11 Integration

11.1 Definitions and Notation

Since each function has a unique derivative function, we can define an **anti-derivative** of a function f as any function F that has f as a derivative. If two functions F and G differ by a constant ($G(x) \equiv F(x) + c$ for constant c), F and G have the same derivative. Therefore, F and G are both anti-derivatives of the same function. Furthermore, it can be shown that any two functions with the same derivative can **only** differ by a constant (though we will not prove this here).[1] This means that any function f has an infinite number of anti-derivative functions parametrised by an additive constant C.

Given a smooth function, $f(x)$, we often wish to find the area defined by the curve of the function. In particular, we wish to find the area enclosed by the curve of the function, the horizontal line of the x axis, and two vertical lines at $x = a$ and $x = b$ on the graph, where a and b are numbers chosen by us for the particular problem we wish to solve.

Our approach will be similar to that for differentiation. We find a suitable estimate for the area for small variations of the input x, which improves as those variations, δx, decrease. Then we take the limit of that estimate as $\delta x \to 0$. In this case, a suitable estimate of the area under the curve is given by partitioning the x axis between a and b (the interval $[a, b]$) into a set of smaller intervals. For some large number n, let $\delta x = (b - a)/n$ and define a set of points x_k as $x_0 = a$, $x_1 = a + \delta x$, $x_2 = a + 2\delta x, \ldots$, $x_k = a + k\delta x$, $\ldots$, $x_n = b$.[2] Our estimate of the area under the curve is then given by the sum of the areas of the rectangles defined by these points and the curve:

$$\sum_{k=0}^{n-1} f(x_k)\delta x \,. \tag{1}$$

Now if we increase n, δx decreases. We get a larger number of narrower rectangles and a better estimate of the true area. So just as with differentiation, we take the limit that $\delta x \to 0$ to arrive at the integral:

$$\int_a^b f(x)\mathrm{d}x \equiv \lim_{\delta x \to 0} \sum_{k=0}^{n-1} f(a + k\delta x)\delta x \,. \tag{2}$$

This process defines the integral of the function between the specific **defined** limits a and b, and so is known as the **definite** integral. As with differentiation, we also wish to generalise this to define a new **function** that encodes information about the integral at arbitrary points. In particular, let's fix the lower limit a,

[1] The formal proof is not difficult and consists of showing that only constant functions differentiate to 0.

[2] The rigorous definition of integration is actually more general than this and allows for unequal partitions, but this simplified approach is sufficient for our purposes. We're only physicists after all...

but allow the upper limit to take arbitrary values y. This then defines a function

$$F(y) = \int_a^y f(x)\mathrm{d}x\,. \tag{3}$$

Notice that we have not used the variable x as the input of this new function F, but have introduced a new variable y. This is because x and y play different roles and it is important to distinguish them. y is the value chosen by us that we can input to the function F to determine the definite integral of f at that point. x ranges over every value between a and y to give a set of values which are summed over to determine the integral: we have no direct influence over x. In fact, we could change the symbol without affecting the calculation. Look again at Eq. 3. In words, it says "evaluate the function f at every value of x between a and y, multiply by the small change δx and sum". We achieve exactly the same if we rephrase this without explicit reference to x: "evaluate the function f at every value between a and y, multiply by a small change in that value and sum". For this reason, the x in this example is known as a **dummy variable** and we are free to relabel it **almost** as we please:

$$F(y) = \int_a^y f(x)\mathrm{d}x = \int_a^y f(t)\mathrm{d}t = \int_a^y f(\theta)\mathrm{d}\theta\,, \tag{4}$$

without changing the meaning. There is, however, one variable name that we **cannot** use: y. Although y is a variable from our perspective, once we have chosen a value for it, as far as the calculation of Eq. 3 is concerned, y is a fixed value. If we try to rephrase the calculation as "evaluate the function f at every value y between a and y..." it is immediately obvious that this does not make sense. As such it is very important to keep the use of the two variables clear and distinct.

11.2 Counter-Example

Consider a particle with a time-dependent velocity given by $v(t)$. Suppose we know the velocity at time 0 and wish to find the displacement of the particle at an arbitrary time t. Notice that the above discussion of dummy variables means we cannot write, for example,

$$s(t) = \int_0^t v(t)\mathrm{d}t \qquad \textbf{Wrong!!}$$

but should instead write, e.g.,

$$s(t) = \int_0^t v(\tau)\mathrm{d}\tau \qquad \textbf{Right!!}$$

11.3 The Fundamental Theorem of Calculus

One of the most important properties of integrals, that follows directly from the definition is that we can combine them as:

$$\int_a^b f(x)\mathrm{d}x + \int_b^c f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x\,. \tag{5}$$

Rather than determine the integrals of specific functions, let's jump straight to a very general result. The **Fundamental Theorem of Calculus** shows that differentiation and integration are inverse operations. To derive this result, we differentiate $F(y)$ with respect to y.

$$\begin{aligned}
\frac{\mathrm{d}F(y)}{\mathrm{d}y} &= \lim_{\delta y \to 0} \frac{\int_a^{y+\delta y} f(x)\mathrm{d}x - \int_a^y f(x)\mathrm{d}x}{\delta y} \\
&= \lim_{\delta y \to 0} \frac{\int_y^{y+\delta y} f(x)\mathrm{d}x}{\delta y} \\
&= \lim_{\delta y \to 0} \frac{f(y)\delta y}{\delta y} \\
&= f(y)
\end{aligned} \tag{6}$$

Differentiating an integral with respect to its upper limit gives back the integrand (the function that was integrated). Notice that this is true regardless of which constant value we choose as the lower limit. Recall that there are an infinite family of anti-derivatives that differ by a constant. This is consistent with the fact the integral is an anti-derivative, since

$$F(y) = \int_a^y f(x)\mathrm{d}x \qquad \text{and} \qquad G(y) = \int_b^y f(x)\mathrm{d}x\,, \tag{7}$$

then

$$F(y) - G(y) = \int_a^y f(x)\mathrm{d}x - \int_b^y f(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x\,, \tag{8}$$

which does not depend on y.

Now suppose we need to find the integral of a particular function between specific limits:

$$\int_{y_1}^{y_2} f(x)\mathrm{d}x\,.$$

Let $F(y)$ be some anti-derivative of $f(y)$. Then, by the fundamental theorem, $F(y)$ must be of the form

$$F(y) = \int_a^y f(x)\mathrm{d}x\,, \tag{9}$$

for some value of a. We then have

$$\int_{y_1}^{y_2} f(x)\mathrm{d}x = \int_a^{y_2} f(x)\mathrm{d}x - \int_a^{y_1} f(x)\mathrm{d}x = F(y_2) - F(y_1)\,, \tag{10}$$

and we have a neat algorithm for determining integrals in terms of an anti-derivative. Notice that it does not matter which anti-derivative we use, since anti-derivatives differ only by a constant, so

$$G(y_2) - G(y_1) = (F(y_2) + c) - (F(y_1) + c) = F(y_2) - F(y_1)\,. \tag{11}$$

An integral is only well-defined with limits. However, given the fundamental theorem above, it is convenient to **define** the **indefinite integral** of a function f as

$$\int f(x)\mathrm{d}x = F(x) + C\,, \tag{12}$$

where F is an anti-derivative and C is an arbitrary (unspecified) constant.[3] With this definition, we have the following standard integrals:

$$\begin{aligned} \int x^n \,\mathrm{d}x &= \frac{x^{n+1}}{n+1} + C \\ \int a^x \,\mathrm{d}x &= \frac{a^x}{\ln a} + C \\ \int \sin x \,\mathrm{d}x &= -\cos x + C \\ \int \cos x \,\mathrm{d}x &= \sin x + C\,, \end{aligned} \tag{13}$$

all of which follow from the fundamental theorem and derivatives of standard functions.

[3] It is worth emphasising, though, that this is a definition and really little more than an abuse of notation. The fundamental concept is the definite integral.