Canonicity of Representable Cylindric Algebras

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1 The problem itself

- Given $\mathcal{C} \in \mathbf{RCA}_\omega$, whether $\mathcal{C}^+$ has a complete, ω-dimensional representation? The conjecture is yes. [6]
- Whether $\mathbf{RCA}_\omega$ is barely canonical. The conjecture is yes.

2 Atomic Representations

A representation of a Boolean algebra $\mathcal{B}$ is an embedding h of $\mathcal{B}$ to some field of sets.

Let $a \in \mathcal{B}$ be an element of a Boolean algebra $\mathcal{B}$, a is called an atom, if for every $b \in \mathcal{B}$ $b < a$ implies $b = 0$. That is, an atom is a minimal non-zero element. $\mathrm{At}(\mathcal{B})$ is the set of all atoms of $\mathcal{B}$.

Let $\mathcal{B}$ be a Boolean algebra and $\mathcal{F}$ a field of sets such that $h : \mathcal{B} \to \mathcal{F}$ is a representation of $\mathcal{B}$, then $\mathcal{B}$ is a complete representation of $\mathcal{B}$, if for every $A \subseteq \mathcal{B}$ we have the following whenever ΣA is defined:

$$h(\Sigma A) = \bigcup h[A]$$

A representation h is called atomic, if $x \in h(1)$ there exists $b \in \mathrm{At}(\mathcal{B})$ such that $x \in h(b)$.

Theorem 1. *Let $\mathcal{B}$ be a Boolean algebra, then $\mathcal{B}$ is atomic iff $\mathcal{B}$ is completely representable. See [5, Corollary 6].*

3 BAOs and Duality

By default, we assume that all operators are at most unary. Here is the rigorous definition:

1. Let $\mathcal{B} = \langle B, +, -, 0, 1\rangle$ be a Boolean algebra. An operator is a function $\Omega : B \to B$ satisfying the following conditions:
   - Normality: $\Omega(0) = 0$
   - Additivity: $\Omega(b + b') = \Omega(b) + \Omega(b')$
2. Let I be an index set, a Boolean algebra with operators (BAO) is an algebra $\langle B, +, -, 0, 1, (\Omega_i)_{i \in I}\rangle$ such that $\langle B, +, -, 0, 1\rangle$ is a Boolean algebra and for each $i \in I$ Ω_i is an operator.

Let $\mathcal{B} = \langle B, +, -, 0, 1, (\Omega_i)_{i \in I}\rangle$ be a BAO, then

1. An opetator Ω is completely additive, if for every $X \subseteq B$ such that ΣX is defined, one has

$$\Omega(\sum X) = \sum_{x \in X} \Omega(x)$$

2. $\mathcal{B}$ is completely additive, if for each $i \in I$ Ω_i is additive,

3. A class $\mathcal{K}$ of BAOs is completely additive, if every $\mathcal{B} \in \mathcal{K}$ is completely additive.

3.1 Atom structures and canonical extensions

Let I be an index set and $(\Omega_i)_{i \in I}$ a set of function symbols

1. A structure is a relational structrure $\mathcal{F} = \langle W, (R_i)_{i \in I} \rangle$ such that R_i is a binary relation symbol for a function symbol $\Omega_{i \in I}$ with the corresponding index,

2. Let $\mathcal{B}$ be an atomic BAO of the signature I, the atom structure of $\mathcal{B}$, written as $\mathfrak{At}\mathcal{B}$, is a structure $\langle \mathrm{At}(\mathcal{B}), (R_i)_{i \in I} \rangle$ such that for all $a, b \in \mathrm{At}(\mathcal{B})$ and for all $i \in I$

$$\mathfrak{At}\mathcal{B} \models R_i(a, b) \text{ iff } \mathcal{B} \models a \leqslant \Omega_i(b)$$

3. Let $\mathcal{F} = \langle W, (R_i)_{i \in I} \rangle$ be an atom structure, the complex algebra of $\mathcal{F}$, written as $\mathfrak{Cm}\mathcal{F}$, is a BAO $\langle \mathcal{P}(W), \cup, -, \varnothing, W, (\Omega_{R_i})_{i \in I} \rangle$ such that for all $X \subseteq W$ and for each $i \in I$:

$$\Omega_{R_i}(X) = \{a \in W \mid \exists b \in X \; \mathcal{F} \models R_i(a, b)\}$$

Let $\mathcal{F} = \langle W, (R_i)_{i \in I} \rangle$ and $\mathcal{F}' = \langle W', ({R'}_i)_{i \in I} \rangle$, then a function $f : \mathcal{F} \to \mathcal{F}'$ is a bounded morphism, if the following holds:

1. xR_iy implies $f(x){R'}_if(y)$;

2. $f(x){R'}_iz$, then there exists $y \in W$ such that xR_iy and $f(y) = z$.

A bounded morphism $f : \mathcal{F} \to \mathcal{F}'$ is a p-morphism, if f is onto. $\mathcal{F} \twoheadrightarrow \mathcal{F}'$ iff there exists a p-morphism from $\mathcal{F}$ onto $\mathcal{F}'$, or $\mathcal{F}'$ is a p-morphic image of $\mathcal{F}$.

Let $\mathcal{F} = \langle W, (R_i)_{i \in I} \rangle$ is an inner substructure [1] of $\mathcal{F}' = \langle W', ({R'}_i)_{i \in I} \rangle$, if $W \subseteq W'$ and the embedding $\mathcal{F} \hookrightarrow \mathcal{F}'$ is a bounded morphism. Let $\mathbb{F}$ be a class atom structures, then $\mathbb{S}(\mathbb{F})$ is the closure of $\mathbb{F}$ under generated subframes.

Let $\mathbb{F}$ be a class of structures, define:

1. $\mathfrak{Cm}(\mathbb{F}) = \{\mathcal{B} \mid \mathcal{B} \cong \mathfrak{Cm}(\mathcal{F}) \text{ for some } \mathcal{F} \in \mathbf{F}\}$.

2. $\mathbf{Up}(\mathbb{F})$ is the class of structures isomorphic to disjoint unions of elements of $\mathbb{F}$.

3. $\mathbf{S}(\mathbb{F})$ is the closure of $\mathbb{F}$ under inner substructures.

Let A be a non-empty subset of a Boolean algebra $\mathcal{B}$, A is a *filter*, if A is closed under finite infima and it is upward closed. A is an ultrafilter, if it has no non-trivial extensions. That is, if $A \subseteq A'$, then $A' = \mathcal{B}$. This is a well-known fact that every filter can be extended to a maximal one using Zorn's lemma.

The following definition is due to, for example, [10, Definition 5.40].

Let $\mathcal{B} = \langle B, +, -, 0, 1, (\Omega_i)_{i \in I} \rangle$ be a BAO and $\mathbf{Spec}(\mathcal{B})$ the set of its ultrafilters. The ultrafilter frame of $\mathcal{B}$ (or the canonical frame) is a relational structure $\mathcal{F}_{\mathcal{B}} = \langle \mathbf{Spec}(\mathcal{B}), R_{\Omega_i} \rangle$ such that for all ultrafilters U_1, U_2 one has

[1] Or alternatively, a generated subframe

$$\mathbf{Spec}(\mathcal{B}) \models R_{\Omega_i}(U_1, U_2) \text{ iff } \{\Omega_i(b) \mid b \in U_1\} \subseteq U_2.$$

Given $\mathcal{B}$ be a BAO, we denoted as $\mathcal{B}^+$ as the complex algebra of the canonical frame $\mathfrak{Cm}(\mathcal{F}_{\mathcal{B}})$, that is, *the canonical extension* of $\mathcal{B}$. A class of BAOs $\mathbf{K}$ is canonical, if it is closed under canonical extensions. That is, $\mathcal{B}^+ \in \mathbf{K}$ whenever $\mathcal{B} \in \mathbf{K}$.

Theorem 2. *Let $\mathcal{A}$, $\mathcal{B}$ be BAOs,*

1. *There exists $\iota : \mathcal{A} \hookrightarrow \mathcal{A}^+$ such that $\iota : a \mapsto \{\gamma \in \mathbf{Spec}(\mathcal{A}) \mid a \in \gamma\}$.*
2. *$i : \mathcal{A} \hookrightarrow \mathcal{B}$ implies $i^+ : \mathcal{A}^+ \hookrightarrow \mathcal{B}^+$*

4 Representable cylindric algebras

Let α be an ordinal. Denote $\{f \mid f\alpha \to U\}$ as ${}^{\alpha}U$. x_i stands for $x(i)$, where $x \in {}^{\alpha}U$ and $i < \alpha$.

A subset of ${}^{\alpha}U$ is an α-ry relation on U. For $i, j < \alpha$, the *i, j-diagonal* D_{ij} is the set of all elements of ${}^{\alpha}U$ such that $y_i = y_j$.

If $i < \alpha$ and X is an α-ry relation on U, then the i-th cylindrification $C_i X$ is the set of all elements of U that agree with some element of X on each coordinate except, perhaps, the i-th one. To be more precise,

$$C_i X = \{y \in {}^{\alpha}U \mid \exists x \in X \forall i < \alpha \, (i \neq j \Rightarrow y_j = x_j)\}.$$

We define the following equivalence relation for $i < \alpha$ and $x, y \in {}^{\alpha}U$:

$$x \equiv_i y \Leftrightarrow \forall j \in \alpha \, (i \neq j \Rightarrow x(i) = y(j))$$

Then one may reformulate the definition of the i-th cylindrification in the following way:

$$C_i X = \{y \in {}^{\alpha}U \mid \exists x \in X \;\; x \equiv_i y\}$$

According to this version of the definiton, one may think of the cylindrification as an $\mathbf{S}5$ modal operator.

A cylindic set algebra of dimension α is an algebra consisting of a set S of α-ry relation on some base set U with the constants and operations $0 = \varnothing$, $1 = {}^{\alpha}U$, $\cap$, $-$, the diagonal elements $(D_{ij})_{i,j<\alpha}$, the cylindrifications $(C_i)_{i<\alpha}$. A generalised cylindric set algebra of dimension α is a subdirect of cylindric algebras that have dimension α. $\mathbf{Cs}_{\alpha}$ denotes the class of all cylindric set algebras of dimension α.

A cylindric algebra of dimension α is an algebra $\mathcal{C} = \langle \mathcal{B}, \{c_i\}_{i<\alpha}, \{d_{ij}\}_{i,j<\alpha} \rangle$ such that

- $\mathcal{B}$ is a Boolean algebra, for each $i, j < \alpha$ c_i is an operator and $d_{ij} \in \mathcal{B}$
- For each $i < \alpha$, $a \leqslant c_i a$, $c_i(a \cdot c_i b) = c_i a \cdot c_i b$ and $d_{ii} = 1$
- For every $i, j < \alpha$, $c_i c_j a = c_j c_i a$
- If $k \neq i, j < \alpha$, then $d_{ij} = c_k(d_{ij} \cdot d_{jk})$
- If $i \neq j$, then $c_i(d_{ij} \cdot a) \cdot c_i(d_{ij} \cdot -a) = 0$

$\mathbf{CA}_{\alpha}$ is the class of all cylindric algebras of dimension α.

One may define a representation of a cylindric algebra explicitly in the following way:

Let $\mathcal{A}$ be a cylindric algebra of dimension α. A representation of $\mathcal{A}$ over the non-empty domain X is a map $f : \mathcal{A} \hookrightarrow 2^{{}^{\alpha}U}$ such that:

1. $f(1) = \bigcup_{i \in I}{}^{\alpha} X_i$ for some disjoint family $\{X_i\}_{i \in I}$ where each $X_i \subseteq X$
2. $h : \mathcal{A} \to 2^{f(1)}$ is a representation of a Boolean reduct
3. for all $\lambda, \eta < \alpha$, $x \in h(d_{\lambda\eta})$ iff $x_\lambda = x_\eta$
4. for all $\lambda < \alpha$ and $a \in \mathcal{A}$, $x \in h(c_\lambda(a))$ iff there is $y \in X$ such that $x[\lambda \mapsto y] \in h(a)$

An α-dimensional cylindric algebra C is representable, if there exists a representation of h. $\mathbf{RCA}_\alpha$ is the class of all representable cylindric algebras that have dimension α. In particular, we are interested in the case $\alpha = \omega$.

It is well known that $\mathbf{RCA}_\alpha$ is a variety, $\mathbf{RCA}_\alpha$ ($\alpha \leqslant 2$) is finitely axiomatisable and $\mathbf{RCA}_\alpha$ ($2 < \alpha < \omega$) has no finite axiomatisation, see [4].

Let $\mathcal{A} \in \mathbf{CA}_\omega$, then $\mathcal{A}$ has a *complete representation*, if its representation preserves all existing suprema. In other words, $\mathcal{A}$ is *completely representable.*

5 $\mathbf{RCA}_\omega$ and canonicity

The following definition of an ω-frame is due to [9]. A cylindric ω-frame is a structure $\mathcal{F} = \langle W, (R_i)_{i<\omega}, (E_{ij})_{i,j<\omega} \rangle$ where $(R_i)_{i<\omega}$ are binary relations and $(E_i j)_{i,j<\omega}$ are unary relations such that, for all $i, j, k < \omega$:

1. Every R_i is an equivalence relation on W,
2. $R_i \circ R_j = R_j \circ R_i$, that is, the set $(R_i)_{i<\omega}$ forms a commutative semigroup under composition.
3. For all $x \in W$, $E_{ii}(x)$ holds.
4. For all $x, y, z \in W$, xR_iy & $E_{ij}(y)$ & xR_iz & $E_{ij}(y)$ implies $y = z$.
5. For all $x \in W$, $E_{ij}(x)$ iff there exists $y \in W$ such that xR_ky, $E_{ik}(y)$, and $E_{kj}(y)$.

$\mathcal{C}\mathfrak{a}_\omega$ is the class of all ω-frames.

If $\mathcal{F} \in \mathcal{C}\mathfrak{a}_\omega$ and $x \in \mathcal{F}$, then $\mathcal{F}^x$ is a generated subframe generated by x, which is defined standardly. Generally, $\mathcal{F}_1$ is a generated subframe of $\mathcal{F}_2$, if $\underline{\mathcal{F}_1} \subseteq \underline{\mathcal{F}_2}$ and $\underline{\mathcal{F}_1}$ is closed under under R_{i2} equivalences for every $i < \omega$. That is:

$$\text{For all } i < \omega \text{ and } x \in \mathcal{F}_1, \text{ we have } R_{i2}(x) \subseteq \mathcal{F}_1 \text{ and, thus, } R_{i1}(x) = R_{i2}(x).$$

We have the following connection betweens ω-frames and their generated subframes, which is standard for modal logic:

Proposition 1. *Let $\mathcal{F} \in \mathcal{C}\mathfrak{a}_\omega$, then*

1. $\mathcal{F} = \coprod_{x \in \mathcal{F}} \mathcal{F}^x$,
2. $\mathfrak{Cm}(\mathcal{F}) \cong \prod_{x \in \mathcal{F}} \mathfrak{Cm}(\mathcal{F}^x)$,
3. $\mathfrak{Cm}(\mathcal{F}^x)$ *is subdirectly irreducible.*

It is known that $\mathcal{C}\mathfrak{a}_\omega$ forms an elementary class, since one can express the conditions of an ω-frame with the first-order language.

The following fact is by Venema, see [9, Proposition 2.1.5]:

Proposition 2. *An ω-frame $\mathcal{F}$ is cylindric iff $\mathfrak{Cm}(\mathcal{F})$ is a cylindric algebra of dimension ω.*

A cylindric ω-frame $\mathcal{F}$ is completely representable, if $\mathfrak{Cm}(\mathcal{F})$ is completely representable as a cylindric algebra of dimension ω.

We are interested in the special case of cylindric ω-frames called Cartesian structure of dimension ω. To be more precise:

Let U be a set and $V \subseteq {}^{\omega}U$ be a non-empty subset of the full Cartesian space of dimension ω, then an α-dimension Cartesian structure generated by V is an ω-frame $\mathfrak{S}(V) = \langle V, (R_i)_{i<\omega}, (E_{ij})_{i,j<\omega} \rangle$ such that:

1. $R_i = \{(w, v) \mid w, v \in V, w_k = w_k, k < \omega, i \neq k\}$
2. $E_{ij} = \{w \in V \mid w_i = w_j\}$

$\mathfrak{S}({}^{\omega}U)$ is the full ω-dimensional Cartesian structure. $\mathcal{F}\mathfrak{ct}_{\omega}$ is the class of all full ω-dimensional Cartesian structures.

Clearly $\mathcal{F}\mathfrak{ct}_{\omega} \subseteq \mathcal{C}\mathfrak{a}_{\omega}$.

We have the following connection between $\mathbf{RCA}_{\omega}$, $\mathbf{IGs}_{\omega}$, and complex algebras of full Cartesian structures:

$$\mathbf{RCA}_{\omega} = \mathbf{IGs}_{\omega} = \mathbf{S}\mathfrak{Cm}\mathbf{Ud}\mathcal{F}\mathfrak{ct}_{\omega} = \mathbf{SP}\mathfrak{Cm}\mathcal{F}\mathfrak{ct}_{\omega}.$$

This follows from the fact that $\mathbf{Cs}_{\omega} = \mathfrak{Cm}\mathcal{F}\mathfrak{ct}_{\omega}$. Every generalised cylindric set algebra is a subdirect product of cylindric set algebras, thus, a generalised cylindric set algebra is a complex algebra of disjoint union of some full Cartesian spaces. But $\mathbf{RCA}_{\omega}$ is the closure of $\mathbf{Cs}_{\omega}$ under isomorphism.

The weak Cartesian space with base U and dimension ω determined by $x \in {}^{\omega}U$ is the set:

$${}^{\omega}U^{(x)} = \{y \in {}^{\omega}U \mid |\{k < \omega \mid x_k \neq y_k\}| < \aleph_0\}$$

$\mathfrak{S}({}^{\omega}U^{(x)})$ is a weak Cartesian structure of dimension ω. $\mathcal{W}\mathfrak{ct}_{\omega}$ is the class of all weak Cartesian structure of dimension ω up to isomorphism. Note that we have $\mathcal{W}\mathfrak{ct}_{\omega} \subseteq \mathcal{C}\mathfrak{a}_{\omega}$.

Every cylindric set algebra is a subalgebra of some complex algebra induced by an ω-dimensional Cartesian structure. In other words,

Lemma 1. *$\mathbf{ICs}_{\omega} = \mathbf{S}\mathfrak{Cm}\mathcal{F}\mathfrak{ct}_{\omega}$.*

In this section, we reproduce the results related to characterisation $\mathbf{RCA}_{\omega}$. The following results are due to Goldblatt [3]. This denotes that a cylindric algebra of dimension algebra is representable iff it is isomorphic to a subalgebra of the complex algebra of disjoint sum of some full ω-dimensional Cartesian structure. Assuming the duality, this is equivalent to the standard definition of representability formulated in terms of sublagebras of subdirect products.

Lemma 2. **??** $\mathbf{RCA}_{\omega} = \mathbf{S}\,\mathfrak{Cm}\mathbb{S}\mathbf{Ud}\mathcal{F}\mathfrak{ct}_{\omega} = \mathbf{S}\,\mathfrak{Cm}\mathbb{S}\mathbf{Ud}\mathcal{W}\mathfrak{ct}_{\omega} = \mathbf{IGws}_{\omega}$

Proof. □

Here we use the following fact related to canonical varieties generated by some class of complex algebras. Let $\mathbf{K}$ be an elementary class of relational structures, then:

If $\mathbf{K}$ is closed under p-morphic images, generated subframes, and disjoint unious,
then $\mathbf{S}\mathfrak{Cm}\mathbf{K}$ is a canonical variety.

One may think of this fact a more abstract version of Fine's theorem which claims that every elementary modal logic is canonical [2]. This version denotes the same fact, but it is formulated in terms of varieties BAOs generated by complex algebras of some atom structures. We provide a more precise formulation of the fact above.

Let $\mathbf{K}$ be a class of frames, denote the closure of $\mathbf{K}$ under ultraproducts as $\mathbf{PuK}$.

Proposition 3. *Let $\mathbf{K}$ be a class of frames, then $\mathbf{PuK} \subseteq \mathbb{HS}\mathbf{UdK}$ implies that $\mathbf{S}\mathfrak{Cm}\mathbb{S}\mathbf{UdK}$ is a canonical variety.*

This is a specialised version of [3, Theorem 4.4] formulated for dimension ω.

Theorem 3. *$\mathbf{RCA}_\omega$ is a canonical variety.*

Proof. We have $\mathbf{RCA}_\omega = \mathbf{S}\mathfrak{Cm}\mathbb{S}\mathbf{Ud}\mathcal{F}\mathfrak{ct}_\omega$. That's enough to show that $\mathbf{Pu}\mathcal{F}\mathfrak{ct}_\omega \subseteq \mathbb{HS}\mathbf{Ud}\mathcal{F}\mathfrak{ct}_\omega$. For that, we need the following claim:

Claim 1. $\mathbf{Pu}\mathcal{F}\mathfrak{ct}_\omega \subseteq$

$\square$

6 Canonicity of $\mathbf{RCA}_n$ for finite n

In this section we consider classes $\mathbf{RCA}_n$, where $n < \omega$ is finite.

We provide the complete proof of the following theorem [7, Theorem 3.4.3].

Theorem 4. *Let $\mathcal{A} \in \mathbf{CA}_n$, then $\mathcal{A}$ is representable iff $\mathcal{A}^+$ is completely representable.*

For that we need such model theoretic notions as saturation and types, see [8, Section 6.3].

Let $\mathcal{M}$ be a first-order structure of a signature L and $S \subseteq \mathcal{M}$. Let $L(S)$ be an extension of L with copies of elements from S as additional constants. We assume that $Cnst(L)$ and S are disjoint.

1. Let $n < \omega$, an n-type over S is a set $\mathcal{T}$ of $L(S)$ formulas $A(\overline{x})$, where $\overline{x}$ is a fixed n-tuple of elements from S. Notation: $\mathcal{T}(\overline{x})$. A type is an n-type for some $n < \omega$.

2. An n-type $\mathcal{T}(\overline{x})$ is realised in $\mathcal{M}$, if there exists $\overline{m} \in \mathcal{M}^n$ such that $\mathcal{M} \models A(\overline{m})$ for every $A \in \mathcal{T}(\overline{x})$. $\mathcal{M}$ omits $\mathcal{T}(\overline{x})$, if $\mathcal{T}(\overline{x})$ is not realised in $\mathcal{M}$.

3. $\mathcal{T}(\overline{x})$ is finitely satisfied in $\mathcal{M}$, if every finite subtype $\mathcal{T}_0(\overline{x}) \subseteq \mathcal{T}(\overline{x})$ is realised in $\mathcal{M}$. We can reformulate that as $\mathcal{M} \models \exists \overline{a} \bigwedge\limits_{A \in \mathcal{T}_0} A(\overline{a})$.

4. Let T be a theory, then a type $\mathcal{T}$ over the empty set of constants is T-consistent, if there exists a model $\mathcal{M} \models T$ such that $\mathcal{T}$ is finitely satisfied in $\mathcal{M}$.

5. Let κ be a cardinal, then $\mathcal{M}$ is κ-saturated, if for every $S \subseteq \mathcal{M}$ with $|S| < \kappa$ every finitely satisfied 1-type $\mathcal{T}$ is realised in $\mathcal{M}$.

By default, a saturated model is an ω-saturated model for us.

The useful facts, they are from [1] and [8]:

Fact 1. *Let $\mathcal{M}$ be an FO-structue and κ a cardinal, then:*

1. *$\mathcal{M}$ is κ-saturated, iff every finitely satisfiable α-type (an arbitrary $\alpha \leqslant \kappa$) with fewer than κ parameters is realised in $\mathcal{M}$.*

2. *If $\mathcal{M}$ is κ-saturated, then $\mathcal{M}$ is λ-saturated for every $\lambda < \kappa$.*

3. *Every consistent theory has a κ-saturated model and every model has an elementary κ-saturated extension.*

4. *Let $(\mathcal{M}_i)_{i<\omega}$ a family of structures of the (at most) countable signature and D a non-principal ultrafilter over ω, then $\Pi_D \mathcal{M}_i$ is ω_1-saturated.*

6.1 Proof of Theorem 4

Let $\mathcal{A} \in \mathbf{CA}_n$, then if $\mathcal{A}$ is completely representable, then h, a complete representation of $\mathcal{A}$, is atomic. That is, $(a_1, \ldots, a_n) \in h(1)$, then $(a_1, \ldots, a_n) \in h(y)$ for some $y \in \mathrm{At}(\mathcal{A})$.

Let $\mathcal{A}$ be a cylindric algebra of dimension $n < \omega$. $L(\mathcal{A})$ is the first-order language that consists of equality plus n-ary predicate letters $(R^n_a)_{a \in \mathcal{A}}$. The $L(\mathcal{A})$-theory $T_{\mathcal{A}}$ consists of the following sentences:

1. $A_+(a,b,c) := \forall x_1, \ldots, x_n \, (R_a(x_1, \ldots, x_n) \leftrightarrow R_b(x_1, \ldots, x_n) \vee R_c(x_1, \ldots, x_n))$. Informally, that means $\mathcal{A} \models a = b + c$.
2. $A_-(a,b) := \forall x_1, \ldots, x_n \, (R_a(x_1, \ldots, x_n) \leftrightarrow \neg R_b(x_1, \ldots, x_n))$. That is, $\mathcal{A} \models a = -b$.
3. $A_{\neq 0}(a) := \exists x_1, \ldots, x_n R_a(x_1, \ldots, x_n)$. That is, $\mathcal{A} \models a \neq 0$.
4. $A_{c_i}(a) := \forall x_1, \ldots, x_n (R_{c_i a}(x_1, \ldots, x_n) \leftrightarrow \exists y_1, \ldots y_n (R_a(y_1, \ldots, y_n) \wedge x_i = y_j))$, for $i < n$ and $j < n$ such that $i \neq j$. Informally, $\mathcal{A} \models c_i a = 1$.
5. $A_{d_{ij}} := \forall x_1, \ldots, x_n (R_{d_{ij}}(x_1, \ldots, x_n) \leftrightarrow x_i = x_j)$, for $i, j < n$.

In fact, we need to show the following implication:

If $\mathcal{A}$ is representable, then A^+ is completely representable.

Assume that $\mathcal{A}$ is representable, then the theory $T(\mathcal{A})$ is consistent, then it has an ω-saturated model $\mathcal{M}$ by Fact 3. We have the following claim:

Claim 2. *The set $U_{x_1,\ldots,x_n} = \{a \in \mathcal{A} | \mathcal{M} \models R_a(x_1, \ldots, x_n)\}$ is an ultrafilter of $\mathcal{A}$, for $x_1, \ldots, x_n \in \mathcal{M}$ with $R_1(x_1, \ldots, x_n)$.*

Those $U_{x_1,\ldots,x_n}$'s allow us to represent atoms of $\mathcal{A}^+$.

We define a representation of $\mathcal{A}^+$ as a map $h : \mathcal{A}^+ \to 2^{\mathcal{M}^n}$ such that:

$$h : S \mapsto \{(x_1, \ldots, x_n) \in 1^{\mathcal{M}} \mid U_{x_1,\ldots,x_n} \in S\}, \text{ for } S \in \mathrm{Spec}(\mathcal{A}).$$

Claim 3. *Let $A_1, A_2 \in \mathrm{Spec}(\mathcal{A})$*

1. $h(0^{\mathcal{A}^+}) = \varnothing$
2. $h(-A_1) = -h(A_1)$
3. $h(1^{\mathcal{A}^+}) = 1^{\mathcal{M}}$
4. *If $S \subseteq \mathrm{Spec}(\mathcal{A})$, then* $h(\bigcup S) = \bigcup\limits_{U \in S} h(U)$

In particular, h is a Boolean homomorphism.

Proof.

1. $h(0^{\mathcal{A}^+}) = h(\varnothing) = \varnothing$.

2. From the definiton of h.

3. $h(-A_1) = -h(A_1)$

   Let $x_1, \ldots, x_n \in 1^{\mathcal{M}}$, then we have:

$$(x_1, \ldots, x_n) \in h(-A_1) \text{ iff } U_{x_1,\ldots,x_n} \in -A_1 \text{ iff } U_{x_1,\ldots,x_n} \notin A_1 \text{ iff } (x_1, \ldots, x_n) \notin h(A_1)$$

4. Let $S = \bigcup_{i \in I} S_i$, where $S_i \in \mathrm{Spec}(\mathcal{A})$ for every $i \in I$. Let $(x_1, \ldots, x_n) \in 1^{\mathcal{M}}$, then we have:

$$(x_1, \ldots, x_n) \in h(\bigcup_{i \in I} S_i) \text{ iff } f_{x_1,\ldots,x_n} \in \bigcup_{i \in I} S_i \text{ iff } \exists i \in I \;\; f_{x_1,\ldots,x_n} \in S_i \text{ iff}$$
$$\exists i \in I \;\; (x_1, \ldots, x_n) \in h(S_i) \text{ iff } (x_1, \ldots, x_n) \in \bigcup_{i \in I} S_i$$

□

Claim 4. *h is injective.*

Proof. Let $U \in \mathrm{Spec}(\mathcal{A})$. The first is to show that $h(U)$ is non-empty. The following n-type:

$$T(x_1, \ldots, x_n) = \{R_a(x_1, \ldots, x_n) \mid a \in U\}$$

if finitely satisfied in $\mathcal{M}$.

Consider $T_0 = \{R_{a_1}(x_1, \ldots, x_n), \ldots, R_{a_k}(x_1, \ldots, x_n)\} \subseteq T$. Then $a_1, \ldots, a_k \in U$ and $a = a_1 \cdot \cdots \cdot a_k \in U$. By the instance of the $A_{\neq 0}(a)$-axiom, we have $\mathcal{M} \models \exists x_1, \ldots, x_n R_a(x_1, \ldots, x_n)$. $a \leqslant a_i$ for $i \leqslant k$, so we have $\mathcal{M} \models \exists x_1, \ldots, x_n R_{a_i}(x_1, \ldots, x_n)$ for every a_i with $i \leqslant k$ by the instance of the $A_+(a_i, a, a)$-axiom. That makes every finite subtype of T satisfiable, thus the whole type is finitely satifiable in $\mathcal{M}$. $\mathcal{M}$ is ω-saturated, then T is realised in $\mathcal{M}$ by some $(x_1, \ldots, x_n) \in \mathcal{M}^n$ and, moreover, $\mathcal{M} \models 1(x_1, \ldots, x_n)$. As we have already said, $U_{x_1,\ldots,x_n}$ is an ultrafilter, but $U_{x_1,\ldots,x_n} \subseteq U$, thus $U = U_{x_1,\ldots,x_n}$, so $(x_1, \ldots, x_n) \in h(U)$.

That makes h one-to-one. □

Claim 5.

1. $h({c_i}^{\mathcal{A}^+} U) = C_i(h(U))$

2. $h(d_{ij}^{\mathcal{A}^+}) = D_{ij} \subseteq \mathrm{Spec}(\mathcal{A})$

Proof.

1. Let $\overline{x} = (x_1, \ldots, x_n) \in \mathcal{M}^n$ and $S \subseteq \mathrm{Spec}(\mathcal{A})$. Assume $(x_1, \ldots, x_n) \in h({c_i}^{\mathcal{A}^+} S)$.

   Let us show that $\overline{x} \in C_i(h(S))$, that is, there exists $\overline{y} = (y_1, \ldots, y_n) \in h(S)$ such that $\overline{x} \equiv_i \overline{y}$.

   Then $\mathcal{M} \models 1(x_1, \ldots, x_n)$ and $U_{x_1,\ldots x_n} \in {c_i}^{\mathcal{A}^+} S$. But $\mathcal{A}^+$ is the complex algebra of the ultrafilter frame $\mathcal{F}_{\mathcal{A}}$. Then we have:

$${c_i}^{\mathcal{A}^+} S = \{U_1 \in \mathrm{Spec}(\mathcal{A}) \mid \exists U' \in S \; U_1 R_i U'\}$$

   Then there must be an ultrafilter $U' \in S$ such that $U_{x_1,\ldots x_n} R_i U'$, that is, $c_i a \in U_{x_1,\ldots x_n}$ whenever $a \in U'$. Hence $\mathcal{M} \models R_{c_i}(x_1, \ldots x_n)$. By the $A_{c_i}(a)$-axiom, we have

$$\mathcal{M} \models \exists z_1, \ldots, z_n (R_a(z_1, \ldots, z_n) \wedge x_i = z_j) \text{ for } i < n \text{ and } j < n \text{ such that } i \neq j.$$

Consider the following n-type with free variables $z_1, \ldots, z_n$ and parameters $x_1, \ldots, x_n \in \mathcal{M}$:

$$T(z_1, \ldots, z_n) = \{R_a(z_1, \ldots, z_n) \wedge x_i = z_j \mid i < n, j < n, i \neq j, a \in U'\}.$$

Let us show that $T(z_1, \ldots, z_n)$ is finitely satifsiable in $\mathcal{M}$. Consider a finite subset of T, say $T_0 = \{R_{b_k}(z_1, \ldots, z_n) \wedge x_i = y_j \mid i < n, j < n, i \neq j, b_k \in U', k < \omega\}$. We put $p = p_1 \cdot \cdots \cdot p_k$ and $p \in U'$ since U' is a filter. Then we have:

$$\mathcal{M} \models \exists z_1, \ldots, z_n (R_b(z_1, \ldots, z_n) \wedge x_i = z_j) \text{ for } i < n \text{ and } j < n \text{ such that } i \neq j$$

Thus, we have, as required:

$$\mathcal{M} \models \exists z_1, \ldots, z_n \bigwedge_{i=1}^{k} (R_{b_k}(z_1, \ldots, z_n) \wedge x_i = z_j) \text{ for } i < n \text{ and } j < n \text{ such that } i \neq j.$$

As above, using ω-saturation, we conclude that T is realised in $\mathcal{M}$ at an n-tuple $(y_1, \ldots, y_n) = \overline{y}$. Then we have:

$$\mathcal{M} \models 1(\overline{y}),\ \overline{x} \equiv_i \overline{y},\ U_{\overline{y}} \supseteq U'$$

Then $U_{\overline{y}} = U'$, then $\overline{y} \in h(S)$. Then $\overline{x} \in C_i(h(S))$.

Suppose for the converse, $\overline{x} = (x_1, \ldots, x_n) \in C_i(h(S))$. We need $\overline{x} \in h(c_i(S))$. Then there exists $\overline{y} = (y_1, \ldots, y_n)$ such that $\overline{x} \equiv_i \overline{y}$ and $\overline{y} \in h(S)$. Then there exists an ultrafilter $U_{y_1,\ldots,y_n} \in S$. Let us show that $\mathcal{M} \models 1(x_1, \ldots, x_n)$ and $U_{x_1,\ldots,x_n} \in c_i U_{y_1,\ldots,y_n}$. Let $a \in U_{y_1,\ldots,y_n}$. Then we have $\mathcal{M} \models R_a(y_1, \models, y_n)$. By the $A_{c_i}(a)$ axiom, we have $\mathcal{M} \models R_{c_i a}(x_1, \ldots, x_n)$. Then $\mathcal{M} \models 1(x_1, \ldots, x_n)$ and $c_i a \in U_{x_1,\ldots,x_n}$, thus, $\overline{x} \in h(c_i(S))$.

2. Let us show that h preserves cylindrifications.

   Let $(x_1, \ldots, x_n) \in \mathcal{M}^n$. Then $(x_1, \ldots, x_n) \in D_{ij}$ iff $\mathcal{M} \models 1(x_1, \ldots, x_n)$ and $x_i = x_j$ iff $U_{x_1,\ldots,x_n} \in d_{ij}^{\mathcal{A}^+} = \{U \in \mathrm{Spec}(\mathcal{A}) \mid d_i j \in U\}$ iff $\mathcal{M} \models d_{ij}^{\mathcal{M}}(x_1, \ldots, x_n)$.

□

7 Representability games

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