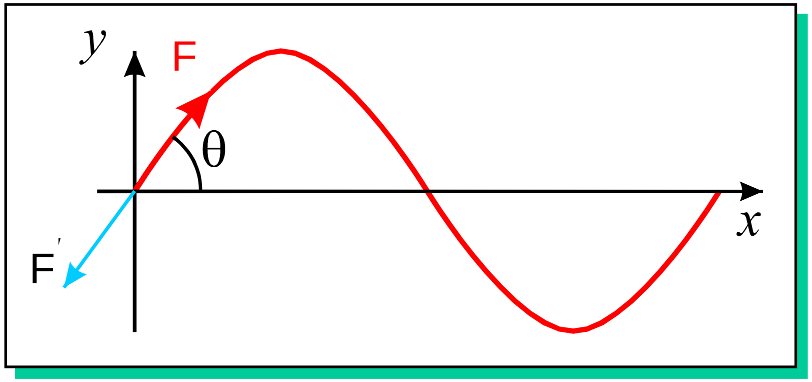
Reflection and transmission of waves on strings

Wave impedance Z: velocity response to driving force

large impedance $\Rightarrow$ small velocity response

driving force is transverse component of string tension F

$$Z = \frac{\text{driving force}}{\text{transverse velocity}}$$



$$F_T = -F\sin\theta \cong -F\tan\theta = -F\frac{\partial y}{\partial x} \Rightarrow Z = -F\frac{\partial y}{\partial x}\bigg/\frac{\partial y}{\partial t}$$

For solutions to the wave equation: $y(x - vt) = y(\alpha)$

$$\frac{\partial y}{\partial t} = -v\frac{\partial y}{\partial \alpha} \quad \& \quad \frac{\partial y}{\partial x} = \frac{\partial y}{\partial \alpha} \quad \Rightarrow \frac{\partial y}{\partial t}\bigg/\frac{\partial y}{\partial x} = -v$$

$$\Rightarrow Z = \frac{F}{v} = \sqrt{F\mu} \quad \text{since} \quad v = \sqrt{\frac{F}{\mu}}$$

wave impedance depends on string tension and mass

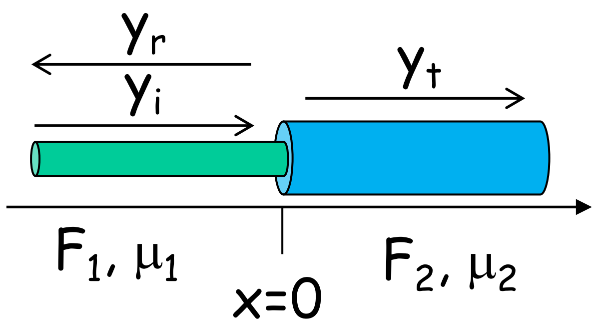
Not examinable

Note that we started with the tranverse force and velocity, and found that their ratio is the same as that of the wave speed and the string tension!

$$\frac{F\sin\theta}{\frac{\partial y}{\partial t}} = \frac{F}{v} = Z$$

i and r on same string so F and μ same, so Z same, ∴ v_1 the same

i + r in opposite directions



Consider two strings of different m and F, i.e. different impedance

$$Z = \sqrt{F\mu}$$

v_2 different value

y(x − vt) where x = 0

1. amplitude must be continuous at x = 0

at any time point t

$$y_i(-v_1 t) + y_r(+v_1 t) = y_t(-v_2 t)$$

2. net force in y at x=0 must be 0

$$F_1 \frac{\partial y_i(-v_1 t)}{\partial x} + F_1 \frac{\partial y_r(+v_1 t)}{\partial x} = F_2 \frac{\partial y_t(-v_2 t)}{\partial x}$$

Assume that all 3 waves have same time dependence! (suggested by 1)

$$y_i(-v_1 t) = g(t) \qquad y_r(+v_1 t) = rg(t) \qquad y_t(-v_2 t) = tg(t)$$

r, t constants

1 is satisfied if 1+r = t *

Not examinable

$$y_i(x - v_1 t) = y_i\left(-v_1\left(-\frac{x}{v_1} + t\right)\right) = g\left(-\frac{x}{v_1} + t\right)$$

$$y_r(x + v_1 t) = y_r\left(v_1\left(\frac{x}{v_1} + t\right)\right) = rg\left(\frac{x}{v_1} + t\right)$$

$$y_t(x - v_2 t) = y_t\left(-v_2\left(-\frac{x}{v_2} + t\right)\right) = tg\left(\frac{x}{v_1} + t\right)$$

$$Z_2 = \frac{F}{v_2} \qquad Z_1 = \frac{F}{v_1}$$

etc., etc., etc,.....

if you really want to see all of it, go to....

~djmorin/**waves**/transverse.pdf on Minerva,
Also a good resource for whole travelling waves topic

Not examinable

.....eventually we obtain:

Reflection coefficient r $$r = \frac{Z_1 - Z_2}{Z_1 + Z_2}$$

wave impedance

$$\boxed{Z = \sqrt{F\mu}}$$

$$Z = \frac{F}{v}$$

Transmission coefficient t $$t = \frac{2Z_1}{Z_1 + Z_2}$$

$$Z_1 = Z_2 \Rightarrow r = 0 \qquad Z_1 = Z_2 \Rightarrow t = 1$$

Complete transmission only if impedances are matched!

Fixed end point $\Rightarrow$ $Z_2 = \infty$ and r = -1, t = 0
Reflected wave has reverse direction and opposite sign (negative amplitude) $y_r(+v_1 t) = -g(t) = -y_i(-v_1 t)$

Freely movable end point (no string) $\Rightarrow$ $Z_2 = 0$ and r = 1
Reflected wave is right way up.

Power reflection and transmission: $$\frac{P_r}{P_i} = r^2 = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2$$

Reflection and transmission of waves on strings

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Also $r = h_r / h_{in}$

$t = h_t / h_{in}$

Reflection coefficient r $\qquad r = \dfrac{Z_1 - Z_2}{Z_1 + Z_2}$

wave impedance

$$\boxed{Z = \sqrt{F\mu}}$$

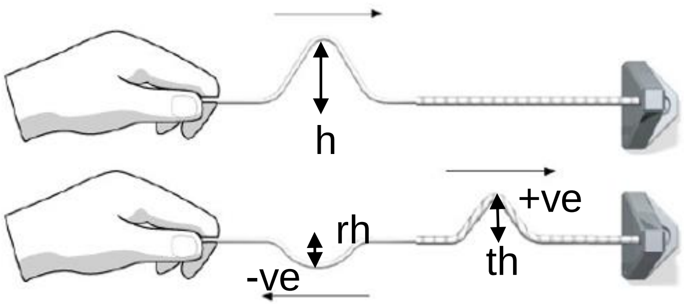
Transmission coefficient t $\qquad t = \dfrac{2Z_1}{Z_1 + Z_2}$

How do we use this? Energy is conserved, that is one method, but the t and r coefficients simply give the heights, or for a harmonic wave the amplitude, of the respective waves.

i.e. if t= 0.7 and r = 0.3 and the initial wave height is h, then the reflected wave height = 0.3h and the transmitted wave = 0.7h

A negative t or r = inverted or 180° out of phase to intial wave.

For harmonic waves, h = A



We’ve looked at what happens when a pulse wave hit a junction of different impedance.

What happens when a continuous harmonic wave hits a wall and is reflected back on itself?

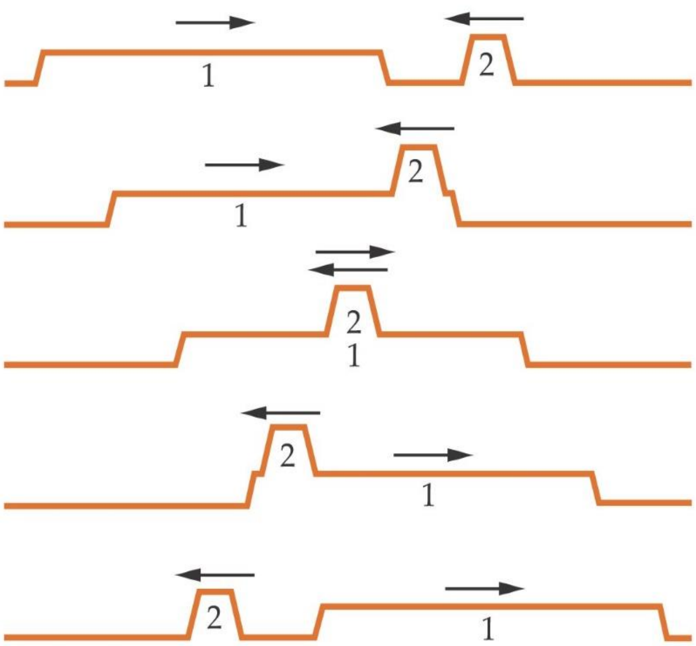
We first need to think about what happens when two waves coincide or overlap.

= SUPERPOSITION

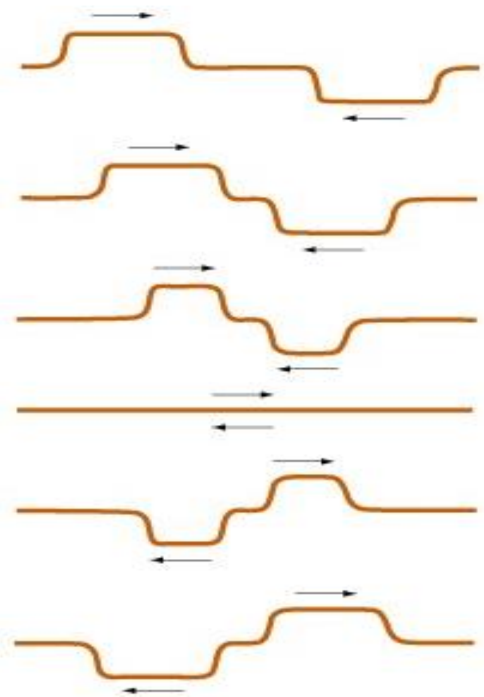
Linear waves respect the **principle of superposition**:

If two or more waves are moving through a medium, the resultant wave function at any point is the sum of the wave functions of the individual waves.

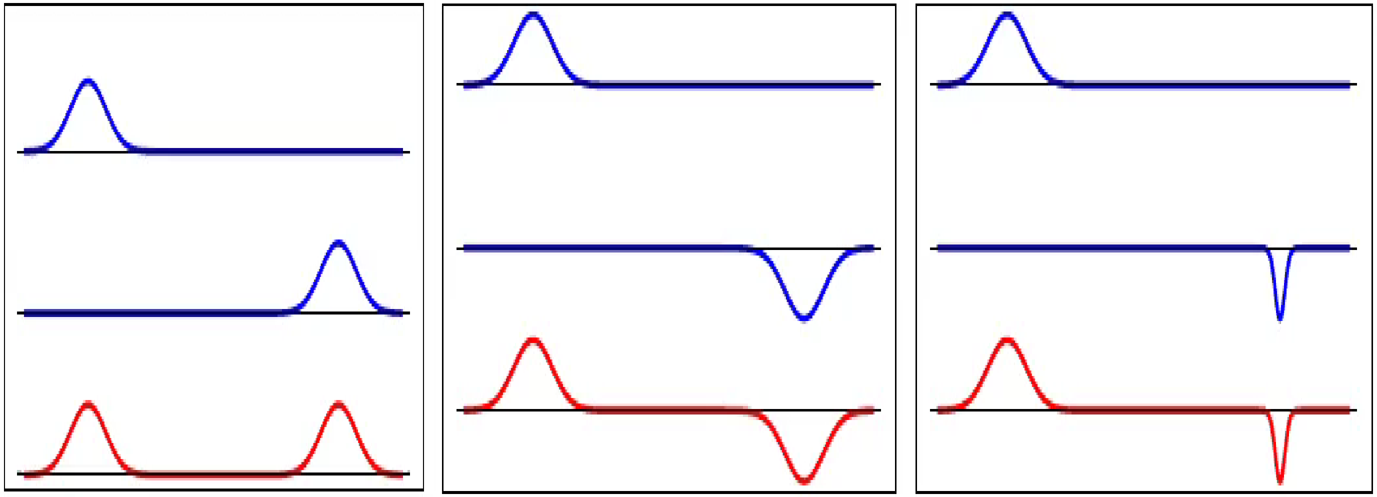
<u>two +ve pulses</u>



<u>one +ve and one -ve pulse</u>



As a consequence, waves travelling in different directions will pass through each other without being altered or destroyed.



http://www.physics.nyu.edu/~ts2/Animation/Superposition.html

Real example at Bell Labs
https://www.youtube.com/watch?v=BWraEDaVXZM

Reflection of harmonic waves

If a harmonic wave travels on a string that is fixed at the end, it is reflected. Incident and reflected waves superimpose.

We have two sinusoidal waves with the same amplitude, frequency and wavelength but travelling in opposite directions:

$$y_1 = A_o \sin(kx - \omega t) \qquad\qquad y_2 = A_o \sin(kx + \omega t)$$

Result is $\quad y = A_o\left[\sin(kx - \omega t) + \sin(kx + \omega t)\right]$

Using the identity $\quad \sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

gives $\quad \boxed{y = 2A_o \sin(kx)\cos(\omega t)}$

So this is another **solution of the wave equation**.

It is the wave function of a standing wave.

Standing waves

Separation of spatial term from the frequency term

$$y = 2A_0 \sin(kx)\cos(\omega t)$$

Amplitude = $2A_0\sin(kx)$

•Every part of the string vibrates in SHM with the same frequency and phase.

•Amplitude depends on x.

[c.f. travelling harmonic wave where all parts oscillate with equal amplitude but different phases.]

$$\boxed{y = 2A_0 \sin(kx)\cos(\omega t)}$$

Zero amplitude occurs at $\sin(kx) = 0$, where $kx = \pi, 2\pi, 3\pi...$

& $k = 2\pi / \lambda$ so zeros are at

$$x = \frac{\lambda}{2}, \frac{2\lambda}{2}, \frac{3\lambda}{2} .... = \frac{n\lambda}{2} \quad \text{with } n = 1, 2, 3,....$$

Positions of zero amplitude are "**nodes**", separated by distance $\lambda/2$.

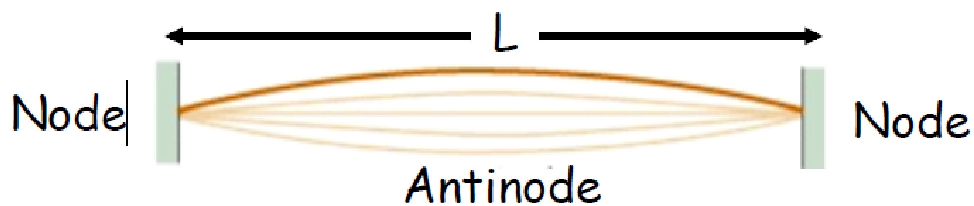
In between are positions of max. amplitude: "**antinodes**", also separated by $\lambda/2$.

"Have spoken so far about ∞ string. Real string (e.g. guitar string) has boundaries. They quantize the allowed λ."

Boundary conditions

Consider a string of length L , fixed at both ends.

Fixed ends must be nodes.



Given that x=0 is a node, demanding another node at x=L fixes λ at one of a quantized set of values.

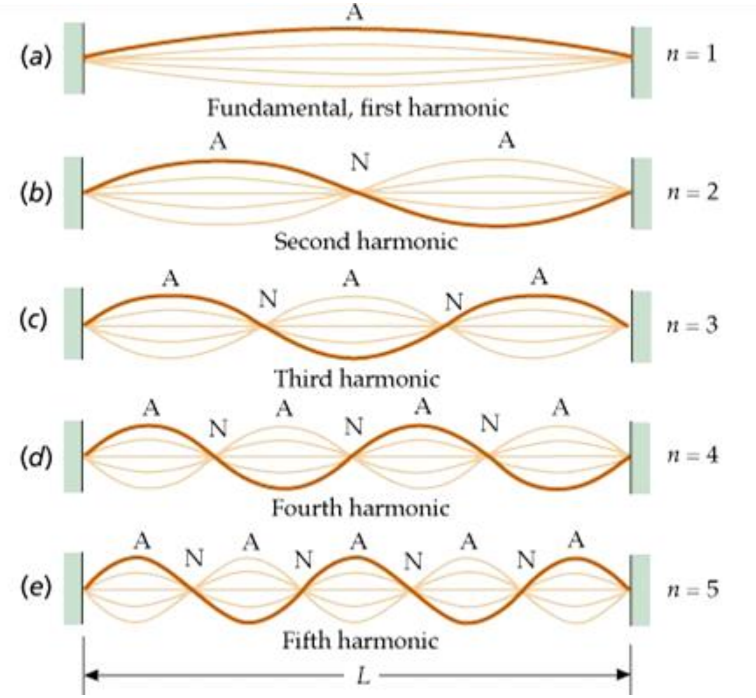
$$\frac{n\lambda}{2} = L \quad \Rightarrow \quad \boxed{\lambda = \frac{2L}{n}}$$

can fit multiple $\frac{1}{2}$-wavelengths between the two fixed nodes.

The **Standing waves** are natural patterns of vibration of the string, each with a different characteristic frequency.

In each of these patterns, all parts of the string move with SHM, in phase or in antiphase.

Normal modes of the rope!



Recall N coupled oscillators had N normal modes. The string is an elastic continuum, i.e. an infinity of coupled oscillators. It has infinitely many discrete normal modes.

Harmonics and Overtones

The lowest frequency mode is called the **fundamental**.

The frequency of each normal mode is an integer multiple of the fundamental frequency.

These integer-multiple frequencies are called **harmonics**.

The normal mode frequencies are also called **overtones**. For a stringed instrument,

The **1st overtone** is the **2nd harmonic**,

the **2nd overtone** is the **3rd harmonic**, etc.

Musical Instruments

When a stretched string is distorted so that the initial shape corresponds to a harmonic, only that particular normal mode is excited. So the string will vibrate at the frequency of that harmonic.

If the string is struck (piano) or plucked (guitar) its initial shape is not that of a single normal mode. Several normal modes are excited, and the resulting vibration will include several of the harmonic frequencies. Typically, the fundamental will have the largest amplitude.

Sound Waves

Tipler 15 & 16 + Main Ch 10 / Gough Ch 5.4

The most important example of **longitudinal waves**, sound waves move through gases, liquids or solids with a speed that depends on the properties of the medium.

Human hearing defines three categories of longitudinal mechanical waves:

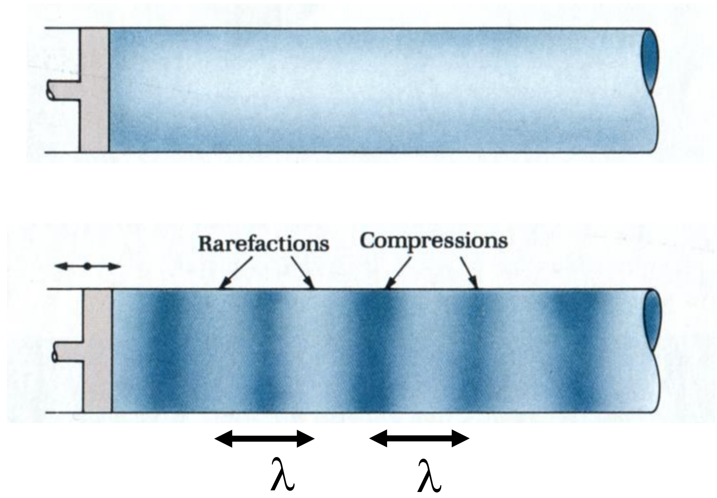
Audible waves: 20 Hz to 20 000 Hz.
Examples - voice, musical instruments, loud speakers.

Infrasonic waves: frequencies below the audible range.
Examples -earthquakes, elephant calls.

Ultrasonic waves: frequencies above the audible range.
Examples - ultrasound scans, bat sonar.

Sound Waves

Longitudinal wave, with regions of rarefactions and compressions moving along the tube

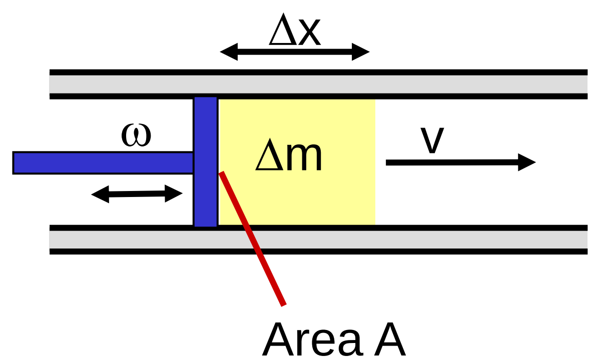


$$\varphi(x,t) = A \sin(kx - \varpi t)$$

Wave propagates in x, element is displaced in x direction by $\varphi(x,t)$. A positive φ corresponds to positive displacement in x-direction.

Energy and intensity of harmonic sound waves

Consider a layer of air in front of an oscillating piston:



The piston transmits energy to the air.

The layer executes SHM $\Rightarrow KE_{av} = PE_{av}$ and $E_{tot} = KE_{max}$

$$E_{tot} = \frac{1}{2}\Delta m\,(\omega s_0)^2$$

ωs_0 is the maximum longitudinal velocity of the medium in front of the piston.

S_0 is the max. displacement of the medium, air. $\therefore = A$

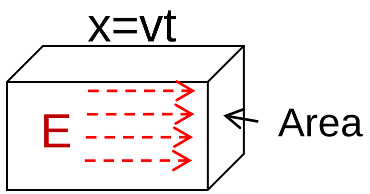
$$(\,s = s_0 \sin\omega\, t \Rightarrow v = \omega s_0 \cos\omega\, t \Rightarrow v_{0\max}\,)$$

$$\Rightarrow \quad \text{Energy density} = \frac{E_{tot}}{\Delta V} = \frac{1}{2}\rho\,\omega^2 s_0^2$$

[s for "sound" (Tipler)]

Power = rate at which energy is transferred to each layer

Energy flows at velocity v



V=vt×Area

$$Energy = Energy\ density \times Area \times vt$$

$$= \frac{1}{2}\rho\omega^2 s_0^2 v \times Area \times t$$

$$Power = \frac{1}{2}\rho\omega^2 s_0^2 v \times Area$$

$$Intensity \equiv \frac{Power}{Area} = \frac{1}{2}\rho v\omega^2 s_0^2$$ (watts / m^2)

[c.f. power on string] $\boxed{P_{av} = \frac{1}{2}\mu v\omega^2 A^2}$

(μ corresponds to ρ, A corresponds to s_0)

The relationship between **Displacement** and **Pressure**

For a wave on a string: $v = \sqrt{\frac{T}{\mu}}$

By analogy, for a pressure wave: $v = \sqrt{\frac{B}{\rho}}$

where $B = -\frac{\Delta p}{\Delta V / V}$ the bulk modulus (of compressibility)

$B = \gamma p \qquad \gamma = C_p / C_V$

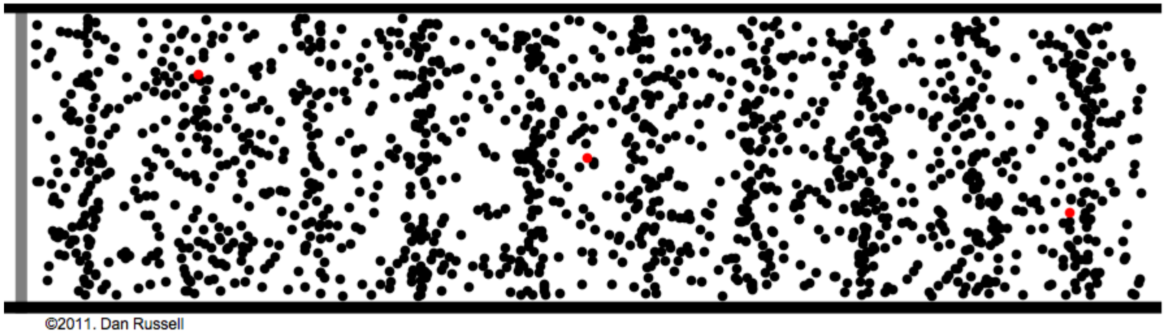
$$\Rightarrow \Delta p = -B\frac{\Delta V}{V} = -B\frac{A\delta s}{A\partial x} = -B\frac{\partial s}{\partial x}$$ where $s = s_0 \sin(kx - \omega t)$

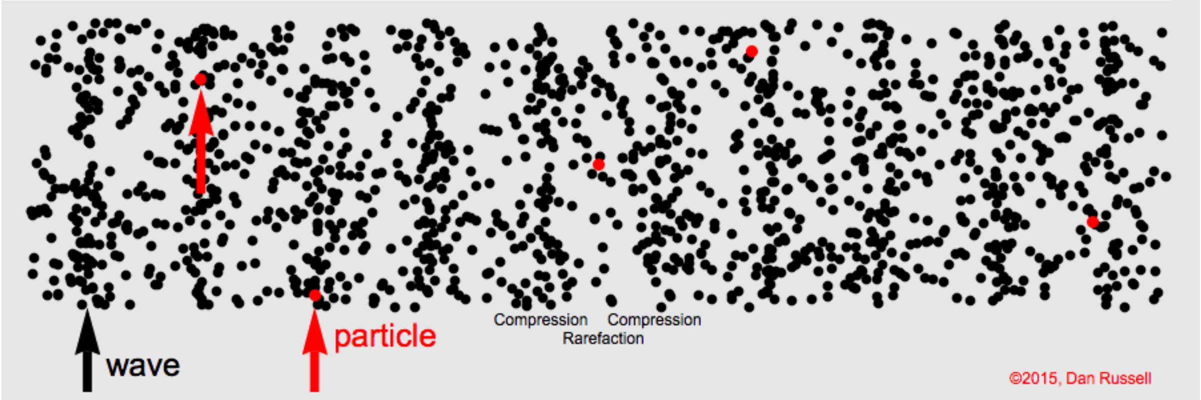
$$\Rightarrow \Delta p = (\pm) B k s_0 \cos(kx - \omega t)$$

s is displacement in x-direction from equilibrium position

pressure wave is 90° out of phase with displacement wave

Longitudinal sound waves
(if the piston moves with SHM it will be a longitudinal harmonic travelling wa





http://www.acs.psu.edu/drussell/Demos/waves/wavemotion.html

Intensity

$$\Rightarrow \Delta p = (\pm) B k s_0 \cos(kx - \omega t)$$

Amplitude of pressure wave is $\Delta p_0 = B k s_0$

$v = \sqrt{\frac{B}{\rho}}$

Substituting $B = \rho v^2$ & $k = \frac{\omega}{v}$

gives $s_0 = \frac{\Delta p_0}{\rho v \omega}$

with $$I = \frac{1}{2}\rho v \omega^2 s_0{}^2 = \frac{1}{2}\rho v \omega^2 \times \left(\frac{\Delta p_0}{\rho v \omega}\right)^2$$

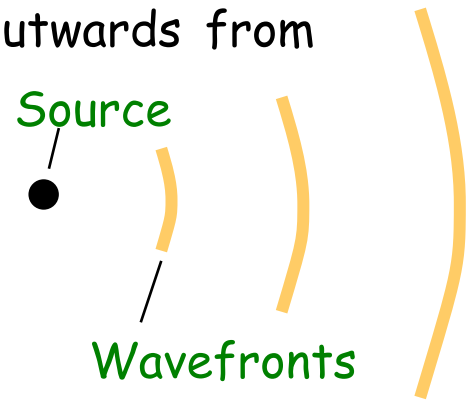
$$\boxed{I = \frac{\Delta p_0^2}{2\rho v}}$$

The intensity of a sound wave is proportional to the square of the pressure amplitude.

Spherical waves

In a uniform medium the wave moves outwards from the source at a constant speed.

Hence, from a point source (a small object that expands and contracts harmonically), sound waves are produced with spherical wavefronts.



Let $\overline{P}$ = acoustic power emitted by a point source.

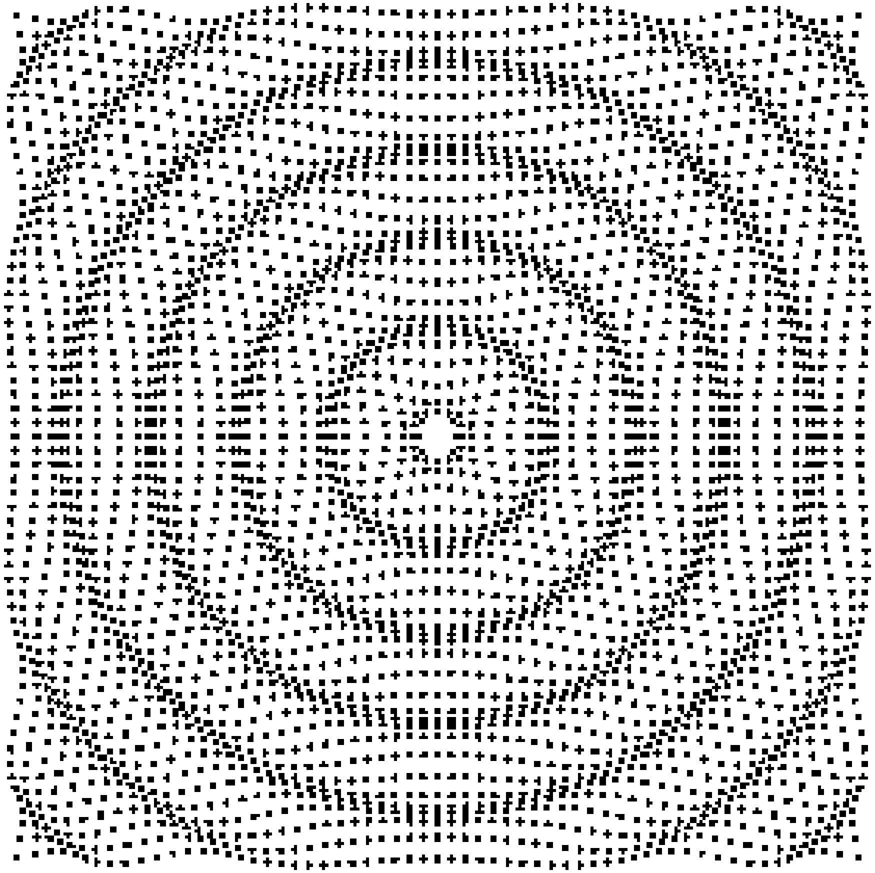
At a distance r from the source this power is distributed over a spherical surface of area $4\pi r^2$.

$$\text{Intensity} \quad I = \frac{P}{A} = \frac{P}{4\pi r^2}$$

Intensity diminishes as the inverse-square of distance from a spherical source.

But Intensity $\propto$ Amplitude^2 so Amplitude $\propto$ distance^{-1}.

Spherical waves



decibels

The human ear detects sound on an approximately logarithmic scale.

We define the **intensity level** of a sound wave by

$$\boxed{\beta = 10 \log_{10}\left(\frac{I}{I_o}\right)}$$

where I is the intensity of the sound, I_o is the threshold of hearing (~10^{-12} W m^{-2}) and β is measured in decibels (dB).

Examples: breathing 10dB busy traffic 80dB

whisper 30dB threshold of pain 120dB

conversation 50dB

Relationship between displacement and pressure amplitude

$$s_0 = \frac{\Delta p_0}{\rho v \omega} \Rightarrow \Delta p_0 = s_0 \rho v \omega = 2\pi f \rho v s_0$$

$\beta = 10 \log_{10}\left(\frac{I}{I_0}\right)$

$\therefore I = 10^{\frac{\beta}{10}} I_0 = 10^{\frac{50}{10}} \cdot 1 \times 10^{-12}$

$= 1 \times 10^{-7}\ \mathrm{W\,m^{-2}}$

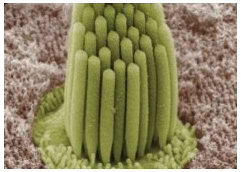
Take $f = 1$ kHz
$\rho = 1.3\ \mathrm{kgm^{-3}}$
$v = 340\ \mathrm{ms^{-1}}$

50 dB corresponds to 10^{-7} W sound intensity I

$$I = \frac{\Delta p_0^2}{2\rho v} \Rightarrow \Delta p_0 = \sqrt{2 I \rho v} = \sqrt{2 \times 10^{-7} \times 1.3 \times 343} \approx 9\mathrm{mPa}$$

$$s_0 = \frac{\Delta p_0}{\rho v \omega} \approx 3nm$$

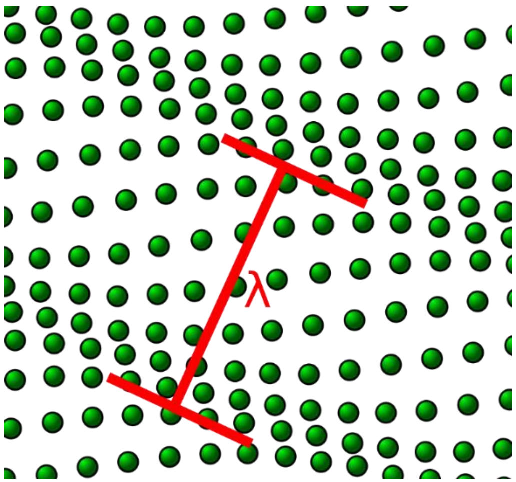
Pain threshold corresponding values are 30 Pa and 11 μm



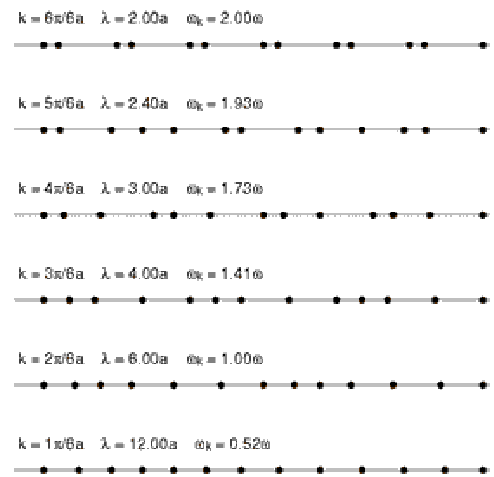
Stereocilia are the mechano-sensing organelles of **hair** cells, which respond to fluid motion for various functions, including hearing and balance. They are about 10–50 micrometers in **length**.

Phonons

A collective excitation in a periodic, elastic arrangement of atoms or molecules (in solids/some liquids). Often referred to as a quasiparticle, it is an excited state in the quantum mechanical quantization of the modes of vibrations for elastic structures of interacting particles. Phonons can be thought of as quantized sound waves, similar to photons as quantized light waves. The phonon is the quantum mechanical description of an elementary vibrational motion in which a lattice of atoms or molecules uniformly oscillates at a single frequency.



Phonon propagating through a square lattice (atom displacements greatly exaggerated)



The first 6 normal modes of a one-dimensional lattice: a linear chain of particles. The shortest wavelength is at top, with progressively longer wavelengths below. In the lowest lines the motion of the waves to the right can be seen.