UNIVERSITY OF LEEDS

Mechanics

Session 10 – Systems of Particles: Collective & Relative Motion

DR BEN HANSON

1

Last Lecture

Tension & Newton's Third Law

We learned:

- That the total energy of a closed system is always conserved
- That moving objects retain energy in the form of kinetic energy
- That conservative forces retain energy in the form of potential energy
- That non-conservative forces change energy into heat, sound, light and other forms of radiation

You should be able to:

- Use the concepts of energy conservation to calculate the positions and velocities of moving objects

2

This Lecture

Systems of Particles

We will:

- Learn how relative motion works in classical physics
- Understand that acceleration, and therefore force, is sometimes measured differently in different reference frames
- Derive the location of the centre of mass of a collection of particles

You will be able to:

- Calculate the relative speed between two moving objects
- Calculate the centre of mass of a collection of particles
- Calculate the velocity (and potentially the acceleration) of a collection of particles using its centre of mass dynamics

Relative Motion

Velocities

Relative Motion

What is it?

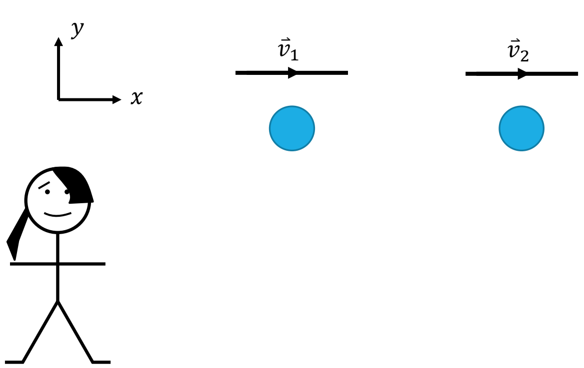
Relative motion is about how one object moves "as perceived by" another object

Everything we've done so far has been relative to a static, external observer. But as you will learn in your relativity lectures, this is not the only way to view things…

Relative Motion

Static Background

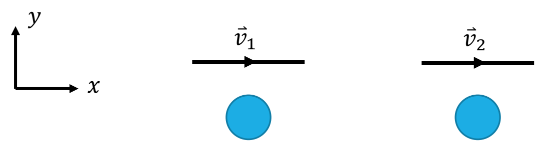
Consider two particles and an observer…

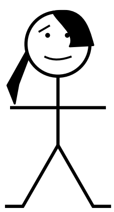


Relative Motion

Static Background

Consider two particles and an observer…



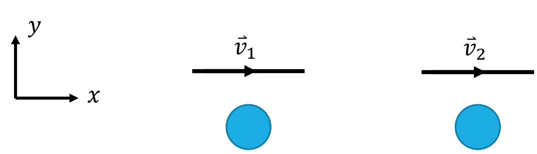


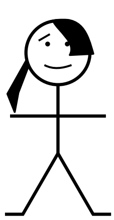
$$|\vec{v}_2| > |\vec{v}_1|$$

Relative Motion

Static Background

Consider two particles and an observer…





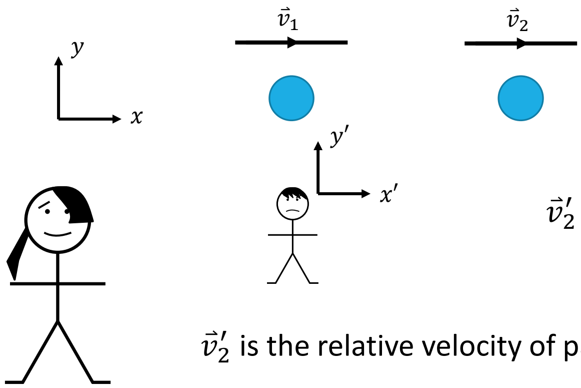
$$|\vec{v}_2| < |\vec{v}_1|$$

These velocities are with respect to our static observer!

Relative Motion

Not Static Background

Now let's add a moving observer…



$$\vec{v}_2' = \vec{v}_2 - \vec{v}_1$$

$$\vec{v}_{12} = \vec{v}_2 - \vec{v}_1$$

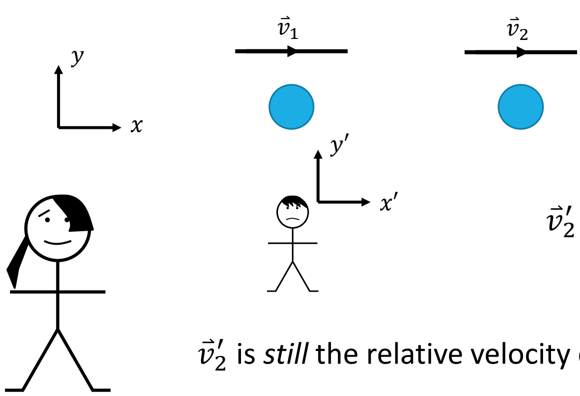
$\vec{v}_2'$ is the relative velocity of particle 2 with respect to particle 1[1]

[1]And our observer, who is moving with the same velocity as particle 1

Relative Motion

Not Static Background

Now let's add a moving observer…



$$\vec{v}_2' = \vec{v}_2 - \vec{v}_1$$

$$\vec{v}_{12} = \vec{v}_2 - \vec{v}_1$$

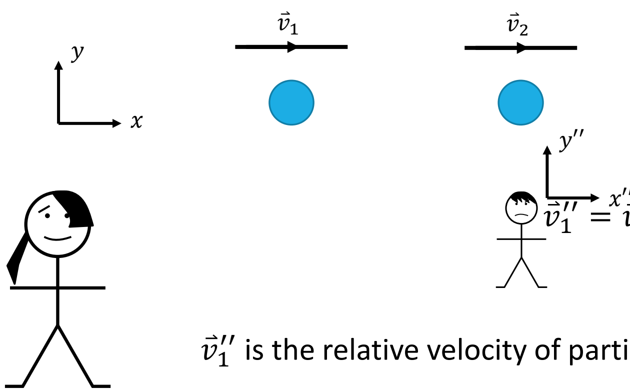
$\vec{v}_2'$ is *still* the relative velocity of particle 2 with respect to particle 1[1]

[1]And our observer, who is moving with the same velocity as particle 1

Relative Motion

Not Static Background

Now let's add a moving observer…



$\vec{v}_1'' = \vec{v}_1 - \vec{v}_2$

$\vec{v}_{12} = \vec{v}_1 - \vec{v}_2$

$\vec{v}_1''$ is the relative velocity of particle 1 with respect to particle 2[1]

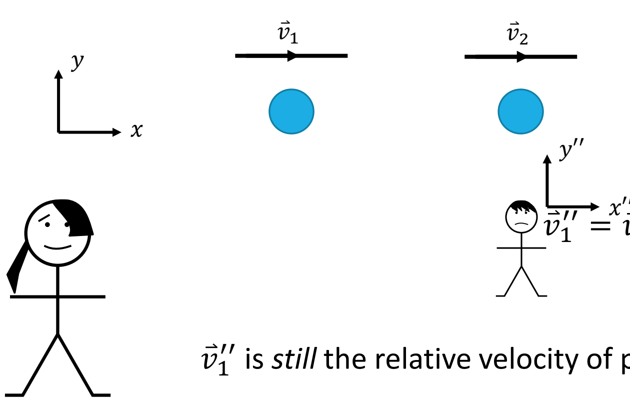
[1]And our observer, who is moving with the same velocity as particle 2

MECHANICS 1 – SYSTEMS OF PARTICLES

Relative Motion

Not Static Background

Now let's add a moving observer…



$\vec{v}_1'' = \vec{v}_1 - \vec{v}_2$

$\vec{v}_{12} = \vec{v}_1 - \vec{v}_2$

$\vec{v}_1''$ is *still* the relative velocity of particle 1 with respect to particle 2[1]

[1]And our observer, who is moving with the same velocity as particle 2

Relative Motion

Acceleration and Forces

Relative Motion

Acceleration and Forces

If velocity can be perceived differently in different “reference frames,” so too can acceleration.

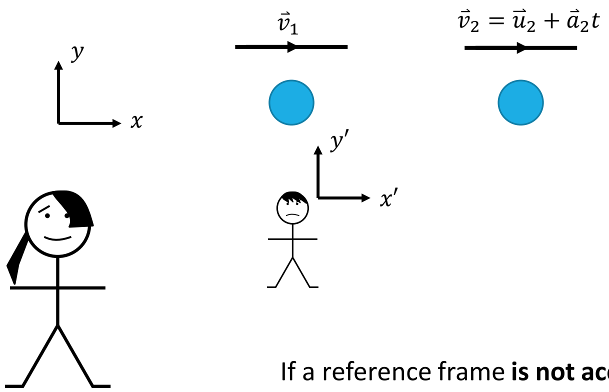
So…what does that mean for forces?

The force we measure on an object is dependent on the reference frame we are in! **This is a tricky concept, so I want you to ask questions as we go.**

Relative Motion

Acceleration and Forces

Let's have an accelerating object…

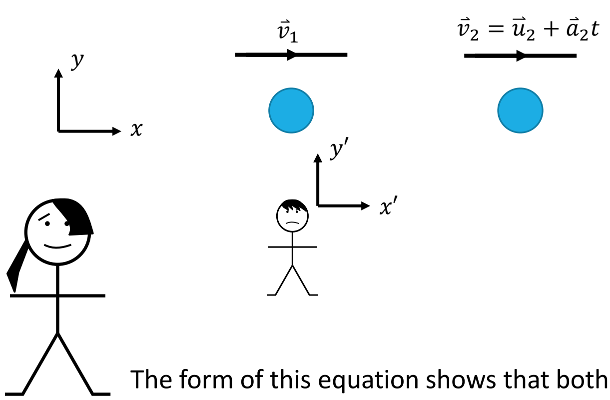


If a reference frame **is not accelerating**, it is an **inertial reference frame**

Relative Motion

Acceleration and Forces

Let's have an accelerating object…



$$\vec{v}_2' = \vec{v}_2 - \vec{v}_1$$
$$\vec{v}_2' = \vec{u}_2 + \vec{a}_2 t - \vec{v}_1$$
$$\vec{v}_2' = (\vec{u}_2 - \vec{v}_1) + \vec{a}_2 t$$
$$\vec{v}_2' = \vec{u}_2' + \vec{a}_2 t$$

The form of this equation shows that both observers see the same acceleration of particle 2!

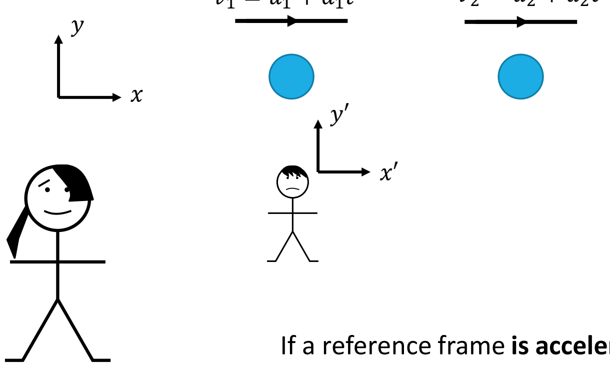
Relative Motion

Acceleration and Forces

Let's have two accelerating objects…

$\vec{v}_1 = \vec{u}_1 + \vec{a}_1 t$

$\vec{v}_2 = \vec{u}_2 + \vec{a}_2 t$



If a reference frame **is accelerating**, it is a **non-inertial reference frame**

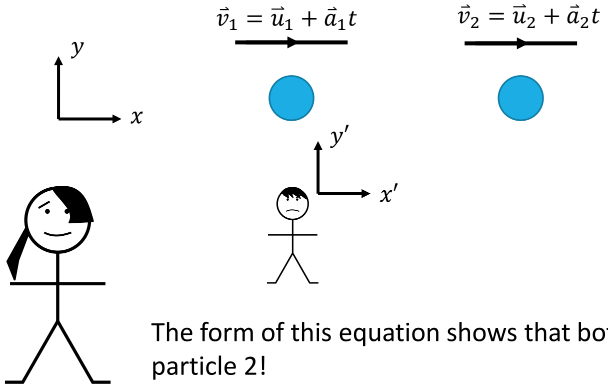
Relative Motion

Acceleration and Forces

Let's have two accelerating objects…

$\vec{v}_1 = \vec{u}_1 + \vec{a}_1 t$

$\vec{v}_2 = \vec{u}_2 + \vec{a}_2 t$



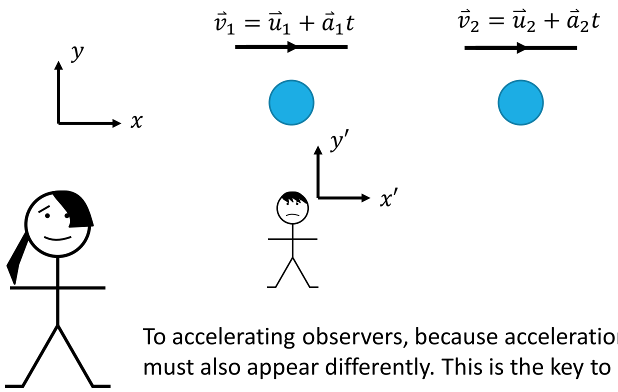
$$\vec{v}_2' = \vec{v}_2 - \vec{v}_1$$
$$\vec{v}_2' = (\vec{u}_2 + \vec{a}_2 t) - (\vec{u}_1 + \vec{a}_1 t)$$
$$\vec{v}_2' = (\vec{u}_2 - \vec{u}_1) + (\vec{a}_2 - \vec{a}_1)\, t$$
$$\vec{v}_2' = \vec{u}_2' + \vec{a}_2' t$$

The form of this equation shows that both observers **do not** see the same acceleration of particle 2!

Relative Motion

Acceleration and Forces

Let's have two accelerating objects…



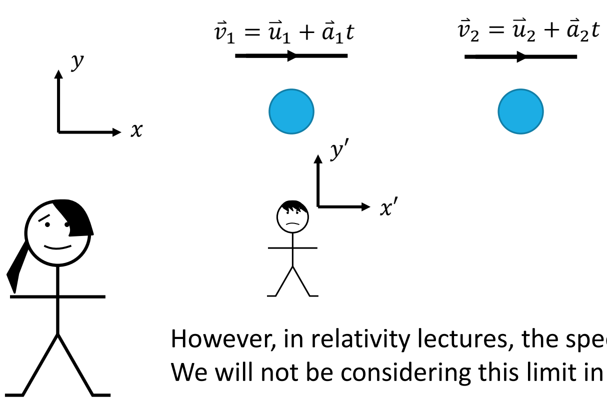
To accelerating observers, because acceleration appears differently, the force causing that acceleration must also appear differently. This is the key to the ideas of both relativity and circular motion

MECHANICS 1 – SYSTEMS OF PARTICLES

Relative Motion

Acceleration and Forces

Let's have two accelerating objects…



However, in relativity lectures, the speed of light must remain constant across frames. We will not be considering this limit in the rest of the Mechanics course. However…

Task 1

Relative Motion (Joint Task with Dr Purdy)

Task 1

Relative Motion (Joint Task with Dr Purdy)

Scenario: Two beginner athletes are competing in the 100m. When they start the race, the first athlete takes the lead, running at a constant speed of $v_1 = 2.7ms^{-1}$ for the entire race as measured by a static, external observer. The second athlete is quite equally matched, running at $v_2 = 2.1ms^{-1}$ for the entire race as measured by a static, external observer. After the race, the athletes continue running at these speeds.

Tasks:

1. Calculate the relative speed of athlete 2 with respect to athlete 1
2. Calculate the relative speed of athlete 1 with respect to athlete 2
3. When athlete 1 reaches the finish line, how far behind is athlete 2 from the perspective of athlete 1?
4. When athlete 1 reaches the finish line, how long will it take athlete 2 to finish the race from the perspective of athlete 1?

Systems of Particles
Centre of Mass

Systems of Particles
Centre of Mass

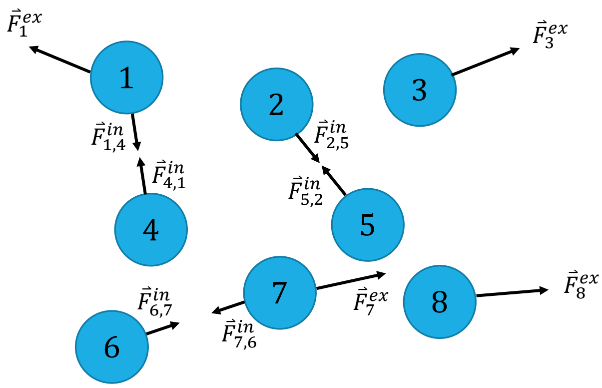
Most things in the universe are not single, isolated objects. Rather, they are large collections of objects (from atoms to stars & galaxies…)

So how can we model many objects simultaneously without having to manually calculate the equilibrium conditions for each and every one of them?
Use the centre of mass!

Systems of Particles

Centre of Mass

Consider many particles…



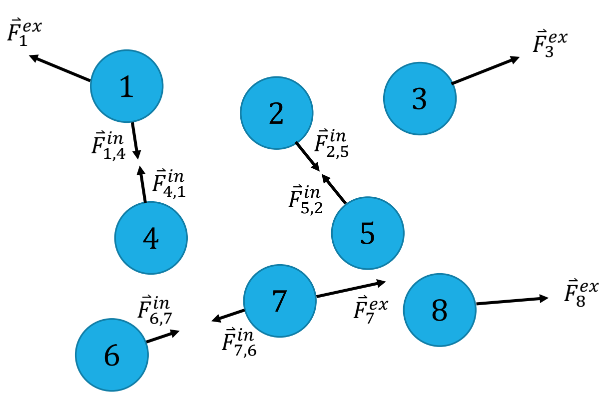
We may define two "types" of force:

1. External force, $\vec{F}_i^{ex}$
   - These forces are external to the system i.e. they are not caused by any of the other particles.
   - Example: External magnetic field
2. Internal force, $\vec{F}_{i,j}^{in}$
   - These forces are internal to the system i.e. they *are* caused by the other particles.
   - Example: Coulomb interaction

Systems of Particles

Centre of Mass

Consider many particles…



Net force on particle i,

$$\vec{F}_i = \vec{F}_i^{ex} + \sum_{j=1}^{N} \vec{F}_{j,i}^{in}$$

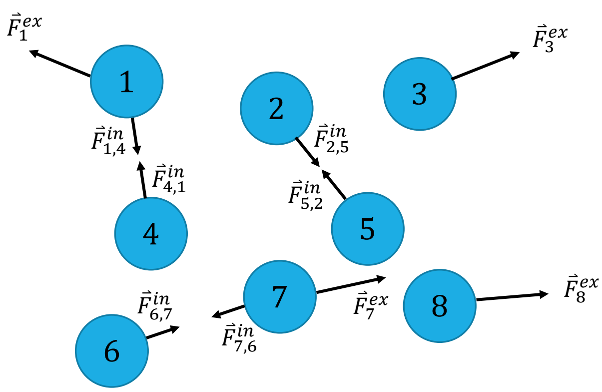
Newton's 2nd law,

$$m_i \vec{a}_i = \vec{F}_i^{ex} + \sum_{j=1}^{N} \vec{F}_{j,i}^{in}$$

Systems of Particles

Centre of Mass

Consider many particles…



Newton's 2[nd] law,

$$m_i\vec{a}_i = \vec{F}_i^{ex} + \sum_{j=1}^{N} \vec{F}_{j,i}^{in}$$

Net force on entire system,

$$\vec{F}_{Net} = \sum_{i=1}^{N} \vec{F}_i$$

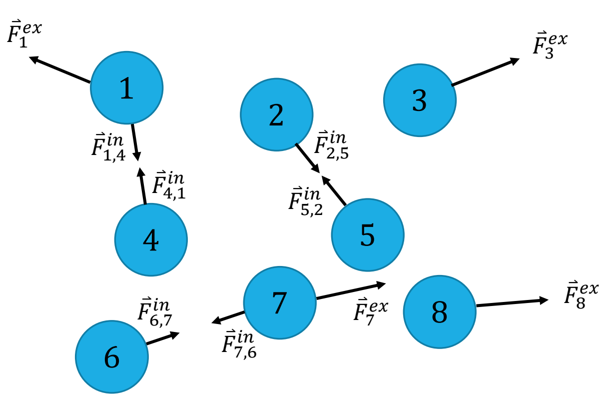
$$\vec{F}_{Net} = \sum_{i=1}^{N} \left(\vec{F}_i^{ex} + \sum_{j=1}^{N} \vec{F}_{j,i}^{in} \right)$$

$$\vec{F}_{Net} = \left(\sum_{i=1}^{N} \vec{F}_i^{ex} \right) + \left(\sum_{i=1}^{N}\sum_{j=1}^{N} \vec{F}_{j,i}^{in} \right)$$

Systems of Particles

Centre of Mass

Consider many particles…



Net force on entire system,

$$\vec{F}_{Net} = \left(\sum_{i=1}^{N} \vec{F}_i^{ex} \right) + \left(\sum_{i=1}^{N}\sum_{j=1}^{N} \vec{F}_{j,i}^{in} \right)$$

Newton's 3[rd] law,

$$\vec{F}_{j,i}^{in} = -\vec{F}_{i,j}^{in}$$

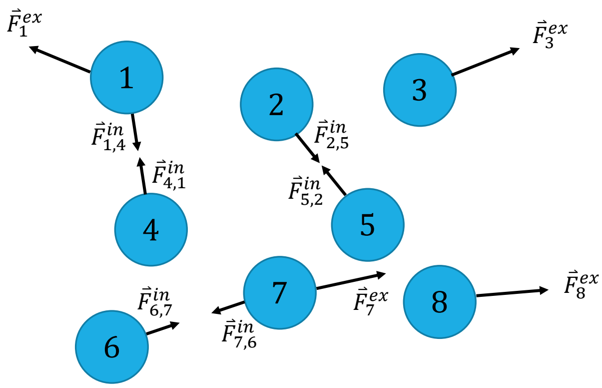
$$\rightarrow \boxed{\vec{F}_{Net} = \sum_{i=1}^{N} \vec{F}_i^{ex}}$$

Net force on the system is sum of external forces <u>only</u>!

Systems of Particles

Centre of Mass

Consider many particles…



$$\vec{F}_{Net} = \sum_{i=1}^{N} \vec{F}_i^{ex}$$

We need a location, a frame of reference, where the entire collection of particles behaves like a single system with total mass $M_T = \sum_{i=1}^{N} m_i$. Call this the centre of mass, CM.

Definition,

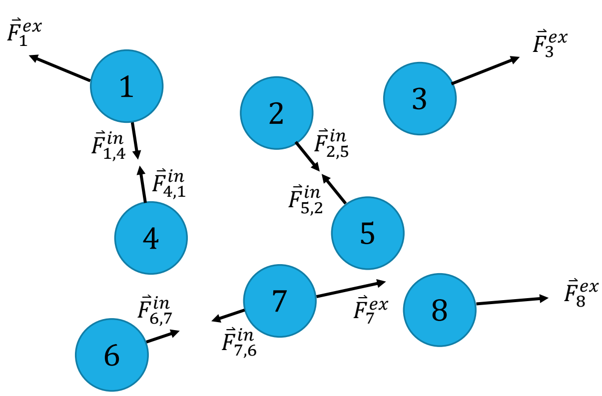
$$\vec{F}_{Net} = M_T \vec{a}_{CM}$$

$$\boxed{\rightarrow \quad M_T \vec{a}_{CM} = \sum_{i=1}^{N} m_i \vec{a}_i}$$

Systems of Particles

Centre of Mass

Consider many particles…



$$M_T \vec{a}_{CM} = \sum_{i=1}^{N} m_i \vec{a}_i$$

Integrate w.r.t time,

$$M_T \vec{v}_{CM} = \sum_{i=1}^{N} m_i \vec{v}_i$$

Integrate w.r.t time again,

$$M_T \vec{r}_{CM} = \sum_{i=1}^{N} m_i \vec{r}_i$$

$$\boxed{\vec{r}_{CM} = \frac{1}{M_T} \sum_{i=1}^{N} m_i \vec{r}_i}$$

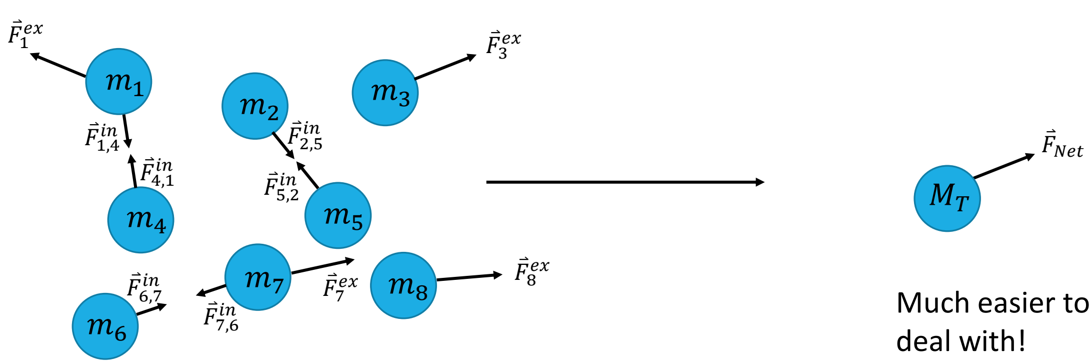
Systems of Particles

Properties & Dynamics of the Centre of Mass

Systems of Particles

Properties of the Centre of Mass

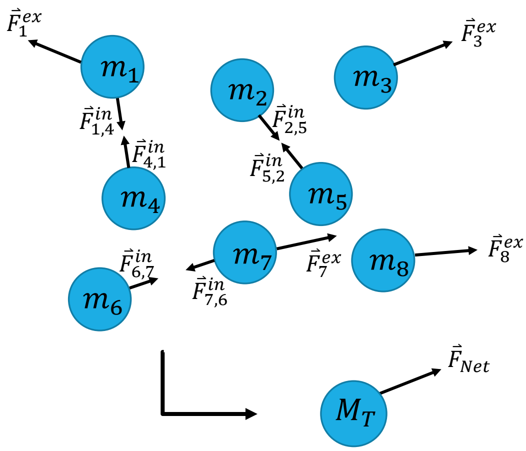
We can replace all particles by a single object subject only to the external forces on the system. This object has a mass M_T that is the sum of all masses in the system, and a location at the centre of mass, $\vec{r}_{CM}$



Much easier to deal with!

Systems of Particles

Properties of the Centre of Mass



Equations:

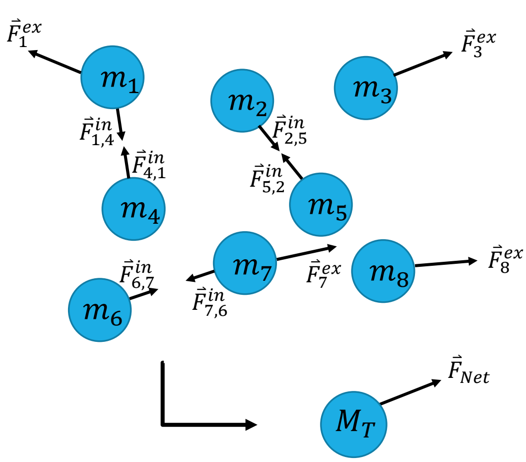
$$\vec{r}_{CM} = \frac{1}{M_T}\sum_{i=1}^{N} m_i \vec{r}_i$$

$$\vec{v}_{CM} = \frac{d\vec{r}_{CM}}{dt} = \frac{1}{M_T}\sum_{i=1}^{N} m_i \vec{v}_i$$

$$\vec{a}_{CM} = \frac{d\vec{v}_{CM}}{dt} = \frac{1}{M_T}\sum_{i=1}^{N} m_i \vec{a}_i$$

Systems of Particles

Properties of the Centre of Mass



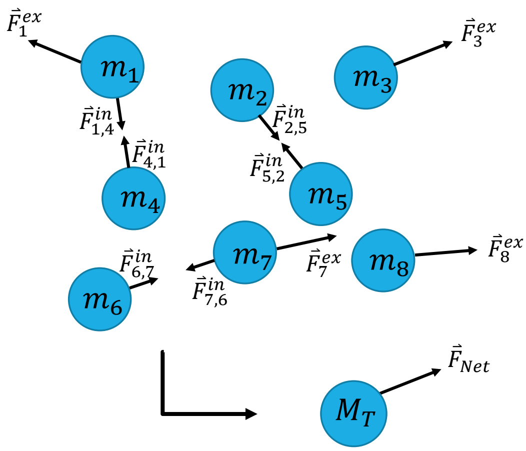
Properties:

$$\vec{v}_{CM} = \frac{1}{M_T}\sum_{i=1}^{N} m_i \vec{v}_i$$

If the velocity of the centre of mass is zero, it implies the sum of the momenta of all of the particles in the system is also zero

Systems of Particles

Properties of the Centre of Mass



Properties:

$$\vec{a}_{CM} = \frac{1}{M_T}\sum_{i=1}^{N} m_i \vec{a}_i$$

If the acceleration of the centre of mass is zero, it implies the sum of all (external) forces on the system is also zero

Thus, the centre of mass frame is an inertial frame if and only if there are no external forces acting on the system!

Task 2

Calculating the Centre of Mass Dynamics

Task 2

Calculating the Centre of Mass Dynamics

Scenario: Three particles have the following properties: Particle 1 has mass $m_1 = 1.5kg$ and is at position $\vec{r}_1 = (5\vec{\imath} + 3\vec{\jmath} - \vec{k})$. Particle 2 has mass $m_2 = 850g$ and is at position $\vec{r}_2 = (\vec{\imath} + \vec{\jmath} + \vec{k})$. Particle 3 has mass $m_3 = 1.75kg$ and is at position $\vec{r}_3 = (9\vec{k})$.

Tasks:

1. Calculate the centre of mass of the system

Task 3

More Calculating the Centre of Mass Dynamics

Task 3

More Calculating the Centre of Mass Dynamics

Scenario: Three particles have the following properties: Particle 1 has mass $m_1 = 1.5kg$ and is at position $\vec{r}_1 = (5t\vec{\imath} + 3t^2\vec{\jmath} - \vec{k})$. Particle 2 has mass $m_2 = 850g$ and is at position $\vec{r}_2 = t(\vec{\imath} + \vec{\jmath} + \vec{k})$. Particle 3 has mass $m_3 = 1.75kg$ and is at position $\vec{r}_3 = (9t^3\vec{k})$.

Tasks:

1. Calculate the centre of mass of the system as a function of time
2. Calculate the velocity of the centre of mass of the system as a function of time
3. Calculate the acceleration of the centre of mass of the system as a function of time

Task 4

Even More Calculating the Centre of Mass Dynamics

Task 4

Even More Calculating the Centre of Mass Dynamics

Scenario: Three charged particles have the following properties: Particle 1 has mass $m_1 = 1.5kg$, charge $q_1 = 8\mu C$, and is at position $\vec{r}_1 = \left(5\vec{\imath} + 3\vec{\jmath} - \vec{k}\right)$. Particle 2 has mass $m_2 = 850g$, charge $q_2 = 6\mu C$, and is at position $\vec{r}_2 = \left(\vec{\imath} + \vec{\jmath} + \vec{k}\right)$. Particle 3 has mass $m_3 = 1.75kg$, charge $q_3 = 3\mu C$, and is at position $\vec{r}_3 = \left(9\vec{k}\right)$. Each particle i is also subject to an external force $\vec{F}_i = 4N\,\vec{k}$

Tasks:

1. Calculate the acceleration of the centre of mass