15 Line Integrals

A line integral is, in some ways, simpler than the surface and volume integrals we have been considering, since it is only 1D. What makes it slightly harder to visualise, however, is that we are no longer integrating over some straight section of the real line, but over some curve in 2D or 3D space. A curve can be expressed as a function from some real interval to the 2D or 3D space we are interested in. Strictly, the **function** is a **path** in space, while the **curve** is the **range** of that function. This distinction matters if we want to parametrise the same curve in different ways (i.e. with different paths), as we will explore later. As examples, the function $p : [0,1] \to \mathbb{R}^3, p(\lambda) = (2\lambda, \lambda^2, -\lambda)$ is a path that describes a section of a parabola starting at the point $(0,0,0)$ and ending at $(2,1,-1)$, while $q : \mathbb{R} \to \mathbb{R}^3, q(\lambda) = (\cos\lambda, \sin\lambda, \lambda)$ describes a helix winding around the z axis.

Given a function $f : \mathbb{R}^3 \to \mathbb{R}$ defined for arbitrary points in 3D space, we can integrate the function along a curve C by breaking the integration variable (position, $\mathbf{r}$) into its Cartesian components:

$$\begin{aligned}\int_C f(x,y,z)\mathrm{d}\mathbf{r} &= \int_C f(x,y,z)\left(\mathrm{d}x\hat{\mathbf{i}} + \mathrm{d}y\hat{\mathbf{j}} + \mathrm{d}z\hat{\mathbf{k}}\right) \\ &= \hat{\mathbf{i}}\int_C f(x,y,z)\mathrm{d}x + \hat{\mathbf{j}}\int_C f(x,y,z)\mathrm{d}y + \hat{\mathbf{k}}\int_C f(x,y,z)\mathrm{d}z\,,\end{aligned} \tag{1}$$

where we are able to take the unit vectors outside the integral because they are constant.

Depending on the situation, it may be possible to evaluate the integral directly from the form above. Other times, if we are able to write the curve in terms of a path, then we can compute the integral using substitution:

$$\int_C f(x,y,z)\mathrm{d}\mathbf{r} = \int_{\lambda_0}^{\lambda_1} f(x(\lambda),y(\lambda),z(\lambda))\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\lambda}\mathrm{d}\lambda\,. \tag{2}$$

15.1 Computing Line Integrals without Parametrisation

If we can express the curve in a non-parametric form, we may want to compute it directly. For example, suppose a ball is rolling on a parabolic track whose height is given by $y = 3x^2$. We need to compute the integral between the bottom of the parabola and $x = 2$ of some function f that depends on both the x and y coordinates as $f(x,y) = x^3y$. In this situation, there is no need to find a path to describe the curve in terms of some parameter, since we already have a perfectly good description of the system in terms of x and y.

$$
\begin{aligned}
\int_C f(x,y) &= \widehat{\mathbf{i}}\int_C f(x,y)\mathrm{d}x + \widehat{\mathbf{j}}\int_C f(x,y)\mathrm{d}y \\
&= \widehat{\mathbf{i}}\int_{(0,0)}^{(2,12)} x^3 y\mathrm{d}x + \widehat{\mathbf{j}}\int_{(0,0)}^{(2,12)} x^3 y\mathrm{d}y \\
&= \widehat{\mathbf{i}}\int_0^2 x^3 \times 3x^2\mathrm{d}x + \widehat{\mathbf{j}}\int_0^{12} \left(\frac{y}{3}\right)^{\frac{3}{2}} \times y\mathrm{d}y \\
&= \widehat{\mathbf{i}}\left[\frac{1}{2}x^6\right]_0^2 + \widehat{\mathbf{j}}\left[\frac{2}{7\times 3^{3/2}}y^{7/2}\right]_0^{12} \\
&= 32\widehat{\mathbf{i}} + 329.1\widehat{\mathbf{j}}\,.
\end{aligned} \tag{3}
$$

Notice that there are various ways we could have computed this integral. Above, we have used x as the integration variable for the first term and y for the second term. Instead we could have used the chain rule to replace $\mathrm{d}y$ with $6x\mathrm{d}x$ in the second term and computed both integrals with respect to x. Equally, we could have written $\mathrm{d}x$ as $\mathrm{d}y/\sqrt{3y}$ in the first term. This leads us neatly into the second method of computation, in which we express the curve as a path in terms of a parameter.

15.2 Parametrisation

In the previous section, if we had computed the integral using x as the integration variable in both terms, then arguably we are already using a parametrisation of the curve. It just happens to be that we have used one of the coordinates as the parameter. More generally, we can parametrise a curve however we like, but the method of integration is the same: we use substitution to write the integral entirely in terms of the parameter. For example, we could have written the parabola in the previous example as $x = \lambda^2$, $y = 3\lambda^4$, $\lambda \in [0, \sqrt{2}]$. In this case, the integral would have become

$$
\int_C f(x,y)\mathrm{d}\mathbf{r} = \int_C f(x(\lambda), y(\lambda))\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\lambda}\mathrm{d}\lambda\,, \tag{4}
$$

where

$$
f(x(\lambda), y(\lambda)) = \left(\lambda^2\right)^3 \times 3\lambda^4 = 3\lambda^{10} \tag{5}
$$

and

$$
\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\lambda} = \frac{\mathrm{d}x}{\mathrm{d}\lambda}\widehat{\mathbf{i}} + \frac{\mathrm{d}y}{\mathrm{d}\lambda}\widehat{\mathbf{j}} = 2\lambda\widehat{\mathbf{i}} + 12\lambda^3\widehat{\mathbf{j}}\,. \tag{6}
$$

It is now simply a case of substituting everything and computing the integral. Of course, in this case, we do not need to, since we already know the answer is $32\widehat{\mathbf{i}} + 329.1\widehat{\mathbf{j}}$ – the result cannot depend on our parametrisation.

Notice that our first approach is the integral of a scalar f with respect to a vector $\mathbf{r}$, while the second is the integral of a vector $f\mathrm{d}\mathbf{r}/\mathrm{d}\lambda$ with respect to a scalar λ.

15.3 Scalar Line Integrals

Once we have the idea of a parametrised line integral, we can also compute scalar line integrals. These are integrals of scalar quantities with respect to a scalar integration variable but over a non-trivial curve in space. The best way to see this is with an example.

15.4 Example

A metal spring is defined by the helical path $\mathbf{p} = \mathbf{p}(\lambda) = (2\cos\lambda, 2\sin\lambda, 2\lambda)$, for $\lambda \in [0, 8\pi]$. What is the length of the spring (if uncoiled)?

A short section of the spring has length

$$\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}\,, \tag{7}$$

which we can express in terms of the parameter through the chain rule

$$\begin{aligned} \mathrm{d}s &= \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\lambda}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\lambda}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}\lambda}\right)^2}\ \mathrm{d}\lambda \\ &= \sqrt{4\sin^2(\lambda) + 4\cos^2(\lambda) + 2^2}\ \mathrm{d}\lambda \\ &= \sqrt{8}\ \mathrm{d}\lambda\,. \end{aligned} \tag{8}$$

Since λ varies from 0 to 8π, it is clear that the length of the spring is $8\sqrt{8}\pi$. In a more general case, we could determine this by integrating $\int \mathrm{d}s$.

15.5 Example

The spring was made in a faulty machine and its local density depends on which side of the machine it was in contact with. The linear density is given by $\rho(x, y, z) = x^2 - 3z$. What is the mass of the spring?

15.6 Re-parametrisation

Notice that the curve in the previous examples could have been parametrised differently. As a trivial example of this, we could have used the parameter $\tau \in [0, 4\pi]$ and parametrised the curve as $\mathbf{p} = \mathbf{r}(\tau) = (2\cos(2\tau), 2\sin(2\tau), \tau)$. A less obvious but equally valid re-parametrisation for the first short section of the curve is $(\sqrt{\xi}, \sqrt{4-\xi}, 2\cos^{-1}(\sqrt{\xi/4}))$. We would have to do some awkward things to account for the positive and negative roots if we wanted to express the whole curve with this parametrisation, but we can see how it could be done in principle. The parametrisation that we use cannot alter the calculation of the mass. Nor will it thanks to the extra factor introduced by the substitution of one parameter for another, via the chain rule, e.g. $\mathrm{d}\lambda/\mathrm{d}\tau$. From these examples, hopefully it is clear that the parametrisation does not necessarily have any obvious physical significance: it is simply a means of specifying the curve. Other than strictly increasing continuously from one end of the path to the other, there are very few restrictions on what the parameter can be.

Notice also that uncoiling the spring would not change its mass. This may help to understand what it is that a line integral represents: we can uncoil the spring and lay it out along a straight line from 0 to $8\sqrt{8}\pi$. The mass of the spring is still given by an integral but now it is an ordinary 1D integral.

15.7 Scalar Products

Often, we want to calculate line integrals of the form

$$\int_C \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r}\,, \tag{9}$$

where the integrand $\mathbf{F}(\mathbf{r})$ is vector-valued, for example when calculating the work done by a force as an object moves along some path C. The method for calculating these integrals is no different to the scalar case. We begin by writing the differential in terms of its Cartesian components:

$$\mathrm{d}\mathbf{r} = \mathrm{d}x\widehat{\mathbf{i}} + \mathrm{d}y\widehat{\mathbf{j}} + \mathrm{d}z\widehat{\mathbf{k}} \tag{10}$$

and taking the scalar product

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C F_x \mathrm{d}x + \int_C F_y \mathrm{d}y + \int_C F_z \mathrm{d}z\,. \tag{11}$$

Since $\mathbf{F}(\mathbf{r})$ can be expressed in terms of x, y and z, we can substitute these into the three separate integrals above. For the specified curve, if we can write one or more of the coordinates in terms of the others, then we can use further substitutions to write everything in terms of the integration variable. On the other hand, if we have a path for the curve, we can use further substitution to write everything in terms of a parameter and compute the integrals that way.

15.8 Exercise (taken from Riley, Hobson and Bence)

For the function $\mathbf{F}(x, y) = (x + y)\widehat{\mathbf{i}} + (y - x)\widehat{\mathbf{j}}$, compute the integrals

$$I = \int_C \mathbf{F} \cdot \mathbf{r} \tag{12}$$

over

1. the parabola $y^2 = x$ from $(1, 1)$ to $(4, 2)$,
2. the curve $x = 2t^2 + t + 1$, $y = 1 + t^2$ from $(1, 1)$ to $(4, 2)$.

Notice that the result is **path-dependent**: in general it matters which route we take from one point to another when computing integrals. This is not something we need to consider with 1D integrals, since there is only one possible route.