Finite model property for residuated semigroups and related remarks

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1 Finite networks for atomic formulas

Let $PV = \{p_i \,|\, i < \omega\}$ be the set of propositional variables (or atomic types). The set of formulas is generated by the following grammar:

$$\varphi, \psi ::= p \mid (\varphi \bullet \psi) \mid (\varphi \backslash \psi) \mid (\varphi / \psi)$$

The Lambek calculus is defined as a Gentzen-style sequent calculus:

$$\frac{}{p \to p}\ \mathrm{ax}$$

$$\frac{\Gamma \to \varphi \qquad \Delta, \psi, \Theta \to \theta}{\Delta, \Gamma, \varphi\backslash\psi, \Theta \to \theta}\ \backslash\to \qquad\qquad \frac{\varphi, \Pi \to \psi}{\Pi \to \varphi\backslash\psi}\ \to\backslash, \Pi \text{ is non-empty}$$

$$\frac{\Gamma \to \varphi \qquad \Delta, \psi, \Theta \to \theta}{\Delta, \psi/\varphi, \Gamma, \Theta \to \theta}\ /\to \qquad\qquad \frac{\Pi, \varphi \to \psi}{\Pi \to \psi/\varphi}\ \to/, \Pi \text{ is non-empty}$$

$$\frac{\Gamma, \varphi, \psi, \Delta \to \theta}{\Gamma, \varphi \bullet \psi, \Delta \to \theta}\ \bullet\to \qquad\qquad \frac{\Gamma \to \varphi \qquad \Delta \to \varphi}{\Gamma, \Delta \to \varphi \bullet \psi}\ \to\bullet$$

1.1 Completeness

Theorem 1. *Let RS be the class of all residuated semigroups, then $\Gamma \to \varphi$ iff $RS \models \Gamma \to \varphi$*

2 Representability networks for at most countable residuated semigroups

2.1 Relational residuated semigroups as Kripke models

One can introduce Kripke-style relational semantics for the Lambek calculus as follows. Let W be a non-empty set and R a transitive relation on W. We consider models of the kind $M = (R, \vartheta)$, where $\vartheta : PV \to 2^R$. The truth definition is inductive:

- $\mathcal{M}, (x, y) \models p_i$ iff $(x, y) \in \vartheta(p_i)$,
- $\mathcal{M}, (x, y) \models \varphi \bullet \psi$ iff there exists $z \in W$ such that $(x, z), (z, y) \in R$ and $\mathcal{M}, (x, z) \models \varphi$ and $\mathcal{M}, (z, y) \models \phi$

- $\mathcal{M}, (x, y) \models \varphi \backslash \psi$ iff for all $z \in W$ such that if $(z, x) \in R$ and $\mathcal{M}, (z, x) \models \varphi$, then $\mathcal{M}, (z, y) \models \psi$
- $\mathcal{M}, (x, y) \models \varphi / \psi$ iff for all $z \in W$ such that if $(y, z) \in R$ and $\mathcal{M}, (y, z) \models \psi$, then $\mathcal{M}, (x, z) \models \varphi$
- $\mathcal{M}, (x, y) \models \varphi_1, \varphi_2, \ldots, \varphi_n \to \varphi$ iff $\mathcal{M}, (x, y) \models \varphi_1 \bullet \varphi_2 \bullet \cdots \bullet \varphi_n$ implies $\mathcal{M}, (x, y) \models \varphi$.

According to the definition above, to refute a sequent $\varphi_1, \varphi_2, \ldots, \varphi_n \to \varphi$, we have to find a transitive binary relation R, some valuation $\vartheta : PV \to 2^R$ and $(x, y) \in R$ such that $\mathcal{M}, (x, y) \models \varphi_1 \bullet \varphi_2 \bullet \cdots \bullet \varphi_n$, but $\mathcal{M}, (x, y) \not\models \varphi$. Alternatively, one can reformulate that condition as

$$(x, y) \in ||\varphi_1||; ||\varphi_2||; \ldots ; ||\varphi_n||, \text{ but } (x, y) \notin ||(x, y)||$$

2.2 Relational representation of residuated semigroups: a game-theoretic approach

Let $\mathcal{A}$ be a residuated semigroup, an $\mathcal{A}$-prenetwork is a triple $\mathcal{N} = (V, E, l)$, where where (V, E) is a directed graph and $l : E \to \mathcal{A}$ is a labelling function. A prenetwork is a network if the following conditions hold:

- E has no loops and it is transitive,
- $l(x, z) \leqslant l(x, y); l(y, z)$, whenever $(x, y), (y, z) \in E$, for all $x, y, z \in V$,
- For all $a \in \mathcal{A}$, for all $x \in U$, there is some $u \in U$ such that $l(u, x) = a$,
- For all $a \in \mathcal{A}$, for all $y \in U$, there is some $v \in U$ such that $l(y, v) = a$,
- For all $a, b, c \in \mathcal{A}$, for all $x, y \in U$, if $c \leqslant a; b$, $(x, y) \in E$ and $l(x, y) = c$, then there exists $z \in U$ such that $l(x, y) = a$ and $l(y, z) = b$.

Let $n \leqslant \omega$, define a game $\mathcal{G}(\mathcal{A})_n$ for two players $\forall$ and $\exists$ by induction on n.

1. **step** 0

   $\forall$ picks a pair of elements $a, b \in \mathcal{A}$ such that $a \nleqslant b$. $\exists$ must respond with a network $\mathcal{N}_0 = (\{x, y\}, \{(x, y)\}, l_0 : (x, y) \mapsto a)$:

   $$x \xrightarrow{\quad a \quad} y$$

2. **step** $n + 1 < \omega$

   Suppose the networks:

   $$\mathcal{N}_0 \subseteq \mathcal{N}_1 \subseteq \cdots \subseteq \mathcal{N}_n$$

   have been already constructed.

   There are four different options:

   (a) **Composition move**

   $\forall$ picks $x, y, z \in \mathcal{N}_n$ such that $b = l_n(x, y)$ and $c = l_n(y, z)$:

   $$\begin{tikzcd} & y \arrow[dr, "c"] & \\ x \arrow[ur, "b"] & & z \end{tikzcd}$$

$\exists$ has to respond with $\mathcal{N}_{n+1} = (V_n, E_n \cup \{(x,z)\}, l_{n+1})$ where $l_{n+1}(x,z) = b;c$ and $l_{n+1}(x', y') = l_n(x', y')$ for $(x', y') \in E_n$.

$$\begin{tikzcd} & y \arrow[dr, "c"] & \\ x \arrow[ur, "b"] \arrow[rr, dashed, "b;c"'] & & z \end{tikzcd}$$

(b) **Witness move** $\forall$ picks $(x,y) \in E_n$ such that $l_n(x,y) = a$ and $a \leqslant b;c$:

$$x \xrightarrow{a} y$$

$\exists$ has to respond with $\mathcal{N}_{n+1} = (V_n \cup \{z\}, E_n \cup \{(x,z),(z,y)\}, l_{n+1})$, where

$$l_{n+1}(x,z) = b$$
$$l_{n+1}(y,z) = c$$
$$l_{n+1}(p) = l_n \text{ for others } p \in E_n$$

The latter can be visualised with the following triangle:

$$\begin{tikzcd} x \arrow[rr, "a"] \arrow[dr, dashed, "b"'] & & y \\ & z \arrow[ur, dashed, "c"'] & \end{tikzcd}$$

(c) **Left redisual move**

(d) **Right residual move**

Theorem 2. *Let $\mathcal{A}$ be a at most countable residuated semigroup, then*

1. *$\exists$ has a winning stragery in $\mathcal{G}_\omega(\mathcal{A})$*

$$rep(a) = \{(x,y) \mid l(x,y) \leqslant a\}$$

   (a) $rep(a;b) = rep(a);rep(b)$

   (b) $rep(a\backslash b) = rep(a)\backslash rep(b)$

   (c) $rep(a/b) = rep(a)/rep(b)$

2. *$\mathcal{A}$ is representable.*

TODO: check if the representability class is closed under products, subalgebras and ultraproducts. Check the criterion for the Horn formulas. Closed under H? can't be defined by equations?

3 Step-by-step construction for the FMP

3.1 Example 1

Consider a sequent $\varphi_1 \bullet \varphi_2 \to \varphi_1 \bullet (\psi_1 \backslash \psi_2)$, which is known to unprovable in the Lambek calculus. We would like to construct a finite model on a network such that there exists $(u,v) \not\models \varphi_1 \bullet \varphi_2 \to \varphi_1 \bullet (\psi_1 \backslash \psi_2)$, that is, $(u,v) \in ||\varphi_1||;||\varphi_2||$ and $(u,v) \notin ||\varphi_1||;(||\psi_1||\backslash||\psi_2||)$.

The construction is step-by-step:

1. **Step 0**

   We construct the edge (u, v) labelled by a pair of sets λ_0 and δ_0:

$$u \xrightarrow[\delta_0]{\lambda_0} v$$

$$\lambda_0(u, v) = \{\varphi_1 \bullet \varphi_2, -(\varphi_1 \bullet (\psi_1 \backslash \psi_2))\}$$
$$\delta_0(u, v) = \{\varphi_1 \bullet \varphi_2\}$$

2. **Step 1**

   We have to create a witness for $\varphi_1 \bullet \varphi_2 \in \delta_0(u, v)$ by adding a vertice w:

$$\begin{tikzcd}
 & w \arrow[dr, "\lambda_1", "\delta_1"'] & \\
u \arrow[ur, "\lambda_1", "\delta_1"'] \arrow[rr, "\lambda_0", "\delta_0"'] & & v
\end{tikzcd}$$

$$\lambda_1(u, w) = \{\varphi_1\}$$
$$\lambda_1(w, v) = \{\varphi_2, -(\psi_1 \backslash \psi_2)\}$$
$$\delta_1(u, w) = \varnothing$$
$$\delta_1(w, v) = \{-(\psi_1 \backslash \psi_2)\}$$

3. **Step 2**

   We create a witness w' for $-(\psi_1 \backslash \psi_2) \in \delta_1(w, v)$

$$\begin{tikzcd}
w \arrow[dr, "\lambda_1", "\delta_1"'] & w' \arrow[l, "\lambda_2"', "\delta_2"] \arrow[d, "\lambda_2"', "\delta_2"] \\
u \arrow[u, "\lambda_1", "\delta_1"'] \arrow[r, "\lambda_0", "\delta_0"'] & v
\end{tikzcd}$$

   By putting:

$$\lambda_2(w', w) = \{\psi_1\}$$
$$\lambda_2(w', v) = \{-\psi_2\}$$
$$\delta_2(w', w) = \varnothing$$
$$\delta_2(w', v) = \varnothing$$

Now let us describe $||\varphi_1||; ||\varphi_2||$ as follows:
TODO:

3.2 Example 2

Let us consider a sequent $LC \not\vdash \varphi_1 \backslash (\varphi_2 \bullet \varphi_3) \to \varphi_1 \backslash \psi$

1. **Step 0**

   Introduce a wedge (u, v) with labelling λ_0 and δ_0

   $$u \xrightarrow[\delta_0]{\lambda_0} v$$

   where

   $$\lambda_0(u, v) = \{\varphi_1 \backslash (\varphi_2 \bullet \varphi_3), -(\varphi_1 \backslash \psi)\}$$
   $$\delta_0(u, v) = \{-(\varphi_1 \backslash \psi)\}$$

2. **Step 1**

   We create a witness w for $-(\varphi_1 \backslash \psi) \in \delta_0(u, v)$

   $$\begin{tikzcd} & w \arrow[dl, "\lambda_1"', "\delta_1"] \arrow[dr, "\lambda_1", "\delta_1"'] & \\ u \arrow[rr, "\lambda_0", "\delta_0"'] & & v \end{tikzcd}$$

   where

   $$\lambda_1(w, u) = \{\varphi_1\}$$
   $$\lambda_1(w, v) = \{-\psi, \varphi_1 \bullet (\varphi_1 \backslash (\varphi_2 \bullet \varphi_3)), \varphi_2 \bullet \varphi_3\}$$
   $$\delta_1(w, u) = \varnothing$$
   $$\delta_1(w, v) = \{\varphi_2 \bullet \varphi_3\}$$

3. **Step 2**

   We have to create a witness w' for $\varphi_2 \bullet \varphi_3 \in \delta_1(w, v)$

   $$\begin{tikzcd} w \arrow[r, "\lambda_2", "\delta_2"'] \arrow[d, "\lambda_1"', "\delta_1"] \arrow[dr, "\lambda_1", "\delta_1"'] & w' \arrow[d, "\lambda_2"', "\delta_2"] \\ u \arrow[r, "\lambda_0", "\delta_0"'] & v \end{tikzcd}$$

   $$\lambda_2(w, w') = \{\varphi_2\}$$
   $$\lambda_2(w', v) = \{\varphi_3\}$$
   $$\delta_2(w, w') = \varnothing$$
   $$\delta_2(w', v) = \varnothing$$

3.3 General case

Consider a sequent $\Gamma \to \varphi$ such that $LC \nvdash \Gamma \to \varphi$. We have to construct a finite relational model over some network such that $(u, v) \not\models \Gamma \to \varphi$, that is, $(u, v) \models \Gamma$ and $(u, v) \not\models \varphi$.

References