

Model-theoretic aspects of relativised cylindric set algebras

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1 Intro

... It is known that the equational theory of \mathbf{RCA}_ω for $\alpha \leq \omega$ is decidable [6]. ...

2 The problems themselves

1. Suppose $\mathcal{C} \in \mathbf{RCA}_\omega$, whether \mathcal{C}^+ has a complete, ω -dimensional representation? [3]
2. Is the class \mathbf{IG}_ω (the isomorphism-closure of the ω -dimensional cylindric relativised set algebras in which the unit is closed under substitutions and permutations) a variety, or even a pseudo-elementary class? Is it closed under ultraproducts? [3]

3 Boolean algebras with operators and cylindric algebras

Definition 1.

1. Let $\mathcal{B} = \langle B, +, -, 0, 1 \rangle$ be a Boolean algebra. An operator is an n -ary function $\Omega : B^n \rightarrow B$ satisfying the following conditions:

- Normality: for all $b_0, \dots, b_{n-1} \in B$, if $b_i = 0$ for some $i < n$, then

$$\Omega(b_0, \dots, b_{i-1}, 0, b_{i+1}, \dots, b_{n-1}) = 0$$

- Additivity: for all $b_0, \dots, b_{n-1}, b, b' \in B$ we have

$$\begin{aligned} \Omega(b_0, \dots, b_{i-1}, (b + b'), b_{i+1}, \dots, b_{n-1}) = \\ \Omega(b_0, \dots, b_{i-1}, b, b_{i+1}, \dots, b_{n-1}) + \Omega(b_0, \dots, b_{i-1}, b', b_{i+1}, \dots, b_{n-1}) \end{aligned}$$

2. Let I be an index set, a Boolean algebra with operators (BAO) is an algebra $\langle B, +, -, 0, 1, \{\Omega_i\}_{i \in I} \rangle$ such that $\langle B, +, -, 0, 1 \rangle$ is a Boolean algebra and for each $i \in I$ Ω_i is an operator.

Definition 2. Let $\mathcal{B} = \langle B, +, -, 0, 1, \{\Omega_i\}_{i \in I} \rangle$ be a BAO, then

1. An operator Ω is completely additive, if for each $b_0, \dots, b_{n-1} \in B$ and $X \subseteq B$, one has

$$\Omega(b_0, \dots, b_{i-1}, \sum X, b_{i+1}, \dots, b_{n-1}) = \sum_{x \in X} \Omega(b_0, \dots, b_{i-1}, x, b_{i+1}, \dots, b_{n-1})$$

2. \mathcal{B} is completely additive, if for each $i \in I$ Ω_i is additive,
3. A class \mathcal{K} of BAOs is completely additive, if every $\mathcal{B} \in \mathcal{K}$ is completely additive.

3.1 Atom structures and canonical extensions

Definition 3. Let I be an index set and $\{\Omega_i\}_{i \in I}$ a set of function symbols

1. An atom structure is a relational structure $\mathcal{F} = \langle W, \{R_i\}_{i \in I} \rangle$ such that R_i is a $n+1$ -ary relation symbol, if Ω_i is an n -ary function symbol,
2. Let \mathcal{B} be an atomic BAO of the signature I , the atom structure of \mathcal{B} , written as $\mathbf{At}\mathcal{B}$, is an atom structure $\langle \mathbf{At}(\mathcal{B}), \{R_i\}_{i \in I} \rangle$ such that for each $a, b_0, \dots, b_{n+1} \in \mathbf{At}(\mathcal{B})$ and for each $i \in I$

$$\mathbf{At}\mathcal{B} \models R_i(a, b_0, \dots, b_{n+1}) \text{ iff } \mathcal{B} \models a \leq \Omega_i(b_0, \dots, b_{n+1})$$

3. Let $\mathcal{F} = \langle W, \{R_i\}_{i \in I} \rangle$ be an atom structure, the complex algebra of \mathcal{F} , written as $\mathbf{Cm}\mathcal{F}$, is a BAO $\langle \mathcal{P}(W), \cup, -, \emptyset, W, \{\Omega_{R_i}\}_{i \in I} \rangle$ such that for all $X_0, \dots, X_{n-1} \subseteq W$ and for each $i \in I$

$$\Omega_{R_i}(X_0, \dots, X_{n-1}) = \{a \in W \mid \exists b_0 \in X_0 \dots \exists b_{n-1} \in X_{n-1} \mathcal{F} \models R_i(a, b_0, \dots, b_{n-1})\}$$

The following duality is due to Thomason [7].

Fact 1.

1. Let \mathcal{B} be a complete atomic BAO, then $\mathcal{B} \cong \mathbf{Cm}(\mathbf{At}(\mathcal{B}))$,
2. Let \mathcal{F} be an atom structure, then $\mathcal{F} \cong \mathbf{At}(\mathbf{Cm}(\mathcal{B}))$.

Let A be a non-empty subset of a Boolean algebra \mathcal{B} , A is a *filter*, if A is closed under finite infima and upwardly closed. A is an *ultrafilter*, if it has no non-trivial extensions. That is, if $A \subseteq A'$, then $A' = \mathcal{B}$.

Definition 4. Let $\mathcal{B} = \langle \mathcal{B}, +, -, 0, 1, \{\Omega_i\}_{i \in I} \rangle$ be a BAO and $\mathbf{Uf}(\mathcal{B})$ the set of its ultrafilters. The ultrafilter frame of \mathcal{B} (or canonical frame) is a relational structure $\mathcal{F}_{\mathcal{B}} = \langle \mathbf{Uf}(\mathcal{B}), R_{\Omega_i} \rangle$ such that for each ultrafilters $\beta_0, \dots, \beta_{n-1}, \gamma$ one has

$$\mathbf{Uf}(\mathcal{B}) \models R_{\Omega_i}(\beta_0, \dots, \beta_{n-1}, \gamma) \text{ iff } \{\Omega_i(b_0, \dots, b_{n-1}) \mid b_0 \in \beta_0, \dots, b_{n-1} \in \beta_{n-1}\} \subseteq \gamma.$$

Definition 5. Let \mathcal{B} be a BAO, then

1. The canonical extension of \mathcal{B} is a complex algebra of the canonical frame $\mathbf{Cm}(\mathcal{F}_{\mathcal{B}})$ denoted as \mathcal{B}^+ ,
2. The class of BAOs is canonical, if it is closed under canonical extensions.

Theorem 1. Let \mathcal{A}, \mathcal{B} be BAOs,

1. There exists $\iota : \mathcal{A} \hookrightarrow \mathcal{A}^+$ such that $\iota : a \mapsto \{\gamma \in \mathbf{Uf}(\mathcal{A}) \mid a \in \gamma\}$.
2. If $i : \mathcal{A} \hookrightarrow \mathcal{B}$, then this embedding might be extended to the embedding $i^+ : \mathcal{A}^+ \hookrightarrow \mathcal{B}^+$

Fact 2.

3.2 (Representable) cylindric algebras and cylindric set algebras

Let α be an ordinal. Let ${}^\alpha U$ be the set of all functions mapping α to a non-empty set U . We denote $x(i) = x_i$ for $x \in {}^\alpha U$ and $i < \alpha$.

Definition 6.

1. A subset of ${}^\alpha U$ is an α -ry relation on U . For $i, j < \alpha$, the i, j -diagonal D_{ij} is the set of all elements of U such that $y_i = y_j$.

If $i < \alpha$ and X is an α -ry relation on U , then the i -th cylindrification $C_i X$ is the set of all elements of U that agree with some element of X on each coordinate except the i -th one. To be more precise, $C_i X = \{y \in {}^\alpha U \mid \exists x \in X \forall i < \alpha (i \neq j \Rightarrow y_j = x_j)\}$.

2. A cylindric set algebra of dimension α is an algebra consisting of a set S of α -ry relation on some base set U with the constants and operations $0 = \emptyset$, $1 = {}^\alpha U$, \cap , $-$, the diagonal elements $\{D_{ij}\}_{i,j < \alpha}$, the cylindrifications $\{C_i\}_{i < \alpha}$.

A generalised cylindric set algebra of dimension α is a subdirect of cylindric algebras that have dimension α

3. A cylindric algebra of dimension α is an algebra $\mathcal{C} = \langle \mathcal{B}, \{c_i\}_{i < \alpha}, \{d_{ij}\}_{i,j < \alpha} \rangle$ such that

- \mathcal{B} is a Boolean algebra, for each $i, j < \alpha$ c_i is an operator and $d_{ij} \in \mathcal{B}$
- For each $i < \alpha$, $a \leq c_i a$, $c_i(a \wedge c_i b) = c_i a \wedge c_i b$ and $d_{ii} = 1$
- For every $i, j < \alpha$, $c_i c_j a = c_j c_i a$
- If $k \neq i, j < \alpha$, then $d_{ij} = c_k(d_{ij} \wedge d_{jk})$
- If $i \neq j$, then $c_i(d_{ij} \wedge a) \wedge c_i(d_{ij} \wedge -a) = 0$

\mathbf{CA}_α is the class of all cylindric algebras of dimension α

4. An α -dimensional cylindric algebra C is representable, if it is isomorphic to a generalised cylindric set algebra of dimension α . Such an isomorphism is a representation of C .

\mathbf{RCA}_α is the class of all representable cylindric algebras that have dimension α . In particular, we are interested in the case when $\alpha = \omega$.

It is well known that \mathbf{RCA}_α is a variety, \mathbf{RCA}_α ($\alpha \leq 2$) is finitely axiomatisable and \mathbf{RCA}_α ($2 < \alpha < \omega$) has no finite axiomatisation, see [2].

Let $\mathcal{A} \in \mathbf{C}_\omega$, then \mathcal{A} has a *complete representation*, if this representation preserves all existing suprema.

Let us concretise the definition of a canonical extension for α -dimensional cylindric algebras:

Definition 7. Let $\mathcal{C} = \langle C, +, -, 0, 1, \{d_{ij}\}_{i,j < \alpha}, \{c_i\}_{i < \omega} \rangle$ be a BAO of type \mathbf{CA}_α . Let $\mathbf{Uf}(\mathcal{C})$ be the set of all ultrafilters of \mathcal{C} , the Boolean part of \mathcal{C} .

Let us define $\mathbf{C}_i : \mathbf{Uf}(\mathcal{C}) \rightarrow \mathbf{Uf}(\mathcal{C})$ for each $i, j < \alpha$ as

1. $\mathbf{C}_i \mathcal{X} = \{\mathcal{F} \in \mathbf{Uf}(\mathcal{C}) \mid \exists \mathcal{F}' \in \mathbf{Uf}(\mathcal{C}) (a \in \mathcal{F} \Rightarrow c_i a \in \mathcal{F}' R)\}$,
2. $D_{ij} = \{\mathcal{F} \in \mathbf{Uf}(\mathcal{C}) \mid d_{ij} \in \mathcal{F}\}$.

The structure $\mathcal{C}^+ = \langle \mathbf{Uf}(\mathcal{C}), \cup, -, \emptyset, C, \mathbf{C}_{i < \alpha}, \{D_{ij}\}_{i,j < \alpha} \rangle$ is called the canonical extension of \mathcal{C} .

Let us discuss the connection between representability and canonical extensions.

The following definitions and facts are due to Henkin, Monk, and Tarski [1].

Let $\mathcal{A} \in \mathbf{CA}_\alpha$ and $x \in \mathcal{A}$. Recall that *the dimension of x* is the set of all ordinals $\gamma < \alpha$ such that $c_\gamma x \neq x$. More formally,

$$\Delta x = \{\gamma \mid \gamma < \alpha \ \& \ c_\gamma x \neq x\}$$

Let us discuss some metamathematical intuitions standing behind the notion of a dimension. Let Θ be a first-order theory and $\mathcal{C}/\equiv_\Theta$ its Lindenbaum-Tarski algebra. Let φ be a formula in the signature of Θ . Then $\Delta(\varphi/\Theta)$ consists of all $\kappa < \alpha$ such that $\exists x_\kappa \varphi \leftrightarrow \varphi$ is not valid in Θ . That is, $\Delta(\varphi/\Theta)$ contains ordinals κ for which x_κ is free in φ . Moreover, $\Delta(\varphi/\Theta)$ consists only of those ordinals for which x_κ is free in every $\psi \in \varphi/\Theta$.

In particular, an element x is called *zero-dimensional* if $\Delta x = 0$. Zero-dimensional elements reflect equivalence classes of sentences in the Lindenbaum-Tarski algebra of a given first-order theory. Thus, the set of zero-dimensional elements form a Boolean algebras of sentences associated with Θ .

Definition 8. Let \mathcal{A} be an α -dimensional cylindric algebra. Let α be an ordinal and Γ a subset α , then an element $x \in \mathcal{A}$ is Γ -closed if $\Delta x \cdot \Gamma = \emptyset$. Alternatively, x is a Γ -cylinder.

$\text{Cl}_\Gamma \mathcal{A}$ is the set of all Γ -closed elements.

Metamathematically, Γ -closed elements reflect universal closures (is it correct?).

Let $\mathcal{C} = \langle C, +, -, 0, 1, \{d_{ij}\}_{i,j < \beta}, \{c_c\}_{c < \beta} \rangle$ be a β -dimensional cylindric algebra and $\alpha \leq \beta$ an ordinal. The α -th reduct of \mathcal{C} , denoted as $\mathfrak{Rd}_\alpha \mathcal{C}$, is an algebra having the form

$$\mathfrak{Rd}_\alpha \mathcal{C} = \langle C, +, -, 0, 1, \{d_{ij}\}_{i,j < \alpha}, \{c_c\}_{c < \alpha} \rangle$$

\mathcal{B} is a subreduct of \mathcal{C} , denoted as $\mathcal{B} \sqsubseteq^r \mathcal{C}$, if $\mathcal{B} \subseteq \mathfrak{Rd}_\gamma \mathcal{C}$ for some $\gamma \leq \beta$.

Definition 9. Let \mathcal{C} be a β -dimensional cylindric algebra and α an ordinal such that $\alpha \leq \beta$. The neat α -reduct of \mathcal{C} , denoted as $\mathfrak{Nr}_\alpha \mathcal{C}$, is the subalgebra \mathcal{A} of $\mathfrak{Rd}_\alpha \mathcal{C}$ with $\mathcal{A} = \text{Cl}_\alpha \mathcal{C}$ where $\alpha + \kappa = \beta$.

Let \mathbb{K} be a class of β -dimensional cylindric algebras, then we put

$$\mathbf{Nr}_\alpha \mathbb{K} = \{\mathfrak{Nr}_\alpha \mathcal{C} \mid \mathcal{C} \in \mathbb{K}\}$$

An algebra \mathcal{B} is a neat subreduct of \mathcal{C} , or \mathcal{B} is neatly embeddable to \mathcal{C} if there exists an ordinal $\gamma \leq \alpha$ such that $\mathcal{C} \subseteq \mathfrak{Rd}_\gamma \mathcal{B}$.

One may define neat reducts alternatively as follows. Let \mathcal{C} be a β -dimensional cylindric algebra and α an ordinal such that $\alpha \leq \beta$. The neat α -reduct of \mathcal{C} is the α -dimensional cylindric algebra having the form

$$\mathfrak{Nr}_\alpha \mathcal{C} = \langle \{a \in \mathcal{C} \mid \forall j (\alpha \leq j \ \& \ j < \beta \Rightarrow c_j a = a)\}, +, -, 0, 1, \{d_{ij}\}_{i,j < \alpha}, \{c_\gamma\}_\gamma \rangle$$

We will adapting the following proof of the fact that **RRA** is canonical¹ to our case. This proof is due to Monk, but that was describe in McKenzie's thesis [5].

1. A relation algebra \mathcal{A} is representable iff \mathcal{A} neatly embeds to some omega-dimensional cylindric algebra,
2. If \mathcal{A} neatly embeds in \mathcal{A} then \mathcal{A}^+ neatly embeds in \mathcal{B}^+ ,

¹This idea is by Ian Hodkinson

3. \mathbf{CA}_α is closed under canonical extensions,
4. Voilá.

We describe the following sketch of that proof published by Maddux in [4].

Lemma 1 (Henkin, Monk, Tarski).

Let \mathcal{A} be a BAO of type \mathbf{CA}_α and \mathcal{B} be a β -dimensional cylindric algebra such that $\beta \leq \alpha$ and \mathcal{A} neatly embeds to \mathcal{B} by a complete embedding. Then \mathcal{A}^+ neatly embeds to \mathcal{B}^+ by a complete embedding.

Proof. □

Theorem 2 (This assumption is by Ian Hodkinson).

Let \mathcal{A} be a BAO of type \mathbf{CA}_ω such that \mathcal{A} neatly embeds into $\mathbf{CA}_{\omega+\omega}$ by a complete embedding. Then \mathcal{A} is completely representable as \mathbf{CA}_ω .

Proof. Hmmmmm, I believe so. □

Lemma 1 and Theorem 2 imply the following theorem.

Theorem 3. *Let $\mathcal{C} \in \mathbf{RCA}_\omega$, then $\mathcal{C}^+ \in \mathbf{RCA}_\omega$. That is, \mathbf{RCA}_ω is closed under canonical extensions.*

Proof. :monkahmm: □

References

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