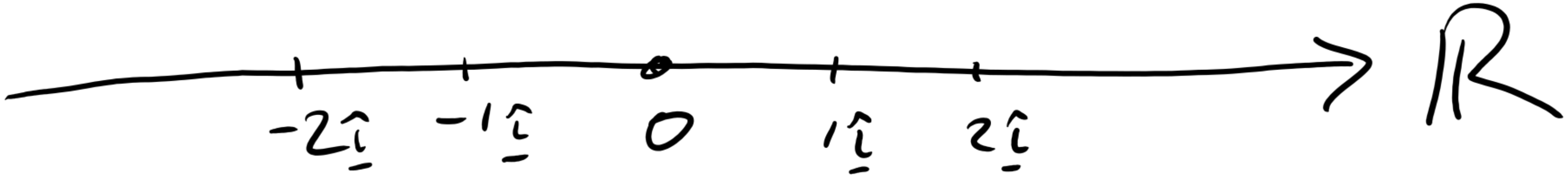
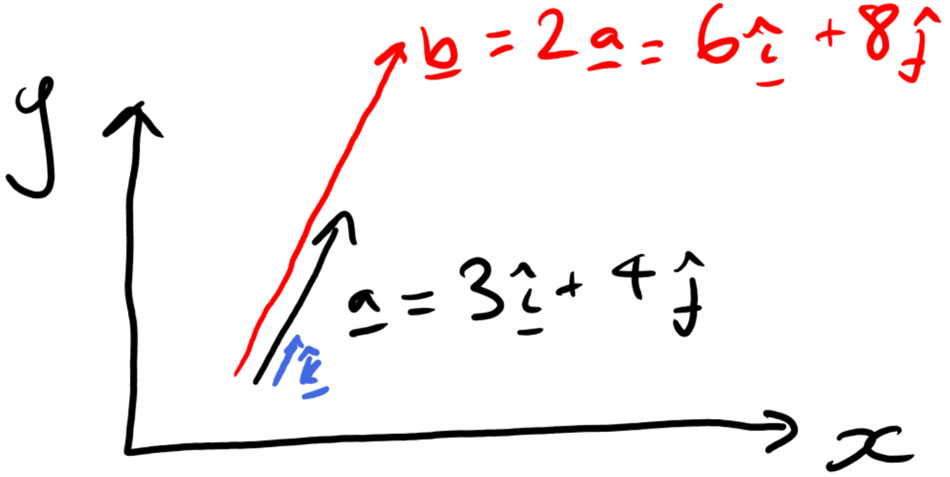
More On Vectors

Multiplication of Vectors

- 1D vectors are similar to ordinary real numbers…





$$\underline{\hat{k}} = \frac{3\underline{\hat{i}} + 4\underline{\hat{j}}}{|3\underline{\hat{i}} + 4\underline{\hat{j}}|}$$

$$\underline{a} = 5\underline{\hat{k}}$$

$$\underline{b} = 10\underline{k}$$

$$\underline{a} \cdot \underline{b} = 50$$

Multiplication of Vectors

- Desired properties:
  - Parallel vectors just multiply lengths

    $\underline{a} \cdot \underline{b} = |\underline{a}| \times |\underline{b}|$ if parallel

  - Perpendicular vectors multiply to 0

    independent subspaces

  - Distributive

$$\lambda(\underline{a} + \underline{b}) = \lambda \underline{a} + \lambda \underline{b}$$

$$\underline{a} \cdot (\underline{b} + \underline{c}) = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c}$$

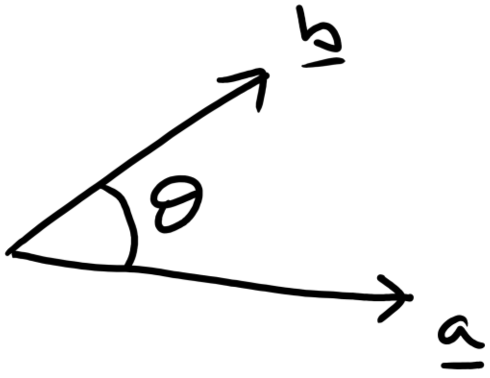
Scalar Product

$$\underline{a} \cdot \underline{b} = \underline{a} \cdot (\underline{b}_{\parallel} + \underline{b}_{\perp})$$

$$= \underline{a} \cdot \underline{b}_{\parallel} + \underbrace{\cancel{\underline{a} \cdot \underline{b}_{\perp}}}_{0}$$

$$= |\underline{a}| \times |\underline{b}_{\parallel}|$$

$$= |\underline{a}| \times |\underline{b}| \cos\theta$$



Special Cases and Useful Properties

$$\underline{a} \cdot \underline{a} = |\underline{a}| \times |\underline{a}| \cos(0)$$

$$= |\underline{a}|^2$$

$$|\underline{a}| \equiv \sqrt{\underline{a} \cdot \underline{a}} \quad \text{def}^n.$$

$$\hat{\underline{i}} \cdot \hat{\underline{i}} = \hat{\underline{j}} \cdot \hat{\underline{j}} = \hat{\underline{k}} \cdot \hat{\underline{k}} = 1$$

$$\hat{\underline{i}} \cdot \hat{\underline{j}} = \hat{\underline{j}} \cdot \hat{\underline{k}} = \hat{\underline{k}} \cdot \hat{\underline{i}} = 0$$

Scalar Product in Cartesian Coordinates

$$\underline{a} = a_1 \hat{\underline{i}} + a_2 \hat{\underline{j}} + a_3 \hat{\underline{k}}$$

$$\underline{b} = b_1 \hat{\underline{i}} + b_2 \hat{\underline{j}} + b_3 \hat{\underline{k}}$$

$$\underline{a} \cdot \underline{b} = (a_1 \hat{\underline{i}} + a_2 \hat{\underline{j}} + a_3 \hat{\underline{k}}) \cdot (b_1 \hat{\underline{i}} + b_2 \hat{\underline{j}} + b_3 \hat{\underline{k}})$$

$$= a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3 \neq \begin{pmatrix} a_1 b_1 \\ a_2 b_2 \\ a_3 b_3 \end{pmatrix}$$

Example

$$\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \\ -6 \end{pmatrix} = 2 \times 4 + (-3) \times 2 + 5 \times (-6)$$

$$= 8 - 6 - 30$$

$$= -28$$

Angle Between Vectors

$$\underline{a} \cdot \underline{b} = |\underline{a}| \times |\underline{b}| \cos\theta$$

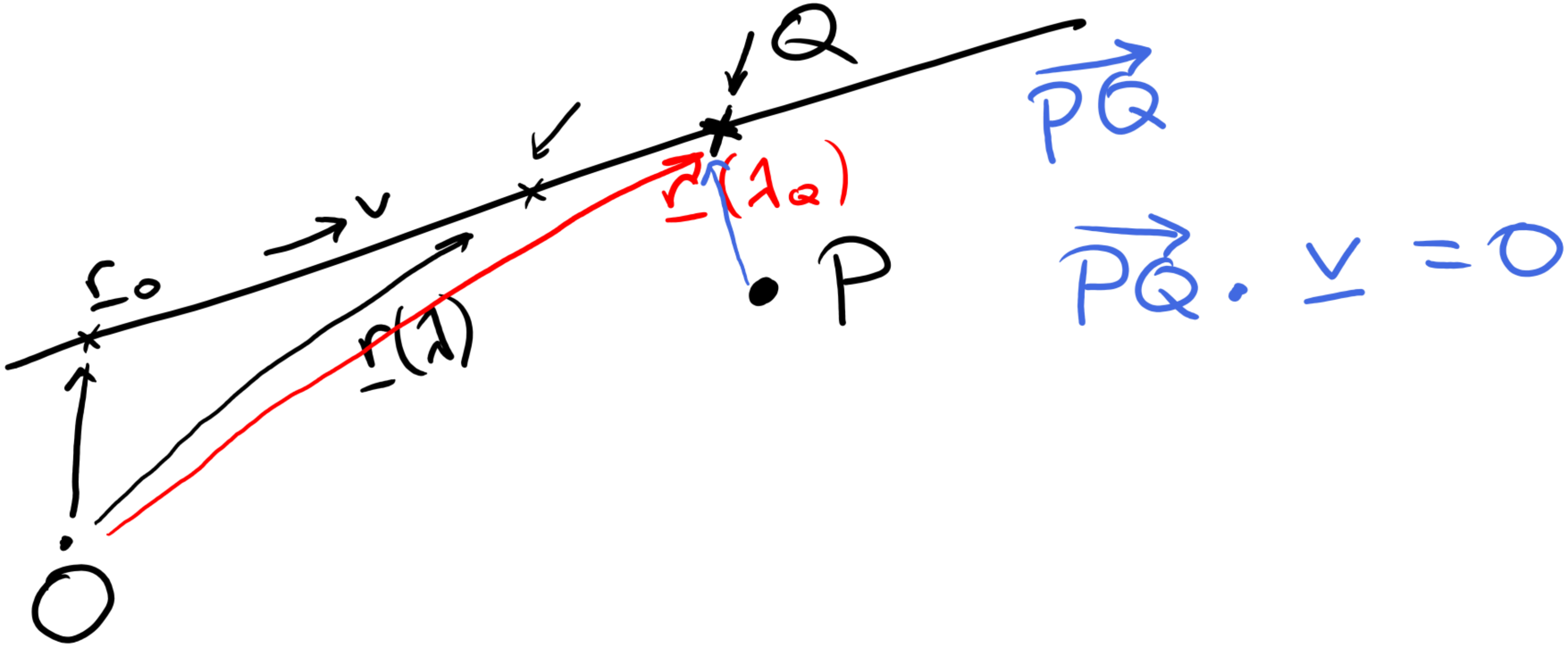
$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{\underline{a} \cdot \underline{b}}{|\underline{a}| \times |\underline{b}|}\right)$$

$$\underline{u} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \qquad \underline{v} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

$$\theta = \cos^{-1}\left(\frac{9}{\sqrt{13}\,\sqrt{34}}\right) = 65^\circ$$

Equation of a Line

$$\underline{r}(\lambda) = \underline{r}_0 + \lambda \underline{v}$$



Cross Product

- Desired properties:
  - Parallel vectors multiply to 0
  - Perpendicular vectors just multiply lengths
  - Distributive

Cross Product

- Definition $\underline{a} \times \underline{b} = |\underline{a}| \times |\underline{b}| \sin(\theta)\, \underline{\hat{n}}$

$\underline{\hat{n}}$ unit vector $\perp$ to both $\underline{a}$ and $\underline{b}$

"right-hand rule"

only in 3D

Special Cases

$$\hat{\underline{i}} \times \hat{\underline{i}} = \hat{\underline{j}} \times \hat{\underline{j}} = \hat{\underline{k}} \times \hat{\underline{k}} = 0$$

$$\underline{i} \times \underline{j} = \underline{k} \qquad \hat{\underline{j}} \times \hat{\underline{k}} = \hat{\underline{i}} \qquad \hat{\underline{k}} \times \hat{\underline{i}} = \hat{\underline{j}}$$

$$\underline{j} \times \underline{i} = -\underline{k} \qquad \underline{k} \times \underline{j} = -\underline{i} \qquad \hat{\underline{i}} \times \hat{\underline{k}} = -\hat{\underline{j}}$$

Cross Product in Cartesian Coordinates

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$