be a **K5**-model and Γ_0 a finite set of formulas closed under subformulas. Let us put $\Gamma = \Gamma_0 \cup \mathrm{Sub}(\{\Diamond\Box\psi \mid \Box\psi \in \Gamma_0\}) \cup \Psi$, where $\Psi = \nabla_1\nabla_2\ldots\nabla_n\Box\psi$ for $\Box\psi \in \Gamma_0$ and $\nabla_i \in \{\Diamond, \Box\}$. By Proposition 1, any element of Φ has one of the four forms. Thus, $W \sim_{\equiv_\Gamma}$ has a finite index. We put $\widehat{R} = R^{\max}_\Gamma$. $\Box$

Definition 5. *A first-order formula is called Horn if it has the following form:*

$$\forall x_1, \ldots, x_n (x_{i_1} R x_{j_1} \wedge \cdots \wedge x_{i_s} R x_{j_s} \to x_k R x_l)$$

Definition 6. *Let H be a Horn property and $\langle W, R\rangle$ a Kripke frame. A Horn closure of a binary relation R is the minimal relation R^H containing R and satisfying H.*

Lemma 4. $R^H = \bigcup\limits_{n<\omega} R_n$ *where*

1. $R_0 = R$.
2. $R_{n+1} = R_n \cup \{(a, b) \in W \mid \exists \vec{c} \in W\ P(a, b, \vec{c})\}$, *where P is a premise of H.*

E-closure (an Euclidean Horn closure of a binary relation) has the following equivalent definitions:

Lemma 5. *Let $\mathcal{F} = \langle W, R\rangle$ be a Kripke frame. The following conditions are equivalent:*

1. *R^E is the smallest Euclidean relation containing R.*
2. $R^E = \bigcup\limits_{i<\omega} R_i$, *where*
   - $R_0 = R$
   - $R_{n+1} = R_n \cup (R_n^{-1} \circ R_n)$
3. *xR^Ey iff there exists $n < \omega$ such that either xRy or $\exists z_1, \ldots, z_n$ with z_1Rx and $z_{n-1}Ry$ and for each $1 < i \leqslant n$ one has either $z_{i-1}Rz_i$ or z_iRz_{i-1}.*
4. $R^E = R \cup \bigcup\limits_{i<\omega} (R^{-1} \circ (R \circ R^{-1})^n \circ R)$.

Proof.

1. (1) $\Rightarrow$ (2) Let us show that if R^E is the smallest Euclidean relation containing R, then $R^E = \bigcup\limits_{i<\omega} R_i$. There are two inclusions:
   - $R^E \subseteq \bigcup\limits_{i<\omega} R_i$. Recall that R^E has the form (?):

$$R^E = \bigcap\{R' \mid R \subseteq R', \forall a, b \in W\ R'(a, b) \Rightarrow \exists x \in W\ R'(x, a)\ \&\ R'(x, b)\}$$

- $\bigcup\limits_{i<\omega} R_i \subseteq R^E$. Let us show that $xR_n y$ for each $n < \omega$ implies $xR^E y$ by induction on n.
  If $n = 0$, then xRy, thus, $xR^E y$, since R is a subrelation of R^E. Suppose $n = m+1$ and $xR_{m+1}y$. Let us show that $xR^E y$. From $xR_{m+1}y$, one has $(x, y) \in R^n \cup (R_n^{-1} \circ R_n)$. There are two cases:
  - $xR^n y$, one needs to merely apply the IH.
  - $xR_n^{-1} \circ R_n y$. Then $\exists z \in W\ xR_n^{-1} z \ \&\ zR_n$. That is, $zR_n x$ and $zR_n y$ for some z. R_n is already a subrelation of R^E. Thus, $zR^E x$ and $zR^E y$. That implies $xR^E y$.

2. (2) $\Rightarrow$ (3) Let $(x, y) \in R_m$, let us the statement by induction on m.
   (a) Suppose $m = 0$, then xRy, and the statement is shown putting $n = 0$.
   (b) Suppose $m = p + 1$ and $xR_{p+1}y$. Assume that either xRy or $\exists z_1, \dots, z_p$ with $z_1 Rx$ and $z_{p-1}Ry$ and for each $1 < i \leqslant p$ one has either $z_{i-1}Rz_i$ or $z_i Rz_{i-1}$.
   $xR_{p+1}y$ implies $(x, y) \in R_p \cup (R_p^{-1} \circ R_p)$. If $(x, y) \in R_p$, then we merely apply the IH. Suppose $(x, y) \in R_p^{-1} \circ R_p$, then $(z, x) \in R_p$ and $(z, y) \in R_p$
3. (3) $\Rightarrow$ (4) Suppose either xRy or there exist $n \geqslant 1$ and $z_1, \dots, z_n$ with $z_1 Rx$ and $z_{n-1}Ry$ and for each $1 < i \leqslant n$ one has either $z_{i-1}Rz_i$ or $z_i Rz_{i-1}$. If xRy, then we are done. Otherwise there exists $n \geqslant 1$ with the condition above. Then $(x, y) \in R_{n+1}$ that follows from the condition.
4. (4) $\Rightarrow$ (1)

□

Lemma 6. *Let $\mathcal{F} = \langle W, R\rangle$ be a Kripke frame. Let us define $R^E = \bigcup\limits_{i<\omega} R_i$ where:*

1. $R_0 = R$

2. $R_{n+1} = R_n \cup ({R_n}^{-1} \circ R_n)$

Then R^E is Euclidean.

Proof. Let $(x, y), (x, z) \in R^E$, one needs to show that $(y, z) \in R^E$. Clearly that $(x, y) \in R_i$ and $(x, y) \in R_j$ for some $i, j < \omega$. Thus, we need $(y, z) \in R_m$ for some m depending on i and j.

Let us consider the following cases:

1. $i = 0$ and $j = 0$
   Suppose $(x, y), (x, z) \in R_0 = R$, then $(y, z) \in R^{-1} \circ R$. Thus, $(y, z) \in R_1$
2. $i = 0$ and $j = k + 1$
   Suppose $(x, y) \in R$ and $(x, z) \in R_{k+1} = R_k \cup ({R_k}^{-1} \circ R_k)$. Clearly that $(x, y) \in R_{k+1}$ as well. It is obviously that $(y, z) \in R_{k+2}$ since $(y, x) \in R_{k+1}^{-1}$ and $(x, z) \in R_{k+1}$.
3. The case with $i = k + 1$ and $j = 0$ is similar to the previous one.
4. Suppose $i = m + 1$ and $i = k + 1$. That is, $(x, y) \in R_{m+1} = R_m \cup ({R_m}^{-1} \circ R_m)$ and $(x, z) \in R_{k+1} = R_k \cup ({R_k}^{-1} \circ R_k)$. Consider the following four subcases:
   (a) Suppose $(x, y) \in R_m$ and $(x, z) \in R_k$ and $m \leqslant k$ without loss of generality. $m \leqslant k$ implies $R_m \subseteq R_k$ and $(x, y) \in R_k$ in particular. Thus, $(y, z) \in {R_k}^{-1} \circ R_k$, so $(y, z) \in R_{k+1}$.

(b) The rest of the cases are similar to the first one.

□

Lemma 7. *Let $\mathcal{M} = \langle W, R, \vartheta\rangle$ be an Euclidean model, Γ a set of* Sub*-closed formulas, and $\sim$ an equivalence relation having a finite index that respects Γ, then $\widehat{R} = (R^{min}_{\Phi})^E \subseteq R^{max}_{\Gamma}$, where $\Phi = \Gamma \cup \{\Diamond\Box\varphi \mid \Box\varphi \in \Gamma\}$.*

Thus, **K**5 *admits strict filtrations.*

Proof. Recall that $(R^{min}_{\Phi})^E$ has the form $(R^{min}_{\Phi})^E = \bigcup\limits_{n<\omega} (R^{min}_{\Phi})_n$, where

1. $(R^{min}_{\Phi})_0 = R^{min}_{\Phi}$
2. $(R^{min}_{\Phi})_{m+1} = (R^{min}_{\Phi})_n \cup (((R^{min}_{\Phi})_n)^{-1} \circ (R^{min}_{\Phi})_n)$

One needs to show that for each $n < \omega$ $(R^{min}_{\Phi})_n \subseteq R^{max}_{\Gamma}$. We prove this by induction. Suppose $\mathcal{M}, x \models \Box\varphi$ for $\Box\varphi \in \Phi$ and $[x](R^{min}_{\Phi})^E[y]$. We need $\mathcal{M}, y \models \varphi$.

1. $([x],[y]) \in (R^{min}_{\Phi})_0$, then $([x],[y]) \in R^{min}_{\Phi}$. Then there exist $x' \in [x]$ and $y' \in [y]$ such that $x'Ry'$. So $\mathcal{M}, x' \models \Box\varphi$ and, thus, $\mathcal{M}, y' \models \varphi$. Then $\mathcal{M}, y' \models \varphi$ as well since $y' \in [y]$.
2. $([x],[y]) \in (R^{min}_{\Phi})_{m+1}$, then $([x],[y]) \in (R^{min}_{\Phi})_m \cup (((R^{min}_{\Phi})_m)^{-1} \circ R^{min}_{\Phi})_m)$.

   If $([x],[y]) \in (R^{min}_{\Phi})_m$, then we apply the IH.

   Suppose $([x],[y]) \in (R^{min}_{\Phi})_m)^{-1} \circ (R^{min}_{\Phi})_m$, then there exists $[z] \in W/\sim_{\Phi}$ such that $([z],[x]) \in (R^{min}_{\Phi})_m)$ and $([z],[y]) \in (R^{min}_{\Phi})_m$.

   Then one has the following picture (using Lemma 5):

$$\begin{tikzcd}
{[z]} & {[z_1]} \arrow[l, "R^{min}_{\Phi}"'] \arrow[r, dash, "R'"] & {[z_2]} \arrow[r, dash, "R'"] & \ldots \arrow[r, dash, "R'"] & {[z_{m-1}]} \arrow[r, dash, "R'"] & {[z_m]} \arrow[r, "R^{min}_{\Phi}"] & {[x]}
\end{tikzcd}$$

$$\begin{tikzcd}
{[z]} & {[z'_1]} \arrow[l, "R^{min}_{\Phi}"] \arrow[r, dash, "R'"'] & {[z'_2]} \arrow[r, dash, "R'"'] & \ldots \arrow[r, dash, "R'"'] & {[z_{m-1}]} \arrow[r, dash, "R'"'] & {[z'_m]} \arrow[r, "R^{min}_{\Phi}"'] & {[y]}
\end{tikzcd}$$

   Where R' is either R^{min}_{Φ} or its converse. One has $\mathcal{M}, x \models \Box\varphi$ for $\Box\varphi \in \Phi$, where $\widehat{M}$ is the minimal filtration of $\mathcal{M}$ through Φ. One has $[z_m]R^{min}_{\Phi}[x]$, then $a_m Ra$ for some $a_m \in [z_n]$ and $a \in [x]$. Thus, $\mathcal{M}, a_m \models \Diamond\Box\varphi$ and, thus, $\widehat{\mathcal{M}}, [z_m] \models \Diamond\Box\varphi$.

   Applying Proposition 2 several times, one may show that $\widehat{\mathcal{M}}, [z_1] \models \Diamond\Box\varphi$. One has $[z_1]R^{min}_{\Phi}[z]$, then for some $a \in [z]$ and $a_1 \in [z_1]$ we have $a_1 Ra$.

   Then $\mathcal{M}, a \models \Box\varphi$ and $\widehat{M}, [z] \models \Box\varphi$.

   We have $[z'_1]R^{min}_{\Phi}[z]$, thus, $a'_1 Ra'$ for some $a'_1 \in [z'_1]$ and $a' \in [z]$. Then $\mathcal{M}, a'_1 \models \Diamond\Box\varphi$, and, thus, $\widehat{M}, [z'_1] \models \Diamond\Box\varphi$.

   One may show that $\widehat{M}, [z'_m] \models \Diamond\Box\varphi$ in the same way via Lemma 2. Thus, $\mathcal{M}, z'_m \models \Diamond\Box\varphi$. We also have $\mathcal{M}, z'_m \models \Diamond\Box\varphi \to \Box\varphi$, and, thus, $\mathcal{M}, z'_m \models \Box\varphi$. Then $\widehat{\mathcal{M}}, [z'_m] \models \Box\varphi$.

   One has $[z'_m]R^{min}_{\Phi}[y]$, then $a'_m Ry'$ for some $a'_m \in [z'_m]$ and $y' \in [y]$. Then $\mathcal{M}, y' \models \varphi$. But $y' \sim_{\Phi} y$, so $\mathcal{M}, y \models \varphi$.

□

3 Filtration for K4

Proposition 3. *Let R be a binary relation on $W \neq \varnothing$. Define $R^+ = \bigcup\limits_{i<\omega} R_i$*

1. $R_0 = R$
2. $R_{n+1} = R_n \circ R$

Then R^+ is transitive

Lemma 8. *Let $\mathcal{M} = \langle W, R, \vartheta\rangle$ be a transitive model and $\overline{\mathcal{M}} = \langle \overline{W}, \overline{R}, \overline{\vartheta}\rangle$ its minimal filtration through a finite* Sub-*closed set of formulas Θ.*

Then $\overline{\mathcal{M}}^+ = \langle \overline{W}, (\overline{R})^+, \overline{\vartheta}\rangle$ is a Θ-filtration of $\mathcal{M}$.

Proof. $(\overline{R})^+$ obviously contains R. By the previous proposition, $(\overline{R})^+$ is transitive. Let us show that $(\overline{R})^+ \subseteq R^{max}_{\Theta}$.

Let $\hat{x}, \hat{y} \in \overline{W}$ with $\hat{x}(\overline{R})^+\hat{y}$ and $\Box\varphi \in \Theta$ with $\mathcal{M}, x \models \Box\varphi$. Let us show that $\mathcal{M}, y \models \varphi$.

If $\hat{x}(\overline{R})^+\hat{y}$, then there exist equivalence classes $\hat{x}_1, \dots, \hat{x}_n$ such that

$$\hat{x}\overline{R}\hat{x}_1\overline{R}\dots\overline{R}\hat{x}_n\overline{R}\hat{y}$$

$\mathcal{M}, x \models \Box\varphi$ implies $\mathcal{M}, x \models \Box\Box\varphi$. Thus, $\overline{M}, \hat{x} \models \Box\Box\varphi$.

$\hat{x}\overline{R}\hat{x}_1$, so there are $x_1 \in \hat{x}$ and $x_2 \in \hat{x}_1$ with $x_1 R x_2$. In particular, $\mathcal{M}, x_2 \models \Box\varphi$, so $\overline{\mathcal{M}}, \hat{x}_2 \models \Box\varphi$, and et cetera.

For each $i \in \{1, \dots, n\}$ we have $\mathcal{M}, x_i \models \Box\varphi$ which is shown inductively:

If $\mathcal{M}, x_i \models \Box\varphi$ for $x_i \in \hat{x}_i$, so $\mathcal{M}, x_i \models \Box\Box\varphi$, but there exist $x_i{}' \in \hat{x}_i$ and $x_{i+1} \in \hat{x}_{i+1}$, so $\mathcal{M}, x_{i+1} \models \Box\varphi$.

Finally, we have $\mathcal{M}, x_n \models \Box\varphi$ for $x_n \in \hat{x}_n$, but $\hat{x}_n\overline{R}\hat{y}$, so $\mathcal{M}, y' \models \varphi$ for each $y' \in \hat{y}$. Thus, φ is true at y as well. □

Proof. Let $\hat{x}, \hat{y} \in \overline{W}$ with $\hat{x}(\overline{R})^+\hat{y}$ and $\Box\varphi \in \Theta$ with $\mathcal{M}, x \models \Box\varphi$. Let us show that $\mathcal{M}, y \models \varphi$.

If $\hat{x}(\overline{R})^+\hat{y}$, then there exist equivalence classes $\hat{x}_1, \dots, \hat{x}_n$ such that

$$\hat{x}\overline{R}\hat{x}_1\overline{R}\dots\overline{R}\hat{x}_n\overline{R}\hat{y}$$

Let us show that $\mathcal{M}, \hat{x}_i \models \Box\varphi$ inductively:

1. $n = 1$ We have the following sequence:

   $$\hat{x}\overline{R}\hat{x}_1\overline{R}\hat{y}$$

   $\hat{x}\overline{R}\hat{x}_1$, so there are $x' \in \hat{x}$ and $x'_1 \in \hat{x}_1$ such that $x' R x'_1$. $\Box\varphi$ is true at x', so is $\Box\Box\varphi$. Then $\mathcal{M}, x'_1 \models \Box\varphi$ since $x'_1 \in R(x')$. So $\overline{\mathcal{M}}, \hat{x}_1 \models \Box\varphi$.

2. $n = i + 1$ The case is the following:

   $$\hat{x}\overline{R}\hat{x}_1\overline{R}\dots\overline{R}\hat{x}_i\overline{R}\hat{x}_{i+1}\overline{R}\hat{y}$$

   By IH, $\Box\varphi$ is true at $\hat{x}_i$, so is $\Box\Box\varphi$. Hence, we have $\overline{\mathcal{M}}, \hat{x}_{i+1} \models \Box\varphi$ since $\hat{x}_i\overline{R}\hat{x}_{i+1}$.

That is, for each $0 < n < \omega$, if we have a sequence of equivalence classes with $\hat{x}\overline{R}\hat{x}_1\overline{R}\dots\overline{R}\hat{x}_n\overline{R}\hat{y}$ where $\overline{\mathcal{M}}, \hat{x} \models \Box\varphi$, then $\overline{\mathcal{M}}, \hat{x}_n \models \Box\varphi$.

If $\hat{x}_n\overline{R}\hat{y}$, then there are $x'_n \in \hat{x}_n$ and $y' \in \hat{y}$ with $x'_n R y'$. $\mathcal{M}, x'_n \models \Box\varphi$ implies $\mathcal{M}, y' \models \varphi$, but y' and y are Γ-equivalent and $\varphi \in \Gamma$, so $\mathcal{M}, y \models \varphi$. □

References