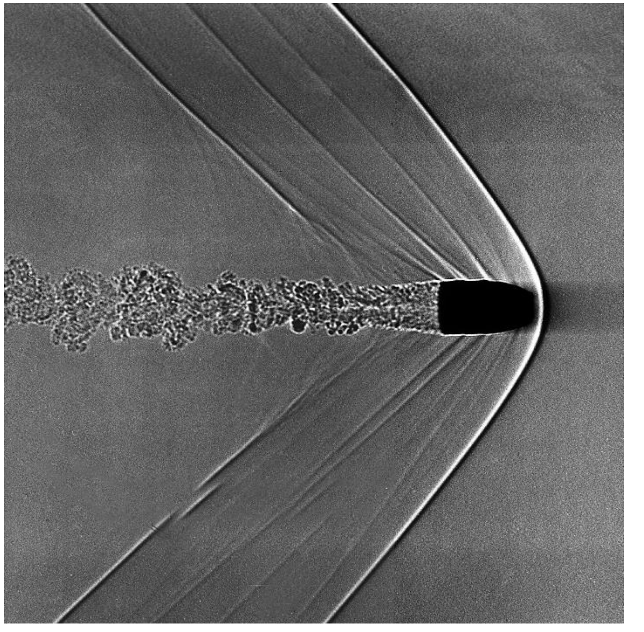
What causes a wave?



What does a wave do?

Waves: general definitions

Variation of some quantity that propagates (typically) and (typically) with some constant characteristic speed

The wave may be a one-off pulse or an oscillatory sequence of pulses (*wave train*).

The pulses cause a local disturbance in some physical quantity followed by a return to equilibrium position (*pulse*) or sustained oscillation (*wave train*). [Can also have a *front* which converts system from one equilibrium state to another – no recovery.]

Quantity may be a positional displacement or some other physical variable, e.g. pressure (sound), temperature (flame), concentration (chemical wave, nerve signal), electric/magnetic field (light) ....

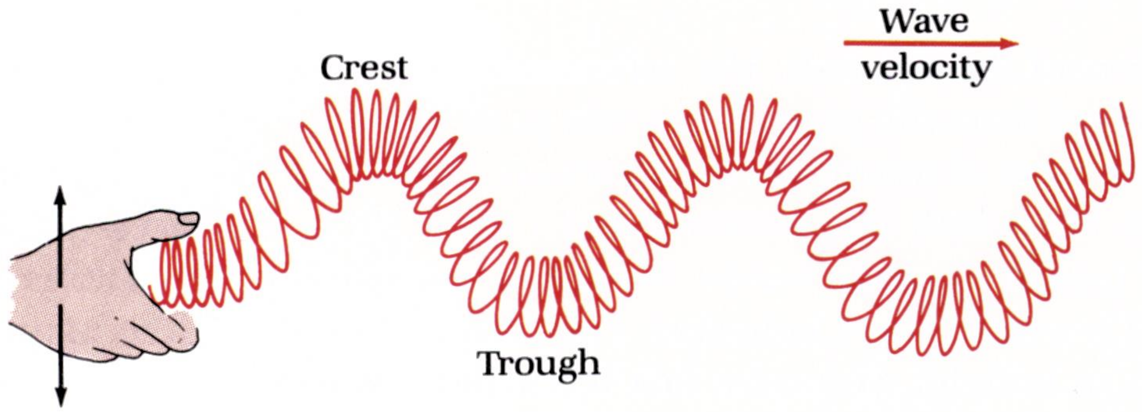
Transport of energy and momentum but not matter

Waves

transport energy and momentum without transporting matter.

Transverse waves :

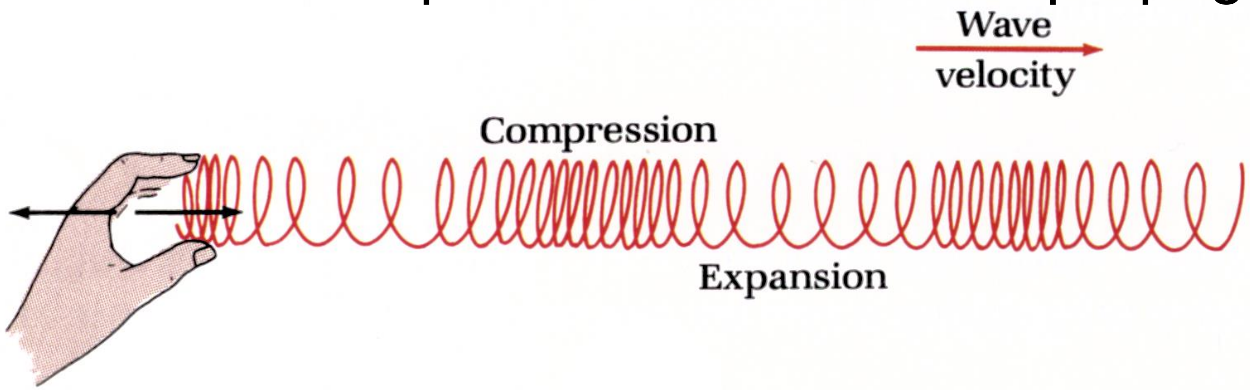
Disturbance is perpendicular to direction of propagation.



e.g.: waves on a string or slinky, electromagnetic waves.

Longitudinal waves :

Disturbance is parallel to direction of propagation.



e.g.: Sound, primary seismic waves.

Transverse waves on a string

On what does wave speed depend?

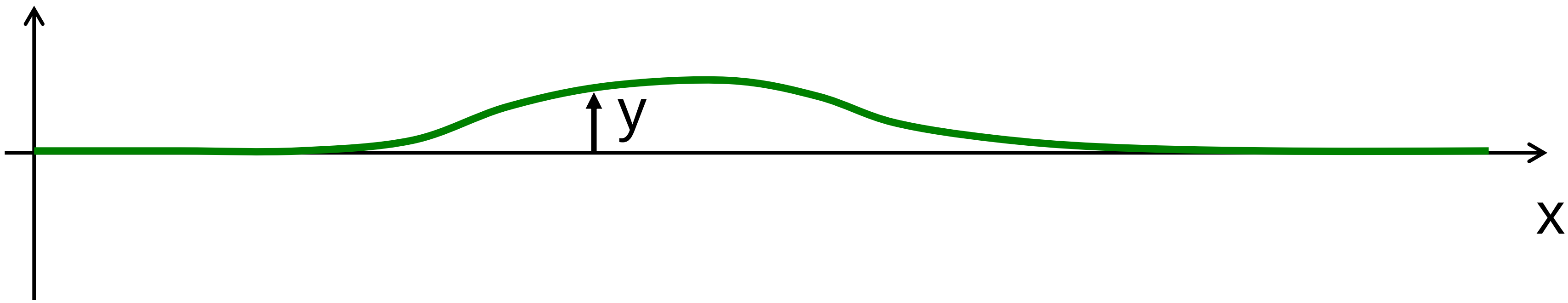
Experimental observations:

(i) the greater the tension in the string, the faster the wave propagates,

(ii) waves travel faster on a light string than a heavy one.

At equilibrium, let the string lie along the x-axis.

Now make small transverse displacements y.



The function y(x) defines the **shape** of the string at a given instant.

Shape will change with time, according to some eqn of motion.

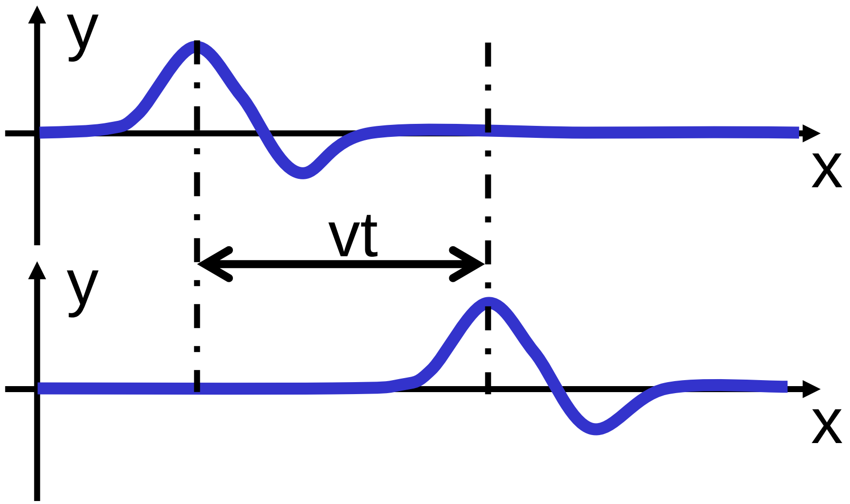
$\Rightarrow$ y is also a function of t.

y(x,t) is sometimes called the **wave function.**

Travelling Waves

If a wave pulse has constant shape, but moves along with speed v, we have a **travelling wave**.

At time t=0, let the shape be given by $y(x,0) = g(x)$:



At some later time t the pulse is a distance vt further along the string:

So we can represent the displacement y for all later times by

$y(x,t) = g(x - vt)$ for a wave moving to right with speed v

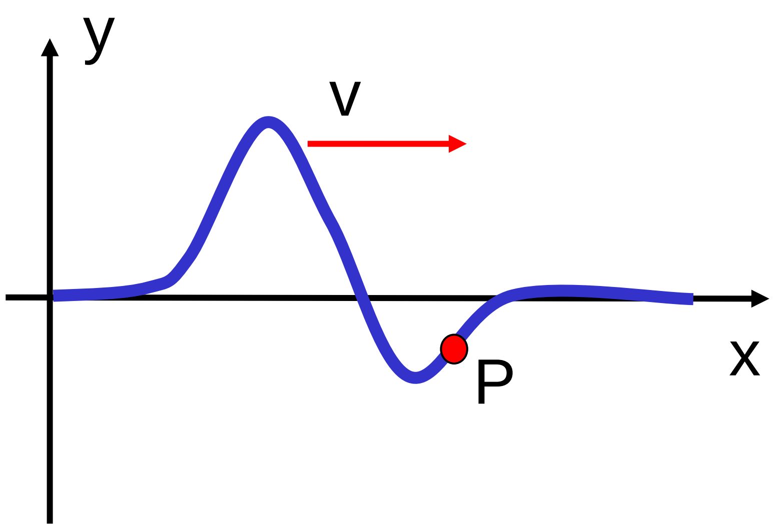
$y(x,t) = g(x + vt)$ for a wave moving to left with speed v.

string is moving in the y-direction
wave is moving in the x-direction

Travelling Waves

•At a fixed time t, y = g(x - vt) describes the shape of the wave at all positions x, ie it is a snapshot of the wave's profile.

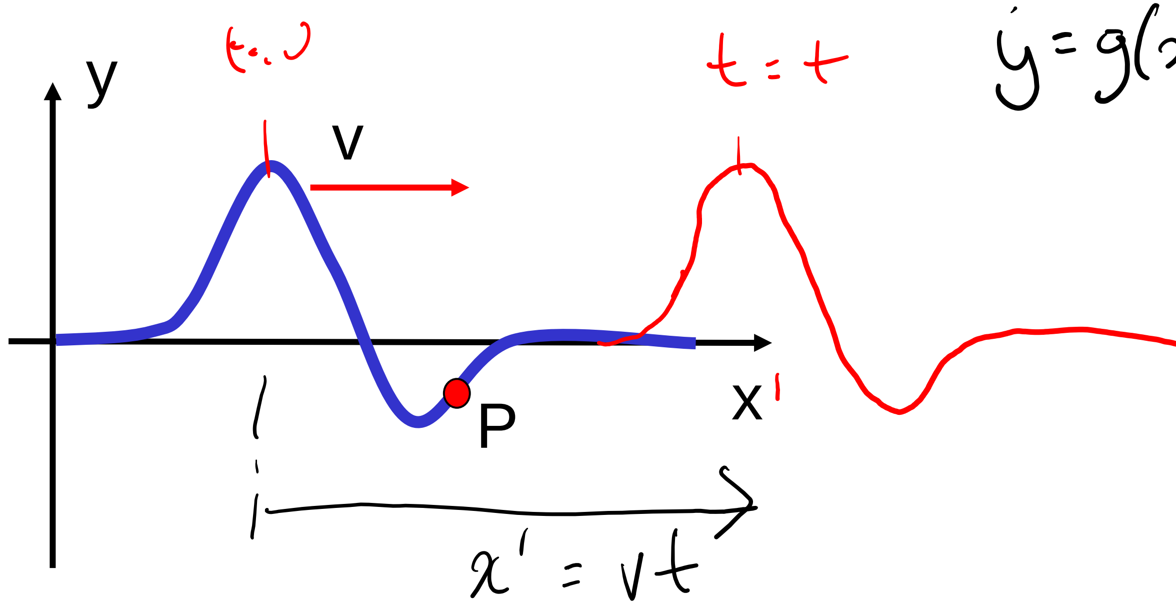
Now consider a point P on the string, at $x = x_p$.



•At fixed x (e.g. at P) the wave function y = g(x - vt) describes the motion of P as a function of t as the wave passes.

Derive a differential equation that describes the wave!

$y = g(x - vt)$



$y = g(x)$ — function of x only, the shape of the wave

The wave moves with time: $y = g(x - \cancel{vt})$

translation of wave shape in +ve x direction at wave speed v, it moves a distance $x = vt$

$$\Downarrow$$

@ $\cancel{t} = 0$, $y = g(x)$

to find displacement of point P in y, need to transform the wave back to $t = 0$

Derive a differential equation that describes the wave!

Coupled oscillators

A.P. French *Vibrations and Waves*: Chapter 6, p135-151

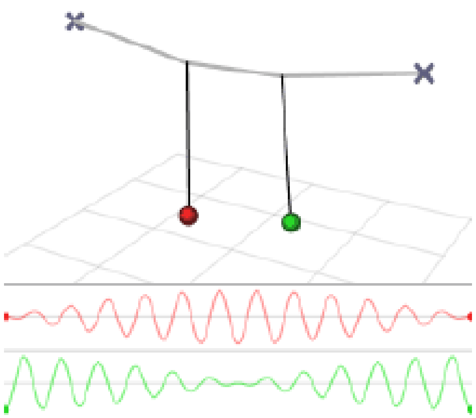
Examples:

Clocks connected physically - e.g. by being fastened to a beam in a wall

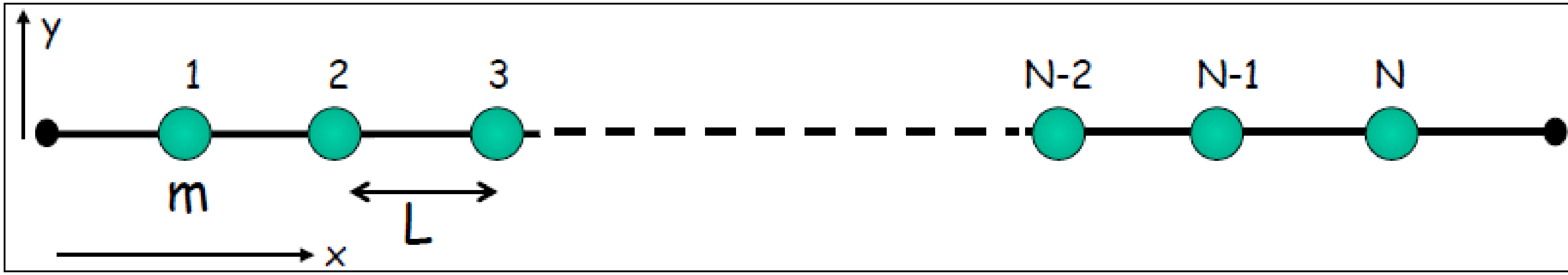
Chemical bonds in a polyatomic molecule

Organisms that can 'sense' each other

(Huygens, 1665) noted that two clocks on his bedroom wall adjusted so the pendulums always swung with the same period but exactly out of phase - *synchronisation*



N coupled oscillators

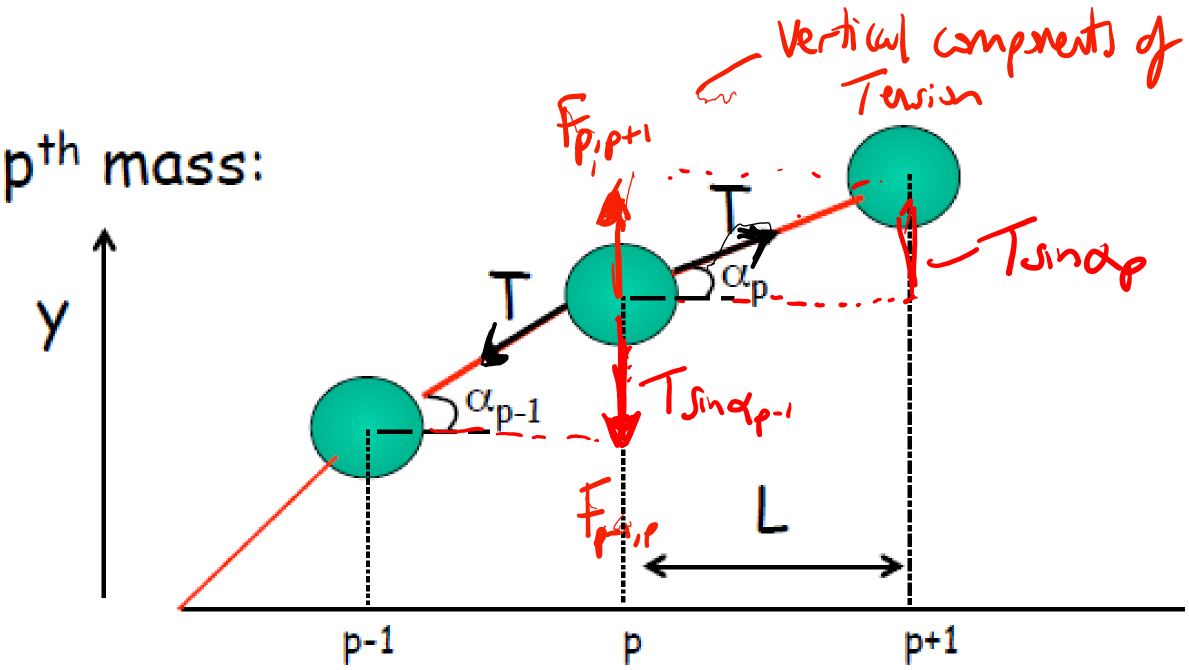


Consider a flexible elastic string to which is attached N identical particles,

- each of mass m,
- equally spaced a distance L apart.
- The ends of the string are fixed a distance L from mass 1 and mass N.
- The tension in the string is T.

Small transverse displacements

Force on the p^{th} mass:

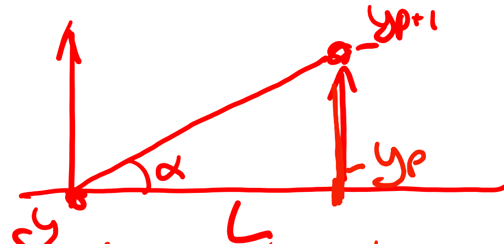


$\sin\theta \simeq \theta$
$\tan\theta \simeq \theta$
so $\tan\theta \simeq \sin\theta$

y-component of force on p is

$$F_p = -T\sin\alpha_{p-1} + T\sin\alpha_p$$

& for small α_p, $\quad \sin\alpha_p \approx \tan\alpha_p = \dfrac{y_{p+1} - y_p}{L}$



$\tan\alpha = \dfrac{y_{p+1} - y_p}{L}$

So
$$F_p = -T\left(\frac{y_p - y_{p-1}}{L}\right) + T\left(\frac{y_{p+1} - y_p}{L}\right)$$

$$\therefore \quad m\frac{d^2 y_p}{dt^2} = -T\left(\frac{y_p - y_{p-1}}{L}\right) + T\left(\frac{y_{p+1} - y_p}{L}\right)$$

$$\boxed{\frac{d^2 y_p}{dt^2} = \frac{T}{mL}\left(y_{p+1} - 2y_p + y_{p-1}\right)}$$

F=ma as with SHM write as $\frac{d^2x}{dt^2}m$ then divide through by m.

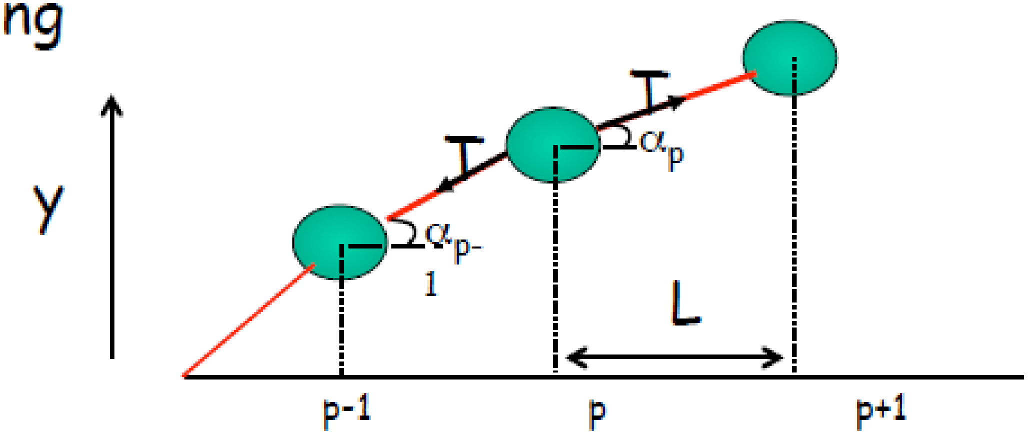
We can write a similar expression for all N particles.

Therefore we have a set of N differential equations: one for each value of p from p=1 to p=N.

NB at fixed ends: $y_0 = 0$ and $y_{N+1} = 0$.

The Wave Equation

Start with system seen previously - N coupled 'transverse' oscillators on a string



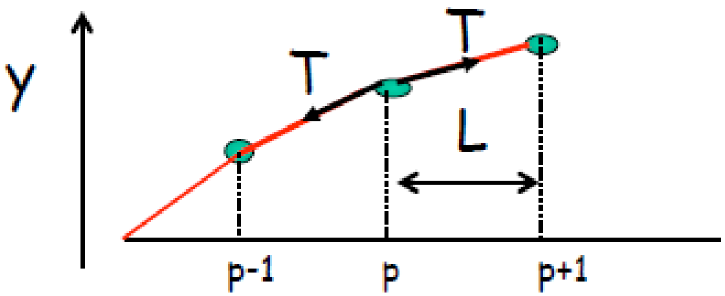
$$\boxed{\frac{d^2 y_p}{dt^2} = \frac{F_T}{m} \frac{(y_{p+1} - 2\, y_p + y_{p-1})}{L}}$$

NB: Using F_T rather than T for the tension to avoid confusion with period T

We can imagine moving from this model to the continuous string by letting N increase to 'fill up' the line.

The Wave Equation

$$\boxed{\frac{d^2 y_p}{dt^2} = \frac{F_T}{m} \frac{(y_{p+1} - 2\, y_p + y_{p-1})}{L}}$$



Will need to replace L by Δx

The mass of the particles tends to zero.

The relevant mass is now the mass of the string segment 'attached to' the point p which is given by $\mu\Delta x$ where μ is the mass per unit length of the string.

The governing equation can now be written as

$$\boxed{\frac{d^2 y_p}{dt^2} = \frac{F_T}{\mu\Delta x} \frac{(y_{p+1} - 2\, y_p + y_{p-1})}{\Delta x}}$$

The Wave Equation

$$\frac{d^2 y_p}{dt^2} = \frac{F_T}{\mu \Delta x} \frac{(y_{p+1} - 2y_p + y_{p-1})}{\Delta x}$$

Can re-write this as

the difference in gradient just before and just after p.

$$\frac{d^2 y_p}{dt^2} = \frac{F_T}{\mu \Delta x}\left\{\frac{(y_{p+1} - y_p)}{\Delta x} - \frac{(y_p - y_{p-1})}{\Delta x}\right\}$$

$$= \frac{F_T}{\mu \Delta x}\left\{\left(\frac{\Delta y}{\Delta x}\right)_{p+1,p} - \left(\frac{\Delta y}{\Delta x}\right)_{p,p-1}\right\} = \frac{F_T}{\mu}\frac{\Delta}{\Delta x}\left(\frac{\Delta y}{\Delta x}\right)$$

we write the rate of change of gradients, the differential like this $\frac{dy}{dx}$ wrt x

In the limit $\Delta x \to 0$, the term on the r.h.s. becomes $\partial^2 y/\partial x^2$

The quantity $(F_T/\mu)^{1/2}$ has units of m s^{-1} and is a wave speed v

WHY ?

Not examinable. Shows explicitly how $F_T/\mu = v^2$.

$$\frac{d^2 y_r}{dt^2} = \frac{F_T}{\mu} \frac{\Delta\left(\frac{\Delta y}{\Delta x}\right)}{\Delta x}$$

the change in gradient with a change in x

in limit as $\Delta x \to 0$

$$* \quad \frac{\partial^2 y_r}{\partial t^2} = \frac{F_T}{\mu} \frac{\partial^2 y}{\partial x^2}$$

$\Rightarrow$ a differential equation known as the WAVE EQUATION

Starting from here, substitute in our travelling wave equation.

function describing a travelling wave $y = g(x - vt)$, let $\alpha = x - vt$, so $y = g(\alpha)$

and y' is the derivative of y w.r.t. α, $y' = \frac{dy}{d\alpha}$

CHAIN RULE

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial \alpha} \frac{\partial \alpha}{\partial t} = y' \frac{\partial \alpha}{\partial t}$$

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial \alpha} \frac{\partial \alpha}{\partial x} = y' \frac{\partial \alpha}{\partial x}$$

Partial derivative of (x-vt) wrt t is simply –v !

where $\frac{\partial \alpha}{\partial t} = \frac{\partial (x - vt)}{\partial t}$

$= -v$

where $\frac{\partial \alpha}{\partial x} = \frac{\partial (x - vt)}{\partial x} = 1$

so, $\frac{\partial y}{\partial t} = -v y'$

, so $\frac{\partial y}{\partial x} = y'$

$\frac{\partial^2 y}{\partial x^2} = y''$

these are 1st derivatives. We need 2nd derivati

$-v$ again

$$\frac{\partial^2 y}{\partial t^2} = -v \frac{\partial y'}{\partial t} = -v \frac{\partial y'}{\partial \alpha} \frac{\partial \alpha}{\partial t} = +v^2 y''$$

y''

$$\therefore \frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

compare with WAVE EQN

$$* \quad v^2 = \frac{F_T}{\mu}$$

The Wave Equation

Final form:

$$\boxed{\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}}$$

See last slide for why this is so.

$$\boxed{v^2 = \frac{F_T}{\mu}}$$

Or more usually

$$v = \sqrt{\frac{F_T}{\mu}}$$

i.e. speed increases with string tension and decreases with string mass

Speed of sound

For sound in fluids such as air or water:

$$v = \sqrt{\frac{B}{\rho}}$$

3D version of 1D travelling wave. B, modulus ≡ Tension, ρ is the mass per unit vol.

where B is the bulk modulus and ρ is the density.

i.e. ∝ work done to expand a vol of gas against P

In gases we can write this as:

$$\gamma = \frac{c_p}{c_V} \quad \begin{matrix} \text{@ const } P \\ \text{@ const } V \end{matrix} \simeq 1.4$$

$$v = \sqrt{\frac{\gamma RT}{M}}$$

Where γ is the ratio of specific heats (= 7/5 for an ideal diatomic gas) and M is the molar mass (~ 30 x 10^{-3} kg mol^{-1} for air)