Notes on filtration of logics containing **K5**

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1 Preliminaries

Definition 1. *An n-normal modal logic is a set of formulas that contains all Boolean tautologies, formulas $\Diamond_i p \vee \Diamond_i q \leftrightarrow \Diamond_i(p \vee q)$ and $\Diamond_i \bot \leftrightarrow \bot$ for $i \leqslant n$, and is closed under modus ponens, substitution, and monotonicity: from $\varphi \to \psi$ infer $\Diamond_i\varphi \to \Diamond_i\psi$ for $i \leqslant n$.*

Definition 2. *An n-Kripke model is a triple $\mathcal{M} = \langle W, R_1, \dots, R_n, \vartheta\rangle$, where $R_i \subseteq W \times W$, $\vartheta : \mathrm{PV} \to 2^W$, and the connectives have the following semantics:*

1. $\mathcal{M}, w \models p \Leftrightarrow w \in \vartheta(p)$
2. $\mathcal{M}, w \models \neg\varphi \Leftrightarrow \mathcal{M}, w \not\models \varphi$
3. $\mathcal{M}, w \models \varphi \vee \psi \Leftrightarrow \mathcal{M}, w \models \varphi$ *or* $\mathcal{M}, w \models \psi$
4. $\mathcal{M}, w \models \Diamond_i\varphi \Leftrightarrow \exists v \in R_i(w)\ \mathcal{M}, v \models \varphi$

By **K5** we mean the logic $\mathbf{K} \oplus A5$, where $A5 = \Diamond p \to \Box\Diamond p$. It is known that **K5** is the modal logic of all Euclidean frames. A frame is called Euclidean if for each x, y, z, xRy and xRz implies yRz.

Proposition 1. *Let $\mathcal{F} = \langle W, R\rangle$ be an Euclidean frame.*

1. *For each $x, y, z \in W$, xRy and xRz implies either yRz or zRy.*
2. *$R \subseteq R; R$, that is, R is dense.*
3. *For each $x \in W$, $R^*(x) = \{x\} \cup R(R(x))$.*
4. *$R^{-1}; R \subseteq R$.*

Proposition 2. **K5** *proves*

1. $\Box^3 p \leftrightarrow \Box^2 p$
2. $\Box^2\Diamond p \leftrightarrow \Box\Diamond p$
3. $\Box\Diamond\Box p \leftrightarrow \Box\Box p$
4. $\Box\Diamond^2 p \leftrightarrow \Box\Diamond p$

Proposition 3. *Let $\mathcal{M}$ be a* $\mathbf{K}5$ *model, xRy for some $x, y \in W$ then one has*

$$\mathcal{M}, x \models \Diamond\Box\varphi \text{ iff } \mathcal{M}, y \models \Diamond\Box\varphi.$$

1.1 Filtrations: general definitions

Let $\mathcal{M} = \langle W, R_1, \ldots, R_n, \vartheta \rangle$ be a Kripke model and Γ a set of formulas closed under subformulas. An equivalence relation $\sim$ is set to have a finite index if the quotient set $W/\sim$ is finite. The equivalence relation $\sim_\Gamma$ induced by Γ is defined as

$$w \sim_\Gamma v \Leftrightarrow \forall \varphi \in \Gamma \, (\mathcal{M}, w \models \varphi \Leftrightarrow \mathcal{M}, v \models \varphi).$$

If Γ is finite, then $\sim_\Gamma$ has a finite index. An equivalence relation $\sim$ respects $\sim_\Gamma$, if $w \sim v$ implies $w \sim_\Gamma v$.

Definition 3. *Let $\mathcal{M} = \langle W, R_1, \ldots, R_n, \vartheta \rangle$ be a Kripke model and Γ be a Sub-closed set formulas. A Γ-filtration of $\mathcal{M}$ is a model $\widehat{\mathcal{M}} = \langle \widehat{W}, \widehat{R_1}, \ldots, \widehat{R_n}, \widehat{\vartheta} \rangle$ such that:*

1. *$\widehat{W} = W/\sim$, where $\sim$ is an equivalence relation having a finite index that respects Γ*
2. *$\widehat{\vartheta}(p) = \{[x]_\sim \mid x \in W \; \& \; x \in \vartheta(p)\}$*
3. *For each $i \in I$ one has $\widehat{R}_i^{min} \subseteq \widehat{R}_i \subseteq \widehat{R}_i^{max}$. $\widehat{R}_{i,\sim}^{min}$ is the i-th minimal filtered relation on $\widehat{W}$ defined as*

$$\hat{x} \widehat{R}_{i,\sim}^{min} \hat{y} \Leftrightarrow \exists x' \sim x \, \exists y' \sim y \; x R_i y$$

$\widehat{R}_{\Gamma,i}^{max}$ is the i-th maximal filtered relation on $\widehat{W}$ induced by Γ defined as

$$\hat{x} \widehat{R}_{\Gamma,i}^{max} \hat{y} \Leftrightarrow \forall \Box_i \varphi \in \Gamma \, (\mathcal{M}, x \models \Box_i \varphi \Rightarrow \mathcal{M}, y \models \varphi)$$

If Φ is finite subset of Γ and $\sim = \sim_\Phi$, then $\widehat{\mathcal{M}}$ is a definable Γ-filtration of $\mathcal{M}$ through Φ. If $\sim = \sim_\Gamma$, then such a filtration by means of the definiton above is called *strict*.

Lemma 1. *Let Γ be a finite set of formulas closed under subformulas and $\widehat{\mathcal{M}}$ a filtration of $\mathcal{M}$ through Γ, then for each $x \in W$ and for each $\varphi \in \Gamma$ one has*

$$\mathcal{M}, x \models \varphi \Leftrightarrow \widehat{\mathcal{M}}, \hat{x} \models \varphi$$

Definition 4. *Let $\mathbb{F}$ be a class of Kripke frames and Γ a finite set of formulas closed under subformulas. If for every model $\mathcal{M}$ over $\mathcal{F} \in \mathbb{F}$ there exists a model that is a Γ-definable filtration of $\mathcal{M}$, then $\mathbb{F}$ admits definable filtration. A class of models $\mathbb{M}$ admits definable filtration if for every $\mathcal{M} \in \mathbb{M}$ there exists a model belonging to the same class that is a definable Γ-filtration of $\mathcal{M}$.*

Lemma 2.

1. *Let $\mathcal{L}$ be a complete normal modal logic. If* $\mathrm{Frames}(\mathcal{L})$ *admits filtration, then $\mathcal{L}$ has the finite model property.*
2. *If the class of models* $\mathrm{Mod}(\mathcal{L})$ *admits filtration, then $\mathcal{L}$ has the finite model property and Kripke complete as well.*

2 Filtration of Euclidean logics

First of all, let us ensure that a filtration of an Euclidean frame is not necessary finite. Let $[x] \sim_\Gamma [y]$ and $[x] \sim_\Gamma [z]$. Then for some $x' \in [x]$ $y' \in [y]$, one has $x'Ry'$ and $x''Rz'$ for some $x'' \in [x]$ and $z' \in [z]$. Clearly, we cannot claim that $x' = x''$ in general. Thus, minimal filtration does not preserve the required property.

Lemma 3. **K5** *admit filtration.*

Proof. Let $\mathcal{M}$ be a **K5**-model and Γ_0 a finite set of formulas closed under subformulas. Let us put $\Gamma = \Gamma_0 \cup \mathrm{Sub}(\{\Diamond\Box\psi \mid \Box\psi \in \Gamma_0\}) \cup \Psi$, where $\Psi = \nabla_1\nabla_2\ldots\nabla_n\Box\psi$ for $\Box\psi \in \Gamma_0$ and $\nabla_i \in \{\Diamond, \Box\}$. By Proposition 2, any element of Φ has one of the four forms. Thus, $W \sim_{\equiv_\Gamma}$ has a finite index. We put $\widehat{R} = R_\Gamma^{\max}$. $\square$

Definition 5. *A relation R is called Horn, if it is defined with some first-order formula having the form*

$$\forall x_1, \ldots, x_n (x_{i_1} R x_{j_1} \wedge \cdots \wedge x_{i_s} R x_{j_s} \to x_k R x_l)$$

Definition 6. *Let H be a Horn property and $\langle W, R\rangle$ a Kripke frame. A Horn closure of a binary relation R is the minimal relation R^H containing R and satisfying H.*

Lemma 4. $R^H = \bigcup\limits_{n<\omega} R_n$ *where*

1. $R_0 = R$.
2. $R_{n+1} = R_n \cup \{(a,b) \in W \mid \exists \vec{c} \in W\ P(a,b,\vec{c})\}$, *where P is a premise of H.*

E-closure (an Euclidean Horn closure of a binary relation) has the following equivalent definitions:

Lemma 5. *Let $\mathcal{F} = \langle W, R\rangle$ be a Kripke frame. The following conditions are equivalent:*

1. *R^E is the smallest Euclidean relation containing R.*
2. $R^E = \bigcup\limits_{i<\omega} R_i$, *where*
   - $R_0 = R$
   - $R_{n+1} = R_n \cup (R_n^{-1}; R_n)$
3. *xR^Ey iff there exists $n < \omega$ such that $\exists x_1, \ldots, x_n$ with $z_1 R x$ and $z_{n-1} R y$ and for each $1 < i \leqslant n$ one has either $z_{i-1} R z_i$ or $z_i R z_{i-1}$.*
4. $R^E = R \cup \bigcup\limits_{i<\omega} (R^{-1}; (R; R^{-1})^n; R)$.

Proof.

1. (1) $\Rightarrow$ (2) Let us show that if R^E is the smallest Euclidean relation containing R, then $R^E = \bigcup\limits_{i<\omega} R_i$. There are two inclusions:
   - $R^E \subseteq \bigcup\limits_{i<\omega} R_i$. It is enough to show that xR^Ey implies xR^ny for some $n < \omega$.

- $\bigcup_{i<\omega} R_i \subseteq R^E$. Let us show that xR_ny for each $n < \omega$ implies xR^Ey by induction on n. If $n = 0$, then xRy, thus, xR^Ey, since R is a subrelation of R^E. Suppose $n = m + 1$ and $xR_{m+1}y$. Let us show that xR^Ey. From $xR_{m+1}y$, one has $(x, y) \in R^n \cup (R_n^{-1}; R_n)$. There are two cases:
  - xR^ny, one needs to merely apply the IH.
  - $xR_n^{-1}; R_ny$. Then $\exists z \in W\, xR_n^{-1}z \,\&\, zR_n$. That is, zR_nx and zR_ny for some z. R_n is already a subrelation of R^E. Thus, zR^Ex and zR^Ey. That implies xR^Ey.

2. (2) $\Rightarrow$ (3) Let xR^Ey, then $(x, y) \in \bigcup_{i<n} R_n$. That is, $(x, y) \in R_n \cup (R_n^{-1}; R_n)$ for some $n < \omega$. There are two cases.
   - $(x, y) \in R_n$ and that's it.
   - $(x, y) \in R_n^{-1}; R_n$, then $(z, x), (z, y) \in R_n$.
3. (3) $\Rightarrow$ (4)
4. (4) $\Rightarrow$ (1)

$\square$

Lemma 6. *Let $\mathcal{M} = \langle W, R, \vartheta\rangle$ be an Euclidean model, Γ a set of* Sub*-closed formulas, and $\sim$ an equivalence relation having a finite index that respects Γ. Then its E-closure of the minimal filtration of R is a filtration itself.*

Proof. Let us show that $\widehat{R^E} = \bigcup_{n<\omega} (R_n)^{\min}_{\sim}$ is a filtration and $\langle W/\sim, \widehat{R}^E, \widehat{\vartheta}\rangle$ is an Euclidean model.

Let $\widehat{\mathcal{M}} = \langle \widehat{W}, \widehat{R}, \widehat{\vartheta}\rangle$ be a minimal filtration of an Euclidean model through $\sim$.

1. Suppose xRy, then $[x]\widehat{R}[y]$ by the definition of a minimal filtration. $\widehat{R}$ is clearly a subrelation of $\widehat{R}^E$, thus $[x]\widehat{R}^E[y]$.
2. Suppose $[x]\widehat{R^E}[y]$ and $\mathcal{M}, x \models \Box\varphi$ for $\Box\varphi \in \Gamma$. then $([x], [y]) \in \bigcup_{n<\omega} R_n$. Then $([x], [y]) \in R_n$ for some $n < \omega$. There are two cases:
   (a) $n = 0$, then $[x]\widehat{R}[y]$, so, obviously, one has $\mathcal{M}, y \models \varphi$
   (b) $n = m+1$. Suppose $[x]\widehat{R}_{m+1}[y]$. Thus, $[x]\widehat{R}_n \cup (\widehat{R}^{-1}; \widehat{R}_n)[y]$. There are the following two cases:
      i. $([x], [y]) \in \widehat{R}_n$, then the statement holds by IH.
      ii. $([x], [y]) \in \widehat{R}^{-1}; \widehat{R}_n$. Then there exists $[z]$ such that $[x]\widehat{R}^{-1}[z]$ and $[z]\widehat{R}_m[y]$, that is, $[z]\widehat{R}[x]$ and $[z]\widehat{R}_m[y]$. One has $\mathcal{M}, x \models \Box\varphi$, then $\widehat{\mathcal{M}}, [x] \models \Box\varphi$. So $\widehat{\mathcal{M}}, [z] \models \Diamond\Box\varphi$. On the other hand, $\mathcal{M}, z \models \Diamond\Box\varphi \to \Box\varphi$, so $\mathcal{M}, z \models \Box\varphi$ and, from that, $\widehat{\mathcal{M}}, [z] \models \Box\varphi$, and, thus, $\widehat{\mathcal{M}}, [y] \models \varphi$ and $\mathcal{M}, y \models \varphi$ by IH and the definition of a minimal filtration.

$\square$

Corollary 1. K5 *admit strict filtrations.*

2.1 Clusters

Let $\mathcal{F} = \langle W, R \rangle$ be a transitive frame. Let us put $xR^{\bullet}y \Leftrightarrow xRy \mathbin{\&} \neg(xRy)$. A point x is proper if xRx. Let us define the following equivalence relation:

$$x \equiv y \Leftrightarrow xRy \mathbin{\&} yRx \vee x = y.$$

A cluster is an element of the quotient set $W/\equiv$. Given $x \in W$, C_x is a cluster containing x. Thus $C_x = \{x\} \cup \{y \mid xRyx\}$. The original relation lifts to the antisymmetric transitive relation on $W/\equiv$ defined as $C_x R C_y$ iff xRy. A cluster C is called maximal if CRC' implies $C = C'$. A point is R-maximal if C_x is a maximal cluster, that is, $R^{\bullet}(x) = \varnothing$. A degenerated cluster is a singleton $\{x\}$ with $\neg(xRx)$. A cluster is called simple if it has the form $\{x\}$ with xRx. If $\langle W', R' \rangle$ is an inner substructure of $\langle W, R \rangle$, then every R'-cluster is an R-cluster and every R-cluster that intersects W' is a subset of W' and is an R'-cluster itself. Given a Kripke model $\mathcal{M}$, a set of formulas Γ is satisfied by a cluster C if every member of Γ is true at some point of C.

If clusters coincide then their poitns have the same theory in the original model:

Lemma 7. $C_x = C_y$ *implies* $\mathcal{M}, x \models \Box\varphi \Leftrightarrow \mathcal{M}, y \models \Box\varphi$

Let us describe the bulldozing technique allowing one to eliminate nondegenerated clusters [3]. Let $\mathcal{L}$ be a transitive logic and $\mathcal{F}$ its frame. We construct first a frame $\mathcal{F}^0 = \langle W^0, R^0 \rangle$ replacing every nondegenerated frame C of W by $C^0 = \{\langle x, i \rangle \mid x \in C, i < \omega\}$. We also replace each degenerated cluster C by $\{\langle x, 0 \rangle\}$. Elements of these subsets form W^0. The relation R^0 is defined as

$$\langle x, i \rangle R^0 \langle y, j \rangle \Leftrightarrow xR^{\bullet}y \text{ or } (x \equiv y \mathbin{\&} i < j) \text{ or } i = j \mathbin{\&} x <_C y$$

where $<_C$ is an arbitrary strict ordering on the proper cluster C containing x and y.

Each nondegenrated cluster C is replaced by an infinite set C_0 such that $\langle C_0, R_0 \rangle$ is a strict linear order. Moreover, $\langle y, j \rangle$, a copy of y, occurs after $\langle y, j \rangle$, a copy of x.

Bulldozing might be extended to models $\mathcal{M} = \langle W, R, \vartheta \rangle$ defining ϑ^0 as follows

$$\vartheta^0(p_i) = \{\langle x, i \rangle \mid x \in \vartheta(p_i), i < \omega\}.$$

One may show inductively the following fact.

Lemma 8. $\mathcal{M}, x \models \varphi \Leftrightarrow \mathcal{M}^0, \langle x, i \rangle \models \varphi$

Let us concretise the case of transitive Euclidean frames. First of all, we consider clusters in **K**45 frames.

2.2

3 Transitive closure stuff

4 PDLisation of Euclidean logics

5 Transitive closure and fusion

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