

Notes on filtration of logics containing **K5**

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1 Preliminaries

Definition 1. An n -normal modal logic is a set of formulas that contains all Boolean tautologies, formulas $\Diamond_i p \vee \Diamond_i q \leftrightarrow \Diamond_i(p \vee q)$ and $\Diamond_i \perp \leftrightarrow \perp$ for $i \leq n$, and is closed under modus ponens, substitution, and monotonicity: from $\varphi \rightarrow \psi$ infer $\Diamond_i \varphi \rightarrow \Diamond_i \psi$ for $i \leq n$.

Definition 2. An n -Kripke model is a triple $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$, where $R_i \subseteq W \times W$, $\vartheta : PV \rightarrow 2^W$, and the connectives have the following semantics:

1. $\mathcal{M}, w \models p \Leftrightarrow w \in \vartheta(p)$
2. $\mathcal{M}, w \models \neg \varphi \Leftrightarrow \mathcal{M}, w \not\models \varphi$
3. $\mathcal{M}, w \models \varphi \vee \psi \Leftrightarrow \mathcal{M}, w \models \varphi$ or $\mathcal{M}, w \models \psi$
4. $\mathcal{M}, w \models \Diamond_i \varphi \Leftrightarrow \exists v \in R_i(w) \mathcal{M}, v \models \varphi$

By **K5** we mean the logic $\mathbf{K} \oplus A5$, where $A5 = \Diamond p \rightarrow \Box \Diamond p$. It is known that **K5** is the modal logic of all Euclidean frames. A frame is called Euclidean if for each x, y, z , xRy and xRz implies yRz .

Proposition 1. **K5** proves

1. $\Box^3 p \leftrightarrow \Box^2 p$
2. $\Box^2 \Diamond p \leftrightarrow \Box \Diamond p$
3. $\Box \Diamond \Box p \leftrightarrow \Box \Box p$
4. $\Box \Diamond^2 p \leftrightarrow \Box \Diamond p$

Proposition 2. Let \mathcal{M} be a **K5** model, xRy for $x, y \in W$ then one has

$$\mathcal{M}, x \models \Box \Diamond \varphi \text{ iff } \mathcal{M}, y \models \Box \Diamond \varphi.$$

Proof.

1. Suppose $\mathcal{M}, x \models \Box \Diamond \varphi$. Then $\mathcal{M}, y \models \Diamond \varphi$ and $\mathcal{M}, y \models \Box \Diamond \varphi$
2. Suppose $\mathcal{M}, y \models \Box \Diamond \varphi$, then $\mathcal{M}, x \models \Diamond \Box \Diamond \varphi$, so $\mathcal{M}, x \models \Box \Diamond \varphi$.

□

1.1 Filtrations: general definitions

Let $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$ be a Kripke model and Γ a set of formulas closed under subformulas. An equivalence relation \sim is set to have a finite index if the quotient set W / \sim is finite. The equivalence relation \sim_Γ induced by Γ is defined as

$$w \sim_\Gamma v \Leftrightarrow \forall \varphi \in \Gamma (\mathcal{M}, w \models \varphi \Leftrightarrow \mathcal{M}, v \models \varphi).$$

If Γ is finite, then \sim_Γ has a finite index. An equivalence relation \sim respects \sim_Γ , if $w \sim v$ implies $w \sim_\Gamma v$.

Definition 3. Let $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$ be a Kripke model and Γ be a Sub-closed set formulas. A Γ -filtration of \mathcal{M} is a model $\widehat{\mathcal{M}} = \langle \widehat{W}, \widehat{R}_1, \dots, \widehat{R}_n, \widehat{\vartheta} \rangle$ such that:

1. $\widehat{W} = W / \sim$, where \sim is an equivalence relation having a finite index that respects Γ
2. $\widehat{\vartheta}(p) = \{[x]_\sim \mid x \in W \ \& \ x \in \vartheta(p)\}$
3. For each $i \in I$ one has $\widehat{R}_i^{\min} \subseteq \widehat{R}_i \subseteq \widehat{R}_i^{\max}$. $\widehat{R}_{i,\sim}^{\min}$ is the i -th minimal filtered relation on \widehat{W} defined as

$$\widehat{R}_{i,\sim}^{\min} \Leftrightarrow \exists x' \sim x \exists y' \sim y \ x R_i y$$

$\widehat{R}_{\Gamma,i}^{\max}$ is the i -th maximal filtered relation on \widehat{W} induced by Γ defined as

$$\widehat{R}_{\Gamma,i}^{\max} \Leftrightarrow \forall \Box_i \varphi \in \Gamma (\mathcal{M}, x \models \Box_i \varphi \Rightarrow \mathcal{M}, y \models \varphi)$$

If Φ is finite subset of Γ and $\sim = \sim_\Phi$, then $\widehat{\mathcal{M}}$ is a definable Γ -filtration of \mathcal{M} through Φ . If $\sim = \sim_\Gamma$, then such a filtration by means of the definition above is called *strict*.

Lemma 1. Let Γ be a finite set of formulas closed under subformulas and $\widehat{\mathcal{M}}$ a filtration of \mathcal{M} through Γ , then for each $x \in W$ and for each $\varphi \in \Gamma$ one has

$$\mathcal{M}, x \models \varphi \Leftrightarrow \widehat{\mathcal{M}}, \widehat{x} \models \varphi$$

Definition 4. Let \mathbb{F} be a class of Kripke frames and Γ a finite set of formulas closed under subformulas. If for every model \mathcal{M} over $\mathcal{F} \in \mathbb{F}$ there exists a model that is a Γ -definable filtration of \mathcal{M} , then \mathbb{F} admits definable filtration. A class of models \mathbb{M} admits definable filtration if for every $\mathcal{M} \in \mathbb{M}$ there exists a model belonging to the same class that is a definable Γ -filtration of \mathcal{M} .

Lemma 2.

1. Let \mathcal{L} be a complete normal modal logic. If $\text{Frames}(\mathcal{L})$ admits filtration, then \mathcal{L} has the finite model property.
2. If the class of models $\text{Mod}(\mathcal{L})$ admits filtration, then \mathcal{L} has the finite model property and Kripke complete as well.

2 Filtration of Euclidean logics

First of all, let us ensure that a minimal filtration of an Euclidean frame is not necessary Euclidean. Let $[x] \sim_\Gamma [y]$ and $[x] \sim_\Gamma [z]$. Then for some $x' \in [x]$ $y' \in [y]$, one has $x'Ry'$ and $x''Rz'$ for some $x'' \in [x]$ and $z' \in [z]$. Clearly, we cannot claim that $x' = x''$ in general. Thus, minimal filtration does not preserve the required property.

Lemma 3. *K5 admit filtration.*

Proof. Let \mathcal{M} be a **K5**-model and Γ_0 a finite set of formulas closed under subformulas. Let us put $\Gamma = \Gamma_0 \cup \text{Sub}(\{\Diamond\Box\psi \mid \Box\psi \in \Gamma_0\}) \cup \Psi$, where $\Psi = \nabla_1\nabla_2\ldots\nabla_n\Box\psi$ for $\Box\psi \in \Gamma_0$ and $\nabla_i \in \{\Diamond, \Box\}$. By Proposition 1, any element of Φ has one of the four forms. Thus, $W \sim_{\equiv_\Gamma}$ has a finite index. We put $\hat{R} = R_\Gamma^{\max}$. \square

Definition 5. *A first-order formula is called Horn if it has the following form:*

$$\forall x_1, \dots, x_n (x_{i_1}Rx_{j_1} \wedge \dots \wedge x_{i_s}Rx_{j_s} \rightarrow x_kRx_l)$$

Definition 6. *Let H be a Horn property and $\langle W, R \rangle$ a Kripke frame. A Horn closure of a binary relation R is the minimal relation R^H containing R and satisfying H .*

Lemma 4. $R^H = \bigcup_{n < \omega} R_n$ where

1. $R_0 = R$.
2. $R_{n+1} = R_n \cup \{(a, b) \in W \mid \exists \vec{c} \in W \text{ } P(a, b, \vec{c})\}$, where P is a premise of H .

E -closure (an Euclidean Horn closure of a binary relation) has the following equivalent definitions:

Lemma 5. *Let $\mathcal{F} = \langle W, R \rangle$ be a Kripke frame. The following conditions are equivalent:*

1. R^E is the smallest Euclidean relation containing R .
2. $R^E = \bigcup_{i < \omega} R_i$, where
 - $R_0 = R$
 - $R_{n+1} = R_n \cup (R_n^{-1} \circ R_n)$
3. xR^Ey iff there exists $n < \omega$ such that either xRy or $\exists z_1, \dots, z_n$ with z_1Rx and $z_{n-1}Ry$ and for each $1 < i \leq n$ one has either $z_{i-1}Rz_i$ or z_iRz_{i-1} .
4. $R^E = R \cup \bigcup_{i < \omega} (R^{-1} \circ (R \circ R^{-1})^n \circ R)$.

Proof.

1. (1) \Rightarrow (2) Let us show that if R^E is the smallest Euclidean relation containing R , then $R^E = \bigcup_{i < \omega} R_i$. There are two inclusions:

- $R^E \subseteq \bigcup_{i < \omega} R_i$. Recall that R^E has the form (?):

$$R^E = \bigcap \{R' \mid R \subseteq R', \forall a, b \in W \text{ } R'(a, b) \Rightarrow \exists x \in W \text{ } R'(x, a) \ \& \ R'(x, b)\}$$

- $\bigcup_{i < \omega} R_i \subseteq R^E$. Let us show that xR_ny for each $n < \omega$ implies xR^Ey by induction on n .
 If $n = 0$, then xRy , thus, xR^Ey , since R is a subrelation of R^E . Suppose $n = m+1$ and $xR_{m+1}y$. Let us show that xR^Ey . From $xR_{m+1}y$, one has $(x, y) \in R^n \cup (R_n^{-1} \circ R_n)$. There are two cases:
 - xR^ny , one needs to merely apply the IH.
 - $xR_n^{-1} \circ R_ny$. Then $\exists z \in W$ $xR_n^{-1}z$ & zR_ny . That is, zR_nx and zR_ny for some z . R_n is already a subrelation of R^E . Thus, zR^Ex and zR^Ey . That implies xR^Ey .
- 2. (2) \Rightarrow (3) Let $(x, y) \in R_m$, let us the statement by induction on m .
 - (a) Suppose $m = 0$, then xRy , and the statement is shown putting $n = 0$.
 - (b) Suppose $m = p+1$ and $xR_{p+1}y$. Assume that either xRy or $\exists z_1, \dots, z_p$ with z_1Rx and $z_{p-1}Ry$ and for each $1 < i \leq p$ one has either $z_{i-1}Rz_i$ or z_iRz_{i-1} .
 $xR_{p+1}y$ implies $(x, y) \in R_p \cup (R_p^{-1} \circ R_p)$. If $(x, y) \in R_p$, then we merely apply the IH.
 Suppose $(x, y) \in R_p^{-1} \circ R_p$, then $(z, x) \in R_p$ and $(z, y) \in R_p$.
- 3. (3) \Rightarrow (4) Suppose either xRy or there exist $n \geq 1$ and z_1, \dots, z_n with z_1Rx and $z_{n-1}Ry$ and for each $1 < i \leq n$ one has either $z_{i-1}Rz_i$ or z_iRz_{i-1} . If xRy , then we are done. Otherwise there exists $n \geq 1$ with the condition above. Then $(x, y) \in R_{n+1}$ that follows from the condition.
- 4. (4) \Rightarrow (1)

□

Lemma 6. Let $\mathcal{F} = \langle W, R \rangle$ be a Kripke frame. Let us define $R^E = \bigcup_{i < \omega} R_i$ where:

1. $R_0 = R$
2. $R_{n+1} = R_n \cup (R_n^{-1} \circ R_n)$

Then R^E is Euclidean.

Proof. Let $(x, y), (x, z) \in R^E$, one needs to show that $(y, z) \in R^E$. Clearly that $(x, y) \in R_i$ and $(x, z) \in R_j$ for some $i, j < \omega$. Thus, we need $(y, z) \in R_m$ for some m depending on i and j .

Let us consider the following cases:

1. $i = 0$ and $j = 0$
 Suppose $(x, y), (x, z) \in R_0 = R$, then $(y, z) \in R^{-1} \circ R$. Thus, $(y, z) \in R_1$
2. $i = 0$ and $j = k+1$
 Suppose $(x, y) \in R$ and $(x, z) \in R_{k+1} = R_k \cup (R_k^{-1} \circ R_k)$. Clearly that $(x, y) \in R_{k+1}$ as well. It is obviously that $(y, z) \in R_{k+2}$ since $(y, x) \in R_{k+1}^{-1}$ and $(x, z) \in R_{k+1}$.
3. The case with $i = k+1$ and $j = 0$ is similar to the previous one.
4. Suppose $i = m+1$ and $j = k+1$. That is, $(x, y) \in R_{m+1} = R_m \cup (R_m^{-1} \circ R_m)$ and $(x, z) \in R_{k+1} = R_k \cup (R_k^{-1} \circ R_k)$. Consider the following four subcases:
 - (a) Suppose $(x, y) \in R_m$ and $(x, z) \in R_k$ and $m \leq k$ without loss of generality. $m \leq k$ implies $R_m \subseteq R_k$ and $(x, y) \in R_k$ in particular. Thus, $(y, z) \in R_k^{-1} \circ R_k$, so $(y, z) \in R_{k+1}$.

(b) The rest of the cases are similar to the first one.

□

Theorem 1. *Let $\mathcal{M} = \langle W, R, \vartheta \rangle$ be an Euclidean model, Γ a set of Sub-closed formulas, and \sim an equivalence relation having a finite index that respects Γ , then $\hat{R} = (R_\Gamma^{min})^E \subseteq R_\Gamma^{max}$.*

Thus, K5 admits strict filtrations.

Proof. We put $R_\Gamma^{min} := \bar{R}$ for brevity.

Recall that \bar{R}^E has the form $\bigcup_{n < \omega} (\bar{R})_n$, where

1. $\bar{R}_0 = \hat{R}$
2. $\bar{R}_{m+1} = \bar{R}_n \cup ((\bar{R})^{-1} \circ \bar{R}_n)$

One needs to show that for each $n < \omega$ $(\bar{R})_n \subseteq R_\Gamma^{max}$. We prove this by induction. Suppose $\mathcal{M}, y \models \varphi$ for $\Diamond\varphi \in \Phi$ and $\hat{x} \bar{R}^E \hat{x}$. We need $\mathcal{M}, x \models \Diamond\varphi$.

1. The case of $n = 0$ is trivial and it follows directly from the definition of a minimal filtration.
2. Suppose $n = 1$. $\hat{x} \bar{R}_1 \hat{y}$ means that $(\hat{x}, \hat{y}) \in \bar{R} \cup (\bar{R}^{-1} \circ \bar{R})$. Let us assume that $(\hat{x}, \hat{y}) \in \bar{R}^{-1} \circ \bar{R}$. That is, there exists $\hat{z} \in W / \sim_\Gamma$ such that $(\hat{z}, \hat{x}), (\hat{z}, \hat{y}) \in \bar{R}$.
 $\hat{z} \bar{R} \hat{y}$ means that there are $z' \in \hat{z}$ and $y' \in \hat{y}$ such that $z' R y'$. $\mathcal{M}, y \models \varphi$ implies $\mathcal{M}, y' \models \varphi$. Then $\mathcal{M}, z' \models \Diamond\varphi$. $\hat{z} \bar{R} \hat{x}$ implies that there are $z'' \in \hat{z}$ and $x' \in \hat{x}$ with $z'' R x'$.
 $z'' \sim_\Gamma z'$ implies $\mathcal{M}, z'' \models \Diamond\varphi$. $\mathcal{M} \models \mathbf{K5}$, so $\mathcal{M}, z'' \models \Box\Diamond\varphi$. Thus, $\mathcal{M}, x' \models \Diamond\varphi$ since $x' \in R(z'')$. But $x' \in \hat{x}$, then $\mathcal{M}, x \models \Diamond\varphi$.
3. Suppose $(\bar{R})_n \subseteq R_\Gamma^{max}$. Let us show that for $(\bar{R})_{n+1}$. Then either $(\hat{x}, \hat{y}) \in (\bar{R})_n$ or $(\hat{z}, \hat{y}), (\hat{z}, \hat{x}) \in (\bar{R})_n$. The first alternative holds due to the IH. We consider only the second one.

By Lemma..., we may visualise $(\hat{z}, \hat{y}), (\hat{z}, \hat{x}) \in (\bar{R})_n$ with the following sequences for some $\hat{u}_1, \dots, \hat{u}_n, \hat{t}_1, \dots, \hat{t}_n$:

$$\hat{z} \xleftarrow{\bar{R}} \hat{u}_1 \xrightarrow{R'} \hat{u}_2 \xrightarrow{R'} \dots \xrightarrow{R'} \widehat{u_{n-1}} \xrightarrow{R'} \hat{u}_n \xrightarrow{\bar{R}} \hat{y}$$

$$\hat{z} \xleftarrow{\bar{R}} \hat{t}_1 \xrightarrow{R'} \hat{t}_2 \xrightarrow{R'} \dots \xrightarrow{R'} \widehat{t_{n-1}} \xrightarrow{R'} \hat{t}_n \xrightarrow{\bar{R}} \hat{x}$$

where R' is either \bar{R} or \bar{R}^{-1}

$\mathcal{M}, y \models \varphi$ and $\widehat{u_n} \bar{R} \hat{y}$, so for some $u'_n \in \widehat{u_n}$ and $y' \in \hat{y}$ with $u'_n R y'$, so $\mathcal{M}, u'_n \models \Diamond\varphi$ and $\widehat{\mathcal{M}}, \widehat{u_n} \models \Diamond\varphi$.

We have either $\widehat{u_{n-1}} \bar{R} \widehat{u_n}$ or $\widehat{u_n} \bar{R} \widehat{u_{n-1}}$.

If the first holds, then there are $u'_{n-1} \in \widehat{u_{n-1}}$ and $u''_n \in \widehat{u_n}$ with $u'_{n-1} R u''_n$. $\mathcal{M}, u''_n \models \Diamond\varphi$. wtf?

If the second holds, then there $u'_{n-1} \in \widehat{u_{n-1}}$ and $u''_n \in \widehat{u_n}$ with $u''_n R u'_{n-1}$. $\mathcal{M}, u''_n \models \Diamond\varphi$ implies $\mathcal{M}, u''_n \models \Box\Diamond\varphi$, so $\mathcal{M}, u'_{n-1} \models \Diamond\varphi$.

□

3 Filtration for K4

Proposition 3. Let R be a binary relation on $W \neq \emptyset$. Define $R^+ = \bigcup_{i < \omega} R_i$

1. $R_0 = R$

2. $R_{n+1} = R_n \circ R$

Then R^+ is transitive

Lemma 7. Let $\mathcal{M} = \langle W, R, \vartheta \rangle$ be a transitive model and $\overline{\mathcal{M}} = \langle \overline{W}, \overline{R}, \overline{\vartheta} \rangle$ its minimal filtration through a finite Sub-closed set of formulas Θ .

Then $\overline{\mathcal{M}}^+ = \langle \overline{W}, (\overline{R})^+, \overline{\vartheta} \rangle$ is a Θ -filtration of \mathcal{M} .

Proof. $(\overline{R})^+$ obviously contains R . By the previous proposition, $(\overline{R})^+$ is transitive. Let us show that $(\overline{R})^+ \subseteq R_{\Theta}^{max}$.

Let $\hat{x}, \hat{y} \in \overline{W}$ with $\hat{x}(\overline{R})^+ \hat{y}$ and $\Box \varphi \in \Theta$ with $\mathcal{M}, x \models \Box \varphi$. Let us show that $\mathcal{M}, y \models \varphi$.

If $\hat{x}(\overline{R})^+ \hat{y}$, then there exist equivalence classes $\hat{x}_1, \dots, \hat{x}_n$ such that

$$\hat{x} \overline{R} \hat{x}_1 \overline{R} \dots \overline{R} \hat{x}_n \overline{R} \hat{y}$$

$\mathcal{M}, x \models \Box \varphi$ implies $\mathcal{M}, x \models \Box \Box \varphi$. Thus, $\overline{\mathcal{M}}, \hat{x} \models \Box \Box \varphi$.

$\hat{x} \overline{R} \hat{x}_1$, so there are $x_1 \in \hat{x}$ and $x_2 \in \hat{x}_1$ with $x_1 R x_2$. In particular, $\mathcal{M}, x_2 \models \Box \varphi$, so $\overline{\mathcal{M}}, \hat{x}_2 \models \Box \varphi$, and et cetera.

For each $i \in \{1, \dots, n\}$ we have $\mathcal{M}, x_i \models \Box \varphi$ which is shown inductively:

If $\mathcal{M}, x_i \models \Box \varphi$ for $x_i \in \hat{x}_i$, so $\mathcal{M}, x_i \models \Box \Box \varphi$, but there exist $x'_i \in \hat{x}_i$ and $x_{i+1} \in \hat{x}_{i+1}$, so $\mathcal{M}, x_{i+1} \models \Box \varphi$.

Finally, we have $\mathcal{M}, x_n \models \Box \varphi$ for $x_n \in \hat{x}_n$, but $\hat{x}_n \overline{R} \hat{y}$, so $\mathcal{M}, y' \models \varphi$ for each $y' \in \hat{y}$. Thus, φ is true at y as well. \square

Proof. Let $\hat{x}, \hat{y} \in \overline{W}$ with $\hat{x}(\overline{R})^+ \hat{y}$ and $\Box \varphi \in \Theta$ with $\mathcal{M}, x \models \Box \varphi$. Let us show that $\mathcal{M}, y \models \varphi$.

If $\hat{x}(\overline{R})^+ \hat{y}$, then there exist equivalence classes $\hat{x}_1, \dots, \hat{x}_n$ such that

$$\hat{x} \overline{R} \hat{x}_1 \overline{R} \dots \overline{R} \hat{x}_n \overline{R} \hat{y}$$

Let us show that $\mathcal{M}, \hat{x}_i \models \Box \varphi$ inductively:

1. $n = 1$ We have the following sequence:

$$\hat{x} \overline{R} \hat{x}_1 \overline{R} \hat{y}$$

$\hat{x} \overline{R} \hat{x}_1$, so there are $x' \in \hat{x}$ and $x'_1 \in \hat{x}_1$ such that $x' R x'_1$. $\Box \varphi$ is true at x' , so is $\Box \Box \varphi$. Then $\mathcal{M}, x'_1 \models \Box \varphi$ since $x'_1 \in R(x')$. So $\overline{\mathcal{M}}, \hat{x}_1 \models \Box \varphi$.

2. $n = i + 1$ The case is the following:

$$\hat{x} \overline{R} \hat{x}_1 \overline{R} \dots \overline{R} \hat{x}_i \overline{R} \hat{x}_{i+1} \overline{R} \hat{y}$$

By IH, $\Box \varphi$ is true at \hat{x}_i , so is $\Box \Box \varphi$. Hence, we have $\overline{\mathcal{M}}, \hat{x}_{i+1} \models \Box \varphi$ since $\hat{x}_i \overline{R} \hat{x}_{i+1}$.

That is, for each $0 < n < \omega$, if we have a sequence of equivalence classes with $\hat{x} \overline{R} \hat{x}_1 \overline{R} \dots \overline{R} \hat{x}_n \overline{R} \hat{y}$ where $\overline{\mathcal{M}}, \hat{x} \models \Box \varphi$, then $\overline{\mathcal{M}}, \hat{x}_n \models \Box \varphi$.

If $\hat{x}_n \overline{R} \hat{y}$, then there are $x'_n \in \hat{x}_n$ and $y' \in \hat{y}$ with $x'_n R y'$. $\mathcal{M}, x'_n \models \Box \varphi$ implies $\mathcal{M}, y' \models \varphi$, but y' and y are Γ -equivalent and $\varphi \in \Gamma$, so $\mathcal{M}, y \models \varphi$. \square

References