Basic Vector Calculus

Recap of Ordinary Differentiation

- Vary input of function $x \mapsto x + \delta x$
- Output varies by corresponding amount $f(x) \mapsto f(x) + \delta f(x)$

$$f(x+\delta x) - f(x)$$

- For small δx, ratio is approximate rate of change

$$\frac{\delta f}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

- Approximation improves as δx shrinks

$$\frac{df}{dx} = \lim_{\delta x \to 0} \frac{\delta f}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

Vector Differentiation

- Vector valued function, e.g. position vector $\underline{r}(t)$
- Small change in t gives small change in position δt

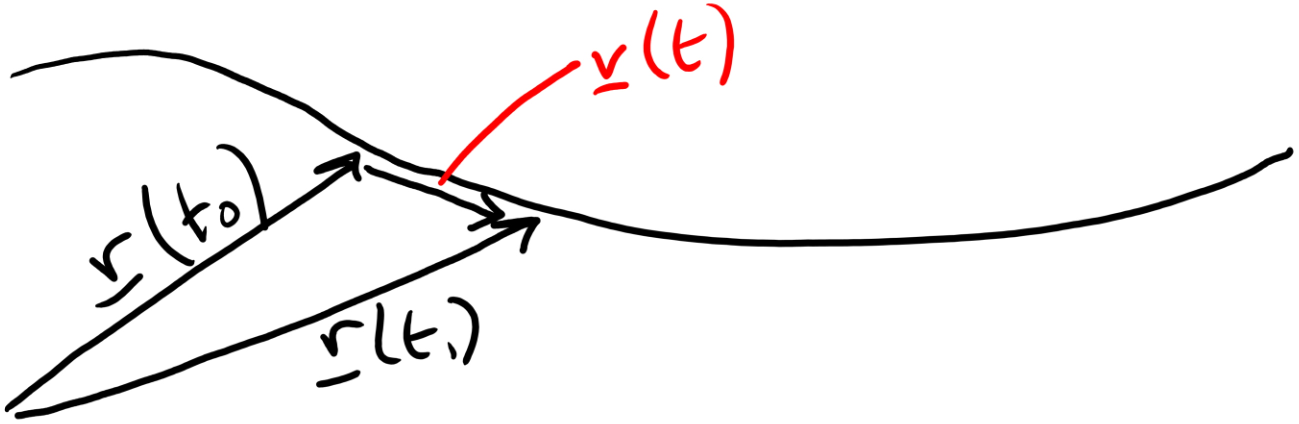
$$\delta\underline{r}(t) = \underline{r}(t+\delta t) - \underline{r}(t)$$

- Ratio is approximate rate of change $\frac{\delta \underline{r}}{\delta t}$
- Approximation improves as δt shrinks

$$\frac{d\underline{r}}{dt} = \lim_{\delta t \to 0} \frac{\underline{r}(t+\delta t) - \underline{r}(t)}{\delta t}$$

vector-valued

Direction of Vector Derivatives



Direction already captured in defⁿ.

Example

- Object with position vector $\underline{r}(t) = (3t + 4)\hat{\underline{i}} + (-t^2)\hat{j}$

- What is the velocity as a function of time? What is the acceleration?

$$\underline{v}(t) = 3\hat{\underline{i}} - 2t\hat{j}$$

$$\underline{a}(t) = -2\hat{j}$$

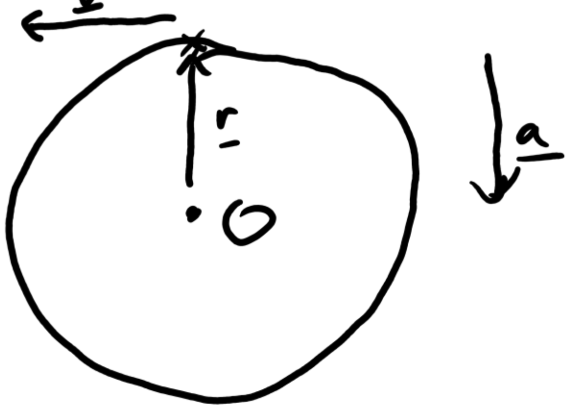
Example

- Object with position vector $\underline{r}(t) = \cos(t)\,\hat{\underline{i}} + \sin(t)\,\hat{\underline{j}}$

- What are the velocity and acceleration? What direction are these in relative to the position?

$$\underline{v}(t) = -\sin(t)\,\hat{\underline{i}} + \cos(t)\,\hat{\underline{j}}$$

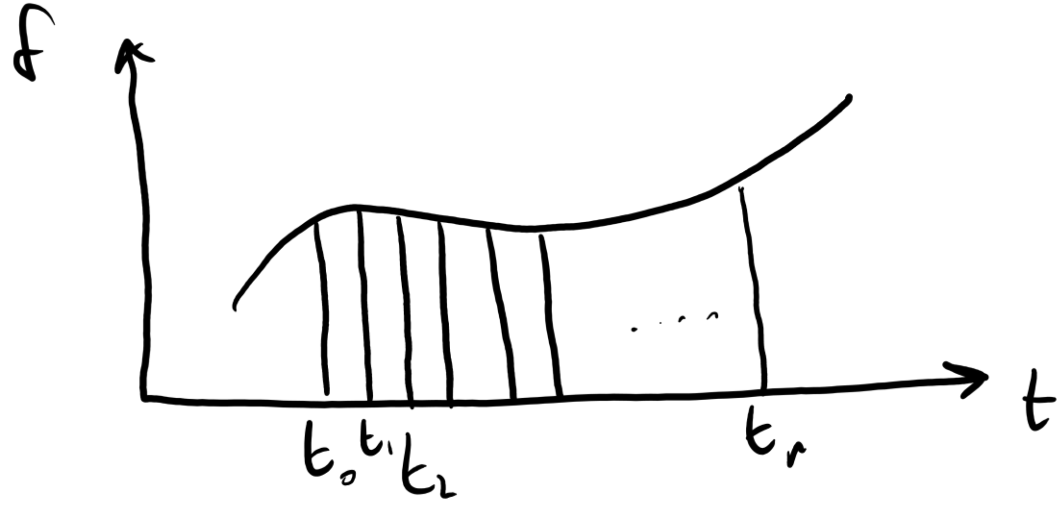
$$\underline{a}(t) = -\cos(t)\,\hat{\underline{i}} - \sin(t)\,\hat{\underline{j}}$$



Recap of Ordinary Integration

- Integral is a sum over infinitesimal quantities

$$\int f(t)\,dt = \lim_{\delta t \to 0} \sum_{k=0}^{n} f(t_k)\,\delta t$$



Integration with Vectors I

- Can integrate a vector-valued function with respect to a scalar

$$\underline{r}(t) = \int \underline{v}(t)\,dt = \lim_{\delta t \to 0} \sum \underline{v}(t)\,\delta t$$

- Result is a vector

Example

- Velocity of a particle $\underline{v}(t) = (12t^2 + 4t)\,\hat{\underline{i}} + \cos(t)\,\hat{\underline{j}}$

- Find general expression for position at time t

$$\underline{r}(t) = \int \underline{v}(t)\,dt = (4t^3 + 2t^2 + c_1)\,\hat{\underline{i}} + (\sin(t) + c_2)\,\hat{\underline{j}}$$

$$= \begin{pmatrix} 4t^3 + 2t^2 \\ \sin(t) \end{pmatrix} + \underline{c}$$

Integration with Vectors II

- Line Integral

Integral over a path in some space

Multiply small changes in position by function of position and summing over some path.

Example

- Path C defined by $\underline{r}(\lambda) = \begin{pmatrix} \lambda^2 \\ 6-\lambda \end{pmatrix} \quad \lambda \in [0,5]$

- Function $\mathbf{F}$ defined over whole space $\underline{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \end{pmatrix}$

- What is the line integral of $\mathbf{F}$ over the curve C?

Integration with Vectors II

- Total distance travelled by an object

$$\underline{r} = \lambda^2 \hat{\underline{i}} + (6-\lambda)\hat{\underline{j}}$$

$$\underline{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \end{pmatrix}$$

$$\int \underline{F} \cdot d\underline{r}$$

$$= \int_C \begin{pmatrix} x+y \\ x-y \end{pmatrix} \cdot d\underline{r} = \int_C \begin{pmatrix} \lambda^2 + 6 - \lambda \\ \lambda^2 - 6 + \lambda \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \end{pmatrix} = \int_C \begin{pmatrix} \lambda^2 + 6 - \lambda \\ \lambda^2 - 6 + \lambda \end{pmatrix} \cdot \begin{pmatrix} 2\lambda \\ -1 \end{pmatrix} d\lambda$$

$$\frac{dx}{d\lambda} = 2\lambda \qquad \frac{dy}{d\lambda} = -1$$

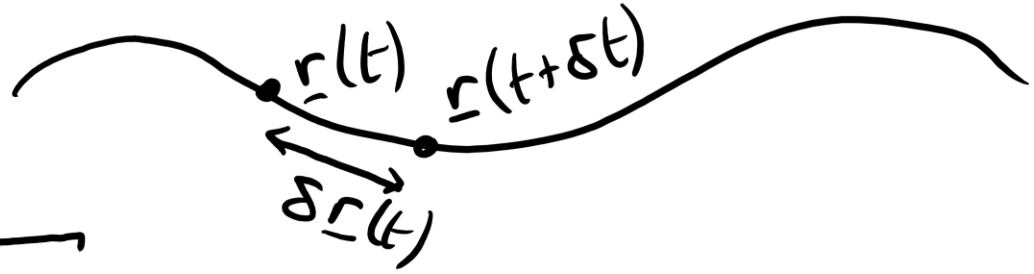
$$dx = 2\lambda \, d\lambda \qquad dy = -d\lambda$$

$$= \int_C \left[(\lambda^2 + 6 - \lambda)(2\lambda) + (\lambda^2 - 6 + \lambda)(-1) \right] d\lambda$$

Example

- Particle travels along path $\underline{r}(t)$

- Find the distance travelled after time t



$$l = \int dl = \int \sqrt{d\underline{r}\cdot d\underline{r}}$$

$$= \int \sqrt{\frac{d\underline{r}}{dt}\cdot\frac{d\underline{r}}{dt}\,dt\,dt} = \int \sqrt{\frac{d\underline{r}}{dt}\cdot\frac{d\underline{r}}{dt}}\,dt$$

$$= \int \sqrt{\underline{v}\cdot\underline{v}}\;dt$$