From previous lecture:

Simple Harmonic Motion(SHM) results when:

$$m\frac{d^2x}{dt^2} \propto -x$$

The displacement x given by:

$$x = A\cos(\omega t + \delta)$$

where ω is the angular frequency and δ is the phase constant
Find unknown δ by substituting in known values given at time t.

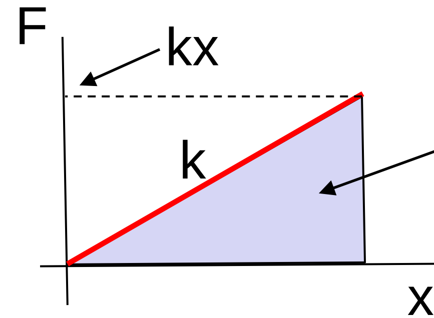
Relationship between ω, f and T:

$$\omega = \sqrt{\frac{k}{m}} = 2\pi f = \frac{2\pi}{T}$$

Pendulum SHM with

$$\omega = \sqrt{\frac{g}{L}}$$

Recall energy stored in a spring



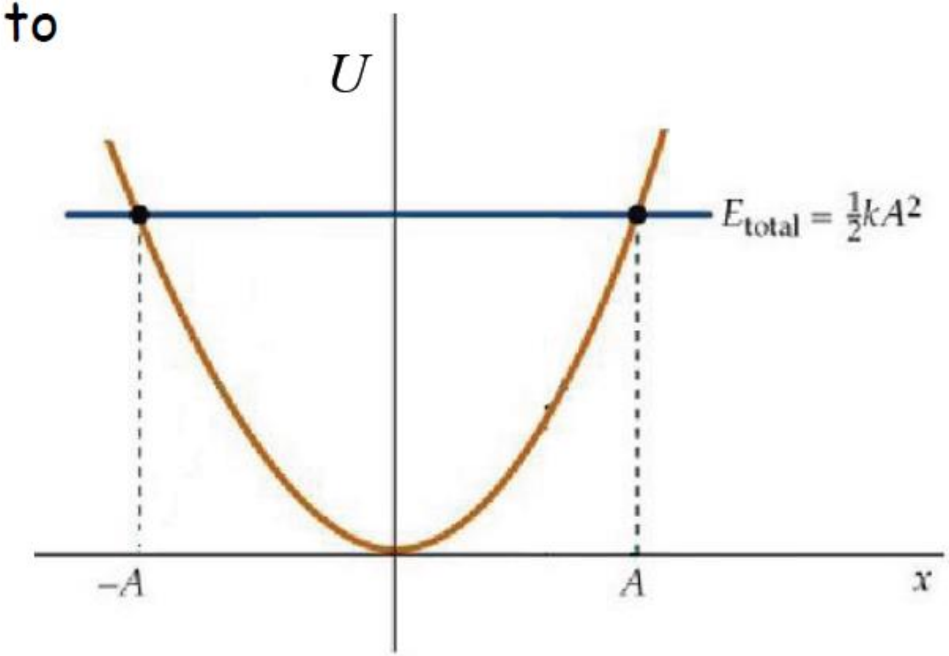
$U = \tfrac{1}{2}x.kx = \tfrac{1}{2}kx^2$

Potential Energy in a Simple Harmonic Oscillator

Potential energy *U* related to force *F* via

$$F = -\frac{dU}{dx} = -kx$$

$$\boxed{U = \tfrac{1}{2}kx^2}$$



- U is quadratic in displacement.
- Force is linear in displacement

Potential Energy in SHM

Spring compression is a conservative force:

$$F = -\frac{dU}{dx} = -kx$$

$$U = -\int F\,dx = \int kx\,dx = \int_0^x kx\,dx = \frac{kx^2}{2}$$

Graphical version of this on previous slide.

using $x = A\cos\omega t$ gives

Recall k = mω^2 from 8 slides back

$$U = \frac{kx^2}{2} = \frac{kA^2\cos^2\omega t}{2} = \frac{m\omega^2A^2\cos^2\omega t}{2} = \frac{m\omega^2x^2}{2}$$

The potential energy is zero at equilibrium point and has maxima at the extremes of displacement

Kinetic Energy in SHM

$$K = \frac{mv^2}{2} = \frac{m(-\omega A \sin \omega\, t)^2}{2} = \frac{m\omega^2 A^2 \sin^2 \omega\, t}{2}$$

since $$v = \frac{dx}{dt} = -A\omega \sin \omega\, t$$

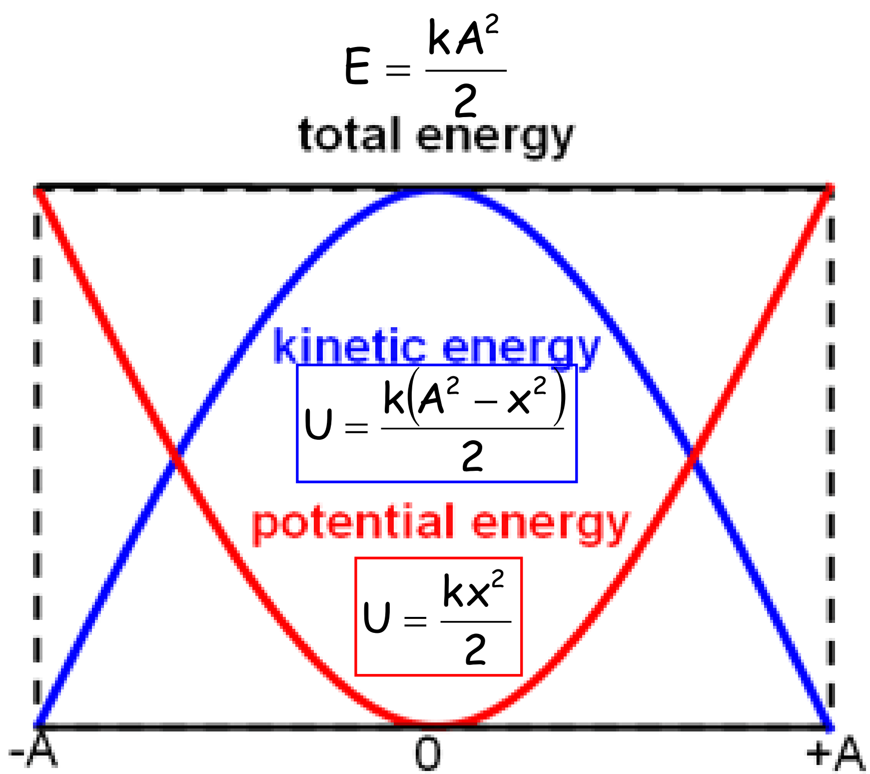
The kinetic energy is zero at the extremes of displacement and has a maximum at the equilibrium point S

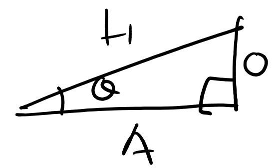
What about the total energy?

$$E = K + U = \frac{m\omega^2 A^2 \sin^2 \omega t}{2} + \frac{m\omega^2 A^2 \cos^2 \omega t}{2} = \frac{m\omega^2 A^2}{2}(\sin^2 \omega t + \cos^2 \omega t) = \frac{m\omega^2 A^2}{2} = \frac{kA^2}{2}$$

because $\sin^2(x) + \cos^2(x) = 1$ (from Pythagoras)

In SHM the total energy is constant with a continuous interchange of kinetic and potential energy

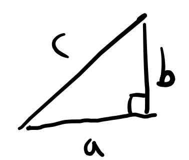




$\sin\theta = \frac{O}{H}$

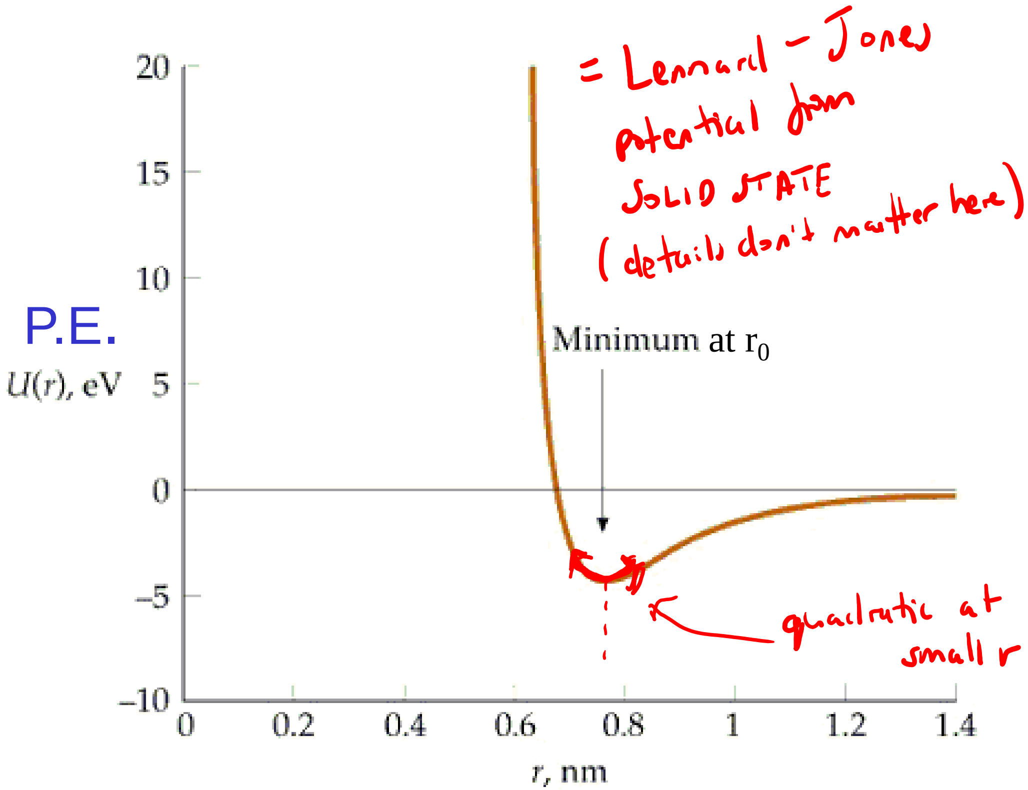
$\cos\theta = \frac{A}{H}$

$$\sin^2\theta + \cos^2\theta = \frac{O^2}{H^2} + \frac{A^2}{H^2} = \frac{O^2 + A^2}{H^2} = \frac{H^2}{H^2} = 1$$



$c^2 = a^2 + b^2$

General oscillations



Interatomic distance

In general, U(*r*) is some complicated function which plateaus at distance, but we can Taylor-expand it about the equilibrium position r_0

Let $x = r - r_0$

(expansion in terms of displacement from eqm, not from origin)

then

$$U(r) = U(r_0) + \cancel{U'(r_0)x} + \tfrac{1}{2}U''(r_0)x^2 + \underbrace{\text{terms of order } x^3}_{\approx 0 \text{ for small } x}$$

Drop because constant

Gradient zero at r_o

Approximately quadratic - Hence *any* interaction (with a minimum) yields SHM for small amplitudes.

Summary of SHM

We have been looking at the behaviour of a series of systems that have governing equations of the form:

$$\alpha \frac{d^2x}{dt^2} = -\beta x$$

where α *and* β are positive quantities independent of *t*.

This can be re-arranged to

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0$$

where $\omega_0 = (\beta/\alpha)^{1/2}$.

The solution of this equation is

$$x = A\cos(\omega_0 t + \delta)$$

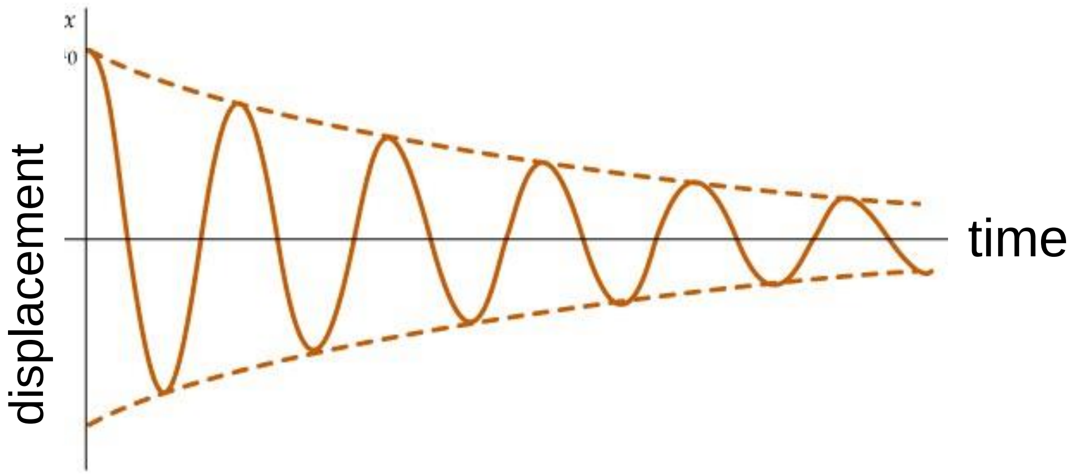
So ω_0 defines the fundamental frequency of this motion

Damped Oscillations: “Ringing”

Tipler 14-4

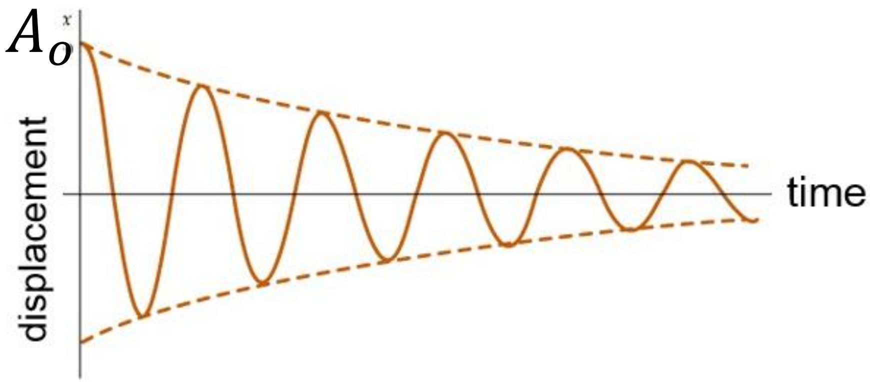
All real oscillations are subject to dissipative forces (friction, viscosity, air resistance, etc.)

These forces remove energy from the oscillating system and reduce the amplitude – sometimes essential in applications!



From earlier, we found the total energy in SHM to be

$$E = \frac{kA^2}{2}$$



Energy is proportional to the amplitude squared, and this mechanical energy will decrease over time due to mechanical losses from friction and fluid drag. This dissipation will be exponential, with a time constant τ

$$A^2 = A_o^2 e^{-\frac{t}{\tau}}$$

Find the equation of motion for damped SHO?

Consider a mass m, subject to dissipation, on the end of a spring with a spring constant k.

For displacement x from equilibrium,

Restoring force = $-kx$.

Drag force $\propto -dx/dt$ with coefficient of resistance b

F direction opposite to motion

by Newton, $F = ma$

$$-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

divide through by m and rearrange

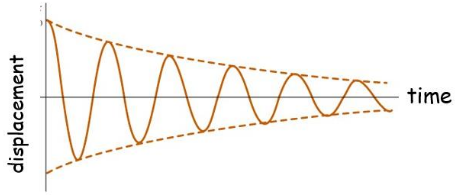
$$\boxed{\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_o^2 x = 0}$$

where $\gamma = b/m$ and $\omega_0^2 = k/m$

The equation of motion for a damped SHO , SOLVE FOR x

Due to friction, motion gradually dies away – likely functional form?
If the damping constant γ is small will look like

as before, guess the function that looks the same



Try : $x(t) = e^{-\beta t} f(t)$

(cos) function times factor shown by dashed lines

β - +ve constant

f(t) - to be determined

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = 0$$

To solve this differential equation we need to find dx/dt and d^2x/dt^2 in terms of f ... does it work?

$$\Rightarrow \quad \frac{dx}{dt} = -e^{-\beta t}\beta f + e^{-\beta t}\frac{df}{dt}$$

$$\& \quad \frac{d^2x}{dt^2} = \beta^2 e^{-\beta t} f - 2\beta e^{-\beta t}\frac{df}{dt} + e^{-\beta t}\frac{d^2f}{dt^2}$$

In detail

$$x(t) = e^{-\beta t} f(t) \qquad ? \qquad \text{PRODUCT RULE}$$

$$\frac{dx}{dt} = e^{-\beta t} \frac{d(f(t))}{dt} + \frac{d}{dt}(e^{-\beta t}) f(t)$$

$$= e^{-\beta t} \frac{df}{dt} - \beta e^{-\beta t} f$$

$$\frac{d^2x}{dt^2} = e^{-\beta t} \frac{d\left(\frac{df}{dt}\right)}{dt} + (-\beta e^{-\beta t}) \frac{df}{dt} - \left(\beta e^{-\beta t} \frac{df}{dt} + \frac{d(-\beta e^{-\beta t})}{dt} f\right)$$

$$= e^{-\beta t} \frac{d^2 f}{dt^2} - \beta e^{-\beta t} \frac{df}{dt} - \beta e^{-\beta t} \frac{df}{dt} + \beta^2 e^{-\beta t} f.$$

$$\frac{d^2 x}{dt^2} = \beta^2 e^{-\beta t} f - 2 e^{-\beta t} \frac{df}{dt} + e^{-\beta t} \frac{d^2 f}{dt^2}$$

Substituting into $$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_o^2 x = 0$$

We get (after some tidying up)

$$\frac{d^2 f}{dt^2} + (\gamma - 2\beta)\frac{df}{dt} + \left(\beta^2 - \gamma\beta + \omega_o^2\right) f = 0$$

Choosing $\beta = \gamma/2$ yields

$$\boxed{\frac{d^2 f}{dt^2} + \left(\omega_o^2 - \frac{\gamma^2}{4}\right) f = 0}$$

SHM

c.f. $\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$

and $\frac{k}{m} = \omega^2$

This equation defines *f* and we can recognise this as being in standard SHM form with a modified frequency

$$\boxed{\omega_d^2 = \left(\omega_o^2 - \frac{\gamma^2}{4}\right)}$$

as damping γ increases the SHM frequency gets smaller.

In detail

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0$$

Substitute $\frac{d^2x}{dt^2}$ and $\frac{dx}{dt}$ into original equation of motion.

$$\beta^2 \cancel{e^{-\beta t}} f - 2\beta \cancel{e^{-\beta t}} \frac{df}{dt} + \cancel{e^{-\beta t}} \frac{d^2f}{dt^2} + \gamma\left(-\cancel{e^{-\beta t}}\beta f + \cancel{e^{-\beta t}}\frac{df}{dt}\right) + \omega_0^2 \cancel{e^{-\beta t}} f = 0$$

$$\frac{d^2f}{dt^2} + (\gamma - 2\beta)\frac{df}{dt} + (\beta^2 - \gamma\beta + \omega_0^2) f = 0.$$

β = positive decay constant

$\beta = \frac{\gamma}{2}$ (related to friction)

$$\frac{d^2f}{dt^2} + \cancel{(\gamma - \gamma)\frac{df}{dt}} + \left(\frac{\gamma^2}{4} - \frac{\gamma^2}{2} + \omega_0^2\right) f = 0$$

$$\frac{d^2f}{dt^2} + \left(\omega_0^2 - \frac{\gamma^2}{4}\right) f = 0.$$

Pulling this all together,

Have found that the evolution of position is given by:

$e^{-\left(\frac{\gamma}{2}\right)t}$ (recall, $\gamma/2 = \beta$)

$$x(t) = Ae^{-\gamma t/2}\cos(\omega_d t + \delta)$$

with

$$\omega_d^2 = \left(\omega_o^2 - \frac{\gamma^2}{4}\right)$$

modified frequency

Can think of this as SHM with a time-dependent amplitude term

$$A(t) = Ae^{-\gamma t/2}$$

Note: if damping constant β (= $\gamma/2$) is gradually increased, ω decreases until 0 at a critical value, β_c. This is critical damping = rapid return to equilibrium with no oscillation. If $\beta > \beta_c$, then SHO will be over-damped.

NOTE $\beta \neq$ b , $\beta = \gamma/2$ is just used for ease of solving for equation of motion, damping coefficient, γ = b/m, so b = γm.