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Mechanics 1

Session 2 – Projectile Trajectories

DR BEN HANSON

Last Lecture

Kinematics

We learned / recapped:

- What mechanics and kinematics are
- The concepts of distance, speed, acceleration, and the links between them

You should be able to:

- Describe what mechanics and kinematics are
- Describe what distance, speed and acceleration are
- Derive the constant-acceleration (SUVAT) equations
- Derive "any" acceleration function from a distance function, or vice versa
- Calculate distances, speeds and accelerations at any future time for a known system

This Lecture

Trajectories

We will learn:

- The physics of projectile trajectories
- How to represent a trajectory as a vector equation
- How to represent the components of the vector using kinematic equations

You will be able to:

- Describe the physical path taken by a projectile in terms of (x, y) coordinates
- Describe the physical path taken by a projectile in terms of a position vector $\underline{r}$
- Calculate the position of a projectile at some time t, given its initial position, velocity and acceleration

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Trajectories

Trajectories

What are they?

A trajectory is basically a “path”; for example:

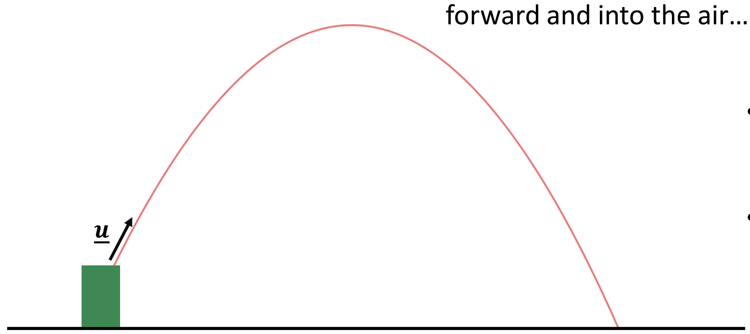
- The path a ball takes when thrown
- The path a particle takes through a gas
- The path a boat takes when floating on a river

Paths are not necessarily one-dimensional. We are going to need vectors!

Trajectories

Two-Dimensional Constant Acceleration

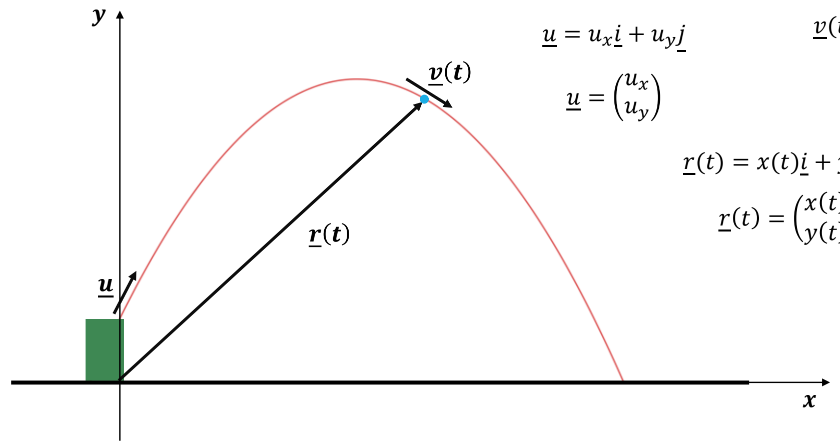
Imagine standing on a hill and throwing a ball forward and into the air…



- Why is this what the trajectory looks like?
- Where does this path come from?

Trajectories

Two-Dimensional Constant Acceleration



$\underline{u} = u_x\underline{i} + u_y\underline{j}$

$\underline{u} = \begin{pmatrix} u_x \\ u_y \end{pmatrix}$

$\underline{v}(t) = v_x(t)\underline{i} + v_y(t)\underline{j}$

$\underline{v}(t) = \begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix}$

$\underline{r}(t) = x(t)\underline{i} + y(t)\underline{j}$

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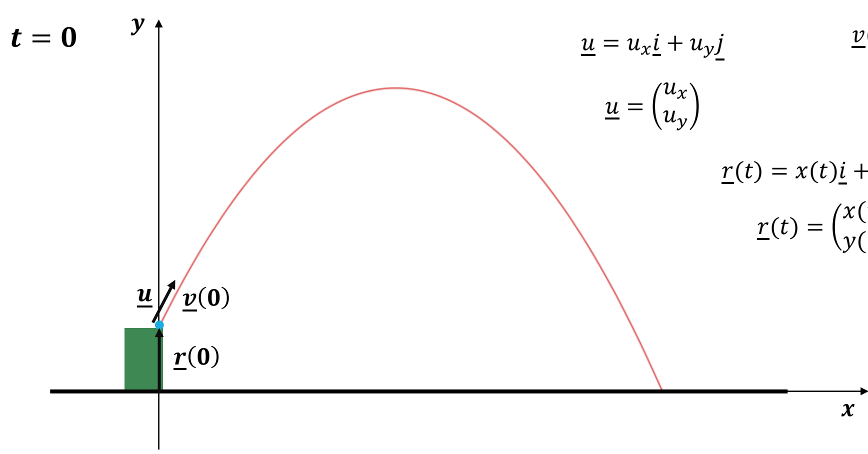
Trajectories

Two-Dimensional Constant Acceleration



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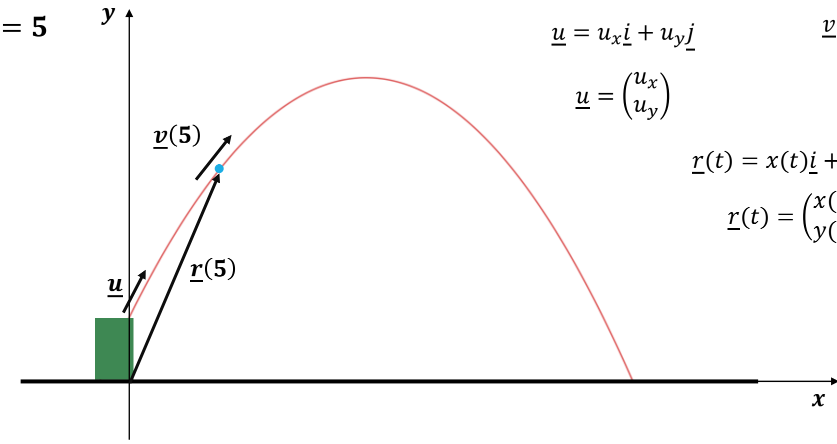
Trajectories

Two-Dimensional Constant Acceleration

$\boldsymbol{t = 5}$



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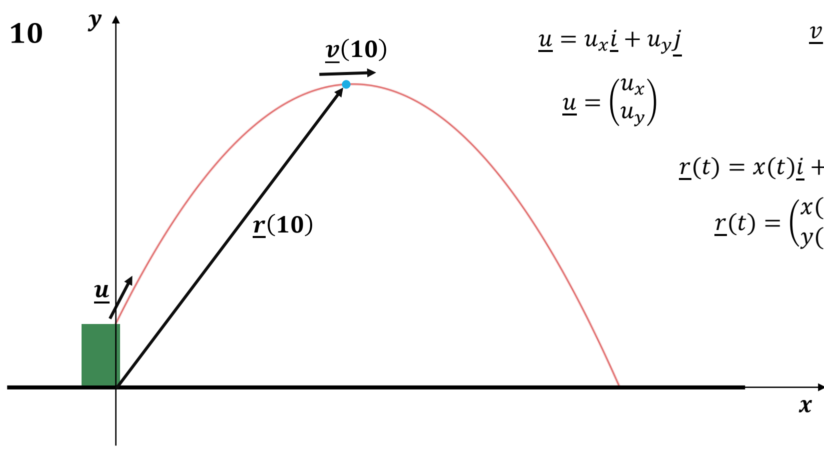
Trajectories

Two-Dimensional Constant Acceleration

$\boldsymbol{t = 10}$



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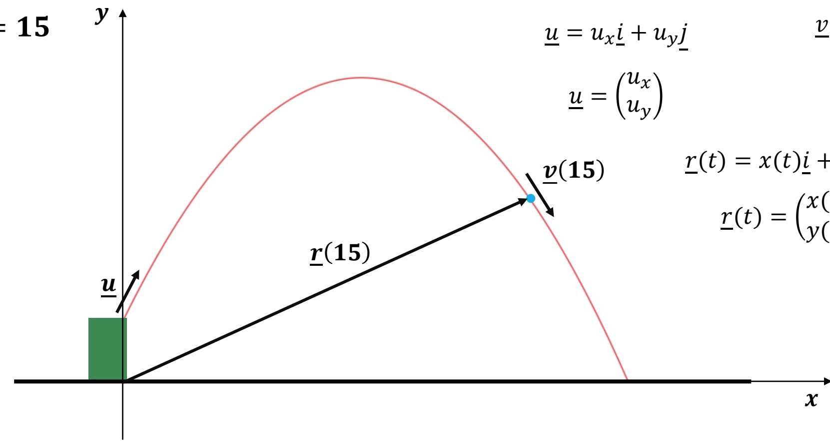
Trajectories

Two-Dimensional Constant Acceleration

$\boldsymbol{t = 15}$



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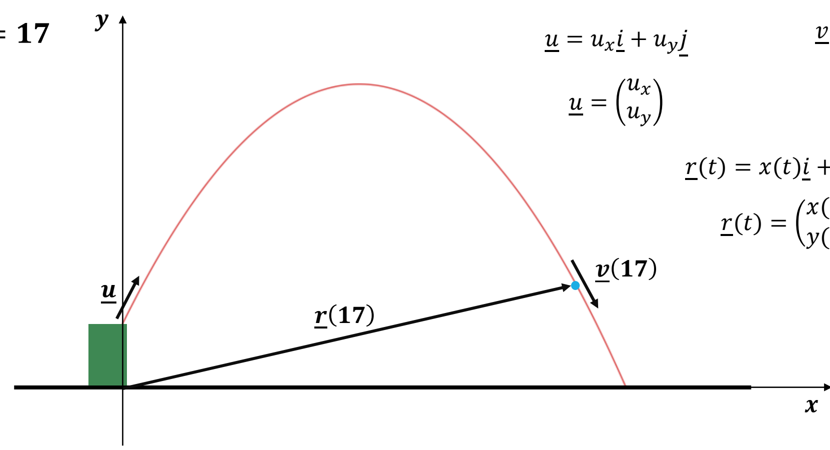
Trajectories

Two-Dimensional Constant Acceleration

$\boldsymbol{t = 17}$



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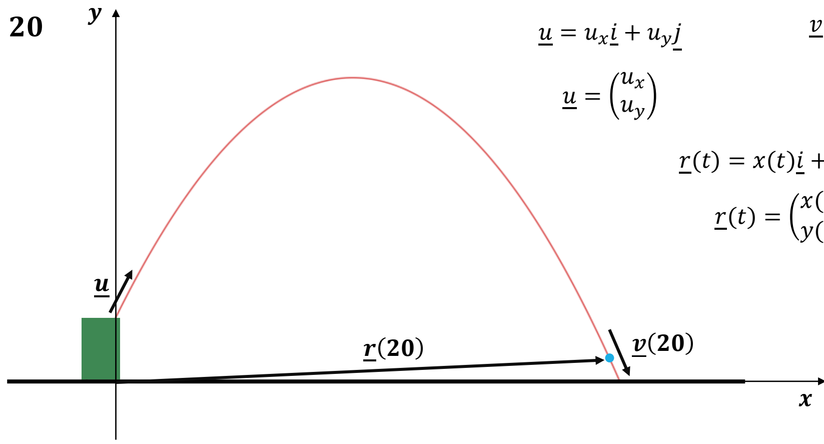
Trajectories

Two-Dimensional Constant Acceleration

$\boldsymbol{t = 20}$



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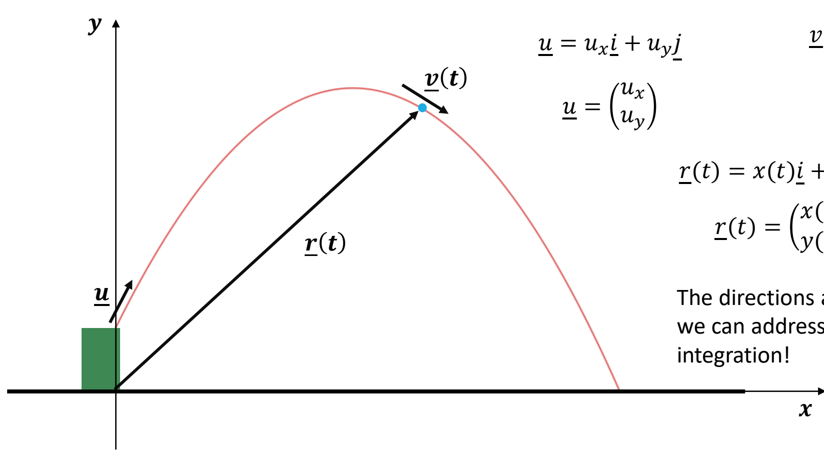
Trajectories

Two-Dimensional Constant Acceleration



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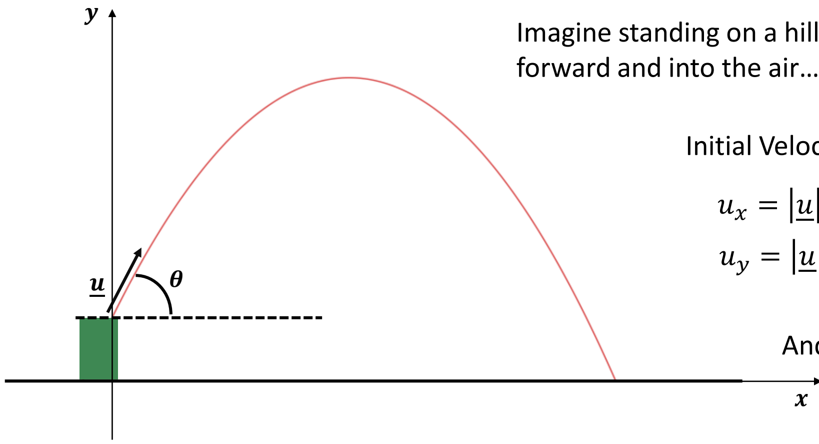
The directions are independent. Thus, we can address them separately with integration!

Trajectories

Two-Dimensional Constant Acceleration



Imagine standing on a hill and throwing a ball forward and into the air…

Initial Velocities	Accelerations
$u_x = \lvert \underline{u} \rvert \cos\theta$	$a_x = 0$
$u_y = \lvert \underline{u} \rvert \sin\theta$	$a_y = -g$

And that's all we'll need!

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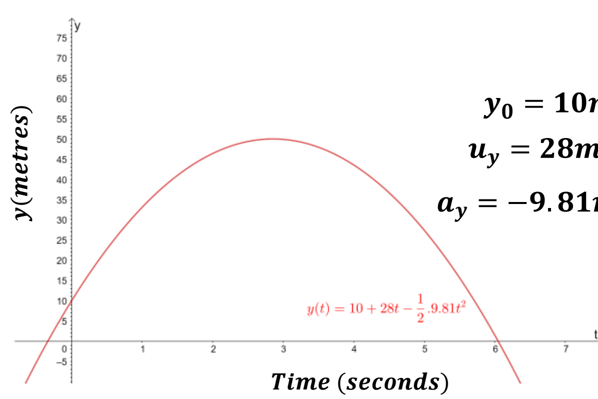
Trajectories

Two-Dimensional Constant Acceleration

$\boldsymbol{y}$ **−direction**

$$\boldsymbol{y(t) = y_0 + u_y t + \frac{1}{2} a_y t^2}$$

$$\boldsymbol{y(t) = y_0 + \lvert \underline{u} \rvert \sin(\theta)\, t - \frac{1}{2} g t^2}$$



$$\boldsymbol{y_0 = 10m}$$
$$\boldsymbol{u_y = 28m/s}$$
$$\boldsymbol{a_y = -9.81m/s^2}$$

In mechanics, I will generally assume that air resistance is negligible. In many real-world cases, this is not true.

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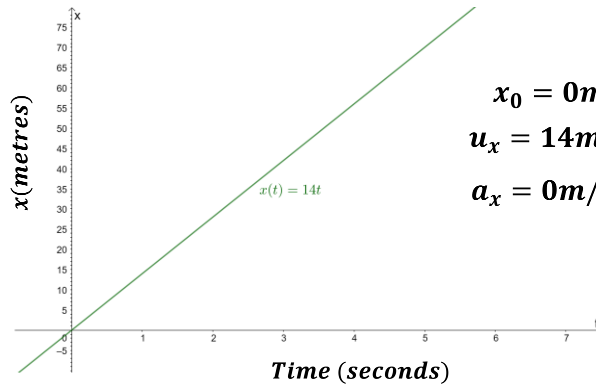
Trajectories

Two-Dimensional Constant Acceleration

x −direction

$$x(t) = x_0 + u_x t + \frac{1}{2} a_x t^2$$

$$x(t) = x_0 + |\underline{u}| \cos(\theta)\, t$$



$$x_0 = 0m$$
$$u_x = 14m/s$$
$$a_x = 0m/s^2$$

In mechanics, I will generally assume that air resistance is negligible. In many real-world cases, this is not true.

MECHANICS 1 – PROJECTILE TRAJECTORIES

Trajectories

Two-Dimensional Constant Acceleration

General Vector Formulation

$$\underline{r}(t) = \left(x_0 + u_x t + \frac{1}{2} a_x t^2\right)\underline{i} + \left(y_0 + u_y t + \frac{1}{2} a_y t^2\right)\underline{j}$$

$$\underline{r}(t) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} u_x \\ u_y \end{pmatrix} t + \frac{1}{2}\begin{pmatrix} a_x \\ a_y \end{pmatrix} t^2$$

$$\boxed{\underline{r}(t) = \underline{r_0} + \underline{u}t + \frac{1}{2}\underline{a}t^2}$$

There are vector forms of all of the constant-acceleration equations. Have a go at deriving them if you feel confident in your vectors and calculus ☺

https://en.wikipedia.org/wiki/Equations_of_motion#Constant_linear_acceleration_in_any_direction

Trajectories

Two-Dimensional Constant Acceleration

General Vector Formulation

$$\underline{\boldsymbol{r}}(\boldsymbol{t}) = \left(\boldsymbol{x_0} + \boldsymbol{u_x t} + \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{a_x t^2}\right)\underline{\boldsymbol{i}} + \left(\boldsymbol{y_0} + \boldsymbol{u_y t} + \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{a_y t^2}\right)\underline{\boldsymbol{j}}$$

$$\underline{\boldsymbol{r}}(\boldsymbol{t}) = \begin{pmatrix}\boldsymbol{x_0}\\ \boldsymbol{y_0}\end{pmatrix} + \begin{pmatrix}\boldsymbol{u_x}\\ \boldsymbol{u_y}\end{pmatrix}\boldsymbol{t} + \frac{\mathbf{1}}{\mathbf{2}}\begin{pmatrix}\boldsymbol{a_x}\\ \boldsymbol{a_y}\end{pmatrix}\boldsymbol{t^2}$$

$$\boxed{\underline{\boldsymbol{r}}(\boldsymbol{t}) = \underline{\boldsymbol{r_0}} + \underline{\boldsymbol{u}}\boldsymbol{t} + \frac{\mathbf{1}}{\mathbf{2}}\underline{\boldsymbol{a}}\boldsymbol{t^2}}$$

A quick and extremely important note. **Acceleration is not always constant!** You can only use the SUVAT equations if the acceleration is constant. If it isn't, you must use calculus to calculate position, velocity and acceleration! If you use the constant acceleration equations where acceleration is not constant, you will get 0 marks in whatever the question is. Sorry!

Task 1

Trajectory Calculations

Task 1

Trajectory Calculations

Scenario: Two athletes are competing in the shot put. The first throws their ball at an angle of 45^o with an initial speed of $35ms^{-1}$. The second throws their ball at an angle of 40^o with an initial speed of $40ms^{-1}$. Both athletes are approximately the same height, $1.7m$.

Tasks:

1. Draw a diagram of the situation (*Hint: You will need to make an approximation of y_0, the initial height of the ball.*)
2. Calculate the time taken for each ball to hit the ground (*Hint:* $y(t) = 0$)
3. Which athlete threw the ball furthest? (*Hint:* Now you know the time taken…)
4. Calculate the maximum height of each throw. Who threw the highest? (*Hint: Consider the velocity at the top*)

Task 2

More Trajectory Calculations (Practice is good!)

Task 2

More Trajectory Calculations

Scenario: An archer is aiming for a target. With their modern compound bow, they can shoot consistently with a speed $u = 90ms^{-1}$ at any angle. The target has a radius $R = 2m$, it's centre is $20m$ above ground level, and they are shooting from a height of $1.5m$.

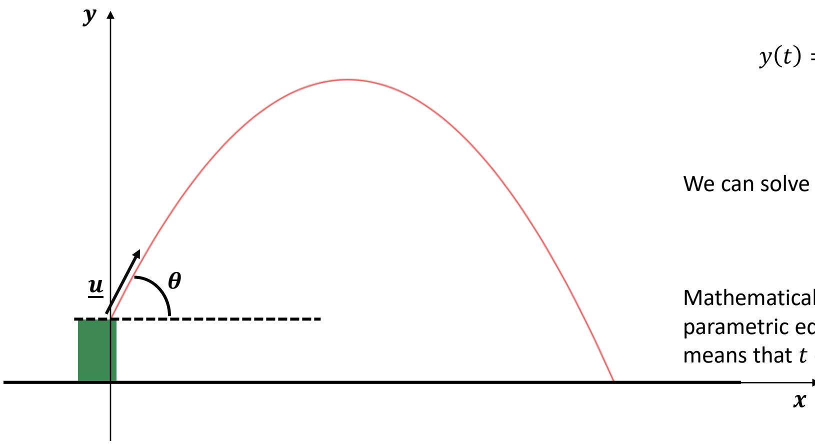
Tasks:

1. Draw a diagram of the situation.
2. The archer takes a shot at an angle $\theta = 15^o$ above the horizontal and hits the target dead centre. How far away could the target be? (*Hint: If you're unsure, just write out the SUVAT equations and see what you can do*)
3. The archer's next shot is $\theta = 10^o$. Could they hit the target?
4. The archer stumbles and shoots at an angle of $\theta = 86.9^o$ above the horizontal, but they still hit the target dead centre! Why is this? (*Hint: Think about how the trajectory changes as you vary the angle*)

Trajectory Path Equation

Extra Bits, Kind of Interesting

Trajectories



Trajectory Path Equation

$$y(t) = y_0 + u_y t - \frac{1}{2} g t^2, \qquad (1)$$

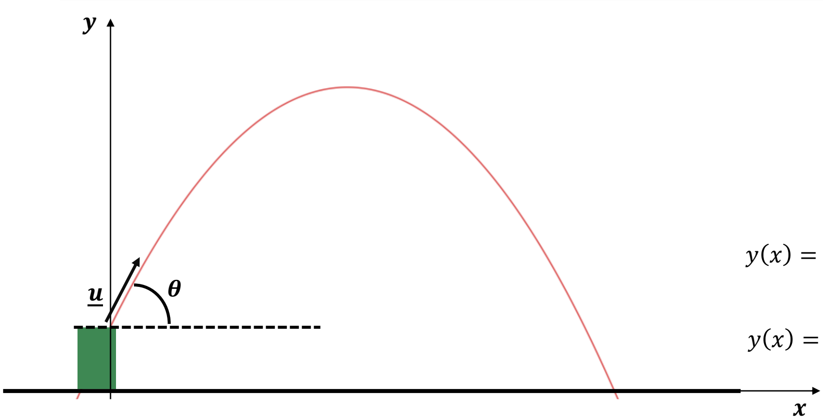
$$x(t) = x_0 + u_x t, \qquad (2)$$

We can solve the whole system with these two, but…

$$y(x) = \ \dots$$

Mathematically, t, is called the parameter of a parametric equation. In clearer language, this means that t can be replaced with something else…

Trajectories



Trajectory Path Equation

$$x(t) = x_0 + u_x t$$

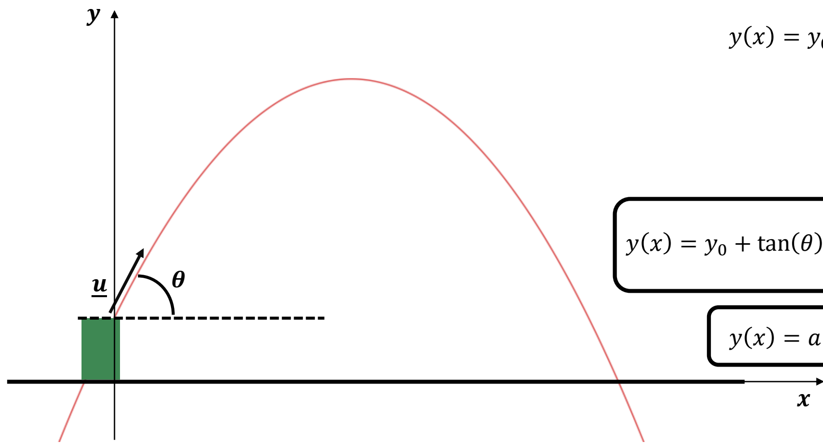
$$t = \frac{x - x_0}{u_x}$$

$$y(t) = y_0 + u_y t - \frac{1}{2} g t^2$$

$$y(x) = y_0 + u_y \left(\frac{x - x_0}{u_x} \right) - \frac{1}{2} g \left(\frac{x - x_0}{u_x} \right)^2$$

$$y(x) = y_0 + \frac{u_y}{u_x}(x - x_0) - \frac{1}{2} \frac{g}{u_x^2} (x - x_0)^2$$

Trajectories



Trajectory Path Equation

$$y(x) = y_0 + \frac{u_y}{u_x}(x - x_0) - \frac{1}{2}\frac{g}{u_x^2}(x - x_0)^2$$

$$u_x = |\underline{u}|\cos(\theta)$$

$$u_y = |\underline{u}|\sin(\theta)$$

$$\boxed{y(x) = y_0 + \tan(\theta)\,(x - x_0) - \frac{1}{2}\frac{g}{|\underline{u}|^2}\sec^2(\theta)\,(x - x_0)^2}$$

$$\boxed{y(x) = a(x - x_0)^2 + b(x - x_0) + c}$$