Modeling the Noise Influence on the Fourier Coefficients After a Discrete Fourier Transform

J. SCHOUKENS AND J. RENNEBOOG

Abstract—**An analysis is made to study the influence of time-domain noise on the results of a discrete Fourier transform (DFT). It is proven that the resulting frequency-domain noise can be modeled using a Gaussian distribution with a covariance matrix which is nearly diagonal, imposing very weak assumptions on the noise in the time domain.**

I. Introduction

THE IMPORTANCE of digital signal processing algorithms has grown very rapidly during the last decade. Also, in the area of spectral analysis techniques there has been an explosive growth in the number of applications, especially after the development of the fast Fourier transform (FFT) algorithm [1]. To use the results of these algorithms in an efficient way, it is necessary to have an idea of the uncertainty on the Fourier coefficients, due to the noise disturbance during the measurement process (e.g., quantization noise). This means that the algorithm results have to be treated as random variables which are described by a probability density function (pdf). Up until this time, not many articles have directed much attention to this problem. Some information about a similar problem, estimating the power spectrum of a random sequence [2], has been found but the problem of the disturbance of the Fourier coefficients after a discrete Fourier transform (DFT) due to time-domain noise is not really studied. This problem is investigated here by examining the pdf of the noise on the Fourier coefficients. It is proven that the noise can be approximately modeled as Gaussian. The mean and covariance matrix, which determine the Gaussian distribution completely, are also calculated.

II. Preliminary Considerations

To study the frequency-domain noise influence, it is necessary to make some assumptions about the character of the time-domain noise. In order to have the most general results, it is necessary to minimize the imposed restrictions. In this article the following assumptions on the noise in the time domain are made [3].

1) The noise on the measurements is additive. This

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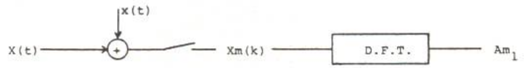

Fig. 1. Graphic representation of the used variables.

means that

$$X(k)_{\text{meas}} = X(k)_{\text{true}} + x(k) \tag{1}$$

with $X(k)_{\text{meas}}$ the kth measurement at time t_k. $X(k)_{\text{true}}$ is the true value and $x(k)$ is the noise contribution.

An example of additive noise is the quantization noise when a signal is measured with a digitizer.

2) The noise is stationary, which means that its statistical properties do not change in time. As a consequence, the autocorrelation of $x(k)$ can be expressed as

$$E[x(k)\, x(l)] = \rho_{xx}(k - l). \tag{2}$$

From this definition, it is immediately seen that the autocorrelation function is symmetric.

3) The mean of the noise is μ_x

$$E[x(k)] = \mu_x. \tag{3}$$

The influence of the noise $x(k)$ on the Fourier coefficients will next be analyzed. The problem of the correct use of the DFT is not studied. This means that attention is not given to the problems of leakage and aliasing which are due to the DFT. Because of the linearity of the DFT and the additive character of the noise, it is possible to separate the study of the stochastical disturbance, due to the noise influence, from the problem of the correct use of the DFT.

The following notations, shown in Fig. 1, will be used.

- $X(t)$ the true, but unknown value of the studied signal,
- $x(t)$ the additive noise,
- $Xm(k)$ the measured value at the time t_k,
- Am_l the measured Fourier coefficient at the lth frequency component (measurement means: the DFT of $Xm(k)$) $Am_l = ARm_l + jAIm_l$, with ARm the real part, AIm the imaginary part,
- A_l the DFT of $X(k)$,
- a_l the DFT of the noise $x(k)$.

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The values are related by

$$Xm(k) = X(k) + x(k)$$
$$Am_l = A_l + a_l. \tag{4}$$

III. Study of the Probability Density Function of the Noise

A very important tool to analyze a stochatic variable is the probability density function [3]. This function describes the amplitude distribution of the noise. The noise on the Fourier coefficients is given by

$$a_i = \sum_{k=0}^{N-1} x(k)\, W_N^{ki} \qquad i = 0, 1, \cdots, N-1 \tag{5a}$$

or

$$aR_i = \sum_{k=0}^{N-1} x(k) \cos \frac{2\pi i}{N} k = \sum_{k=0}^{N-1} c(k)$$
$$aI_i = -\sum_{k=0}^{N-1} x(k) \sin \frac{2\pi i}{N} k = \sum_{k=0}^{N-1} s(k) \tag{5b}$$

with

N the number of samples,
W_N $e^{-j2\pi/N}$,
$x(k)$ a stochatic variable, representing the noise.

The variables $c(k)$ and $s(k)$ can be considered as random variables. From (5b) it is seen that the noise on the Fourier coefficients is derived from a sum of a great number of random variables. (typical values of N are 128, 512, 1024 $\cdots$). The Central Limit Theorem states: "a sum of a great number of random variables with a limited autocorrelation function is a new random variable with an asymptotic Gaussian distribution" [5].

This means that the noise on the Fourier coefficients aR and aI can be considered as having a Gaussian distribution, for N to be sufficiently high. In Fig. 2, the cummulative probability density function is given for aR_l. The number of points is $N = 1024$. The noise sequence $x(k)$ was considered to be white. In the figure, the cummulative distribution is compared to the "best fitted" Gaussian distribution. It is clear that there is good agreement between practice and theory.

A Gaussian distribution is completely characterized by its mean and its covariance matrix. In the following paragraphs the mean and covariance matrix will be calculated.

IV. Study of the Mean of Am

Firstly, the influence of the noise on the mean value of Am is studied. It will be shown that the mean values of the DFT coefficients are not influenced by the noise, except for the dc component. The DFT of the sequence $Xm(k)$ is given by

$$Am_i = \sum_{k=0}^{N-1} Xm(k)\, W_N^{ki} \qquad i = 0, 1, \cdots, N-1$$

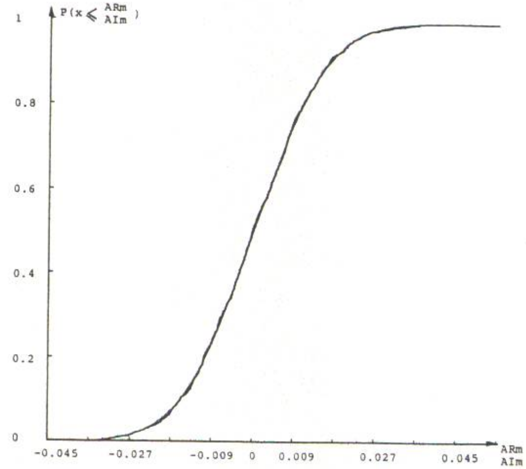

Fig. 2. Comparison of the cumulative distribution function of a DFT with a Gaussian distribution.

with

N the number of samples
W_N $e^{-j2\pi/N}$.

Substitution of Xm gives

$$Am_i = \sum_{k=0}^{N-1} [X(k)\, W_N^{ki} + x(k)\, W_N^{ki}]. \tag{6}$$

The mean of Am_i is found by taking the mathematical expectation $E[\cdots]$

$$\begin{aligned} E[Am_i] &= E\left[\sum_{k=0}^{N-1} (X(k)\, W_N^{ki} + x(k)\, W_N^{ki})\right] \\ &= \sum_{k=0}^{N-1} X(k)\, W_N^{ki} + \sum_{k=0}^{N-1} E[x(k)]\, W_N^{ki} \\ &= A_i + \mu_x \sum_{k=0}^{N-1} W_N^{ki}. \end{aligned} \tag{7}$$

The first term in this expression is the true value of the measured Fourier coefficient Am_i. The value of the second term depends upon the value of i:

$$\text{If} \quad i = 0, \; W_N^0 = 1 \quad \text{and} \quad E[Am_0] = A_0 + N\mu_x \tag{8}$$

$$\text{If} \quad i \neq 0, \; \text{the} \; \sum_{k=0}^{N-1} W_N^{ki} = 0 \quad \text{and} \quad E[Am_i] = A_i$$
$$i = 1, \cdots, N-1. \tag{9}$$

From these results (8) and (9) it is seen that the mean values of the Fourier coefficients go to the exact values except for the dc component. A systematic error, given by $N\mu$, occurs which can be easily derived from the knowledge of the noise properties in the time domain, so that A_0 can be derived.

V. Study of the Covariance Matrix of AR and AI

In this section the covariance matrix of the noise on the Fourier coefficients is studied. It is necessary to distinguish the following situations.

1) Covariance between the real parts of the spectrum

$$E[aR_i aR_j]. \tag{10}$$

2) Covariance between the imaginary parts of the spectrum

$$E[aI_i aI_j]. \tag{11}$$

3) Covariance between the real and imaginary parts

$$E[aR_i aI_j]. \tag{12}$$

$$\sigma_{ij}^2(RR) = E[aR_i aR_j] = E\left[\sum_{K=0}^{N-1} x(k) \cos kif \cdot \sum_{l=0}^{N-1} x(l) \cos ljf\right]$$

with $f = 2\pi/N$

$$= E\left[\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} x(k)\, x(l) \cos kif \cos ljf\right] = \sum_{k=0}^{N-1}\sum_{l=0}^{N-1} \rho(k-l) \cos kif \cos ljf. \tag{13}$$

In the same manner it is found:

$$\sigma_{ij}^2(II) = E[aI_i aI_j] = \sum_{K=0}^{N-1}\sum_{l=0}^{N-1} \rho(k-l) \sin kif \sin ljf \tag{14}$$

$$\sigma_{ij}^2(RI) = E[aR_i aI_j] = -\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} \rho(k-l) \cos kif \sin ljf. \tag{15}$$

A similarity between these expressions can be observed. Now define

$$S^{RR}(k,l) = \cos kif \cos ljf$$
$$S^{II}(k,l) = \sin kif \sin ljf$$
$$S^{RI}(k,l) = -\cos kif \sin ljf.$$

The expressions (12)–(14) can then be written as

$$\sigma_{ij}^2 = \sum_{k=0}^{N-1}\sum_{l=0}^{N-1} \rho(k-l)\, S(k,l). \tag{16}$$

The sum in (16) can be split into three parts

$$\sigma_{ij}^2 = \sum_{k=0}^{N-1} \rho(0)\, S(k,k) + \sum_{k=1}^{N-1}\sum_{l=0}^{k-1} \rho(k-l)\, S(k,l) + \sum_{l=1}^{N-1}\sum_{k=0}^{k-1} \rho(k-l)\, S(k,l). \tag{17}$$

Using the symmetry property of ρ, (17) becomes

$$\sigma_{ij}^2(II) = \sum_{k=0}^{N-1} \rho(0)\, S(k,k) + \sum_{k=1}^{N-1}\sum_{l=0}^{k-1} \rho(k-l)[S(k,l) + S(l,k)] \tag{18}$$

$$= \sum_{k=0}^{N-1} \rho(0)\, S(k,k) + \sum_{m=1}^{N-1} \rho(m) \cdot \sum_{l=0}^{N-m-1} [S(m+l,l) + S(l,m+l)]. \tag{19}$$

To refine these expressions, it is necessary to split up the covariance matrix of the real and imaginary parts in partial matrices

$$C = \begin{pmatrix} C_{RR} & C_{RI} \\ C_{IR} & C_{II} \end{pmatrix}$$

with

$$C_{RR,ij} = E[aR_i aR_j]$$
$$C_{II,ij} = E[aI_i aI_j]$$
$$C_{RI,ij} = E[aR_i aI_j]$$
$$C_{IR,ij} = E[aI_i aR_j]. \tag{20}$$

A. Study of $C_{RR,ij}$

In expression (19) two terms can be distinguished. The first term can be evaluated immediately ($S = S^{RR}$):

$$\sum_{k=0}^{N-1} \rho(0)\, S^{RR}(k,k) = \rho(0) N/2 \delta_{ij}, \qquad i > 0$$

with δ_{ij} the Kronecker delta

$$= \rho(0) N, \qquad i = j = 0.$$

The second is given by

$$\sum_{m=1}^{N-1} \rho(m) \sum_{l=0}^{N-m-1} [S^{RR}(m+l,l) + S^{RR}(l,m+l)]. \tag{21}$$

The expression in the second summation can be transformed to

$$S^{RR}(m+l,l) + S^{RR}(l,m+l) = \cos\frac{mf(i+j)}{2} \cos\left[\frac{mf(i-j)}{2} + lf(i-j)\right] + \cos\frac{mf(i-j)}{2} \cos\left[\frac{mf(i+j)}{2} + lf(i+j)\right]. \tag{22}$$

As can be seen here, to evaluate expression (21) two cases have to be considered: $i \neq j$ and $i = j$.

1) $i \neq j$: Using [4, eq. (1.353/3)] (appendix), the summation in (21) is reduced to:

$$\sum_{l=0}^{N-m-1} [S^{RR}(m+l, l) + S^{RR}(l, m+l)]$$

$$= \cos fm \frac{i+j}{2} \cos \left[fm \frac{i-j}{2} + (N-m-1) f \frac{i-j}{2} \right] \frac{\sin \left[\frac{N-m}{2} f \frac{i-j}{2} \right]}{\sin f \frac{i-j}{2}}$$

$$+ \cos fm \frac{i-j}{2} \cos \left[fm \frac{i+j}{2} + (N-m-1) f \frac{i+j}{2} \right] \frac{\sin \left[\frac{N-m}{2} f \frac{i+j}{2} \right]}{\sin f \frac{i+j}{2}}. \tag{23}$$

Using the fact that $Nf = 2\pi$, (23) can be further reduced to:

$$= -\cos mf \frac{i+j}{2} \cos f \frac{i-j}{2} \frac{\sin mf \frac{i-j}{2}}{\sin f \frac{i-j}{2}} - \cos mf \frac{i-j}{2} \cos f \frac{i+j}{2} \frac{\sin mf \frac{i+j}{2}}{\sin f \frac{i+j}{2}} \quad \text{for } i \neq j \tag{24a}$$

$$= \frac{\sin mjf \sin fj - \sin mif \sin if}{2 \sin f \frac{i-j}{2} \sin f \frac{i+j}{2}}. \tag{24b}$$

2) $i = j$: In this case, (21) becomes

$$\sum_{m=1}^{N-1} \rho(m) \sum_{l=0}^{N-m-1} [\cos mfi + \cos (mf + 2lf)i]. \tag{25}$$

Two parts have to be evaluated:

i) $\sum_{l=0}^{N-m-1} \cos mfi = (N-m) \cos mfi$

ii) $\sum_{l=0}^{N-m-1} \cos (mf + 2lf)i$

$$= \cos [mfi + (N-m-1)fi] \frac{\sin [N-m]fi}{\sin fi}$$

$$i > 0 \quad \text{(using [4, eq. (1.342/2])}$$

$$= -\cos fi \frac{\sin mfi}{\sin fi}. \tag{26}$$

From the previous results, the following equations can be easily derived

$$\sigma_{ii}^2(RR) = \rho(0) \frac{N}{2} + \sum_{m=1}^{N-1} \rho(m) \left[(N-m) \cos mfi - \cos fi \frac{\sin mfi}{\sin fi} \right] \quad i > 0$$

$$\sigma_{oo}^2(RR) = \rho(0) N + 2 \sum_{m=1}^{N-1} \rho(m)(N-m)$$

$$\sigma_{ij}^2(RR) = \sum_{m=1}^{N-1} \rho(m) \frac{\sin mjf \sin fj - \sin mif \sin if}{2 \sin f \frac{i-j}{2} \sin f \frac{i+j}{2}}$$

$$i \neq j \tag{27}$$

B. Study of $C_{II,ij}$

The first part of (19) is the same as in Section V-A. The second part has to be modified:

$$S^{II}(m+l, l) + S^{II}(l, m+l)$$

$$= \cos \frac{mf(i+j)}{2} \cos \left[mf \frac{i-j}{2} + lf(i-j) \right] + \cos \frac{mf(i-j)}{2} \cos \left[mf \frac{i+j}{2} + lf(i+j) \right]. \tag{28}$$

These expressions can be used in the same way as in Section V-A and finally the following results are found:

$$\sigma_{ii}^2(II) = \rho(0) \frac{N}{2} + \sum_{m=1}^{N-1} \rho(m) \left[(N-m) \cos mfi + \cos fi \frac{\sin mfi}{\sin fi} \right] \quad i > 0$$

$$\sigma_{ij}^2(II) = \sum_{m=1}^{N-1} \rho(m) \frac{\sin fi \sin mfj - \sin fj \sin mfi}{2 \sin f \frac{i-j}{2} \sin f \frac{i+j}{2}},$$

$$i \neq 0. \tag{29}$$

C. Study of $C_{IR,ij}$

In this case

$$S^{IR}(m+l, l) + S^{IR}(l, m+l)$$

$$= -\sin \left[mf \frac{i+j}{2} + lf(i+j) \right] \cos mf \frac{i-j}{2} - \sin \left[mf \frac{i-j}{2} + lf(i-j) \right] \cos mf \frac{i+j}{2}. \tag{30}$$

Again, expression (19) can be evaluated. This time the first term of (19) is always zero, the second term reduces to the same result for $i \neq j$ and $i = j$ in

$$\sigma_{ij}^2(IR) = -\sum_{m=1}^{N-1} \rho(m) \sin mfi$$

$$\sigma_{ij}^2(RI) = -\sum_{m=1}^{N-1} \rho(m) \sin mfj \tag{31}$$

VI. Study of the Results

A. $x(k)$ is a White Noise Sequence

If $x(k)$ is a white noise sequence the correlation function is given by

$$\rho(m) = \begin{cases} \sigma_x^2 & m = 0 \\ 0 & m \neq 0 \end{cases} \tag{32}$$

with σ_x^2 the variance of the sequence $x(k)$.

The expressions in (27), (29), and (31) are reduced to:

$$\begin{aligned} \sigma_{ii}^2(RR) &= \sigma_{ii}^2(II) = \sigma_x^2 N/2 \quad i = 1, \cdots, N/2 - 1 \\ \sigma_{oo}^2(RR) &= \sigma_x^2 N \\ \sigma_{ij}^2(RR) &= \sigma_{ij}^2(II) = 0 \qquad i \neq j \\ \sigma_{ij}^2(IR) &= 0 \qquad \forall i, j \end{aligned} \tag{33}$$

This means that the covariance matrix C is reduced to a diagonal matrix

$$C = \frac{N}{2}\sigma_x^2 I \quad \text{with } I \text{ the identity matrix.}$$

To interpret this result, it is necessary to know that the DFT introduces a scale factor $N/2$ for the Fourier coefficients (N for the dc coefficient). To find the original values for the amplitudes, all the amplitudes have to be divided by $N/2$, the variances of the scaled results by $(N/2)^2$. So a scaled covariance matrix is found to be

$$C^s = \frac{\sigma_x^2}{N/2} I. \tag{34}$$

This relation is useful to model, for instance, the influence of the quantization noise. The quantization noise can be modeled as a white noise sequence with a uniform distribution (Fig. 3). The standard deviation is

$$\sigma_x = \frac{\text{quantization step}}{2^n \sqrt{12}}$$

with n being the number of bits of the adc. The standard deviation on the DFT results after scaling by $N/2$, is given by

$$\begin{aligned} \sigma_{\mathrm{DFT}} &= \frac{\sigma_x}{\sqrt{N/2}} \\ &\quad \text{e.g., } N = 1024 \\ &= \sigma_x/22. \end{aligned} \tag{35}$$

This means that it is possible to reduce the influence of the quantization noise to a value below the quantization step by choosing a sufficient number of points. In the example of Fig. 4, this result is illustrated. $y(t)$ is a sum of two sinusoids with amplitude of 0.9 and 0.001 V. This signal was digitized with a quantization step of 0.008 mV ($\sigma_{\mathrm{DFT}} = 0.1$ mV). In Fig. 4 the amplitude spectrum after a DFT is given. It is seen that the small sinusoid with an amplitude less then the quantization step is detected. Because the noise on the DFT has a Gaussian distribution, 99.7 percent has to be in the interval $[-3\sqrt{2}\, s_{\mathrm{DFT}}, 3\sqrt{2}\, s_{\mathrm{DFT}}]$. This means that almost all of the values in Fig. 4 have to be below -68 dB, which coincides very well with the practical results. The value of the dc component is given in (8). In this example $\mu = 0.004$ (half the quantization step) and $20 * \log(\mu) = -48$ dB.

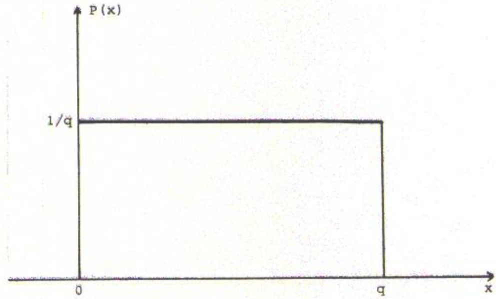

Fig. 3. Modelization of the quantization noise with a uniform distribution (digitalization with truncation).

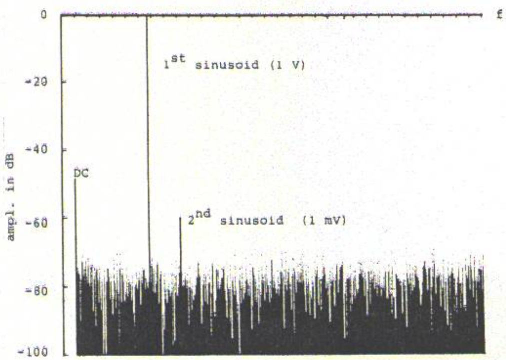

Fig. 4. Detection of a signal below the quantization level.

In Fig. 5, the influence of the quantization noise of an 8-bit digitizer after a DFT is visualized. ($N = 1024$, full scale $+/-1$ V, quantization step $= 8$ mV $\rightarrow \sigma_{\mathrm{DFT}} = 0.1$ mV).

B. $x(k)$ is a Colored Noise Sequence

To study the expressions (27), (29), and (31) for a colored noise sequence, it is necessary to make some assumptions on the noise sequence $x(k)$. The correlation function $\rho(m)$ has to be specified. In this section, it is assumed that the noise sequence $x(k)$ is derived from a white noise sequence $n(k)$ filtered with a first-order low-pass filter with a transfer function $H(z)$. By the proper choice of the low-pass filter, the variance of the colored noise sequence is equal to the variance of the white noise sequence σ^2:

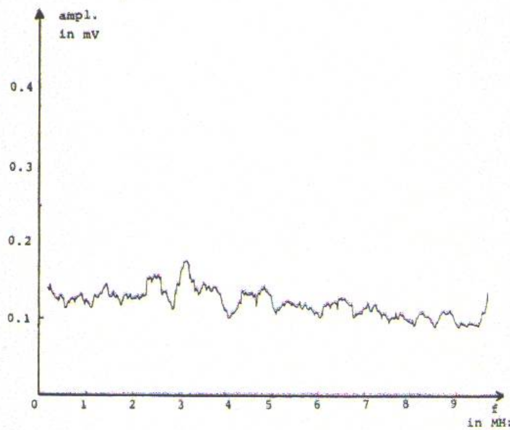

Fig. 5. Measurement of the standard deviation σ_{DFT} of an 8-bit digitizer.

$$H(z) = \frac{\sqrt{1 - e^{-4\pi K}}}{1 - e^{-2\pi K} z^{-1}}$$

the noise described as a discrete sequence,

$$H(s) = \frac{\sqrt{1 - e^{-4\pi K}}}{2\pi f_c + s}$$

Impulse Invariant Transformation,
the noise described as a continous signal

$$\tag{36}$$

with f_c being the 3-dB point of the first-order filter and $K = f_c/f_s$ with f_s as the sampling frequency. K is the ratio of the 3-dB point of the low-pass filter to the sample rate. From $H(z)$, the impulse response $h(k)$ of the low-pass filter is easily derived using the inverse z-transform

$$h(k) = \sqrt{1 - e^{-4\pi K}}\, e^{-2\pi k K}. \tag{37}$$

The response of the filter to the sequence $n(k)$ is given by

$$x(k) = \sum_{i=0}^{\infty} h(i)\, n(k - i) = h * n$$

with $*$ the convolution operator. (38)

With the previous assumptions the correlation function can be calculated as

$$\begin{aligned} \rho(m) &= E[x(k)\, x(k + m)] \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} h(i)\, h(j)\, E[\underbrace{n(k - i)\, n(k + m - j)}_{\sigma^2 \delta_{i,j+m}}] \\ &= \sigma^2 e^{-2\pi m K}. \end{aligned} \tag{39}$$

The variance of the sequence $x(k)$ is given by the value $\rho(0) = \sigma^2$ which is the variance of the white noise sequence. Expression (39) for $\rho(m)$ is substituted into expressions (27), (29), (31). Using [4, eqs. (1.353/1) and 1.353/3] (appendix), the following results are found:

$$\begin{aligned} \sigma_{ij}^2(RR) &= \sigma^2 \left[\delta_{ij} \frac{N}{2} \frac{1 - p^2}{1 - 2p \cos fi + p^2} (1 + \delta_{io}), \qquad (i, j = 0, \cdots, N/2 - 1) \right. \\ &\quad \left. + p \frac{2p(1 + \cos fi \cos fj) - (1 + p^2)(\cos fi + \cos fj)}{(1 - 2p \cos fj + p^2)(1 - 2p \cos fi + p^2)} \right] \\ \sigma_{ij}^2(II) &= \sigma^2 \left[\delta_{ij} \frac{N}{2} \frac{1 - p^2}{1 - 2p \cos fi + p^2}, \qquad (i, j = 1, \cdots, N/2 - 1) \right. \\ &\quad \left. + \frac{2p^2 \sin fi \sin fj}{(1 - 2p \cos fi + p^2)(1 - 2p \cos fj + p^2)} \right] \\ \sigma_{ij}^2(RI) &= \sigma^2 \frac{p \sin fj}{1 - 2p \cos fj + p^2}, \qquad \begin{array}{l} (i = 0, \cdots, N/2 - 1) \\ (j = 1, \cdots, N/2 - 1). \end{array} \end{aligned} \tag{40}$$

with $p = e^{-2\pi K}$ and making the supposition that $p^N \lll 1$. These expressions were verified with simulations (using a discrete noise sequence) and experiments (using a continous noise signal). The results of these tests confirmed completely the theoretical derived results.

VII. Interpretation

For a lot of applications, it is preferable to deal with independent measurements. In the previous sections, it

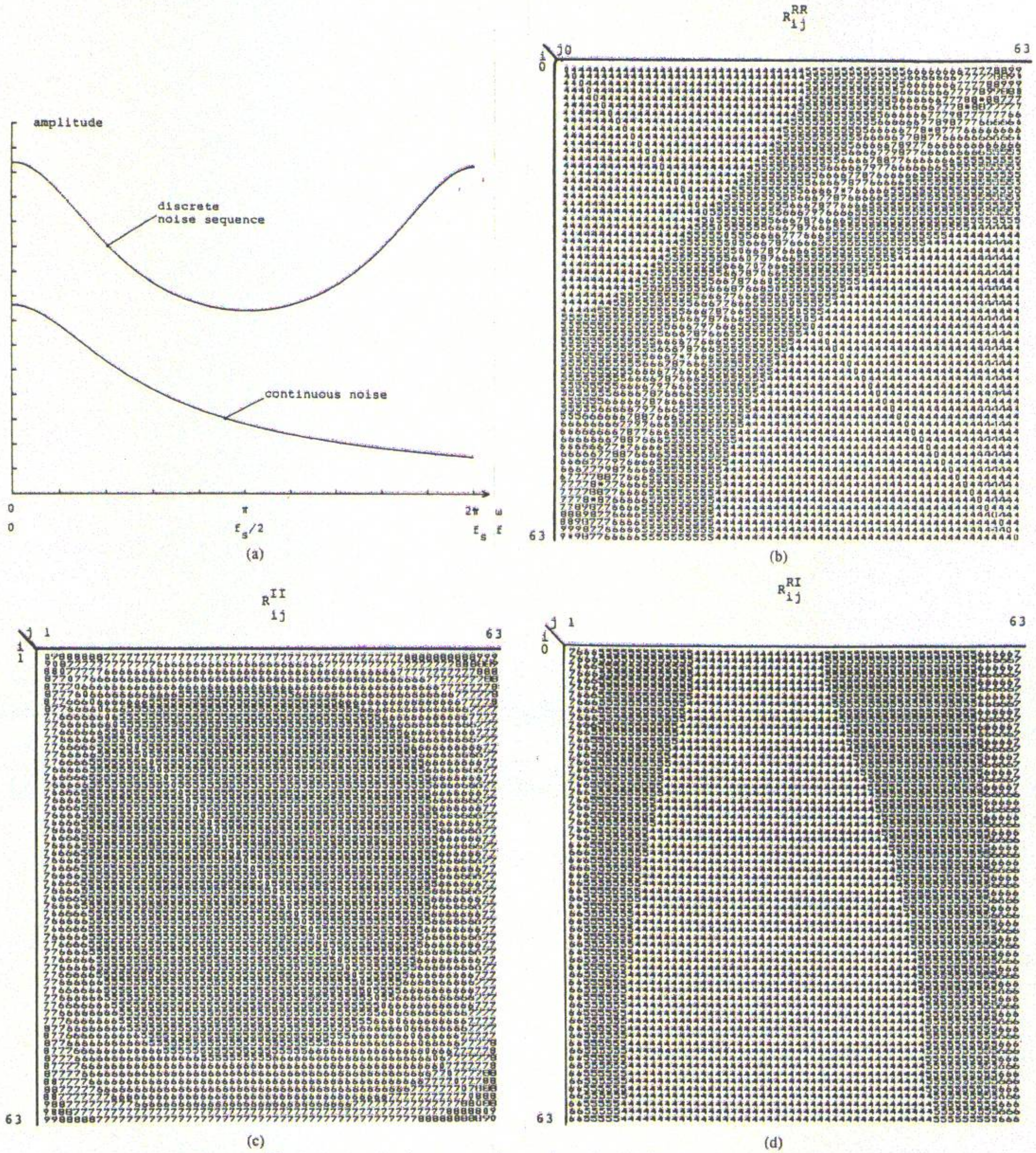

Fig. 6. (a) Amplitude spectrum of an analog colored noise signal and its corresponding discrete noise sequence ($K = 0.2$). (b) Correlation matrix R^{RR} ($K = 0.2$, $N = 128$). (c) Correlation matrix R^{II}. (d) Correlation matrix R^{RI}, with 0 : 0 dB $> |\rho| >$ −10 dB; 1 : −10 dB $> |\rho| >$ −20 dB; 9 : −90 dB $> |\rho| >$ −100 dB; x : −100 dB $> |\rho|$.

was shown that the results of a DFT have a Gaussian distribution. If the covariance matrix of this distribution is diagonal, then it follows that the results are statistically independent. To study the nondiagonal terms in the covariance matrix, the correlation matrix R is defined from the covariance matrix C:

$$R_{ij} = \frac{C_{ij}}{\sqrt{C_{ii} C_{jj}}} \qquad -1 \le R_{ij} \le 1. \tag{41}$$

In Fig. 6(b) a grafical interpretation of R_{ij} for $K = 0.2$ and

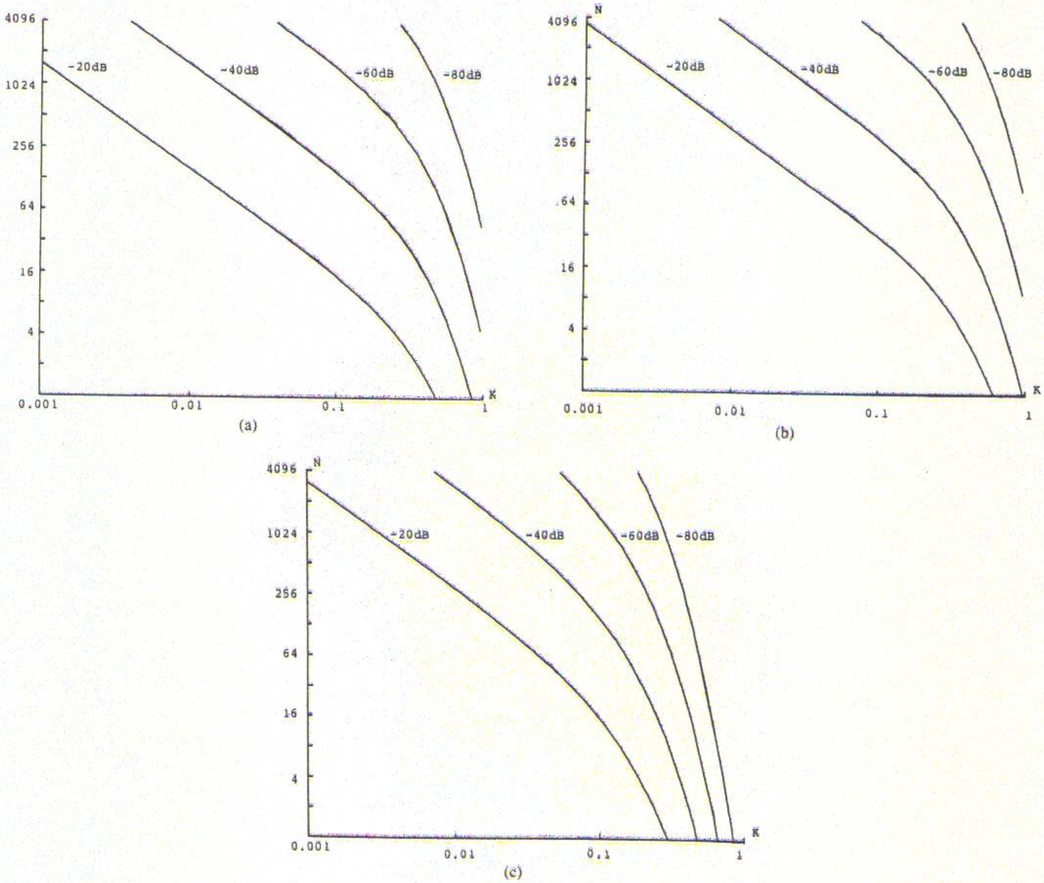

Fig. 7. (a) Relation between $|\rho^{RR}_{\max}|$ and K, N. (b) Relation between $|\rho^{II}_{\max}|$ and K, N. (c) Relation between $|\rho^{RI}_{\max}|$ and K, N.

$N = 128$ is given. The amplitude spectra of the analog and the corresponding discrete noise sequence is given in Fig. 6(a) (notice the influence of the aliasing effect). From Fig. 6(b)-(d) it is seen that the correlation matrix is nearly diagonal, even for highly correlated noise. Much more useful expressions are at the upper limits of (42) for the correlation coefficients:

$$|\rho_{ij}^{RR}| < \left|\frac{4p}{N(1-p^2)-4p}\right|$$

$$|\rho_{ij}^{II}| < \frac{4p^2}{N(1-p^2)} \quad \text{with } p = e^{-2\pi K}$$

$$|\rho_{ij}^{RI}| < \frac{2p}{N(1-p)}. \tag{42}$$

From these expressions it is seen that for sufficiently high values of N, the nondiagonal elements are negligible. In the previous example, the upper limits are

$$|\rho_{ij}^{RR}| < 0.0095 \qquad |\rho_{ij}^{II}| < 0.0028 \qquad |\rho_{ij}^{RI}| < 0.0062$$
$$< -40 \text{ dB} \qquad < -51 \text{ dB} \qquad < -44 \text{ dB}.$$

In Fig. 7(a)-(c) the relations (42) are plotted for $|\rho_{\max}| = C$ with $C = 0.1$, 0.01, 0.001, and 0.0001 as a function of N and K.

VIII. Conclusion

It has been proven that the distribution of the DFT components of additive noise is Gaussian regardless of the distribution of the noise. It is also shown that the covariance matrix of the DFT components is approximately a diagonal matrix. Formulas are given to estimate the maximal values of the nondiagonal elements. The mean of the distribution is zero except for the dc component where there is a known bias. The results of this study are very useful for the design of experiments (e.g., the number of points N necessary to reach a given signal-to-noise ratio can be derived). They are also important for parameter estimation problems where the knowledge of the pdf of the noise on the measurements is very important [6].

Appendix

In this Appendix, the formulas from [4] used in this article, are given as

1.353.1

$$\sum_{k=1}^{n-1} p^k \sin kx = \frac{p \sin x - p^n \sin nx + p^{n+1} \sin (n-1)x}{1 - 2p \cos x + p^2}$$

1.353.3

$$\sum_{k=0}^{n-1} p^k \cos kx = \frac{1 - p \cos x - p^n \cos nx + p^{n+1} \cos (n-1)x}{1 - 2p \cos x + p^2}.$$

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