Characterising representable positive relation algebras with Priestley duality

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1 Distributive lattice representation and Priestley duality

Given a bounded distributive lattice $\mathcal{L}$, a proper subset $F \subset \mathcal{L}$ is said to be a *filter* if it is upward closed and closer under finite infima. A filter F is *prime* if $a + b \in F$ implies either $a \in F$ or $b \in F$. The spectrum of $\mathcal{L}$, denoted as $\mathrm{Spec}(\mathcal{L})$, is the set of all prime filters.

A filter is *complete* if whenever ΠT exists for $T \subseteq F$, then $\Pi T \in F$. A filter is *completely prime* if whenever ΣT exists for $T \subseteq F$, then there exists $t \in T$ such that $t \in F$. The dual definitions are for ideals.

Proposition 1. *Let $h : \mathcal{L} \to \mathcal{R}$ be a representation, then then*

$$h^{-1}[x] = \{a \in \mathcal{L} \mid x \in h(a)\} \in \mathrm{Spec}(\mathcal{L})$$

Recall that a *Priestley space* is a triple $\mathcal{X} = (X, \tau, \leqslant)$ such that (X, τ) is a compact topological space, $(X, \leqslant)$ is a bounded poset such that if $x \nleqslant y$, then there exists a clopen U such that $x \in U$ and $y \notin U$. Given a bounded distributive lattice $\mathcal{L}$, define the map $\phi : \mathcal{L} \to 2^{\mathrm{Spec}(\mathcal{L})}$ such that

$$\phi : a \mapsto \{F \in \mathrm{Spec}(\mathcal{L}) \mid a \in F\}$$

Fact 1.

1. *The sets $\phi(a)$ and $-\phi(a)$ form the subbasis of the topology τ on* $\mathrm{Spec}(\mathcal{L})$.
2. $(\mathrm{Spec}(\mathcal{L}), \tau, \subseteq)$ *is a Priestley space.*

Given a Priestley space $\mathcal{X} = (X, \tau, \leqslant)$, the set $\mathrm{ClOp}(\mathcal{X})$ consists of all clopens of $\mathcal{X}$. The structure $(\mathrm{ClOp}(\mathcal{X}), \cap, \cup, \varnothing, X)$ is a distributive lattice.

Fact 2. *Let $\mathcal{L}$ be a distributive lattice and let $\mathcal{X}$ be a Priestley space:*

1. $\mathcal{L} \hookrightarrow \mathcal{L}^+ = (2^{\mathrm{Spec}(\mathcal{L})}, \cap, \cup, \varnothing, \mathrm{Spec}(\mathcal{L}))$,
2. $\mathcal{L} \cong \mathrm{ClOp}(\mathrm{Spec}(\mathcal{L}))$,
3. $\mathcal{X} \cong \mathrm{Spec}(\mathrm{ClOp}(\mathcal{X}))$,
4. *The categories of Priestley spaces and bounded distributive lattices are dually equivalent.*

1.1 Completely representable distibutive lattices

Let $\mathcal{L}$ be a bounded distributive lattice, then a set $S \subseteq 2^{\mathcal{L}}$ is said to be distinguishing if for every $a, b \in \mathcal{L}$ such that $a \neq b$ there exists $s \in S$ such that either $a \in s$ and $b \notin b$ or vice versa.

Theorem 1. *Let $\mathcal{L}$ be a bounded distributive lattice, then*

1. *$\mathcal{L}$ is completely representable iff $\mathcal{L}$ has a distinguishing set of complete, completely prime filters,*
2. *$(\mathcal{L}_+)^+$ is completely representable.*

TODO: read [EH12]

2 Representatiting positive relation algebras

Definition 1. *A positive relation algebra is a algebra $\mathcal{R} = (R, \cdot, +, ;, \smile, 0, 1, \mathbf{1'})$ such that*

1. *$(R, \cdot, +, 0, 1)$ is a bounded distributive lattice,*
2. *$(R, ;, \mathbf{1'})$ is a monoid,*
3. *for all $a, b, c \in R$*
   (a) *$a; (b + c) = a; b + a; c$,*
   (b) *$a^{\smile\smile} = a$,*
   (c) *$(a + b)^{\smile} = a^{\smile} + b^{\smile}$,*
   (d) *$(a; b)^{\smile} = b^{\smile}; a^{\smile}$,*
   (e) *$(a; b) \cdot c^{\smile} = 0 \leftrightarrow (b; c) \cdot c^{\smile} = 0$.*

A positive relation algebra $\mathcal{R}$ is *representable* if there exists a one-to-one function $h : \mathcal{R} \to 2^{X \times X}$ over the base set $X \neq \varnothing$ such that:

- $f(a \cdot b) = f(a) \cap f(b)$,
- $f(a + b) = f(a) + f(b)$,
- $f(0) = \varnothing$,
- $f(1) = \bigcup_{a \in \mathcal{R}} f(a)$,
- $f(\mathbf{1'}) = \Delta_X$,
- $f(a; b) = \{(x, z) \mid \exists y \in X\ ((x, y) \in f(a)\ \&\ (y, z) \in f(b))\} = f(a)|f(b)$,
- $f(a^{\smile}) = \{(y, x) \mid (x, y) \in f(a)\}$.

A positive relation algebra is *completely representable* if it is representable and its bounded distributive lattice reduct is completely representable.

2.1 Priestley duality for positive relation algebras

Given a positive relation algebra $\mathcal{R} = (R, \cdot, +, ;, \smile, \mathbf{1'}, 0, 1)$, let $A, B \subseteq \mathcal{R}$, define the usual pointwise product operation:

$$A; B = \{a; b \mid a \in A, b \in B\}$$

We modify this operation with the upward closure:

$$A \bullet B = \uparrow (A; B)$$

We use the $\bullet$ operation to analyse the following properties of filters in positive relation algebras:

Fact 3. *Let $\mathcal{R}$ be a positive relation algebra, then*

1. *If $F \subseteq \mathcal{R}$ is a filter, then for each $X, Y \subseteq \mathcal{R}$ $X; Y \subseteq F$ iff $X \bullet Y \subseteq F$*
2. *If $F_1, F_2 \subseteq \mathcal{R}$ are filters, so is $F_1 \bullet F_2$,*
3. *Let F_1, F_2 be filters and $F_3 \in \mathrm{Spec}(\mathcal{R})$ such that $F_1 \bullet F_2 \subseteq F_3$, then there are $F_1', F_2' \in \mathrm{Spec}(\mathcal{R})$ such that $F_1 \subseteq F_1'$, $F_2 \subseteq F_2'$ and $F_1' \bullet F_2' \subseteq F_3$.*

First of all, given a positive relation algebra $\mathcal{R} = (R, \cdot, +, ;, \smile, 0, 1, \mathbf{1'})$, we define its canonical extension $\mathcal{R}^+$ on the subsets of the spectrum $\mathrm{Spec}(\mathcal{R})$ by piggybacking Priestley representation of bounded distibutive lattices.

TODO:

A PRA-space is a structure $(X, \tau, \leqslant, R, I, E)$ where $X = (X, \tau, \leqslant)$ is a Priestley space and $R \subseteq X^3$, $I \subseteq X^2$ and $E \subseteq X$ such that:

1. For all $x, y, z \in X$ such that $R(x, y, z)$:
   - $x \geqslant x'$ for $x' \in X$ implies $R(x', y, z)$,
   - $y \leqslant y'$ for $y' \in X$ implies $R(x, y', z)$,
   - $z \leqslant z'$ for $z' \in X$ implies $R(x, y, z')$.
2. For all $x, y, z, w \in X$ there exists $u \in X$ such that $R(x, y, u)$ and $R(u, z, w)$ iff there exists $v \in X$ such that $R(y, z, v)$ and $R(x, v, w)$,
3. If $A, B \subseteq X$ are upward closed, so is $R[A, B, \_]$,
4. $I(A)$ is upward closed clopen whenever A is upward closed clopen,
5. $I(x)$ is closed for each $x \in X$,
6. For all $x, y, z \in X$, $x \leqslant y$ and $I(x, z)$ imply $I(y, z)$,
7. For all $x, y \in X$ there exists $z \in X$ such that $x = y$ iff $I(z, y)$ and $I(x, z)$,
8. For all $x, y, z \in X$ there exists $u \in X$ such that $I(u, z)$ and $R(x, y, u)$ iff there exist $u, w \in X$ such that $R(v, w, z)$, $I(y, v)$ and $I(x, w)$.
9. For all $x, y, u, v \in X$, $R(u, v, y)$ and $I(x, u)$ implies $R(x, y, v)$,
10. E is upward closed clopen such that for each clopen $A \subseteq X$ one has

$$R[A, E, \_] = R[E, A, \_] = A$$

Theorem 2. *Let $\mathcal{R}$ be a positive relation algebra and $\mathcal{X}$ a PRA-space, then*

1. $\mathcal{R} \hookrightarrow \mathcal{R}^+ = (2^{\mathrm{Spec}(R)}, \cup, \cap, \varnothing, \mathrm{Spec}(R), \bullet, \iota, \epsilon)$,
2. $\mathcal{R} \cong \mathrm{ClOp}(\mathrm{Spec}(\mathcal{R}))$,
3. $\mathcal{X} \cong \mathrm{Spec}(\mathrm{ClOp}(\mathcal{X}))$,
4. *The categories of positive relation algebras and PRA-spaces are dually equivalent.*

Proof.

1. We check only the cases of $\bullet$, ι and ϵ.
2. Note that the dual space of a positive relation algebra $\mathcal{R}$ is a PRA-space.
   TODO: check the claim above
3. The dual space of a PRA-space is a positive relation algebra.
   TODO: check the claim above
4.

$\square$

3 The main result

For that we need such model theoretic notions as saturation and types, see [Hod93, Section 6.3].

Definition 2. *Let $\mathcal{M}$ be a first-order structure of a signature L and $S \subseteq \mathcal{M}$. Let $L(S)$ be an extension of L with copies of elements from S as additional constants. We assume that $Cnst(L)$ and S are disjoint.*

1. *Let $n < \omega$, an n-type over S is a set $\mathcal{T}$ of $L(S)$ formulas $A(\overline{x})$, where $\overline{x}$ is a fixed n-tuple of elements from S. Notation: $\mathcal{T}(\overline{x})$. A type is an n-type for some $n < \omega$.*
2. *An n-type $\mathcal{T}(\overline{x})$ is realised in $\mathcal{M}$, if there exists $\overline{m} \in \mathcal{M}^n$ such that $\mathcal{M} \models A(\overline{m})$ for every $A \in \mathcal{T}(\overline{x})$. $\mathcal{M}$ omits $\mathcal{T}(\overline{x})$, if $\mathcal{T}(\overline{x})$ is not realised in $\mathcal{M}$.*
3. *$\mathcal{T}(\overline{x})$ is finitely satisfied in $\mathcal{M}$, if every finite subtype $\mathcal{T}_0(\overline{x}) \subseteq \mathcal{T}(\overline{x})$ is realised in $\mathcal{M}$. We can reformulate that as $\mathcal{M} \models \exists \overline{a} \bigwedge\limits_{A \in \mathcal{T}_0} A(\overline{a})$.*
4. *Let T be a theory, then a type $\mathcal{T}$ over the empty set of constants is T-consistent, if there exists a model $\mathcal{M} \models T$ such that $\mathcal{T}$ is finitely satisfied in $\mathcal{M}$.*
5. *Let κ be a cardinal, then $\mathcal{M}$ is κ-saturated, if for every $S \subseteq \mathcal{M}$ with $|S| < \kappa$ every finitely satisfied 1-type $\mathcal{T}$ is realised in $\mathcal{M}$.*

By default, a saturated model is an ω-saturated model for us.
The useful facts, they are from [CK90] and [Hod93]:

Fact 4. *Let $\mathcal{M}$ be an FO-structue and κ a cardinal, then:*

1. *$\mathcal{M}$ is κ-saturated iff every finitely satisfiable α-type (an arbitrary $\alpha \leqslant \kappa$) with fewer than κ parameters is realised in $\mathcal{M}$.*
2. *If $\mathcal{M}$ is κ-saturated, then $\mathcal{M}$ is λ-saturated for every $\lambda < \kappa$.*
3. *Every consistent theory has a κ-saturated model and every model has an elementary κ-saturated extension.*
4. *Let $(\mathcal{M}_i)_{i<\omega}$ a family of structures of the (at most) countable signature and D a non-principal ultrafilter over ω, then $\Pi_D \mathcal{M}_i$ is ω_1-saturated.*

Let $\mathcal{A}$ be a positive relation algebra, define the first-order relational language of the form

$$\mathcal{L}(\mathcal{A}) = (=, \{R_a^2\}_{a \in \mathcal{A}})$$

The $\mathcal{L}(\mathcal{A})$-theory $T_{\mathcal{A}}$ consists of the following statements:

- $\sigma_1 = \forall x \forall y (\mathbf{1'}(x, y) \leftrightarrow (x = y))$
- $\sigma_+(R, S, T) = \forall x \forall y (R(x, y) \leftrightarrow S(x, y) \vee T(x, y))$
- $\sigma_.(R, S, T) = \forall x \forall y (R(x, y) \leftrightarrow S(x, y) \wedge T(x, y))$
- $\sigma_;(R, S, T) = \forall x \forall y (R(x, y) \leftrightarrow \exists z (S(x, z) \wedge T(z, y)))$
- $\sigma_{\smile}(R, S) = \forall x \forall y (R(x, y) \leftrightarrow S(y, x))$
- $\sigma_{\neq 0} = \exists x \exists y R(x, y)$ for any R_a such that $a \neq 0$
- $\sigma_0 = \neg \exists x \exists y 0(x, y)$
- $\sigma_{\mathbf{1}} = \forall x \forall y (R(x, y) \rightarrow \mathbf{1}(x, y))$

Proposition 2. *$T_{\mathcal{A}}$ is satisfiable whenever $\mathcal{A}$ is representable.*

Theorem 3. *Let $\mathcal{A}$ be a positive relation algebra, then $\mathcal{R}$ is representable iff $\mathcal{R}^+$ is completely representable.*

Proof. The right-to-left implication is easy. If $\mathcal{R}^+$ is representable (no completeness needed here), so is $\mathrm{ClOp}(\mathrm{Spec}(\mathcal{L}))$ as a subalgebra of $\mathcal{R}^+$. But, by Priestley duality for positive relation algebras, $\mathrm{ClOp}(\mathrm{Spec}(\mathcal{L})) \cong \mathcal{L}$, so $\mathcal{L}$ is representable.

Assume that $\mathcal{R}$ is representable, then $T_{\mathcal{R}}$ is satisfiable, let $M \models T_{\mathcal{R}}$ and M is ω-saturated. Define a map $h : \mathcal{R}^+ \to 2^{M \times M}$ as

$$h : F \mapsto \{(x, y) \in 1^M \mid f_{x,y} \in F\}$$

where

$$f_{x,y} = \{a \in \mathcal{R} \mid M \models R_a(x, y)\}$$

Claim 1. *$f_{x,y} \in \mathrm{Spec}(\mathcal{R})$ whenever $M \models 1(x, y)$.*

Proof. Let $a \in f_{x,y}$ and $a \leqslant b$.

Then $M \models R_a(x, y)$. But $a \leqslant b$ iff $a \cdot b = a$, so, from the axiom $R.(R_a, R_a, R_b)$ we have $M \models R_b(x, y)$.

Let $a, b \in f_{x,y}$, so $M \models R_a(x, y) \wedge R_b(x, y)$, so $R_{a \cdot b}(x, y)$ by the axiom $R.(R_a, R_a, R_b)$ again.

Let $a + b \in f_{x,y}$, so $M \models R_{a+b}(x, y)$, then we have either $M \models R_a(x, y)$ or $M \models R_b(x, y)$, so either $a \in f_{x,y}$ or $b \in f_{x,y}$. □

Claim 2. *h is one-to-one.*

Claim 3. *h is a complete representation of the bounded lattice reduct.*

Proof. Follows from the fact that $\mathrm{Spec}(\mathcal{L})$ is a Hausdorff space. (????) $\square$

Claim 4. *h preserves the structure of an involutive monoid.*

$\square$

Theorem 4. **RPRA** *is a canonical variety. (is it a variety at all?)*

4 Union-free reducts of positive relation algebras

TODO: [BJ11].

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