3 - Soluções - Determinantes.

3.1 $|A| = 3$ $|B| = -2$ $|C| = 1$ $|D| = 12$ $|E| = 14$ $|F| = 13$ $|G| = 9$ $|H| = 0$ $|I| = -26$
$|J| = 3$ $|K| = -99$ $|L| = -20$ $|M| = -6$

3.2 A é invertível $\Leftrightarrow r(A) = 4 \Leftrightarrow |A| \neq 0 \Leftrightarrow x(x-1)^2(x-2) \neq 0 \Leftrightarrow x \neq 0 \wedge x \neq 1 \wedge x \neq 2.$

3.3 (a) $\det A = 12k - 24$.
$r(A) = 5 \Leftrightarrow \det(A) \neq 0 \Leftrightarrow 12k - 24 \neq 0 \Leftrightarrow k \neq 2$

(b) A matriz B obtém-se da matriz A através das operações elementares:
- Troca da linha 2 com a linha 5;
- Multiplicação da coluna 2 por 2;

logo $\det(B) = -2\det(A)$.

(c) Por exemplo:
- Somar a linha 1 à linha 2;
- Multiplicar a coluna 5 por 3.

3.4 (a) $\det(A^{-1}) = \dfrac{1}{\det(A)} = \dfrac{1}{2}$

(b) $\det(AB) = \det(A)\det(B) = -\dfrac{10}{3}$

(c) $\det(ABC) = \det(A)\det(B)\det(C) = 0$

(d) $\det(A^T B^{-1}) = \det(A^T)\det(B^{-1}) = \det(A)\dfrac{1}{\det(B)} = -\dfrac{6}{5}$

(e) $\det(5A) = 5^3\det(A) = 250$

3.5 $\det(2A^{-1}) = 4 \Leftrightarrow 2^4 \dfrac{1}{\det(A)} = 2^2 \Leftrightarrow \det(A) = \dfrac{2^4}{2^2} \Leftrightarrow \det(A) = 4$

$\det(A^3(B^T)^{-1}) = 4 \Leftrightarrow (\det(A))^3 \dfrac{1}{\det(B)} = 4 \Leftrightarrow \det(B) = \dfrac{4^3}{4} \Leftrightarrow \det(B) = 16$

3.6 (a) $\det(A) = (-2)(2 + a)$

(b) $\det(C^T(BC)^{-1}) = \dfrac{1}{32} \Leftrightarrow \det(C^T)\dfrac{1}{\det(BC)} = \dfrac{1}{32} \Leftrightarrow \det(C)\dfrac{1}{\det(B)\det(C)} = \dfrac{1}{32} \Leftrightarrow \det(B) = 32 \Leftrightarrow 2^4 \det(A) = 32 \Leftrightarrow \det(A) = 2 \Leftrightarrow (-2)(2 + a) = 2 \Leftrightarrow a = -3$

3.7 $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$$\det(B) = \begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix} \underset{L_1 \leftrightarrow L_3}{=} - \begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} \underset{L_2 \leftrightarrow L_3}{=} + \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \det(A)$$

$$\det(C) = \begin{vmatrix} -a & -b & -c \\ 2d & 2e & 2f \\ g & h & i \end{vmatrix} \underset{\substack{(P1) \\ \text{slides}}}{=} (-1) \begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{vmatrix} \underset{\substack{(P1) \\ \text{slides}}}{=} -2 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -2\det(A)$$

$$\det(D) = \begin{vmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{vmatrix} \underset{L_1'=L_1-L_2}{=} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \det(A)$$

$$\det(E) = \begin{vmatrix} a & b & c \\ 7d-3a & 7e-3b & 7f-3c \\ g & h & i \end{vmatrix} \underset{L_2'=L_2+3L_1}{=} \begin{vmatrix} a & b & c \\ 7d & 7e & 7f \\ g & h & i \end{vmatrix} \underset{\substack{(P1) \\ \text{slides}}}{=} 7 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7\det(A)$$

$$\begin{aligned} \det(F) &= \begin{vmatrix} d+a & 5g-4d & -2a \\ e+b & 5h-4e & -2b \\ f+c & 5i-4f & -2c \end{vmatrix} \underset{\substack{(P1) \\ \text{slides}}}{=} -2 \begin{vmatrix} d+a & 5g-4d & a \\ e+b & 5h-4e & b \\ f+c & 5i-4f & c \end{vmatrix} \underset{C_1'=C_1-C_3}{=} -2 \begin{vmatrix} d & 5g-4d & a \\ e & 5h-4e & b \\ f & 5i-4f & c \end{vmatrix} \underset{C_2'=C_2+4C_1}{=} \\ &= -2 \begin{vmatrix} d & 5g & a \\ e & 5h & b \\ f & 5i & c \end{vmatrix} \underset{\substack{(P1) \\ \text{slides}}}{=} -2\cdot 5 \begin{vmatrix} d & g & a \\ e & h & b \\ f & i & c \end{vmatrix} \underset{\substack{(P8) \\ \text{slides}}}{=} -10 \begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix} = -10\det(B) = -10\det(A) \end{aligned}$$

3.8 $\det(A) = -6$

$\det(3A^{-1}BX) = \det(B^TA) \Leftrightarrow 3^4 \dfrac{1}{\det(A)} \det(B)\det(X) = \det(B^T)\det(A) \Leftrightarrow \det(X) = \dfrac{1}{3^4}(\det(A))^2 \Leftrightarrow$ $\det(X) = \dfrac{4}{9}$

3.9
$$\begin{aligned} \det(A) &= x \begin{vmatrix} y & 1 & x & y \\ x & o & y & x \\ a & 0 & 0 & b \\ z & 1 & z & x \end{vmatrix} \underset{L_3}{=} xa \begin{vmatrix} 1 & x & y \\ 0 & y & x \\ 1 & x & z \end{vmatrix} - xb \begin{vmatrix} y & 1 & x \\ x & 0 & y \\ z & 1 & x \end{vmatrix} \underset{|M|=|M^T|}{=} xa \begin{vmatrix} 1 & 0 & 1 \\ x & y & x \\ y & x & z \end{vmatrix} - xb \begin{vmatrix} y & x & z \\ 1 & 0 & 1 \\ x & y & x \end{vmatrix} = \\ &= -xa \begin{vmatrix} x & y & x \\ 1 & 0 & 1 \\ y & x & z \end{vmatrix} + xb \begin{vmatrix} x & y & x \\ 1 & 0 & 1 \\ y & x & z \end{vmatrix} = xb\det(B) - xa\det(B) = x(b-a)\det(B) \end{aligned}$$

3.10 $\det(A) \underset{C_3}{=} -x \underbrace{\begin{vmatrix} b & -1 & a \\ a^2 & 0 & ab \\ b & -a & -1 \end{vmatrix}}_{D_1} + y \underbrace{\begin{vmatrix} b & -1 & a \\ -2b & 1+a & 1-a \\ 1 & -a & -1 \end{vmatrix}}_{D_2}$

$$D_1 = a \begin{vmatrix} b & -1 & a \\ a & 0 & b \\ b & -a & -1 \end{vmatrix} = -a \begin{vmatrix} b & 1 & a \\ a & 0 & b \\ b & a & -1 \end{vmatrix} \underset{|M|=|M^T|}{=} -a \begin{vmatrix} b & a & b \\ 1 & 0 & a \\ a & b & -1 \end{vmatrix} \underset{\substack{L_1 \leftrightarrow L_2 \\ L_2 \leftrightarrow L_3}}{=} -a \begin{vmatrix} 1 & 0 & a \\ a & b & -1 \\ b & a & b \end{vmatrix} = -a\det(B)$$

$$D_2 \underset{L_2'=L_2+L_3}{=} \begin{vmatrix} b & -1 & a \\ -b & a & -a \\ 1 & -a & -1 \end{vmatrix} \underset{L_2=-L_1}{=} 0$$

$$\det(A) = -xD_1 + yD_2 = xa\det(B)$$

3.11 (a) $\det(A) = ad - bc \neq 0$ logo A é invertível.

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{ad-bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(b) $\det(B) = 21 \neq 0$ logo B é invertível.

$$B^{-1} = \frac{1}{\det(B)}\operatorname{adj}(B) = \frac{1}{21}\begin{bmatrix} 1 & -2 & 4 \\ 9 & 3 & -6 \\ -4 & 8 & 5 \end{bmatrix}^T = \frac{1}{21}\begin{bmatrix} 1 & 9 & -4 \\ -2 & 3 & 8 \\ 4 & -6 & 5 \end{bmatrix}$$

(c) $\det(C) = 8 \neq 0$ logo C é invertível.

$$C^{-1} = \frac{1}{\det(C)}\operatorname{adj}(C) = \frac{1}{8}\begin{bmatrix} -3 & 2 & -2 \\ -2 & 4 & 4 \\ 7 & -2 & 2 \end{bmatrix}^T = \frac{1}{8}\begin{bmatrix} -3 & -2 & 7 \\ 2 & 4 & -2 \\ -2 & 4 & 2 \end{bmatrix}$$

(d) $\det(D) = 1 \times 1 \times 1 \times 1 = 1 \neq 0$ logo D é invertível.

$$D^{-1} = \frac{1}{\det(D)}\operatorname{adj}(D) = \frac{1}{1}\begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ a^2 & a & 1 & 0 \\ a^3 & a^2 & a & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & a & a^2 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3.12 (a) $\det(A) = 1 \neq 0$ logo A é invertível

$$C_3(A^{-1}) = \frac{1}{\det(A)}C_3(\operatorname{adj}(A)) = \frac{1}{1}\begin{bmatrix} 5 & -2 & -3 \end{bmatrix}^T = \begin{bmatrix} 5 \\ -2 \\ -3 \end{bmatrix}$$

(b) $\det(B) = 30 \neq 0$ logo B é invertível

$$(B^{-1})_{(3,4)} = \frac{1}{\det(B)}\operatorname{adj}(B)_{(3,4)} = \frac{1}{30} = \frac{1}{30}\begin{vmatrix} 2 & 0 & 1 \\ 4 & 1 & -1 \\ 2 & -1 & 4 \end{vmatrix} = 0$$

3.13 $\begin{vmatrix} a & 0 & b \\ a & a & 4 \\ 0 & a & 0 \end{vmatrix} = ba^2 - 2a^2 = a^2(b-2) \neq 0 \Leftrightarrow a \neq 0 \wedge b \neq 2$

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{a^2(b-2)}\begin{bmatrix} -2a & ab & -ab \\ -2a & 2a & -4a+ab \\ a^2 & -a^2 & a^2 \end{bmatrix} = \frac{1}{a(b-2)}\begin{bmatrix} -2 & b & -b \\ -2 & 2 & -4+b \\ a & -a & a \end{bmatrix}$$

3.14 (a) $\left(\dfrac{\begin{vmatrix} 19 & 8 \\ 11 & 7 \end{vmatrix}}{\begin{vmatrix} 7 & 8 \\ 6 & 7 \end{vmatrix}}, \dfrac{\begin{vmatrix} 7 & 19 \\ 6 & 11 \end{vmatrix}}{\begin{vmatrix} 7 & 8 \\ 6 & 7 \end{vmatrix}} \right) = (45, -37)$ (b) $\left(\dfrac{\begin{vmatrix} 2 & 0 & 1 \\ 3 & 1 & 0 \\ 4 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}}, \dfrac{\begin{vmatrix} 1 & 2 & 1 \\ 1 & 3 & 0 \\ 0 & 4 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}}, \dfrac{\begin{vmatrix} 1 & 0 & 2 \\ 1 & 1 & 3 \\ 0 & 1 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}} \right) = \left(\frac{1}{2}, \frac{5}{2}, \frac{3}{2} \right)$

3.15 (a) $\begin{vmatrix} -2 & 1 & 1 \\ 1 & 3 & -1 \\ 3 & -1 & 3 \end{vmatrix} = -32 \neq 0$ logo (S_1) é de Cramer. $x = \dfrac{\begin{vmatrix} 2 & 1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{vmatrix}}{-32} = -\dfrac{1}{2}$

(b) $\begin{vmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 2 & -1 & 0 \\ 1 & -1 & -2 & 0 \end{vmatrix} = 10 \neq 0$ logo (S_2) é de Cramer. $\qquad y = \dfrac{\begin{vmatrix} 1 & 1 & -1 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & -1 & 0 \\ 1 & 1 & -2 & 0 \end{vmatrix}}{10} = \dfrac{1}{10},$

(c) $\begin{vmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{vmatrix} = 1 \neq 0$ logo (S_3) é de Cramer. $\qquad z = \dfrac{\begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 1 & -1 & 1 \\ 0 & 1 & 3 & 1 \end{vmatrix}}{1} = 1,$

(d) $\begin{vmatrix} 1 & 1 & -1 & -1 \\ 2 & 1 & 0 & 1 \\ 1 & 0 & 1 & -2 \\ 0 & 2 & -1 & 3 \end{vmatrix} = -12 \neq 0$ logo (S_3) é de Cramer. $\qquad w = \dfrac{\begin{vmatrix} 1 & 1 & -1 & 2 \\ 2 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{vmatrix}}{-12} = -\dfrac{3}{4}$

3.16 (a) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ a & 1 & 0 \end{vmatrix} = a - a^2 \neq 0 \Leftrightarrow a \neq 0 \wedge a \neq 1 \qquad y = \dfrac{\begin{vmatrix} 1 & a+1 & 1 \\ 1 & 1 & 1 \\ a & a+a^2 & 0 \end{vmatrix}}{a - a^2} = \dfrac{a}{1-a}, a \neq 0 \wedge a \neq 1$

(b) $\begin{vmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & a & 1 \end{vmatrix} = 2 - 2a \neq 0 \Leftrightarrow a \neq 1$

$$\left(\frac{\begin{vmatrix} b & 1 & 0 \\ 0 & 2 & 1 \\ 2 & a & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & a & 1 \end{vmatrix}}, \frac{\begin{vmatrix} 2 & b & 0 \\ 3 & 0 & 1 \\ 1 & 2 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & a & 1 \end{vmatrix}}, \frac{\begin{vmatrix} 2 & 1 & b \\ 3 & 2 & 0 \\ 1 & a & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & a & 1 \end{vmatrix}} \right) = \left(\frac{2b+2-ab}{2-2a}, \frac{-2b-4}{2-2a}, \frac{2-2b+3ab}{2-2a} \right), a \neq 1$$