

# Notes on filtrations for logics that contain **K5**

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## 1 Preliminaries

Let  $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$  be a Kripke model and  $\Gamma$  a set of formulas closed under subformulas. An equivalence relation  $\sim$  is set to have a finite index if the quotient set  $W / \sim$  is finite. The equivalence relation  $\sim_\Gamma$  induced by  $\Gamma$  is defined as

$$w \sim_\Gamma v \Leftrightarrow \forall \varphi \in \Gamma (\mathcal{M}, w \models \varphi \Leftrightarrow \mathcal{M}, v \models \varphi).$$

If  $\Gamma$  is finite, then  $\sim_\Gamma$  has a finite index. An equivalence relation  $\sim$  respects  $\sim_\Gamma$ , if  $w \sim v$  implies  $w \sim_\Gamma v$ .

The following definition of a filtration is due to, e.g., [6].

**Definition 1.** Let  $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$  be a Kripke model and  $\Gamma$  be a Sub-closed set formulas. A  $\Gamma$ -filtration of  $\mathcal{M}$  is a model  $\widehat{\mathcal{M}} = \langle \widehat{W}, \widehat{R}_1, \dots, \widehat{R}_n, \widehat{\vartheta} \rangle$  such that:

1.  $\widehat{W} = W / \sim$ , where  $\sim$  is an equivalence relation that respects  $\Gamma$
2.  $\widehat{\vartheta}(p) = \{[x]_\sim \mid x \in W \text{ \& } x \in \vartheta(p)\}$
3. For each  $i \in I$  one has  $\widehat{R}_i^{\min} \subseteq \widehat{R}_i \subseteq \widehat{R}_i^{\max}$ .  $\widehat{R}_{i,\sim}^{\min}$  is the  $i$ -th minimal filtered relation on  $\widehat{W}$  defined as

$$\widehat{x} \widehat{R}_{i,\sim}^{\min} \widehat{y} \Leftrightarrow \exists x' \sim x \exists y' \sim y x R_i y$$

$\widehat{R}_{\Gamma,i}^{\max}$  is the  $i$ -th maximal filtered relation on  $\widehat{W}$  induced by  $\Gamma$  defined as

$$\widehat{x} \widehat{R}_{\Gamma,i}^{\max} \widehat{y} \Leftrightarrow \forall \Box_i \varphi \in \Gamma (\mathcal{M}, x \models \Box_i \varphi \Rightarrow \mathcal{M}, y \models \varphi)$$

Alternatively, one may reformulate the condition of the maximal filtered relations using  $\Diamond$ 's as follows. We will use this formulation occasionally:

$$\widehat{x} \widehat{R}_{\Gamma,i}^{\max} \widehat{y} \Leftrightarrow \forall \Diamond_i \varphi \in \Gamma (\mathcal{M}, y \models \varphi \Rightarrow \mathcal{M}, x \models \Diamond_i \varphi)$$

If  $\Phi$  is an extension of  $\Gamma$  and  $\sim = \sim_\Phi$ , then  $\widehat{\mathcal{M}}$  is a definable  $\Gamma$ -filtration of  $\mathcal{M}$  through  $\Phi$ . If  $\sim = \sim_\Gamma$ , then such a filtration by means of the definition above is called *strict*.

A class of models  $\mathbb{M}$  admits strict filtrations for models (ASF), if for every Sub-closed set  $\Gamma$  and for every  $\mathcal{M} \in \mathbb{M}$  there exists a model  $\widehat{\mathcal{M}}$  such that  $\widehat{\mathcal{M}} \in \mathbb{M}$  and  $\widehat{\mathcal{M}}$  is a filtration of  $\mathcal{M}$  through  $\Gamma$ .

A class of frames  $\mathbb{F}$  admits strict filtrations for frames, if for every Sub-closed set  $\Gamma$  and for every frame  $\mathcal{F} \in \mathbb{F}$  and every model  $\mathcal{M}$  over  $\mathcal{F}$  there exists a  $\Gamma$  filtration of  $\mathcal{M}$ , and the underlying frame of this filtration belongs to  $\mathbb{F}$ .

If  $\mathcal{L}$  is canonical, then the ASF property for frames and ASF property for models are equivalent, see [4, Theorem 2.10].

The key lemma about filtrations is the following, see [1, Theorem 2.39]:

**Lemma 1.** *Let  $\Gamma$  be a finite set of formulas closed under subformulas and  $\widehat{\mathcal{M}}$  a filtration of  $\mathcal{M}$  through  $\Gamma$ , then for each  $x \in W$  and for each  $\varphi \in \Gamma$  one has*

$$\mathcal{M}, x \models \varphi \Leftrightarrow \widehat{\mathcal{M}}, \hat{x} \models \varphi$$

**Definition 2.**

1. *Let  $\mathbb{F}$  be a class of Kripke frames and  $\Gamma$  a finite set of formulas closed under subformulas. If for every model  $\mathcal{M}$  over  $\mathcal{F} \in \mathbb{F}$  there exists a model that is a  $\Gamma$ -definable filtration of  $\mathcal{M}$ , then  $\mathbb{F}$  admits definable filtration.*
2. *A class of models  $\mathbb{M}$  admits definable filtration if for every  $\mathcal{M} \in \mathbb{M}$  there exists a model belonging to the same class that is a definable  $\Gamma$ -filtration of  $\mathcal{M}$ .*

**Lemma 2.**

1. *Let  $\mathcal{L}$  be a complete normal modal logic. If  $\text{Frames}(\mathcal{L})$  admits filtration, then  $\mathcal{L}$  has the finite model property.*
2. *If the class of models  $\text{Mod}(\mathcal{L})$  admits filtration, then  $\mathcal{L}$  has the finite model property and it is Kripke complete as well.*

**Definition 3.** *A first-order formula is called Horn if it has the following form (see [3]):*

$$\forall x_1, \dots, x_n (x_{i_1} R x_{j_1} \wedge \dots \wedge x_{i_s} R x_{j_s} \rightarrow A), \text{ where } A \text{ is either } x_k R x_l \text{ or } \perp.$$

**Definition 4.** *Let  $H$  be a Horn property and  $\langle W, R \rangle$  a Kripke frame. A Horn closure of a binary relation  $R$  is the minimal relation  $R^H$  containing  $R$  and satisfying  $H$ .*

**Lemma 3.**  $R^H = \bigcup_{n < \omega} R_n$  where

1.  $R_0 = R$ .
2.  $R_{n+1} = R_n \cup \{(a, b) \in W \mid \exists \vec{c} \in W \text{ } P(a, b, \vec{c})\}$ , where  $P$  is a premise of  $H$ .

## 2 Filtrations for K5

The **K5**-closure (an Euclidean Horn closure of a binary relation) has the following equivalent definitions:

**Lemma 4.** *Let  $\mathcal{F} = \langle W, R \rangle$  be a Kripke frame and  $R \subseteq R^{\mathbf{K5}}$ . The following conditions are equivalent:*

1.  $R^{\mathbf{K5}}$  is the smallest Euclidean relation containing  $R$ , that is, the Horn closure of  $R$ .
2.  $R^{\mathbf{K5}} = \bigcup_{i < \omega} R_i$ , where
  - $R_0 = R$
  - $R_{n+1} = R_n \cup (R_n^{-1} \circ R_n)$

3.  $xR^{\mathbf{K5}}y$  iff there exists  $n < \omega$  such that either  $xRy$  or  $\exists z_1, \dots, z_n$  with  $z_1Rx$  and  $z_{n-1}Ry$  and for each  $1 < i \leq n$  one has either  $z_{i-1}Rz_i$  or  $z_iRz_{i-1}$ .
4.  $R^{\mathbf{K5}} = R \cup \bigcup_{i < \omega} (R^{-1} \circ (R \cup R^{-1})^n \circ R)$ .

*Proof.*

1. (1)  $\Rightarrow$  (2) Let us show that if  $R^E$  is the smallest Euclidean relation containing  $R$ , then  $R^E = \bigcup_{i < \omega} R_i$ . There are two inclusions:
- $R^E \subseteq \bigcup_{i < \omega} R_i$ . Recall that  $R^E$  has the form (?):  

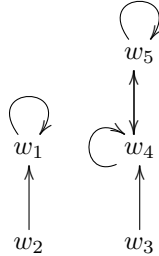
$$R^E = \bigcap \{R' \mid R \subseteq R', \forall a, b \in W \ R'(a, b) \Rightarrow \exists x \in W \ R'(x, a) \ \& \ R'(x, b)\}$$
  - $\bigcup_{i < \omega} R_i \subseteq R^E$ . Let us show that  $xR_ny$  for each  $n < \omega$  implies  $xR^E y$  by induction on  $n$ .  
 If  $n = 0$ , then  $xRy$ , thus,  $xR^E y$ , since  $R$  is a subrelation of  $R^E$ . Suppose  $n = m + 1$  and  $xR_{m+1}y$ . Let us show that  $xR^E y$ . From  $xR_{m+1}y$ , one has  $(x, y) \in R^n \cup (R_n^{-1} \circ R_n)$ . There are two cases:
    - $xR^n y$ , one needs to merely apply the IH.
    - $xR_n^{-1} \circ R_n y$ . Then  $\exists z \in W \ xR_n^{-1}z \ \& \ zR_n y$ . That is,  $zR_n x$  and  $zR_n y$  for some  $z$ .  $R_n$  is already a subrelation of  $R^E$ . Thus,  $zR^E x$  and  $zR^E y$ . That implies  $xR^E y$ .
2. (2)  $\Rightarrow$  (3) Let  $(x, y) \in R_m$ , let us the statement by induction on  $m$ .
- (a) Suppose  $m = 0$ , then  $xRy$ , and the statement is shown putting  $n = 0$ .
- (b) Suppose  $m = p + 1$  and  $xR_{p+1}y$ . Assume that either  $xRy$  or  $\exists z_1, \dots, z_p$  with  $z_1Rx$  and  $z_{p-1}Ry$  and for each  $1 < i \leq p$  one has either  $z_{i-1}Rz_i$  or  $z_iRz_{i-1}$ .  
 $xR_{p+1}y$  implies  $(x, y) \in R_p \cup (R_p^{-1} \circ R_p)$ . If  $(x, y) \in R_p$ , then we merely apply the IH. Suppose  $(x, y) \in R_p^{-1} \circ R_p$ , then  $(z, x) \in R_p$  and  $(z, y) \in R_p$ .
3. (3)  $\Rightarrow$  (4) Suppose either  $xRy$  or there exist  $n \geq 1$  and  $z_1, \dots, z_n$  with  $z_1Rx$  and  $z_{n-1}Ry$  and for each  $1 < i \leq n$  one has either  $z_{i-1}Rz_i$  or  $z_iRz_{i-1}$ . If  $xRy$ , then we are done. Otherwise there exists  $n \geq 1$  with the condition above. Then  $(x, y) \in R_{n+1}$  that follows from the condition.
4. (4)  $\Rightarrow$  (1)

□

**Theorem 1.** *K5 does not admit strict filtrations.*

*Proof.* Let us consider a **K5** model whose Euclidean closure of the minimal filtration does not give us a filtration.

Let us consider a frame called  $\mathcal{F}_{bad}$ . We define this frame with the following graph:



Let us define a valuation  $\vartheta$  such that  $\vartheta(p) = \{w_5\}$  and  $\vartheta(q) = \{w_1\}$ . Let us consider a minimal filtration of  $\mathcal{M}_{bad}$  through the Sub-closure of  $\Gamma = \{\neg p, \neg \Diamond p\}$ .

Clearly that  $w_2 \sim_\Gamma w_3$ , since  $\neg p$  and  $\neg \Diamond p$  are true both at  $w_2$  and  $w_3$ .

Moreover,  $R_{min} \cup (R_{min}^{-1} \circ R_{min})$  is not a subset of  $R_{max}$  since  $(\hat{w}_1, \hat{w}_5) \in (R_{min}^{-1} \circ R_{min})$ , but  $\Diamond p$  is not true at  $w_5$ .

Let us also note that strict filtrations of this model is not Euclidean. Suppose by contrary that  $\hat{R}^\mathcal{E}$  is a strict filtration of that model. So  $R_{min}^E \subseteq \hat{R}^\mathcal{E}$ , since  $R_{min}^E$  is the minimal Euclidean relation containing  $R_{min}$ . On the other hand,  $R_{min}^E \not\subseteq R_{max}$ , so is not  $\hat{R}^\mathcal{E}$ .  $\square$

**Theorem 2.** **K5** admits definable filtrations.

**Theorem 3.** **K45** admits strict filtrations.

*Proof.* Let  $\mathcal{M} = \langle W, R, \vartheta \rangle$  be a transitive Euclidean model and  $\overline{\mathcal{M}} = \langle \overline{W}, \overline{R}, \overline{\vartheta} \rangle$  its minimal filtration through  $\Gamma$ , where  $\Gamma$  is finite and Sub-closed. Let us put  $\hat{R} = \overline{R}^+ \cup \overline{R}^{\mathbf{K5}}$ . Let us show that  $\overline{R}^+ \cup \overline{R}^{\mathbf{K45}} \subseteq \overline{R}^{max}$ .

That is, if  $\mathcal{M}, y \models \varphi$  for  $\Diamond \varphi \in \Gamma$  and  $\hat{x}\hat{R}\hat{y}$ , then  $\mathcal{M}, x \models \Diamond \varphi$ .

Let  $\hat{x}\hat{R}\hat{y}$ .

1. Suppose  $(\hat{x}, \hat{y}) \in \overline{R}$ , then  $\mathcal{M}, x \models \Diamond \varphi$  holds trivially by the definition of the minimal filtration.
2. Let us consider the case when  $(\hat{x}, \hat{y}) \in \overline{R}^{\mathbf{K5}}$ . The second alternative is the same as for the **K4**-case, see [2, p. 141].

Suppose the statement holds  $\overline{R}_n$  and  $(\hat{x}, \hat{y}) \in \overline{R}_{n+1} = \overline{R}_n \cup (\overline{R}_n^{-1} \circ \overline{R}_n)$ . We consider the case of  $(\hat{x}, \hat{y}) \in (\overline{R}_n^{-1} \circ \overline{R}_n)$ .

Then there exists  $\hat{z}$  such that  $(\hat{z}, \hat{x}), (\hat{z}, \hat{y}) \in \overline{R}_n$ .

By IH,  $\mathcal{M}, z \models \Diamond \varphi$ .

$(\hat{z}, \hat{y}) \in \overline{R}_n$  iff there are  $\hat{u}_1, \dots, \hat{u}_n$  such that

$$\hat{z} \xleftarrow{\hat{R}} \hat{u}_1 \xrightarrow{\hat{R}'} \hat{u}_2 \xrightarrow{\hat{R}'} \dots \xrightarrow{\hat{R}'} \hat{u}_{n-1} \xrightarrow{\hat{R}'} \hat{u}_n \xrightarrow{\hat{R}} \hat{y}$$

where  $\hat{R}'$  is either  $\hat{R}$  or  $\hat{R}^{-1}$ .

As it is known,  $\Diamond \Diamond \varphi \rightarrow \Box \Diamond \varphi \in \mathbf{K45}$ .

$\hat{u}_1 \hat{z}$ , that is,  $u'_1 R z'$  for some  $u'_1 \in \hat{u}_1$  and  $z' \in \hat{z}$ . That is,  $\mathcal{M}, u'_1 \models \Diamond \Diamond \varphi$ , so  $\mathcal{M}, u'_1 \models \Diamond \varphi$  and  $\overline{\mathcal{M}}, \hat{u}_1 \models \Diamond \varphi$ .

We have  $\hat{u}_1 \hat{R}' \hat{u}_2$ . Suppose  $\mathcal{M}, u''_1 \models \Diamond \varphi$  and  $u''_1 R u'_2$ . We also have  $\mathcal{M}, u''_1 \models \Box \Diamond \varphi$ , thus,  $\mathcal{M}, u'_2 \models \Diamond \varphi$ .

Suppose  $\hat{u}_2 \hat{R} \hat{u}_1$  and  $u'_2 R u''_1$ , then  $\mathcal{M}, u'_2 \models \Diamond \varphi$ .

Similarly, we have  $\mathcal{M}, u_i \models \Diamond \varphi$  iff  $\mathcal{M}, u_{i+1} \models \Diamond \varphi$ , whenever  $\hat{u}_i \hat{R}' \hat{u}_{i+1}$ .

Finally, we have  $\hat{u}_n \hat{R} \hat{x}$ . Thus,  $u'_n R x'$  for some  $u'_n \in \hat{u}_n$  and  $x' \in \hat{x}$ .  $\mathcal{M}, u'_n \models \Diamond \varphi$ , so  $\mathcal{M}, u'_n \models \Box \Diamond \varphi$ . Then  $\mathcal{M}, x' \models \Diamond \varphi$ .  $\square$

### 3 i,j-Euclideaness

A binary relation  $R \subseteq W \times W$  is called  $i, j$ -Euclidean for  $i, j < \omega$ , if for each  $x, y, z$  such that  $xR^i y$  and  $xR^j z$  implies  $xRz$ .

**Proposition 1.** *Let  $\mathcal{F} = \langle W, R \rangle$ , then  $\mathcal{F}$  is  $i, j$ -Euclidean iff  $\mathcal{F} \models \Diamond^i p \rightarrow \Box^j \Diamond p$ . As a corollary, the logic  $\mathbf{K5}^{i,j} = \mathbf{K} \oplus \Diamond^i p \rightarrow \Box^j \Diamond p$  is Kripke-complete.*

Let  $R$  be a binary relation on  $W \neq \emptyset$ , then the  $i, j$ -Euclidean closure of  $R$  (where  $i, j < \omega$ ), denoted as  $R^{\mathbf{K5}^{i,j}}$ , is a binary relation defined recursively as follows:

1.  $R_0 = R$
2.  $R_{n+1} = R_n \cup (((R_n)^{-1})^i \circ R_n^j)$
3.  $R^{E_{i,j}} = \bigcup_{k < \omega} R_k$

Note that the following property holds not for every  $\mathbf{K5}^{i,j}$ , consider  $\mathbf{KB}$  as an example (clearly,  $\mathbf{KB} = \mathbf{K5}^{0,1}$ ).

**Lemma 5.**  $\mathbf{K5}^{i,j}$  has  $n$  non-equivalent sequences of modalities, if . *TODO: define  $n$ .*

**Theorem 4.**  $\mathbf{K5}^{i,j}$  admits definable filtrations.

*Proof.* Let  $\mathcal{M} = \langle W, R, \vartheta \rangle$  and  $\Gamma$  a finite Sub-closed of formulas. We extend  $\Gamma$  as

$$\Delta = \Gamma \cup \text{Sub}\{\Diamond^i \psi \mid \Box \psi \in \Gamma\} \cup \text{Sub}\{\Box^j \Diamond \psi \mid \Diamond \psi \in \Gamma\}$$

Let  $(\hat{x}, \hat{y}) \in R^{E_{i,j}}$ ,  $\mathcal{M}, x \models \Box \psi$  for  $\Box \psi$  in  $\Delta$ . If  $n = 0$ , then the statement is obvious. Suppose  $n = 1$ . Then  $(\hat{x}, \hat{y}) \in R_{\min} \cup (((R_{\min})^{-1})^i \circ R_{\min}^j)$ . Consider the second alternative. Then there exists  $\hat{z}$  such that  $(\hat{x}, \hat{z}) \in ((R_{\min})^{-1})^i$  and  $(\hat{z}, \hat{y}) \in R_{\min}^j$ .

That is, there are  $\hat{x}_1, \hat{x}_2, \dots, \hat{x}_i$  and  $\hat{y}_1, \hat{y}_2, \dots, \hat{y}_j$  such that

$$\hat{z} \xrightarrow{R_{\min}} \hat{x}_1 \xrightarrow{R_{\min}} \hat{x}_2 \xrightarrow{R_{\min}} \dots \xrightarrow{R_{\min}} \hat{x}_i \xrightarrow{R_{\min}} \hat{x}$$

$$\hat{z} \xrightarrow{R_{\min}} \hat{y}_1 \xrightarrow{R_{\min}} \hat{y}_2 \xrightarrow{R_{\min}} \dots \xrightarrow{R_{\min}} \hat{y}_j \xrightarrow{R_{\min}} \hat{y}$$

???

□

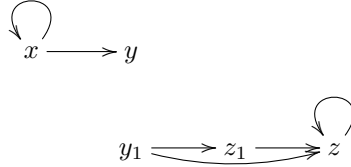
### 4 The case of 2-transitivity

Let us define the logic  $\mathcal{L}$  as  $\mathbf{K} \oplus \Diamond \Diamond p \rightarrow \Diamond p$ . Let  $R$  be a binary relation, the  $\mathcal{L}$ -closure of  $R$  is defined (denoted as  $R^{\star}$ ) as the following union:

$$R^{\star} = R \cup R^3 \cup R^5 \cup \dots \cup R^{2k+1} \cup \dots$$

**Theorem 5.**  $\mathcal{L}$  does not admit strict filtrations.

*Proof.* Consider the following frame  $\mathcal{F} = \langle W, R \rangle$ :



Clearly that  $\mathcal{F}$  is an  $\mathcal{L}$ -frame. We define the valuation  $\vartheta$  as follows:

$$\begin{aligned}\vartheta(p) &= \{x\} \\ \vartheta(q) &= \{y, y_1\} \\ \vartheta(r) &= \{z\}\end{aligned}$$

Let us put  $\Gamma = \text{Sub}\{p, q, \Diamond r\}$ . We factorise  $W$  through  $\sim_\Gamma$  and consider a model  $\widehat{\mathcal{M}} = \langle W / \sim_\Gamma, \widehat{R}, \widehat{\vartheta} \rangle$ , where  $\widehat{R} = (\widehat{R}_{\min})^\Delta$ . We have  $(\hat{x}, \hat{z}) \in \widehat{R} \circ \widehat{R} \circ \widehat{R}$ , but  $\Diamond r$  is not true at  $x$ .  $\square$

## 5 Finite “canonical” models

Let  $\mathcal{L}$  be a normal modal logic,  $\mathcal{M}_\mathcal{L}$  its canonical model, and  $\Gamma$  a finite Sub-closed set of formulas. Let us put  $\Gamma' = \text{Sub}(\varphi) \cup \{\neg\psi \mid \psi \in \text{Sub}(\varphi)\}$ .

A subset  $\Delta \subseteq' \Gamma$  is a *finite  $\mathcal{L}$ -consistent set* if  $\neg \bigwedge \Delta \notin \mathcal{L}$ . A subset  $\Delta$  is maximal, if (the following are obviously equivalent):

1.  $\Delta$  is maximal amongst finite  $\mathcal{L}$ -consistent sets,
2. For each  $\psi \in \text{Sub}(\varphi)$  either  $\psi \in \Delta$  or  $\neg\psi \in \Delta$ .

Every finite  $\mathcal{L}$ -theory is clearly can be extended to some maximal one. It is the finite version of Lindenbaum’s lemma.

**Definition 5.** Let  $\mathcal{L}$  be a modal logic and  $\Gamma$  be a finite Sub-closed set of formulas. A finite “canonical” model is a triple  $\mathcal{M}_\mathcal{L}^\Gamma = \langle W_\mathcal{L}^\Gamma, R_\mathcal{L}^\Gamma, \vartheta_\mathcal{L}^\Gamma \rangle$ , where

1.  $W_\mathcal{L}^\Gamma$  is the set all maximal theories that extend finite  $\mathcal{L}$ -theories
2.  $R_\mathcal{L}^\Gamma$  is a relation such that  $\langle W_\mathcal{L}^\Gamma, R_\mathcal{L}^\Gamma \rangle$  is an  $\mathcal{L}$ -frame and

$$\forall \Box\psi \in \text{Sub}(\varphi) \quad \forall \Delta_1 \in W_\mathcal{L}^\Gamma \quad (\Box\psi \in \Delta_1 \Leftrightarrow \forall \Delta_2 \in R_\mathcal{L}^\Gamma(\Delta_1) \quad \psi \in \Delta_2)$$

3.  $\vartheta_\mathcal{L}^\Gamma(p) = \{\Delta \in W_\mathcal{L}^\Gamma \mid p \in \Delta\}$  for every variable  $p \in \Gamma$ .

**Lemma 6.** Let  $\mathcal{L}$  be a modal logic and  $\varphi \notin \mathcal{L}$ , then  $\mathcal{M}_\mathcal{L}^{\text{Sub}(\varphi)} \not\models \varphi$ .

**Lemma 7.** Let  $\mathcal{L}$  be a modal logic and  $\Gamma$  a finite Sub-closed set of formulas, then if  $\mathcal{L}$  admits strict filtrations, then there exists a finite “canonical” model  $\mathcal{M}_\mathcal{L}^\Gamma$  such that  $\mathcal{M}_\mathcal{L}^\Gamma \models \mathcal{L}$ .

*Proof.* ( $\Rightarrow$ ) Let  $\Gamma$  be a finite Sub-closed of formulas.  $\mathcal{L}$  admits strict filtrations, so the filtration of the canonical model  $\mathcal{M}_\mathcal{L}$  through  $\Gamma$  is also an  $\mathcal{L}$ -model. The underlying set of  $\mathcal{M}_\mathcal{L} / \sim_\Gamma$  consists of maximal  $\mathcal{L}$  theories up to  $\Gamma$ -equivalence and this quotient set is finite.

It is readily checked that the quotient model  $\mathcal{M}_\mathcal{L} / \sim_\Gamma$  satisfies Definition 5.  $\square$

The converse implication does not have to true generally. **GL** might be an example of a logic that has the “finite canonical” model property with no filtrations.

## 6 Fusion stuff

**Definition 6.** Let  $\mathcal{L}_1$  and  $\mathcal{L}_2$  be modal logics, then the fusion  $\mathcal{L}_1 * \mathcal{L}_2$  is the minimal bimodal logic that contains  $\mathcal{L}_1$  and  $\mathcal{L}_2$  [5].

**Lemma 8.**

- Let  $\Gamma$  be a finite and Sub-closed set of formulas and  $\mathcal{M} = \langle W, R, \vartheta \rangle$ . Consider  $\Gamma' = \Gamma \cup \{\Diamond\Box\psi \mid \Box\psi \in \Gamma\}$ . Let  $\Delta$  be any finite and Sub-closed extension of  $\Gamma'$ . Then a model  $\widehat{\mathcal{M}} = \langle W / \sim_\Delta, (R_\Delta^{min})^E, \widehat{\vartheta} \rangle$  is a filtration of  $\mathcal{M}$  through  $\Delta$ .
- The same for  $\mathbf{K} \oplus \Diamond\Diamond\Diamond p \rightarrow \Diamond p$

*Proof.* Recall that  $(R_\Delta^{min})^E$  is defined inductively as:

1.  $R_\Delta^0 = R_\Delta^{min}$
2.  $R_\Delta^{n+1} = R_\Delta^n \cup (R_\Delta^{n-1} \circ R_\Delta^n)$
3.  $(R_\Delta^{min})^E = \bigcup_{k < \omega} R_\Delta^k$

If  $R_\Delta^{min}$  is already a subrelation of  $R^{max}$ , so the base case is self-evident.

Suppose the statement holds for  $R_\Delta^n$ ,  $(\hat{x}, \hat{y}) \in R_\Delta^{n+1}$ , that is, there exists  $\hat{z}$  such that  $(\hat{z}, \hat{x}), (\hat{z}, \hat{y}) \in R_\Delta^n$ . Let  $\mathcal{M}, x \models \Box\psi$  (for  $\Box\psi \in \Gamma$ ).

We rewrite  $(\hat{z}, \hat{x}) \in R_\Delta^n$  as the following sausage (for some  $\widehat{u}_1, \widehat{u}_2, \dots, \widehat{u}_{n-1}, \widehat{u}_n$ ):

$$\hat{z} \xleftarrow{R_\Delta^{min}} \widehat{u}_1 \xrightarrow{R'} \widehat{u}_2 \xrightarrow{R'} \dots \xrightarrow{R'} \widehat{u}_{n-1} \xrightarrow{R'} \widehat{u}_n \xrightarrow{R_\Delta^{min}} \hat{x}$$

where  $R'$  is either  $R_\Delta^{min}$  or its converse. We have  $\mathcal{M}, z \models \Box\psi$ , since  $\mathcal{M}, x \models \Box\psi$  implies  $\mathcal{M}, u_n \models \Diamond\Box\psi$ . After that we apply the following property of  $\mathbf{K5}$ -models:

For each  $a, b \in M_i$  such that  $aR_ib$  we have  $\mathcal{M}_i, a \models \Diamond\Box p_\psi$  iff  $\mathcal{M}_i, b \models \Diamond\Box p_\psi$

Note that we always stay within  $\Delta$  since  $\Gamma' \subseteq \Delta$ . Finally, we conclude  $\mathcal{M}, x \models \psi$  from IH and  $\mathcal{M}, z \models \Box\psi$ .

The second item is shown similarly. □

**Theorem 6.**

1.  $\mathbf{K5} * \mathbf{K5}$  admits definable filtrations.
2.  $\mathbf{K5} * \dots * \mathbf{K5}$  admits definable filtrations.
3. If  $\mathcal{L}$  admits strict filtrations, then  $\mathbf{K5} * \mathcal{L}$  admits definable filtrations
4. If  $\mathcal{L}_1, \dots, \mathcal{L}_n$  admit strict filtrations, then  $\mathbf{K5} * \dots * \mathbf{K5} * \mathcal{L}_1 * \dots * \mathcal{L}_n$
5. Let  $\mathcal{L} = \mathbf{K} \oplus \Diamond\Diamond\Diamond p \rightarrow \Diamond p$  (here and below), then  $\mathcal{L} * \mathcal{L}$  admits definable filtrations.
6. Let  $\mathcal{L} = \mathbf{K} \oplus \Diamond\Diamond\Diamond p \rightarrow \Diamond p$  and  $\mathcal{L}_1$  a logic that admits strict filtrations, then  $\mathcal{L} * \mathcal{L}_1$

*Proof.*

1. Let  $\Gamma$  be a finite Sub-closed set of bimodal formulas,  $\mathcal{F} = \langle W, R_1, R_2 \rangle$  a  $\mathbf{K5} * \mathbf{K5}$ -frame, and  $\vartheta$  a valuation on  $\mathcal{F}$ . Denote  $\langle \mathcal{F}, \vartheta \rangle$  as  $\mathcal{M}$ .

We introduce the set of fresh variables  $V = \{p_\psi \mid \psi \in \Gamma\}$  and define a new model  $\mathcal{M}' = \langle \mathcal{F}, \vartheta' \rangle$  as follows:

$$\text{For all } \psi \in \Gamma, \mathcal{M}, x \models \psi \Leftrightarrow \mathcal{M}', x \models \psi \Leftrightarrow \mathcal{M}', x \models p_\psi.$$

Consider these modifications of  $\Gamma$  and  $V$ :

$$\begin{aligned}\Gamma' &= \Gamma \cup \{\Diamond_1 \Box_1 \psi \mid \Box_1 \psi \in \Gamma\} \cup \{\Diamond_2 \Box_2 \psi \mid \Box_2 \psi \in \Gamma\} \\ \Delta &= V \cup \text{Sub}(\{\Diamond \Box p_\psi \mid \Box_i \psi \in \Gamma, in = 1, 2\})\end{aligned}$$

Let us define an equivalence relation  $\sim_\Delta$  induced by  $\Delta$ .

Consider  $\mathcal{M}_i = \langle W, R_i, \vartheta' \rangle$ , a reduct of  $\mathcal{M}'$ , we have:

- (a)  $\mathcal{M}_i, x \models \Box p_\psi$  iff  $\mathcal{M}, x \models \Box_i \psi$
- (b)  $\mathcal{M}_i, x \models \Diamond \Box p_\psi$  iff  $\mathcal{M}, x \models \Diamond_i \Box_i \psi$

So  $\sim = \sim_{\Gamma'}$  by the construction. Let us put  $\widehat{W} = W / \sim_{\Gamma'}$ . Lemma 8 implies the following claim:

**Claim 1.** *Let  $\widehat{R}_i = (R_\Delta^{min})^E$  and  $\widehat{\vartheta}(p) = \{[x]_{\sim_i} \mid \mathcal{M}_i, x \models p\}$  for  $p \in \Delta_1$ , define  $\widehat{\mathcal{M}}_i = \langle \widehat{W}, \widehat{R}_i, \widehat{\vartheta} \rangle$ . Then  $\widehat{\mathcal{M}}_i \models \mathbf{K5}$  and  $\widehat{\mathcal{M}}_i$  is a filtration of  $\mathcal{M}_i$  through  $\Delta$ .*

Finally, we consider a model  $\widehat{\mathcal{M}} = \langle \widehat{W}, \widehat{R}_1, \widehat{R}_2, \widehat{\vartheta} \rangle$ , where  $\widehat{R}_{\Gamma'_i} = R_{i\Gamma'}^{min^E}$  and  $\vartheta(p)$  is defined as usual for  $p \in \Gamma$ .  $\widehat{\mathcal{M}}$  is a filtration of  $\mathcal{M}$  through  $\Gamma'$ .

Let  $\hat{x}\widehat{R}_{\Gamma'_i}\hat{y}$  and  $\mathcal{M}, x \models \Box_i \psi$  for  $\Box_i \psi \in \Gamma$ . Then  $\mathcal{M}_i, x \models \Box p_\psi$ , so  $\widehat{\mathcal{M}}_i, \hat{x} \models \Box p_\psi$ . By the claim above,  $\widehat{\mathcal{M}}_i$  is a filtration of  $\mathcal{M}_i$  through  $\Delta$ , so  $\mathcal{M}_i, y \models p_\psi$ . Then  $\mathcal{M}, y \models \psi$ .

2. Likewise
3. The argument is the same as in the proof of the first item, except for **Claim 1** that has the following formulation: Let  $\widehat{R}_1 = (R_\Delta^{min})^E$  and  $\widehat{R}_2 = (R_\Delta^{min})^{\mathcal{L}_1}$ . Define a valuation as usual as  $\widehat{\vartheta}(p) = \{[x]_{\sim_i} \mid \mathcal{M}_i, x \models p\}$  for  $p \in \Delta_1$ , define  $\widehat{\mathcal{M}}_1 = \langle \widehat{W}, \widehat{R}_1, \widehat{\vartheta} \rangle$  and  $\widehat{\mathcal{M}}_2 = \langle \widehat{W}, \widehat{R}_2, \widehat{\vartheta} \rangle$ . Then  $\widehat{\mathcal{M}}_1 \models \mathbf{K5}$  and  $\widehat{\mathcal{M}}_1 \models \mathbf{L}$  and  $\widehat{\mathcal{M}}_i$  is a filtration of  $\mathcal{M}_i$  through  $\Delta$ .
4. Likewise
5. The argument is similar to the proof of first item, but filtrations are slightly different. Let  $\mathcal{M} = \langle W, R_1, R_2, \vartheta \rangle$  be a  $\mathcal{L} * \mathcal{L}$  model and  $\Gamma$  a Sub-closed set of formulas. As above, we define a set  $V$  and a model  $\mathcal{M}'$ . Define extensions of  $\Gamma$  and  $V$ :

$$\begin{aligned}\Gamma' &= \Gamma \cup \{\Diamond_1 \Diamond_1 \psi \mid \Diamond_1 \psi \in \Gamma\} \cup \{\Diamond_2 \Diamond_2 \psi \mid \Diamond_2 \psi \in \Gamma\} \\ \Delta &= V \cup \text{Sub}(\{\Diamond \Diamond p_\psi \mid \Diamond \psi \in \Gamma, i = 1, 2\})\end{aligned}$$

As above  $\sim_\Delta = \Gamma'$  and  $\widehat{\mathcal{M}}' = \langle W / \sim_\Delta, \widehat{R}_i, \widehat{\vartheta} \rangle$  are filtrations of reducts of  $\mathcal{M}'$  through  $\Delta$ . Then  $\widehat{\mathcal{M}} = \langle W / \sim_\Delta, \widehat{R}_1, \widehat{R}_2, \widehat{\vartheta} \rangle$  is a required filtration of the original  $\mathcal{M}$ .

6. Extend  $\Gamma$  with  $\{\Diamond_i \Diamond_i \psi \mid \Diamond_i \psi \in \Gamma, i = 1, 2\}$  and  $V$  with  $\{\Diamond \Diamond p_\psi \mid \Diamond_i \psi, i = 1, 2\}$

□

**Theorem 7.** *Let  $\mathcal{L}_1$  and  $\mathcal{L}_2$  be modal logics that admit definable filtrations. If  $\text{Mod}(\mathcal{L}_1)$  and  $\text{Mod}(\mathcal{L}_2)$  admit definable filtrations, so does  $\text{Mod}(\mathcal{L}_1 * \mathcal{L}_2)$ .*

*Proof.* Let  $\mathcal{M} = \langle W, R_1, R_2, \vartheta \rangle$  be an  $\mathcal{L}_1 * \mathcal{L}_2$ . We define a notation  $\nabla = \{\neg \Diamond, \Diamond \neg, \Diamond\}$ .

Both logics admit definable filtrations, so for every finite Sub-closed set  $\Gamma$  and for every  $\mathfrak{M}$ , an  $\mathcal{L}_1$ -model (or an  $\mathcal{L}_2$  one) there exists there exists  $\Delta$ , a extension of  $\Gamma$  having the form:



$$\begin{aligned}\Delta_1 &= \Gamma \cup \text{Sub}\{\nabla_1 \nabla_2 \dots \nabla_n \diamond \psi \mid \diamond \psi \in \Gamma\} \text{ (for } \mathcal{L}_1) \\ \Delta_2 &= \Gamma \cup \text{Sub}\{\nabla_1 \nabla_2 \dots \nabla_k \diamond \psi \mid \diamond \psi \in \Gamma\} \text{ (for } \mathcal{L}_2)\end{aligned}$$

such that  $\widehat{\mathfrak{M}} = \langle W / \sim_{\Delta_i}, \widehat{R}, \vartheta \rangle$  is a filtration of  $\mathfrak{M}$  through the corresponding  $\Delta_i$ .

Let  $V$  be a set of fresh variables indexed over  $\Gamma$  as in the proof for a fusion of **K5** with something else. Let  $\mathcal{M}'$  be a model defined as previously. We extend  $V$  and  $\Gamma$  in the following way:

$$\begin{aligned}\Gamma' &= \Gamma \cup \text{Sub}\{\nabla_{11} \nabla_{21} \dots \nabla_{n1} \diamond_1 \psi \mid \diamond_1 \psi \in \Gamma\} \cup \text{Sub}\{\nabla_{12} \nabla_{22} \dots \nabla_{n2} \diamond_2 \psi \mid \diamond_2 \psi \in \Gamma\} \\ \Delta &= V \cup \text{Sub}\{\nabla_1 \nabla_2 \dots \nabla_n \diamond p_\psi \mid \nabla_{n+1} \psi \in \Gamma'\} \cup \text{Sub}\{\nabla_1 \nabla_2 \dots \nabla_k \diamond p_\psi \mid \diamond_2 \psi \in \Gamma\}.\end{aligned}$$

By the construction,  $\sim_{\Gamma'} = \sim_{\Delta}$ . So we have filtrations for the corresponding reducts of  $\mathcal{M}'$  through  $\Delta$  as well as for the original  $\mathcal{M}$ .  $\square$

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