Simple model for Purcell filter designed at UCSB (v.1)

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In this write-up we explain a simple model for the Purcell filter designed at USCB. Using this simple model we show that the Purcell rate can be strongly suppressed for realistic experimental parameters for circuit QED systems.

We consider a superconducting transmon qubit coupled to a transmission line resonator 1, which is in turn coupled to a second resonator 2 through a small capacitor. We assume that the two resonators are practically at resonance (in reality the first resonator frequency changes due to the qubit; since the change is quite small we assume $\omega_{\rm r} = \omega_{\rm r_1} = \omega_{\rm r_2}$) and the second resonator is coupled to the environment. The schematic of the model is shown in Fig. 1. The system interaction Hamiltonian, in frame rotating at the resonator frequency ω_r, is given by ($\hbar = 1$)

$$H = \frac{\Delta}{2}\sigma_z + g(a^\dagger\sigma_- + a\sigma_+) + \mathcal{G}(a^\dagger b + ab^\dagger), \qquad (1)$$

where $\Delta = \omega_{\rm q} - \omega_{\rm r}$ with $\omega_{\rm q}$ is the qubit frequency, g is the qubit-resonator coupling constant and $\mathcal{G}$ is the coupling constant between the two resonators (the resonator's Hamiltonians are zero because of the rotating frame). Here $\sigma_\pm$ are the qubit raising and lowering operators, a and b are annihilation operators for the first and the second resonators, respectively.

The master equation including the damping of photons to the environment of the second resonator has the form

$$\dot\rho = -i[H,\rho] + \kappa\left(b\rho b^\dagger - \frac{1}{2}b^\dagger b\rho - \frac{1}{2}\rho b^\dagger b\right), \qquad (2)$$

where κ is the damping rate of photons through the second resonator. Note that we ignore dissipation (damping) from the first resonator, as the capacitive coupling to the second resonator has minimal loss.

Let us first consider the simple case to understand how the qubit relaxation rate gets modified in this Purcell filter model. For this purpose, we consider an initial condition in which the qubit is in the excited state $|e\rangle$ and the two resonators are in their respective vacuum states. We introduce the bare basis spanned by Hilbert space of the three subsystems as: $|e\rangle = |e00\rangle, |1\rangle = |g10\rangle, |2\rangle = |g01\rangle, |g\rangle = |g00\rangle$, where in $|jnm\rangle$, j represents the qubit states and n and m represent the first and second resonator Fock states, respectively. The relevant equations of evolution for the density matrix elements in bare basis

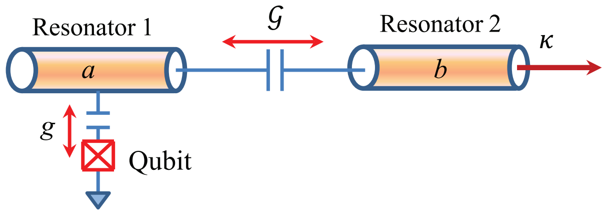

FIG. 1. Schematics of the Purcell filter model proposed at UCSB. The qubit is capacitively coupled to the resonator 1, which in turn is coupled to the resonator 2 with a coupling constant $\mathcal{G}$. The photon decays through the second resonator at a rate κ.

are:

$$\dot\rho_{ee} = ig(\rho_{e1} - \rho_{1e}), \qquad (3)$$
$$\dot\rho_{e1} = -i\Delta\rho_{e1} - ig(\rho_{11} - \rho_{ee}) + i\mathcal{G}\rho_{e2}, \qquad (4)$$
$$\dot\rho_{e2} = -\left(\frac{\kappa}{2} + i\Delta\right)\rho_{e2} + i\mathcal{G}\rho_{e1}, \qquad (5)$$
$$\dot\rho_{11} = -ig(\rho_{e1} - \rho_{1e}) - i\mathcal{G}(\rho_{21} - \rho_{12}), \qquad (6)$$
$$\dot\rho_{12} = -\frac{\kappa}{2}\rho_{12} - i\mathcal{G}(\rho_{22} - \rho_{11}), \qquad (7)$$
$$\dot\rho_{22} = -\kappa\rho_{22} - i\mathcal{G}(\rho_{12} - \rho_{21}). \qquad (8)$$

The only channel of decay for the first resonator is the damping of photons through the second resonator. The effective decay rate for the first resonator can be derived using Eqs. (6)-(8). We assume the photon damping rate of the second resonator is larger than the coupling between the two resonators, $\kappa \gg \mathcal{G}$. With this assumption, the probability ρ_{22} of finding the photon in the second resonator decays at a rate much larger than the rate it is fed by coherent Rabi oscillation $\mathcal{G}$. Thus, this probability remains very small and its the steady state value [from (8)] reads, $\rho_{22} = -i\mathcal{G}(\rho_{12} - \rho_{21})/\kappa$. Inserting this result into (7), one can show that

$$\begin{aligned}\dot\rho_{12} - \dot\rho_{21} &= -\frac{\kappa}{2}\left(1 + \frac{4\mathcal{G}^2}{\kappa^2}\right)(\rho_{12} - \rho_{21}) + 2i\mathcal{G}\rho_{11}\\ &\simeq -\frac{\kappa}{2}(\rho_{12} - \rho_{21}) + 2i\mathcal{G}\rho_{11}, \end{aligned}\qquad (9)$$

where we dropped $4\mathcal{G}^2/\kappa^2$ compared to 1, which is consistent with our assumption $\kappa \gg \mathcal{G}$. Substituting the steady state solution of (9), $\rho_{12} - \rho_{21} = -4i\mathcal{G}\rho_{11}/\kappa$ into

TABLE I. Purcell rate suppression factor for realistic experimental parameters for cQED schemes.

$\kappa/2\pi$(MHz)	200	200	200
$\Delta/2\pi$(MHz)	300	400	500
η	0.1	0.06	0.04

(6), we obtain the equation for probability of finding the photon in the first resonator

$$\dot{\rho}_{11} = -\kappa_{\rm r}\rho_{11} - ig(\rho_{e1} - \rho_{1e}). \tag{10}$$

where $\kappa_{\rm r} = 4\mathcal{G}^2/\kappa$ is the first resonator decay rate due to the damping of the photon through the second resonator.

We next derive the Purcell rate using Eqs. (3)-(5) and (10). Inserting the steady state solution of (5)

$$\rho_{e2} = \frac{i\mathcal{G}}{\kappa/2 + i\Delta}\rho_{e1}$$

into Eq. (4), we can write the coupled equations for the coherences as

$$\dot{\rho}^{(+)}_{e1} = -\frac{\kappa_{\rm qb}}{2}\rho^{(+)}_{e1} - i\Delta_{\rm eff}\rho^{(-)}_{e1}, \tag{11}$$

$$\dot{\rho}^{(-)}_{e1} = -\frac{\kappa_{\rm qb}}{2}\rho^{(-)}_{e1} - i\Delta_{\rm eff}\rho^{(+)}_{e1} - 2ig(\rho_{11} - \rho_{ee}), \tag{12}$$

where $\rho^{(\pm)}_{e1} = \rho_{e1} \pm \rho_{1e}$ and the effective detuning $\Delta_{\rm eff}$ and frequency dependent damping rate $\kappa_{\rm qb}$ that the qubit "sees" are, respectively

$$\Delta_{\rm eff} = \Delta\left(1 - \frac{\mathcal{G}^2}{\Delta^2 + \kappa^2/4}\right), \tag{13}$$

$$\kappa_{\rm qb}(\Delta) = \frac{4\mathcal{G}^2/\kappa}{1 + (2\Delta/\kappa)^2} = \frac{\kappa_{\rm r}}{1 + (2\Delta/\kappa)^2}. \tag{14}$$

Notice that the Purcell filter works when the qubit "sees" very suppressed damping rate. Here what the qubit "sees" is the frequency dependent effective rate $\kappa_{\rm qb}$, which can be smaller than that of the first resonator damping rate $\kappa_{\rm r}$, $\kappa_{\rm qb} \ll \kappa_{\rm r}$, when $\Delta > \kappa$.

Now let us put together all four coupled equations which are important to derive the Purcell rate: Eqs. (3), (10), (11), and (12)

$$\dot{\rho}_{ee} = ig\rho^{(-)}_{e1}, \tag{15}$$

$$\dot{\rho}_{11} = -\kappa_{\rm r}\rho_{11} - ig\rho^{(-)}_{e1}, \tag{16}$$

$$\dot{\rho}^{(+)}_{e1} = -\frac{\kappa_{\rm qb}}{2}\rho^{(+)}_{e1} - i\Delta_{\rm eff}\rho^{(-)}_{e1}, \tag{17}$$

$$\dot{\rho}^{(-)}_{e1} = -\frac{\kappa_{\rm qb}}{2}\rho^{(-)}_{e1} - i\Delta_{\rm eff}\rho^{(+)}_{e1} - 2ig(\rho_{11} - \rho_{ee}). \tag{18}$$

It is useful to compare these equations with the corresponding equation for standard Purcell rate calculation (single resonator with damping rate $\kappa_{\rm r}$). In the equations used to calculate the standard Purcell rate, the probability of finding the photon in the first resonator ρ_{11} as well as the coherence terms $\rho^{\pm}_{e1}$ decay at rate $\kappa_{\rm r}$ and $\kappa_{\rm r}/2$, respectively. For Purcell filter model only ρ_{11} decay at rate $\kappa_{\rm r}$. The coherence terms $\rho^{(\pm)}_{e1}$, which involve the qubit state, decay at rate $\kappa_{\rm qb}/2 = (\kappa_{\rm r}/2)/[1 + (2\Delta/\kappa)^2]$, which is less than $\kappa_{\rm r}/2$ by a factor $[1 + (2\Delta/\kappa)^2]^{-1}$ (assuming $\Delta > \kappa$). This is the reason why the qubit "sees" a diminished damping rate. Now following the standard derivation of the Purcell rate, we insert the steady state solution of Eqs. (16) and (17)

$$\rho_{11} = -\frac{ig}{\kappa_{\rm r}}\rho^{(-)}_{e1},$$

$$\rho^{(+)}_{e1} = -\frac{2i\Delta_{\rm eff}}{\kappa_{\rm qb}}\rho^{(-)}_{e1}$$

into (18) to obtain

$$\dot{\rho}^{(-)}_{e1} = -\frac{1}{2\kappa_{\rm qb}}\left(\kappa^2_{\rm qb} + 4g^2\frac{\kappa_{\rm qb}}{\kappa_{\rm r}} + 4\Delta^2_{\rm eff}\right)\rho^{(-)}_{e1} + 2ig\rho_{ee}. \tag{19}$$

Since $\kappa \gg \mathcal{G}$, the effective detuning (13) can be approximated as $\Delta_{\rm eff} \approx \Delta$. If we further assume that $\Delta > g$, the first two terms in the bracket in Eq. (19) can be dropped compared to $4\Delta^2$. Thus, Eq. (19) reduces to

$$\dot{\rho}^{(-)}_{e1} \simeq -\frac{\Delta^2}{\kappa_{\rm qb}}\rho^{(-)}_{e1} + 2ig\rho_{ee}. \tag{20}$$

Inserting the steady state solution of Eq. (20), $\rho^{(-)}_{e1} = i(g\kappa_{\rm qb}/\Delta^2)\rho_{ee}$ into (15), we get

$$\dot{\rho}_{ee} \simeq -\left[\frac{1}{1 + (2\Delta/\kappa)^2}\right]\frac{g^2\kappa_{\rm r}}{\Delta^2}\rho_{ee}. \tag{21}$$

Thus the Purcell relaxation rate for this simple Purcell filter model is

$$\Gamma_{\rm p} = \left[\frac{1}{1 + (2\Delta/\kappa)^2}\right]\frac{g^2\kappa_{\rm r}}{\Delta^2}. \tag{22}$$

If we consider a single resonator with a damping rate $\kappa_{\rm r}$, the corresponding "naive" Purcell rate is $\Gamma_0 = g^2\kappa_{\rm r}/\Delta^2$. Therefore, Purcell filter gives rise to suppression of the Purcell rate by factor

$$\eta = \frac{\Gamma_{\rm p}}{\Gamma_0} = \frac{1}{1 + (2\Delta/\kappa)^2}. \tag{23}$$

For realistic experimental parameters, more than 90% suppression of the Purcell rate can be achieved as compared to without the filter; see Table .