$\boxed{08/24/11}$

Idea of Daniel Sank

$$|\Psi_{in}\rangle = \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle e^{-|\alpha|^2/2} \otimes |g\rangle$$

Now evolve for time t in resonance with qubit.

$$H_{int} = \hbar g \left(a^\dagger \sigma_- + a \sigma_+\right)$$

$$|n\rangle|g\rangle \to \cos(\sqrt{n} g t)|n\rangle|g\rangle - i \sin(\sqrt{n} g t)|n-1\rangle|e\rangle$$

Measure $|g\rangle$ or $|e\rangle$

If $|g\rangle$, then $|\cancel{\alpha}\rangle \to \frac{1}{Norm} \sum_n \frac{\alpha^n}{\sqrt{n!}} \cos(\sqrt{n} g t)|n\rangle$

If $|e\rangle$, then $|\cancel{\alpha}\rangle \to \frac{1}{Norm} \sum_n \frac{\alpha^n}{\sqrt{n!}} \sin(\sqrt{n} g t)|n-1\rangle$

~~Start with~~ Consider case $|e\rangle$

$$|\Psi_{fin}\rangle = \frac{1}{Norm} \sum_n \frac{\alpha^n}{\sqrt{n!}} \sin(\sqrt{n} g t)|n-1\rangle$$

If $ngt \ll 1$ for all n, then $\sin \approx \sqrt{n} g t$, and then

$$|\Psi_{fin}\rangle = \frac{1}{Norm} \sum \frac{\alpha^n}{\sqrt{(n-1)!}} |n-1\rangle = \frac{\alpha \cdot |\alpha\rangle}{Norm} = |\alpha\rangle$$

~~only~~ Nothing changes

Let us take into account the next order:

$$\sin\sqrt{n}gt \approx \sqrt{n}gt - \frac{(\sqrt{n}gt)^3}{6} = \sqrt{n}gt\left(1-\frac{ng^2t^2}{6}\right)$$

$$|\Psi_{fin}\rangle = \frac{1}{Norm}\sum_n \frac{\alpha^n}{\sqrt{n!}}\sqrt{n}gt\left(1-\frac{ng^2t^2}{6}\right)|n-1\rangle =$$

$$= \frac{1}{Norm}\sum_n \frac{\alpha\cdot\alpha^{n-1}}{\sqrt{(n-1)!}}\sqrt{g}t\left(1-\frac{ng^2t^2}{6}|n-1\rangle\right) =$$

$$= \frac{1}{Norm}\sum_n \frac{\alpha^n}{\sqrt{n!}}\left(1-\frac{(n+1)g^2t^2}{6}\right)|n\rangle$$

This means $\alpha \to \alpha\left(1-\frac{g^2t^2}{6}\right)$ $\quad (gt \ll 1)$

Consider average energy

$$\langle E\rangle = \frac{\sum\limits_{n=1}^{\infty}\hbar(n-1)\frac{|\alpha^2|^n}{n!}\sin^2(\sqrt{n}gt)}{\sum\limits_{n=1}^{\infty}\frac{|\alpha^2|^n}{n!}\sin^2(\sqrt{n}gt)}$$

difficult to do exactly

Now assume that $|g\rangle$ is measured

$$|\Psi_{fin}\rangle = \frac{1}{Norm}\sum_n \frac{\alpha^n}{\sqrt{n!}}\cos(\sqrt{n}gt)\,|n\rangle$$

If $ngt \ll 1$ so that $\cos \approx 1$, then $|\Psi_{fin}\rangle = |\alpha\rangle$ in first order

Next order: $\cos(\sqrt{n}gt) \approx 1 - \frac{1}{2}ng^2t^2$,

$$|\Psi_{fin}\rangle \approx \frac{1}{Norm}\sum_n \frac{\alpha^n}{\sqrt{n!}}\left(1-\frac{1}{2}ng^2t^2\right)|n\rangle, \quad \alpha \to \left(1-\frac{g^2t^2}{2}\right)\alpha$$

So, in both cases α decreases, but for measured $|e\rangle$ it is 3 times smaller decrease than for $|g\rangle$

~~What is qubit energy on average?~~

Average energy of oscillator $\langle E\rangle = \hbar\omega|\alpha|^2$

For measured $|e\rangle$ $\quad \langle E\rangle \to \hbar\omega|\alpha|^2\left(1-\frac{g^2t^2}{3}\right)$

$|g\rangle \quad \langle E\rangle \to \hbar\omega|\alpha|^2\left(1-g^2t^2\right)$

What is qubit energy on average?

$$\langle E_{qb}\rangle = \hbar\omega\cdot \text{prob}(|e\rangle) = \hbar\omega \sum \frac{|\alpha|^{2n}}{n!}e^{-|\alpha|^2}\sin^2(\sqrt{n}\,gt) =$$

$$\approx \hbar\omega\cdot g^2t^2 \underbrace{\sum n\frac{|\alpha|^{2n}}{n!}e^{-|\alpha|^2}}_{|\alpha|^2} = \hbar\omega|\alpha|^2\cdot g^2t^2$$

So, energy conservation (on average) is OK

qubit: 0 with probability $1-|\alpha|^2g^2t^2$; $\hbar\omega$ with $p=|\alpha|^2g^2t^2$ $\Big\}$ on average $\hbar\omega\cdot|\alpha|^2g^2t^2$

resonator: $\hbar\omega|\alpha|^2(1-g^2t^2)$ with probability $1-|\alpha|^2g^2t^2$; $\hbar\omega|\alpha|^2\left(1-\frac{g^2t^2}{3}\right)$ with $p=|\alpha|^2g^2t^2$ $\Big\}$ on average $\hbar\omega|\alpha|^2(1-g^2t^2)$ (in this order)

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Daniel's paradox still holds:

from the point of view of Schrödinger equation average energy of resonator decreases because of populating $|e\rangle$:

in units of $\hbar\omega$ $\quad \frac{\Delta E_{res}}{\hbar\omega} = -\,\mathrm{prob}(|e\rangle)$

However, from two actual scenarios; in the case $|e\rangle$, resonator energy practically does not change ($\ll \hbar\omega$), so total energy increases by $\approx \hbar\omega$

In case $|g\rangle$ total energy slighty decreases, that compensates (on average) the increase in the case $|e\rangle$.

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What if incoherent (statistical) mixture ~~of~~ Fock states?
(I guess same thing)

$$\rho_{in} = \sum_n \underbrace{\frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2}}_{\text{probability}} |ng\rangle\langle ng|$$

Then after interaction with qubit still the same mixture,

but $|ng\rangle\langle ng| \to \left(\cos|ng\rangle - i\sin|n-1,e\rangle\right)\left(\cos\langle ng| + i\sin\langle n-1,e|\right)$

If measure $|e\rangle$, then resonator becomes $|n-1\rangle$, but now have to do classical Bayes update

$$\rho_{fin,res} = \frac{1}{Norm} \sum_n \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2} \cdot \sin^2 \cdot |n-1\rangle\langle n-1|$$

that exactly corresponds to the coherent case (if also ~~think~~ ~~about~~ convert info mixture)

Similarly for the case when $|g\rangle$ is measured

$\Rightarrow$ yes, classical interpretation is possible (bad news)
however, we know (checked expt.) that this is a coherent state (not classical mixture) $\Rightarrow$ still interesting