Model-theoretic aspects of relativised cylindric set algebras

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1 Intro

... It is known that the equational theory of $\mathbf{RCA}_\omega$ for $\alpha \leqslant \omega$ is decidable [6]. ...

2 The problems themselves

1. Suppose $\mathcal{C} \in \mathbf{RCA}_\omega$, whether $\mathcal{C}^+$ has a complete, ω-dimensional representation? [3]
2. Is the class $\mathbf{IG}_\omega$ (the isomorphism-closure of the ω-dimensional cylindric relativised set algebras in which the unit is closed under substitutions and permutations) a variety, or even a pseudo-elementary class? Is it closed under ultraproducts? [3]

3 Boolean algebras with operators and cylindric algebras

Definition 1.

1. *Let $\mathcal{B} = \langle B, +, -, 0, 1 \rangle$ be a Boolean algebra. An operator is an n-ary function $\Omega : B^n \to B$ satisfying the following conditions:*
   - *Normality: for all $b_0, \dots, b_{n-1} \in B$, if $b_1 = 0$ for some $i < n$, then*
   $$\Omega(b_0, \dots, b_{i-1}, 0, b_{i+1}, \dots, b_{n-1}) = 0$$
   - *Additivity: for all $b_0, \dots, b_{n-1}, b, b' \in B$ we have*
   $$\Omega(b_0, \dots, b_{i-1}, (b + b'), b_{i+1}, \dots, b_{n-1}) =$$
   $$\Omega(b_0, \dots, b_{i-1}, b, b_{i+1}, \dots, b_{n-1}) + \Omega(b_0, \dots, b_{i-1}, b', b_{i+1}, \dots, b_{n-1})$$
2. *Let I be an index set, a Boolean algebra with operators (BAO) is an algebra $\langle B, +, -, 0, 1, \{\Omega_i\}_{i \in I} \rangle$ such that $\langle B, +, -, 0, 1 \rangle$ is a Boolean algebra and for each $i \in I$ Ω_i is an operator.*

Definition 2. *Let $\mathcal{B} = \langle B, +, -, 0, 1, \{\Omega_i\}_{i \in I} \rangle$ be a BAO, then*

1. *An opetator Ω is completely additive, if for each $b_0, \dots, b_{n-1} \in B$ and $X \subseteq B$, one has*
$$\Omega(b_0, \dots, b_{i-1}, \sum X, b_{i+1}, \dots, b_{n-1}) = \sum_{x \in X} \Omega(b_0, \dots, b_{i-1}, x, b_{i+1}, \dots, b_{n-1})$$
2. *$\mathcal{B}$ is completely additive, if for each $i \in I$ Ω_i is additive,*
3. *A class $\mathcal{K}$ of BAOs is completely additive, if every $\mathcal{B} \in \mathcal{K}$ is completely additive.*

3.1 Atom structures and canonical extensions

Definition 3. *Let I be an index set and $\{\Omega_i\}_{i \in I}$ a set of function symbols*

1. *An atom structure is a relational structrure $\mathcal{F} = \langle W, \{R_i\}_{i \in I} \rangle$ such that R_i is a $n+1$-ary relation symbol, if $\Omega_{i \in I}$ is an n-ary function symbol,*
2. *Let $\mathcal{B}$ be an atomic BAO of the signature I, the atom structure of $\mathcal{B}$, written as $\mathbf{At}\mathcal{B}$, is an atom structure $\langle \mathrm{At}(\mathcal{B}), \{R_i\}_{i \in I} \rangle$ such that for each $a, b_0, \ldots, b_{n+1} \in \mathrm{At}(\mathcal{B})$ and for each $i \in I$*

$$\mathbf{At}\mathcal{B} \models R_i(a, b_0, \ldots, b_{n+1}) \text{ iff } \mathcal{B} \models a \leqslant \Omega_i(b_0, \ldots, b_{n+1})$$

3. *Let $\mathcal{F} = \langle W, \{R_i\}_{i \in I} \rangle$ be an atom structure, the complex algebra of $\mathcal{F}$, written as $\mathbf{Cm}\mathcal{F}$, is a BAO $\langle \mathcal{P}(W), \cup, -, \varnothing, W, \{\Omega_{R_i}\}_{i \in I} \rangle$ such that for all $X_0, \ldots, X_{n-1} \subseteq W$ and for each $i \in I$*

$$\Omega_{R_i}(X_0, \ldots, X_{n-1}) = \{a \in W \mid \exists b_0 \in X_0 \ldots \exists b_{n-1} \in X_{n-1}\ \mathcal{F} \models R_i(a, b_0, \ldots, b_{n-1})\}$$

The following duality is due to Thomason [7].

Fact 1.

1. *Let $\mathcal{B}$ be a complete atomic BAO, then $\mathcal{B} \cong \mathbf{Cm}(\mathbf{At}(\mathcal{B}))$,*
2. *Let $\mathcal{F}$ be an atom structure, then $\mathcal{F} \cong \mathbf{At}(\mathbf{Cm}(\mathcal{B}))$.*

Let A be a non-empty subset of a Boolean algebra $\mathcal{B}$, A is a *filter*, if A is closed under finite infima and upwardly closed. A is an ultrafilter, if it has no non-trivial extensions. That is, if $A \subseteq A'$, then $A' = \mathcal{B}$.

Definition 4. *Let $\mathcal{B} = \langle B, +, -, 0, 1, \{\Omega_i\}_{i \in I} \rangle$ be a BAO and $\mathbf{Uf}(\mathcal{B})$ the set of its ultrafilters. The ultrafilter frame of $\mathcal{B}$ (or canonical frame) is a relational structure $\mathcal{F}_{\mathcal{B}} = \langle \mathbf{Uf}(\mathcal{B}), R_{\Omega_i} \rangle$ such that for each ultrafilters $\beta_0, \ldots, \beta_{n-1}, \gamma$ one has*

$$\mathbf{Uf}(\mathcal{B}) \models R_{\Omega_i}(\beta_0, \ldots, \beta_{n-1}, \gamma) \text{ iff } \{\Omega(b_0, \ldots, b_{n-1}) \mid b_0 \in \beta_0, \ldots, b_{n-1} \in \beta_{n-1}\} \subseteq \gamma.$$

Definition 5. *Let $\mathcal{B}$ be a BAO, then*

1. *The canonical extension of $\mathcal{B}$ is a complex algebra of the canonical frame $\mathbf{Cm}(\mathcal{F}_{\mathcal{B}})$ denoted as $\mathcal{B}^+$,*
2. *The class of BAOs is canonical, if it is closed under canonical extensions.*

Theorem 1. *Let $\mathcal{A}$, $\mathcal{B}$ be BAOs,*

1. *There exists $\iota : \mathcal{A} \hookrightarrow \mathcal{A}^+$ such that $\iota : a \mapsto \{\gamma \in \mathbf{Uf}(\mathcal{A}) \mid a \in \gamma\}$.*
2. *If $i : \mathcal{A} \hookrightarrow \mathcal{B}$, then this embedding might be extented to the embedding $i^+ : \mathcal{A}^+ \hookrightarrow \mathcal{B}^+$*

Fact 2.

3.2 (Representable) cylindric algebras and cylindric set algebras

Let α be an ordinal. Let ${}^{\alpha}U$ be the set of all functions mapping α to a non-empty set U. We denote $x(i) = x_i$ for $x \in {}^{\alpha}U$ and $i < \alpha$.

Definition 6.

1. *A subset of ${}^{\alpha}U$ is an α-ry relation on U. For $i, j < \alpha$, the i, j-diagonal D_{ij} is the set of all elements of U such that $y_i = y_j$.*

   If $i < \alpha$ and X is an α-ry relation on U, then the i-th cylindrification $C_i X$ is the set of all elements of U that agree with some element of X on each coordinate except the i-th one. To be more precise, $C_i X = \{y \in {}^{\alpha}U \mid \exists x \in X \forall i < \alpha \, (i \neq j \Rightarrow y_j = x_j)\}$.

2. *A cylindic set algebra of dimension α is an algebra consisting of a set S of α-ry relation on some base set U with the constants and operations $0 = \varnothing$, $1 = {}^{\alpha}U$, $\cap$, $-$, the diagonal elements $\{D_{ij}\}_{i,j<\alpha}$, the cylindrifications $\{C\}_{i<\alpha}$.*

   A generalised cylindric set algebra of dimension α is a subdirect of cylindric algebras that have dimension α

3. *A cylindric algebra of dimension α is an algebra $\mathcal{C} = \langle \mathcal{B}, \{c_i\}_{i<\alpha}, \{d_{ij}\}_{i,j<\alpha} \rangle$ such that*

   - *$\mathcal{B}$ is a Boolean algebra, for each $i, j < \alpha$ c_i is an operator and $d_{ij} \in \mathcal{B}$*
   - *For each $i < \alpha$, $a \leqslant c_i a$, $c_i(a \wedge c_i b) = c_i a \wedge c_i b$ and $d_{ii} = 1$*
   - *For every $i, j < \alpha$, $c_i c_j a = c_j c_i a$*
   - *If $k \neq i, j < \alpha$, then $d_{ij} = c_k(d_{ij} \wedge d_{jk})$*
   - *If $i \neq j$, then $c_i(d_{ij} \wedge a) \wedge c_i(d_{ij} \wedge -a) = 0$*

   $\mathbf{CA}_{\alpha}$ *is the class of all cylindric algebras of dimension α*

4. *An α-dimensional cylindric algebra C is representable, if it is isomorphic to a generalised cylindric set algebra of dimension α. Such is isomorphism is a representation of C.*

   $\mathbf{RCA}_{\alpha}$ *is the class of all representable cylindric algebras that have dimension α. In particular, we are interested in the case when $\alpha = \omega$.*

It is well known that $\mathbf{RCA}_{\alpha}$ is a variety, $\mathbf{RCA}_{\alpha}$ ($\alpha \leqslant 2$) is finitely axiomatisable and $\mathbf{RCA}_{\alpha}$ ($2 < \alpha < \omega$) has no finite axiomatisation, see [2].

Let $\mathcal{A} \in \mathbf{C}_{\omega}$, then $\mathcal{A}$ has a *complete representation*, if this representation preserves all existing suprema.

Let us concretise the definition of a canonical extension for α-dimensional cylindric algebras:

Definition 7. *Let $\mathcal{C} = \langle C, +, -, 0, 1, \{d_{ij}\}_{i,j<\alpha}, \{c_i\}_{i<\omega} \rangle$ $\mathcal{A}$ be a BAO of type $\mathbf{CA}_{\alpha}$ Let $\mathbf{Uf}(\mathcal{C})$ be the set of all ultrafilters of $\mathfrak{B}\mathcal{C}$, the Boolean part of $\mathcal{C}$.*

Let us define $\mathbf{C}_i : \mathbf{Uf}(\mathcal{C}) \to \mathbf{Uf}(\mathcal{C})$ for each $i, j < \alpha$ as

1. *$\mathbf{C}_i \mathcal{X} = \{\mathcal{F} \in \mathbf{Uf}(\mathcal{C}) \mid \exists \mathcal{F}' \in \mathbf{Uf}(\mathcal{C}) \, (a \in \mathcal{F} \Rightarrow c_i a \in \mathcal{F}' R)\}$,*
2. *$D_{ij} = \{\mathcal{F} \in \mathbf{Uf}(\mathcal{C}) \mid d_{ij} \in \mathcal{F}\}$.*

The structure $\mathcal{C}^+ = \langle \mathbf{Uf}(\mathcal{C}), \cup, -, \varnothing, C, \mathbf{C}_{i<\alpha}, \{D_{ij}\}_{i,j<\alpha} \rangle$ is called the canonical extension of $\mathcal{C}$.

Let us discuss the connection between representability and canonical extensions.

The following definitions and facts are due to Henkin, Monk, and Tarski [1].

Let $\mathcal{A} \in \mathbf{CA}_\alpha$ and $x \in \mathcal{A}$. Recall that *the dimension of* x is the set of all ordinals $\gamma < \alpha$ such that $c_\gamma x \neq x$. More formally,

$$\Delta x = \{\gamma \mid \gamma < \alpha \,\&\, c_\gamma x \neq x\}$$

Let us discuss some metamathematical intuitions standing behind the notion of a dimension. Let Θ be a first-order theory and $\mathcal{C}/\equiv_\Theta$ its Lindenbaum-Tarski algebra. Let φ be a formula in the signature of Θ. Then $\Delta(\varphi/\Theta)$ consists of all $\kappa < \alpha$ such that $\exists x_\kappa \varphi \leftrightarrow \varphi$ is not valid in Θ. That is, $\Delta(\varphi/\Theta)$ contains ordinals κ for which x_κ is free in φ. Moreover, $\Delta(\varphi/\Theta)$ consists only of those ordinals for which x_κ is free in every $\psi \in \varphi/\Theta$.

In particular, an element x is called *zero-dimensional* if $\Delta x = 0$. Zero-dimensional elements reflect equivalence classes of sentences in the Lindenbaum-Tarski algebra of a given first-order theory. Thus, the set of zero-dimensional elements form a Boolean algebras of sentences associated with Θ.

Definition 8. *Let $\mathcal{A}$ be an α-dimensional cylindric algebra. Let α be an ordinal and Γ a subset α, then an element $x \in \mathcal{A}$ is Γ-closed if $\Delta x \cdot \Gamma = \varnothing$. Alternatively, x is a Γ-cylinder.*

$\mathrm{Cl}_\Gamma \mathcal{A}$ *is the set of all Γ-closed elements.*

Metamathematically, Γ-closed elements reflect universal closures (is it correct?).

Let $\mathcal{C} = \langle C, +, -, 0, 1, \{d_{ij}\}_{i,j<\beta}, \{c\}_{c<\beta}\rangle$ be a β-dimensional cylindic algebra and $\alpha \leqslant \beta$ an ordinal. The *α-th reduct* of $\mathcal{C}$, denoted as $\mathfrak{Rd}_\alpha \mathcal{C}$, is an algebra having the form

$$\mathfrak{Rd}_\alpha \mathcal{C} = \langle C, +, -, 0, 1, \{d_{ij}\}_{i,j<\alpha}, \{c\}_{c<\alpha}\rangle$$

$\mathcal{B}$ is a subreduct of $\mathcal{C}$, denoted as $\mathcal{B} \subseteq^r \mathcal{C}$, if $\mathcal{B} \subseteq \mathfrak{Rd}_\gamma \mathcal{C}$ for some $\gamma \leqslant \beta$.

Definition 9. *Let $\mathcal{C}$ be a β-dimensional cylindic algebra and α an ordinal such that $\alpha \leqslant \beta$. The neat α-reduct of $\mathcal{C}$, denoted as $\mathfrak{Nr}_\alpha \mathcal{C}$, is the subalgbera $\mathcal{A}$ of $\mathfrak{Rd}_\alpha \mathcal{C}$ with $\mathcal{A} = \mathrm{Cl}_\kappa \mathcal{C}$ where $\alpha + \kappa = \beta$.*

Let $\mathbb{K}$ be a class of β-dimensional cylindic algebras, then we put

$$\mathbf{Nr}_\alpha \mathbb{K} = \{\mathfrak{Nr}_\alpha \mathcal{C} \mid \mathcal{C} \in \mathbb{K}\}$$

An algebra $\mathcal{B}$ is a neat subreduct of $\mathcal{C}$, or $\mathcal{B}$ is neatly embeddable to $\mathcal{C}$ if there exists an ordinal $\gamma \leqslant \alpha$ such that $\mathcal{C} \subseteq \mathfrak{Rd}_\gamma \mathcal{B}$.

One may define neat reducts alternatively as follows. Let $\mathcal{C}$ be a β-dimensional cylindic algebra and α an ordinal such that $\alpha \leqslant \beta$. The neat α-reduct of $\mathcal{C}$ is the α-dimensional cylindric algebra having the form

$$\mathfrak{Nr}_\alpha \mathcal{C} = \langle \{a \in \mathcal{C} \mid \forall j(\alpha \leqslant j \,\&\, j < \beta \Rightarrow c_j a = a)\}, +, -, 0, 1, \{d_{ij}\}_{i,j<\alpha}, \{c_\gamma\}_\gamma \rangle$$

We will adapting the following proof of the fact that **RRA** is canonical [1] to our case. This proof is due to Monk, but that was describe in McKenzie's thesis [5].

1. A relation algebra $\mathcal{A}$ is representable iff $\mathcal{A}$ neatly embeds to some omege-dimensional cylinric algebra,
2. If $\mathcal{A}$ neatly embeds in $\mathcal{A}$ then $\mathcal{A}^+$ neatly embeds in $\mathcal{B}^+$,

[1] This idea is by Ian Hodkinson

3. $\mathbf{CA}_\alpha$ is closed under canonical extensions,

4. Voilá.

We descibe the following sketch of that proof published by Maddux in [4].

Theorem 2. **RRA** *is closed under canonical extensions.*

Proof. $\square$

Lemma 1 (Henkin, Monk, Tarski)**.**
Let $\mathcal{A}$ be a BAO of type $\mathbf{CA}_\alpha$ and $\mathcal{B}$ be a β-dimensional cylindric algebra such that $\beta \leqslant \alpha$ and $\mathcal{A}$ neatly embeds to $\mathcal{B}$ by a complete embedding. Then $\mathcal{A}^+$ neatly embeds to $\mathcal{B}^+$ by a complete embedding.

Proof. $\square$

Theorem 3 (This assumption is by Ian Hodkinson)**.**
Let $\mathcal{A}$ be a BAO of type $\mathbf{CA}_\omega$ such that $\mathcal{A}$ neatly embeds into $\mathbf{CA}_{\omega+\omega}$ by a complete embedding. Then $\mathcal{A}$ is completely representable as $\mathbf{CA}_\omega$.

Proof. Hmmmmm, I believe so. $\square$

Lemma 1 and Theorem 2 imply the following theorem.

Theorem 4. *Let $\mathcal{C} \in \mathbf{RCA}_\omega$, then $\mathcal{C}^+ \in \mathbf{RCA}_\omega$. That is, $\mathbf{RCA}_\omega$ is closed under canonical extensions.*

Proof. :monkahmm: $\square$

References

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