

The finite base property for some subreducts of representable relation algebras

Daniel Rogozin

1 The Relation Algebras Background

We describe the basic definitions and results about relation algebras [10] [15].

Definition 1.

A relation algebra is an algebra $\mathcal{R} = \langle R, 0, 1, +, -, \cdot, \smile, \mathbf{1} \rangle$ such that $\langle R, 0, 1, +, - \rangle$ is a Boolean algebra and the following equations hold, for each $a, b, c \in R$:

1. $a; (b; c) = (a; b); c$
2. $(a + b); c = (a; c) + (b; c)$
3. $a; \mathbf{1} = a$
4. $a^{\smile\smile} = a$
5. $(a + b)^{\smile} = a^{\smile} + b^{\smile}$
6. $(a; b)^{\smile} = b^{\smile}; a^{\smile}$
7. $a^{\smile}; -(a; b) \leq -b$

where $a \leq b$ iff $a + b = b$. **RA** denotes the class of all relation algebras.

A relation algebra is called symmetric, if every element is self-converse. A relation algebra is called integral, if

$$a; b = 0 \Rightarrow a = 0 \text{ or } b = 0.$$

Definition 2. A proper relation algebra is an algebra $\mathcal{R} = \langle R, 0, 1, \cup, -, |, \smile, \mathbf{1} \rangle$ such that $R \subseteq \mathcal{P}(W)$, where W is an equivalence relation; $0 = \emptyset$; $1 = W$; $\cap, \cup, -$ are set-theoretic intersection, union, and complement respectively; $|$ is relation composition, \smile is relation converse, $\mathbf{1}$ is a diagonal relation restricted to W , that is:

1. $a|b = \{\langle x, z \rangle \mid \exists y \langle x, y \rangle \in a \text{ \& } \langle y, z \rangle \in b\}$
2. $a^{\smile} = \{\langle x, y \rangle \mid \langle y, x \rangle \in a\}$
3. $\mathbf{1} = \{\langle x, y \rangle \mid x = y\}$

The class of all proper relation algebras is denoted as **PRA**. **Rs** is the class of all relation set algebras, proper relation algebra with a diagonal subrelation as an identity. **RRA** is the class of all representable relation algebras, that is, the closure of **PRA** under isomorphic copies. That is, **RRA** = **IPRA**.

Note that the (quasi)equational theories of those classes coincide, that is

$$\mathbf{IPRA} = \mathbf{RRA} = \mathbf{SPRs}$$

Moreover, \mathbf{RRA} is a variety, but it cannot be defined by any set of first-order formulas [16].

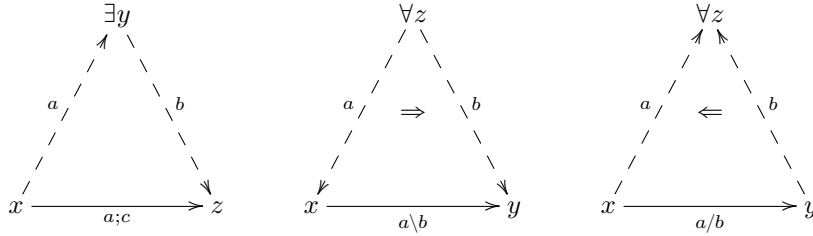
One may express residuals in every $\mathcal{R} \in \mathbf{RA}$ as follows, for every $a, b \in \mathcal{R}$:

1. $a \setminus b = -(a^\smile; -b)$
2. $a / b = -(-a; b^\smile)$

Those residuals have the following interpretation in $\mathcal{R} \in \mathbf{PRA}$ (as well as in \mathbf{RRA}), for every $a, b \in \mathcal{R}$:

1. $a \setminus b = \{\langle x, y \rangle \mid \forall z (z, x) \in a \Rightarrow (z, y) \in b\}$
2. $a / b = \{\langle x, y \rangle \mid \forall z (y, z) \in b \Rightarrow (x, z) \in a\}$

One may illustrate composition and residuals in \mathbf{PRA} and \mathbf{RRA} via the following triangles:



Given a subset of definable operations in \mathbf{RA} τ , we denote the class of subalgebras of the τ -reducts by $\mathbf{R}(\tau)$. The algebras containing to this class are defined as restrictions of elements belonging to \mathbf{Rs} to operations of τ . By $\mathbf{Q}(\tau)$ we mean a quasivariety generated by $\mathbf{R}(\tau)$. As in [12], we put $\mathbf{Q}(\tau)$ as the closure of $\mathbf{R}(\tau)$ under subalgebras and products assuming that $\mathbf{R}(\tau)$ is already closed under ultraproducts.

2 The Finite Base Property

We recall the underlying definitions according to [10, Section 19]

Definition 3. Let \mathbf{K} be a class of algebras of a signature Ω , \mathbf{K} has the finite algebra property, if if any first-order Ω -sentence that is true in all finite algebras in \mathbf{K} is true in every algebra in \mathbf{K} .

The finite base property is a version of the finite algebra property if \mathbf{K} is a class of representable algebras:

Definition 4. Let \mathbf{K} be a class of representable algebras of a signature Ω

1. \mathbf{K} has the finite base property if any first-order Ω -sentence that is true in every algebra in \mathbf{K} having a representation over a finite base set is valid in \mathbf{K} .
2. \mathbf{K} has the finite algebra on finite base property if any finite algebra in \mathbf{K} has a representation with finite base.
3. \mathbf{K} has the finite algebra property for equations/quasi-identities if any equation/quasi-identity that is true in all finite algebras is true in every algebra in \mathbf{K} . The finite base property for equations/quasi-identities is defined similarly.

The following statements were shown in [3]. This lemma connects finite base property with finite algebra on finite base and finite algebra properties as follows:

Lemma 1. *Let \mathbf{K} be a class of representable Ω -algebras:*

1. *If \mathbf{K} has the finite algebra property, then it has the finite algebra and the finite base properties for equations/quasi-identities.*
2. *The finite algebra on finite base and the finite algebra properties implies the finite base property for \mathbf{K} . The same holds for equations/quasi-identities.*
3. *If any representation of an infinite algebra has an infinite base, then the finite base property implies the finite algebra one for \mathbf{K} .*
4. *Suppose Ω is finite and any subalgebra of a representable algebra is representable on the same base. Then the finite base property implies the finite algebra on finite base property.*

3 The Relation Residuated Semigroups Background

3.1 The underlying definitions and results

A relation structure (**RS**) is an arbitrary algebra of the signature $\Omega = \langle \circ, \backslash, /, \leq \rangle$, where $\circ, \backslash, /$ are binary function symbols and \leq is a binary relation symbol.

Definition 5. *A residuated semigroup is an algebra $\mathcal{S} = \langle S, \circ, \leq, \backslash, / \rangle$ such that $\langle S, \circ, \leq \rangle$ is an ordered residuated semigroup and the following equivalences hold for each $a, b, c \in S$:*

$$b \leq a \backslash c \Leftrightarrow a \circ b \leq c \Leftrightarrow a \leq c / b$$

ORS is the class of all residuated semigroups.

See this paper to have a proof of the following theorem [7]:

Theorem 1. *Every finite residuated semigroup is isomorphic to some residuated subsemigroup of some finite residuated lattice.*

Definition 6. *Let A be a set of binary relations on some base set W such that $R = \cup A$ is transitive and $\{x, y \mid xRy\} = W$. A relation residuated semigroup is an algebra $\mathcal{A} = \langle A, ;, \backslash, /, \subseteq \rangle$ where for each $r, s \in A$*

1. $r; s = \{ \langle a, c \rangle \mid \exists b \in W (\langle a, b \rangle \in r \ \& \ \langle b, c \rangle \in s) \}$
2. $r \backslash s = \{ \langle a, c \rangle \mid \forall b \in W (\langle b, a \rangle \in r \Rightarrow \langle b, c \rangle \in s) \}$
3. $r / s = \{ \langle a, c \rangle \mid \forall b \in W (\langle c, b \rangle \in s \Rightarrow \langle a, b \rangle \in r) \}$
4. $r \leq s$ iff $r \subseteq s$.

Relation residuated semigroup are also called representable relativised relational structure (**RRS**).

See [6]

Theorem 2. *Every complete residuated semigroup \mathcal{A} (quantale) is isomorphic to relational complete residuated semigroup on the underlying set of \mathcal{A} .*

Definition 7. Let $\mathcal{A} = \langle A, \leq, ;, \backslash, / \rangle$ be a residuated semigroup. A representation of \mathcal{A} is an inclusion map $h : \mathcal{A} \rightarrow \mathcal{A}'$, where $\mathcal{A}' \in \mathbf{RRS}$ such that:

1. $\text{dom}(\mathcal{A}') = \{\hat{a} \mid a \in A\}$, where $\hat{a} = \{(b, c) \mid b \leq a; c\}$.

Such a map preserves order, residuals, and composition. Andr  ka and Mikul  s proved the following representation theorem for **ORS** in [4] that implies relational completeness of the Lambek calculus, the logic of **ORS**:

Theorem 3. **ORS** = **IRRS**, where **IRRS** is a closure of **RRS** under isomorphic copies.

Corollary 1. Every finite representable residuated semigroup is isomorphic to representable residuated subsemigroup of some finite residuated lattice.

Theorem 4. Let $\mathcal{A} \in \mathbf{RRS}$ and $|\mathcal{A}| < \omega$, then there exists a set W , a set A of binary relations on W , $R = \cup A$ with $\text{dom}(R) = A$ such that $\mathcal{A} \cong \langle A, |, \backslash, /, \subseteq \rangle$.

Theorem 5. The Lambek calculus is complete w.r.t finite relational models (has the fmp).

4 Join-semilattice ordered semigroups

Definition 8. A join-semilattice ordered semigroup (**OS**⁺) is an algebra $\mathcal{S} = \langle S, ;, + \rangle$ such that $\langle S, ; \rangle$ is a semigroup, $\langle S, + \rangle$ is a join-semilattice and the following equations hold for each $a, b, c \in S$:

1. $a; (b + c) = (a; b) + (a; c)$
2. $(a + b); c = (a; c) + (b; c)$

This class is clearly a variety since **OS**⁺ has the equational definition so far as $+$ is defined as an associative, idempotent, and commutative operation.

Let A be a set of binary relations on some base set W such that $R = \cup A$ is transitive and $\{x, y \mid xRy\} = W$ as in Definition 6. A representable join semilattice-ordered semigroup is an algebra isomorphic to some join semilattice-ordered semigroup having the form $\mathcal{A} = \langle A, |, \cup \rangle$ such that $;$ is a relation composition as above and \cup is the set-theoretic union. If \mathcal{A} is representable, then $\mathcal{A} \in \mathbf{I}(\mathbf{R}(\cup, |))$. Let us recall some of underlying facts about representable join semilattice-ordered semigroups [2]:

Proposition 1.

1. Let $\mathcal{A} = \langle A, +, ; \rangle$ be a join semilattice-ordered semigroup such that, for all $a, b \in A$:
 - (a) If $a \not\leq b$, then there exists an atom $c \leq a$ and $c \not\leq b$.
 - (b) If $c \leq a; b$ and c is an atom, then there exists an atom $a' \leq a$ such that $c \leq a' \cdot b$.
 then \mathcal{A} is representable.
2. Let $\mathcal{A} = \langle A, ; \rangle$ be a posemigroup, then \mathcal{A} is representable and such a representation preserve any existing finite suprema and infima, if
 - (a) The set of atoms is closed under $;$.
 - (b) \mathcal{A} has enough atoms, that is, if $x \in \text{At}(A)$ and $z, w \in A$, then $x \leq z; w$ implies there exist atoms $z_1 \leq z$ and $w_1 \leq w$ such that $x \leq z_1; w_1$. If $z \not\leq w$, then there exists an atom x such that $x \leq z$ and $x \not\leq w$.

Recall that a class of structures \mathbf{K} is called finitely axiomatisable iff both \mathbf{K} and its complement are closed ultraproducts and isomorphic copies.

It is known that the class of all representable join-semilattice ordered semigroups has no finite axiomatisation [1]. In other words,

Theorem 6. $\mathbf{R}(\cup, |)$ is not finitely axiomatisable.

Let us provide a proof of this fact using the rainbow technique [10] to show that the complement of \mathbf{ROS}^+ is not closed ultraproducts. This construction sometimes exploits the similar construction used by Andr  ka [2] and by Maddux [14]. We note that representability is not decidable for finite relation algebras [9] and this result has several generalisations [11]. Moreover, representability is undecidable for lattice-ordered semigroups and ordered complemented semigroups [17]. We use (more or less) a standard way of showing that the class of certain reducts of representable relation algebras has no finite axiomatisation, see [13] [8].

First of all, we recall several definitions such as colourings. We provide a sequence of symmetric, integral, finite relation algebras $\{\mathfrak{A}_n\}_{n < \omega}$ such that $\mathfrak{A}_n \notin \mathbf{RRA}$. The statement has been proved by Andr  ka [2] and reproduced here [5].

Given $n < \omega$, the set of atoms $\text{At}(\mathfrak{A}_n)$ consists of the following elements:

- identity: $\mathbf{1}$, an atom with no colour,
- white: \mathbf{w} ,
- greens: \mathbf{g}_i for $1 \leq i \leq n$,
- yellows: \mathbf{y}_i for $1 \leq i \leq n$,
- ivory: \mathbf{i} ,
- reds: \mathbf{r}_i for $1 \leq i \leq n$,
- blacks: \mathbf{b}_i for $1 \leq i \leq n$.

We have the following steps:

Step 1. Let \mathcal{A}_n be the upper semilattice presented with the set $\text{At}(\mathfrak{A}_n) \cup \{0\}$ and the following relations for each $x \in \text{At}(\mathfrak{A}_n)$ and for each $1 \leq i \leq n$.

1. $\mathbf{w} \leq \mathbf{g}_i + \mathbf{y}_i$,
2. $\mathbf{i} \leq \mathbf{y}_i + \mathbf{r}_i$,
3. $x + 0 = x$

Step 2. We define S , the set of two element subsets of \mathcal{A}_n :

$$S = \{\{\mathbf{w}, \mathbf{r}_1\}\} \cup \{\{\mathbf{g}_i, \mathbf{b}_i\} \mid 1 \leq i \leq n\} \cup \{\{\mathbf{y}_i, \mathbf{r}_i\} \mid 1 \leq i < n\} \cup \{\{\mathbf{y}_n, \mathbf{i}\}\}.$$

Step 3. The operations on \mathfrak{A}_n :

1. $\mathbf{1} = \sum \text{At}(\mathcal{A}_n) \cup \{0\}$,
2. $0 = \emptyset$,
3. $x = x^\smile$,

4. $0; x = 0; x = 0,$
5. $1; x = 1; x = x,$
6. $x; y = \begin{cases} \mathbf{i}, & \text{if } \{x, y\} \in S \\ 1, & \text{otherwise} \end{cases}$
unless $x, y \in \{0, 1\}.$

Step 4. Define the following quasi-identity:

$$q_n = \bigwedge_{1 \leq i \leq n} ((x \leq x'_i + x''_i) \wedge (y \leq y'_i + y''_i)) \rightarrow \\ x; y \leq x; y'_1 + \sum_{1 \leq i < n} (x'_i; y''_i + x''_i; y'_{i+1}) + x'_n; y''_n + x''_n; y$$

Lemma 2. 1. q_n is valid in **RRA** for each $n < \omega$.

2. q_n fails in \mathfrak{A}_n .

Proof. The valuation ϑ defined as:

1. $\vartheta(x) = \mathbf{w}$
2. $\vartheta(x'_i) = \mathbf{g}_i$
3. $\vartheta(x''_i) = \mathbf{y}_i$
4. $\vartheta(y) = \mathbf{i}$
5. $\vartheta(y'_i) = \mathbf{r}_i$
6. $\vartheta(y''_i) = \mathbf{b}_i$

falsifies q_n in \mathfrak{A}_n □

The second is to show that any non-trivial ultraproduct $\prod_D \mathfrak{A}_n \in \mathbf{RRA}$, where D is an ultrafilter over $\mathcal{P}(\omega)$. We show that via the rainbow technique. Let us define networks and games according to [8].

Definition 9. Let \mathcal{A} be a relation algebra. A network is a complete directed finite graph with edges labelled by elements of \mathcal{A} . Such a graph have the following form. $N = \langle E_N, l_N \rangle$, where $E_N = U_N \times U_N$ for some finite base set and $l_N : E_N \rightarrow \text{At}(\mathcal{A})$ is function mapping each edge to some atom of \mathcal{A} . This function obey the following requirements:

1. $l_N(x, y) \leq 1$ iff $x = y$
2. $l_N(x, y); l_N(y, z) \geq l_N(x, z)$

Given two networks $N = \langle E_N, l_N \rangle$ and $N' = \langle E_{N'}, l_{N'} \rangle$, N is a subnetwork of N' ($N \subseteq N'$, or N' refines N) if $E_N \subseteq E_{N'}$ and for each $x, y \in U_N$, $l_{N'}(x, y) = l_N(x, y)$.

Definition 10. Let $n < \omega$. We define a game $\mathcal{G}_n(\mathcal{A})$ for two players \forall (Abelard) and \exists (Héloïse). Abelard and Héloïse build a finite chain of networks $N_0 \subseteq N_1 \cdots \subseteq N_n$ as follows. In the first round \forall picks an atom α and \exists plays a network N_0 containing an edge (m_0, n_0) such that $l_n(m_0, n_0) = \alpha$. If $\alpha \leq 1$, then $m_0 = n_0$, otherwise $m_0 \neq n_0$. If $m_0 \neq n_0$, the edges (m_0, n_0) and (n_0, m_0) belong to Abelard. Suppose N_{i-1} for $i < n$ has been played, then

- \forall chooses an edge $(m, n) \in E_{N_{i-1}}$ and atoms $x, y \in \text{At}(\mathcal{A})$ such that $l_{N_{i-1}}(m, n) \leq x; y$.
- \exists provides a network $N_i = \langle E_{N_i}, l_{N_i} \rangle \supseteq N_{i-1}$ such that there exists $l \in U_{N_i}$ such that $l_{N_i}(m, l) = x$ and $l_{N_i}(l, n) = y$.

If $(m, n) \in E_i$ such that $m \neq n$ and $m, n \in U_{N_{i-1}}$, then the owner of this edge is the same as in the previous round. The edges (m, l) and (l, n) and their converses belong to Abelard. The rest of the irreflexive edges belongs to Héloïse. \exists wins a match of the game $\mathcal{G}_n(\mathcal{A})$ if she can provide a network N_i for each move of \forall for each $i \leq n$. \exists has a winning strategy if she can win all matches.

This lemma has been proved by Hirsch and Hodkinson here [8]. This lemma provide a criterion of representability for relation algebras.

Lemma 3. *Let \mathcal{A} be an atomic relation algebra. Then \exists has a winning strategy in $\mathcal{G}_n(\mathcal{A})$ for each $n < \omega$ iff \mathcal{A} is elementary equivalent to some completely representable relation algebra. If \exists has a winning strategy, then \mathcal{A} is representable since **RRA** is elementary.*

Lemma 4. *Any non-trivial ultraproduct of $\{\mathfrak{A}_n\}_{n < \omega}$ is representable, that is, belongs to **RRA**. The same statement for non-trivial ultraproduct of reducts $\{\mathfrak{S}_n\}_{n < \omega}$ that belongs to **R**($\cup, |$).*

Lemma 5. *TODO: one needs to realise when \exists has a winning strategy*

4.1 The finite algebra on finite base for **R**($\cup, |$) (or its failure)

References

- [1] Hajnal Andréka. On the ‘union-relation composition’ reducts of relation algebras. In *Abstracts Amer. Math. Soc.*, volume 10, page 174, 1989.
- [2] Hajnal Andréka. Representations of distributive lattice-ordered semigroups with binary relations. *Algebra Universalis*, 28(1):12–25, 1991.
- [3] Hajnal Andréka, Ian Hodkinson, and István Németi. Finite algebras of relations are representable on finite sets. *The Journal of Symbolic Logic*, 64(1):243–267, 1999.
- [4] Hajnal Andréka and Szabolcs Mikulás. Lambek calculus and its relational semantics: completeness and incompleteness. *Journal of Logic, Language and Information*, 3(1):1–37, 1994.
- [5] Hajnal Andréka and Szabolcs Mikulás. Axiomatizability of positive algebras of binary relations. *Algebra universalis*, 66(1-2):7, 2011.
- [6] Carolyn Brown and Doug Gurr. A representation theorem for quantales. *Journal of Pure and Applied Algebra*, 85(1):27–42, 1993.
- [7] Robert Goldblatt. A kripke-joyal semantics for noncommutative logic in quantales. *Advances in modal logic*, 6:209–225, 2006.
- [8] Robin Hirsch and Ian Hodkinson. Step by step—building representations in algebraic logic. *The Journal of Symbolic Logic*, 62(1):225–279, 1997.
- [9] Robin Hirsch and Ian Hodkinson. Representability is not decidable for finite relation algebras. *Transactions of the American Mathematical Society*, 353(4):1403–1425, 2001.
- [10] Robin Hirsch and Ian Hodkinson. *Relation algebras by games*. Elsevier, 2002.

- [11] Robin Hirsch and Marcel Jackson. Undecidability of representability as binary relations. *The Journal of Symbolic Logic*, 77(4):1211–1244, 2012.
- [12] Robin Hirsch and Szabolcs Mikulás. Positive fragments of relevance logic and algebras of binary relations. *The Review of Symbolic Logic*, 4(1):81–105, 2011.
- [13] Ian Hodkinson and Szabolcs Mikulás. Axiomatizability of reducts of algebras of relations. *Algebra Universalis*, 43(2-3):127–156, 2000.
- [14] Roger D. Maddux. Nonfinite axiomatizability results for cylindric and relation algebras. *Journal of Symbolic Logic*, 54(3):951–974, 1989.
- [15] Roger D Maddux. *Relation algebras*, volume 13. Elsevier Science Limited, 2006.
- [16] Donald Monk et al. On representable relation algebras. *The Michigan mathematical journal*, 11(3):207–210, 1964.
- [17] Murray Neuzerling. Undecidability of representability for lattice-ordered semigroups and ordered complemented semigroups. *Algebra universalis*, 76(4):431–443, 2016.