17 Limits and l'Hôpital's/Bernoulli's Rule

Often, we want to understand the behaviour of a function in the limit as the argument tends to a particular value. This can be a finite value at which the behaviour is unclear in some way, $\lim_{x\to a}$, or it could be as the argument grows arbitrarily large, $\lim_{x\to\pm\infty}$.

We have already seen one example of unclear behaviour when calculating the derivative of $\sin(x)$. There, we needed to consider the limit

$$\lim_{\theta\to 0}\left(\frac{\sin(\theta)}{\theta}\right), \tag{1}$$

which is not immediately obvious, because we cannot simply evaluate the function $\sin(x)/x$ at this point. Since we cannot divide by 0, the domain of the function cannot include 0 and the function has no definition at this point. However, that does not mean that the function does not have a well defined **limit**. To make any further progress, we should define the concept of a limit.

17.1 Limits

17.1.1 One-Sided Limits

The general definition of a limit allows it to be applied to all sorts of functions, and gets us into the realm of **metric spaces**. This is a fascinating branch of mathematics but is beyond what we need to consider here. The following definition is not the most general, but is sufficient for our purposes of considering functions of real numbers.

Consider a function f and a point a. Suppose there exists some finite value L_+ with the following property: for **any** chosen value ε, we can find a corresponding positive value δ such that $|f(a+x)-L_+|<\varepsilon$ for all $x\in(0,\delta)$. Then we say that L_+ is a **limit from above** of f at a. We can denote this

$$\lim_{x\to a^+} f(x) = L_+\,. \tag{2}$$

If, on the other hand, for any value $n>0$, there exists a $\delta>0$ such that $f(a+x)>n$ for all $x\in(0,\delta)$, then we say that the limit of f from above at a is ∞. We can denote this as

$$\lim_{x\to a^+} f(x) = \infty\,. \tag{3}$$

Likewise, if for any $n<0$, there exists a $\delta>0$ such that $f(a+x)<n$ for all $x\in(0,\delta)$ then the limit from above is $-\infty$:

$$\lim_{x\to a^+} f(x) = -\infty\,. \tag{4}$$

Note that an **infinite limit** from above is not the same as having **no limit** from above. For example, the function $1/x$ has

$$\lim_{x\to 0+}\frac{1}{x} = +\infty\,, \tag{5}$$

whereas the function $\sin(1/x)$ has no limit from above at $x = 0$. In the first case, as we approach 0 from above, the function consistently tends towards ∞. In the second case, as we approach 0 from above, the behaviour of the function becomes more and more erratic.

Limits from below are defined similarly. If there exists a finite value L_- such that, for any ε, there is a corresponding $\delta > 0$ with $|f(a-x) - L_-| < \varepsilon$ for all $x \in (0, \delta)$, then L_- is a limit from below.

17.1.2 Two-Sided Limits

It is possible for one or other of the one-sided limits to exist without the other. For example, in the somewhat contrived function

$$f : \mathbb{R} \setminus \{0\} \to \mathbb{R}, f(x) = \begin{cases} x^2 & x < 0 \\ \sin(1/x) & x > 0 \end{cases}, \tag{6}$$

we have

$$\lim_{x \to 0^-} f(x) = 0\,, \tag{7}$$

while the limit from above does not exist.

Similarly, even when both one-sided limits exist, they may not be equal. For example, for the function

$$f(x) = \frac{1}{1 + 2^{1/x}}\,, \tag{8}$$

we have

$$\lim_{x \to 0^+} f(x) = 0 \qquad \text{but} \qquad \lim_{x \to 0^-} f(x) = 1\,, \tag{9}$$

while for the function $1/x$, we have

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty \qquad \text{but} \qquad \lim_{x \to 0^-} \frac{1}{x} = -\infty\,. \tag{10}$$

In none of these cases can we define a sensible two-sided limit. Only when both one-sided limits exist and $L_+ = L_-$ can we define **the** limit $L = L_+ = L_-$ as $x \to a$. We then say that

$$\lim_{x \to a} f(x) = L\,. \tag{11}$$

17.2 Properties of Limits

Most of the time, we will not need the full formal definition of a limit. One useful application of it, however, is in proving a few key properties of limits. Suppose the functions f and g both have limits from above at a point a, and that

$$\lim_{x \to a^+} f(x) = L_f \quad \text{and} \quad \lim_{x \to a^+} g(x) = L_g\,. \tag{12}$$

Then by definition, for a given ε, there exist δ_f and δ_g such that

$$|f(x) - L_f| < \varepsilon \forall x < \delta_f \quad \text{and} \quad |g(x) - L_g| < \varepsilon \forall x < \delta_g\,. \tag{13}$$

Let $\delta = \min(\delta_f, \delta_g)$. Then for all $x < \delta$, we must have both $x < \delta_f$ and $x < \delta_g$, so

$$\begin{aligned} |(f(x)+g(x)) - (L_f + L_g)| &= |(f(x) - L_f) + (g(x) - L_g)| \\ &\leq |f(x) - L_f| + |g(x) - L_g| \\ &< 2\varepsilon\,. \end{aligned} \tag{14}$$

This proves that the limit from above of $h(x) = f(x) + g(x)$ at a exists and is equal to $L_f + L_g$. Note that the factor of 2 in the last line does not matter, since we could have chosen $\varepsilon/2$ as our arbitrary value in Eq. 12: the definition of L_f and L_g would still guarantee that suitable δ values exist.

Similarly considerations demonstrate the following useful facts. If the relevant limits of both f and g exist, then

$$\begin{aligned} \lim_{x\to a}(f(x)+g(x)) &= \left(\lim_{x\to a} f(x)\right) + \left(\lim_{x\to a} g(x)\right) \\ \lim_{x\to a}(f(x)\times g(x)) &= \left(\lim_{x\to a} f(x)\right) \times \left(\lim_{x\to a} g(x)\right) \\ \lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) &= \frac{\left(\lim_{x\to a} f(x)\right)}{\left(\lim_{x\to a} g(x)\right)} \end{aligned} \tag{15}$$

These results may seem trivially straightforward. However, it is worth emphasising that they follow from the rigorous definition of a limit, because there are equally "obvious" results that are incorrect. For example

$$\lim_{x\to a}(f(x)^g(x)) \neq \left(\lim_{x\to a} f(x)\right)^{(\lim_{x\to a} g(x))}, \tag{16}$$

as the following counter-example demonstrates:

$$\begin{aligned} \lim_{x\to\infty}\left(1+\frac{1}{x}\right) &= 1 \\ \lim_{x\to\infty} x &= \infty \\ \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x &= e\,. \end{aligned} \tag{17}$$

17.3 Continuity

Now that we have the concept of a limit defined rigorously, we can also define what it means for a function to be continuous. A function f is continuous at the point a if

$$\lim_{x\to a} f(x) = f(a)\,. \tag{18}$$

This definition implies that the two-sided limit exists, which in turn implies that both one-sided limits exist and are equal.

A function is discontinuous at a if the two-sided limit does not exist, **or** if the limit does exist but is not equal to the value of the function at that point. For example, $1/(1+2^{1/x})$ is discontinuous because it has different one-sided limits. On the other hand, the function

$$f(x) = \begin{cases} x^2 & x < 0 \\ 1 & x = 0 \\ x^2 & x > 0 \end{cases} \tag{19}$$

has a well-defined two-sided limit at 0:

$$\lim_{x \to 0} f(x) = 0 \tag{20}$$

but is discontinuous, since the limit is different from the value of the function at that point.

Likewise, $1/x$ has a discontinuity at $x = 0$, since its one-sided limits are different, and $1/x^2$ is also discontinuous at 0: despite the one-sided limits agreeing, the function is **not defined** at $x = 0$ and so there is no sense in which the function can be continuous.

17.4 Removable Singularities

A singularity is an isolated point at which a function is, in some way, "badly behaved". Often this means that the function has an asymptote at that value. However, there are situations in which a singularity is really just an artifact of the way in which we have expressed the function. For example, the function $f(x) = (x^3+x)/(x^2+x)$ **as written** cannot be defined at $x = 0$ since then the denominator would be 0: it has a singularity at 0. However, dividing through by x, it is immediately obvious that f is equivalent to the function $(x^2+1)/(x+1)$ and that the value of $f(0)$ "should be" 1. However, notice that we cannot divide through as we did **at** 0. Really what we have done is to divide through by x **everywhere else** and then take the limit as $x \to 0$. In situations like this, where there the two-sided limit exists, the singularity is removable, since we can simply define the function at this point to take the value of the limit.

17.5 l'Hôpital's/Bernoulli's Rule

The last of the rules in Eq. 15 does not help us if the relevant limits of both functions are 0. In this case, we have the indeterminate form 0/0. This does not necessarily mean the limit does not exist: it merely tells us that whatever the limit is, it obeys $0L = 0$. But this could be anything! Clearly we need a different approach in this situation. This is where Taylor series come to the rescue. Expanding the functions f and g around the point a, we have

$$L = \lim_{x \to a} \left(\frac{f(a) + f^{(1)}(a)(x-a) + \frac{1}{2} f^{(2)}(a)(x-a)^2 + \ldots}{g(a) + g^{(1)}(a)(x-a) + \frac{1}{2} g^{(2)}(a)(x-a)^2 + \ldots} \right). \tag{21}$$

Since the value of both f and g at a is 0, the first terms in the numerator and denominator vanish. We are then able to divide top and bottom through by $(x-a)$ to arrive at

$$\begin{aligned} L &= \lim_{x\to a}\left(\frac{f^{(1)}(a)+\frac{1}{2}f^{(2)}(a)(x-a)+\ldots}{g^{(1)}(a)+\frac{1}{2}g^{(2)}(a)(x-a)+\ldots}\right) \\ &= \lim_{x\to a}\left(\frac{f^{(1)}(x)}{g^{(1)}(x)}\right). \end{aligned} \tag{22}$$

If both $\lim f^{(1)}(x)$ and $\lim g^{(1)}(x)$ are non-zero, then we can now easily determine the value of L. If they are both 0, then we can remove them and divide again by $(x-a)$ to arrive at $L = f^{(2)}(x)/g^{(2)}(x)$. We can repeat this step as many times as necessary until we no longer have an indeterminate form.

17.6 Example

Find

$$\lim_{x\to 0}\frac{\sin(x)}{x}. \tag{23}$$

17.7 Other Indeterminate Forms

l'Hôpital's/Bernoulli's Rule also applies for other indeterminate forms, like $0\times\infty$ and ∞/∞. For example, if $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = \infty$, then

$$\lim_{x\to a}\left(f(x)\times g(x)\right) = \lim_{x\to a}\left(\frac{f(x)}{1/g(x)}\right) \tag{24}$$

and we can differentiate both f and $1/g$ to find the limit. Similarly, for 0^0, we can take logarithms and apply the rule, as in

$$\lim_{x\to 0}(1+x)^{1/x} = \exp\left(\lim_{x\to 0}\left(\frac{1}{x}\ln(1+x)\right)\right) = \exp\left(\lim_{x\to 0}\frac{(1+x)^{-1}}{1}\right) = \exp(1) = e\,. \tag{25}$$

17.8 Exercise

Find the following limit:

$$\lim_{x\to 0}\left(\frac{\tan(x)-x}{\cos(x)-1}\right).$$