UNIVERSITY OF LEEDS

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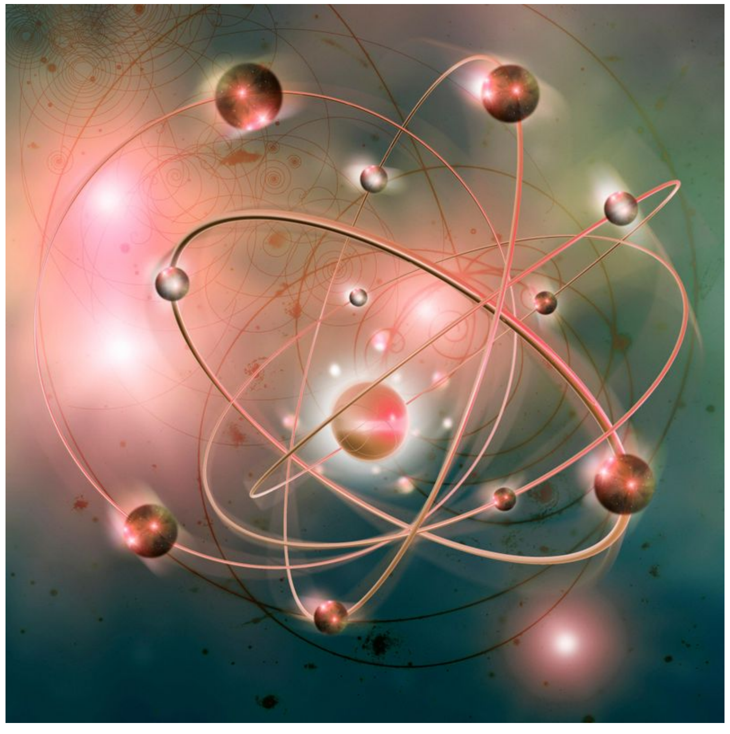
2024/25

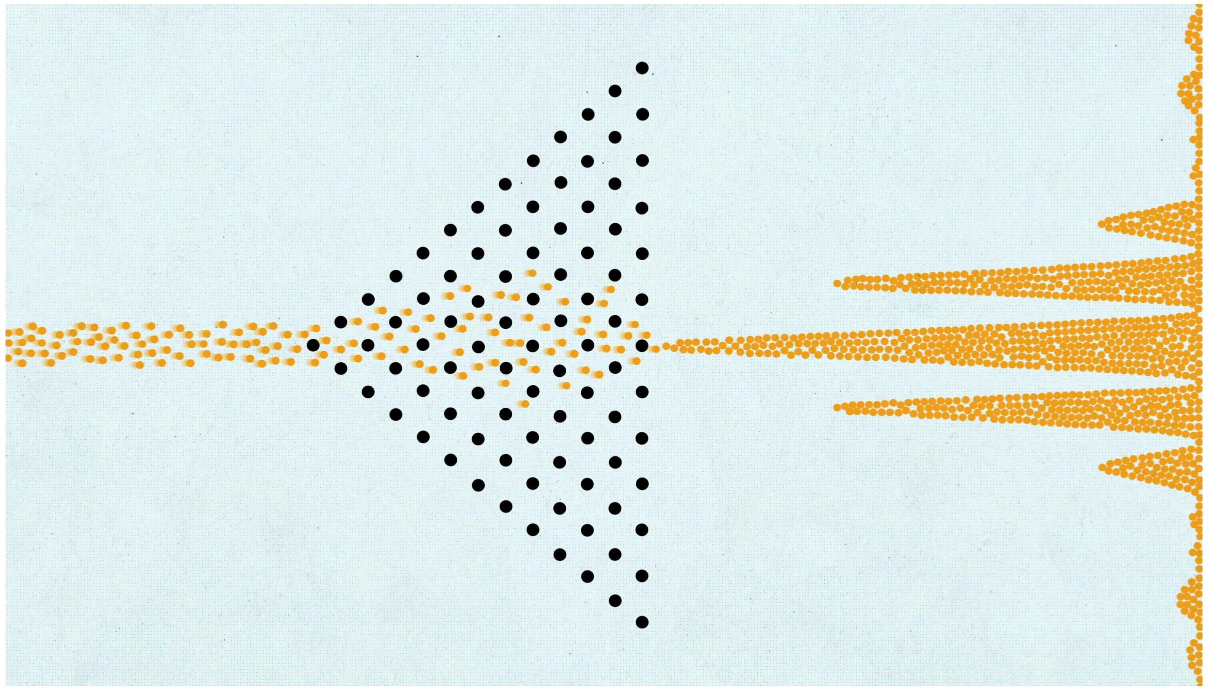
Probability theory



What is probability?

The many faces of probability

- The probability of rain tomorrow is 60%
- The probability of a person being hit by a meteorite is larger than 10^(-90)
  The probability of Leeds University deciding to relocate to Paris is smaller than 0.01%
- The probability of this Hydrogen atom radiating a photon is 0.25



Logical statements

- A = “Scotland wins the 2026 World Cup”
- B= “2+2=4”
- C =“It is raining in Sydney right now ”
- D=“Tyrannosaurus Rex had feathers”
- E=“This patient has Covid”
- F=“The Earth is an infinite flat plane”

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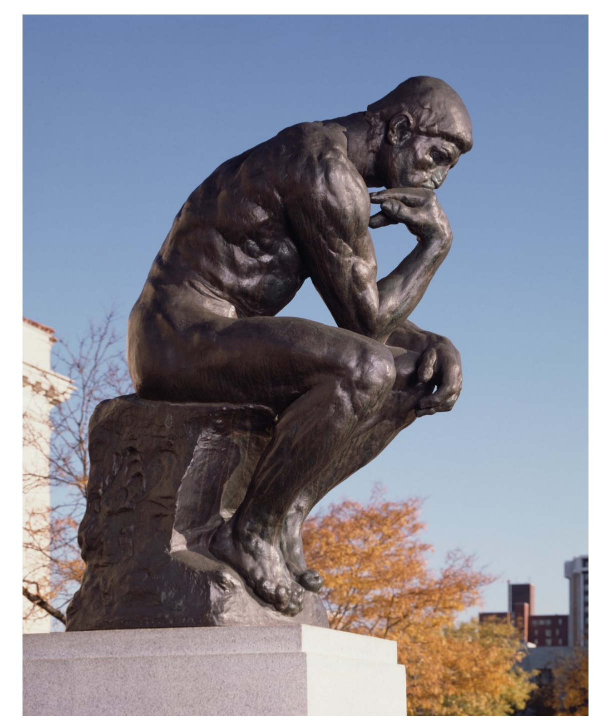
Probability is extended logic

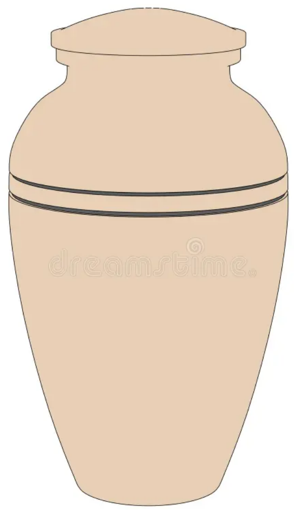
$$A \longrightarrow P(A) = 0.2$$

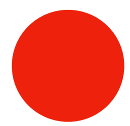
Probability is about <u>logical statements.</u>

P(A) = a measure of a person's confidence that A is **True.**



Example





- You have an urn filled with **100 balls**, some red and some green.
- You can't see inside; all you know is that someone determined the number of red balls by picking a number between 0 and 100 randomly.
- You reach into the urn and pull out a ball. It's red.

If you now pull out a second ball, is it more likely to be:

(1) Red,

(2) Green,

(3) Equally likely?

“To be or not to be”

$A = \text{Scotland wins the 2026 World Cup}$

$\bar{A} = \text{Scotland does not win the 2026 World Cup}$

$$\boxed{P(A) + P(\bar{A}) = 1}$$

$\bar{A} = \text{Logical Negation of } A$

Multiple statements

$$W = \text{ Scotland win the 2026 World Cup}$$
$$F = \text{ Scotland get to the finals of the 2026 World Cup}$$

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$$P(W) = 0.01$$

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Conditional Probability:

$$P(W|F) = P(W \text{ given that } F \text{ occurs})$$

Multiple statements

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$$P(W|F) = 0.5$$

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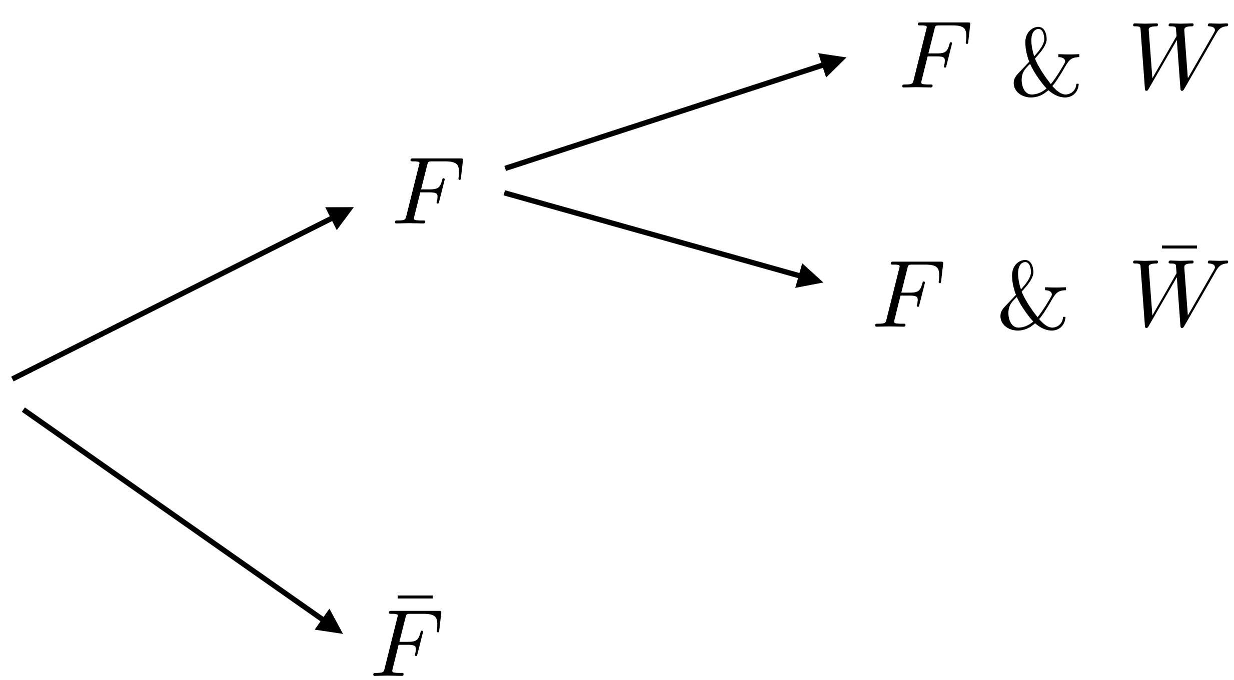
$$P(F|W) = 1$$

Probability tree diagrams

F = “get to finals”
W = “win”



Probability tree diagrams

F = “get to finals”
W = “win”

$$\begin{tikzcd}
 & & F \;\&\; W \\
 & F \arrow[ur] \arrow[dr] & \\
 & & F \;\&\; \bar{W} \\
 {} \arrow[uur] \arrow[ddr] & & \\
 & & \bar{F} \;\&\; W \\
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\end{tikzcd}$$

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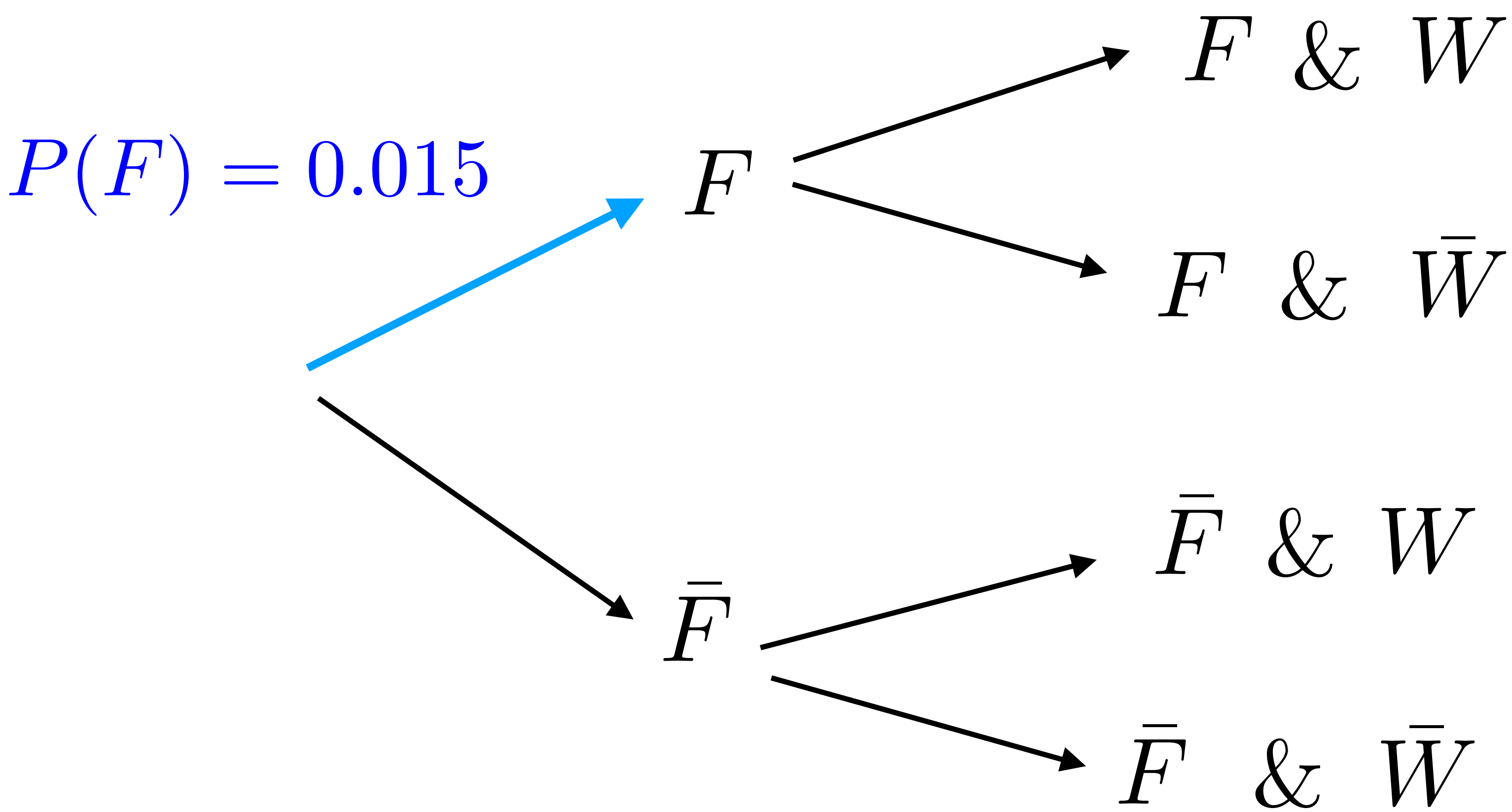

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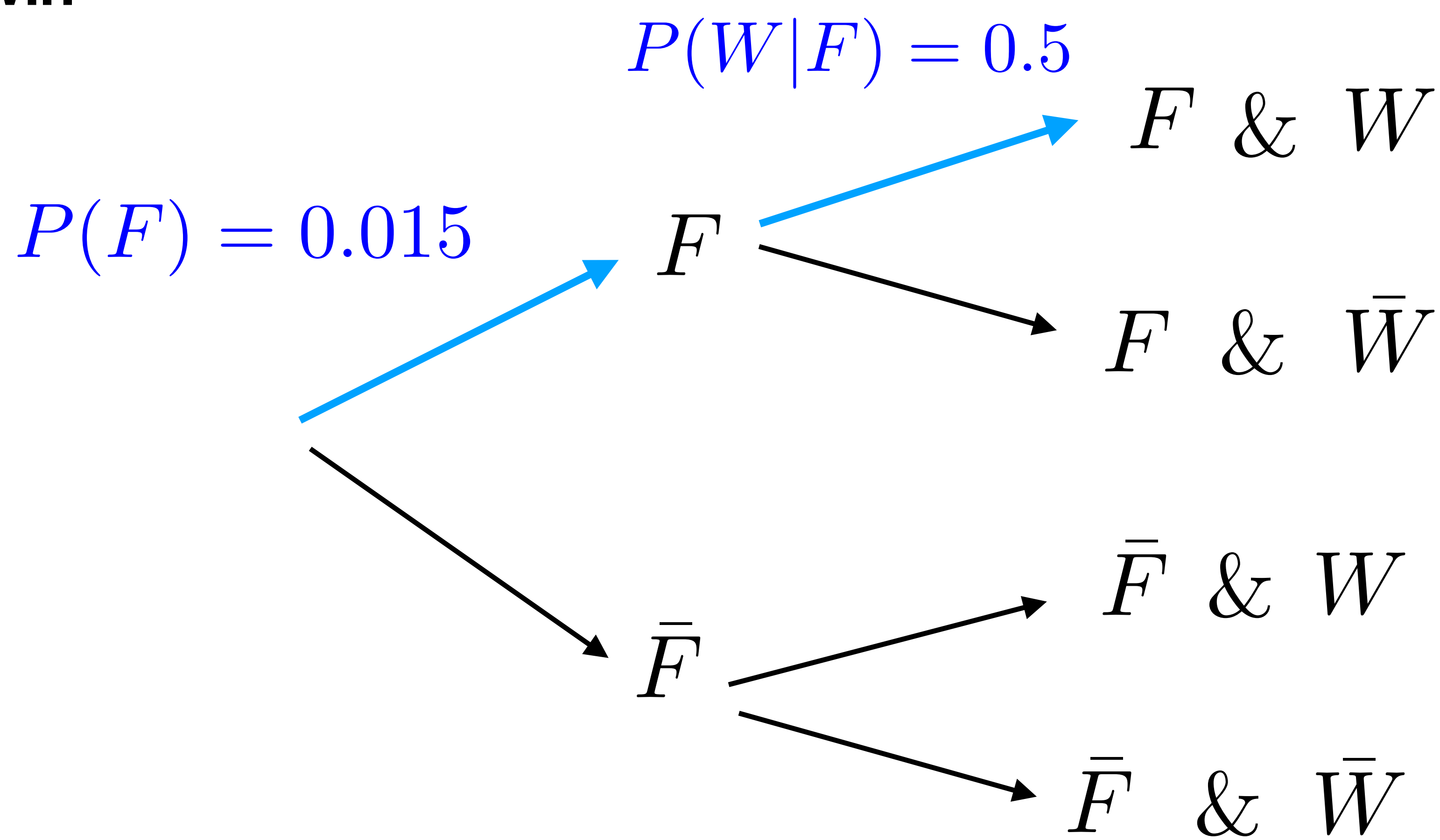
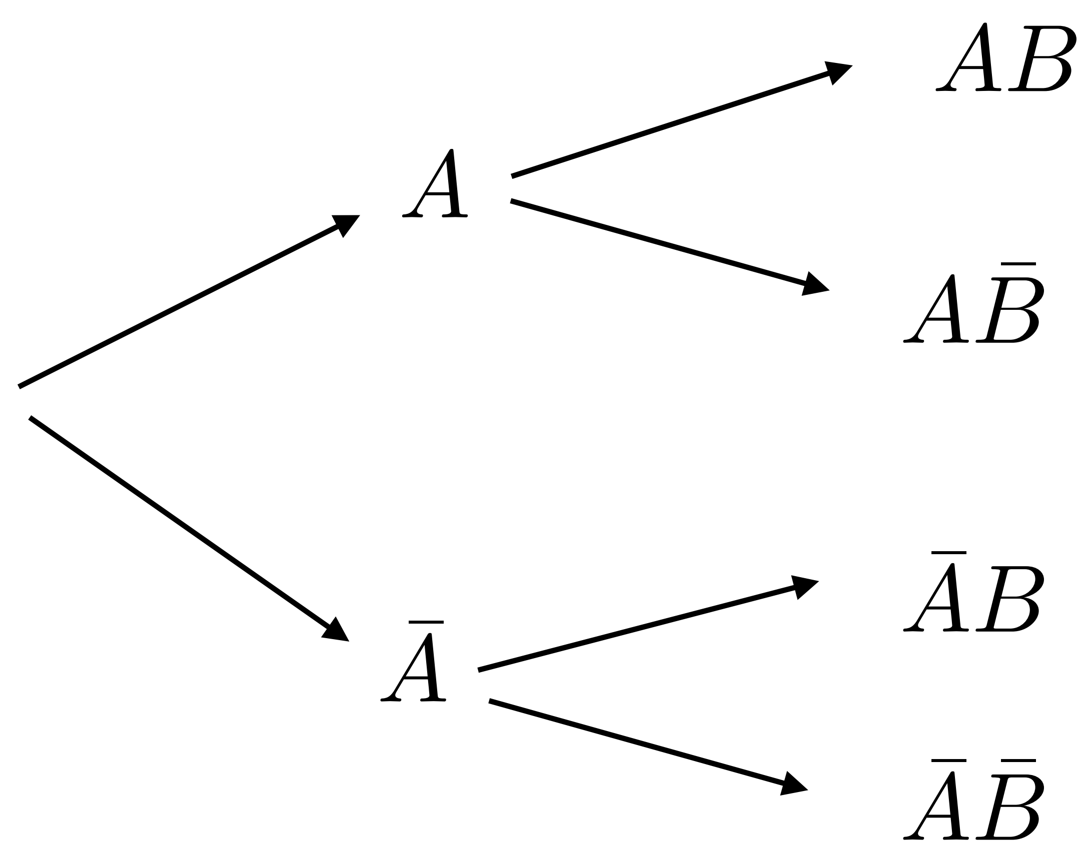


Probability tree diagrams

$AB =$ "A and B"

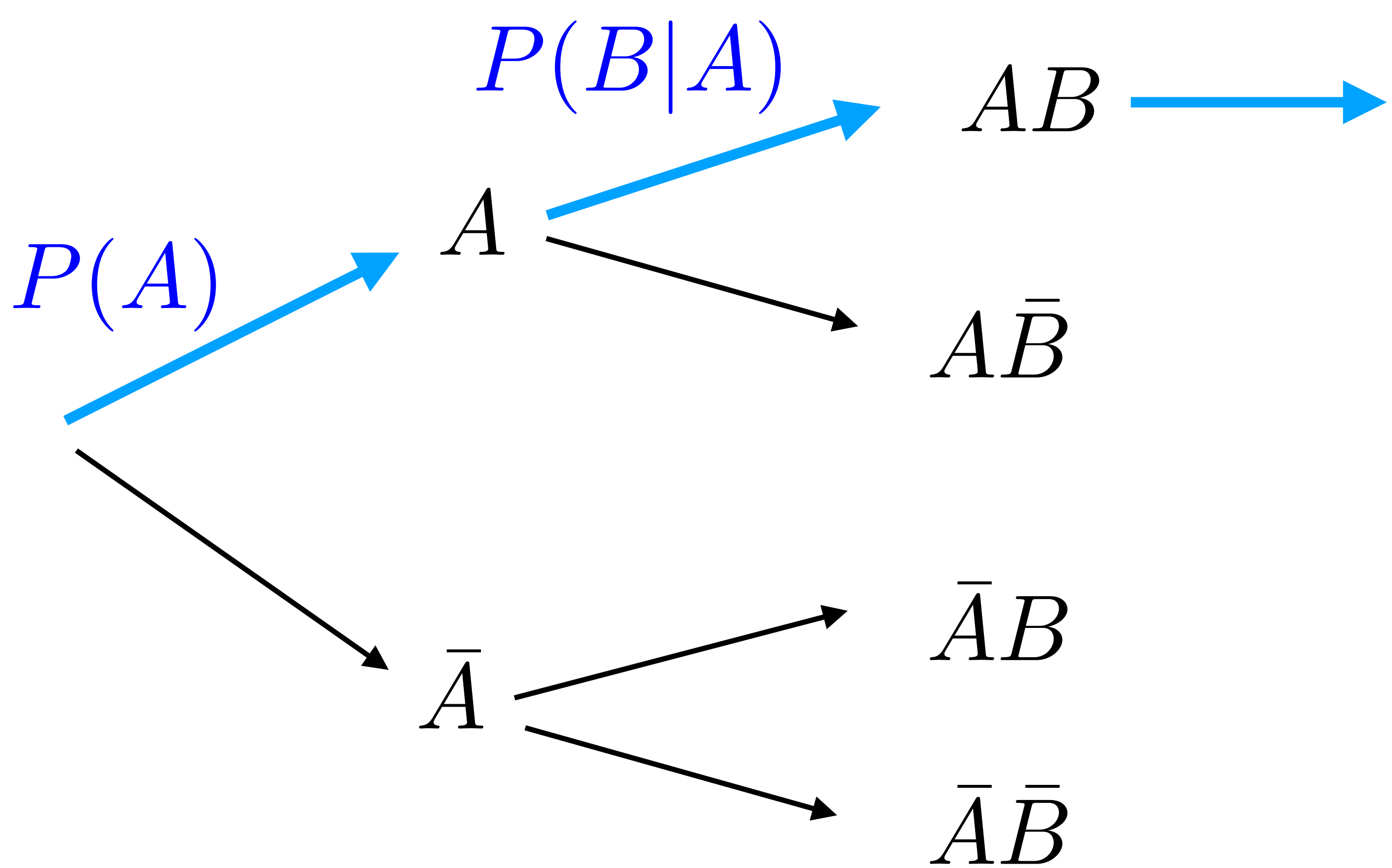
$A\bar{B} =$ "A and NOT B"

etc...



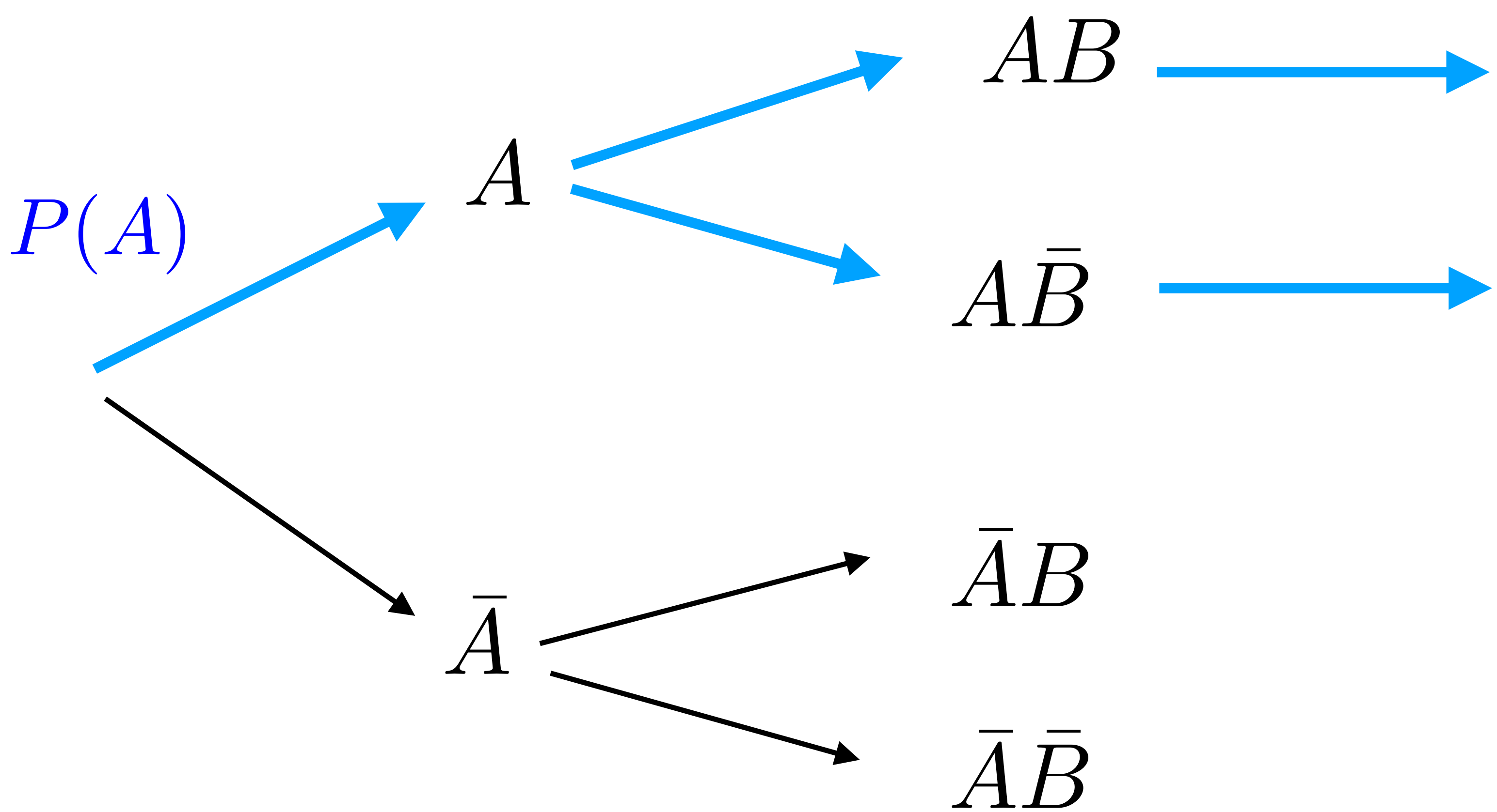
Probability tree diagrams

$$\boxed{P(AB) = P(A)P(B|A)}$$



“Conserved flow of water”

$$\boxed{P(AB) + P(A\bar{B}) = P(A)}$$

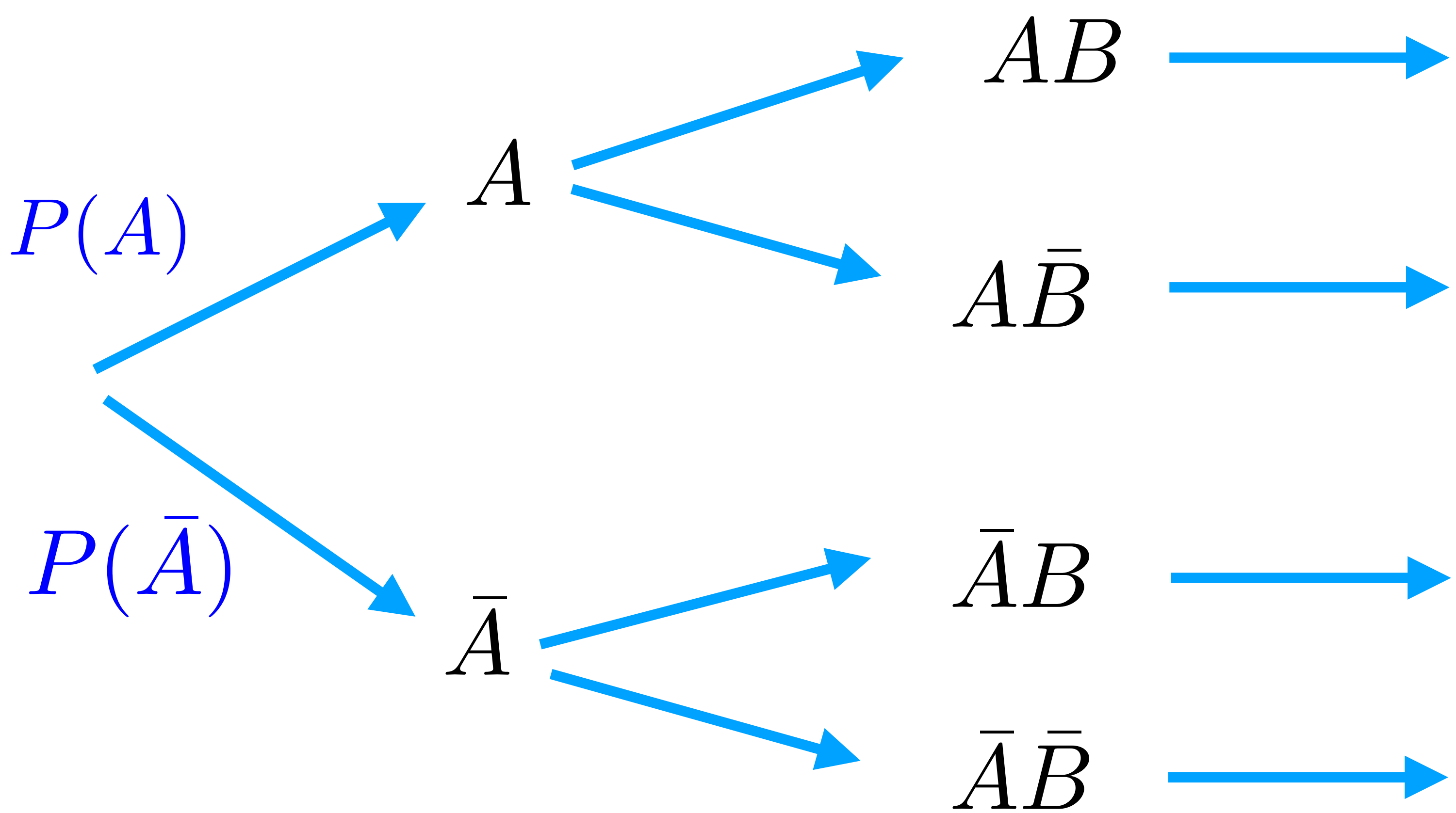


“Conserved flow of water”

$$P(AB) + P(A\bar{B}) + P(\bar{A}B) + P(\bar{A}\bar{B}) = ??$$

“Conserved flow of water”

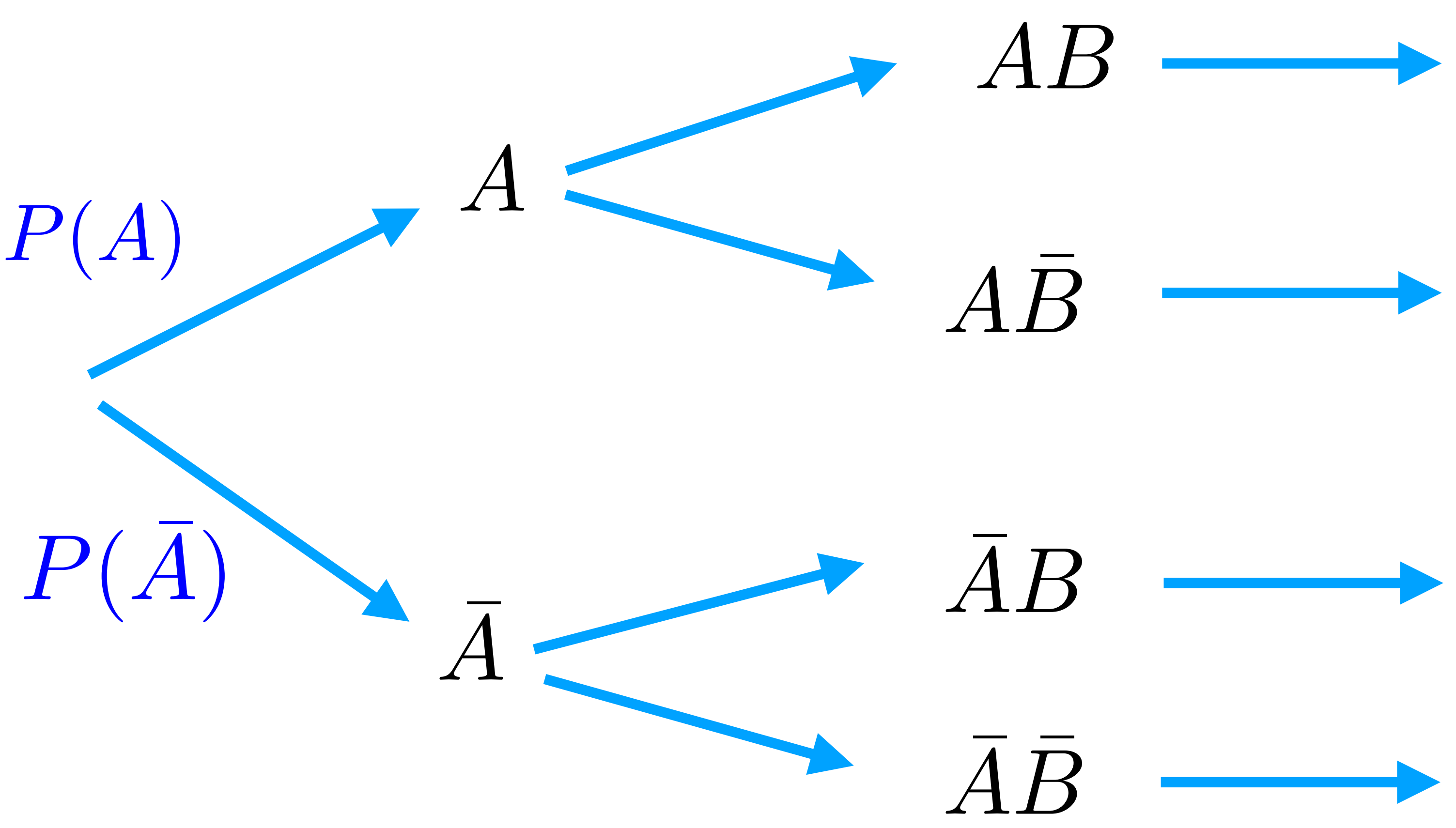
$$P(AB) + P(A\bar{B}) + P(\bar{A}B) + P(\bar{A}\bar{B}) = P(A) + P(\bar{A})$$



“Conserved flow of water”

$$P(AB) + P(A\bar{B}) + P(\bar{A}B) + P(\bar{A}\bar{B}) = P(A) + P(\bar{A})$$

$$= 1$$



The Fundamental Equations of Probability Theory

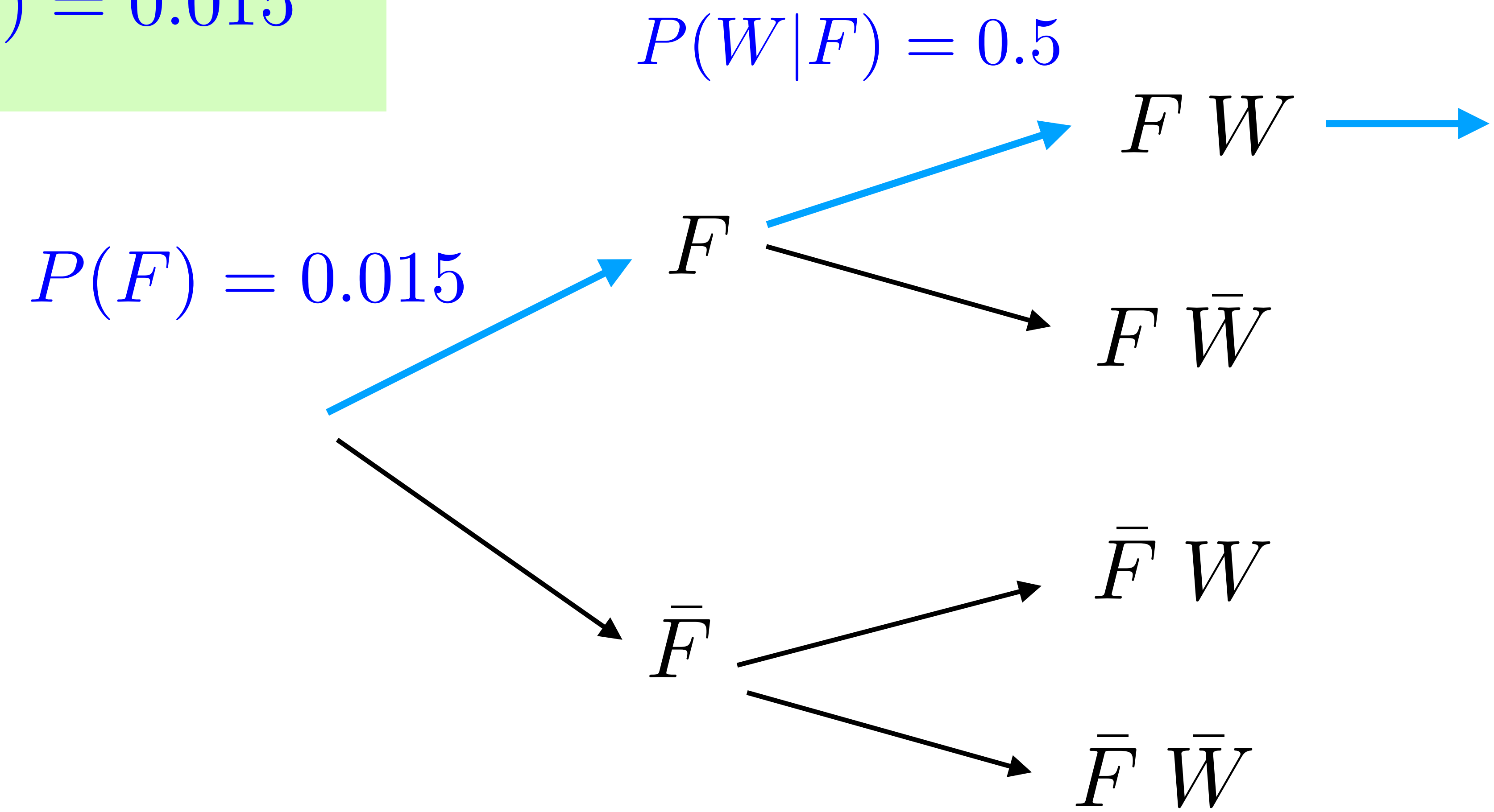
$$\boxed{P(AB) + P(A\bar{B}) = P(A)}$$

"To be or not to be"

$$\boxed{P(AB) = P(A)P(B|A)}$$

Conditional Probability

$$P(W) = 0.01$$
$$P(F) = 0.015$$



$$P(W) = 0.01$$
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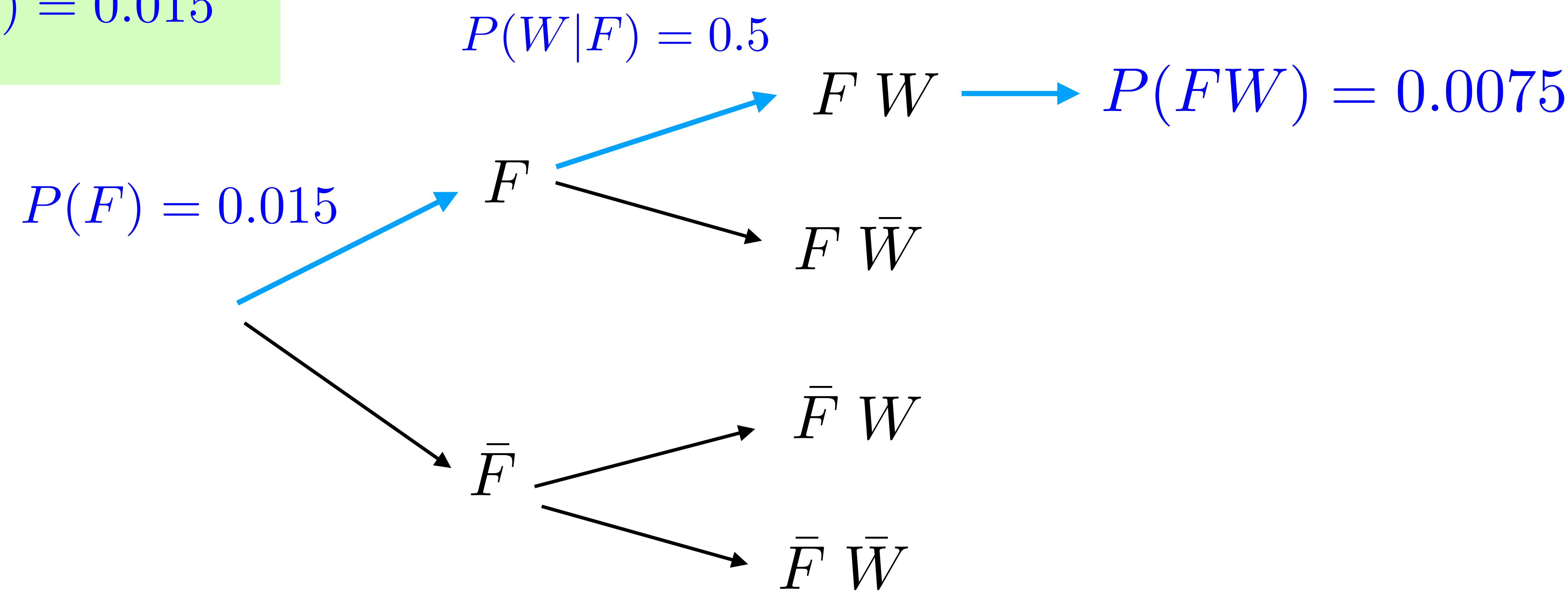
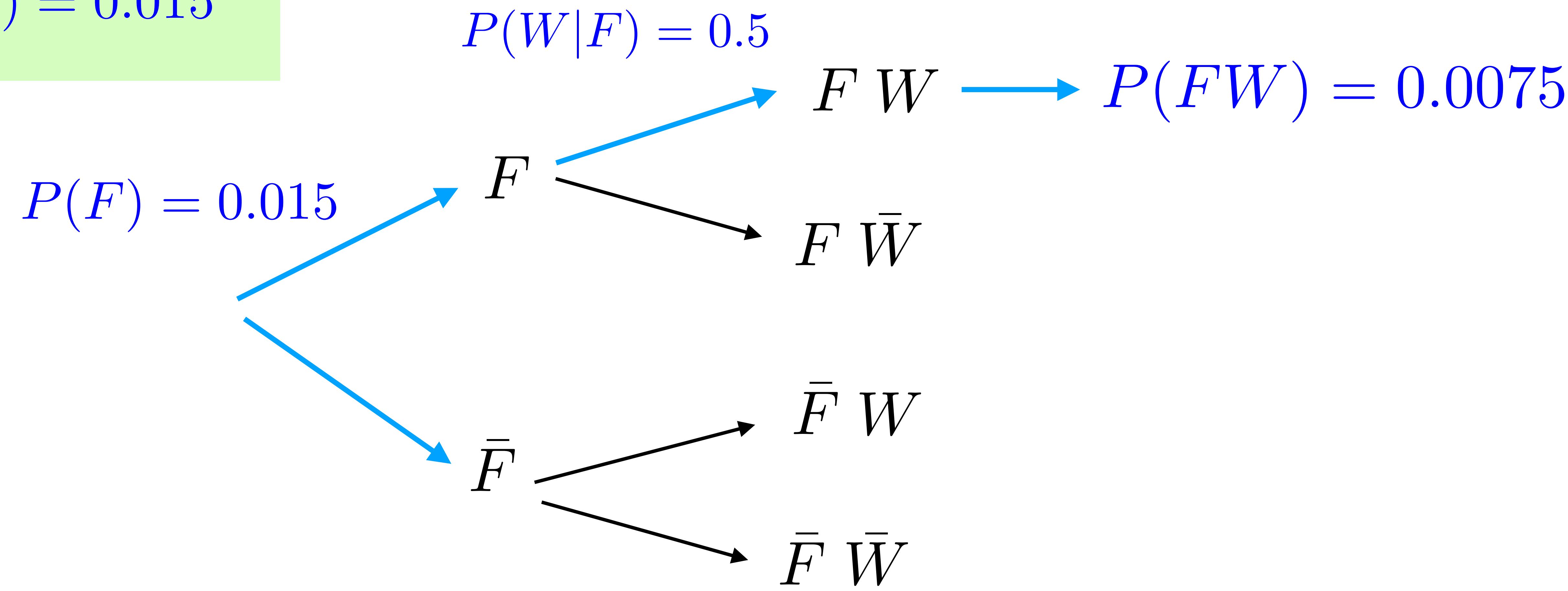


$P(W) = 0.01$

$P(F) = 0.015$



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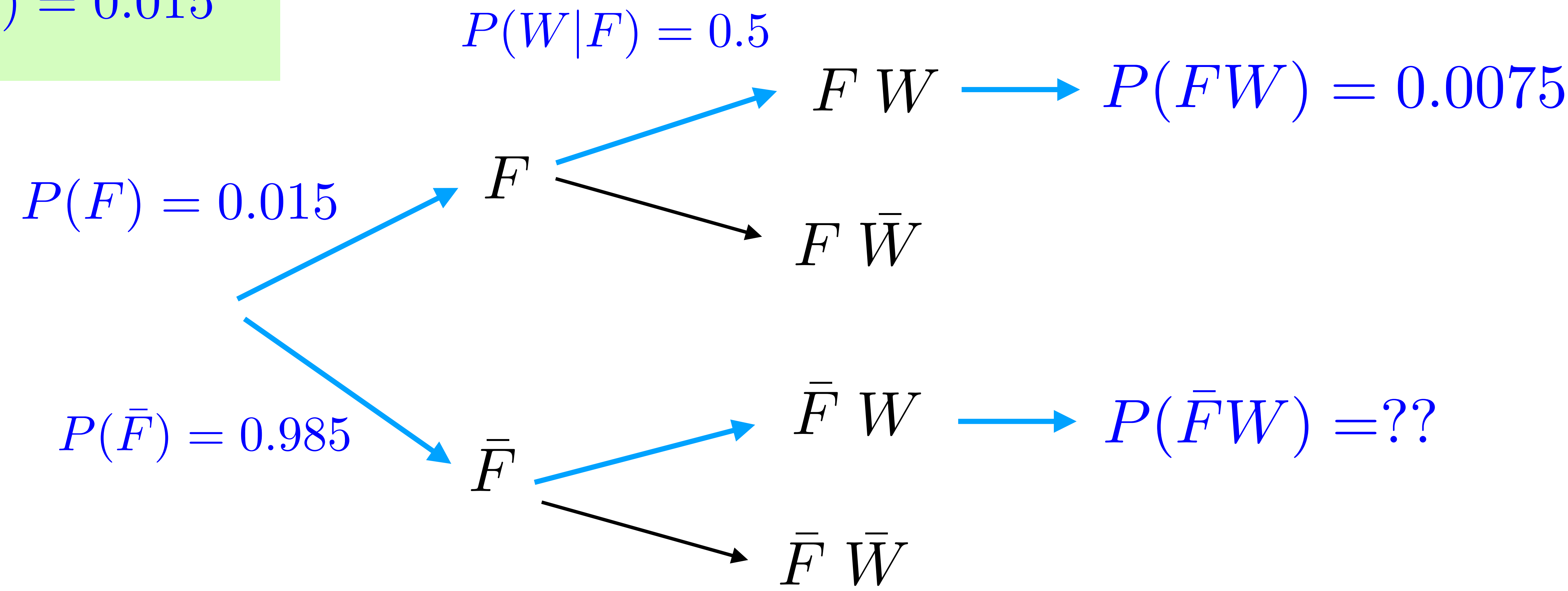

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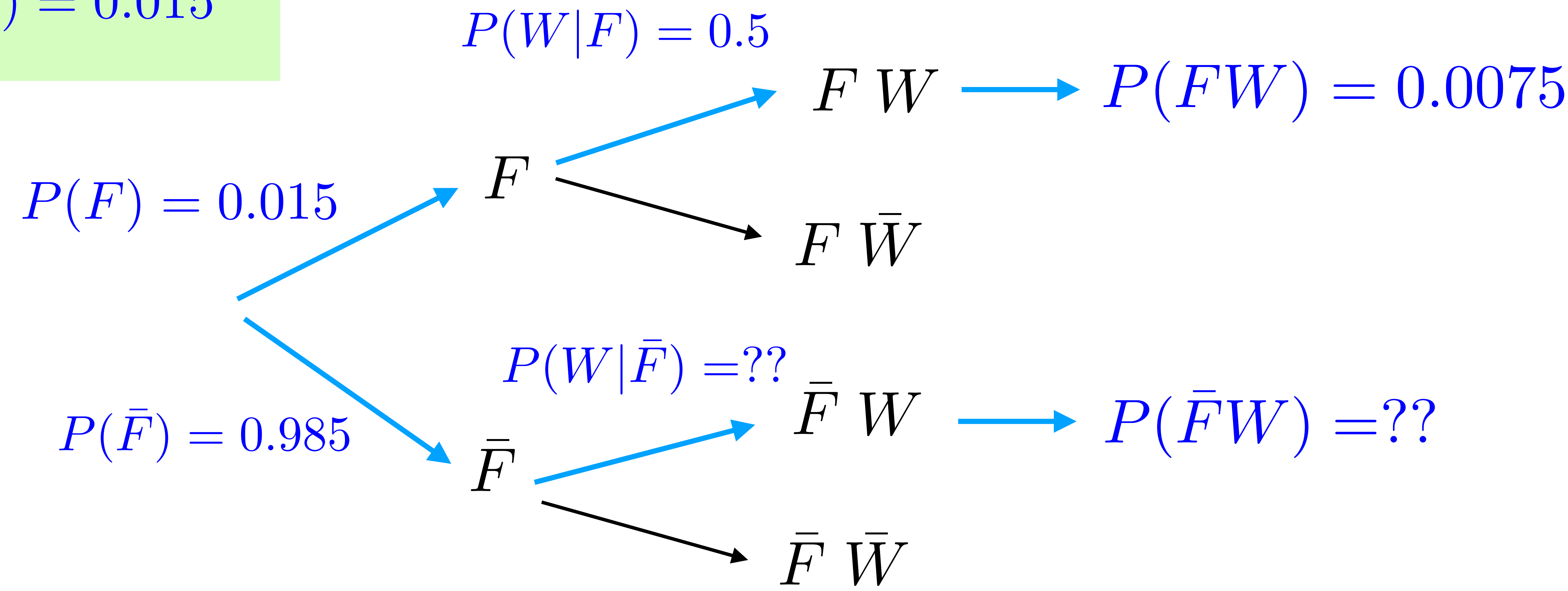

$$P(W) = P(FW) + P(\bar{F}W) = 0.01$$

$$P(W) = 0.01$$
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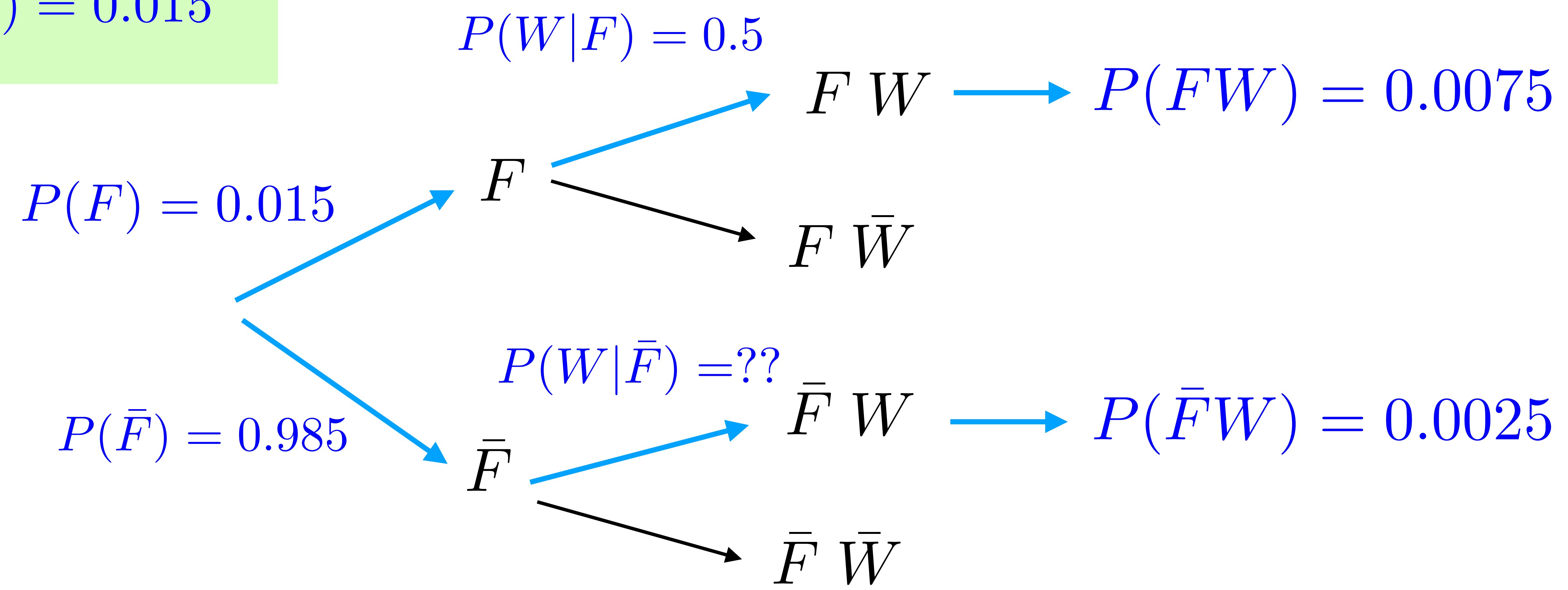
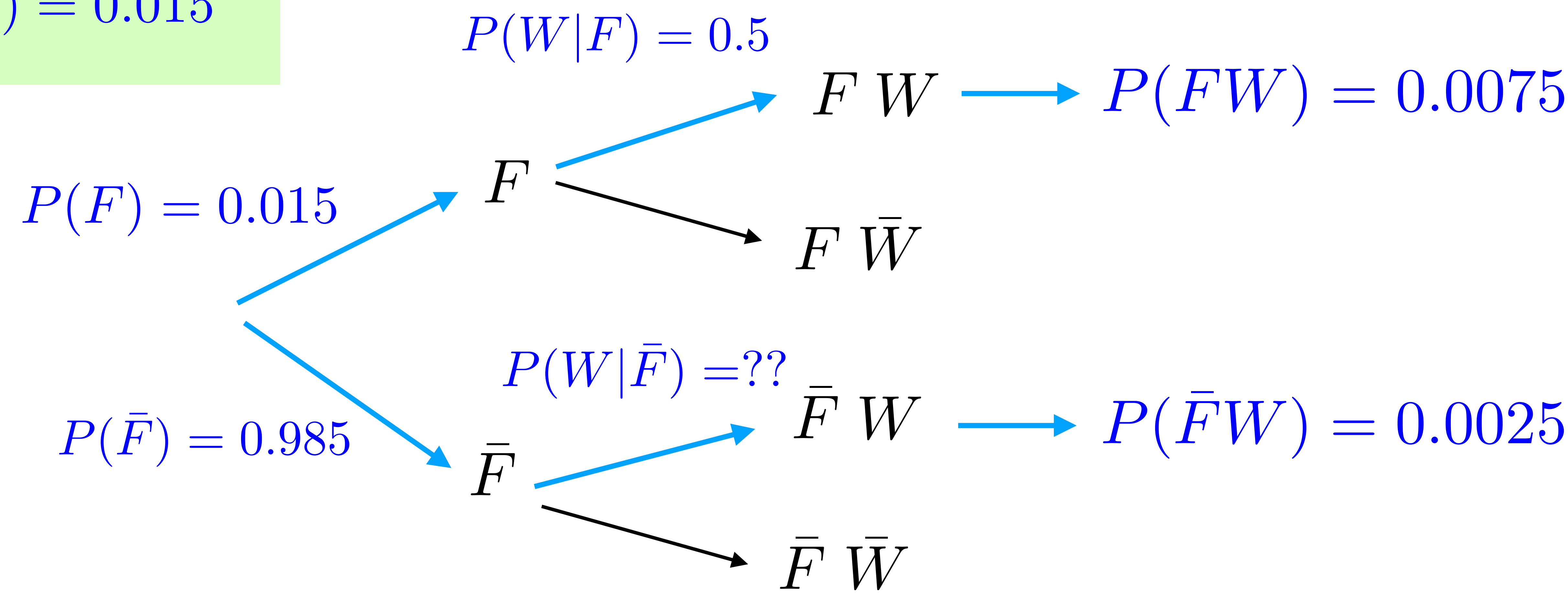


$$P(W) = P(FW) + P(\bar{F}W) = 0.01$$
$$\Rightarrow P(\bar{F}W) = 0.0025$$

$P(W) = 0.01$

$P(F) = 0.015$



$$P(\bar{F}W) = P(W|\bar{F})P(\bar{F}) = 0.0025$$

Being consistent...

$$P(W) = 0.01$$
$$P(F) = 0.015$$

$P(W|F) = 0.5$

$F\,W \longrightarrow P(FW) = 0.0075$

$P(F) = 0.015$

F

$F\,\bar{W}$

$P(W|\bar{F}) = 0.002538071$

$\bar{F}\,W \longrightarrow P(\bar{F}W) = 0.0025$

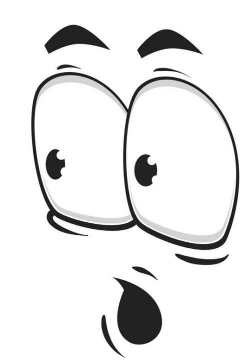
$P(\bar{F}) = 0.985$

$\bar{F}$

$\bar{F}\,\bar{W}$

$$P(\bar{F}W) = P(W|\bar{F})P(\bar{F}) = 0.0025$$

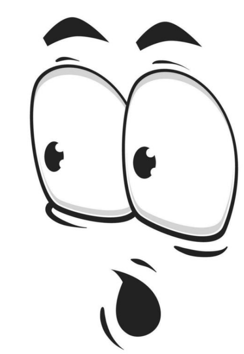
$$\Rightarrow P(W|\bar{F}) = 0.002538071$$



Being consistent is hard

$$W = \text{ Scotland win the 2026 World Cup}$$
$$F = \text{ Scotland get to the finals of the 2026 World Cup}$$

$$P(W) = 0.01$$
$$P(F) = 0.015$$
$$P(W|F) = 0.5$$



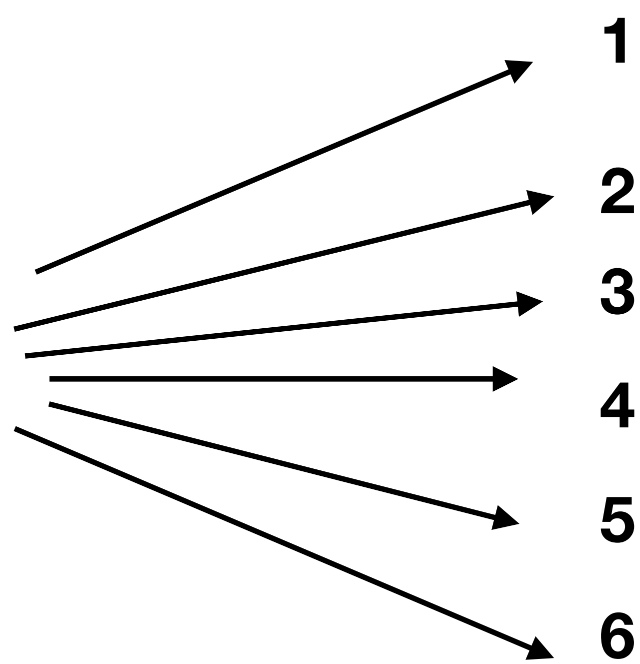
Having these probabilities implies that you expect Scotland to win the World Cup 0.2538% of the time that it <u>doesn't</u> make the final !!

Exercise: prove that $P(F|W) = 1$ is not consistent with the above!

Simpler Examples

Example

We role an unbiased die. What is the probability of getting a prime number?

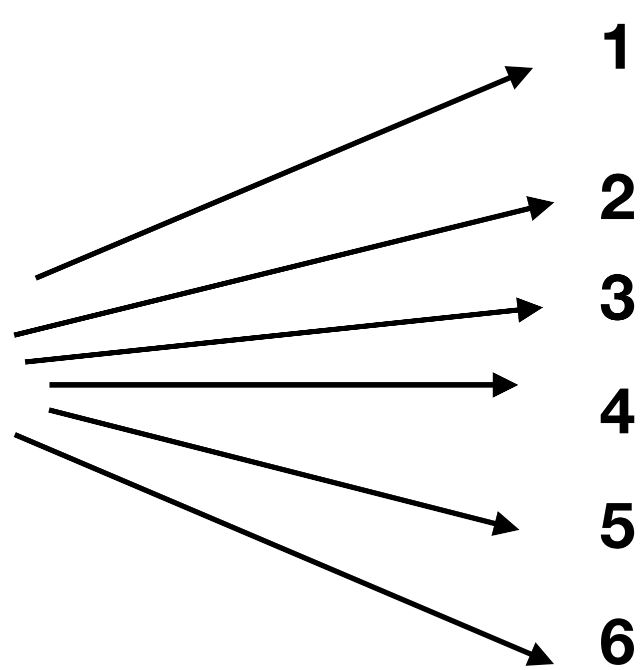


Example

We role an unbiased die. What is the probability of getting a prime number?

P(1)=P(2)=P(3)=P(4)=P(5)=P(6) = 1/6

A = “we get a prime number”
= “We get 2,3, or 5”



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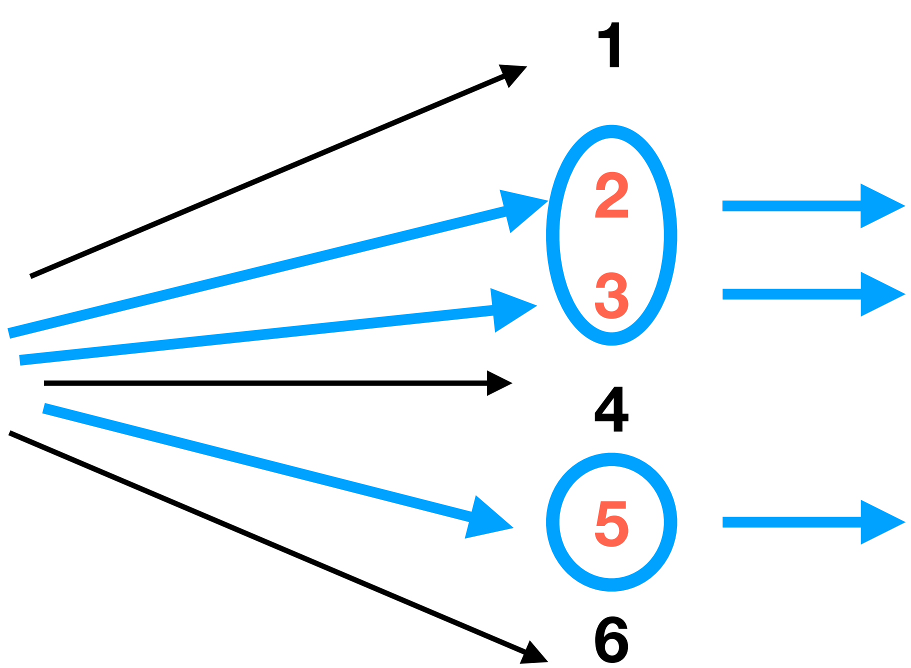


$$P(A) = 1/6 + 1/6 + 1/6 = 3/6$$

Example

A program outputs a **string of 1's and 0's**.
Each bit is *independent* of the other.
For each bit, the probability of a 1 is P(1)=1/2 and the probability of a 0 is P(0)=1/2.

If there are N=100 bits in the string, what is the probability of:

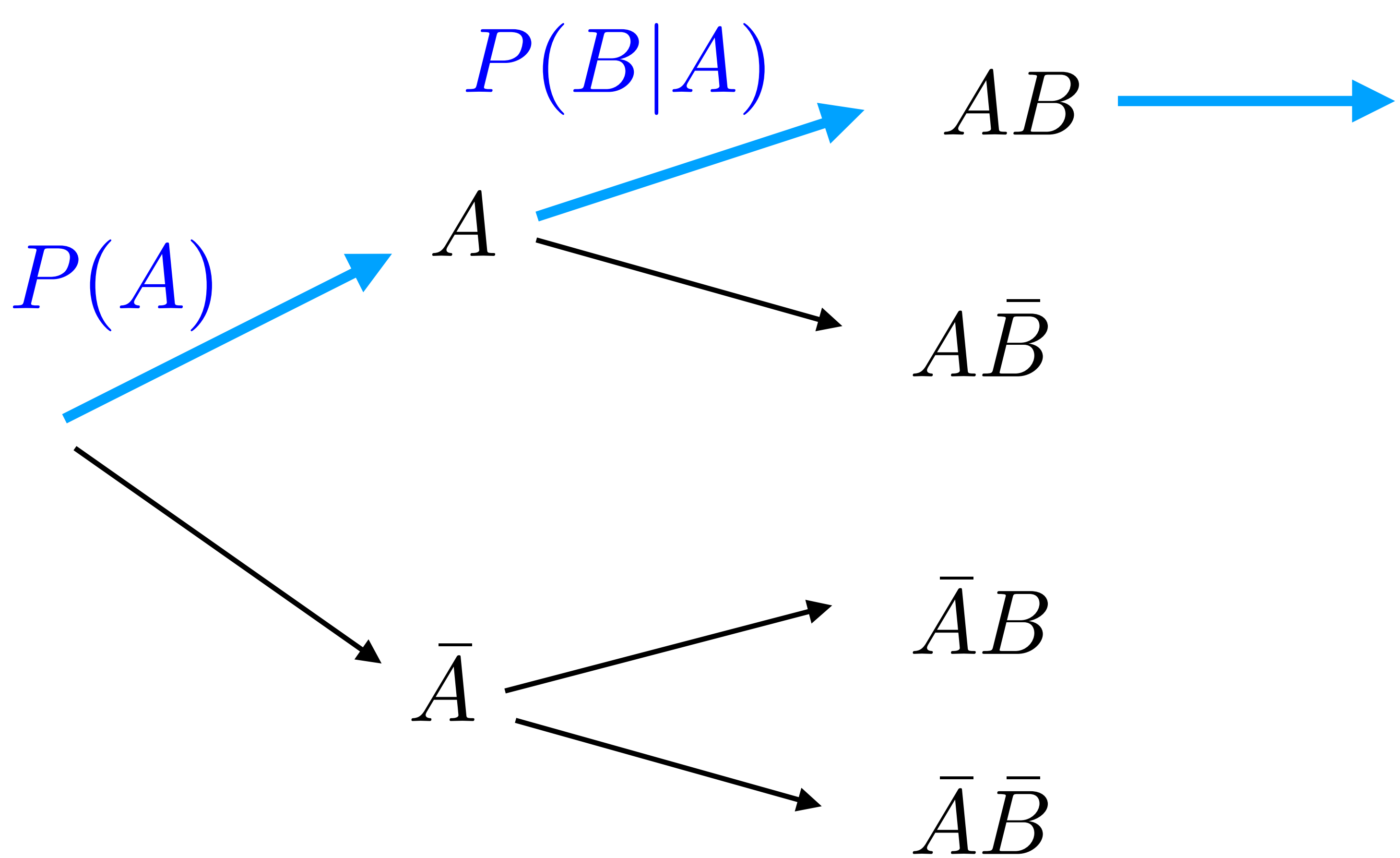
A = The string is 000000….00000
B= The string is 01010101…..01 (alternating 0 and 1)
C = The string has exactly 50 one's in it and 50 zero's.

B independent of A

Definition:

Independent: $P(B|A) = P(B)$

$$\boxed{P(AB) = P(A)P(B|A)}$$



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Definition:

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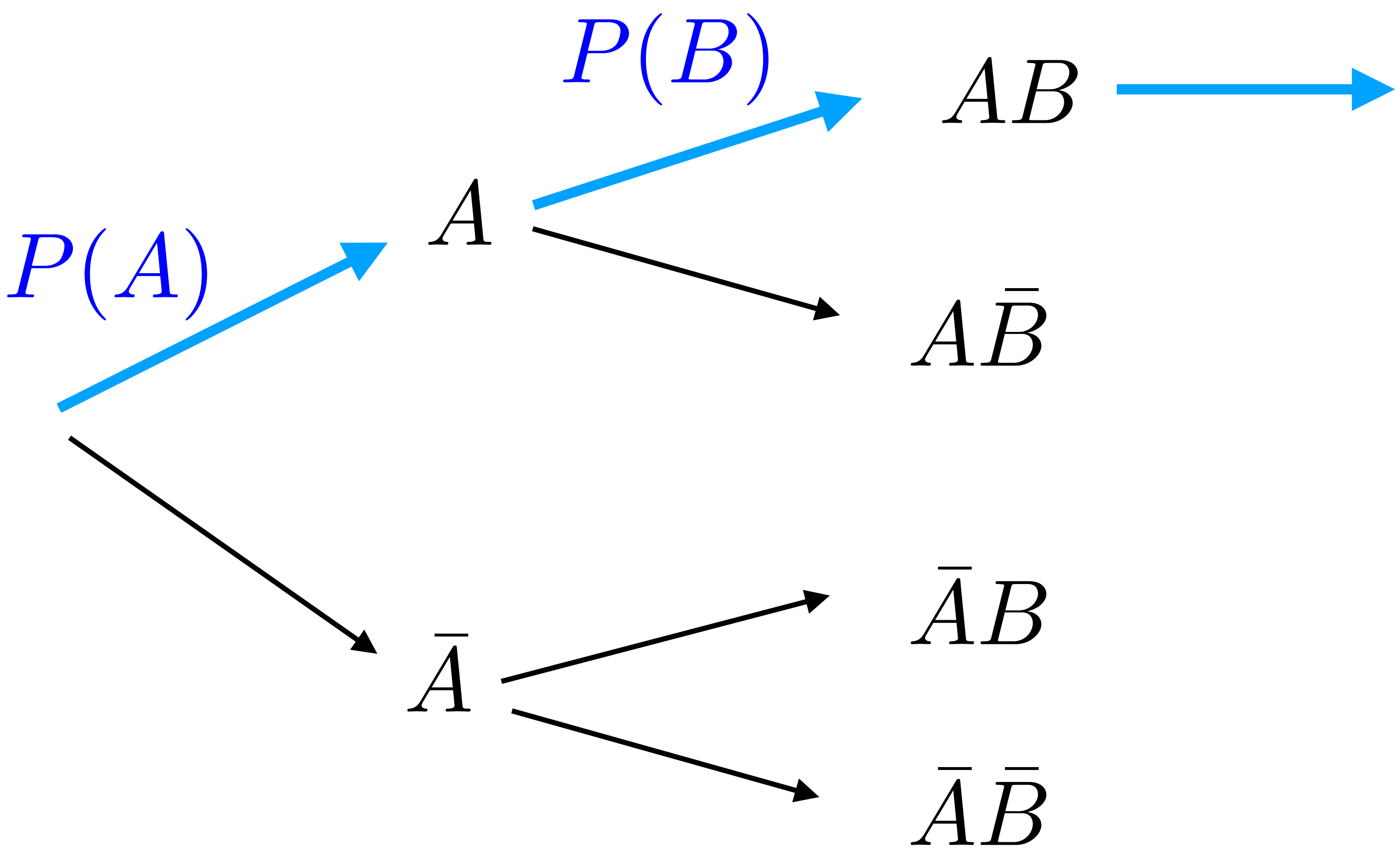


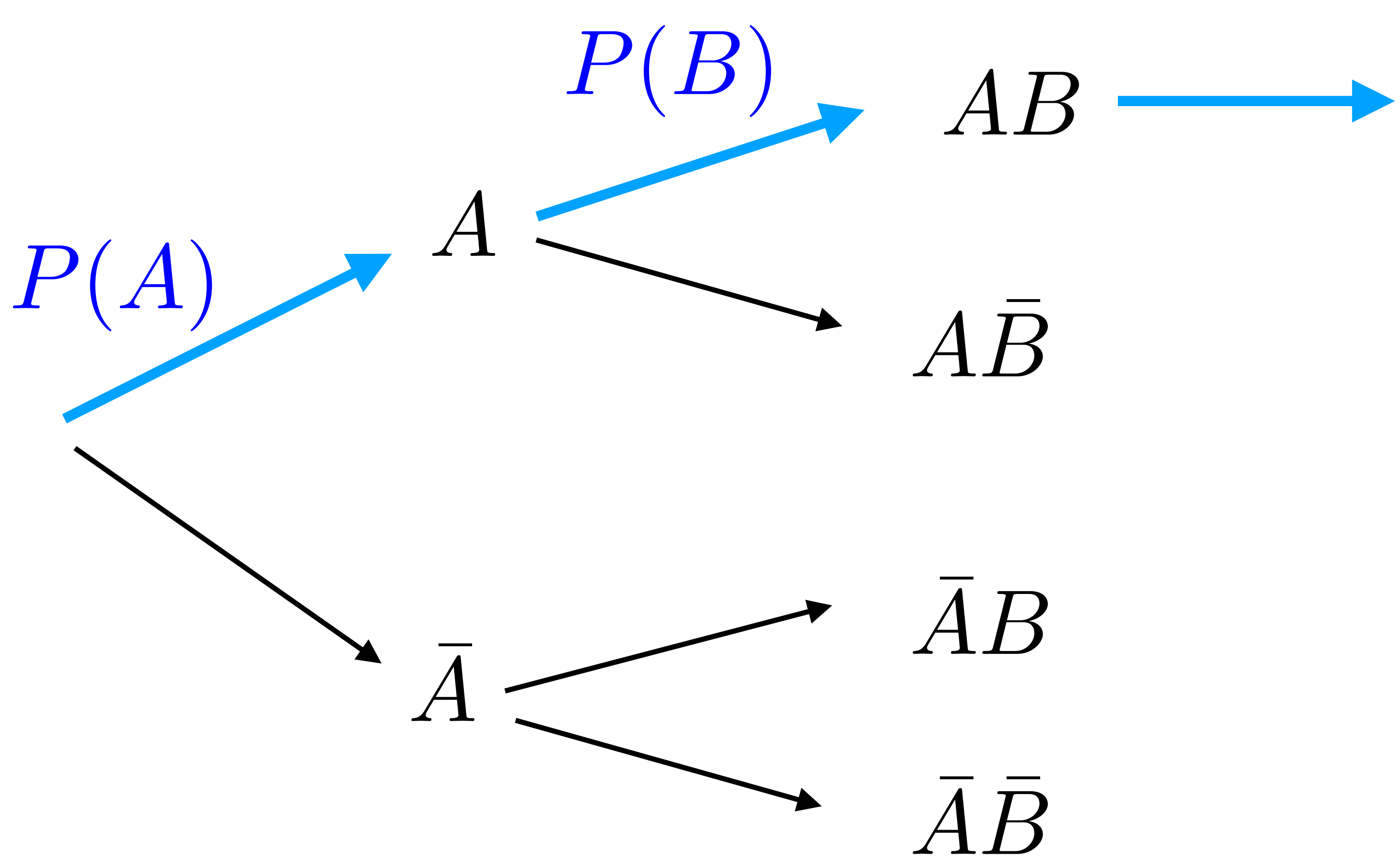
B independent of A

Definition:

Independent: $P(B|A) = P(B)$

Independent: “Just multiply”

$$P(AB) = P(A)P(B)$$



Multiple independent cases

$$X_1, X_2, X_3, \ldots, X_N \text{ all independent}$$

$$\Rightarrow P(X_1 X_2 X_3 \ldots X_N) = P(X_1)P(X_2)P(X_3)\ldots P(X_N)$$

E.g. $X_k =$ the k'th bit in the string is 1

Example

100 independent, equally likely bits

A = The string is 000000….00000
B= The string is 01010101…..01 (alternating 0 and 1)

$$P(A) = (1/2) \times (1/2) \times (1/2) \cdots \times (1/2) = \frac{1}{2^{100}}$$

$$P(B) = (1/2) \times (1/2) \times (1/2) \cdots \times (1/2) = \frac{1}{2^{100}}$$

Example

C = The string has exactly 50 one's in it and 50 zero's.

$$P(C): \text{ how many strings are in A?}$$

$$\text{choose 50 slots for the one's from 100 slots: } \binom{N}{50} = \binom{100}{50}$$

Example

C = The string has exactly 50 one's in it and 50 zero's.

$$P(C): \text{ how many strings are in A?}$$

$$\text{choose 50 slots for the one's from 100 slots: } \binom{N}{50} = \binom{100}{50}$$

$$P(C) = \binom{100}{50} \times \frac{1}{2^{100}} \approx 0.0796$$

The binomial distribution

N independent, random bits.
Probability(1) =p and Probability(0)=1-p.

If there are N bits in the string, what is the probability of:

A = The string having exactly k one's.

$$\boxed{P(\text{“exactly } k \text{ one's”}) = \binom{N}{k} p^k (1-p)^{N-k}}$$

The number of combinations

Probability for one such string

Example

A new app is launched, and there is a 20% chance that a user who downloads the app will make an in-app purchase. If 1000 users download the app, what is the probability that exactly 200 users will make an in-app purchase?

Example

A new app is launched, and there is a 20% chance that a user who downloads the app will make an in-app purchase. If 1000 users download the app, what is the probability that exactly 200 users will make an in-app purchase?

- Repeated, independent events (1,000 downloads)
- Binary notion of “success” and “failure” (1=purchase or 0= no purchase)

Binomial Distribution !!

N= 1000
p= 0.2
k= 200

$$P(k=200) = \binom{1000}{200}(0.2)^{200}(0.8)^{1000-200} \approx 0.032$$

Checklist for solving probability problems

1. If the problem involves nontrivial details, write out statements A= …, B=….. These must be **True** or **False** statements, and you want P(A), say.

2. Draw out **Probability Trees** showing the different possibilities..

3. Apply the core formulae and transform to what you want.

4. If something is repeated many times. Check if it a **Binomial Distribution** problem. If so then then thing you're counting is k. Use the Binomial formula.

Side note: The relation between the binomial distribution and binomial expansion

$$P(\text{“exactly } k \text{ one's” }) = \binom{N}{k} p^k (1-p)^{N-k}$$

$$\sum_{k=0}^{N} P(\text{“exactly } k \text{ one's” }) = \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k}$$

(Binomial expansion)

$$= (p + (1-p))^N$$

$$= 1^N = 1$$

The probability formula makes sense: all the probabilities sum to 1