UNIVERSITY OF LEEDS

Mechanics 1

Session 12: Circular Motion – Velocity in Radial Co-ordinates

DR BEN HANSON

Last Lecture

Momentum

We learned that:

- Understand what momentum is conceptually
- See that momentum is a vector
- Understand why momentum is always conserved following collisions
- See that everything becomes mathematically easier in the centre of mass reference frame

You should be able to:

- Use the concept of momentum conservation to calculate the subsequent kinetic properties (velocities) following a collision

This Lecture

Circular Motion 1 – A New Coordinate System

We will:
- Recap everything we've done so far
- Learn that each of the concepts we have studied so far has a parallel in circular motion
- Recall what we know of circular motion from previous studies (A-levels etc)
- Derive the fundamental kinematic equations of motion in *circular* co-ordinates

You will be able to:
- Understand that circular motion is best studied in *circular* co-ordinates
- Understand that unlike the Cartesian unit vectors $\underline{i}, \underline{j}$, circular unit vectors rotate
- Reproduce the derivation of the kinematic equations for velocity and acceleration in circular co-ordinates

This Lecture

Don't be afraid…

The methods we will cover today will be very new to you all. I promise you, you are all capable of understanding this.

- You have already shown in your coursework that you understand the concepts we are about to discuss
- Don't be afraid to ask questions. If I don't see your hand, shout out!

The Story So Far

A Cartesian View of Physics

The Story So Far

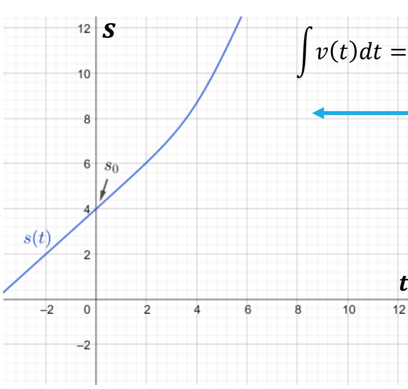
Kinematics

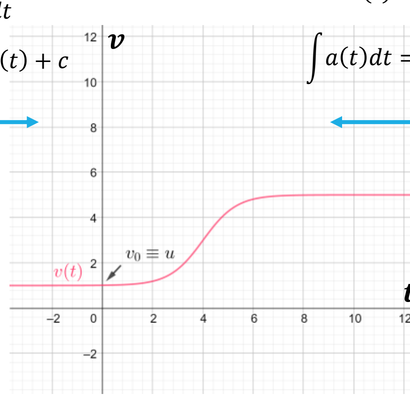
$$v(t) = \frac{ds}{dt}$$

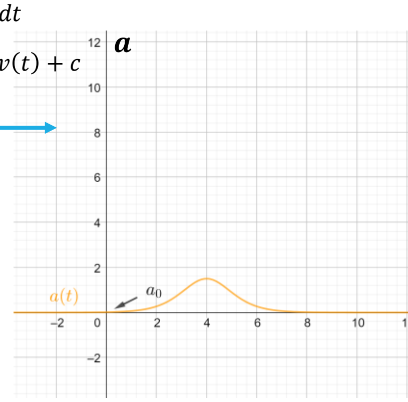
$$\int v(t)dt = s(t) + c$$

$$a(t) = \frac{dv}{dt}$$

$$\int a(t)dt = v(t) + c$$

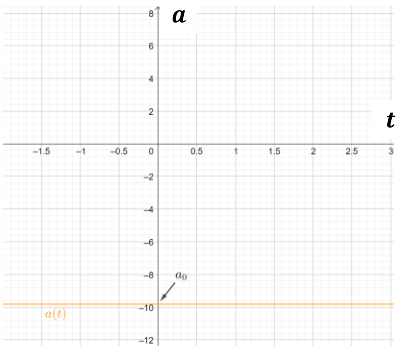




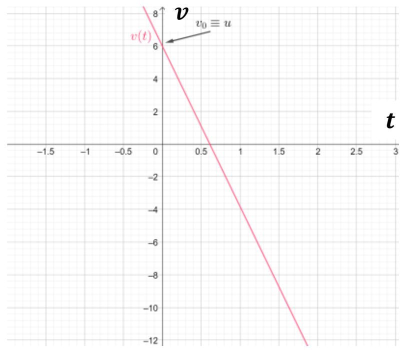


The Story So Far

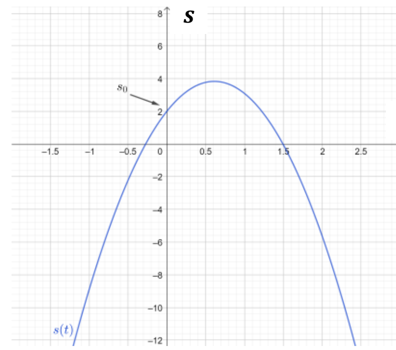
Kinematics



$$a(t) = a$$



$$v(t) = u + at$$



$$s(t) = s_0 + ut + \frac{1}{2}at^2$$

The Story So Far

Newton’s Laws of Motion

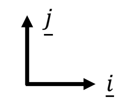
1. A body remains at rest, or in motion at a constant speed in a straight line, unless acted upon by a force.

2. The impulse acting on an object is equal to its change in momentum
   a) => The net force on an object is equal to the rate of change of its momentum
   b) => If the mass is constant, the net force on an object is equal to its mass multiplied by its acceleration: $\underline{F} = m\underline{a}$

3. If body A applies a force to body B, then body B applies and equal and opposite force to body A

The Story So Far

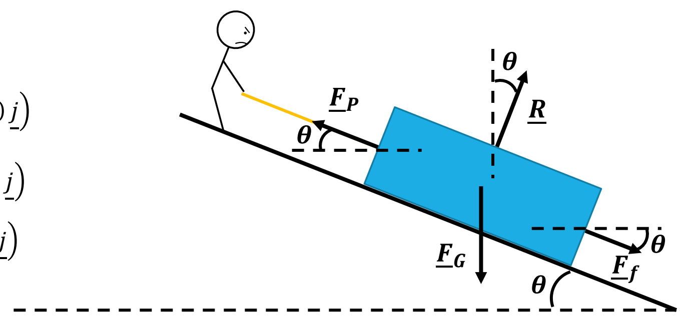
Resolving forces in x and y



Calculate net force, $\underline{F}_{Net} = \sum_i \underline{F}_i$

$$\underline{F}_{Net} = \underline{F}_G + \underline{F}_P + \underline{F}_f + \underline{R}$$

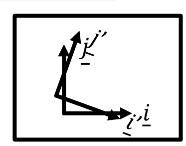
$$\underline{F}_{Net} = -mg\underline{j} + \left(-|\underline{F}_P|\cos(\theta)\underline{i} + |\underline{F}_P|\sin(\theta)\,\underline{j}\right)$$
$$+ \left(|\underline{F}_f|\cos(\theta)\,\underline{i} - |\underline{F}_f|\sin(\theta)\,\underline{j}\right)$$
$$+ \left(|\underline{R}|\sin(\theta)\,\underline{i} + |\underline{R}|\cos(\theta)\,\underline{j}\right)$$



The Story So Far

Resolving forces in x' and y'

This is going to be a vital idea in circular motion!



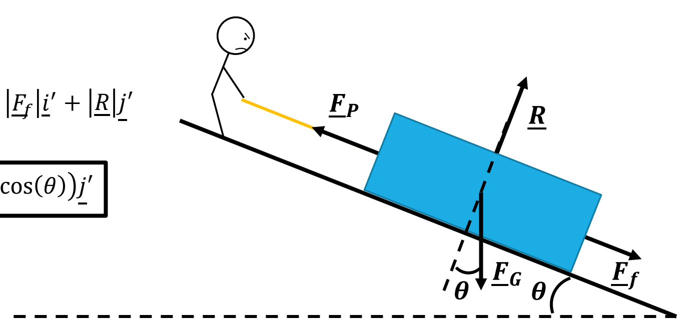
Calculate net force, $\underline{F}_{Net} = \sum_i \underline{F}_i$

$$\underline{F}_{Net} = \underline{F}_G + \underline{F}_P + \underline{F}_f + \underline{R}$$

$$\underline{F}_{Net} = \left(mg\sin(\theta)\,\underline{i}' - mg\cos(\theta)\underline{j}'\right) - |\underline{F}_P|\underline{i}' + |\underline{F}_f|\underline{i}' + |\underline{R}|\underline{j}'$$

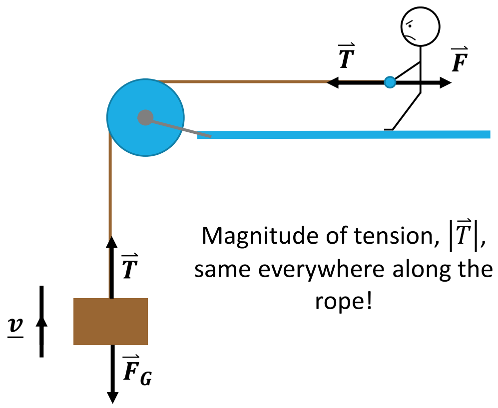
$$\boxed{\underline{F}_{Net} = \left(|\underline{F}_f| + mg\sin(\theta) - |\underline{F}_p|\right)\underline{i}' + \left(|\underline{R}| - mg\cos(\theta)\right)\underline{j}'}$$

Simpler, minimal angles, easier to manipulate and solve ☺



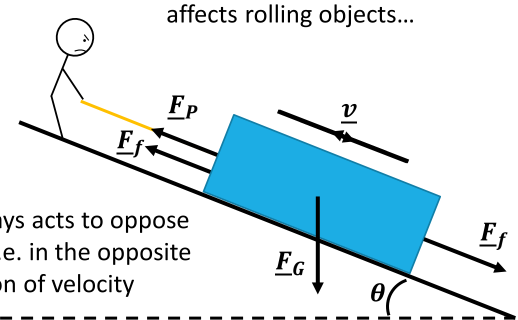
The Story So Far

Tension & Friction



Magnitude of tension, $|\vec{T}|$, same everywhere along the rope!

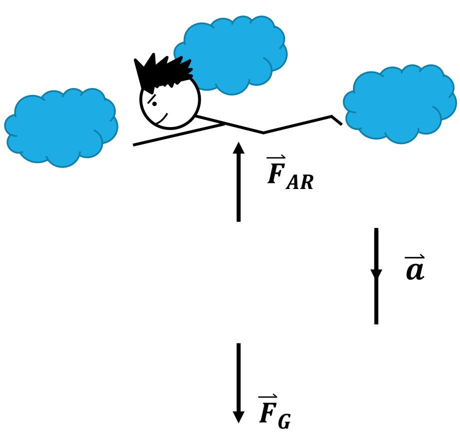
Thus, friction magnitude and direction depends on the other forces acting. I wonder how that affects rolling objects…



Friction always acts to oppose the motion i.e. in the opposite direction of velocity

The Story So Far

Energy



$$F_{AR}(t) = ma(t)$$

$$\frac{dv}{dt} = g - \frac{b}{m}v(t)^2$$

Non-linear differential equation. Extremely complex!

Let’s think about things in terms of energies instead!

The Story So Far

Work Done, Kinetic & Potential Energy

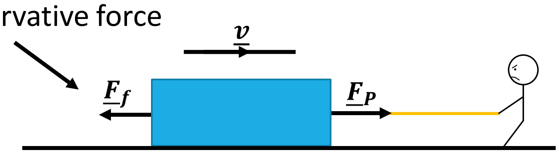
$$W = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}_{Net} \cdot d\vec{r}$$

Forces do work! Or, they transfer energy.

Kinetic Energy, $W = \Delta K = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$

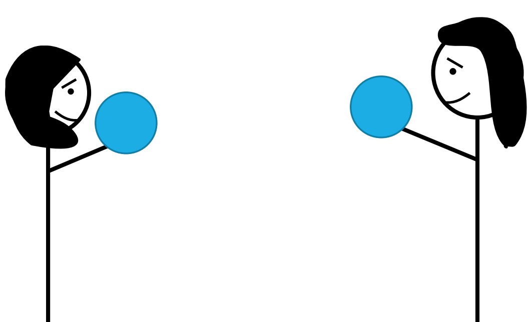
Potential Energy, $\Delta U = -W_{Con}$

Non-conservative force



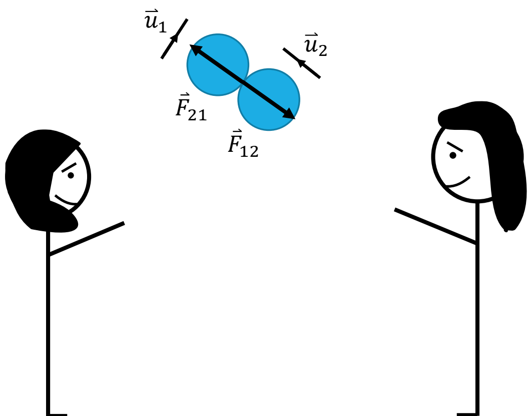
The Story So Far

Momentum



The Story So Far

Momentum



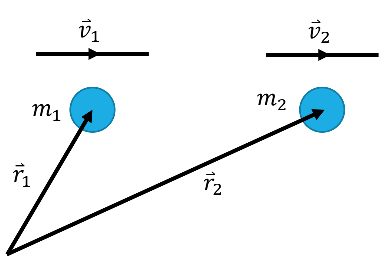
Momentum conserved:
Momentum before = momentum after

$$m_1\vec{u}_1 + m_2\vec{u}_2 = m_1\vec{v}_1 + m_2\vec{v}_2$$

$$\sum_{i=1}^{N} m_i\vec{u}_i = \sum_{i=1}^{N} m_i\vec{v}_i$$

The Story So Far

Momentum



$$M_T\vec{r}_{CM} = \sum_{i=1}^{N} m_i\vec{r}_i \qquad M_T\vec{v}_{CM} = \sum_{i=1}^{N} m_i\vec{v}_i$$

$$\vec{r}_{CM} = \frac{1}{M_T}\sum_{i=1}^{N} m_i\vec{r}_i \qquad \vec{v}_{CM} = \frac{1}{M_T}\sum_{i=1}^{N} m_i\vec{v}_i$$

Momentum in centre of mass frame is always zero!

The reference frame, the co-ordinate system we choose:

1. **Does not change** the underlying physical equations (momentum still conserved etc)
2. **Does change** how we measure things!

Extremely important in circular co-ordinates!

The Story So Far

To be continued…

Annnndddddd….breathe

You've learned all this in just 5 weeks! Amazing ☺

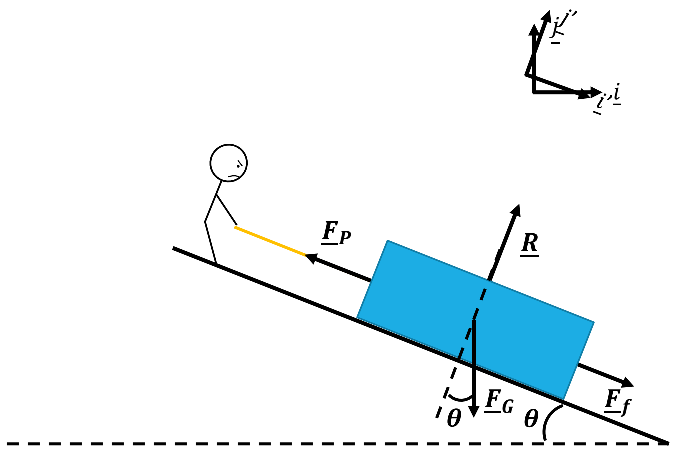
Keep practicing and reflecting on these ideas. That's how you learn (spaced repetition)

Linear Motion

Changing Co-ordinate Systems

Linear Motion

Changing Co-ordinate Systems



Changing to "ideal" co-ordinate systems:

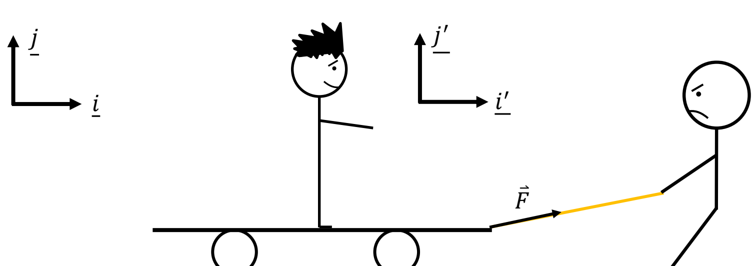
1. Doesn't change the underlying physics
2. Makes calculations easier
3. Can change how we measure things (positions, velocities (momentum), accelerations (forces))

Linear Motion

Changing Co-ordinate Systems

Changing to "ideal" co-ordinate systems:

1. Doesn't change the underlying physics
2. Makes calculations easier
3. Can change how we measure things (positions, velocities (momentum), accelerations (forces))

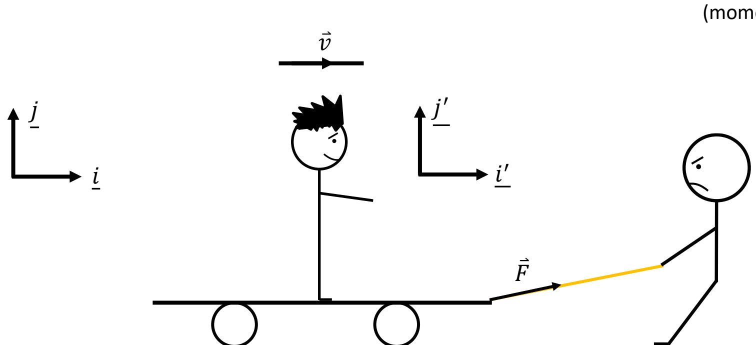


Linear Motion

Changing Co-ordinate Systems

Changing to "ideal" co-ordinate systems:

1. Doesn't change the underlying physics
2. Makes calculations easier
3. Can change how we measure things (positions, velocities (momentum), accelerations (forces))



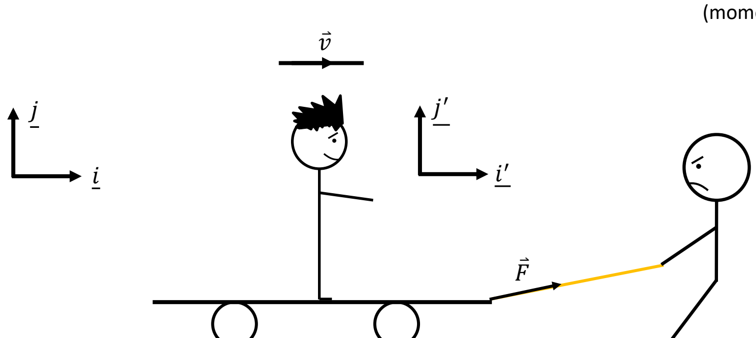
MECHANICS 1: CIRCULAR MOTION - VELOCITY IN RADIAL CO-ORDINATES

Linear Motion

Changing Co-ordinate Systems

Changing to "ideal" co-ordinate systems:

1. Doesn't change the underlying physics
2. Makes calculations easier
3. Can change how we measure things (positions, velocities (momentum), accelerations (forces))

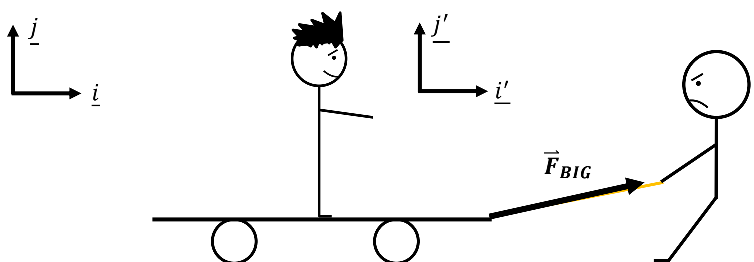


Linear Motion

Changing Co-ordinate Systems

Changing to "ideal" co-ordinate systems:

1. Doesn't change the underlying physics
2. Makes calculations easier
3. Can change how we measure things (positions, velocities (momentum), accelerations (forces))

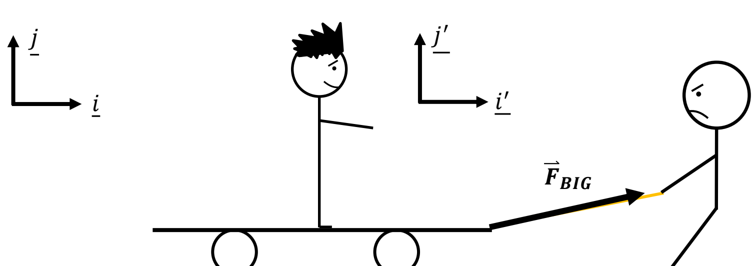


Linear Motion

Changing Co-ordinate Systems

Changing to "ideal" co-ordinate systems:

1. Doesn't change the underlying physics
2. Makes calculations easier
3. Can change how we measure things (positions, velocities (momentum), accelerations (forces))



Circular Motion

Changing Co-ordinate Systems

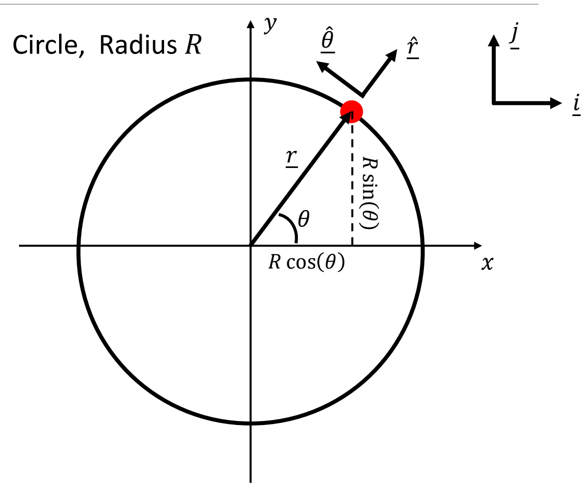
Circular Motion

Changing Co-ordinate Systems

	Cartesian $(\underline{i}, \underline{j})$	**Radial** $(\underline{\hat{r}}, \underline{\hat{\theta}})$
Position, $\underline{r} =$	$R\cos(\theta)\,\underline{i} + R\sin(\theta)\,\underline{j}$	$R\underline{\hat{r}}$
Velocity, $\underline{v} =$		
Acceleration, $\underline{a} =$		

$$\underline{r} = R\underline{\hat{r}} = R\cos(\theta)\,\underline{i} + R\sin(\theta)\,\underline{j}$$

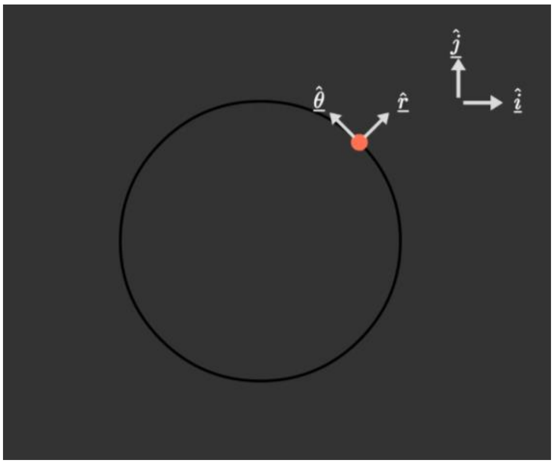
$$\boxed{\underline{\hat{r}} = \cos(\theta)\,\underline{i} + \sin(\theta)\,\underline{j}}$$



Angles always measured in radians when considering circular motion!

Circular Motion

Changing Co-ordinate Systems



Changing to "ideal" co-ordinate systems:

1. Doesn't change the underlying physics
2. Makes calculations easier
3. Can change how we measure things (positions, velocities (momentum), accelerations (forces))

- $\underline{i}$ and $\underline{j}$ are constant unit vectors
- $\hat{\underline{r}}$ and $\hat{\underline{\theta}}$ are unit vectors that vary with θ
- θ varies with time

- $\boxed{\hat{\underline{r}} \text{ and } \hat{\underline{\theta}} \text{ vary with time!}}$

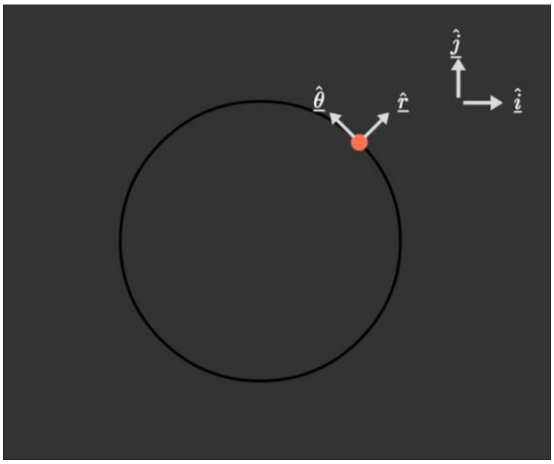
Angles always measured in radians when considering circular motion!

Task 1

Some Conceptual Questions

Task 1

Some Conceptual Questions



Scenario: An object is undergoing pure circular motion, as in the video to the left. It's angular speed ($\omega = d\theta/dt$) is constant.

Questions:

1. Is the object accelerating?
2. What is this acceleration doing to the object?
3. In which direction is the force acting that causes this acceleration?
4. If I applied a force to this system which acted in the $\underline{\hat{\theta}}$ direction, what would happen?
5. If I were in a rotating reference frame (i.e. in a car on a roundabout, stood on the surface of the Earth etc), would that be an inertial reference frame?

Angles always measured in radians when considering circular motion!

MECHANICS 1: CIRCULAR MOTION - VELOCITY IN RADIAL CO-ORDINATES

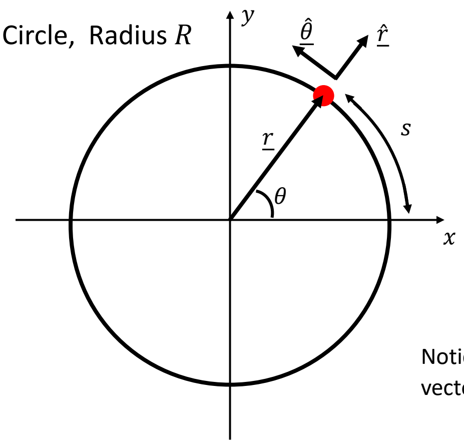
Circular Motion

Kinematics in Circular Coordinates

Circular Motion

Kinematics in Circular Coordinates

Circle, Radius R



Kinematic Properties

	Angular	Linear
Distance	θ	$s = R\theta$
Speed	$\omega = \dfrac{d\theta}{dt}$	$v_\theta = \dfrac{ds}{dt} = R\omega$
Acceleration	$\alpha = \dfrac{d\omega}{dt}$	$a_\theta = \dfrac{dv}{dt} = R\alpha$

Notice the subscripts. These are important! They are the components of the vectors $\underline{v}$ and $\underline{a}$ in the $\hat{\underline{\theta}}$ direction. In other words,

Vectors are same

$$\left\{ \begin{array}{ll} \underline{v} = v_\theta \hat{\underline{\theta}} + v_r \hat{\underline{r}} & \underline{a} = a_\theta \hat{\underline{\theta}} + a_r \hat{\underline{r}} \\ \underline{v} = v_x \underline{i} + v_y \underline{j} & \underline{a} = a_x \underline{i} + a_y \underline{j} \end{array} \right\}$$

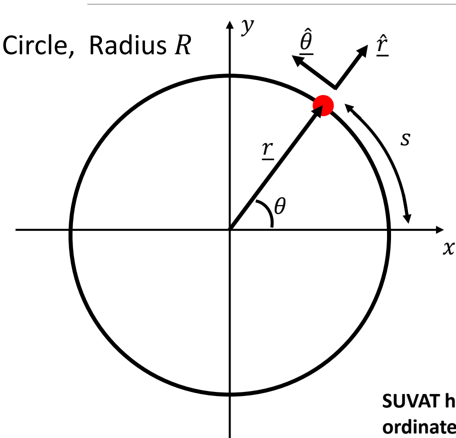
Representation is different

Angles always measured in radians when considering circular motion!

Circular Motion

Kinematics in Circular Coordinates

Circle, Radius R



A SUVAT example

Distance SUVAT, $\quad s = s_0 + u_\theta t + \frac{1}{2} a_\theta t^2$

$s - s_0 = R\theta, \quad R\theta = u_\theta t + \frac{1}{2} a_\theta t^2$

$u_\theta = R\omega_0, \quad R\theta = R\omega_0 t + \frac{1}{2} a_\theta t^2$

$a_\theta = R\alpha, \quad R\theta = R\omega_0 t + \frac{1}{2} R\alpha t^2$

Cancel R, $\quad \boxed{\theta = \omega_0 t + \frac{1}{2} \alpha t^2}$

SUVAT holds in angular co-ordinates! Physics is conserved ☺

Angles always measured in radians when considering circular motion!

Circular Motion

Kinematics in Circular Coordinates

It is extremely, extremely important that we recognise that the acceleration,

$$a_\theta = R\alpha = R\frac{d\omega}{dt} = R\frac{d^2\theta}{dt^2}$$

is not the centripetal acceleration that causes rotation. This component of acceleration points in the $\underline{\hat{\theta}}$ direction!

Circular Motion

Velocity in Circular Coordinates (for constant R)

This Lecture

Don't be afraid…

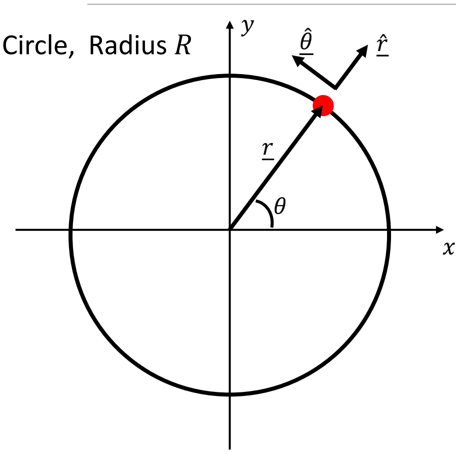
Ok Ben, the SUVAT equations are the same for angles, but…haven't you been saying that everything is vectors and calculus?

Absolutely! Let's derive not the speed, but the *velocity*

Circular Motion

Velocity in Circular Coordinates (constant R)

Circle, Radius R



Position, $\underline{r} = R\underline{\hat{r}}$

Velocity, $\underline{v} = \frac{d\underline{r}}{dt}$

$\underline{v} = \frac{d}{dt}(R\underline{\hat{r}})$

Only direction changing with time!

R constant, $\underline{v} = R\frac{d}{dt}(\underline{\hat{r}})$

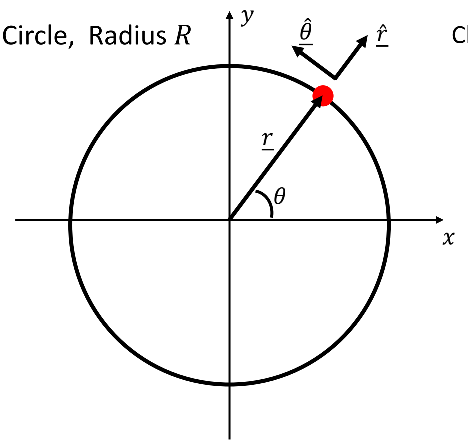
Change to Cartesian, $\underline{v} = R\frac{d}{dt}\left(\cos(\theta)\,\underline{i} + \sin(\theta)\,\underline{j}\right)$

Angles always measured in radians when considering circular motion!

Circular Motion

Velocity in Circular Coordinates (constant R)

Circle, Radius R



Change to Cartesian, $\underline{v} = R\dfrac{d}{dt}\left(\cos(\theta)\,\underline{i} + \sin(\theta)\,\underline{j}\right)$

Chain Rule, $\underline{v} = R\dfrac{d\theta}{dt}\dfrac{d}{d\theta}\left(\cos(\theta)\,\underline{i} + \sin(\theta)\,\underline{j}\right)$

$\underline{v} = R\omega\dfrac{d}{d\theta}\left(\cos(\theta)\,\underline{i} + \sin(\theta)\,\underline{j}\right)$

Differentiate, $\underline{v} = R\omega\,\underbrace{\left(-\sin(\theta)\,\underline{i} + \cos(\theta)\,\underline{j}\right)}_{\underline{\hat{\theta}}}$

Substitute, $\boxed{\underline{v} = R\omega\underline{\hat{\theta}}}$ $\boxed{\underline{v} = v_\theta\underline{\hat{\theta}}}$

$|\underline{v}| = R\omega$ as expected! Crucially, it is in the $\underline{\hat{\theta}}$ direction i.e. the velocity vector is tangent to the circle

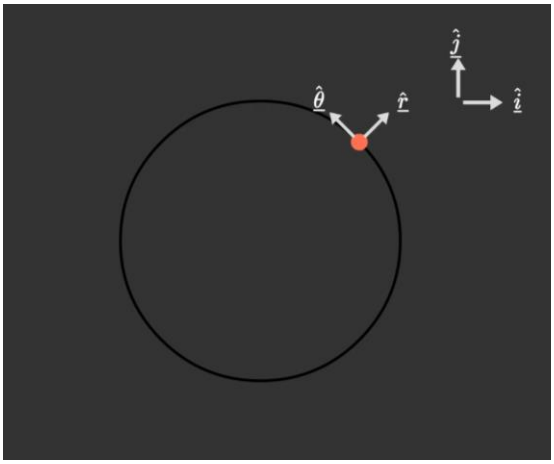
Angles always measured in radians when considering circular motion!

Task 2

Around and Around the Circle We Go

Task 2

Around and Around the Circle We Go



Scenario: A car is driving around a roundabout with radius $R = 12m$. Unfortunately, the driver does not know when to leave the roundabout so just keeps driving around at a constant speed.

Tasks:

1. The speedometer in the car reads $20mph$. Calculate the angular speed, ω. (Hint: $20mph$ is about $9ms^{-1}$)
2. What is the linear velocity vector of the car? (Hint: In the UK, which way do we drive around roundabouts?)
3. Calculate the time period, the total time taken for a single revolution of the circle.
4. Calculate the frequency, the number of revolutions of the circle the car does per second.

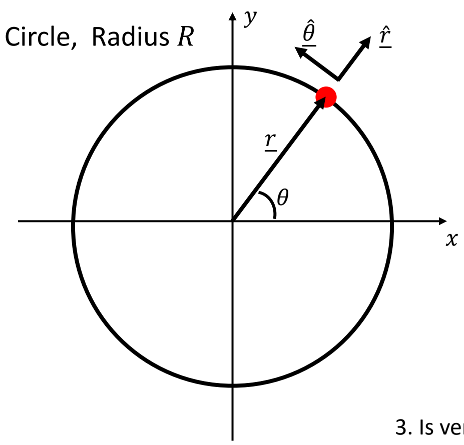
Angles always measured in radians when considering circular motion!

Task 3

Acceleration in Circular Co-ordinates

Task 3

Acceleration in Circular Coordinates



Scenario: You have seen my derivation of the velocity vector, $\underline{v} = \frac{d\underline{r}}{dt}$, in circular coordinates.

Tasks:

1. Derive the acceleration vector, $\underline{a} = \frac{d\underline{v}}{dt}$, in circular coordinates with constant R and ω $\left(\text{Hint: } \frac{d}{d\theta}\hat{\underline{r}} = \hat{\underline{\theta}}, \frac{d}{d\theta}\hat{\underline{\theta}} = -\hat{\underline{r}}\right)$
2. Imagine that the radius, R, were not constant (often the case for orbiting astronomical objects). What sort of motion would this be? What shape?
3. Derive the velocity vector, $\underline{v} = \frac{d\underline{r}}{dt}$, and the acceleration vector, $\underline{a} = \frac{d\underline{v}}{dt}$, if both R and ω are not constant.

3. Is very optional, but if you're feeling particularly mathsy, go for it! You will see the origin of the "Coriolis Force"

Angles always measured in radians when considering circular motion!