Model-theoretic aspects of relativised cylindric set algebras

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1 Model-theoretic and universal algebraic preliminaries

1.1 Ultraproducts

Here are the required notions and facts from model theory and universal algebra [5] [6] [9].

Let A be a non-empty set, an *ultrafilter* on A is a set of subsets $U \subseteq \mathcal{P}(\mathcal{P})(A)$ such that A is closed under intersections, $\subseteq$-upwardly closed, and either $X \in U$ or $-X \in U$, where $X \subseteq A$. An ultrafilter is called principal if it has the form $\uparrow X = \{Y \in \mathcal{P}(A) \mid X \subseteq Y\}$.

Let $\Omega = \langle \mathrm{Cnst}, \mathrm{Fn}, \mathrm{Pred} \rangle$ be a signature and Λ an index set, and let $\{M_\lambda\}_{\lambda \in \Lambda}$ be an indexed set of Ω structures. The Ω-structure

$$M = \prod_{\lambda \in \Lambda} M_\lambda$$

is called a *product* that defined as follows. Its domain is the Cartesian product of the domains of M_λ. $a \in M$ is a function $\Lambda \to \bigcup\limits_{\lambda \in \Lambda} \mathrm{dom}(M_\lambda)$ such that $a(\lambda) \in M_\lambda$ for each $\lambda \in \Lambda$. Given $\lambda \in \Lambda$ and $a_\lambda \in M_\lambda$, we denote the function mapping λ to a_λ as $\langle a_\lambda \mid \lambda \in \Lambda \rangle$. We define the interpretation of Ω-symbols as

1. If $c \in \mathrm{Cnst}$, then $c^M = \langle c^{M_\lambda} \mid \lambda \in \Lambda \rangle$
2. If $f \in \mathrm{Fn}$ is an n-ary function symbol, then $f^M(\overline{a}) = \langle f^{M_\lambda}(\overline{a}) \mid \lambda \in \Lambda \rangle$, where $\overline{n} \in M^n$
3. If $R \in \mathrm{Pred}$ is an n-ary relation symbol and $\overline{n} \in M^a$, then $R^M(\overline{a}) = \langle R^{M_\lambda}(\overline{a}) \mid \lambda \in \Lambda \rangle$

Given $\lambda \in \Lambda$, we define the λth projection as $\pi_\lambda : M \to M_\lambda$ such that $\pi_\lambda(a) = a(\lambda)$.

Let Λ be an index set and D an ultrafilter on the Boolean algebra $\langle \mathcal{P}(\Lambda), \cup, -, \Lambda, \varnothing \rangle$. Consider the product $M = \prod\limits_{\lambda \in \Lambda} M_\lambda$ of the Ω-structures $\{M_\lambda\}_{\lambda \in \Lambda}$ and the equivalence relation on $\mathrm{dom}(M)$ defined as

$$a_1 \sim a_2 \Leftrightarrow \{\lambda \in \Lambda \mid a_1(\lambda) = a_2(\lambda)\} \in D$$

Let us denote $\mathrm{dom}(M)/\sim$ as U and $[a]_\sim$ as a/D, where $a \in \mathrm{dom}(M)$. We also denote the *ultraproduct* of $\{M_\lambda\}_\lambda$ as $\prod\limits_{\lambda \in \Lambda} M_\lambda/D$, or, for brevity, as $\prod_D M_\lambda$. The Ω-symbols have the following interpretation

1. If $c \in \mathrm{Cnst}$, then $c^U = c^M/D$
2. If $f \in \mathrm{Fn}$ is an n-ary function symbol and $\overline{a} \in M^n$, then $f^U(\overline{a}) = f^M(x) = f^M(\overline{a})/D$
3. If $R \in \mathrm{Fn}$ is an n-ary relation symbol and $\overline{a} \in M^n$, then $U \models R(\overline{a}/D)$ iff $\{\lambda \in \Lambda \mid M_\lambda \models R(\overline{a}(\lambda))\} \in D$

The ultraproduct is principal if D is a principal filter.

Definition 1.

1. *Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a set of Ω-structures such that every M_λ is isomorphic to the single structure M, then their ultraproduct over D is called the ultrapower over D. The denotation is $\prod\limits_D M$ or M^Λ/D.*

2. *If $\prod\limits_D M \cong N$ for some structure N, then M is an ultraroot of N.*

Theorem 1 (Łos)**.** *Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be Ω-structures and D an ultrafilter on Λ, and let $U = \prod\limits_D M_\lambda$ be an ultraproduct of $\{M_\lambda\}_{\lambda\in\Lambda}$ over D. For each first-order formula $\varphi(x_1,\ldots,x_n)$ and for each $a_1/D,\ldots,a_n/D \in U$:*

$$U \models \varphi(a_1/D,\ldots,a_n/D) \text{ iff } \{\lambda\in\Lambda \mid \varphi(a_1(\lambda),\ldots,a_n(\lambda))\} \in D$$

The Łos has the following helpful corollary:

Corollary 1. *Let $\prod\limits_D M$ be an ultrapower of M. For $a \in M$, let us define a function $\overline{a} : a \mapsto a/D$. Then such a map is an elementary embedding of M into $\prod\limits_D M$.*

Moreover, any elementary equivalent structures have isomorphic ultrapowers.

Recall that a class of Ω-structures $\mathbf{K}$ is called *elementary*, if $\mathbf{K} = \mathrm{Mod(T)}$ for some first-order theory $\mathbf{T}$. In that case, T is an axiomatisation of $\mathbf{K}$.

Theorem 2. *Let $\mathbf{K}$ be a class Ω-structures, $\mathbf{K}$ is elementary iff $\mathbf{K}$ is closed under isomorphic copies, ultraroots, and ultrapowers.*

1.2 Preliminaries from universal algebra

Definition 2. *Let $\mathbf{K}$ be a class of Ω-structures, then $\mathbf{K}$ is a variety, if it is defined by some set of equations. The variety generated by $\mathbf{K}$ is the smallest variety contatining $\mathbf{K}$. $\mathbf{K}$ is a quasi-variety, if it is defined by some set of quasi-identities.*

Given a class $\mathbf{K}$ of Ω-structures, then $\mathbf{I}(\mathbf{K})$, $\mathbf{S}(\mathbf{K})$, $\mathbf{H}(\mathbf{K})$, and $\mathbf{P}(\mathbf{K})$ are the classes of isomorphic copies, algebras isomorphic to subalgebras belonging to $\mathbf{K}$, algebras isomorphic to homomorphic images belonging to $\mathbf{K}$, and algebras isomorphic to direct products belonging to $\mathbf{K}$. We claim that $\mathbf{I}(\mathbf{K}) \subseteq \mathbf{S}(\mathbf{K})$. $\mathbf{Up}(\mathbf{K})$ is the class of algebras isomorphic to ultraproducts belonging to $\mathbf{K}$.

Theorem 3. *Let $\mathbf{K}$ be a class of Ω-structures*

1. *$\mathbf{K}$ is a variety iff $\mathbf{H}(\mathbf{K}), \mathbf{S}(\mathbf{K}), \mathbf{P}(\mathbf{K}) \subseteq \mathbf{K}$*

2. *$\mathbf{HSP}(\mathbf{K}) = \mathbf{H}(\mathbf{S}(\mathbf{P}(\mathbf{K})))$ is the smallest variety containg $\mathbf{K}$*

3. *$\mathbf{K}$ is a quasi-variety iff it is closed under subalgebras, products, and ultraproducts, iff $\mathbf{SPUp}(\mathbf{K}) = \mathbf{K}$.*

1.3 Subdirect products

Definition 3.

1. *Let $\{A\}_{i\in I}$ be Ω-structures, a subdirect product of $\langle A_i \mid i \in I\rangle$ is a subalgebra B of $\prod_{i\in I} A_i$ such that for each $i \in I$, a projection map $\pi_i : B \to A_i$ is a surjection.*
2. *A subdirect representation of an Ω-structure is an embedding $f : A \to \prod_{i\in I} A_i$ for some I and $\{A_i\}_{i\in I}$ such that $f \circ \pi_i : A \to A_i$ is a surjection.*
3. *An Ω-structure A is subdirectly irreducible if for every subdirect representation $f : A \to \prod_{i\in I} A_i$ there exists a projection π_i such that $f \circ \pi_i$ is an isomorphism.*
4. $\mathbf{Sir}(\mathbf{K})$ *is the class of subdirectly irreducible structures belonging to* $\mathbf{K}$.
5. *A subdirect decomposition of A if there exists a subdirect representation $f : A \to \prod_{i\in I} A_i$ such that every A_i is subdirectly irreducible.*

It is known that every Boolean algebra with operators has a subdirect demcomposition. Moreover, that implies:

Theorem 4.

1. *If $\mathbf{K}$ is a variety, then every element of $\mathbf{K}$ has a subdirect decomposition with some subdirect irreducible elements of $\mathbf{K}$.*
2. *If $\mathbf{K}$ is a variety and ε is an equation, $\mathbf{Sir}(\mathbf{K}) \models \varepsilon \Leftrightarrow \mathbf{K} \models \varepsilon$.*

1.4 Pseudo-elementary classes

2 Cylindric algebras

2.1 (Representable) cylindric algebras and cylindric set algebras

Let α be an ordinal. Let U^α be the set of all functions mapping α to a non-empty set U. We denote $x(i) = x_i$ for $x \in U^\alpha$ and $i < \alpha$.

Definition 4.

1. *A subset of U^α is an α-ry relation on U. For $i, j < \alpha$, the i, j-diagonal D_{ij} is the set of all elements of U such that $y_i = y_j$. If $i < \alpha$ and X is an α-ry relation on U, then the i-th cylindrification C_iX is the set of all elements of U that agree with some element of X on each coordinate except the i-th one. To be more precise, $C_iX = \{y \in U^\alpha \mid \exists x \in X \forall i < \alpha\, (i \neq j \Rightarrow y_j = x_j)\}$.*
2. *A cylindic set algebra of dimension α is an algebra consisting of a set S of α-ry relation on some base set U with the constants and operations $0 = \varnothing$, $1 = U^\alpha$, $\cap$, $-$, the diagonal elements $\{D_{ij}\}_{i,j<\alpha}$, the cylindrifications $\{C\}_{i<\alpha}$. A generalised cylindric set algebra of dimension α is a subdirect of cylindric algebras that have dimension α*
3. *A cylindric algebra of dimension α is an algebra $\mathcal{C} = \langle \mathcal{B}, \{c_i\}_{i<\alpha}, \{d_{ij}\}_{i,j<\alpha}\rangle$ such that*

- *$\mathcal{B}$ is a Boolean algebra, for each $i, j < \alpha$ c_i is an operator and $d_{ij} \in \mathcal{B}$*
- *For each $i < \alpha$, $a \leqslant c_i a$, $c_i(a \wedge c_i b) = c_i a \wedge c_i b$ and $d_{ii} = 1$*
- *For every $i, j < \alpha$, $c_i c_j a = c_j c_i a$*
- *If $k \neq i, j < \alpha$, then $d_{ij} = c_k(d_{ij} \wedge d_{jk})$*
- *If $i \neq j$, then $c_i(d_{ij} \wedge a) \wedge c_i(d_{ij} \wedge -a) = 0$*

$\mathbf{CA}_\alpha$ *is the class of all cylindric algebras of dimension α*

4. *An α-dimensional cylindric algebra C is representable, if it is isomorphic to a generalised cylindric set algebra of dimension α. Such is isomorphism is a representation of C.* $\mathbf{RCA}_\alpha$ *is the class of all representable cylindric algebras that have dimension α.*

2.2 Substitution in cylindric algebras

Definition 5. *Given a cylindric algebra of dimension α C, let x be a term of its signature, the substitution operator s^i_j have the following definition:*

$$s^i_j x = \begin{cases} x, \textit{if } i = j \\ c_i(d_{ij} \wedge x), \textit{otherwise} \end{cases}$$

Proposition 1. *Let α be an ordinal and let $i, j, k, l < \alpha$. The following facts hold in* $\mathbf{CA}_\alpha$

1. *$s^i_j x \leqslant c_i x$.*
2. *$s^i_j(x \wedge y) = s^i_j x \wedge s^i_j y$, $s^i_j(x \vee y) = s^i_j x \vee s^i_j y$, $-s^i_j x = s^i_j(-x)$. Moreover, s^i_j is completely additive.*
3. *$i \neq k, l$ implies $s^i_j d_{ik} = d_{jk}$ and $s^i_j d_{kl} = d_{kl}$.*
4. *$d_{jk} \wedge s^i_j = d_{jk} \wedge s^i_k$.*
5. *$s^i_j c_i x = c_i x$.*
6. *$k \neq i, j$ implies $s^i_j c_i x = c_i s^i_j x$.*
7. *$c_j s^i_j x = c_i s^j_i x$.*
8. *$i \neq j$ implies $c_i s^i_j x = s^i_j x$.*
9. *$i \neq k$ implies $s^i_j s^i_k = s^i_k x$.*
10. *If either $i \notin \{k, l\}$ and $k \notin \{i, j\}$, or $j = l$, then $s^i_j s^k_l x = s^k_l s^i_j x$.*
11. *$s^i_j s^j_i x = s^i_j x$.*
12. *$s^i_k s^j_i x = s^i_k s^j_k x = s^j_k s^i_j x$*

3 $\mathbf{IG}_\omega$ and ultraproducts

4 $\mathbf{IG}_\omega$ is (not) a variety; is (not) (pseudo-)elementary

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