

Finite model property for residuated semigroups and related remarks

Daniel Rogozin

1 Finite networks for atomic formulas

Let $PV = \{p_i \mid i < \omega\}$ be the set of propositional variables (or atomic types). The set of formulas is generated by the following grammar:

$$\varphi, \psi ::= p \mid (\varphi \bullet \psi) \mid (\varphi \backslash \psi) \mid (\varphi / \psi)$$

The Lambek calculus is defined as a Gentzen-style sequent calculus:

$$\begin{array}{c} \overline{p \rightarrow p} \text{ ax} \\[10pt] \frac{\Gamma \rightarrow \varphi \quad \Delta, \psi, \Theta \rightarrow \theta}{\Delta, \Gamma, \varphi \backslash \psi, \Theta \rightarrow \theta} \backslash \rightarrow \qquad \frac{\varphi, \Pi \rightarrow \psi}{\Pi \rightarrow \varphi \backslash \psi} \rightarrow \backslash, \Pi \text{ is non-empty} \\[10pt] \frac{\Gamma \rightarrow \varphi \quad \Delta, \psi, \Theta \rightarrow \theta}{\Delta, \psi / \varphi, \Gamma, \Theta \rightarrow \theta} / \rightarrow \qquad \frac{\Pi, \varphi \rightarrow \psi}{\Pi \rightarrow \psi / \varphi} \rightarrow /, \Pi \text{ is non-empty} \\[10pt] \frac{\Gamma, \varphi, \psi, \Delta \rightarrow \theta}{\Gamma, \varphi \bullet \psi, \Delta \rightarrow \theta} \bullet \rightarrow \qquad \frac{\Gamma \rightarrow \varphi \quad \Delta \rightarrow \varphi}{\Gamma, \Delta \rightarrow \varphi \bullet \psi} \rightarrow \bullet \end{array}$$

1.1 Completeness

Theorem 1. *Let RS be the class of all residuated semigroups, then $\Gamma \rightarrow \varphi$ iff $RS \models \Gamma \rightarrow \varphi$*

2 Representability networks for at most countable residuated semigroups

2.1 Relational residuated semigroups as Kripke models

One can introduce Kripke-style relational semantics for the Lambek calculus as follows. Let W be a non-empty set and R a transitive relation on W . We consider models of the kind $M = (R, \vartheta)$, where $\vartheta : PV \rightarrow 2^R$. The truth definition is inductive:

- $\mathcal{M}, (x, y) \models p_i$ iff $(x, y) \in \vartheta(p_i)$,
- $\mathcal{M}, (x, y) \models \varphi \bullet \psi$ iff there exists $z \in W$ such that $(x, z), (z, y) \in R$ and $\mathcal{M}, (x, z) \models \varphi$ and $\mathcal{M}, (z, y) \models \psi$

- $\mathcal{M}, (x, y) \models \varphi \backslash \psi$ iff for all $z \in W$ such that if $(z, x) \in R$ and $\mathcal{M}, (z, x) \models \varphi$, then $\mathcal{M}, (z, y) \models \psi$
- $\mathcal{M}, (x, y) \models \varphi / \psi$ iff for all $z \in W$ such that if $(y, z) \in R$ and $\mathcal{M}, (y, z) \models \psi$, then $\mathcal{M}, (x, z) \models \varphi$
- $\mathcal{M}, (x, y) \models \varphi_1, \varphi_2, \dots, \varphi_n \rightarrow \varphi$ iff $\mathcal{M}, (x, y) \models \varphi_1 \bullet \varphi_2 \bullet \dots \bullet \varphi_n$ implies $\mathcal{M}, (x, y) \models \varphi$.

According to the definition above, to refute a sequent $\varphi_1, \varphi_2, \dots, \varphi_n \rightarrow \varphi$, we have to find a transitive binary relation R , some valuation $\vartheta : PV \rightarrow 2^R$ and $(x, y) \in R$ such that $\mathcal{M}, (x, y) \models \varphi_1 \bullet \varphi_2 \bullet \dots \bullet \varphi_n$, but $\mathcal{M}, (x, y) \not\models \varphi$. Alternatively, one can reformulate that condition as

$$(x, y) \in \|\varphi_1\|; \|\varphi_2\|; \dots; \|\varphi_n\|, \text{ but } (x, y) \notin \|(x, y)\|$$

2.2 Relational representation of residuated semigroups: a game-theoretic approach

Let \mathcal{A} be a residuated semigroup, an \mathcal{A} -prenetwork is a triple $\mathcal{N} = (V, E, l)$, where where (V, E) is a directed graph and $l : E \rightarrow \mathcal{A}$ is a labelling function. A prenetwork is a network if the following conditions hold:

- E has no loops and it is transitive,
- $l(x, z) \leq l(x, y); l(y, z)$, whenever $(x, y), (y, z) \in E$, for all $x, y, z \in V$,
- For all $a \in \mathcal{A}$, for all $x \in U$, there is some $u \in U$ such that $l(u, x) = a$,
- For all $a \in \mathcal{A}$, for all $y \in U$, there is some $v \in U$ such that $l(y, v) = a$,
- For all $a, b, c \in \mathcal{A}$, for all $x, y \in U$, if $c \leq a; b$, $(x, y) \in E$ and $l(x, y) = c$, then there exists $z \in U$ such that $l(x, y) = a$ and $l(y, z) = b$.

Let $n \leq \omega$, define a game $\mathcal{G}(\mathcal{A})_n$ for two players \forall and \exists by induction on n .

1. step 0

\forall picks a pair of elements $a, b \in \mathcal{A}$ such that $a \not\leq b$. \exists must respond with a network $\mathcal{N}_0 = (\{x, y\}, \{(x, y)\}, l_0 : (x, y) \mapsto a)$:

$$x \xrightarrow{a} y$$

2. step $n + 1 < \omega$

Suppose the networks:

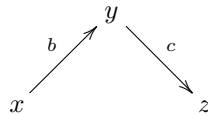
$$\mathcal{N}_0 \subseteq \mathcal{N}_1 \subseteq \dots \subseteq \mathcal{N}_n$$

have been already constructed.

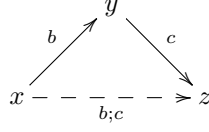
There are four different options:

(a) Composition move

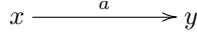
\forall picks $x, y, z \in \mathcal{N}_n$ such that $b = l_n(x, y)$ and $c = l_n(y, z)$:



\exists has to respond with $\mathcal{N}_{n+1} = (V_n, E_n \cup \{(x, z)\}, l_{n+1})$ where $l_{n+1}(x, z) = b; c$ and $l_{n+1}(x', y') = l_n(x', y')$ for $(x', y') \in E_n$.



(b) **Witness move** \forall picks $(x, y) \in E_n$ such that $l_n(x, y) = a$ and $a \leq b; c$:



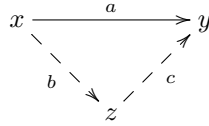
\exists has to respond with $\mathcal{N}_{n+1} = (V_n \cup \{z\}, E_n \cup \{(x, z), (z, y)\}, l_{n+1})$, where

$$l_{n+1}(x, z) = b$$

$$l_{n+1}(y, z) = c$$

$$l_{n+1}(p) = l_n \text{ for others } p \in E_n$$

The latter can be visualised with the following triangle:



(c) **Left residual move**

(d) **Right residual move**

Theorem 2. *Let \mathcal{A} be a at most countable residuated semigroup, then*

1. \exists has a winning strategy in $\mathcal{G}_\omega(\mathcal{A})$

$$rep(a) = \{(x, y) \mid l(x, y) \leq a\}$$

$$(a) \ rep(a; b) = rep(a); rep(b)$$

$$(b) \ rep(a \setminus b) = rep(a) \setminus rep(b)$$

$$(c) \ rep(a/b) = rep(a)/rep(b)$$

2. \mathcal{A} is representable.

TODO: check if the representability class is closed under products, subalgebras and ultra-products. Check the criterion for the Horn formulas. Closed under H? can't be defined by equations?

3 Step-by-step construction for the FMP

3.1 Example 1

Consider a sequent $\varphi_1 \bullet \varphi_2 \rightarrow \varphi_1 \bullet (\psi_1 \setminus \psi_2)$, which is known to unprovable in the Lambek calculus. We would like to construct a finite model on a network such that there exists $(u, v) \models \varphi_1 \bullet \varphi_2 \rightarrow \varphi_1 \bullet (\psi_1 \setminus \psi_2)$, that is, $(u, v) \in \|\varphi_1\|; \|\varphi_2\|$ and $(u, v) \notin \|\varphi_1\|; (\|\psi_1\| \setminus \|\psi_2\|)$.

The construction is step-by-step:

1. Step 0

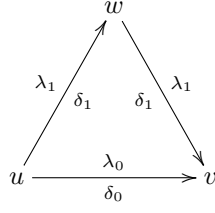
We construct the edge (u, v) labelled by a pair of sets λ_0 and δ_0 :

$$u \xrightarrow[\delta_0]{\lambda_0} v$$

$$\begin{aligned}\lambda_0(u, v) &= \{\varphi_1 \bullet \varphi_2, -(\varphi_1 \bullet (\psi_1 \setminus \psi_2))\} \\ \delta_0(u, v) &= \{\varphi_1 \bullet \varphi_2\}\end{aligned}$$

2. Step 1

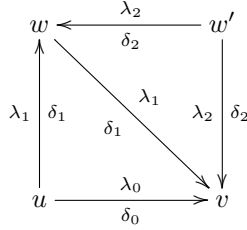
We have to create a witness for $\varphi_1 \bullet \varphi_2 \in \delta_0(u, v)$ by adding a vertice w :



$$\begin{aligned}\lambda_1(u, w) &= \{\varphi_1\} \\ \lambda_1(w, v) &= \{\varphi_2, -(\psi_1 \setminus \psi_2)\} \\ \delta_1(u, w) &= \emptyset \\ \delta_1(w, v) &= \{-(\psi_1 \setminus \psi_2)\}\end{aligned}$$

3. Step 2

We create a witness w' for $-(\psi_1 \setminus \psi_2) \in \delta_1(w, v)$



By putting:

$$\begin{aligned}\lambda_2(w', w) &= \{\psi_1\} \\ \lambda_2(w', v) &= \{-\psi_2\} \\ \delta_2(w', w) &= \emptyset \\ \delta_2(w', v) &= \emptyset\end{aligned}$$

Now let us describe $||\varphi_1||; ||\varphi_2||$ as follows:

TODO:

3.2 Example 2

Let us consider a sequent $LC \not\vdash \varphi_1 \setminus (\varphi_2 \bullet \varphi_3) \rightarrow \varphi_1 \setminus \psi$

1. Step 0

Introduce a wedge (u, v) with labelling λ_0 and δ_0

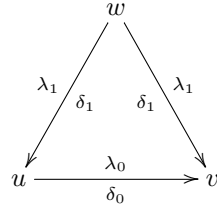
$$u \xrightarrow[\delta_0]{\lambda_0} v$$

where

$$\begin{aligned}\lambda_0(u, v) &= \{\varphi_1 \setminus (\varphi_2 \bullet \varphi_3), -(\varphi_1 \setminus \psi)\} \\ \delta_0(u, v) &= \{-(\varphi_1 \setminus \psi)\}\end{aligned}$$

2. Step 1

We create a witness w for $-(\varphi_1 \setminus \psi) \in \delta_0(u, v)$

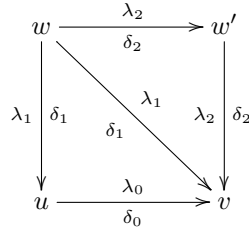


where

$$\begin{aligned}\lambda_1(w, u) &= \{\varphi_1\} \\ \lambda_1(w, v) &= \{-\psi, \varphi_1 \bullet (\varphi_1 \setminus (\varphi_2 \bullet \varphi_3)), \varphi_2 \bullet \varphi_3\} \\ \delta_1(w, u) &= \emptyset \\ \delta_1(w, v) &= \{\varphi_2 \bullet \varphi_3\}\end{aligned}$$

3. Step 2

We have to create a witness w' for $\varphi_2 \bullet \varphi_3 \in \delta_1(w, v)$



$$\begin{aligned}\lambda_2(w, w') &= \{\varphi_2\} \\ \lambda_2(w', v) &= \{\varphi_3\} \\ \delta_2(w, w') &= \emptyset \\ \delta_2(w', v) &= \emptyset\end{aligned}$$

3.3 General case

Consider a sequent $\Gamma \rightarrow \varphi$ such that $LC \not\vdash \Gamma \rightarrow \varphi$. We have to construct a finite relational model over some network such that $(u, v) \Vdash \Gamma \rightarrow \varphi$, that is, $(u, v) \models \Gamma$ and $(u, v) \not\models \varphi$.

References