Notes on filtration of logics containing **K5**

Daniel Rogozin

1 Preliminaries

Definition 1. *An n-normal modal logic is a set of formulas that contains all Boolean tautologies, formulas $\Diamond_i p \vee \Diamond_i q \leftrightarrow \Diamond_i (p \vee q)$ and $\Diamond_i \bot \leftrightarrow \bot$ for $i \leqslant n$, and is closed under modus ponens, substitution, and monotonicity: from $\varphi \to \psi$ infer $\Diamond_i \varphi \to \Diamond_i \psi$ for $i \leqslant n$.*

Definition 2. *An n-Kripke model is a triple $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta \rangle$, where $R_i \subseteq W \times W$, $\vartheta : \mathrm{PV} \to 2^W$, and the connectives have the following semantics:*

1. $\mathcal{M}, w \models p \Leftrightarrow w \in \vartheta(p)$
2. $\mathcal{M}, w \models \not\varphi \Leftrightarrow \mathcal{M}, w \not\models \varphi$
3. $\mathcal{M}, w \models \varphi \vee \psi \Leftrightarrow \mathcal{M}, w \models \varphi$ *or* $\mathcal{M}, w \models \psi$
4. $\mathcal{M}, w \models \Diamond_i \varphi \Leftrightarrow \exists v \in R_i(w)\ \mathcal{M}, v \models \varphi$

By **K5** we mean the logic $\mathbf{K} \oplus A5$, where $A5 = \Diamond p \to \Box\Diamond p$. It is known that **K5** is the modal logic of all Euclidean frames. A frame is called Euclidean if for each x, y, z, xRy and xRz implies yRz.

Proposition 1. *Let $\mathcal{F} = \langle W, R \rangle$ be an Euclidean frame.*

1. *For each $x, y, z \in W$, xRy and xRz implies either yRz or zRy.*
2. *For each $x \in W$, $R^*(x) = \{x\} \cup R(R(x))$.*
3. $R^{-1} \circ R \subseteq R$.

Proposition 2. **K5** *proves*

1. $\Box^3 p \leftrightarrow \Box^2 p$
2. $\Box^2 \Diamond p \leftrightarrow \Box\Diamond p$
3. $\Box\Diamond\Box p \leftrightarrow \Box\Box p$
4. $\Box\Diamond^2 p \leftrightarrow \Box\Diamond p$

Proposition 3. *Let $\mathcal{M}$ be a* $\mathbf{K}5$ *model, xRy for $x, y \in W$ then one has*

$$\mathcal{M}, x \models \Diamond\Box\varphi \text{ iff } \mathcal{M}, y \models \Diamond\Box\varphi.$$

Proof.

1. Suppose $\mathcal{M}, x \models \Diamond\Box\varphi$. One also has $\mathcal{M}, x \models \Diamond\Box\varphi \to \Box\Diamond\Box\varphi$, so $\mathcal{M}, x \models \Box\Diamond\Box\varphi$. Thus, $\mathcal{M}, y \models \Diamond\Box\varphi$ since $y \in R(x)$.
2. Suppose $\mathcal{M}, y \models \Diamond\Box\varphi$, then $\mathcal{M}, y \models \Box\varphi$, so $\mathcal{M}, x \models \Diamond\Box\varphi$.

$\Box$

1.1 Filtrations: general definitions

Let $\mathcal{M} = \langle W, R_1, \ldots, R_n, \vartheta \rangle$ be a Kripke model and Γ a set of formulas closed under subformulas. An equivalence relation $\sim$ is set to have a finite index if the quotient set $W/\sim$ is finite. The equivalence relation $\sim_\Gamma$ induced by Γ is defined as

$$w \sim_\Gamma v \Leftrightarrow \forall \varphi \in \Gamma\, (\mathcal{M}, w \models \varphi \Leftrightarrow \mathcal{M}, v \models \varphi).$$

If Γ is finite, then $\sim_\Gamma$ has a finite index. An equivalence relation $\sim$ respects $\sim_\Gamma$, if $w \sim v$ implies $w \sim_\Gamma v$.

Definition 3. *Let $\mathcal{M} = \langle W, R_1, \ldots, R_n, \vartheta \rangle$ be a Kripke model and Γ be a Sub-closed set formulas. A Γ-filtration of $\mathcal{M}$ is a model $\widehat{\mathcal{M}} = \langle \widehat{W}, \widehat{R_1}, \ldots, \widehat{R_n}, \widehat{\vartheta} \rangle$ such that:*

1. *$\widehat{W} = W/\sim$, where $\sim$ is an equivalence relation having a finite index that respects Γ*
2. *$\widehat{\vartheta}(p) = \{[x]_\sim \mid x \in W \;\&\; x \in \vartheta(p)\}$*
3. *For each $i \in I$ one has $\widehat{R}_i^{min} \subseteq \widehat{R}_i \subseteq \widehat{R}_i^{max}$. $\widehat{R}_{i,\sim}^{min}$ is the i-th minimal filtered relation on $\widehat{W}$ defined as*

$$\hat{x} \widehat{R}_{i,\sim}^{min} \hat{y} \Leftrightarrow \exists x' \sim x\, \exists y' \sim y\; x R_i y$$

$\widehat{R}_{\Gamma,i}^{max}$ is the i-th maximal filtered relation on $\widehat{W}$ induced by Γ defined as

$$\hat{x} \widehat{R}_{\Gamma,i}^{max} \hat{y} \Leftrightarrow \forall \Box_i \varphi \in \Gamma\, (\mathcal{M}, x \models \Box_i \varphi \Rightarrow \mathcal{M}, y \models \varphi)$$

If Φ is finite subset of Γ and $\sim = \sim_\Phi$, then $\widehat{\mathcal{M}}$ is a definable Γ-filtration of $\mathcal{M}$ through Φ. If $\sim = \sim_\Gamma$, then such a filtration by means of the definiton above is called *strict*.

Lemma 1. *Let Γ be a finite set of formulas closed under subformulas and $\widehat{\mathcal{M}}$ a filtration of $\mathcal{M}$ through Γ, then for each $x \in W$ and for each $\varphi \in \Gamma$ one has*

$$\mathcal{M}, x \models \varphi \Leftrightarrow \widehat{\mathcal{M}}, \hat{x} \models \varphi$$

Definition 4. *Let $\mathbb{F}$ be a class of Kripke frames and Γ a finite set of formulas closed under subformulas. If for every model $\mathcal{M}$ over $\mathcal{F} \in \mathbb{F}$ there exists a model that is a Γ-definable filtration of $\mathcal{M}$, then $\mathbb{F}$ admits definable filtration. A class of models $\mathbb{M}$ admits definable filtration if for every $\mathcal{M} \in \mathbb{M}$ there exists a model belonging to the same class that is a definable Γ-filtration of $\mathcal{M}$.*

Lemma 2.

1. *Let $\mathcal{L}$ be a complete normal modal logic. If* Frames($\mathcal{L}$) *admits filtration, then $\mathcal{L}$ has the finite model property.*
2. *If the class of models* Mod($\mathcal{L}$) *admits filtration, then $\mathcal{L}$ has the finite model property and Kripke complete as well.*

2 Filtration of Euclidean logics

First of all, let us ensure that a filtration of an Euclidean frame is not necessary finite. Let $[x] \sim_\Gamma [y]$ and $[x] \sim_\Gamma [z]$. Then for some $x' \in [x]$ $y' \in [y]$, one has $x'Ry'$ and $x''Rz'$ for some $x'' \in [x]$ and $z' \in [z]$. Clearly, we cannot claim that $x' = x''$ in general. Thus, minimal filtration does not preserve the required property.

Lemma 3. **K5** *admit filtration.*

Proof. Let $\mathcal{M}$ be a **K5**-model and Γ_0 a finite set of formulas closed under subformulas. Let us put $\Gamma = \Gamma_0 \cup \mathrm{Sub}(\{\Diamond\Box\psi \mid \Box\psi \in \Gamma_0\}) \cup \Psi$, where $\Psi = \nabla_1\nabla_2 \dots \nabla_n\Box\psi$ for $\Box\psi \in \Gamma_0$ and $\nabla_i \in \{\Diamond, \Box\}$. By Proposition 2, any element of Φ has one of the four forms. Thus, $W \sim_{\equiv_\Gamma}$ has a finite index. We put $\widehat{R} = R_\Gamma^{\max}$. $\Box$

Definition 5. *A first-order formula is called Horn if it has the following form:*

$$\forall x_1, \dots, x_n (x_{i_1} R x_{j_1} \wedge \dots \wedge x_{i_s} R x_{j_s} \to x_k R x_l)$$

Definition 6. *Let H be a Horn property and $\langle W, R \rangle$ a Kripke frame. A Horn closure of a binary relation R is the minimal relation R^H containing R and satisfying H.*

Lemma 4. $R^H = \bigcup\limits_{n<\omega} R_n$ *where*

1. $R_0 = R$.
2. $R_{n+1} = R_n \cup \{(a, b) \in W \mid \exists \vec{c} \in W\ P(a, b, \vec{c})\}$, *where P is a premise of H.*

E-closure (an Euclidean Horn closure of a binary relation) has the following equivalent definitions:

Lemma 5. *Let $\mathcal{F} = \langle W, R \rangle$ be a Kripke frame. The following conditions are equivalent:*

1. *R^E is the smallest Euclidean relation containing R.*
2. $R^E = \bigcup\limits_{i<\omega} R_i$, *where*
   - $R_0 = R$
   - $R_{n+1} = R_n \cup (R_n^{-1} \circ R_n)$
3. *xR^Ey iff there exists $n < \omega$ such that either xRy or $\exists z_1, \dots, z_n$ with $z_1 R x$ and $z_{n-1} R y$ and for each $1 < i \leqslant n$ one has either $z_{i-1} R z_i$ or $z_i R z_{i-1}$.*
4. $R^E = R \cup \bigcup\limits_{i<\omega} (R^{-1} \circ (R \circ R^{-1})^n \circ R)$.

Proof.

1. (1) $\Rightarrow$ (2) Let us show that if R^E is the smallest Euclidean relation containing R, then $R^E = \bigcup\limits_{i<\omega} R_i$. There are two inclusions:
   - $R^E \subseteq \bigcup\limits_{i<\omega} R_i$. Recall that R^E has the form (?):

$$R^E = \bigcap \{R' \mid R \subseteq R', \forall a, b \in W\ R'(a, b) \Rightarrow \exists x \in W\ R'(x, a)\ \&\ R'(x, b)\}$$

- $\bigcup_{i<\omega} R_i \subseteq R^E$. Let us show that xR_ny for each $n < \omega$ implies xR^Ey by induction on n. If $n = 0$, then xRy, thus, xR^Ey, since R is a subrelation of R^E. Suppose $n = m+1$ and $xR_{m+1}y$. Let us show that xR^Ey. From $xR_{m+1}y$, one has $(x,y) \in R^n \cup (R_n^{-1} \circ R_n)$. There are two cases:
  - xR^ny, one needs to merely apply the IH.
  - $xR_n^{-1} \circ R_ny$. Then $\exists z \in W \; xR_n^{-1}z \;\&\; zR_n$. That is, zR_nx and zR_ny for some z. R_n is already a subrelation of R^E. Thus, zR^Ex and zR^Ey. That implies xR^Ey.

2. (2) $\Rightarrow$ (3) Let $(x,y) \in R_m$, let us the statement by induction on m.

   (a) Suppose $m = 0$, then xRy, and the statement is shown putting $n = 0$.

   (b) Suppose $m = p+1$ and $xR_{p+1}y$. Assume that either xRy or $\exists z_1, \ldots, z_p$ with z_1Rx and $z_{p-1}Ry$ and for each $1 < i \leqslant p$ one has either $z_{i-1}Rz_i$ or z_iRz_{i-1}.
   $xR_{p+1}y$ implies $(x,y) \in R_p \cup (R_p^{-1} \circ R_p)$. If $(x,y) \in R_p$, then we merely apply the IH. Suppose $(x,y) \in R_p^{-1} \circ R_p$, then $(z,x) \in R_p$ and $(z,y) \in R_p$

3. (3) $\Rightarrow$ (4) Suppose either xRy or there exist $n \geqslant 1$ and $z_1, \ldots, z_n$ with z_1Rx and $z_{n-1}Ry$ and for each $1 < i \leqslant n$ one has either $z_{i-1}Rz_i$ or z_iRz_{i-1}. If xRy, then we are done. Otherwise there exists $n \geqslant 1$ with the condition above. Then $(x,y) \in R_{n+1}$ that follows from the condition.

4. (4) $\Rightarrow$ (1)

$\square$

Lemma 6. *Let $\mathcal{M} = \langle W, R, \vartheta \rangle$ be an Euclidean model, Γ a set of* Sub*-closed formulas, and $\sim$ an equivalence relation having a finite index that respects Γ, then $\widehat{R} = (R_\Phi^{min})^E \subseteq R_\Gamma^{max}$, where $\Phi = \Gamma \cup \{\Diamond\Box\varphi \mid \Box\varphi \in \Gamma\}$.*

Thus, **K**5 *admits strict filtrations.*

Proof. Recall that $(R_\Phi^{min})^E$ has the form $(R_\Phi^{min})^E = \bigcup_{n<\omega} (R_\Phi^{min})_n$, where

1. $(R_\Phi^{min})_0 = R_\Phi^{min}$
2. $(R_\Phi^{min})_{m+1} = (R_\Phi^{min})_n \cup (((R_\Phi^{min})_n)^{-1} \circ (R_\Phi^{min})_n)$

One needs to show that for each $n < \omega$ $(R_\Phi^{min})_n \subseteq R_\Gamma^{max}$. We prove this by induction. Suppose $\mathcal{M}, x \models \Box\varphi$ for $\Box\varphi \in \Phi$ and $[x](R_\Phi^{min})^E[y]$. We need $\mathcal{M}, y \models \varphi$.

1. $([x],[y]) \in (R_\Phi^{min})_0$, then $([x],[y]) \in R_\Phi^{min}$. Then there exist $x' \in [x]$ and $y' \in [y]$ such that $x'Ry'$. So $\mathcal{M}, x' \models \Box\varphi$ and, thus, $\mathcal{M}, y' \models \varphi$. Then $\mathcal{M}, y' \models \varphi$ as well since $y' \in [y]$.

2. $([x],[y]) \in (R_\Phi^{min})_{m+1}$, then $([x],[y]) \in (R_\Phi^{min})_m \cup (((R_\Phi^{min})_m)^{-1} \circ R_\Phi^{min})_m)$.

   If $([x],[y]) \in (R_\Phi^{min})_m$, then we apply the IH.

   Suppose $([x],[y]) \in (R_\Phi^{min})_m)^{-1} \circ (R_\Phi^{min})_m$, then there exists $[z] \in W/\sim_\Phi$ such that $([z],[x]) \in (R_\Phi^{min})_m)$ and $([z],[y]) \in (R_\Phi^{min})_m$.

   Then one has the following picture (using Lemma 5):

$$[z] \xleftarrow{R_\Phi^{min}} [z_1] \overset{R'}{\text{———}} [z_2] \overset{R'}{\text{———}} \dots \overset{R'}{\text{———}} [z_{m-1}] \overset{R'}{\text{———}} [z_m] \xrightarrow{R_\Phi^{min}} [x]$$

$$[z] \xleftarrow[R_\Phi^{min}]{} [z_1^{'}] \underset{R'}{\text{———}} [z_2^{'}] \underset{R'}{\text{———}} \dots \underset{R'}{\text{———}} [z_{m-1}] \underset{R'}{\text{———}} [z_m^{'}] \xrightarrow[R_\Phi^{min}]{} [y]$$

Where R' is either R_Φ^{min} or its converse. One has $\mathcal{M}, x \models \Box\varphi$ for $\Box\varphi \in \Phi$, where $\widehat{M}$ is the minimal filtration of $\mathcal{M}$ through Φ. One has $[z_m]R_\Phi^{min}[x]$, then $a_m R a$ for some $a_m \in [z_n]$ and $a \in [x]$. Thus, $\mathcal{M}, a_m \models \Diamond\Box\varphi$ and, thus, $\widehat{\mathcal{M}}, [z_m] \models \Diamond\Box\varphi$.

Applying Proposition 3 several times, one may show that $\widehat{\mathcal{M}}, [z_1] \models \Diamond\Box\varphi$. One has $[z_1]R_\Phi^{min}[z]$, then for some $a \in [z]$ and $a_1 \in [z_1]$ we have $a_1 R a$.

Then $\mathcal{M}, a \models \Box\varphi$ and $\widehat{M}, [z] \models \Box\varphi$.

We have $[z_1^{'}]R_\Phi^{min}[z]$, thus, $a_1^{'} R a^{'}$ for some $a_1^{'} \in [z_1^{'}]$ and $a^{'} \in [z]$. Then $\mathcal{M}, a_1^{'} \models \Diamond\Box\varphi$, and, thus, $\widehat{M}, [z_1^{'}] \models \Diamond\Box\varphi$.

One may show that $\widehat{M}, [z_m^{'}] \models \Diamond\Box\varphi$ in the same way via Lemma 3. Thus, $\mathcal{M}, z_m^{'} \models \Diamond\Box\varphi$. We also have $\mathcal{M}, z_m^{'} \models \Diamond\Box\varphi \to \Box\varphi$, and, thus, $\mathcal{M}, z_m^{'} \models \Box\varphi$. Then $\widehat{\mathcal{M}}, [z_m^{'}] \models \Box\varphi$.

One has $[z_m^{'}]R_\Phi^{min}[y]$, then $a_m^{'} R y'$ for some $a_m^{'} \in [z_m^{'}]$ and $y' \in [y]$. Then $\mathcal{M}, y' \models \varphi$. But $y' \sim_\Phi y$, so $\mathcal{M}, y \models \varphi$.

$\square$

2.1 Clusters

Let $\mathcal{F} = \langle W, R \rangle$ be a transitive frame. Let us put $xR^\bullet y \Leftrightarrow xRy \,\&\, \neg(xRy)$. A point x is proper if xRx. Let us define the following equivalence relation:

$$x \equiv y \Leftrightarrow xRy \,\&\, yRx \vee x = y.$$

A cluster is an element of the quotient set $W/\equiv$. Given $x \in W$, C_x is a cluster containing x. Thus $C_x = \{x\} \cup \{y \mid xRyx\}$. The original relation lifts to the antisymmetric transitive relation on $W/\equiv$ defined as $C_x R C_y$ iff xRy. A cluster C is called maximal if CRC' implies $C = C'$. A point is R-maximal if C_x is a maximal cluster, that is, $R^\bullet(x) = \varnothing$. A degenerated cluster is a singleton $\{x\}$ with $\neg(xRx)$. A cluster is called simple if it has the form $\{x\}$ with xRx. If $\langle W', R' \rangle$ is an inner substructure of $\langle W, R \rangle$, then every R'-cluster is an R-cluster and every R-cluster that intersects W' is a subset of W' and is an R'-cluster itself. Given a Kripke model $\mathcal{M}$, a set of formulas Γ is satisfied by a cluster C if every member of Γ is true at some point of C.

If clusters coincide then their poitns have the same theory in the original model:

Lemma 7. *$C_x = C_y$ implies $\mathcal{M}, x \models \Box\varphi \Leftrightarrow \mathcal{M}, y \models \Box\varphi$*

Let us describe the bulldozing technique allowing one to eliminate nondegenerated clusters [3]. Let $\mathcal{L}$ be a transitive logic and $\mathcal{F}$ its frame. We construct first a frame $\mathcal{F}^0 = \langle W^0, R^0 \rangle$ replacing every nondegenerated frame C of W by $C^0 = \{\langle x, i \rangle \mid x \in C, i < \omega\}$. We also replace each degenerated cluster C by $\{\langle x, 0 \rangle\}$. Elements of these subsets form W^0. The relation R^0 is defined as

$$\langle x, i \rangle R^0 \langle y, j \rangle \Leftrightarrow xR^\bullet y \text{ or } (x \equiv y \,\&\, i < j) \text{ or } i = j \,\&\, x <_C y$$

where $<_C$ is an arbitrary strict ordering on the proper cluster C containing x and y.

Each nondegenrated cluster C is replaced by an infinite set C_0 such that $\langle C_0, R_0 \rangle$ is a strict linear order. Moreover, $\langle y, j \rangle$, a copy of y, occurs after $\langle y, j \rangle$, a copy of x.

Bulldozing might be extended to models $\mathcal{M} = \langle W, R, \vartheta \rangle$ defining ϑ^0 as follows

$$\vartheta^0(p_i) = \{\langle x, i \rangle \mid x \in \vartheta(p_i), i < \omega\}.$$

One may show inductively the following fact.

Lemma 8. $\mathcal{M}, x \models \varphi \Leftrightarrow \mathcal{M}^0, \langle x, i \rangle \models \varphi$

Let us concretise the case of transitive Euclidean frames. First of all, we consider clusters in **K**45 frames.

2.2

3 Transitive closure stuff

4 PDLisation of Euclidean logics

5 Transitive closure and fusion

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