UNIVERSITY OF LEEDS

Mechanics 1

Session 15: Circular Motion – The Moment of Inertia

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Last Lecture

Circular Motion – Force & Torque

We:

- Described different causes of centripetal force for different physical systems
- Derived the concept of torque, and how torques cause changes in angular speed

You should be able to:

- Calculate accelerations and forces in circular coordinates
- Calculate the torques about any axis

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This Lecture

Circular Motion – The Moment of Inertia

We will:
- Recall that torque "causes" angular acceleration, just as force "causes" linear acceleration
- Consider the idea of "rotational mass", otherwise known as the moment of inertia

You will be able to:
- Calculate the moment of inertia of a single object undergoing circular motion
- Calculate the moment of inertia of a set of objects undergoing collective circular motion
- Calculate the moment of inertia of a single, continuous object rotating about some axis

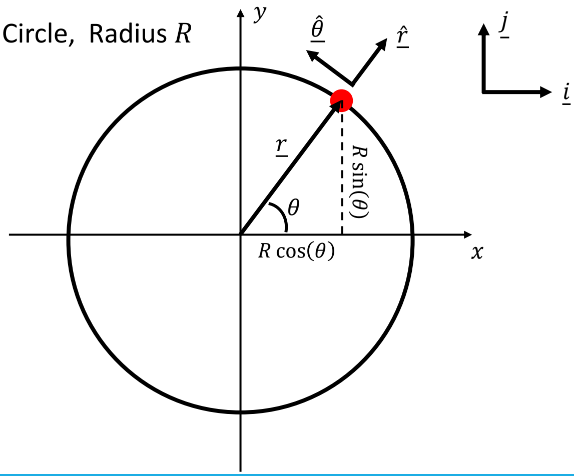
Theta Hat $(\underline{\hat{\theta}})$

Why does it have that equation?

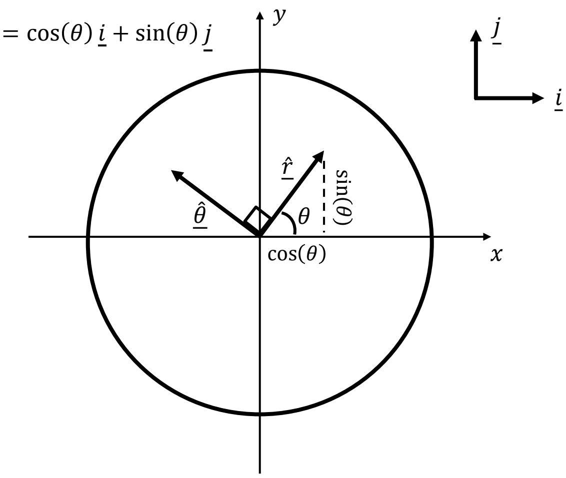
Theta Hat $(\underline{\hat{\theta}})$

Why does it have that equation?

Circle, Radius R

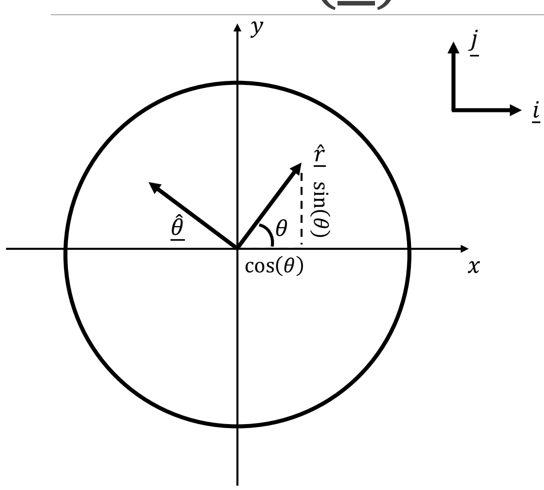


$\underline{\hat{r}} = \cos(\theta)\,\underline{i} + \sin(\theta)\,\underline{j}$



Theta Hat $(\underline{\hat{\theta}})$

Why does it have that equation?



$$\underline{\hat{r}} = \cos(\theta)\,\underline{i} + \sin(\theta)\,\underline{j}$$

$$\underline{\hat{\theta}} = -\sin(\theta)\,\underline{i} + \cos(\theta)\,\underline{j}$$

When rotating 90^o $\left(\frac{\pi}{2}\ radians\right)$:

1. Vector components also get rotated
2. *$\underline{i} \to \underline{j}, \quad \underline{j} \to -\underline{i}$*
3. *$\underline{\hat{\theta}}$ defined as $\frac{\pi}{2}\ radians$ w.r.t $\underline{\hat{r}}$*
4. Hence, $\underline{\hat{\theta}} = -\sin(\theta)\,\underline{i} + \cos(\theta)\,\underline{j}$
5. Crucially, $\frac{d\underline{\hat{r}}}{d\theta} = \underline{\hat{\theta}}$, $\frac{d\underline{\hat{\theta}}}{d\theta} = \underline{-\hat{r}}$

Torque

What is it actually doing?

Torque

What is it actually doing?

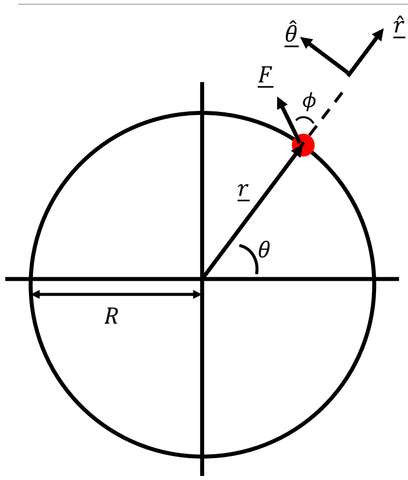
Torque causes angular acceleration…but why? Up to now we've only seen $\vec{\tau} = \vec{r} \times \vec{F}$, and there's no angular acceleration here…is there?

And on top of that, if torque causes angular acceleration, shouldn't we have something like $F = ma$?
Maybe $\tau = m\alpha$?
It's almost this, <u>but not quite</u>, as we'll see very shortly…

$\tau \neq m\alpha$. Torque <u>does not equal</u> mass times angular acceleration! I'm making a rhetorical point here, nothing more ☺

Moment of Inertia
Of a Single Particle

Moment of Inertia
Of a Single Particle



$$\vec{\tau} = \vec{r} \times \vec{F}$$

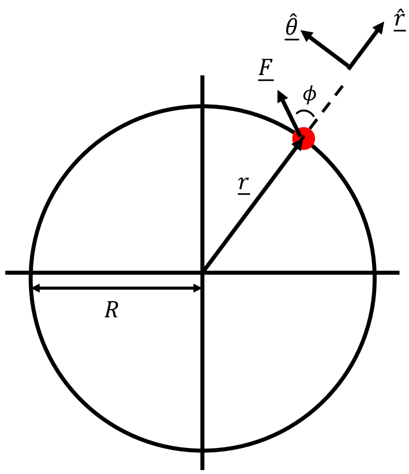
$$\vec{\tau} = |\vec{r}||\vec{F}| \sin(\phi)\hat{\underline{n}}$$

$\boldsymbol{\phi \neq \theta}$

$$\rightarrow |\vec{\tau}| = |\vec{r}||\vec{F}| \sin(\phi)$$

Moment of Inertia

Of a Single Particle



$$|\vec{\tau}| = |\vec{r}||\vec{F}|\sin(\phi)$$

$$|\vec{F}|\sin(\phi) = F_\theta, \qquad |\vec{\tau}| = |\vec{r}|F_\theta$$

$$|\vec{r}| = R, \qquad |\vec{\tau}| = RF_\theta$$

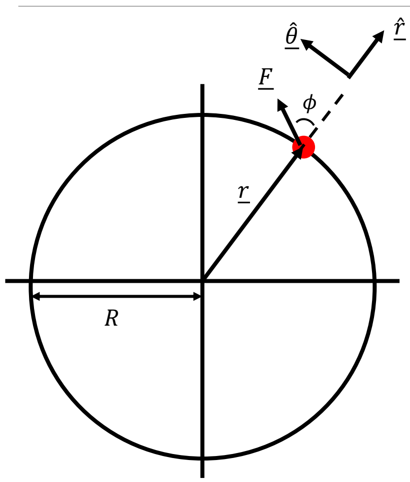
$$F_\theta = ma_\theta, \qquad |\vec{\tau}| = Rma_\theta$$

$$a_\theta = R\alpha\,, \qquad \boxed{|\vec{\tau}| = mR^2\alpha}$$

$$\boxed{|\vec{\tau}| = I\alpha}$$

Moment of Inertia

Of a Single Particle



For a single particle rotating around an axis with position vector $\hat{\underline{r}}$ and applied force $\underline{F}$:

The torque vector, $\vec{\tau} = \vec{r} \times \vec{F}$

The torque magnitude, $|\vec{\tau}| = I\alpha$

The Moment of Inertia, $I = mR^2$

Task 1

The Moment of Inertia of that Rope Thing Ben Built

Task 1

The Moment of Inertia of that Rope Thing Ben Built

Scenario: Ben is swinging that rope around again to display circular motion. The mass of the knot at the end of the rope $m = 3kg$, and we'll assume the rest of the rope is massless (for now!). Initially, Ben is swinging the rope in a circular path with radius $R = 1.2m$

Tasks:

1. Calculate the moment of inertia of the object
2. Calculate the moment of inertia of the object if I now decrease its radius to $R = 0.8m$, keeping the knot mass the same $m = 3kg$
3. Calculate the moment of inertia of the object if I now increase the knot mass to $m = 6.75kg$, keeping the radius $R = 0.8m$

Moment of Inertia

Of Multiple Particles (same axis)

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Moment of Inertia

Of Multiple Particles (same axis)

With multiple masses, we've seen that we can just add up all the (external) forces, and they can be viewed as acting on an object with total mass ($M_T = \sum_i m_i$) located at the centre of mass ($\vec{r}_{cm} = \frac{1}{M_T}\sum_i m_i \vec{r}_i$)

Is the same true for rotational mass i.e. multiple moments of inertia? Let's find out…

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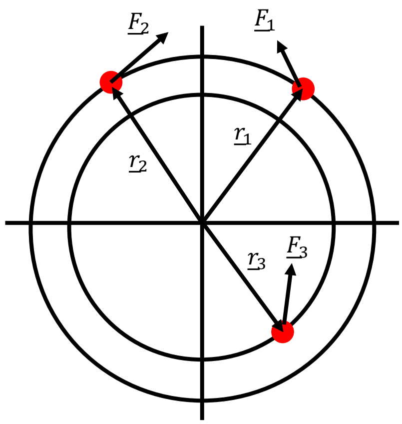
Moment of Inertia

Of Multiple Particles (same axis)



$$\vec{\tau}_{Net} = \sum_{i}^{N} \vec{r}_i \times \vec{F}_i$$

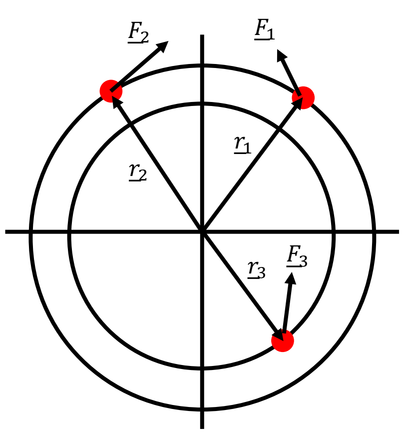
$$\vec{\tau}_{Net} = \sum_{i}^{N} |\vec{r}_i||\vec{F}_i| \sin(\phi_i)\underline{\hat{n}}$$

All same axis!

$$\rightarrow |\vec{\tau}_{Net}| = \sum_{i}^{N} |\vec{r}_i||\vec{F}_i| \sin(\phi_i)$$

Moment of Inertia

Of Multiple Particles (same axis)



$$|\vec{\tau}_{Net}| = \sum_{i}^{N} |\vec{r}_i||\vec{F}_i| \sin(\phi_i)$$

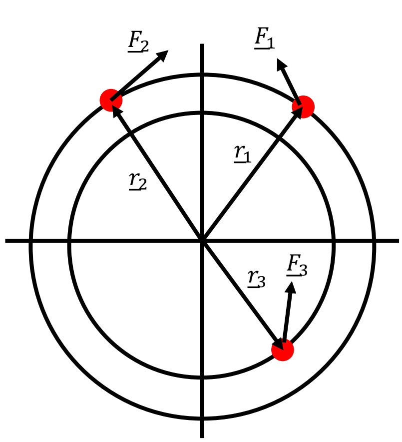
$$|\vec{F}_i| \sin(\phi_i) = F_{\theta,i}, \qquad |\vec{\tau}_{Net}| = \sum_{i}^{N} |\vec{r}_i| F_{\theta,i}$$

$$|\vec{r}_i| = R_i, \qquad |\vec{\tau}_{Net}| = \sum_{i}^{N} R_i F_{\theta,i}$$

$$F_{\theta,i} = m_i a_{\theta,i}, \qquad |\vec{\tau}_{Net}| = \sum_{i}^{N} R_i m_i a_{\theta,i}$$

Moment of Inertia

Of Multiple Particles (same axis)



$$F_{\theta,i} = m_i a_{\theta,i},$$

$$|\vec{\tau}_{Net}| = \sum_i^N R_i m_i a_{\theta,i}$$

$$a_{\theta,i} = R_i \alpha_i \ ,$$

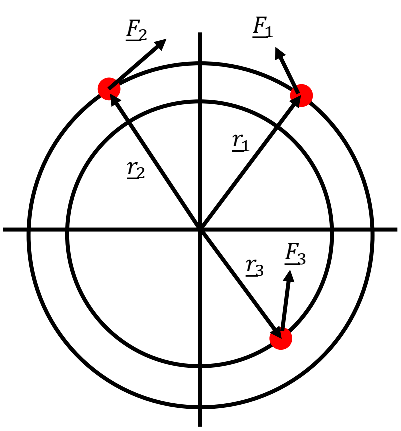
$$|\vec{\tau}_{Net}| = \sum_i^N m_i R_i^2 \alpha_i$$

$$\boxed{|\vec{\tau}_{Net}| = \sum_i^N I_i \alpha_i}$$

Torques (about the same axis) sum like forces do! Each torque on each particle corresponds to its own moment of inertia and angular acceleration!

Moment of Inertia

Of Multiple Particles (same axis)



If, <u>and only if</u>, all $\alpha_i = \alpha$ i.e. all rotational accelerations equal,

$$|\vec{\tau}_{Net}| = \sum_i^N I_i \alpha$$

$$|\vec{\tau}_{Net}| = \alpha \sum_i^N I_i$$

Define net moment of inertia,

$$I_{Net} \alpha = \alpha \sum_i^N I_i$$

Cancel,

$$\boxed{I_{Net} = \sum_i^N I_i}$$

This is not a general result! I'm showing that all α_i must be equal to define a "net" moment of inertia

Task 2

Conceptualising the Moment of Inertia

MECHANICS 1: CIRCULAR MOTION – THE MOMENT OF INERTIA

Task 2

Conceptualising the Moment of Inertia

Tasks:

1. With your neighbours, discuss the following:
    1. Given that $\tau = I\alpha$ is the rotational equivalent of $F = ma$, explain what would happen if, for the same applied torque, I decreased the moment of inertia (i.e. the "rotational mass").
    2. Olympic divers, after jumping from the diving board, often want to do as many spins as they can before they hit the water. Using the concept of moment of inertia, explain why they often tuck their legs and arms as close to their bodies as they can.
    3. If things spin more quickly with low moments of inertia, why do our wheelchairs, our bicycles, our cars, not have really, really, really small wheels?

Task 3

If it has mass, then it has a moment of inertia!

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If it has mass, then it has a moment of inertia!

Scenario:
A fairground ride is made up of 8 seats attached to a central axis via cylindrical metal bars. 4 of the seats are at a radius $R_1 = 5m$, and the other 4 are at a radius $R_2 = 7m$. The ride rotates all passengers at the same angular acceleration α. Each seat has mass $m_s = 450kg$.

Tasks:
1. Draw a diagram of the situation
2. Calculate the moment of inertia if 2 of each type of seat (so 4 total) are populated by passengers of mass $m_p = 65kg$. Assume the metal bars are massless.
3. The fairground ride wants to reach an angular speed $\omega = 3.14\ rads.s^{-1}$ in 10 seconds. What torque is required to achieve this?
4. Now, consider that the metal bars are not massless at all, but have mass $m_b = 200kg$. The moment of inertia of a metal bar rotating about the end, $I = \frac{1}{3}m_b L^2$, where L is the length of the bar[1]. Recalculate your answers to parts 2 and 3. How much more torque is now required?
5. The force applied to these rides by their engines is within the central axis itself. The axis has radius $R_{ax} = 1.2m$. What is the minimum force the engine must apply to achieve the required torque in part 4?

[1]We will see why this is next lecture