

Complete representability for canonical extensions of representable cylindric algebras

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1 The problem itself

Given $\mathcal{C} \in \mathbf{RCA}_\omega$, whether \mathcal{C}^+ has a complete, ω -dimensional representation? [4]

2 Atomic Representations

A representation of a Boolean algebra \mathcal{B} is an embedding h of \mathcal{B} to some field of sets.

Let $a \in \mathcal{B}$ be an element of a Boolean algebra \mathcal{B} , a is called an atom, if for every $b \in \mathcal{B}$ $b < a$ implies $b = 0$. That is, an atom is a minimal non-zero element. $\text{At}(\mathcal{B})$ is the set of all atoms of \mathcal{B} .

Let \mathcal{B} be a Boolean algebra and \mathcal{F} a field of sets such that $h : \mathcal{B} \rightarrow \mathcal{F}$ is a representation of \mathcal{B} , then \mathcal{B} is a complete representation of \mathcal{B} , if for every $A \subseteq \mathcal{B}$ we have the following whenever ΣA is defined:

$$h(\Sigma A) = \bigcup h[A]$$

A representation h is called atomic, if $x \in h(1)$ there exists $b \in \text{At}(\mathcal{B})$ such that $x \in h(b)$.

Theorem 1. *Let \mathcal{B} be a Boolean algebra, then \mathcal{B} is atomic iff \mathcal{B} is completely representable. See [3, Corollary 6].*

3 BAOs and Duality

By default, we assume that all operators are at most unary. Here is the rigorous definition:

Definition 1.

1. Let $\mathcal{B} = \langle B, +, -, 0, 1 \rangle$ be a Boolean algebra. An operator is a function $\Omega : B \rightarrow B$ satisfying the following conditions:
 - Normality: $\Omega(0) = 0$
 - Additivity: $\Omega(b + b') = \Omega(b) + \Omega(b')$
2. Let I be an index set, a Boolean algebra with operators (BAO) is an algebra $\langle B, +, -, 0, 1, (\Omega_i)_{i \in I} \rangle$ such that $\langle B, +, -, 0, 1 \rangle$ is a Boolean algebra and for each $i \in I$ Ω_i is an operator.

Definition 2. Let $\mathcal{B} = \langle B, +, -, 0, 1, (\Omega_i)_{i \in I} \rangle$ be a BAO, then

1. An operator Ω is completely additive, if for every $X \subseteq B$ such that ΣX is defined, one has

$$\Omega(\sum X) = \sum_{x \in X} \Omega(x)$$

2. \mathcal{B} is completely additive, if for each $i \in I$ Ω_i is additive,
3. A class \mathcal{K} of BAOs is completely additive, if every $\mathcal{B} \in \mathcal{K}$ is completely additive.

3.1 Atom structures and canonical extensions

Definition 3. Let I be an index set and $(\Omega_i)_{i \in I}$ a set of function symbols

1. A structure is a relational structure $\mathcal{F} = \langle W, (R_i)_{i \in I} \rangle$ such that R_i is a binary relation symbol for a function symbol $\Omega_{i \in I}$ with the corresponding index,
2. Let \mathcal{B} be an atomic BAO of the signature I , the atom structure of \mathcal{B} , written as $\mathfrak{At}\mathcal{B}$, is a structure $\langle \text{At}(\mathcal{B}), (R_i)_{i \in I} \rangle$ such that for all $a, b \in \text{At}(\mathcal{B})$ and for all $i \in I$

$$\mathfrak{At}\mathcal{B} \models R_i(a, b) \text{ iff } \mathcal{B} \models a \leq \Omega_i(b)$$

3. Let $\mathcal{F} = \langle W, (R_i)_{i \in I} \rangle$ be an atom structure, the complex algebra of \mathcal{F} , written as $\mathfrak{Cm}\mathcal{F}$, is a BAO $\langle \mathcal{P}(W), \cup, -, \emptyset, W, (\Omega_{R_i})_{i \in I} \rangle$ such that for all $X \subseteq W$ and for each $i \in I$:

$$\Omega_{R_i}(X) = \{a \in W \mid \exists b \in X \mathcal{F} \models R_i(a, b)\}$$

Definition 4. Let $\mathcal{F} = \langle W, (R_i)_{i \in I} \rangle$ and $\mathcal{F}' = \langle W', (R'_i)_{i \in I} \rangle$, then a function $f : \mathcal{F} \rightarrow \mathcal{F}'$ is a bounded morphism, if the following holds:

1. $xR_i y$ implies $f(x)R'_i f(y)$;
2. $f(x)R'_i z$, then there exists $y \in W$ such that $xR_i y$ and $f(y) = z$.

A bounded morphism $f : \mathcal{F} \rightarrow \mathcal{F}'$ is a p -morphism, if f is onto. $\mathcal{F} \twoheadrightarrow \mathcal{F}'$ iff there exists a p -morphism from \mathcal{F} onto \mathcal{F}' , or \mathcal{F}' is a p -morphic image of \mathcal{F} .

Definition 5. Let $\mathcal{F} = \langle W, (R_i)_{i \in I} \rangle$ is an inner substructure¹ of $\mathcal{F}' = \langle W', (R'_i)_{i \in I} \rangle$, if $W \subseteq W'$ and the embedding $\mathcal{F} \hookrightarrow \mathcal{F}'$ is a bounded morphism.

Let \mathbb{F} be a class of structures, define:

1. $\mathfrak{Cm}(\mathbb{F}) = \{\mathcal{B} \mid \mathcal{B} \cong \mathfrak{Cm}(\mathcal{F}) \text{ for some } \mathcal{F} \in \mathbb{F}\}$.
2. $\mathbf{Up}(\mathbb{F})$ is the class of structures isomorphic to disjoint unions of elements of \mathbb{F} .
3. $\mathbf{S}(\mathbb{F})$ is the closure of \mathbb{F} under inner substructures.

Let A be a non-empty subset of a Boolean algebra \mathcal{B} , A is a *filter*, if A is closed under finite infima and it is upward closed. A is an *ultrafilter*, if it has no non-trivial extensions. That is, if $A \subseteq A'$, then $A' = \mathcal{B}$. This is a well-known fact that every filter can be extended to a maximal one using Zorn's lemma.

The following definition is due to, for example, [6, Definition 5.40].

Definition 6. Let $\mathcal{B} = \langle B, +, -, 0, 1, (\Omega_i)_{i \in I} \rangle$ be a BAO and $\mathbf{Spec}(\mathcal{B})$ the set of its ultrafilters. The ultrafilter frame of \mathcal{B} (or the canonical frame) is a relational structure $\mathcal{F}_{\mathcal{B}} = \langle \mathbf{Spec}(\mathcal{B}), R_{\Omega_i} \rangle$ such that for all ultrafilters U_1, U_2 one has

¹Or alternatively, a generated subframe

$$\mathbf{Spec}(\mathcal{B}) \models R_{\Omega_i}(U_1, U_2) \text{ iff } \{\Omega_i(b) \mid b \in U_1\} \subseteq U_2.$$

Given \mathcal{B} be a BAO, we denoted as \mathcal{B}^+ as the complex algebra of the canonical frame $\mathfrak{Cm}(\mathcal{F}_{\mathcal{B}})$, that is, *the canonical extension* of \mathcal{B} . A class of BAOs \mathbf{K} is canonical, if it is closed under canonical extensions. That is, $\mathcal{B}^+ \in \mathbf{K}$ whenever $\mathcal{B} \in \mathbf{K}$.

Theorem 2. *Let \mathcal{A}, \mathcal{B} be BAOs,*

1. *There exists $\iota : \mathcal{A} \hookrightarrow \mathcal{A}^+$ such that $\iota : a \mapsto \{\gamma \in \mathbf{Spec}(\mathcal{A}) \mid a \in \gamma\}$.*
2. *$i : \mathcal{A} \hookrightarrow \mathcal{B}$ implies $i^+ : \mathcal{A}^+ \hookrightarrow \mathcal{B}^+$*

4 Representable cylindric algebras

Let α be an ordinal. Let ${}^\alpha U$ be the set of all functions mapping α to a non-empty set U . We denote $x(i) = x_i$ for $x \in {}^\alpha U$ and $i < \alpha$.

A subset of ${}^\alpha U$ is an α -ry relation on U . For $i, j < \alpha$, the i, j -diagonal D_{ij} is the set of all elements of ${}^\alpha U$ such that $y_i = y_j$.

If $i < \alpha$ and X is an α -ry relation on U , then the i -th cylindrification $C_i X$ is the set of all elements of U that agree with some element of X on each coordinate except, perhaps, the i -th one. To be more precise,

$$C_i X = \{y \in {}^\alpha U \mid \exists x \in X \forall i < \alpha (i \neq j \Rightarrow y_j = x_j)\}.$$

We define the following equivalence relation for $i < \alpha$ and $x, y \in {}^\alpha U$:

$$x \equiv_i y \Leftrightarrow \forall j \in \alpha (i \neq j \Rightarrow x(j) = y(j))$$

Then one may reformulate the definition of the i -th cylindrification in the following way:

$$C_i X = \{y \in {}^\alpha U \mid \exists x \in X \ x \equiv_i y\}$$

According to this version of the definition, one may think of the cylindrification as an **S5** modal operator.

Definition 7. *A cylindric set algebra of dimension α is an algebra consisting of a set S of α -ry relation on some base set U with the constants and operations $0 = \emptyset$, $1 = {}^\alpha U$, \cap , $-$, the diagonal elements $(D_{ij})_{i,j < \alpha}$, the cylindrifications $(C_i)_{i < \alpha}$. A generalised cylindric set algebra of dimension α is a subdirect of cylindric algebras that have dimension α . \mathbf{Cs}_α denotes the class of all cylindric set algebras of dimension α .*

Definition 8. *A cylindric algebra of dimension α is an algebra $\mathcal{C} = \langle \mathcal{B}, \{c_i\}_{i < \alpha}, \{d_{ij}\}_{i,j < \alpha} \rangle$ such that*

- \mathcal{B} is a Boolean algebra, for each $i, j < \alpha$ c_i is an operator and $d_{ij} \in \mathcal{B}$
- For each $i < \alpha$, $a \leq c_i a$, $c_i(a \cdot c_i b) = c_i a \cdot c_i b$ and $d_{ii} = 1$
- For every $i, j < \alpha$, $c_i c_j a = c_j c_i a$
- If $k \neq i, j < \alpha$, then $d_{ij} = c_k(d_{ij} \cdot d_{jk})$
- If $i \neq j$, then $c_i(d_{ij} \cdot a) \cdot c_i(d_{ij} \cdot -a) = 0$

\mathbf{CA}_α is the class of all cylindric algebras of dimension α .

One may define a representation of a cylindric algebra explicitly in the following way:

Definition 9. Let \mathcal{A} be a cylindric algebra of dimension α . A representation of \mathcal{A} over the non-empty domain X is a map $f : \mathcal{A} \hookrightarrow 2^{\alpha U}$ such that:

1. $f(1) = \bigcup_{i \in I} {}^\alpha X_i$ for some disjoint family $\{X_i\}_{i \in I}$ where each $X_i \subseteq X$
2. $h : \mathcal{A} \rightarrow 2^{f(1)}$ is a representation of a Boolean reduct
3. for all $\lambda, \eta < \alpha$, $x \in h(d_{\lambda\eta})$ iff $x_\lambda = x_\eta$
4. for all $\lambda < \alpha$ and $a \in \mathcal{A}$, $x \in h(c_\lambda(a))$ iff there is $y \in X$ such that $x[\lambda \mapsto y] \in h(a)$

An α -dimensional cylindric algebra C is representable, if there exists a representation of h . \mathbf{RCA}_α is the class of all representable cylindric algebras that have dimension α . In particular, we are interested in the case $\alpha = \omega$.

It is well known that \mathbf{RCA}_α is a variety, \mathbf{RCA}_α ($\alpha \leq 2$) is finitely axiomatisable and \mathbf{RCA}_α ($2 < \alpha < \omega$) has no finite axiomatisation, see [2].

Let $\mathcal{A} \in \mathbf{CA}_\omega$, then \mathcal{A} has a *complete representation*, if its representation preserves all existing suprema. In other words, \mathcal{A} is completely representable.

5 \mathbf{RCA}_ω and canonicity

The following definition of an ω -frame is due to [5].

Definition 10. A cylindric ω -frame is a structure $\mathcal{F} = \langle W, (R_i)_{i < \omega}, (E_{ij})_{i, j < \omega} \rangle$ where $(R_i)_{i < \omega}$ are binary relations and $(E_{ij})_{i, j < \omega}$ are unary relations such that, for all $i, j, k < \omega$:

1. Every R_i is an equivalence relation on W ,
2. $R_i \circ R_j = R_j \circ R_i$, that is, the set $(R_i)_{i < \omega}$ forms a commutative semigroup under composition.
3. For all $x \in W$, $E_{ii}(x)$ holds.
4. For all $x, y, z \in W$, $xR_i y \ \& \ E_{ij}(y) \ \& \ xR_i z \ \& \ E_{ij}(y)$ implies $y = z$.
5. For all $x \in W$, $E_{ij}(x)$ iff there exists $y \in W$ such that $xR_k y$, $E_{ik}(y)$, and $E_{kj}(y)$.

The following fact is by Venema, see [5, Proposition 2.1.5]:

Proposition 1. An ω -frame \mathcal{F} is cylindric iff $\mathfrak{Cm}(\mathcal{F})$ is a cylindric algebra of dimension ω .

A cylindric ω -frame \mathcal{F} is completely representable, if $\mathfrak{Cm}(\mathcal{F})$ is completely representable as a cylindric algebra of dimension ω .

We are interested in the special case of cylindric ω -frames called Cartesian structure of dimension ω . To be more precise:

Definition 11. Let U be a set and $V \subseteq {}^\omega U$ be a non-empty subset of the full Cartesian space of dimension ω , then an α -dimension Cartesian structure generated by V is an ω -frame $\mathfrak{G}(V) = \langle V, (R_i)_{i < \omega}, (E_{ij})_{i, j < \omega} \rangle$ such that:

1. $R_i = \{(w, v) \mid w, v \in V, w_k = v_k, k < \omega, i \neq k\}$
2. $E_{ij} = \{w \in V \mid w_i = w_j\}$

$\mathfrak{S}({}^\omega U)$ is the full ω -dimensional Cartesian structure. \mathcal{Fct}_ω is the class of all full ω -dimensional Cartesian structures.

Every cylindric set algebra is a subalgebra of some complex algebra induced by an ω -dimensional Cartesian structure. That is, we have the following characterisation:

Lemma 1. $\mathbf{ICS}_\omega = \mathbf{S}\mathfrak{Cm}\mathcal{Fct}_\omega$.

In this section, we reproduce the results related to characterisation \mathbf{RCA}_ω . The following results are due to Goldblatt [1].

Lemma 2. $\mathbf{RCA}_\omega = \mathbf{S}\mathfrak{CmUd}\mathcal{Fct}_\omega$

Theorem 3. \mathbf{RCA}_ω is a canonical variety.

6 Representability games

References

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