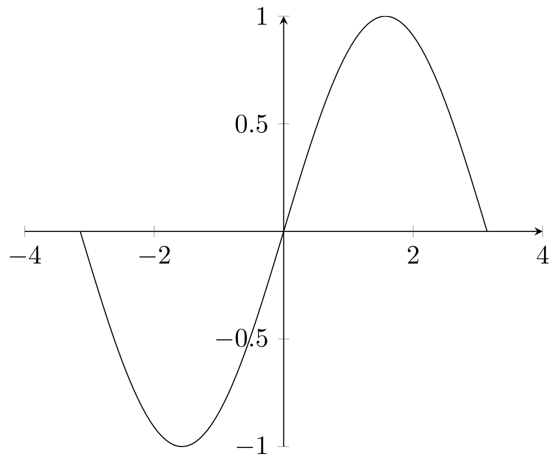
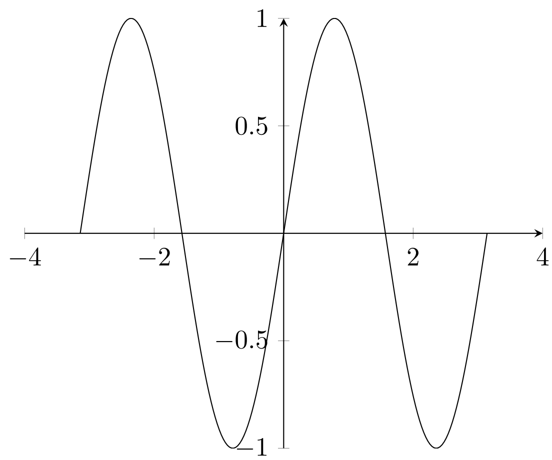
22 Fourier Series

22.1 Some Properties of Sine and Cosine Functions

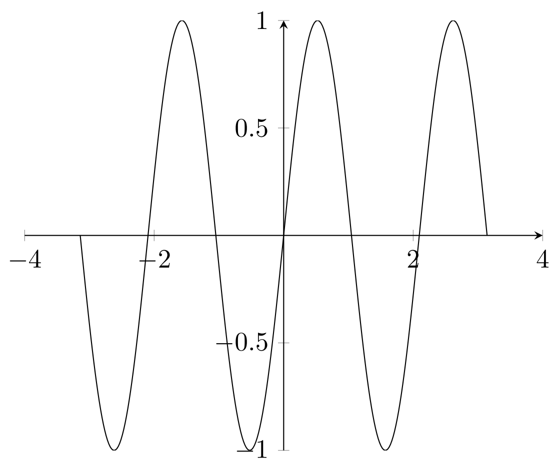
Consider the following functions and their graphs:



(a) $\sin(x)$



(b) $\sin(2x)$



(c) $\sin(3x)$

These functions are periodic with periods 2π, π and $2\pi/3$, respectively. Since these periods are all multiples of 2π, we know in particular that all of these functions obey the property $f(x) = f(x+2\pi)\ \forall x$. As such, any linear combination of these functions will have the same property. For example, the function $f(x) = 0.1\sin(x) - 0.5\sin(2x) + \sin(3x)$ is periodic with period 2π, with each period of the form shown by the graph in Fig. 2.

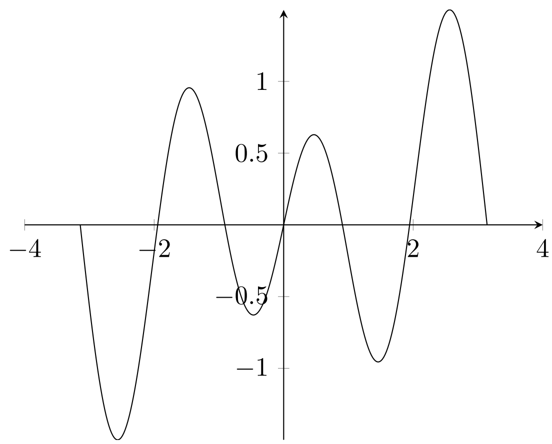


Figure 2: Graph of $f(x) = 0.1\sin(x) - 0.5\sin(2x) + \sin(3x)$ for $x \in [-\pi, \pi]$

Taking this idea further, we can take arbitrarily complicated linear combinations of functions of the form $\sin(nx)$. Doing so will always result in a periodic function. We can also give our function a different period, L, by simply scaling the argument and using functions of the general form:

$$\sin\left(\frac{2\pi n x}{L}\right). \tag{1}$$

All of the sin functions above are **odd** functions – that is, they obey the property that $f(-x) = -f(x)\ \forall x$. This means that any linear combination of these functions will also be odd. If we were instead to take linear combinations of $\cos(n\pi x/L)$ functions, we would again get a function with period L but now it would be an **even** function – one that obeys $f(-x) = f(x)\ \forall x$. There is one more type of function we wish to consider that is even and (trivially) periodic: the constant function $f(x) = a$.

We will return to these periodic functions shortly, but first we will demonstrate a very general fact about odd and even functions. Consider a function f that is neither odd nor even. That is, there is neither rotational symmetry nor reflection symmetry in the graph of the function. Now let

$$f_{\mathrm{e}}(x) = \frac{f(x) + f(-x)}{2} \quad \text{and} \quad f_{\mathrm{o}}(x) = \frac{f(x) - f(-x)}{2}. \tag{2}$$

Then it is easy to verify that f_{e} is even and f_{o} is odd, and also that $f(x) = f_{\mathrm{e}}(x) + f_{\mathrm{o}}(x)\ \forall x$. That is, we can express **any** function, f, as a (unique) sum of an even function and an odd function.

22.2 Fourier Series

We can show that the set of all functions of period L is a vector space. For example, adding any two of these functions pointwise (defined as as $(f+g)(x) = f(x) + g(x)\ \forall x$) gives a new function that also has period L. Similarly, multiplying a period-L function by a constant value λ gives back another period-L function. Notice also that the zero function, $z(x) = 0\ \forall x$ is trivially a member of this set and obeys $f + z = f\ \forall f$. Likewise, we can verify that the other properties of a vector space are also satisfied.

Recall we previously showed that the set of monomials made a suitable basis for the vector space of functions with a finite domain, allowing us to write any such function as a Taylor series. We can show that a suitable basis for the set of functions with period L is given by the set

$$\{1, \sin(2\pi n x/L), \cos(2\pi n x/L)\}\ \ \forall n \in \mathbb{N}\,. \tag{3}$$

Just as we did with Taylor series, rather than prove this directly, we will assume it is true and then determine what the coefficients would need to be for this to work. That is, let us assume that the periodic function $f(x)$ can be written as a series of the form:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nx}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nx}{L}\right) , \tag{4}$$

with coefficients a_n for even contributions and b_n for odd contributions. The factor of 2 in the first term is included by convention, since it simplifies some of the following expressions. This representation of the function is called a **Fourier series**.

22.3 Dirichlet Conditions

As well as periodicity, there are a few other conditions that must be met for a function to be expressible as a Fourier series. Together, these are known as the Dirichlet conditions:

1. The function must be periodic
2. The function must have a finite number of discontinuities per period (this can be 0)
3. The function must have a finite number of maxima and minima per period
4. The function must be absolutely integrable over a period:

$$\int_0^L |f(x)| \,\mathrm{d}x \in \mathbb{R} \tag{5}$$

In practice, we will almost never need to consider functions that do not meet these conditions.

22.4 Calculating Coefficients: the Inversion Formulae

When we derived an expression for the coefficients of the Taylor series, we manipulated the general expression to set all but one term to 0. We wish to do something similar to determine the coefficients of the Fourier series. To do so, we need to make use of the following identities:

$$\begin{aligned}
\int_0^L \sin\left(\frac{2\pi nx}{L}\right) \mathrm{d}x &= 0 \;\forall n \\
\int_0^L \cos\left(\frac{2\pi nx}{L}\right) \mathrm{d}x &= 0 \;\forall n \\
\int_0^L \sin\left(\frac{2\pi mx}{L}\right) \cos\left(\frac{2\pi nx}{L}\right) \mathrm{d}x &= 0 \;\forall m \forall n \\
\int_0^L \sin\left(\frac{2\pi mx}{L}\right) \sin\left(\frac{2\pi nx}{L}\right) \mathrm{d}x &= \frac{L}{2}\delta_{mn} \;\forall m \forall n \\
\int_0^L \cos\left(\frac{2\pi mx}{L}\right) \cos\left(\frac{2\pi nx}{L}\right) \mathrm{d}x &= \frac{L}{2}\delta_{mn} \;\forall m \forall n \,,
\end{aligned} \tag{6}$$

where δ_{mn} is the Kronecker delta symbol, defined by

$$\delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases} . \tag{7}$$

Notice that the last two expressions above are just a concise way of saying that

$$
\begin{aligned}
\int_0^L \sin\left(\frac{2\pi m x}{L}\right)\sin\left(\frac{2\pi n x}{L}\right)\mathrm{d}x = 0 \quad \text{for} m \neq n \\
\int_0^L \sin^2\left(\frac{2\pi n x}{L}\right)\mathrm{d}x = \frac{L}{2} \\
\int_0^L \cos\left(\frac{2\pi m x}{L}\right)\cos\left(\frac{2\pi n x}{L}\right)\mathrm{d}x = 0 \quad \text{for} m \neq n \\
\int_0^L \sin^2\left(\frac{2\pi n x}{L}\right)\mathrm{d}x = \frac{L}{2}
\end{aligned}
\tag{8}
$$

This is a specific case of a more general idea: the sin and cos functions are **orthogonal** in the vector space of periodic functions with period L. When dealing with spaces of functions, the inner product (the generalisation of the scalar product) often takes the form

$$\langle f \mid g\rangle = \int f^*(x)g(x)\mathrm{d}x\,. \tag{9}$$

You will meet this idea again in different contexts, but most notably in quantum mechanics.

Multiplying both sides of Eq. 4 by $\sin 2\pi m x L$ for a chosen value of $m > 0$ and integrating over a period, then by the above identities, we have

$$
\begin{aligned}
\int_0^L f(x)\sin\left(\frac{2\pi m x}{L}\right)\mathrm{d}x &= \frac{a_0}{2}\int_0^L \sin\left(\frac{2\pi m x}{L}\right)\mathrm{d}x \\
&\quad + \sum_{n=1}^{\infty} a_n \int_0^L \sin\left(\frac{2\pi m x}{L}\right)\cos\left(\frac{2\pi n x}{L}\right)\mathrm{d}x \\
&\quad\quad + \sum_{n=1}^{\infty} b_n \int_0^L \sin\left(\frac{2\pi m x}{L}\right)\sin\left(\frac{2\pi n x}{L}\right)\mathrm{d}x \\
&= 0 + \sum_{n=1}^{\infty} 0 + \sum_{n=1}^{\infty} b_n \frac{L}{2}\delta_{mn} \\
&= \frac{b_m L}{2}\,,
\end{aligned}
\tag{10}
$$

so we can identify

$$b_m = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{2\pi m x}{L}\right)\mathrm{d}x\,. \tag{11}$$

In a similar fashion, we can show that

$$a_0 = \frac{2}{L}\int_0^L f(x)\mathrm{d}x \tag{12}$$

and, for $m > 0$,

$$a_m = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{2\pi m x}{L}\right)\mathrm{d}x\,. \tag{13}$$

Since $\cos(0) = 1$ and $\sin(0) = 0$, actually we can summarise all of Fourier series coefficients as

$$
\begin{aligned}
a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{2\pi n x}{L}\right)\mathrm{d}x \\
b_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{2\pi n x}{L}\right)\mathrm{d}x
\end{aligned}
\tag{14}
$$

for all $n \geq 0$.

22.5 Example

Find the Fourier series for the function $f(x)$ where f is periodic with period 2 defined by

$$f(x) = x^2 \quad \text{for} \quad x \in [-1, 1]\,. \tag{15}$$

22.6 Shorcuts

The following facts often make the calculation of Fourier coefficients simpler:

1. An even function multiplied by an even function is even
2. An odd function multiplied by even function is odd
3. An odd function multiplied by an odd function is even
4. A symmetric integral of an odd function about the origin is 0
5. A symmetric integral of an even function about the origin is twice the value of a one-sided integral
6. Since the functions we are interested in are periodic, we can choose a different period to integrate over

We will not prove these facts here, but they are reasonably straightforward to prove, so you should convince yourself that they are true.

Algebraically, we have

$$\begin{aligned}
(\text{even} \times \text{even}) &= \text{even} \\
(\text{even} \times \text{odd}) &= \text{odd} \\
(\text{odd} \times \text{odd}) &= \text{even} \\
\int_{-a}^{a} (\text{odd})\, \mathrm{d}x &= 0 \\
\int_{-a}^{a} (\text{even})\, \mathrm{d}x &= 2\int_{0}^{a} (\text{even})\, \mathrm{d}x \\
\int_{0}^{L} f(x) \ldots \mathrm{d}x &= \int_{a}^{a+L} f(x) \ldots \mathrm{d}x
\end{aligned} \tag{16}$$

To demonstrate the utility of the above statements, let's calculate another example.

22.7 Example

Consider the square wave described by

$$f(x) = \begin{cases} -1 & -1 < x < 0 \\ 1 & 0 < x < 1 \end{cases} \tag{17}$$

with period 2, as shown in Fig. 3.

Find the Fourier series describing this wave. To simplify the calculation, we first recognise that this is an odd function. Therefore, we know immediately that $a_n = 0$ for all n. We can also simplify the calculation of b_n using

$$b_n = 2\int_0^1 f(x) \sin(\pi n x)\, \mathrm{d}x\,. \tag{18}$$

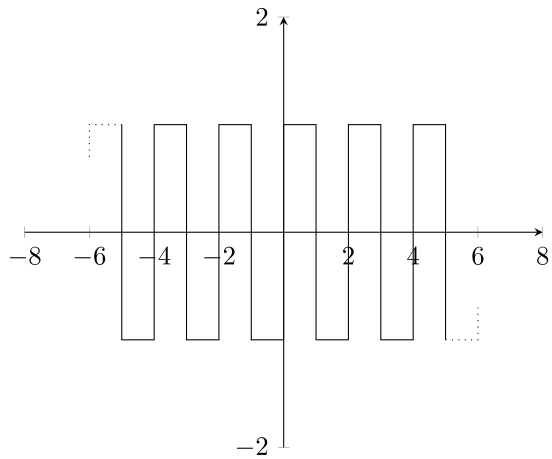


Figure 3: A square wave

22.8 Alternative Forms for Inversion Formulae

Using symmetry arguments, we can demonstrate that all of the following are valid forms of the inversion formulae.

$$
\begin{aligned}
a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{2\pi n x}{L}\right)\mathrm{d}x \\
&= \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\cos\left(\frac{2\pi n x}{L}\right)\mathrm{d}x \\
&= f(x)\frac{4}{L}\int_0^{\frac{L}{2}} f(x)\cos\left(\frac{2\pi n x}{L}\right)\mathrm{d}x \quad \text{(even } f\text{)}
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{2\pi n x}{L}\right)\mathrm{d}x \\
&= \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\sin\left(\frac{2\pi n x}{L}\right)\mathrm{d}x \\
&= \frac{4}{L}\int_0^{\frac{L}{2}} f(x)\sin\left(\frac{2\pi n x}{L}\right)\mathrm{d}x \quad \text{(odd } f\text{)}
\end{aligned}
$$

22.9 Why Do We Care About Fourier Series?

A good way to appreciate what a Fourier series is really doing for us is to consider a complicated wave pattern, such as the one shown in Fig. 5a, as might be produced by a musical instrument. In the figure, the horizontal axis is time and the vertical axis represents the local variation in air pressure as the sound wave travels past some point. Calculating the Fourier series for this curve, we find that the coefficients are $b_1 = 1, b_2 = 0.04, b_3 = 0.97, b_4 = 0.1, b_5 = 0.5, b_6 = 0.1, b_7 = 0.25, b_8 = 0.05\ldots$. We could plot these coefficients on a different graph as in Fig. 5b.

This second graph arguably tells us much more about the wave being produced. In this case, the horizontal axis is proportional to the **frequency** of the contribution to the wave and the vertical axis is proportional to the magnitude of that contribution. Notice that the points on the graph in this case are discrete. Mathematically, that must be the case, since the Fourier series has discrete terms indexed by integers. However, we now also have a physical intuition for why this must be the case: the contributions to the musical instrument's overall waveform are **harmonics** of the fundamental wave.

Notice that frequency is the inverse of time, so finding the Fourier coefficients has transformed us to a **reciprocal domain**. Notice also that all of the same information is still present, since we can always write out the Fourier series and reproduce the original function. So another way to think of this transformation is as an alternative way to encode all of the information about a function.

So far, we have only considered functions that are periodic. Suppose we want to understand a time-varying wave that only exists for a finite time. How should we proceed in this case? The answer is simple: cheat! Since the domain is finite, nothing outside the domain matters. So we can just construct a periodic function with period equal to the width of the domain. That way,

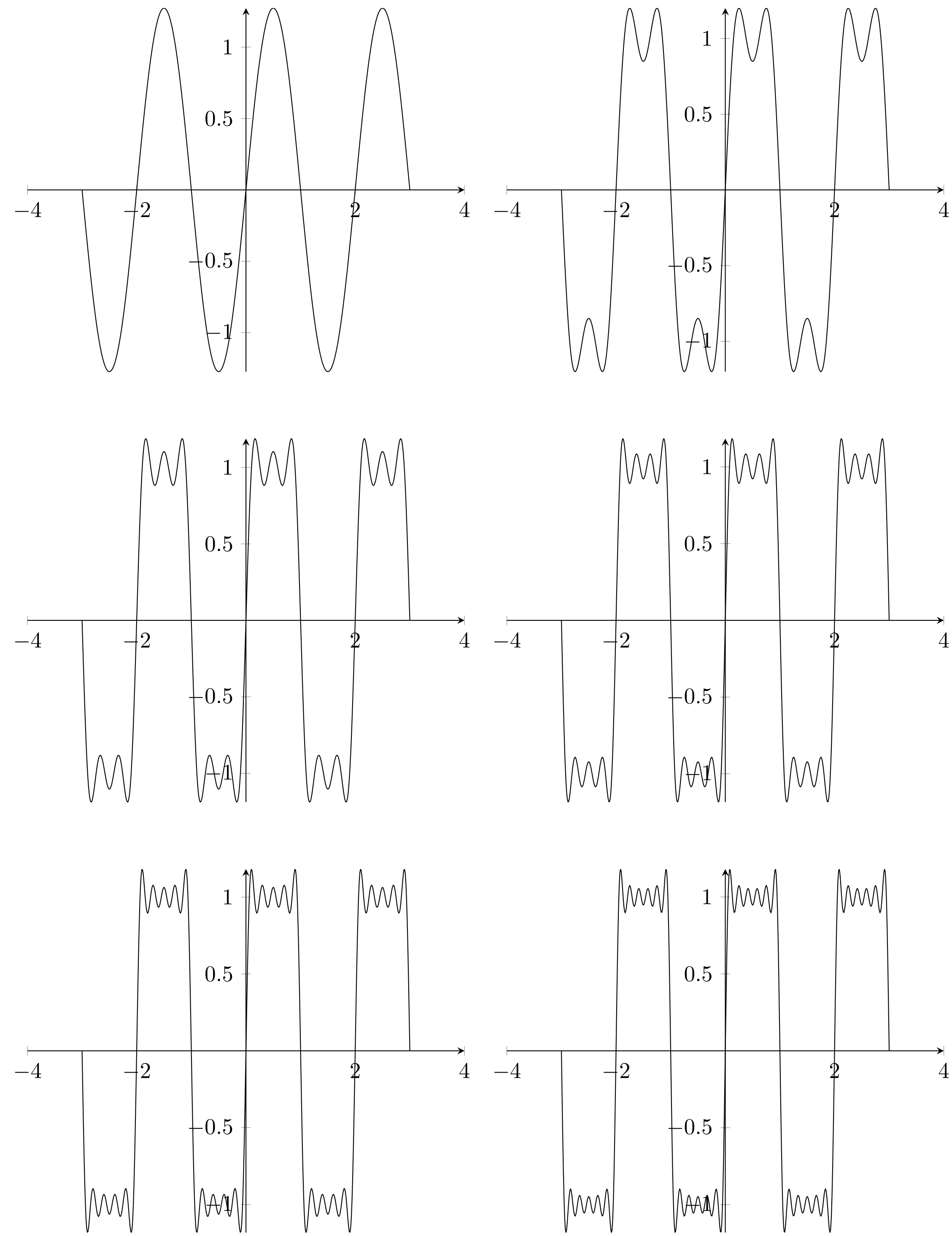
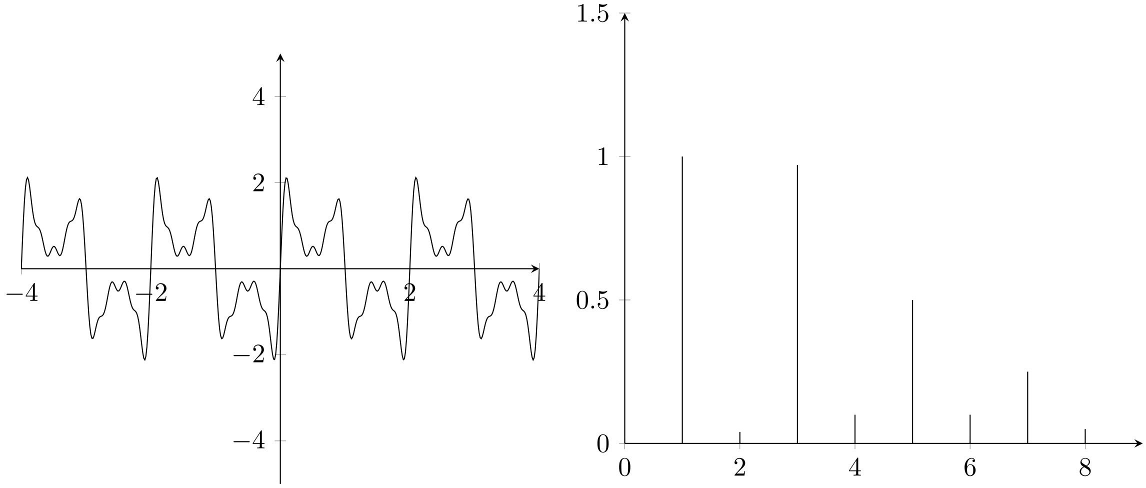


Figure 4: Approximations to a square wave with 1,2,3 and 4 terms of the Fourier series.

we can still use an ordinary Fourier series to understand the behaviour of the function within the domain of interest. We can see this by again considering a musical instrument. Any note played on an instrument only takes up a finite time, yet we can still pull out useful information about the frequency spectrum of the note. When we did this above, there was an implicit assumption that the wave was infinite in duration! Even with no periodicity, we can use the same idea to find the frequency spectrum. In this case, the "period" is the full domain.



(a) A sound wave produced by a musical instrument displayed as air pressure against time

(b) The same sound wave displayed as the contribution of each frequency

So a Fourier series allows us to transform from a domain to a reciprocal domain, such as from time to frequency. However, this works equally well with domains other than time. For example, a variation in brightness from one side of a greyscale image to the other can be represented on a spatial axis. Finding the Fourier coefficients will re-encode all of this information about the brightness into the reciprocal domain. The reciprocal domain is no longer a frequency, however, but a **spatial frequency** proportional to the wavenumber, k. Since the human eye cannot resolve high-frequency changes in images, we can often ignore the Fourier coefficients beyond some threshold. Setting these to 0 and reconstructing the image through the Fourier series will not noticeably affect the resulting image. This idea is used in file compression standards such as JPEG.

22.10 Finite Domains

We have already said that we can extend a function with a finite domain to a periodic function with period equal to the original domain. However, when we do this, we will usually end up with a Fourier series with all terms (odd and even). Depending on the situation, we may sometimes prefer to keep only odd or even terms. In this case, we can take the period to be twice the original domain and extend with either odd or even symmetry. For example, if a function is defined on the domain $[0, a]$ and we know that $f(0) = f(a) = 0$, it may be beneficial to model it in terms of sine functions only. In this case, we can extend first to the domain $[-a, a]$ defining $f(x) = -f(-x)$ for $x \in [-a, 0]$, and only then extending to periodic. This guarantees that all a_n coefficients are 0. We call the resulting Fourier series the **sine series**. Similarly, extending with $f(x) = f(-x)$ for $x \in [-a, 0]$ would give the **cosine series**.

22.11 Exercise

For the function $f : [0, 3] \to \mathbb{R}, \quad f(x) = 1 - x$, graph the

1. even extension
2. odd extension

and find the cosine and sine series.

22.12 Calculus with Fourier Series

Recall that there are some functions for which we cannot write down the integral in a simple closed form. If the function is periodic or has a finite domain, it may be useful to write it as a Fourier series and then integrate. This is because the coefficients of the Fourier series are constant, so the only parts that will be affected by differentiation or integration are the sines and cosines, and we know how to deal with these. While the result will be in the form of an infinite series, for which we still have no neat closed form, often it will be enough to consider only a finite set of terms and neglect higher orders. In addition, Fourier series can be very useful for solving differential equations on finite domains. For example, given a function that describes the shape of a string, we can use Fourier series and the wave equation to determine the later evolution of that string. Notice that if the string is held at both ends, as on a violin, this would be a good time to use the sine series.