

The finite base property for some relation algebras subreducts

Daniel Rogozin

1 The Relation Algebras Background

We describe the basic definitions and results about relation algebras [8] [16].

Definition 1.

1. A relation algebra is an algebra $\mathcal{R} = \langle R, 0, 1, \wedge, \vee, \neg, ;, \smile, \mathbf{1} \rangle$ such that $\langle R, 0, 1, \wedge, \vee, \neg \rangle$ is a Boolean algebra and the following equations hold, for each $a, b, c \in R$:

- (a) $a; (b; c) = (a; b); c$
- (b) $(a \vee b); c = (a; c) \vee (b; c)$
- (c) $a; \mathbf{1} = a$
- (d) $a^{\smile\smile} = a$
- (e) $(a \vee b)^{\smile} = a^{\smile} \vee b^{\smile}$
- (f) $(a; b)^{\smile} = b^{\smile}; a^{\smile}$
- (g) $a^{\smile}; (\neg(a; b)) \leq \neg b$

where $a \leq b$ iff $a \wedge b = a$ iff $a \vee b = b$. **RA** denotes the class of all relation algebras.

2. A proper relation algebra is an algebra $\mathcal{R} = \langle R, 0, 1, \wedge, \vee, \neg, ;, \smile, \mathbf{1} \rangle$ such that $R \subseteq \mathcal{P}(W)$, where W is an equivalence relation; $0 = \emptyset$; $1 = W$; \wedge, \vee, \neg are set-theoretic intersection, union, and complement respectively; $;$ is relation composition, \smile is relation converse, $\mathbf{1}$ is a diagonal relation restricted to W , that is:

- (a) $a; b = \{ \langle x, z \rangle \mid \exists y \langle x, y \rangle \in a \ \& \ \langle y, z \rangle \in b \}$
- (b) $a^{\smile} = \{ \langle x, y \rangle \mid \langle y, x \rangle \in a \}$
- (c) $\mathbf{1} = \{ \langle x, y \rangle \mid x = y \}$

The class of all proper relation algebras is denoted as **PRA**. **Rs** is the class of all relation set algebras, proper relation algebra with a diagonal subrelation as an identity. **RRA** is the class of all representable relation algebras, that is, the closure of **PRA** under isomorphic copies. That is, **RRA** = **IPRA**.

Note that the (quasi)equational theories of those classes coincide, that is

$$\mathbf{IPRA} = \mathbf{RRA} = \mathbf{SPRs}$$

Moreover, **RRA** is a variety, but it cannot be defined by any set of first-order formulas [19] [].

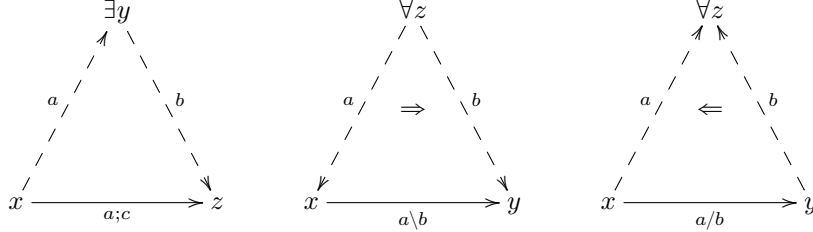
One may express residuals in every $\mathcal{R} \in \mathbf{RA}$ as follows, for every $a, b \in \mathcal{R}$:

1. $a \setminus b = \neg(a^\smile; \neg b)$
2. $a/b = \neg(\neg a; b^\smile)$

Those residuals have the following interpretation in $\mathcal{R} \in \mathbf{PRA}$ (as well as in \mathbf{RRA}), for every $a, b \in \mathcal{R}$:

1. $a \setminus b = \{\langle x, y \rangle \mid \forall z (z, x) \in a \Rightarrow (z, y) \in b\}$
2. $a/b = \{\langle x, y \rangle \mid \forall z (y, z) \in b \Rightarrow (x, z) \in a\}$

One may illustrate composition and residuals in \mathbf{PRA} and \mathbf{RRA} via the following triangles:



Given a subset of definable operations in \mathbf{RA} τ , we denote the class of subalgebras of the τ -reducts by $\mathbf{R}(\tau)$. The algebras containing to this class are defined as restrictions of elements belonging to \mathbf{Rs} to operations of τ . By $\mathbf{Q}(\tau)$ we mean a quasivariety generated by $\mathbf{R}(\tau)$. As in [11], we put $\mathbf{Q}(\tau)$ as the closure of $\mathbf{R}(\tau)$ under subalgebras and products assuming that $\mathbf{R}(\tau)$ is already closed under ultraproducts.

2 The Finite Base Property

We recall the underlying definitions according to [8, Section 19]

Definition 2. Let \mathbf{K} be a class of algebras of a signature Ω , \mathbf{K} has the finite algebra property, if if any first-order Ω -sentence that is true in all finite algebras in \mathbf{K} is true in every algebra in \mathbf{K} .

The finite base property is a version of the finite algebra property if \mathbf{K} is a class of representable algebras:

Definition 3. Let \mathbf{K} be a class of representable algebras of a signature Ω

1. \mathbf{K} has the finite base property if any first-order Ω -sentence that is true in every algebra in \mathbf{K} having a representation over a finite base set is valid in \mathbf{K} .
2. \mathbf{K} has the finite algebra on finite base property if any finite algebra in \mathbf{K} has a representation with finite base.
3. \mathbf{K} has the finite algebra property for equations/quasi-identities if any equation/quasi-identity that is true in all finite algebras is true in every algebra in \mathbf{K} . The finite base property for equations/quasi-identities is defined similarly.

The following statements were shown in [3]. This lemma connects finite base property with finite algebra on finite base and finite algebra properties as follows:

Lemma 1. Let \mathbf{K} be a class of representable Ω -algebras:

1. If \mathbf{K} has the finite algebra property, then it has the finite algebra and the finite base properties for equations/quasi-identities.
2. The finite algebra on finite base and the finite algebra properties implies the finite base property for \mathbf{K} . The same holds for equations/quasi-identities.
3. If any representation of an infinite algebra has an infinite base, then the finite base property implies the finite algebra one for \mathbf{K} .
4. Suppose Ω is finite and any subalgebra of a representable algebra is representable on the same base. Then the finite base property implies the finite algebra on finite base property.

3 The Relation Residuated Semigroups Background

3.1 The underlying definitions and results

A *relation structure* (**RS**) is an arbitrary algebra of the signature $\Omega = \langle \cdot, \backslash, /, \leq \rangle$, where $\cdot, \backslash, /$ are binary function symbols and \leq is a binary relation symbol.

Definition 4. A *residuated semigroup* is an algebra $\mathcal{S} = \langle S, \cdot, \leq, \backslash, / \rangle$ such that $\langle S, \cdot, \leq \rangle$ is an ordered residuated semigroup and the following equivalences hold for each $a, b, c \in S$:

$$b \leq a \backslash c \Leftrightarrow a \cdot b \leq c \Leftrightarrow a \leq c / b$$

ORS is the class of all residuated semigroups.

Definition 5. Let A be a set of binary relations on some base set W such that $R = \cup A$ is transitive and $\{x, y \mid xRy\} = W$. A *relation residuated semigroup* is an algebra $\mathcal{A} = \langle A, ;, \backslash, /, \subseteq \rangle$ where for each $r, s \in A$

1. $r; s = \{ \langle a, c \rangle \mid \exists b \in W (\langle a, b \rangle \in r \ \& \ \langle b, c \rangle \in s) \}$
2. $r \backslash s = \{ \langle a, c \rangle \mid \forall b \in W (\langle b, a \rangle \in r \Rightarrow \langle b, c \rangle \in s) \}$
3. $r / s = \{ \langle a, c \rangle \mid \forall b \in W (\langle c, b \rangle \in s \Rightarrow \langle a, b \rangle \in r) \}$

Relation residuated semigroup are also called representable relativised relational structure (**RRS**).

Andréka and Mikuláš proved the following representation theorem for **ORS** in [4] that implies relational completeness of the Lambek calculus, the logic of **ORS**:

Theorem 1. $\mathbf{ORS} = \mathbf{IRRS}$, where **IRRS** is a closure of **RRS** under isomorphic copies.

3.2 The finite base property for RRS

Definition 6. A *relativised representation*

Definition 7. The *standard translation*

TODO: take a look at relativised representations and loosely guarded fragments in general
 TODO: realise whether it makes sense to use the technique similar to [8, Theorem 19.13] used for weakly associative algebras.

Theorem 2. Let \mathcal{A} be a finite residuated semigroup and $|\mathcal{A}| < \omega$, then \mathcal{A} has a finite relativised representation.

Theorem 3. *Let \mathcal{A} be a finite representable residuated semigroup, then \mathcal{A} is isomorphic to representable residuated semigroup a domain of which is finite.*

Proof. That might follow from the previous theorem, Theorem 1, and something else. \square

Corollary 1. *The Lambek calculus has the fmp and the universal theory of **IRRS** is NP-complete.*

The hypothetical plan is the following one:

1. Define properly relativised representation for residuated semigroups, that should look like ternary Kripke frames for the basic Lambek calculus or arrow logic.
2. Define the standard translation to such first-order relation structures. TODO: take a look at loosely guarded fragment stuff.
3. Every finite residuated semigroup has a finite relativised representation.
4. If every Π_1 -statement φ of the language of residuated semigroups that is valid in every residuated semigroup is valid in algebra having a finite relativised representation (one may use here Theorem 1 somehow), then φ is valid in **ORS** as well as in **IRRS**.
5. Every finite residuated semigroup should have a finite relativised representation.
6. Construct a finitely based relation residuated semigroup from that (an analogue of complex algebra or smth like that). This item is the most non-trivial one.
7. As a corollary, the first-order universal first-order theory of **IRRS** should be decidable and (it seems so) NP-complete (that should follow from the results in [23]). The Lambek calculus is decidable that was shown syntactically via cut elimination and subformula property. Here we would have an alternative way of showing decidability for some substructural logics.

4 Join-semilattice ordered semigroups

Definition 8. *A join-semilattice ordered semigroup (\mathbf{OS}^\vee) is an algebra $\mathcal{S} = \langle S, \cdot, \vee \rangle$ such that $\langle S, \cdot \rangle$ is a semigroup, $\langle S, \vee \rangle$ is a join-semilattice and the following equations hold for each $a, b, c \in S$:*

1. $a \cdot (b \vee c) = (a \cdot b) \vee (a \cdot c)$
2. $(a \vee b) \cdot c = (a \cdot c) \vee (b \cdot c)$

This class is clearly a variety since \mathbf{OS}^\vee has the equational definition so far as \vee is defined as an associative, idempotent, and commutative operation.

Let A be a set of binary relations on some base set W such that $R = \cup A$ is transitive and $\{x, y \mid xRy\} = W$ as in Definition 5. A relation join-semilattice ordered semigroup (\mathbf{ROS}^\vee) is an algebra of binary relations $\mathcal{A} = \langle A, |, \cup \rangle$ such that $|$ is a relation composition as above and \cup is the set-theoretic union.

Recall that a class of structures \mathbf{K} is called finitely axiomatisable iff both \mathbf{K} and its complement are closed ultraproducts and isomorphic copies.

It is known that the class of all representable join-semilattice ordered semigroups has no finite axiomatisation [1]. In other words,

Theorem 4. *The equational and quasiequational theories of $R(;;, \vee)$ is not finitely based.*

Let us provide a proof of this fact using the rainbow technique [8] to show that the complement of \mathbf{ROS}^\vee is not closed ultraproducts. This is (more or less) a standard way, see [14]. We note that representability is not decidable for finite relation algebras [7]. Moreover, representability is undecidable for lattice-ordered semigroups and ordered complemented semigroups [21].

First of all, we recall several definitions such as colourings. We define a sequence of relation algebras $\{\mathfrak{A}_n\}_{n < \omega}$ each of which belongs to \mathbf{RA} . We need these algebras to show that their $\{;;, \vee\}$ -reducts are not representable. That is, we are seeking to show that

Given $n < \omega$, the set of atoms $\text{At}(\mathfrak{A}_n)$ consists of the following elements:

- identity: $\mathbf{1}$, an atom with no colour
- greens: \mathbf{g}_i for $0 \leq i \leq 2^n$
- yellow: \mathbf{y}
- black: \mathbf{b}
- whites: \mathbf{b}_{ij} for $0 \leq i \leq j \leq 2^n$
- reds: \mathbf{r}_i for $0 < i \leq 2^n$

We claim that every atom is self-converse ($a^\smile = a$). Given $x, y, z \in \mathfrak{A}_n$, a triple (x, y, z) is an inconsistent triangle if

$$x \wedge (y; z) = y \wedge (z; x) = z \wedge (x; y) = 0$$

We define the set of inconsistent triangles explicitly as follows.

- $(\mathbf{g}_i, \mathbf{g}_i, \mathbf{g}_i)$ for $0 \leq i \leq 2^n$
- $(\mathbf{y}, \mathbf{y}, \mathbf{y})$ for $0 \leq i \leq 2^n$
- $(\mathbf{g}_i, \mathbf{g}_i, \mathbf{w}_{kj})$ for $0 \leq i \leq 2^n$ and $0 \leq k \leq j \leq 2^n$
- $(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_k)$ unless $i + k = j$ or $i + k = j$ or $j + k = i$
- $(\mathbf{g}_i, \mathbf{g}_{i+1}, \mathbf{r})$ unless $j = 1$
- $(\mathbf{g}_i, \mathbf{y}, \mathbf{w}_{jk})$ unless $i \in \{j, k\}$

$(\mathbf{g}_i, \mathbf{g}_i, \mathbf{w}_{kj})$ stands for $\mathbf{g}_i \wedge (\mathbf{g}_i; \mathbf{w}_{kj}) = \mathbf{g}_i \wedge (z; \mathbf{g}_i) = \mathbf{w}_{kj} \wedge (\mathbf{g}_i; \mathbf{g}_i) = 0$, and so on.

Lemma 2. *For each $n < \omega$, \mathfrak{A}_n does not belong \mathbf{RRA} . The $(\vee, ;)$ -reduct \mathfrak{S}_n of \mathfrak{A}_n is not representable as well. For each $n < \omega$, there is an equation valid in set algebras failing in \mathfrak{S}_n .*

Proof. See [14] to have a proof that $\mathfrak{A}_n \notin \mathbf{RRA}$.

We prove that \mathfrak{S}_n is not representable by contradiction. Suppose h is an isomorphism of \mathfrak{S}_n to a set relation of relations having similarity type $\{;;, \vee\}$. Let 0 be a zero element of \mathfrak{A}_n . \square

TODO: define games and networks. Take a look at [?].

Lemma 3. *Any non-trivial ultraproduct of $\{\mathfrak{A}_n\}_{n < \omega}$ is representable.*

Lemma 4. *TODO: one needs to realise when \exists has a winning strategy*

4.1 The finite algebra on finite base for RJSOS (or its failure)

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