Model-theoretic aspects of relativised cylindric set algebras

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1 Intro

... It is known that the equational theory of $\mathbf{RCA}_\omega$ for $\alpha \leqslant \omega$ is decidable [17]. ...

2 The problems themselves

1. Suppose $\mathcal{C} \in \mathbf{RCA}_\omega$, whether $\mathcal{C}^+$ has a complete, ω-dimensional representation?
2. Is the class $\mathbf{IG}_\omega$ (the isomorphism-closure of the ω-dimensional cylindric relativised set algebras in which the unit is closed under substitutions and permutations) a variety, or even a pseudo-elementary class? Is it closed under ultraproducts?

3 Boolean algebras with operators and cylindric algebras

Definition 1.

1. *Let $\mathcal{B} = \langle B, +, -, 0, 1\rangle$ be a Boolean algebra. An operator is an n-ary function $\Omega : B^n \to B$ satisfying the following conditions:*
   - *Normality: for all $b_0, \ldots, b_{n-1} \in B$, if $b_1 = 0$ for some $i < n$, then*
   $$\Omega(b_0, \ldots, b_{i-1}, 0, b_{i+1}, \ldots, b_{n-1}) = 0$$
   - *Additivity: for all $b_0, \ldots, b_{n-1}, b, b' \in B$ we have*
   $$\Omega(b_0, \ldots, b_{i-1}, (b + b'), b_{i+1}, \ldots, b_{n-1}) =$$
   $$\Omega(b_0, \ldots, b_{i-1}, b, b_{i+1}, \ldots, b_{n-1}) + \Omega(b_0, \ldots, b_{i-1}, b', b_{i+1}, \ldots, b_{n-1})$$
2. *Let I be an index set, a Boolean algebra with operators (BAO) is an algebra $\langle B, +, -, 0, 1, \{\Omega_i\}_{i \in I}\rangle$ such that $\langle B, +, -, 0, 1\rangle$ is a Boolean algebra and for each $i \in I$ Ω_i is an operator.*

Definition 2. *Let $\mathcal{B} = \langle B, +, -, 0, 1, \{\Omega_i\}_{i \in I}\rangle$ be a BAO, then*

1. *An opetator Ω is completely additive, if for each $b_0, \ldots, b_{n-1} \in B$ and $X \subseteq B$, one has*
$$\Omega(b_0, \ldots, b_{i-1}, \sum X, b_{i+1}, \ldots, b_{n-1}) = \sum_{x \in X} \Omega(b_0, \ldots, b_{i-1}, x, b_{i+1}, \ldots, b_{n-1})$$
2. *$\mathcal{B}$ is completely additive, if for each $i \in I$ Ω_i is additive,*
3. *A class $\mathcal{K}$ of BAOs is completely additive, if every $\mathcal{B} \in \mathcal{K}$ is completely additive.*

3.1 Atom structures and canonical extensions

Definition 3. *Let I be an index set and $\{\Omega_i\}_{i\in I}$ a set of function symbols*

1. *An atom structure is a relational structrure $\mathcal{F} = \langle W, \{R_i\}_{i\in I}\rangle$ such that R_i is a $n+1$-ary relation symbol, if $\Omega_{i\in I}$ is an n-ary function symbol,*
2. *Let $\mathcal{B}$ be an atomic BAO of the signature I, the atom structure of $\mathcal{B}$, written as $\mathbf{At}\mathcal{B}$, is an atom structure $\langle \mathrm{At}(\mathcal{B}), \{R_i\}_{i\in I}\rangle$ such that for each $a, b_0, \ldots, b_{n+1} \in \mathrm{At}(\mathcal{B})$ and for each $i \in I$*

$$\mathbf{At}\mathcal{B} \models R_i(a, b_0, \ldots, b_{n+1}) \text{ iff } \mathcal{B} \models a \leqslant \Omega_i(b_0, \ldots, b_{n+1})$$

3. *Let $\mathcal{F} = \langle W, \{R_i\}_{i\in I}\rangle$ be an atom structure, the complex algebra of $\mathcal{F}$, written as $\mathbf{Cm}\mathcal{F}$, is a BAO $\langle \mathcal{P}(W), \cup, -, \varnothing, W, \{\Omega_{R_i}\}_{i\in I}\rangle$ such that for all $X_0, \ldots, X_{n-1} \subseteq W$ and for each $i \in I$*

$$\Omega_{R_i}(X_0, \ldots, X_{n-1}) = \{a \in W \mid \exists b_0 \in X_0 \ldots \exists b_{n-1} \in X_{n-1}\ \mathcal{F} \models R_i(a, b_0, \ldots, b_{n-1})\}$$

The following duality is due to Thomason [19].

Fact 1.

1. *Let $\mathcal{B}$ be a complete atomic BAO, then $\mathcal{B} \cong \mathbf{Cm}(\mathbf{At}(\mathcal{B}))$,*
2. *Let $\mathcal{F}$ be an atom structure, then $\mathcal{F} \cong \mathbf{At}(\mathbf{Cm}(\mathcal{B}))$.*

Let A be a non-empty subset of a Boolean algebra $\mathcal{B}$, A is a *filter*, if A is closed under finite infima and upwardly closed. A is an ultrafilter, if it has no non-trivial extensions. That is, if $A \subseteq A'$, then $A' = \mathcal{B}$.

Definition 4. *Let $\mathcal{B} = \langle B, +, -, 0, 1, \{\Omega_i\}_{i\in I}\rangle$ be a BAO and $\mathbf{Uf}(\mathcal{B})$ the set of its ultrafilters. The ultrafilter frame of $\mathcal{B}$ (or canonical frame) is a relational structure $\mathcal{F}_{\mathcal{B}} = \langle \mathbf{Uf}(\mathcal{B}), R_{\Omega_i}\rangle$ such that for each ultrafilters $\beta_0, \ldots, \beta_{n-1}, \gamma$ one has*

$$\mathbf{Uf}(\mathcal{B}) \models R_{\Omega_i}(\beta_0, \ldots, \beta_{n-1}, \gamma) \text{ iff } \{\Omega(b_0, \ldots, b_{n-1}) \mid b_0 \in \beta_0, \ldots, b_{n-1} \in \beta_{n-1}\} \subseteq \gamma.$$

Definition 5. *Let $\mathcal{B}$ be a BAO, then*

1. *The canonical extension of $\mathcal{B}$ is a complex algebra of the canonical frame $\mathbf{Cm}(\mathcal{F}_{\mathcal{B}})$ denoted as $\mathcal{B}^+$,*
2. *The class of BAOs is canonical, if it is closed under canonical extensions.*

Theorem 1. *Let $\mathcal{A}$, $\mathcal{B}$ be BAOs,*

1. *There exists $\iota : \mathcal{A} \hookrightarrow \mathcal{A}^+$ such that $\iota : a \mapsto \{\gamma \in \mathbf{Uf}(\mathcal{A}) \mid a \in \gamma\}$.*
2. *If $i : \mathcal{A} \hookrightarrow \mathcal{B}$, then this embedding might be extented to the embedding $i^+ : \mathcal{A}^+ \hookrightarrow \mathcal{B}^+$*

Fact 2.

3.2 (Representable) cylindric algebras and cylindric set algebras

Let α be an ordinal. Let ${}^{\alpha}U$ be the set of all functions mapping α to a non-empty set U. We denote $x(i) = x_i$ for $x \in {}^{\alpha}U$ and $i < \alpha$.

Definition 6.

1. *A subset of ${}^{\alpha}U$ is an α-ry relation on U. For $i, j < \alpha$, the i, j-diagonal D_{ij} is the set of all elements of U such that $y_i = y_j$.*

   If $i < \alpha$ and X is an α-ry relation on U, then the i-th cylindrification $C_i X$ is the set of all elements of U that agree with some element of X on each coordinate except the i-th one. To be more precise, $C_i X = \{y \in {}^{\alpha}U \mid \exists x \in X \forall i < \alpha \ (i \neq j \Rightarrow y_j = x_j)\}$.

2. *A cylindic set algebra of dimension α is an algebra consisting of a set S of α-ry relation on some base set U with the constants and operations $0 = \varnothing$, $1 = {}^{\alpha}U$, $\cap$, $-$, the diagonal elements $\{D_{ij}\}_{i,j<\alpha}$, the cylindrifications $\{C\}_{i<\alpha}$.*

   A generalised cylindric set algebra of dimension α is a subdirect of cylindric algebras that have dimension α

3. *A cylindric algebra of dimension α is an algebra $\mathcal{C} = \langle \mathcal{B}, \{c_i\}_{i<\alpha}, \{d_{ij}\}_{i,j<\alpha} \rangle$ such that*

   - *$\mathcal{B}$ is a Boolean algebra, for each $i, j < \alpha$ c_i is an operator and $d_{ij} \in \mathcal{B}$*
   - *For each $i < \alpha$, $a \leqslant c_i a$, $c_i(a \wedge c_i b) = c_i a \wedge c_i b$ and $d_{ii} = 1$*
   - *For every $i, j < \alpha$, $c_i c_j a = c_j c_i a$*
   - *If $k \neq i, j < \alpha$, then $d_{ij} = c_k(d_{ij} \wedge d_{jk})$*
   - *If $i \neq j$, then $c_i(d_{ij} \wedge a) \wedge c_i(d_{ij} \wedge -a) = 0$*

   $\mathbf{CA}_{\alpha}$ *is the class of all cylindric algebras of dimension α*

4. *An α-dimensional cylindric algebra C is representable, if it is isomorphic to a generalised cylindric set algebra of dimension α. Such is isomorphism is a representation of C.*

   $\mathbf{RCA}_{\alpha}$ *is the class of all representable cylindric algebras that have dimension α. In particular, we are interested in the case when $\alpha = \omega$.*

It is well known that $\mathbf{RCA}_{\alpha}$ is a variety, $\mathbf{RCA}_{\alpha}$ ($\alpha \leqslant 2$) is finitely axiomatisable and $\mathbf{RCA}_{\alpha}$ ($2 < \alpha < \omega$) has no finite axiomatisation, see [7].

Let $\mathcal{A} \in \mathbf{C}_{\omega}$, then $\mathcal{A}$ has a *complete representation*, if this representation preserves all existing suprema.

Let us concretise the definition of a canonical extension for α-dimensional cylindric algebras:

Definition 7. *Let $\mathcal{C} = \langle C, +, -, 0, 1, \{d_{ij}\}_{i,j<\omega}, \{c_i\}_{i<\omega} \rangle$ be an ω-dimensional cylindric algebra. Let $\mathbf{Uf}(\mathcal{C})$ be the set of all ultrafilters of $\mathfrak{B}\mathcal{C}$, a Boolean reduct of $\mathcal{C}$.*

Let us define $\mathbf{C}_i : \mathbf{Uf}(\mathcal{C}) \to \mathbf{Uf}(\mathcal{C})$ for each $i, j < \omega$ as

1. $\mathbf{C}_i \mathcal{X} = \{\mathcal{F} \in \mathbf{Uf}(\mathcal{C}) \mid \exists \mathcal{F}' \in \mathbf{Uf}(\mathcal{C}) \ (a \in \mathcal{F} \Rightarrow c_i a \in \mathcal{F}' R)\}$,
2. $D_{ij} = \{\mathcal{F} \in \mathbf{Uf}(\mathcal{C}) \mid d_{ij} \in \mathcal{F}\}$.

The structure $\mathcal{C}^+ = \langle \mathbf{Uf}(\mathcal{C}), \cup, -, \varnothing, C, \mathbf{C}_{i<\omega}, \{D_{ij}\}_{i,j<\omega} \rangle$ is called the canonical extension of $\mathcal{C}$.

Let us discuss the connection between representability and canonical extensions.

The following definitions and facts are due to Henkin, Monk, and Tarski [6].

Let $\mathcal{C} = \langle C, +, -, 0, 1, \{d_{ij}\}_{i,j<\beta}, \{c\}_{c<\beta}\rangle$ be a β-dimensional cylindic algebra and $\alpha \leqslant \beta$ an ordinal. The *α-th reduct* of $\mathcal{C}$, denoted as $\mathfrak{Rd}_\alpha\mathcal{C}$, is an algebra having the form

$$\mathfrak{Rd}_\alpha\mathcal{C} = \langle C, +, -, 0, 1, \{d_{ij}\}_{i,j<\alpha}, \{c\}_{c<\alpha}\rangle$$

$\mathcal{B}$ is a subreduct of $\mathcal{C}$, denoted as $\mathcal{B} \subseteq^r \mathcal{C}$, if $\mathcal{B} \subseteq \mathfrak{Rd}_\gamma\mathcal{C}$ for some $\gamma \leqslant \beta$.

Definition 8. *Let $\mathcal{C}$ be a β-dimensional cylindic algebra and α an ordinal such that $\alpha \leqslant \beta$. The neat α-reduct of $\mathcal{C}$, denoted as $\mathfrak{Nr}_\alpha\mathcal{C}$, is the subalgbera $\mathcal{A}$ of $\mathfrak{Rd}_\alpha\mathcal{C}$ with $\mathcal{A} = \mathrm{Cl}_\kappa$ where $\alpha + \kappa = \beta$.*

Let $\mathbb{K}$ be a class of β-dimensional cylindic algebras, then we put

$$\mathbf{Nr}_\alpha\mathbb{K} = \{\mathfrak{Nr}_\alpha\mathcal{C} \mid \mathcal{C} \in \mathbb{K}\}$$

An algebra $\mathcal{B}$ is a neat subreduct of $\mathcal{C}$, or $\mathcal{B}$ is neatly embeddable to $\mathcal{C}$ if there exists an ordinal $\gamma \leqslant \alpha$ such that $\mathcal{C} \subseteq \mathfrak{Rd}_\gamma\mathcal{B}$.

3.3 Substitution in cylindric algebras

Definition 9. *Given a cylindric algebra of dimension α C, let x be a term of its signature, the substitution operator s^i_j have the following definition:*

$$s^i_j x = \begin{cases} x, \text{if } i = j \\ c_i(d_{ij} \wedge x), \text{otherwise} \end{cases}$$

Proposition 1. *Let α be an ordinal and let $i, j, k, l < \alpha$. The following facts hold in $\mathbf{CA}_\alpha$*

1. $s^i_j x \leqslant c_i x$.
2. $s^i_j(x \wedge y) = s^i_j x \wedge s^i_j y$, $s^i_j(x \vee y) = s^i_j x \vee s^i_j y$, $-s^i_j x = s^i_j(-x)$. *Moreover, s^i_j is completely additive.*
3. $i \neq k, l$ *implies* $s^i_j d_{ik} = d_{jk}$ *and* $s^i_j d_{kl} = d_{kl}$.
4. $d_{jk} \wedge s^i_j = d_{jk} \wedge s^i_k$.
5. $s^i_j c_i x = c_i x$.
6. $k \neq i, j$ *implies* $s^i_j c_i x = c_i s^i_j x$.
7. $c_j s^i_j x = c_i s^j_i x$.
8. $i \neq j$ *implies* $c_i s^i_j x = s^i_j x$.
9. $i \neq k$ *implies* $s^i_j s^i_k = s^i_k x$.
10. *If either* $i \notin \{k, l\}$ *and* $k \notin \{i, j\}$, *or* $j = l$, *then* $s^i_j s^k_l x = s^k_l s^i_j x$.
11. $s^i_j s^j_i x = s^i_j x$.
12. $s^i_k s^j_i x = s^i_k s^j_k x = s^j_k s^i_j x$

4 Model-theoretic and universal algebraic preliminaries

4.1 Ultraproducts

Here are the required notions and facts from model theory and universal algebra [10] [12] [18].

Let Λ be an index set and D an ultrafilter on the Boolean algebra $\langle \mathcal{P}(\Lambda), \cup, -, \Lambda, \varnothing \rangle$. Consider the product $M = \prod\limits_{\lambda \in \Lambda} M_\lambda$ of the Ω-structures $\{M_\lambda\}_{\lambda \in \Lambda}$ and the equivalence relation on $\mathrm{dom}(M)$ defined as

$$a_1 \sim a_2 \Leftrightarrow \{\lambda \in \Lambda \mid a_1(\lambda) = a_2(\lambda)\} \in D$$

Let us denote $\mathrm{dom}(M)/\sim$ as U and $[a]_\sim$ as a/D, where $a \in \mathrm{dom}(M)$. We also denote the *ultraproduct* of $\{M_\lambda\}_\lambda$ as $\prod\limits_{\lambda \in \Lambda} M_\lambda/D$, or, for brevity, as $\prod_D M_\lambda$. The Ω-symbols have the following interpretation

1. If $c \in \mathrm{Cnst}$, then $c^U = c^M/D$
2. If $f \in \mathrm{Fn}$ is an n-ary function symbol and $\overline{a} \in M^n$, then $f^U(\overline{a}) = f^M(x) = f^M(\overline{a})/D$
3. If $R \in \mathrm{Fn}$ is an n-ary relation symbol and $\overline{a} \in M^n$, then $U \models R(\overline{a}/D)$ iff $\{\lambda \in \Lambda \mid M_\lambda \models R(\overline{a}(\lambda))\} \in D$

The ultraproduct is principal if D is a principal filter.

Definition 10.

1. *Let $\{M_\lambda\}_{\lambda \in \Lambda}$ be a set of Ω-structures such that every M_λ is isomorphic to the single structure M, then their ultraproduct over D is called the ultrapower over D. The denotation is $\prod\limits_D M$ or M^Λ/D.*
2. *If $\prod\limits_D M \cong N$ for some structure N, then M is an ultraroot of N.*

Theorem 2 (Łos). *Let $\{M_\lambda\}_{\lambda \in \Lambda}$ be Ω-structures and D an ultrafilter on Λ, and let $U = \prod\limits_D M_\lambda$ be an ultraproduct of $\{M_\lambda\}_{\lambda \in \Lambda}$ over D. For each first-order formula $\varphi(x_1, \ldots, x_n)$ and for each $a_1/D, \ldots, a_n/D \in U$:*

$$U \models \varphi(a_1/D, \ldots, a_n/D) \text{ iff } \{\lambda \in \Lambda \mid \varphi(a_1(\lambda), \ldots, a_n(\lambda))\} \in D$$

The Łos has the following helpful corollary:

Corollary 1. *Let $\prod\limits_D M$ be an ultrapower of M. For $a \in M$, let us define a function $\overline{a} : a \mapsto a/D$. Then such a map is an elementary embedding of M into $\prod\limits_D M$.*

Moreover, any elementary equivalent structures have isomorphic ultrapowers.

Recall that a class of Ω-structures $\mathbf{K}$ is called *elementary*, if $\mathbf{K} = \mathrm{Mod}(\mathrm{T})$ for some first-order theory $\mathbf{T}$. In that case, T is an axiomatisation of $\mathbf{K}$.

Theorem 3. *Let $\mathbf{K}$ be a class Ω-structures, $\mathbf{K}$ is elementary iff $\mathbf{K}$ is closed under isomorphic copies, ultraroots, and ultrapowers.*

5 $\mathbf{IG}_\omega$ and ultraproducts

6 $\mathbf{IG}_\omega$ is (not) (pseudo-)elementary; is (not) a variety

References

[1] Hajnal Andréka, Robert Goldblatt, and István Németi. Relativised quantification: Some canonical varieties of sequence-set algebras. *The Journal of Symbolic Logic*, 63(1):163–184, 1998.

[2] Hajnalka Andréka. A finite axiomatization of locally square cylindric-relativized set algebras. *Studia Scientiarum Mathematicarum Hungarica*, 38(1-4):1–11, 2001.

[3] Dov M Gabbay and Valentin B Shehtman. Products of modal logics. part 2: Relativised quantifiers in classical logic. *Logic Journal of the IGPL*, 8(2):165–210, 2000.

[4] Robert Goldblatt. Varieties of complex algebras. *Annals of pure and applied logic*, 44(3):173–242, 1989.

[5] Robert Goldblatt. Elementary generation and canonicity for varieties of boolean algebras with operators. *Algebra Universalis*, 34(4):551–607, 1995.

[6] L Henkin, J D Monk, and A Tarski. *Cylindric Algebras Part I.* North-Holland, 1971.

[7] Leon Henkin, J. Donald Monk, and Alfred Tarski. Cylindric algebras. part ii. *Journal of Symbolic Logic*, 53(2):651–653, 1988.

[8] Leon Henkin, J.Donald Monk, and Alfred Tarski. Representable cylindric algebras. *Annals of Pure and Applied Logic*, 31:23 – 60, 1986.

[9] Robin Hirsch and Ian Hodkinson. Relation algebras from cylindric algebras, i. *Annals of Pure and Applied logic*, 112(2-3):225–266, 2001.

[10] Robin Hirsch and Ian Hodkinson. *Relation algebras by games.* Elsevier, 2002.

[11] Robin Hirsch and Ian Hodkinson. Completions and complete representations. In *Cylindric-like Algebras and Algebraic Logic*, pages 61–89. Springer, 2013.

[12] Wilfrid Hodges et al. *A shorter model theory.* Cambridge university press, 1997.

[13] Ian Hodkinson. Atom structures of cylindric algebras and relation algebras. *Annals of Pure and Applied Logic*, 89(2):117 – 148, 1997.

[14] Agi Kurucz. Representable cylindric algebras and many-dimensional modal logics. In *Cylindric-like Algebras and Algebraic Logic*, pages 185–203. Springer, 2013.

[15] Anatolij Ivanovic Mal'Cev. *Algebraic systems*, volume 192. Springer Science & Business Media, 2012.

[16] Maarten Marx and Yde Venema. *Multi-dimensional modal logic.* Springer, 1997.

[17] Istvan Németi. A fine-structure analysis of first-order logic. *Arrow Logic and Multimodal Logics, Studies in Logic, Language and Information*, pages 221–247, 1996.

[18] Hanamantagouda P Sankappanavar and Stanley Burris. A course in universal algebra. *Graduate Texts Math*, 78, 1981.

[19] Steven K Thomason. Categories of frames for modal logic. *The journal of symbolic logic*, 40(3):439–442, 1975.

[20] Yde Venema. A modal logic for quantification and substitution. *Logic Journal of the IGPL*, 2(1):31–45, 1994.

[21] Yde Venema. Cylindric modal logic. *Journal of Symbolic Logic*, pages 591–623, 1995.

[22] Yde Venema. Atom structures. *Advances in Modal Logic*, 1:63–72, 1996.

[23] Yde Venema. *Cylindric Modal Logic*, pages 249–269. Springer Berlin Heidelberg, Berlin, Heidelberg, 2013.