Notes on filtrations for logics that contain **K5**

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1 Preliminaries

Definition 1. *A normal modal logic is a set of formulas that contains all Boolean tautologies, formulas $\Diamond p \vee \Diamond q \leftrightarrow \Diamond(p \vee q)$ and $\Diamond\bot \leftrightarrow \bot$, and is closed under Modus Ponens, Substitution, and Monotonicity: from $\varphi \to \psi$ infer $\Diamond\varphi \to \Diamond\psi$.*

Definition 2. *An Kripke model is a triple $\mathcal{M} = \langle W, R, \vartheta \rangle$, where $R \subseteq W \times W$, $\vartheta : \mathrm{PV} \to 2^W$, and the connectives have the following semantics:*

1. $\mathcal{M}, w \models p \Leftrightarrow w \in \vartheta(p)$
2. $\mathcal{M}, w \models \neg\varphi \Leftrightarrow \mathcal{M}, w \not\models \varphi$
3. $\mathcal{M}, w \models \varphi \vee \psi \Leftrightarrow \mathcal{M}, w \models \varphi$ *or* $\mathcal{M}, w \models \psi$
4. $\mathcal{M}, w \models \Diamond\varphi \Leftrightarrow \exists v \in R(w)\ \mathcal{M}, v \models \varphi$

A modal logic $\mathcal{L}$ is called locally finite, if every fragment of $\mathcal{L}$ with a finite number of variables is finite. Equivalently, its weak canonical model generated by $k < \omega$ variables is finite. Moreover, if $\mathcal{L}$ is locally finite, then $\mathcal{L} = Log(\mathcal{F}_{\mathcal{L}} \upharpoonright k \mid k < \omega)$, where $\mathcal{F}_{\mathcal{L}} \upharpoonright k$ is a weak canonical frame generated by k variables.

Definition 3. *The modal depth of a formula is defined recursively by the following function:*

$$\begin{gathered} md(p_i) = 0 \\ md(\neg\varphi) = md(\varphi) \\ md(\varphi \vee \psi) = \max(md(\varphi), md(\psi)) \\ md(\Diamond\varphi) = 1 + md(\psi) \end{gathered}$$

Let $\mathcal{L}$ be a modal logic, then the modal depth of $\mathcal{L}$, denoted as $md(\mathcal{L})$, is the minimal n such that any $\varphi \in \mathcal{L}$ is equivalent to some ψ such that $md(\psi) = n$. If such an n does not exists, then $md(\mathcal{L}) = \infty$.

1.1 Filtrations

Let $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$ be a Kripke model and Γ a set of formulas closed under subformulas. An equivalence relation $\sim$ is set to have a finite index if the quotient set $W/\sim$ is finite. The equivalence relation $\sim_\Gamma$ induced by Γ is defined as

$$w \sim_\Gamma v \Leftrightarrow \forall \varphi \in \Gamma\ (\mathcal{M}, w \models \varphi \Leftrightarrow \mathcal{M}, v \models \varphi).$$

If Γ is finite, then $\sim_\Gamma$ has a finite index. An equivalence relation $\sim$ respects $\sim_\Gamma$, if $w \sim v$ implies $w \sim_\Gamma v$.

The following definition of a filtration is due to, e.g., [8].

Definition 4. *Let $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta\rangle$ be a Kripke model and Γ be a Sub-closed set formulas. A Γ-filtration of $\mathcal{M}$ is a model $\widehat{\mathcal{M}} = \langle \widehat{W}, \widehat{R_1}, \dots, \widehat{R_n}, \widehat{\vartheta}\rangle$ such that:*

1. *$\widehat{W} = W/\sim$, where $\sim$ is an equivalence relation that respects Γ*
2. *$\widehat{\vartheta}(p) = \{[x]_\sim \mid x \in W \ \&\ x \in \vartheta(p)\}$*
3. *For each $i \in I$ one has $\widehat{R}_i^{min} \subseteq \widehat{R}_i \subseteq \widehat{R}_i^{max}$. $\widehat{R}_{i,\sim}^{min}$ is the i-th minimal filtered relation on $\widehat{W}$ defined as*

$$\hat{x}\widehat{R}_{i,\sim}^{min}\hat{y} \Leftrightarrow \exists x' \sim x\ \exists y' \sim y\ xR_iy$$

$\widehat{R}_{\Gamma,i}^{max}$ is the i-th maximal filtered relation on $\widehat{W}$ induced by Γ defined as

$$\hat{x}\widehat{R}_{\Gamma,i}^{max}\hat{y} \Leftrightarrow \forall \Box_i\varphi \in \Gamma\ (\mathcal{M}, x \models \Box_i\varphi \Rightarrow \mathcal{M}, y \models \varphi)$$

Alternatively, one may reformulate the condition of the maximal filtered relations using $\Diamond$'s as follows. We will use this formulation occasionally:

$$\hat{x}\widehat{R}_{\Gamma,i}^{\max}\hat{y} \Leftrightarrow \forall \Diamond_i\varphi \in \Gamma\ (\mathcal{M}, y \models \varphi \Rightarrow \mathcal{M}, x \models \Diamond_i\varphi)$$

If Φ is an extension of Γ and $\sim = \sim_\Phi$, then $\widehat{\mathcal{M}}$ is a definable Γ-filtration of $\mathcal{M}$ through Φ. If $\sim = \sim_\Gamma$, then such a filtration by means of the definiton above is called *strict*.

A class of models $\mathbb{M}$ admits strict filtrations for models (ASF), if for every Sub-closed set Γ and for every $\mathcal{M} \in \mathbb{M}$ there exists a model $\widehat{\mathcal{M}}$ such that $\widehat{\mathcal{M}} \in \mathbb{M}$ and $\widehat{\mathcal{M}}$ is a filtration of $\mathcal{M}$ through Γ.

A class of frames $\mathbb{F}$ admits strict filtrations for frames, if for every Sub-closed set Γ and for every frame $\mathcal{F} \in \mathbb{F}$ and every model $\mathcal{M}$ over $\mathcal{F}$ there exists a Γ filtration of $\mathcal{M}$, and the underlying frame of this filtration belongs to $\mathbb{F}$.

If $\mathcal{L}$ is canonical, then the ASF property for frames and ASF property for models are equivalent, see [6, Theorem 2.10].

The key lemma about filtrations is the following, see [1, Theorem 2.39]:

Lemma 1. *Let Γ be a finite set of formulas closed under subformulas and $\widehat{\mathcal{M}}$ a filtration of $\mathcal{M}$ through Γ, then for each $x \in W$ and for each $\varphi \in \Gamma$ one has*

$$\mathcal{M}, x \models \varphi \Leftrightarrow \widehat{\mathcal{M}}, \hat{x} \models \varphi$$

Definition 5.

1. *Let $\mathbb{F}$ be a class of Kripke frames and Γ a finite set of formulas closed under subformulas. If for every model $\mathcal{M}$ over $\mathcal{F} \in \mathbb{F}$ there exists a model that is a Γ-definable filtration of $\mathcal{M}$, then $\mathbb{F}$ admits definable filtration.*
2. *A class of models $\mathbb{M}$ admits definable filtration if for every $\mathcal{M} \in \mathbb{M}$ there exists a model belonging to the same class that is a definable Γ-filtration of $\mathcal{M}$.*

Lemma 2.

1. *Let $\mathcal{L}$ be a complete normal modal logic. If $\mathrm{Frames}(\mathcal{L})$ admits filtration, then $\mathcal{L}$ has the finite model property.*
2. *If the class of models $\mathrm{Mod}(\mathcal{L})$ admits filtration, then $\mathcal{L}$ has the finite model property and it is Kripke complete as well.*

1.2 Horn closure

Definition 6. *A first-order formula is called Horn if it has the following form (see [3]):*

$$\forall x_1, \ldots, x_n (x_{i_1} R x_{j_1} \wedge \cdots \wedge x_{i_s} R x_{j_s} \rightarrow A), \text{ where } A \text{ is either } x_k R x_l \text{ or } \bot.$$

Definition 7. *Let H be a Horn property and $\langle W, R\rangle$ a Kripke frame. A Horn closure of a binary relation R is the minimal relation R^H containing R and satisfying H.*

Lemma 3. $R^H = \bigcup\limits_{n<\omega} R_n$ *where*

1. $R_0 = R$.
2. $R_{n+1} = R_n \cup \{(a, b) \in W \mid \exists \vec{c} \in W\ P(a, b, \vec{c})\}$, *where P is a premise of H.*

1.3 Bisimulations

A model $\mathcal{M}$ is called *weak* model, if its valuation is restriced to some finite subset of variables, namely $PV \upharpoonright k$, if the cardinality of this subset equals k.

Definition 8. *Let $\mathcal{M}_1 = \langle W_1, R_1, \vartheta_1\rangle$ and $\mathcal{M}_2 = \langle W_2, R_2, \vartheta_2\rangle$ be weak Kripke models (with the same variables), $x \in W_1$, and $y \in W_2$, then x and y are* 0*-equivalent, if the following holds:*

$$x \equiv_0 y \text{ iff for each } p \in PV \upharpoonright k \text{ (or for each } p \in PV) \text{ one has } \mathcal{M}_1, x \models p \Leftrightarrow \mathcal{M}_2, y \models p$$

Definition 9. *Let $\mathcal{M}_1 = \langle W_1, R_1, \vartheta_1\rangle$ $\mathcal{M}_2 = \langle W_2, R_2, \vartheta_2\rangle$ be Kripke models, then a bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_2$ is a binary relation $E \subseteq W_1 \times W_2$ between two models such that:*

- *(0) $x_1 E x_2$ implies $x_1 \equiv_0 x_1$*
- *(**zig**) $x_1 E y_1$ and $x_1 R_1 z$ implies $y_1 R_2 y_2$ and $z E y_2$ for some $y_2 \in W_2$*
- *(**zag**) $x_1 E y_1$ and $y_1 R_2 y_2$ implies $x_1 R_1 x_2$ and $x_2 E y_2$ for some $x_2 \in W_1$*

Models with designated points $\langle \mathcal{M}_1, x\rangle$ and $\langle \mathcal{M}_2, y\rangle$ are bisimilar, if there exists a bisimulation E between them with xEy. We denote that as $\langle \mathcal{M}_1, x\rangle \underline{\leftrightarrow} \langle \mathcal{M}_2, y\rangle$. Moreover, we have

$$\langle \mathcal{M}_1, x\rangle \underline{\leftrightarrow} \langle \mathcal{M}_2, y\rangle \text{ implies } Th(\mathcal{M}_1, x) = Th(\mathcal{M}_2, y).$$

Definition 10. *Let $\langle \mathcal{M}_1, x\rangle$ and $\langle \mathcal{M}_2, y\rangle$ be models with designated points and $r < \omega$. The r-round bisimulation game $\mathcal{G}(\mathcal{M}_1, \mathcal{M}_2, x, y)$ is played by two players, $\forall$ (Abelard, man) and $\exists$ (Heloise, woman). The rules of the game are the following:*

1. *(**Round 0**) $\exists$ wins, if $x \equiv_0 y$. Otherwise, $\forall$ wins.*
2. *(**Round** $n+1$) Let $n+1 < r$. Suppose rounds $0, \ldots, n$ have been played. The n-th position is a pair (x_n, y_n). $\forall$ chooses x_{n+1} (or $y_n \in W_2$) with $x_n R_1 x_{n+1}$ (or $y_n R_1 y_{n+1}$). $\exists$ chooses y_{n+1} (x_{n+1}) with $y_n R_1 y_{n+1}$ (or $y_n R_1 y_{n+1}$) and $x_{n+1} \equiv_0 y_{n+1}$. $\forall$ ($\exists$) wins, if $\exists$ ($\forall$) cannot move.*

$\exists$ wins the game after r rounds. $\forall$ wins the game, if there exists $k < r$ such that $\forall$ wins in **Round** *k.*

There are two n-equivalences for formulas and games (the first one is due to Fine [4]). These definitions are equivalent, this is due to Stirling.

Definition 11.

- $(\mathcal{M}_1, x) \equiv_n (\mathcal{M}_2, y)$ *iff for every* φ *such that* $md(\varphi) \leqslant n$ *we have* $\mathcal{M}_1, x \models \varphi$ *iff* $\mathcal{M}_2, y \models \varphi$.
- $(\mathcal{M}_1, x) \equiv_n (\mathcal{M}_2, y)$ *iff* $\exists$ *has a winning strategy in* $\mathcal{G}(\mathcal{M}_1, \mathcal{M}_2, x, y)$.

Lemma 4. *(Stabilisation lemma)*
Let n *be a natural number and* $\mathcal{L}$ *a modal logic. If* $\equiv_n = \equiv_{n+1}$ *in a weak canonical model* $\mathcal{M}_{\mathcal{L}} \upharpoonright k$*, then* $md(\mathcal{L} \upharpoonright k) \leqslant n$.

Theorem 1 (See [5])**.** *Let* $\mathcal{L}$ *be a modal logic such that* $md(\mathcal{L}) = n$ *for some* $n < \omega$*, then* $\mathcal{L}$ *is locally finite.*

2 Filtrations for K5

The **K5**-closure (the Euclidean Horn closure of a binary relation) has the following equivalent definitions:

Lemma 5. *Let* $\mathcal{F} = \langle W, R \rangle$ *be a Kripke frame and* $R \subseteq R^{\mathbf{K5}}$*. The following conditions are equivalent:*

1. $R^{\mathbf{K5}}$ *is the smallest Euclidean relation containing* R*, that is, the Horn closure of* R*.*
2. $R^{\mathbf{K5}} = \bigcup\limits_{i<\omega} R_i$*, where*
   - $R_0 = R$
   - $R_{n+1} = R_n \cup (R_n^{-1} \circ R_n)$
3. $xR^{\mathbf{K5}}y$ *iff there exists* $n < \omega$ *such that either* xRy *or* $\exists z_1, \ldots, z_n$ *with* z_1Rx *and* $z_{n-1}Ry$ *and for each* $1 < i \leqslant n$ *one has either* $z_{i-1}Rz_i$ *or* z_iRz_{i-1}.
4. $R^{\mathbf{K5}} = R \cup \bigcup\limits_{i<\omega} (R^{-1} \circ (R \cup R^{-1})^n \circ R)$.

Proof.

1. (1) $\Rightarrow$ (2) Let us show that if R^E is the smallest Euclidean relation containing R, then $R^E = \bigcup\limits_{i<\omega} R_i$. There are two inclusions:
   - $R^E \subseteq \bigcup\limits_{i<\omega} R_i$. Recall that R^E has the form (?):
   $$R^E = \bigcap \{R' \mid R \subseteq R', \forall a, b \in W \ R'(a,b) \Rightarrow \exists x \in W \ R'(x,a) \ \& \ R'(x,b)\}$$
   - $\bigcup\limits_{i<\omega} R_i \subseteq R^E$. Let us show that xR_ny for each $n < \omega$ implies xR^Ey by induction on n. If $n = 0$, then xRy, thus, xR^Ey, since R is a subrelation of R^E. Suppose $n = m+1$ and $xR_{m+1}y$. Let us show that xR^Ey. From $xR_{m+1}y$, one has $(x,y) \in R^n \cup (R_n^{-1} \circ R_n)$. There are two cases:
     - xR^ny, one needs to merely apply the IH.
     - $xR_n^{-1} \circ R_ny$. Then $\exists z \in W \ xR_n^{-1}z \ \& \ zR_n$. That is, zR_nx and zR_ny for some z. R_n is already a subrelation of R^E. Thus, zR^Ex and zR^Ey. That implies xR^Ey.

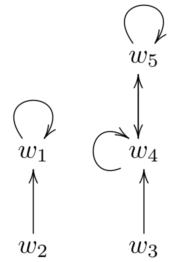
2. (2) $\Rightarrow$ (3) Let $(x, y) \in R_m$, let us the statement by induction on m.
   (a) Suppose $m = 0$, then xRy, and the statement is shown putting $n = 0$.
   (b) Suppose $m = p + 1$ and $xR_{p+1}y$. Assume that either xRy or $\exists z_1, \ldots, z_p$ with $z_1 Rx$ and $z_{p-1}Ry$ and for each $1 < i \leqslant p$ one has either $z_{i-1}Rz_i$ or $z_i R z_{i-1}$.
   $xR_{p+1}y$ implies $(x, y) \in R_p \cup (R_p^{-1} \circ R_p)$. If $(x, y) \in R_p$, then we merely apply the IH. Suppose $(x, y) \in R_p^{-1} \circ R_p$, then $(z, x) \in R_p$ and $(z, y) \in R_p$
3. (3) $\Rightarrow$ (4) Suppose either xRy or there exist $n \geqslant 1$ and $z_1, \ldots, z_n$ with z_1Rx and $z_{n-1}Ry$ and for each $1 < i \leqslant n$ one has either $z_{i-1}Rz_i$ or z_iRz_{i-1}. If xRy, then we are done. Otherwise there exists $n \geqslant 1$ with the condition above. Then $(x, y) \in R_{n+1}$ that follows from the condition.
4. (4) $\Rightarrow$ (1)

$\square$

Theorem 2. **K**5 *does not admit strict filtrations.*

Proof. Let us consider a **K**5 model whose Euclidean closure of the minimal filtration does not give us a filtration.

Let us consider a frame called $\mathcal{F}_{bad}$. We define this frame with the following graph:



Let us define a valuation ϑ such that $\vartheta(p) = \{w_5\}$ and $\vartheta(q) = \{w_1\}$. Let us consider a minimal filtration of $\mathcal{M}_{bad}$ through the Sub-closure of $\Gamma = \{\neg p, \neg\Diamond p\}$.

Clearly that $w_2 \sim_\Gamma w_3$, since $\neg p$ and $\neg\Diamond p$ are true both at w_2 and w_3.

Moreover, $R_{min} \cup (R_{min}^{-1} \circ R_{min})$ is not a subset of R_{max} since $(\hat{w_1}, \hat{w_5}) \in (R_{min}^{-1} \circ R_{min})$, but $\Diamond p$ is not true at w_5.

Let us also note that strict filtrations of this model is not Euclidean. Suppose by contrary that $\widehat{R}^{\mathcal{E}}$ is a strict filtraction of that model. So $R_{min}^{E} \subseteq \widehat{R}^{\mathcal{E}}$, since R_{min}^{E} is the minimal Euclidean relation containing R_{min}. On the other hand, $R_{min}^{E} \nsubseteq R_{max}$, so is not $\widehat{R}^{\mathcal{E}}$. $\square$

Theorem 3. **K5** *admits definable filtrations.*

Theorem 4. **K**45 *admits strict filtrations.*

Proof. Let $\mathcal{M} = \langle W, R, \vartheta \rangle$ be a transitive Euclidean model and $\overline{\mathcal{M}} = \langle \overline{W}, \overline{R}, \overline{\vartheta} \rangle$ its minimal filtration through Γ, where Γ is finite and Sub-closed. Let us put $\widehat{R} = \overline{R}^{+} \cup \overline{R}^{\mathbf{K5}}$. Let us show that $\overline{R}^{+} \cup \overline{R}^{\mathbf{K45}} \subseteq \overline{R}^{max}$.

That is, if $\mathcal{M}, y \models \varphi$ for $\Diamond\varphi \in \Gamma$ and $\hat{x}\widehat{R}\hat{y}$, then $\mathcal{M}, x \models \Diamond\varphi$.

Let $\hat{x}\widehat{R}\hat{y}$.

1. Suppose $(\hat{x}, \hat{y}) \in \overline{R}$, then $\mathcal{M}, x \models \Diamond\varphi$ holds trivially by the definition of the minimal filtration.

2. Let us consider the case when $(\hat{x}, \hat{y}) \in \overline{R}^{\mathbf{K}5}$. The second alternative is the same as for the **K**4-case, see [2, p. 141].

   Suppose the statement holds $\overline{R}_n$ and $(\hat{x}, \hat{y}) \in \overline{R}_{n+1} = \overline{R}_n \cup (\overline{R}_n^{-1} \circ \overline{R}_n)$. We consider the case of $(\hat{x}, \hat{y}) \in (\overline{R}_n^{-1} \circ \overline{R}_n)$.

   Then there exists $\hat{z}$ such that $(\hat{z}, \hat{x}), (\hat{z}, \hat{y}) \in \overline{R}_n$.

   By IH, $\mathcal{M}, z \models \Diamond\varphi$.

   $(\hat{z}, \hat{y}) \in \overline{R}_n$ iff there are $\hat{u}_1, \ldots, \hat{u}_n$ such that

$$\hat{z} \xleftarrow[\widehat{R}]{} \hat{u}_1 \xrightarrow[\widehat{R}']{} \hat{u}_2 \xrightarrow[\widehat{R}']{} \ldots \xrightarrow[\widehat{R}']{} \hat{u}_{n-1} \xrightarrow[\widehat{R}']{} \hat{u}_n \xrightarrow[\widehat{R}]{} \hat{y}$$

   where $\widehat{R}'$ is either $\widehat{R}$ or $\widehat{R}^{-1}$.

As it is known, $\Diamond\Diamond\varphi \to \Box\Diamond\varphi \in \mathbf{K}45$.

$\hat{u}_1 \widehat{\hat{z}}$, that is, $u_1' R z'$ for some $u_1' \in \hat{u}_1$ and $z' \in \hat{z}$. That is, $\mathcal{M}, u_1' \models \Diamond\Diamond\varphi$, so $\mathcal{M}, u_1' \models \Diamond\varphi$ and $\overline{M}, \hat{u}_1 \models \Diamond\varphi$.

We have $\hat{u}_1 \widehat{R}' \hat{u}_2$. Suppose $\mathcal{M}, u_1'' \models \Diamond\varphi$ and $u_1'' R u_2'$. We also have $\mathcal{M}, u_1'' \models \Box\Diamond\varphi$, thus, $\mathcal{M}, u_2' \models \Diamond\varphi$.

Suppose $\hat{u}_2 \widehat{R} \hat{u}_1$ and $u_2' R u_1''$, then $\mathcal{M}, u_2' \models \Diamond\varphi$.

Similarly, we have $\mathcal{M}, u_i \models \Diamond\varphi$ iff $\mathcal{M}, u_{i+1} \models \Diamond\varphi$, whenever $\hat{u}_i \widehat{R}' \hat{u}_{i+1}$.

Finally, we have $\hat{u}_n \widehat{R} \hat{x}$. Thus, $u_n' R x'$ for some $u_n' \in \hat{u}_n$ and $x' \in \hat{x}$. $\mathcal{M}, u_n' \models \Diamond\varphi$, so $\mathcal{M}, u_n' \models \Box\Diamond\varphi$. Then $\mathcal{M}, x' \models \Diamond\varphi$. $\Box$

3 i,j-Euclideaness

A binary relation $R \subseteq W \times W$ is called i, j-Euclidean for $i, j < \omega$, if for each x, y, z such that $xR^i y$ and $xR^j z$ implies xRz.

Proposition 1. *Let $\mathcal{F} = \langle W, R \rangle$, then $\mathcal{F}$ is i, j-Euclidean iff $\mathcal{F} \models \Diamond^i p \to \Box^j \Diamond p$. As a corollary, the logic $\mathbf{K}5^{i,j} = \mathbf{K} \oplus \Diamond^i p \to \Box^j \Diamond p$ is Kripke-complete.*

Let R be a binary relation on $W \neq \varnothing$, then the i, j-Euclidean closure of R (where $i, j < \omega$), denoted as $R^{\mathbf{K}5_{i,j}}$, is a binary relation defined recursively as follows:

1. $R_0 = R$
2. $R_{n+1} = R_n \cup (((R_n)^{-1})^i \circ {R_n}^j)$
3. $R^{E_{i,j}} = \bigcup\limits_{k<\omega} R_k$

Note that the following property holds not for every $\mathbf{K5}^{i,j}$, consider **KB** as an example (clearly, $\mathbf{KB} = \mathbf{K5}^{0,1}$), whose modal depth is ∞.

Theorem 5. *$md(\mathbf{K5}^{i,j}) = 1 + j + i$, if $i \neq 0$.*

Proof. $\Box$

Theorem 6. *$\mathbf{K5}^{i,j}$ admits definable filtrations.*

Proof. Let $\mathcal{M} = \langle W, R, \vartheta\rangle$ and Γ a finite Sub-closed of formulas. We extend Γ as

$$\Delta = \Gamma \cup \mathrm{Sub}\{\Diamond^i \psi \mid \Box\psi \in \Gamma\} \cup \mathrm{Sub}\{\Box^j \Diamond\psi \mid \Diamond\psi \in \Gamma\}$$

Let $(\hat{x}, \hat{y}) \in R^{E_{i,j}}$, $\mathcal{M}, x \models \Box\psi$ for $\Box\psi$ in Δ. If $n = 0$, then the statement is obvious. Suppose $n = 1$. Then $(\hat{x}, \hat{y}) \in R_{min} \cup (((R_{min})^{-1})^i \circ {R_{min}}^j)$. Consider the second alternative. Then there exists $\hat{z}$ such that $(\hat{x}, \hat{z}) \in ((R_{min})^{-1})^i$ and $(\hat{z}, \hat{y}) \in {R_{min}}^j$.

That is, there are $\hat{x_1}, \hat{x_2}, \dots, \hat{x_i}$ and $\hat{y_1}, \hat{y_2}, \dots, \hat{y_j}$ such that

$$\hat{z} \xrightarrow{R_{min}} \hat{x_1} \xrightarrow{R_{min}} \hat{x_2} \xrightarrow{R_{min}} \dots \xrightarrow{R_{min}} \hat{x_i} \xrightarrow{R_{min}} \hat{x}$$

$$\hat{z} \xrightarrow{R_{min}} \hat{y_1} \xrightarrow{R_{min}} \hat{y_2} \xrightarrow{R_{min}} \dots \xrightarrow{R_{min}} \hat{y_j} \xrightarrow{R_{min}} \hat{y}$$

??? $\Box$

4 The case of 2-transitivity

Let us define the logic $\mathcal{L}$ as $\mathbf{K} \oplus \Diamond\Diamond\Diamond p \to \Diamond p$. Let R be a binary relation, the $\mathcal{L}$-closure of R is defined (denoted as $R^{\text{✡}}$) as the following union:

$$R^{\text{✡}} = R \cup R^3 \cup R^5 \cup \dots \cup R^{2k+1} \cup \dots$$

Theorem 7. *$\mathcal{L}$ does not admit strict filtrations.*

Proof. Consider the following frame $\mathcal{F} = \langle W, R\rangle$:

$$\begin{tikzcd}
x \arrow[loop above] \arrow[r] & y & & \\
& y_1 \arrow[r] \arrow[rr, bend right] & z_1 \arrow[r] & z \arrow[loop above]
\end{tikzcd}$$

Clearly that $\mathcal{F}$ is an $\mathcal{L}$-frame. We define the valuation ϑ as follows:

$$\vartheta(p) = \{x\}$$
$$\vartheta(q) = \{y, y_1\}$$
$$\vartheta(r) = \{z\}$$

Let us put $\Gamma = \mathrm{Sub}\{p, q, \Diamond r\}$. We factorise W through $\sim_\Gamma$ and consider a model $\widehat{\mathcal{M}} = \langle W/\sim_\Gamma, \widehat{R}, \widehat{\vartheta}\rangle$, where $\widehat{R} = (\widehat{R}_{min})^{\text{✡}}$. We have $(\hat{x}, \hat{z}) \in \widehat{R} \circ \widehat{R} \circ \widehat{R}$, but $\Diamond r$ is not true at x. $\Box$

5 Fusion stuff

Definition 12. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be modal logics, then the fusion $\mathcal{L}_1 * \mathcal{L}_2$ is the minimal bimodal logic that contains $\mathcal{L}_1$ and $\mathcal{L}_2$ [7].*

Lemma 6. *Let Γ be a finite and* Sub*-closed set of formulas.*

1. *$\mathcal{M} = \langle W, R, \vartheta\rangle$ be a* **K**5*-model. Consider $\Gamma' = \Gamma \cup \{\Diamond\Box\psi \mid \Box\psi \in \Gamma\}$. Let Δ be any finite and* Sub*-closed extension of Γ' that contains all $\Diamond\Box\psi$ for each $\Box\psi \in \Delta$. Then a model $\widehat{M} = \langle W/\sim_\Delta, (R^{min}_\Delta)^{\mathbf{K5}}, \widehat{\vartheta}\rangle$ is a filtration of $\mathcal{M}$ through Δ.*

2. *Let $\mathcal{L} = \mathbf{K} \oplus \Diamond\Diamond\Diamond p \rightarrow \Diamond p$, then we have the similar statement for $\mathcal{L}$, where Δ is an extension of $\Gamma' = \{\Diamond\Diamond\psi \mid \Diamond\psi\}$ that contains all $\Diamond\Diamond\varphi$ for $\Diamond\varphi \in \Delta$.*

   Proof.
   Recall that $(R^{min}_\Delta)^E$ is defined inductively as:

   (a) $R^0_\Delta = R^{min}_\Delta$
   (b) $R^{n+1}_\Delta = R^n_\Delta \cup (R^{n\,-1}_\Delta \circ R^n_\Delta)$
   (c) $(R^{min}_\Delta)^E = \bigcup_{k<\omega} R^k_\Delta$

   If R^{min}_Δ is already a subrelation of R^{max}, so the base case is self-evident.

   Suppose the statement holds for R^n_Δ, $(\hat{x}, \hat{y}) \in R^{n+1}_\Delta$ such that $\mathcal{M}, y \models \psi$ for $\Diamond\psi \in \Delta$.

   According to the third item of Lemma 5, this is the same as there exist $\widehat{z_0}, \widehat{z_1}, \ldots, \widehat{z_{n-1}}, \widehat{z_n}$ such that $\widehat{z_1} R^{min}_\Delta \hat{x}$, $\widehat{z_n} R^{min}_\Delta \hat{y}$, and for each $i \in n+1$ we have either $\widehat{z_i} R^{min}_\Delta \widehat{z_{i+1}}$ or $\widehat{z_{i+1}} R^{min}_\Delta \widehat{z_i}$.

   We visualise this with the following graph:

   $$\hat{x} \xleftarrow{R^{min}_\Delta} \widehat{z_0} \xleftrightarrow{R'} \widehat{z_1} \xleftrightarrow{R'} \ldots \xleftrightarrow{R'} \widehat{z_{n-1}} \xrightarrow{R'} \widehat{z_n} \xrightarrow{R^{min}_\Delta} \hat{y}$$

   where R' is either R^{min}_Δ or its converse. We have $\mathcal{M}, z \models \Box\psi$, $\mathcal{M}, y \models \psi$, so $\mathcal{M}, z_n \models \Diamond\psi$. Since $\mathcal{M}$ is a $\mathbf{K}5$-model, we have $\mathcal{M}, z_n \models \Box\Diamond\psi$.

   After that we apply the following property of $\mathbf{K}5$-models:

   Let $\mathcal{M} \models \mathbf{K}5$ and φ a formula, then for each $a, b \in \mathcal{M}$ such that aRb we have
   $\mathcal{M}, a \models \Box\Diamond\varphi$ iff $\mathcal{M}, b \models \Box\Diamond\varphi$

   So we have $\mathcal{M}, z_0 \models \Box\Diamond\varphi$. Note that we always stay within Δ. Thus, $\mathcal{M}, x \models \Diamond\varphi$.

2. Let us prove the second item. Let $\mathcal{M}$ be an $\mathcal{L}$-model. The $\mathcal{L}$-closure of the minimal filtered relation modulo Δ, namely ${R^{min}_\Delta}^{\mathcal{L}}$ has the following form:

   $$ {R^{min}_\Delta}^{\mathcal{L}} = \bigcup_{k<\omega} {R^{min}_\Delta}^{2k+1}$$

   We reformulate this closure equivalently as follows:

   (a) $R_0 = R^{min}_\Delta$
   (b) $R_{n+1} = R_n \cup ((R^{min}_\Delta)^2 \circ R_n)$
   (c) ${R^{min}_\Delta}^{\mathcal{L}} = \bigcup\limits_{k<\omega}$

   The base case is self-evident. Suppose the statement holds for R_n and $(\hat{x}, \hat{y}) \in (R^{min}_\Delta)^2 \circ R_n$, that is, there exists $\hat{z}$ such that $(\hat{x}, \hat{z}) \in (R^{min}_\Delta)^2$ and $(\hat{z}, \hat{y}) \in R_n$. By IH, we have $\mathcal{M}, z \models \Diamond\varphi$.

   We have the following:

   $$\hat{x} \xrightarrow{R^{min}_\Delta} \widehat{y_1} \xrightarrow{R^{min}_\Delta} \hat{z}$$

The sequence of implications if the following:

$$\widehat{\mathcal{M}}, \hat{z} \models \Diamond\varphi \Rightarrow \widehat{\mathcal{M}}, \widehat{y_1} \models \Diamond\Diamond\varphi \Rightarrow \mathcal{M}, x \models \Diamond\Diamond\Diamond\varphi \Rightarrow \mathcal{M}, x \models \Diamond\varphi$$

$\square$

Theorem 8.

1. $\mathbf{K}5 * \mathbf{K}5$ *admits definable filtrations.*
2. $\mathbf{K}5 * \cdots * \mathbf{K}5$ *admits definable filtrations.*
3. *If* $\mathcal{L}$ *admits strict filtrations, then* $\mathbf{K}5 * \mathcal{L}$ *admits definable filtrations*
4. *If* $\mathcal{L}_1, \ldots, \mathcal{L}_n$ *admit strict filtrations, then* $\mathbf{K}5 * \cdots * \cdots * \mathbf{K}5 * \mathcal{L}_1 * \cdots * \mathcal{L}_n$
5. *Let* $\mathcal{L} = \mathbf{K} \oplus \Diamond\Diamond\Diamond p \to \Diamond p$ *(here and below), then* $\mathcal{L} * \mathcal{L}$ *admits definable filtrations.*
6. *Let* $\mathcal{L} = \mathbf{K} \oplus \Diamond\Diamond\Diamond p \to \Diamond p$ *and* $\mathcal{L}_1$ *a logic that admits strict filtrations, then* $\mathcal{L} * \mathcal{L}_1$

Proof.

1. Let Γ be a finite Sub-closed set of bimodal formulas, $\mathcal{F} = \langle W, R_1, R_2 \rangle$ a $\mathbf{K}5 * \mathbf{K}5$-frame, and ϑ a valuation on $\mathcal{F}$. Denote $\langle \mathcal{F}, \vartheta \rangle$ as $\mathcal{M}$.

   We introduce the set of fresh variables $V = \{p_\psi | \psi \in \Gamma\}$ and define a new model $\mathcal{M}' = \langle \mathcal{F}, \vartheta' \rangle$ as follows:

   $$\text{For all } \psi \in \Gamma,\ \mathcal{M}, x \models \psi \Leftrightarrow \mathcal{M}', x \models \psi \Leftrightarrow \mathcal{M}', x \models p_\psi.$$

   Consider these modifications of Γ and V:

   $$\Gamma' = \Gamma \cup \{\Diamond_1 \Box_1 \psi \mid \Box_1 \psi \in \Gamma\} \cup \{\Diamond_2 \Box_2 \psi \mid \Box_2 \psi \in \Gamma\}$$
   $$\Delta = V \cup \mathrm{Sub}(\{\Diamond \Box p_\psi \mid \Box_i \psi \in \Gamma, in = 1, 2\})$$

   Let us define an equivalence relation $\sim_\Delta$ induced by Δ.

   Consider $\mathcal{M}_i = \langle W, R_i, \vartheta' \rangle$, a reduct of $\mathcal{M}'$, we have:

   (a) $\mathcal{M}_i, x \models \Box p_\psi$ iff $\mathcal{M}, x \models \Box_i \psi$
   (b) $\mathcal{M}_i, x \models \Diamond \Box p_\psi$ iff $\mathcal{M}, x \models \Diamond_i \Box_i \psi$

   So $\sim = \sim_{\Gamma'}$ by the construction. Let us put $\widehat{W} = W / \sim_{\Gamma'}$. Lemma 6 implies the following claim:

   Claim 1. *Let* $\widehat{R}_i = (R_\Delta^{min})^E$ *and* $\widehat{\vartheta(p)} = \{[x]_{\sim_i} \mid \mathcal{M}_i, x \models p\}$ *for* $p \in \Delta_1$*, define* $\widehat{\mathcal{M}}_i = \widehat{W}, \widehat{R}_i, \hat{\vartheta}$*. Then* $\widehat{\mathcal{M}}_i \models \mathbf{K}5$ *and* $\widehat{\mathcal{M}}_i$ *is a filtration of* $\mathcal{M}_i$ *through* Δ*.*

   Finally, we consider a model $\widehat{\mathcal{M}} = \langle \widehat{W}, \widehat{R}_1, \widehat{R}_2, \vartheta \rangle$, where $\widehat{R_{\Gamma'}}_i = {R_i}_{\Gamma'}^{min^E}$ and $\vartheta(p)$ is defined as usual for $p \in \Gamma$. $\widehat{\mathcal{M}}$ is a filtration of $\mathcal{M}$ through Γ'.

   Let $\hat{x} \widehat{R_{\Gamma'}}_i \hat{y}$ and $\mathcal{M}, x \models \Box_i \psi$ for $\Box_i \psi \in \Gamma$. Then $\mathcal{M}_i, x \models \Box p_\psi$, so $\widehat{\mathcal{M}_i}, \hat{x} \models \Box p_\psi$. By the claim above, $\widehat{\mathcal{M}_i}$ is a filtration of $\mathcal{M}_i$ through Δ, so $\mathcal{M}_i, y \models p_\psi$. Then $\mathcal{M}, y \models \psi$.

2. Likewise

3. The argument is the same as in the proof of the first item, except for **Claim** 1 that has the following formulation: Let $\widehat{R}_1 = (R_\Delta^{min})^E$ and $\widehat{R}_2 = (R_\Delta^{min})^{\mathcal{L}_1}$ Define a valuation as usual as $\widehat{\vartheta(p)} = \{[x]_{\sim_i} \mid \mathcal{M}_i, x \models p\}$ for $p \in \Delta_1$, define $\widehat{\mathcal{M}}_1 = \langle \widehat{W}, \widehat{R}_1, \widehat{\vartheta}\rangle$ and $\widehat{\mathcal{M}}_2 = \langle \widehat{W}, \widehat{R}_2, \widehat{\vartheta}\rangle$. Then $\widehat{\mathcal{M}}_1 \models \mathbf{K}5$ and $\widehat{\mathcal{M}}_1 \models \mathbf{L}$ and $\widehat{\mathcal{M}}_i$ is a filtration of $\mathcal{M}_i$ through Δ.

4. Likewise

5. The argument is similar to the proof of first item, but filtrations are slightly different. Let $\mathcal{M} = \langle W, R_1, R_2, \vartheta\rangle$ be a $\mathcal{L} * \mathcal{L}$ model and Γ a Sub-closed set of formulas. As above, we define a set V and a model $\mathcal{M}'$. Define extensions of Γ and V:

$$\Gamma' = \Gamma \cup \{\Diamond_1\Diamond_1\psi \mid \Diamond_1\psi \in \Gamma\} \cup \{\Diamond_2\Diamond_2\psi \mid \Diamond_2\psi \in \Gamma\}$$
$$\Delta = V \cup \mathrm{Sub}(\{\Diamond\Diamond p_\psi \mid \Diamond\psi \in \Gamma, i = 1, 2\})$$

   As above $\sim_\Delta = \Gamma'$ and $\widehat{\mathcal{M}'} = \langle W/\sim_\Delta, \widehat{R_i}, \widehat{\vartheta}\rangle$ are filtrations of reducts of $\mathcal{M}'$ through Δ. Then $\widehat{\mathcal{M}} = \langle W/\sim_\Delta, \widehat{R_1}, \widehat{R_2}, \widehat{\vartheta}\rangle$ is a required filtration of the original $\mathcal{M}$.

6. Extend Γ with $\{\Diamond_i\Diamond_i\psi \mid \Diamond_i\psi \in \Gamma, i = 1, 2\}$ and V with $\{\Diamond\Diamond p_\psi \mid \Diamond_i\psi, i = 1, 2\}$

□

Theorem 9. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be modal logics that admit definable filtrations. If* $\mathrm{Mod}(\mathcal{L}_1)$ *and* $\mathrm{Mod}(\mathcal{L}_2)$ *admit definable filtrations, so does* $\mathrm{Mod}(\mathcal{L}_1 * \mathcal{L}_2)$*.*

Proof. Let $\mathcal{M} = \langle W, R_1, R_2, \vartheta\rangle$ be an $\mathcal{L}_1 * \mathcal{L}_2$. We define a notation $\nabla = \{\neg\Diamond, \Diamond\neg, \Diamond\}$.

Both logics admit definable filtrations, so for every finite Sub-closed set Γ and for every $\mathfrak{M}$, an $\mathcal{L}_1$-model (or an $\mathcal{L}_2$ one) there exists there exists Δ, a extension of Γ having the form:

$$\Delta_1 = \Gamma \cup \mathrm{Sub}\{\nabla_1\nabla_2 \dots \nabla_n\Diamond\psi \mid \Diamond\psi \in \Gamma\} \text{ (for } \mathcal{L}_1)$$
$$\Delta_2 = \Gamma \cup \mathrm{Sub}\{\nabla_1\nabla_2 \dots \nabla_k\Diamond\psi \mid \Diamond\psi \in \Gamma\} \text{ (for } \mathcal{L}_2)$$

such that $\widehat{\mathfrak{M}} = \langle W/\sim_{\Delta_i}, \widehat{R}, \vartheta\rangle$ is a filtration of $\mathfrak{M}$ through the corresponding Δ_i.

Let V be a set of fresh variables indexed over Γ as in the proof for a fusion of **K5** with something else. Let $\mathcal{M}'$ be a model defined as previously. We extend V and Γ in the following way:

$$\Gamma' = \Gamma \cup \mathrm{Sub}\{\nabla_{11}\nabla_{21} \dots \nabla_{n1}\Diamond_1\psi \mid \Diamond_1\psi \in \Gamma\} \cup \mathrm{Sub}\{\nabla_{12}\nabla_{22} \dots \nabla_{n2}\Diamond_2\psi \mid \Diamond_2\psi \in \Gamma\}$$
$$\Delta = V \cup \mathrm{Sub}\{\nabla_1\nabla_2 \dots \nabla_n\Diamond p_\psi \mid \nabla_{n+1 1}\psi \in \Gamma'\} \cup \mathrm{Sub}\{\nabla_1\nabla_2 \dots \nabla_k\Diamond p_\psi \mid \Diamond_2\psi \in \Gamma\}.$$

By the construction, $\sim_{\Gamma'} = \sim_\Delta$. So we have filtrations for the corresponding reducts of $\mathcal{M}'$ through Δ as well as for the original $\mathcal{M}$. □

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