

# Varieties of representable relation algebra reducts

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## 1 Question

Let  $\tau$  be a subsignature of operations expressible in the signature of  $RA$ .  $\mathbf{R}(\tau)$  is a variety iff ??.  
Investigate discriminators for representable relation algebra reducts.

## 2 Varieties and discriminators

Standardly, a class of algebras is called *variety*, if it can be determined by some equational theory, or, equivalently, it is closed under homomorphic images, subalgebras and direct products. Given a class of algebras  $\mathcal{K}$ ,  $\mathbf{V}(\mathcal{K})$  is a variety generated by  $\mathcal{K}$  or, equivalently,  $\mathbf{HSP}(\mathcal{K})$ , the closure of  $\mathcal{K}$  under homomorphic images, subalgebras and direct products.

Let  $\{A_i \mid i \in I\}$  be an indexed family of algebras, then a subalgebra  $A \subseteq \prod_{i \in I} A_i$  is a subdirect product if  $\pi_i(A) = A_i$ . An embedding  $\alpha : A \rightarrow \prod_{i \in I} A_i$  is subdirect if  $\alpha(A)$  is subdirect product. An algebra is subdirectly irreducible if for every subdirect embedding  $\alpha : A \rightarrow \prod_{i \in I} A_i$  there exists  $i \in I$  such that  $\pi_i \circ \alpha : A \rightarrow A_i$  is an isomorphism.

An equivalence relation  $\theta$  on an algebra  $A$  is called *congruence*, if  $\theta$  respects any operation.  $\mathbf{Con}(A)$  is the set of all congruences on  $A$ . An algebra is called *simple*, if  $\mathbf{Con}(A/\theta) = \{\Delta, \nabla\}$ , where  $\Delta$  and  $\nabla$  are trivial congruences. One can obtain a simple algebra by factorising it through the maximal congruence [SB81, Theorem 8].

One can equivalently define subdirectly irreducible algebras using congruences as follows. an algebra is subdirectly irreducible iff either  $A$  is trivial or there exists a minimal congruence in  $\mathbf{Con}(A) \setminus \{\Delta\}$ .

Recall that a Stone space is a compact Hausdorff zero-dimensional topological space. A subdirect product  $A \subseteq \prod_{x \in X} A_x$  over a Stone space  $X$  if

1. for all  $a, b \in A$   $\{x \in X \mid a(x) = b(x)\}$  is clopen.
2. for all  $a, b \in A$  and for all clopen  $Y \subseteq X$   $a \upharpoonright_Y \cup a \upharpoonright_{X \setminus Y} \in A$ .

A variety  $\mathcal{V}$  is *arithmetical*, if it is congruence-permutable and congruence-distributive, or, equivalently, there exists a ternary term  $p$  such that:

$$\mathcal{V} \models p(x, y, x) \approx p(x, y, y) \approx p(y, y, x) \approx x \quad (1)$$

A ternary term  $t(x, y, z)$  for an algebra  $A$  if, for all  $a, b, c \in A$ :

$$t(a, b, c) = \begin{cases} c & \text{if } a = b \\ a & \text{otherwise} \end{cases} \quad (2)$$

A variety  $\mathcal{V}$  is called discriminator if there exists a class  $\mathcal{K}$  such that  $\mathcal{V} = \mathbf{V}(\mathcal{K})$  and there exists a term  $t(x, y, z)$ , which is a discriminator term for every member of  $\mathcal{K}$ . It is known that if

an algebra  $A$  has a discriminator term, then  $A$  is simple [SB81, Lemma 9.2]. Moreover, we have the following property of discriminator terms, see [SB81, Theorem 9.4].

**Theorem 1.** *Let  $t(x, y, z)$  be a discriminator term for every member of a class  $\mathcal{K}$ :*

1.  $\mathbf{V}(\mathcal{K})$  is an arithmetical variety.
2. Every indecomposable member of  $\mathbf{V}(\mathcal{K})$  is simple.
3. Simple algebras are precisely members of  $\text{ISP}_U(\mathcal{K}_+)$ .
4. Every member of  $\mathbf{V}(\mathcal{K})$  is isomorphic to a Boolean product of simple algebras.

### 3 BAOs, relation algebras and their reducts

#### 3.1 Discriminator varieties of BAOs

Let  $B$  be a Boolean algebra, an operator is an  $n$ -ary function  $f : A^n \rightarrow A$  such that, for all  $x_1, \dots, x_n, x, y \in B$ :

- $f(x_1, \dots, x + y, \dots, x_n) = f(x_1, \dots, x, \dots, x_n) + f(x_1, \dots, y, \dots, x_n)$
- $f(x_1, \dots, 0, \dots, x_n) = 0$

A Boolean algebra with operators is an algebra  $M = (B, (f_i)_{i \in I})$ , where each  $f_i$  is an operator.

In the case of BAOs, one can define discriminator simpler, as a unary term  $d(x)$  such that, for all  $a \in M$ , where  $M$  is a BAO:

$$d(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{otherwise} \end{cases} \quad (3)$$

One can characterise discriminator varieties as follows, see [AGM<sup>+</sup>98, Lemma 2.1]:

**Theorem 2.** *Let  $\mathcal{V}$  be a variety of BAOs and  $d(x)$  a unary term, then the following are equivalent:*

1.  $d$  is a discriminator variety.
2. The following equations are valid in  $\mathcal{V}$ :
  - (a)  $x \leq d(x)$
  - (b)  $d(d(x)) \leq d(x)$
  - (c)  $d(-d(x)) \leq -d(x)$
  - (d)  $f(x_0, \dots, x_{n-1}) \leq d(x_i)$  for all  $n > 0$  and for every operator  $f$  of  $M$

#### 3.2 Relation algebras and their reducts

In this subsection, we consider relation algebras, a kind of BAOs.

**Definition 1.**

*A relation algebra is an algebra  $\mathcal{R} = (R, 0, 1, +, -, ;, \smile, \mathbf{1}')$  such that  $(R, 0, 1, +, -)$  is a Boolean algebra and the following hold:*

1.  $(R, ;, \mathbf{1}')$  is a monoid

2.  $(a + b); c = (a; c) + (b; c)$
3.  $a^{\smile\smile} = a$
4.  $(a + b)^{\smile} = a^{\smile} + b^{\smile}$
5.  $(a; b)^{\smile} = b^{\smile}; a^{\smile}$
6.  $a^{\smile}; (-a; b) \leq -b$

where  $a \leq b$  iff  $a + b = b$ . **RA** denotes the class of all relation algebras.

**Definition 2.** A proper relation algebra is an algebra  $\mathcal{R} = (R, \emptyset, W, \cup, -, |, \smile, \mathbf{1})$  such that  $R \subseteq \mathcal{P}(W)$ , where  $W \subseteq X \times X$  is an equivalence relation;  $|$  is relation composition,  $\smile$  is relation converse, **Id** is a diagonal subset of  $W$ , that is:

1.  $a|b = \{(x, z) \mid \exists y (x, y) \in a \ \& \ (y, z) \in b\}$
2.  $a^{\smile} = \{(x, y) \mid (y, x) \in a\}$
3. **Id** =  $\{(x, y) \mid x = y\}$

The class of all proper relation algebras is denoted as **PRA**. **Rs** is the class of all relation set algebras, proper relation algebra with a diagonal subrelation as an identity. **RRA** is the class of all representable relation algebras, that is, the closure of **PRA** under isomorphic copies.

### 3.3 Varieties

#### 3.3.1 Boolean algebras with residuated operators

Let  $\mathcal{A}_0$  be a Boolean algebra, a unary operation on  $f$  is called *residuated* if there exists a *residual* operation  $g$  such that for all  $a, b \in \mathcal{A}_0$ :

$$f(a) \leq b \text{ iff } a \leq f(b)$$

Equivalently,  $f$  is residuated if there exists a conjugate operation  $h$  such that for all  $a, b \in \mathcal{A}_0$ :

$$f(a) \cdot b = 0 \text{ iff } a \wedge h(b) = 0$$

$f$  is *self-conjugate* whenever  $f$  is equal to its conjugate operation.

**Theorem 3.** Let  $f$  be a residuated operator on  $\mathcal{A}_0$ , then

1.  $f$  is normal and completely additive,
2.  $f$  and  $h$  are conjugate operations on  $\mathcal{A}_0$  iff they are normal and the following holds for all  $a, b \in \mathcal{A}_0$

$$f(a) \cdot b \leq f(a \cdot h(b)) \text{ and } a \cdot h(b) \leq h(f(a) \cdot b)$$

Two  $n$ -ary operators  $f$  and  $h$  are conjugate in the  $i$ -th argument if  $f_{\bar{a},i}$  is conjugate to  $g_{\bar{a},i}$ . Let  $\mathcal{A}$  be a BAO,  $\mathcal{A}$  is called a *Boolean algebra with residuated operators* if for each operator  $f$  of  $\mathcal{A}$  and for each  $i < ar(f)$  there exists an  $ar(f)$ -ry term  $t$ , which is conjugate to  $f$  in the  $i$ -th argument. **RA** is an example, where  $\smile$  is self-conjugate and composition has conjugate terms  $x; y^{\smile}$  and  $x^{\smile}; y$ .

An  $r$ -algebra is a Boolean algebra with three residuated binary operations  $\bullet$ ,  $\triangleright$  and  $\triangleleft$ , where  $\triangleright$  and  $\triangleleft$  are right and left conjugates to  $\bullet$ , that is, the following conditions are equivalent:

1.  $(x \bullet y) \cdot z = 0$
2.  $(x \triangleright z) \cdot y = 0$
3.  $(z \triangleleft y) \cdot x = 0$

$r$ -algebras generalises relation algebras since every relation algebra is a Boolean residuated monoid, where  $x \triangleright y = x^\smile; y$  and  $x \triangleleft y = x; y^\smile$ .

### 3.3.2 Residuated monoids

### 3.3.3 Positive relation algebras

### 3.3.4 Domain algebras

## 3.4 Non-varieties

## 3.5 Unknown

# 4 Decidability aspects

## 4.1 Current results

## 4.2 Problems

# References

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