Boom and Bust Mathematics (from http://www.unc.edu/depts/cmse/math/Verhulst.html)

It's the summer of 1997: rabies is devastating rabies populations in the Triangle. Although this troubling disease certainly seems like something out of the ordinary, the mathematics predicts that wild swings in populations are possible even in the most normal of times. The mathematics involved is amazingly simple.

The founder of population mathematics was **Pierre Verhulst** (1804-1849), a very talented Belgian mathematician dogged by poor health throughout his short life. Prior to his work, the famous British economist Robert Malthus had predicted that animal and human populations were fated to grow exponentially. But Verhulst understood that real populations are capped: in every habitat and for every species there is a **carrying capacity**. A population exceeding this capacity must go down rather than up.

Verhulst reasoned something like this. Let's write p(n) for the population in year n and let M be the carrying capacity.

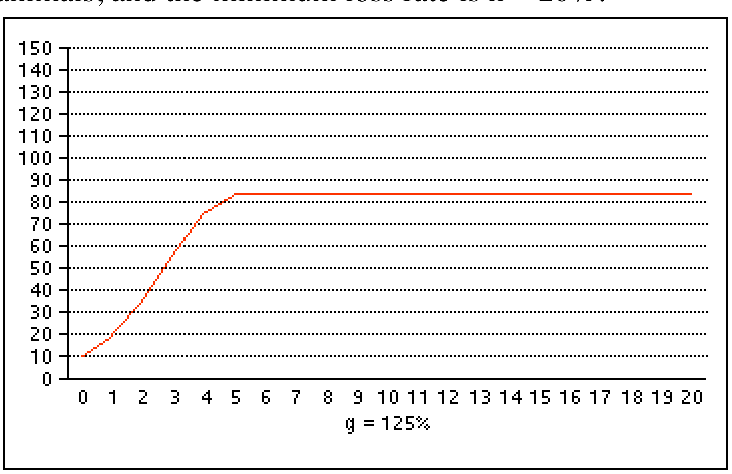
1. The population grows at a rate g, a certain percentage per year, as animals are born or wander into the area. (For big, long-lived species like humans g will be small, like 5%, but for most small, short-lived animals it will be much larger, usually more than 100%.) The population grows every year by gp(n) animals.
2. Some animals will die or wander away every year, even if there is plenty of room left in the habitat. Let h be this minimum loss rate. This is also a percentage; it must be less than 100% unless all the animals disappear. The population loses hp(n) animals every year.
3. Verhulst added to this model a new negative term which has the effect of removing excess population when the population grows beyond the carrying capacity M. The Verhulst term is the product of gp(n) -- the number of new animals -- and p(n)/M -- the population expressed as a fraction of the carrying capacity. The product, $gp(n)^2/M$, is our estimate of how many animals will be lost every year due to overcrowding of the habitat.

This is our completed Verhulst formula predicting next year's population from this year's:

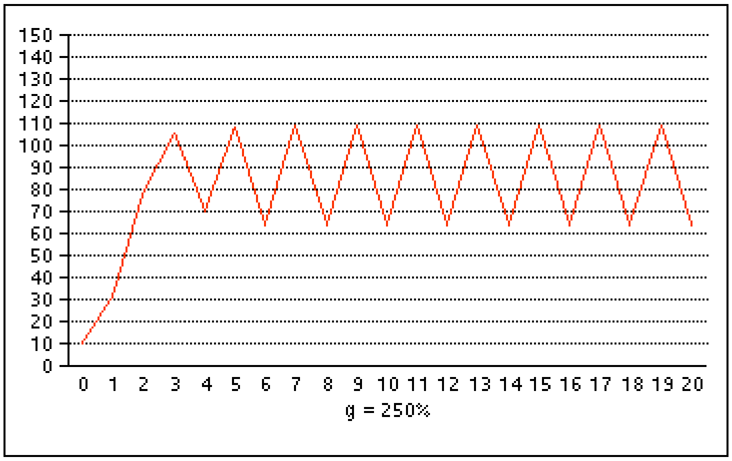
$$p(n+1) = (1+g-h)p(n) - gp(n)^2/M.$$

Mathematicians call a formula of this kind **recursive**. Recursion comes from Latin roots meaning "run again." When we have a recursive formula, we start with an initial value p(0) and "run" the formula again and again, as many times as we wish, to compute p(1), p(2), p(3), etc.

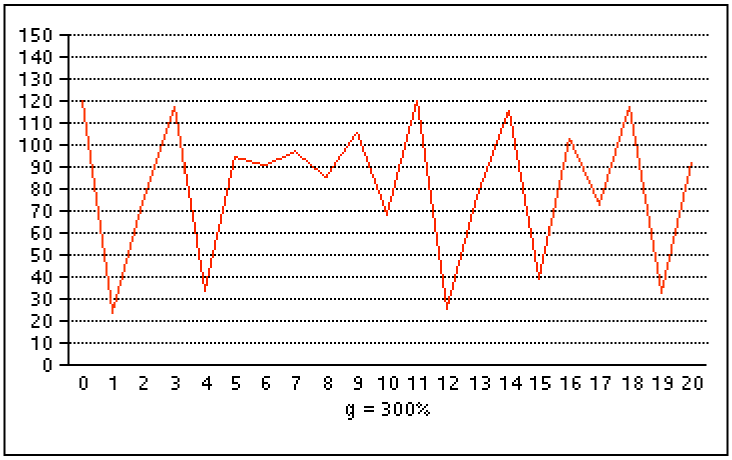
It used to be that recursive formulas involved huge amounts of calculation, but today we can handle them quite easily using spreadsheets or graphing calculators. Here are graphs of a population from year 0 to year 20 drawn by a Claris Works spreadsheet. In each case, we begin with p(0) = 10 animals, the carrying capacity is M = 100 animals, and the minimum loss rate is h = 20%.



If g = 125% the population rises quickly, then more slowly, and levels off at a stable value somewhat below the carrying capacity. This sort of growth typically occurs when species are released into a new habitat.



If g = 250%, we see the same kind of initial growth, but instead of reaching a stable value the population swings wildly and regularly back and forth between a level just above the carrying capacity and one that is comfortably below. This population behavior is common among animals with relatively high population growth rates.



If g = 300%, we see huge, uncontrolled swings in the population, and these variations are not at all regular. Both mathematicians and biologists were amazed that such a simple model can exhibit such complex behavior, behavior we now call **chaotic**.

The lesson here is that population variation is perfectly natural as species attempt to adjust to the constraints of their habitats. When we see sudden declines in a population, we seek an explanation such as disease or pollution. But populations can boom when there are too few animals in the habitat and crash when there are too many, without any special intervention or disaster.

The Verhulst model is very widely used. It's a good topic for students in middle and high schools, an example of using technology to explore "real life" mathematics experimentally.

Internet Sources

Pierre François Verhulst

Biographical information on Verhulst. This page is part of the MacTutor History of Mathematics Archive, a site maintained by the University of St. Andrews in Scotland (yes, the location of the famous golf course).

FEEDBACK: We'd be happy to have your comments and suggestions.

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http://www.unc.edu/depts/cmse/math/Verhulst.html

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