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2024/25

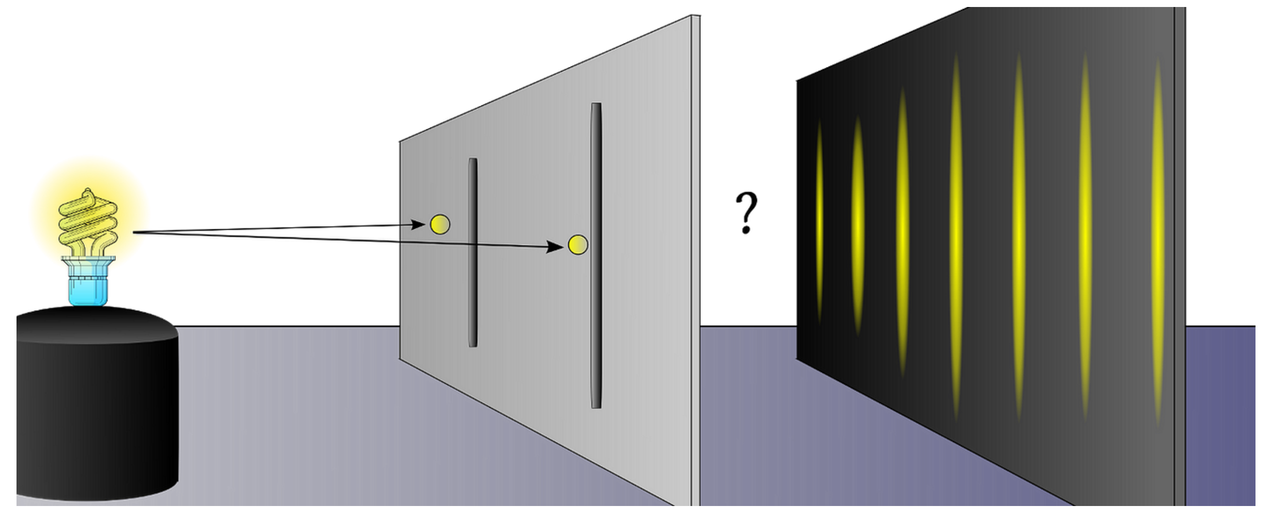
Binomial & Taylor Expansions

Lecturer: David Jennings
(Theoretical Physics Group)

Outline of Course

1. Binomial and Taylor Series.
2. Probability Theory Fundamentals.
3. Heisenberg Uncertainty Relation.
4. The Photoelectric Effect.
5. The Bohr Atom.
6. The Double Slit Experiment.
7. De Broglie Wavelength
8. Compton Scattering.
9. Particle in a Box.
10. Complementarity.

The binomial expansion



- Crucial tool in algebra, calculus, and especially probability.
- *"If I flip a biased coin N times what is the probability of exactly k heads?"*

Expanding powers

$$(x+y)^2 = x^2 + 2xy + y^2$$

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$$= (x+y)(x^2 + 2xy + y^2)$$

Expanding powers

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$\begin{aligned}(x+y)^3 &= (x+y)(x+y)^2 \\ &= (x+y)(x^2+2xy+y^2) \\ &= x^3 + 2x^2y + xy^2 + yx^2 + 2xy^2 + y^3 \\ &= x^3 + 3x^2y + 3xy^2 + y^3\end{aligned}$$

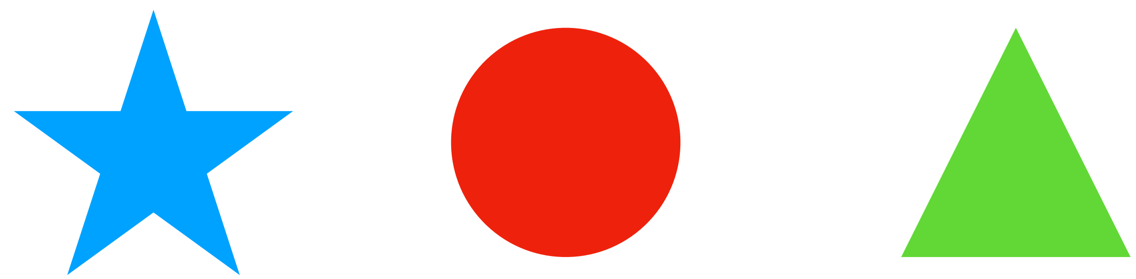
Expanding powers

$$(x+y)^4 = ???$$

$$(x+y)^5 = ???$$

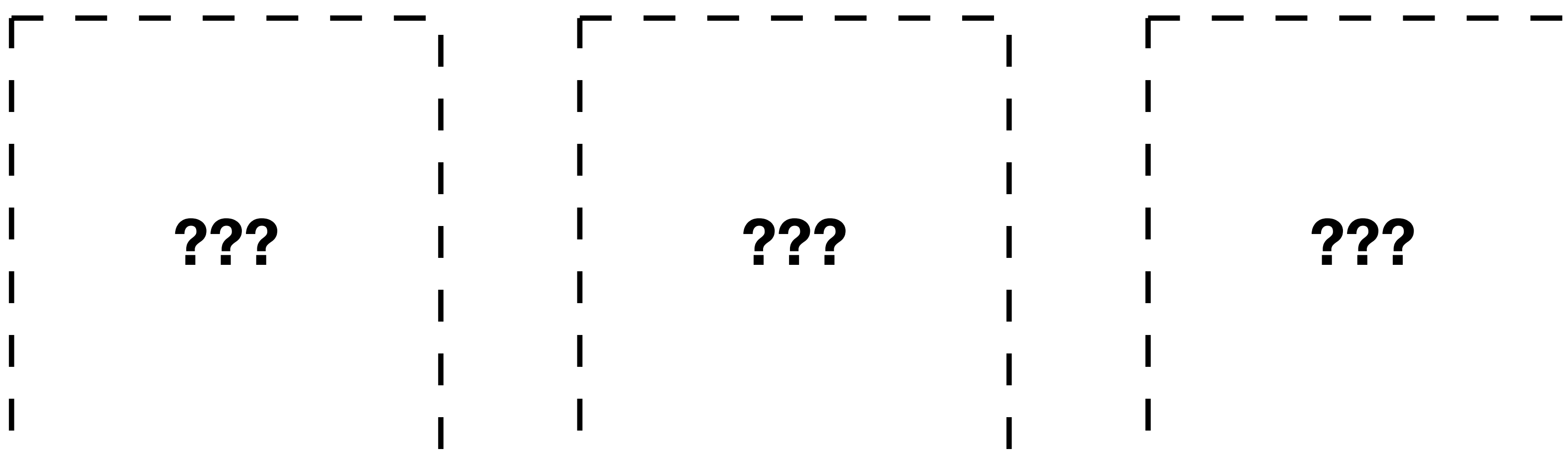
$$(x+y)^{20} = ???$$

Combinatorics

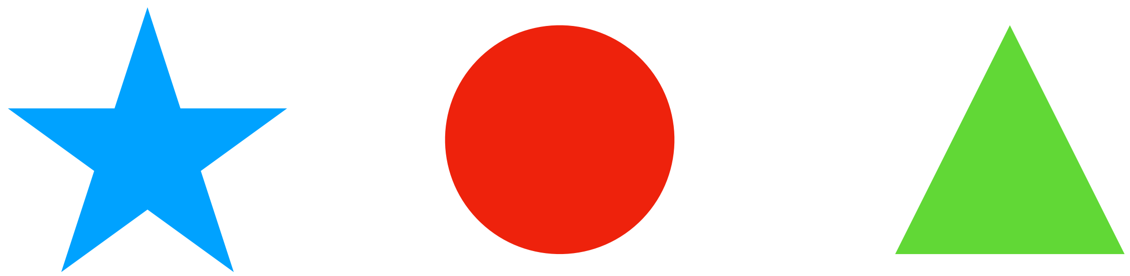


3 objects can be arranged in a line in $3 \times 2 \times 1$ ways

$= 6$ ways



Combinatorics



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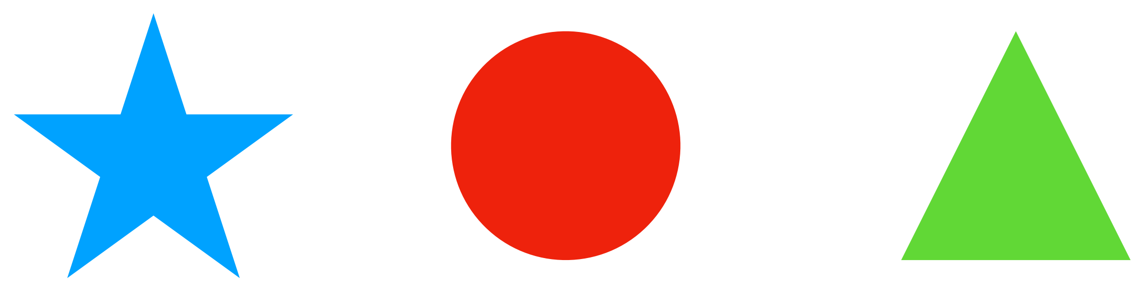
3 options | **???** | **???**

Combinatorics



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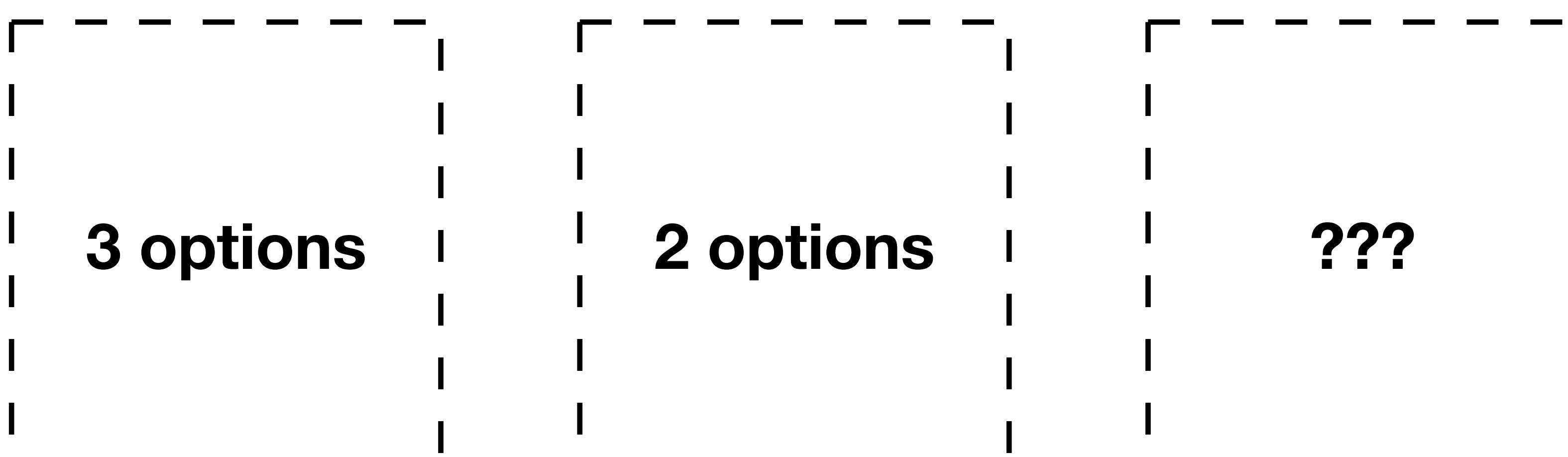
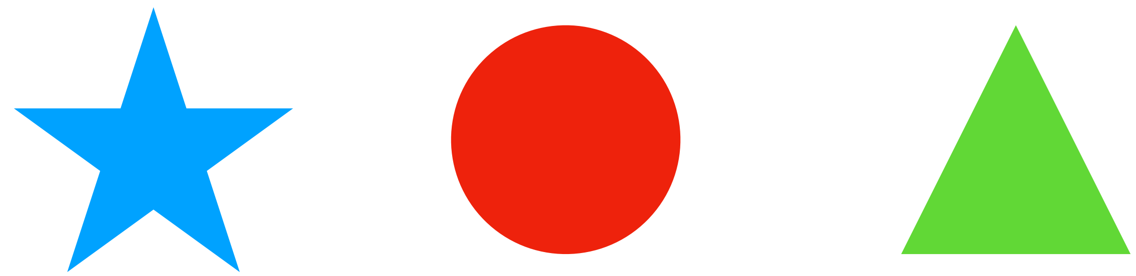


Combinatorics



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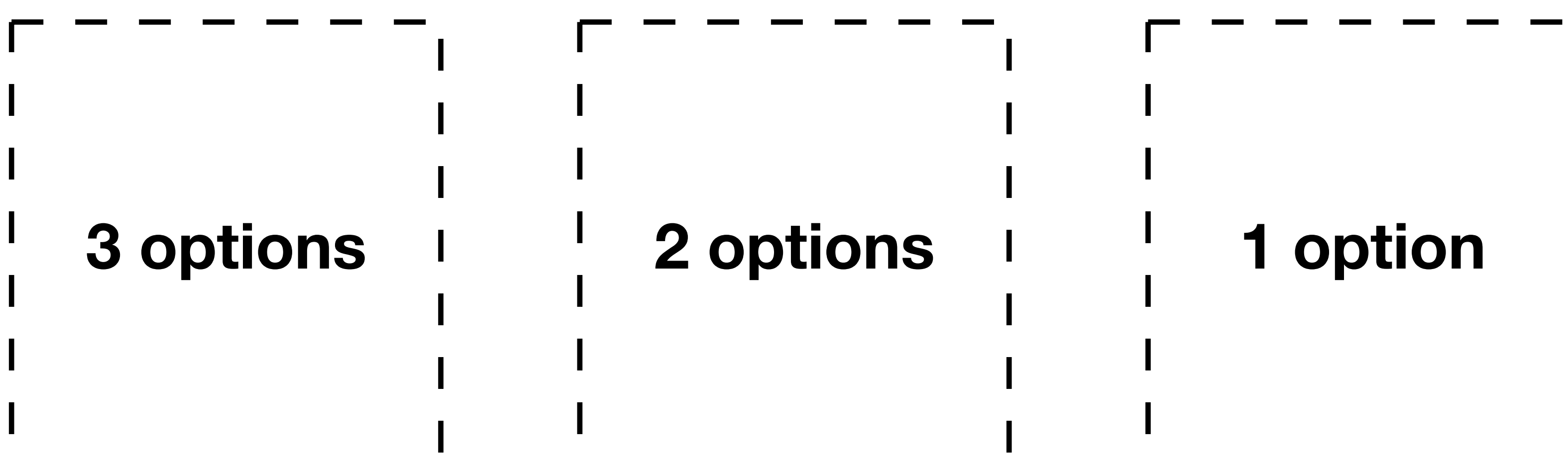

3 x 2 x 1 = 6

Combinatorics

n objects can be arranged in a line in:

$$n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1 \text{ ways}$$

“n factorial ways”

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

Combinatorics

$$1! = 1$$

$$2! = 2 \times 1 = 2$$

$$3! = 3 \times 2 \times 1 = 6$$

$$4! = 4 \times 3 \times 2 \times 1 = 24$$

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

$$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$$

n! grows very quickly

Combinatorics

If we have a bag with **n** objects and choose **k** objects from the bag, then how many possible combinations are there?

"n choose k" : $$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Combinatorics

Example: A room contains 10 students. How many distinct triples of people could be selected?

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$$= \frac{10!}{3! \times 7!}$$

$$= \frac{10!}{3! \times 7!} = \frac{3628800}{6 \times 5040} = 120$$

Combinatorics

Example: you have to travel to Edinburgh twice in a particular week, and have to stay for the entire day each time. How many options do you have?

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$$\text{Number of choices} = \binom{7}{2} = \frac{7!}{2!(7-2)!}$$

Combinatorics

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Note that $7! = 7 \times 6 \times 5!$

Combinatorics

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$$= \frac{7!}{2! \times 5!}$$

Note that $7! = 7 \times 6 \times 5!$

$$\text{Number of choices} = \binom{7}{2} = \frac{7 \times 6}{2!} = 7 \times 3 = 21$$

Therefore you have 21 different choices for the travel

Key properties

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n \times (n-1) \times (n-2) \cdots \times (n-k+1)}{k!}$$

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\binom{n}{1} = n$$

$0! = 1$ by definition $\Rightarrow$

$$\binom{n}{0} = \frac{n!}{0!n!} = 1$$

$$\binom{n}{n} = \frac{n!}{n!0!} = 1$$

The Binomial Expansion Formula

For any power n we have that:

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \dots$$

$$\dots + \binom{n}{n-2}x^2y^{n-2} + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

The Binomial Expansion Formula

Example: Expand $(x + y)^5$ in powers of x and y

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$$= \binom{5}{0}x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^2 + \binom{5}{2}x^2y^3 + \binom{5}{1}x^1y^4 + \binom{5}{0}y^5$$

$$= x^5 + 5x^4y + \frac{5\times 4}{2\times 1}x^3y^2 + \frac{5\times 4}{2\times 1}x^2y^3 + 5xy^4 + y^5$$

$$= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

The Binomial Expansion Formula

Example:

What is the coefficient before the term x^4y^{96} when we expand $(x+y)^{100}$?

The coefficient is given by $\binom{100}{96}$

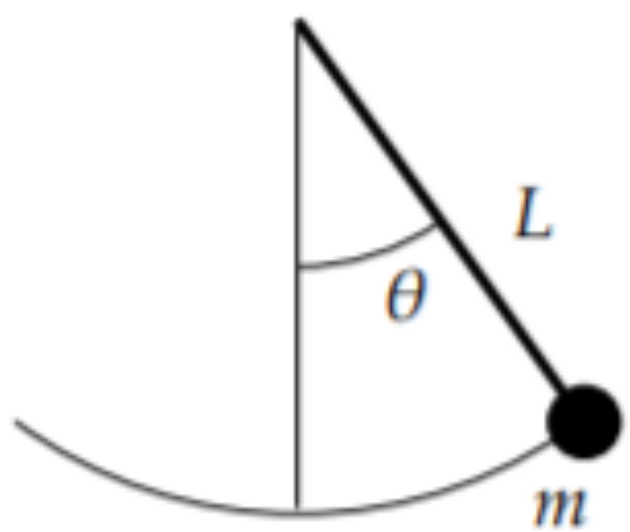
$$\binom{100}{96} = \binom{100}{4}$$

$$= \frac{100 \times 99 \times 98 \times 97}{4 \times 3 \times 2 \times 1}$$

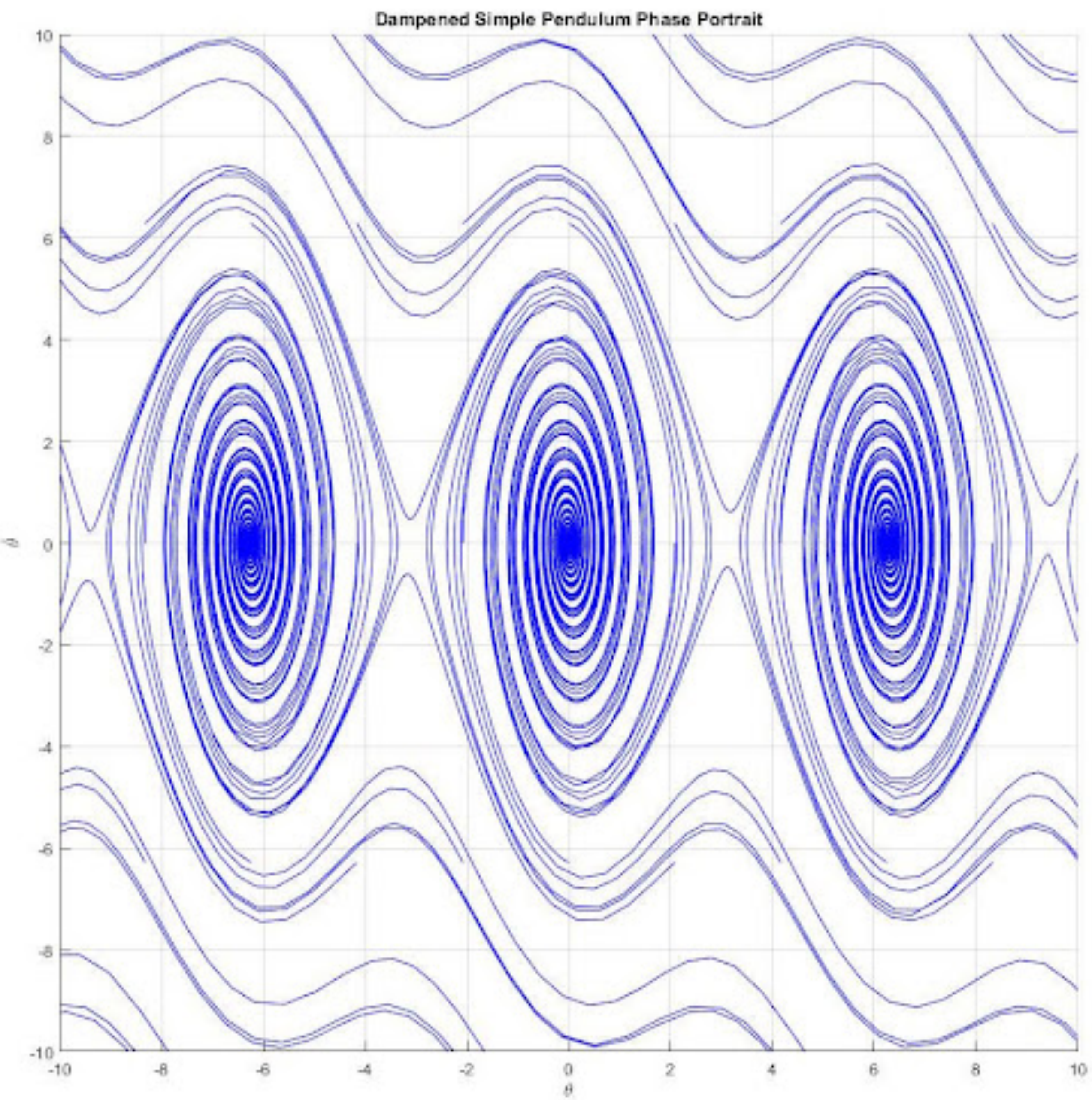
$$= 3,921,225$$

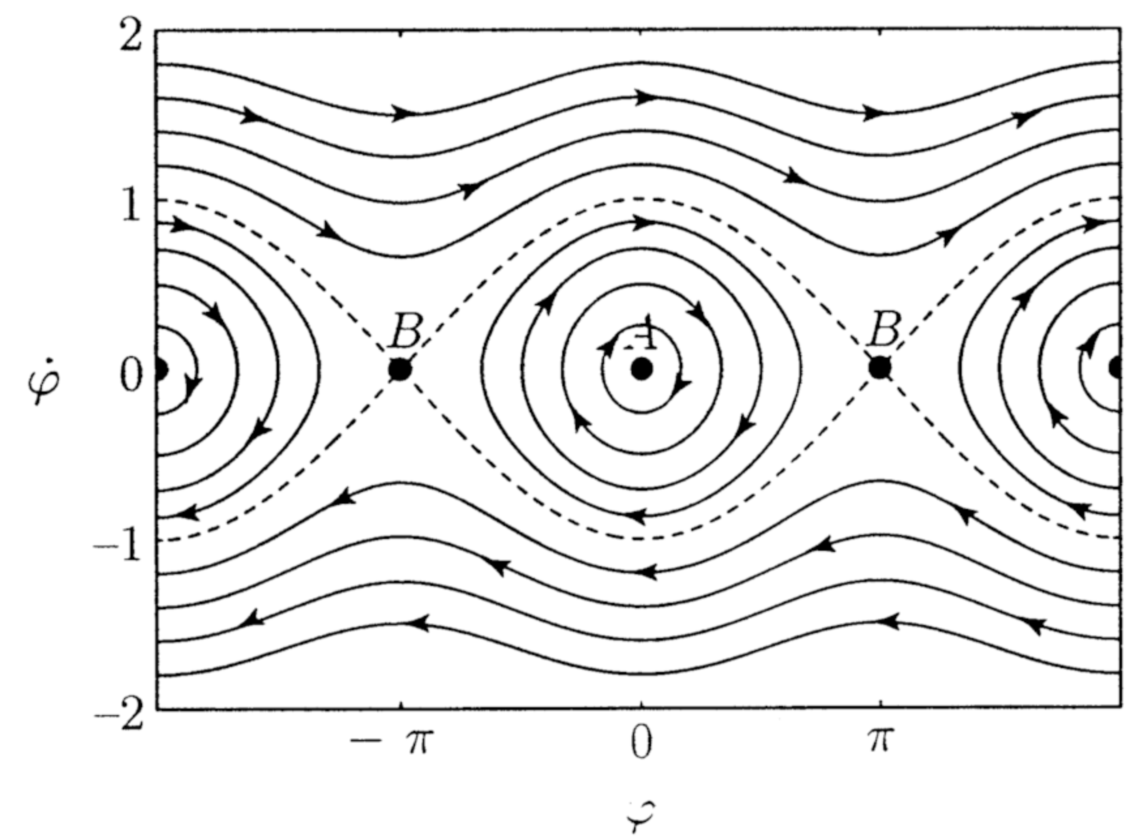
Taylor expansions

A pendulum oscillating



Taylor expansions



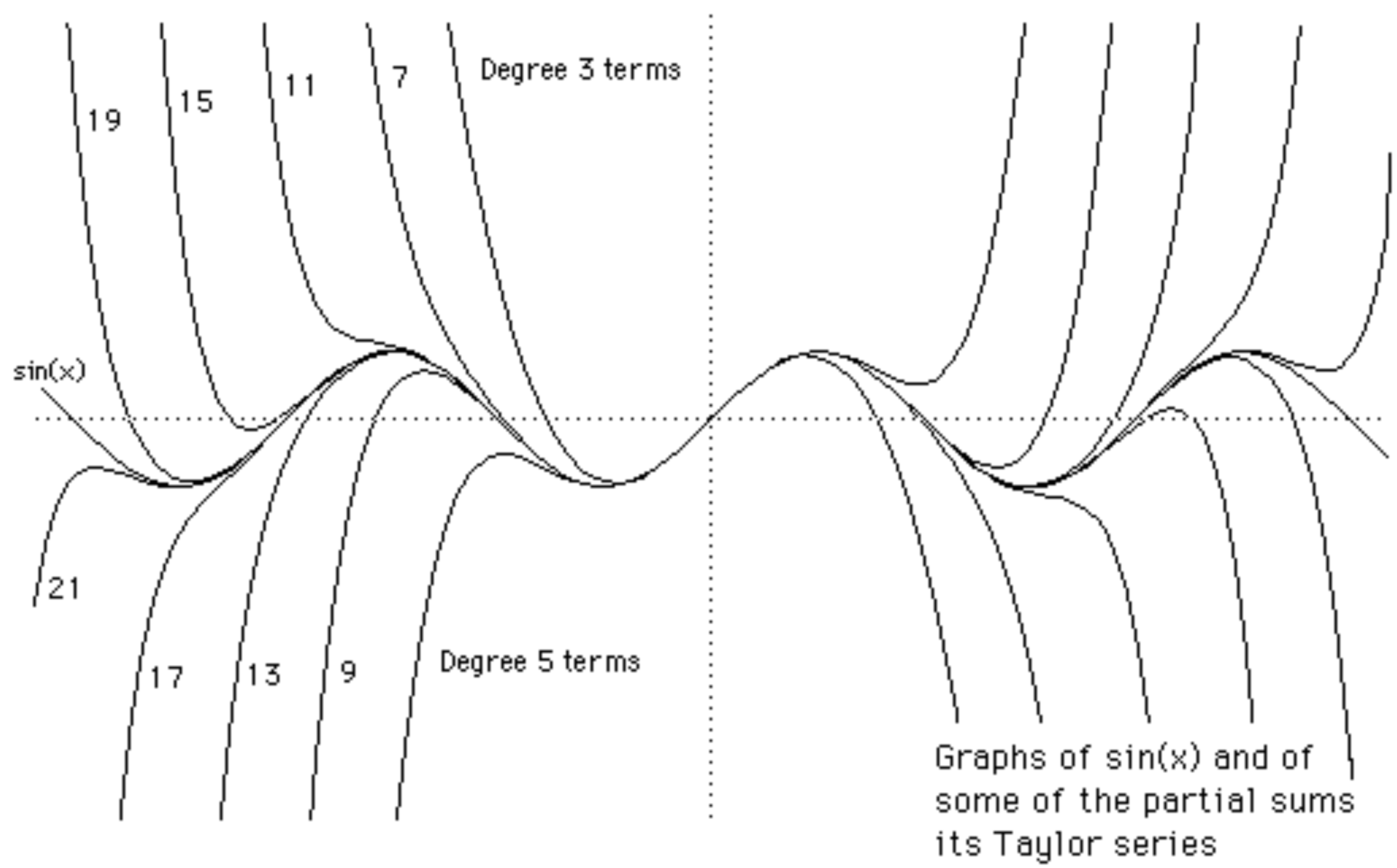


<u>Locally</u> things are simple!

Taylor expansions

We can approximate many many functions by polynomials!

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$



Taylor expansions

Given a function $f(x)$ we use the following notation:

$$f'(x) = \frac{df}{dx}$$

$$f''(x) = \frac{d^2 f}{dx^2}$$

$$f'''(x) = \frac{d^3 f}{dx^3}$$

Taylor expansions

Given a function $f(x)$ we use the following notation:

$$f^{(4)}(x) = \frac{d^4 f}{dx^4}$$

$$f^{(5)}(x) = \frac{d^5 f}{dx^5}$$

$$\vdots$$

$$f^{(235)}(x) = \frac{d^{235} f}{dx^{235}}$$

$$\vdots$$

Taylor expansions

The Taylor series of $f(x)$ expanded about the point $x = a$ is given by:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$+ \frac{f^{(4)}(a)}{4!}(x-a)^4 + \frac{f^{(5)}(a)}{5!}(x-a)^5 + \dots$$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Taylor expansions

In practice, we *truncate* the series to some low power.

The truncated Taylor series to order N is given by:

$$f(x) = \sum_{k=0}^{N} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$f(x) = \frac{f^{(0)}(a)}{0!}(x-a)^0 + \frac{f^{(1)}(a)}{1!}(x-a)^1 + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(N)}(a)}{N!}(x-a)^N$$

$$= f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(N)}(a)}{N!}(x-a)^N$$

Taylor expansions

Compute the Taylor expansion of $f(x) = \cos x$ around the point $x = 0$ up to order 5

$$f^{(0)}(x) = f(x) = \cos(x)$$

$$f^{(0)}(0) = 1$$

$$f^{(1)}(x) = -\sin(x)$$

$$f^{(1)}(0) = 0$$

$$f^{(2)}(x) = -\cos(x)$$

$$f^{(2)}(0) = -1$$

$$f^{(3)}(x) = \sin(x)$$

$$f^{(3)}(0) = 0$$

$$f^{(4)}(x) = \cos(x)$$

$$f^{(4)}(0) = 1$$

$$f^{(5)}(x) = -\sin(x)$$

$$f^{(5)}(0) = 0$$

Taylor expansions

$$f(x) = \frac{f^{(0)}(a)}{0!}(x-a)^0 + \frac{f^{(1)}(a)}{1!}(x-a)^1 + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(N)}(a)}{N!}(x-a)^N$$

$$= f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(N)}(a)}{N!}(x-a)^N$$

Taylor expansions

$$f(x) = \frac{f^{(0)}(a)}{0!}(x-a)^0 + \frac{f^{(1)}(a)}{1!}(x-a)^1 + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(N)}(a)}{N!}(x-a)^N$$

$$= f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(N)}(a)}{N!}(x-a)^N$$

$$= 1 + 0 \times x + \frac{(-1)}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \frac{0}{5!}x^5$$

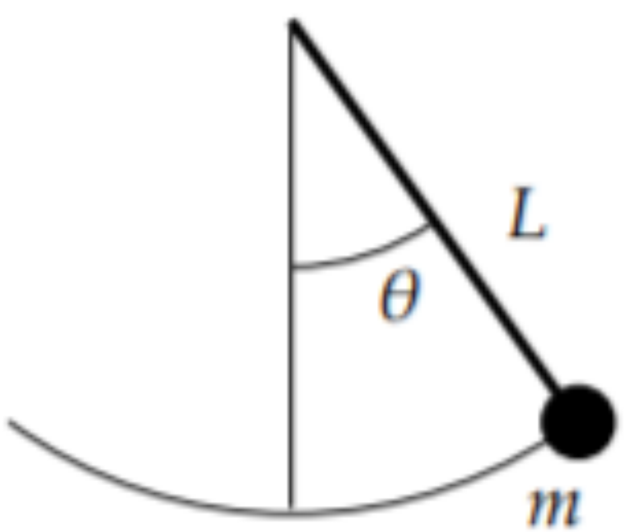
$$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4$$

This means, that near x=0 $\quad \cos x \approx 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4$

Taylor Series = Microscope

We use Taylor Series to “zoom in” on the <u>simple</u> behaviour of a function near a point.

A pendulum oscillating



$$\text{near } \theta = 0$$

$$\sin\theta = \theta + \text{ higher order terms}$$

Relativity

In Relativity you will learn that the total energy of a particle of mass m moving at a velocity v, is given by

$$E = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} mc^2$$

Here c is the speed of light.

Question: How does the energy of a relativistic particle behave for low velocities?

Relativity

$$E(v) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}mc^2$$

Expand this function around v=0.

$$E(v) = E(0) + E'(0)(v-0) + E''(0)(v-0)^2/2 + \ldots$$

Easy to see that: $E(0) = mc^2$

Relativity

$$E'(v) = \frac{d}{dv} E(v)$$

$$= \frac{d}{dv} \left[\left(1 - \frac{v^2}{c^2}\right)^{-1/2} mc^2 \right]$$

$$= mc^2 \frac{d}{dv} \left[\left(1 - \frac{v^2}{c^2}\right)^{-1/2} \right]$$

$$= mc^2 \left[-\frac{1}{2} \left(1 - \frac{v^2}{c^2}\right)^{-3/2} \left(-\frac{2v}{c^2}\right) \right]$$

Relativity

$$\Rightarrow E'(v) = mv\left(1 - \frac{v^2}{c^2}\right)^{-3/2}$$

$$\Rightarrow E'(0) = 0$$

This implies that to first order in velocity we have

$$E(v) = E(0) + E'(0)(v-0) + E''(0)(v-0)^2/2 + \ldots$$

$$\Rightarrow E(v) = mc^2 \text{ to first order in } v$$

What about to second order??

Relativity

$$\text{Need } E''(v) = \frac{d}{dv}E'(v)$$

$$= \frac{d}{dv}\left[mv\left(1-\frac{v^2}{c^2}\right)^{-3/2}\right]$$

$$= \left[m\left(1-\frac{v^2}{c^2}\right)^{-3/2} + mv\frac{d}{dv}\left(1-\frac{v^2}{c^2}\right)^{-3/2}\right]$$

$$= \left[m\left(1-\frac{v^2}{c^2}\right)^{-3/2} + 3\frac{mv^2}{c^2}\left(1-\frac{v^2}{c^2}\right)^{-5/2}\right]$$

Relativity

$$E''(v) = \left[m\left(1 - \frac{v^2}{c^2}\right)^{-3/2} + 3\frac{mv^2}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{-5/2} \right]$$

$$\Rightarrow E''(0) = m$$

$$E(v) = E(0) + E'(0)(v-0) + E''(0)(v-0)^2/2 + \ldots$$

Relativity

$$E''(v) = \left[m\left(1-\frac{v^2}{c^2}\right)^{-3/2} + 3\frac{mv^2}{c^2}\left(1-\frac{v^2}{c^2}\right)^{-5/2}\right]$$

$$\Rightarrow E''(0) = m$$

$$E(v) = \underset{\underset{mc^2}{\downarrow}}{E(0)} + \overset{\overset{\mathbf{0}}{\uparrow}}{E'(0)}(v-0) + \underset{\underset{m}{\downarrow}}{E''(0)}(v-0)^2/2 + \ldots$$

Relativity

$$E''(v) = \left[m\left(1 - \frac{v^2}{c^2}\right)^{-3/2} + 3\frac{mv^2}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{-5/2} \right]$$

$$\Rightarrow E''(0) = m$$

$$E(v) = \underbrace{E(0)}_{mc^2} + \overbrace{E'(0)}^{0}(v-0) + \underbrace{E''(0)}_{m}(v-0)^2/2 + \ldots$$

$$\Rightarrow E(v) = mc^2 + \frac{1}{2}mv^2 \text{ to 2nd order in v}$$