Varieties of representable relation algebra reducts

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1 Varieties and discriminators

Standardly, a class of algebras is called *variety*, if it can be determined by some equational theory, or, equivalently, it is closed under homomorphic images, subalgebras and direct products. Given a class of algebras $\mathcal{K}$, $\mathbf{V}(\mathcal{K})$ is a variety generated by $\mathcal{K}$ or, equivalently, $\mathbf{HSP}(\mathcal{K})$, the closure of $\mathcal{K}$ under homomorphic images, subalgebras and direct products.

Let $\{A_i \mid i \in I\}$ be an indexed family of algebras, then a subalgebra $A \subseteq \Pi_{i \in I} A_i$ is a subdirect product if $\pi_i(A) = A_i$. An embedding $\alpha : A \to \Pi_{i \in I} A_i$ is subdirect if $\alpha(A)$ is subdirect product. An algebra is subdirectly irreducible if for every subdirect embedding $\alpha : A \to \Pi_{i \in I} A_i$ there exists $i \in I$ such that $\pi_i \circ \alpha : A \to A_i$ is an isomorphism.

An equivalence relation θ on an algebra A is called *congruence*, if θ respects any operation. $\mathbf{Con}(A)$ is the set of all congruences on A. An algebra is called *simple*, if $\mathbf{Con}(A/\theta) = \{\Delta, \nabla\}$, where Δ and ∇ are trivial congruences. One can obtain a simple algebra by factorising it through the maximal congruence [SB81, Theorem 8].

One can equivalently define subdirectly irreducible algebras using congruences as follows. an algebra is subdirectly irreducible iff either A is trivial or there exists a minimal congruence in $\mathbf{Con}(A) \backslash \{\Delta\}$.

Recall that a Stone space is a compact Hausdorff zero-dimensional topological space. A subdirect product $A \subseteq \Pi_{x \in X} A_x$ over a Stone space X if

1. for all $a, b \in A$ $\{x \in X \mid a(x) = b(x)\}$ is clopen.
2. for all $a, b \in A$ and for all clopen $Y \subseteq X$ $a \restriction_Y \cup a \restriction_{X \backslash Y} \in A$.

A variety $\mathcal{V}$ is *arithmetical*, if it is congruence-permutable and congruence-distributive, or, equivalently, there exists a ternary term p such that:

$$\mathcal{V} \models p(x, y, x) \approx p(x, y, y) \approx p(y, y, x) \approx x \tag{1}$$

A ternary term $t(x, y, z)$ for an algebra A if, for all $a, b, c \in A$:

$$t(a, b, c) = \begin{cases} c & \text{if } a = b \\ a & \text{otherwise} \end{cases} \tag{2}$$

A variety $\mathcal{V}$ is called discriminator if there exists a class $\mathcal{K}$ such that $\mathcal{V} = \mathbf{V}(\mathcal{K})$ and there exists a term $t(x, y, z)$, which is a discriminator term for every member of $\mathcal{K}$. It is known that if an algebra A has a discriminator term, then A is simple [SB81, Lemma 9.2]. Moreover, we have the following property of discriminator terms, see [SB81, Theorem 9.4].

Theorem 1. *Let $t(x, y, z)$ be a discriminator term for every member of a class $\mathcal{K}$:*

1. $\mathbf{V}(\mathcal{K})$ *is an arithmetical variety.*

2. *Every indecomposable member of* $\mathbf{V}(\mathcal{K})$ *is simple.*

3. *Simple algebras are precisely members of* $ISP_U(\mathcal{K}_+)$.

4. *Every member of* $\mathbf{V}(\mathcal{K})$ *is isomorphic to a Boolean product of simple algebras.*

2 BAOs, relation algebras and their reducts

2.1 Discriminator varieties of BAOs

Let B be a Boolean algebra, an operator is an n-ary function $f : A^n \to A$ such that, for all $x_1, \ldots, x_n, x, y \in B$:

- $f(x_1, \ldots, x + y, \ldots, x_n) = f(x_1, \ldots, x, \ldots, x_n) + f(x_1, \ldots, y, \ldots, x_n)$
- $f(x_1, \ldots, 0, \ldots, x_n) = 0$

A Boolean algebra with operators is an algebra $M = (B, (f_i)_{i \in I})$, where each f_i is an operator.

In the case of BAOs, one can define discriminator simpler, as an unary term $d(x)$ such that, for all $a \in M$, where M is a BAO:

$$d(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{otherwise} \end{cases} \tag{3}$$

One can characterise discriminator varieties as follows, see [AGM+98, Lemma 2.1]:

Theorem 2. *Let* $\mathcal{V}$ *be a variety of BAOs and* $d(x)$ *a unary term, then the following are equivalent:*

1. d *is a discriminator variety.*

2. *The following equations are valid in* $\mathcal{V}$*:*

   (a) $x \leqslant d(x)$

   (b) $d(d(x)) \leqslant d(x)$

   (c) $d(-d(x)) \leqslant -d(x)$

   (d) $f(x_0, \ldots, x_{n-1}) \leqslant d(x_i)$ *for all* $n > 0$ *and for every operator* f *of* M

2.2 Relation algebras and their reducts

In this subsection, we consider relation algebras, a kind of BAOs.

Definition 1.
A relation algebra is an algebra $\mathcal{R} = (R, 0, 1, +, -, ;, ^{\smile}, \mathbf{1}')$ *such that* $(R, 0, 1, +, -)$ *is a Boolean algebra and the following hold:*

1. $(R, ;, \mathbf{1}')$ *is a monoid*

2. $(a + b); c = (a; c) + (b; c)$

3. $a^{\smile\smile} = a$

4. $(a + b)^{\smile} = a^{\smile} + b^{\smile}$

5. $(a;b)^{\smile} = b^{\smile}; a^{\smile}$

6. $a^{\smile}; (-(a;b)) \leqslant -b$

where $a \leqslant b$ *iff* $a + b = b$. **RA** *denotes the class of all relation algebras.*

Definition 2. *A proper relation algebra is an algebra* $\mathcal{R} = (R, \varnothing, W, \cup, -, |, ^{\smile}, \mathbf{1})$ *such that* $R \subseteq \mathcal{P}(W)$*, where* $W \subseteq X \times X$ *is an equivalence relation;* $|$ *is relation composition,* $^{\smile}$ *is relation converse,* **Id** *is a diagonal subset of* W*, that is:*

1. $a|b = \{(x,z) \mid \exists y \, (x,y) \in a \,\&\, (y,z) \in b\}$

2. $a^{\smile} = \{(x,y) \mid (y,x) \in a\}$

3. $\mathbf{Id} = \{(x,y) \mid x = y\}$

The class of all proper relation algebras is denoted as **PRA**. **Rs** *is the class of all relation set algebras, proper relation algebra with a diagonal subrelation as an identity.* **RRA** *is the class of all representable relation algebras, that is, the closure of* **PRA** *under isomorphic copies. That is,* **RRA** = **IPRA**.

2.3 Varieties

2.4 Non-varieties

2.5 Unknown

3 Decidability aspects

3.1 Current results

3.2 Problems

References

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