UNIVERSITY OF LEEDS

Mechanics 1

Session 17: Circular Motion – Rolling

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Last Lecture

Circular Motion – The Continuous Moment of Inertia

We:

- Conceptualised continuous objects as infinitely dense collections of particles
- Understood how to determine the moment of inertia of continuous objects

You should be able to:

- Calculate the moment of inertia of continuous objects with regular structure
- Calculate the moment of inertia of continuous objects with variable density

This Lecture

Circular Motion – Rolling

We will:

- Cover the final topics in our discussion of pure circular motion
- Consider how translational and rotational motion can be combined, and see some examples
- Understand that rolling objects occur when translation and rotation are coupled together, and not independent from one another
- Understand the roll friction plays for a rolling object

You will be able to:

- Calculate the rotational kinetic energy and angular momentum of a rotating object
- Calculate the kinematic properties of a rolling object

Incredibly Important Announcements

Incredibly Important Announcements

My partner has a dog, and he is extremely cute!

He's a sausage dog. Sausage…rolls? Is that too much of a stretch?

Rotational Kinetic Energy

A Quick Overview

Rotational Kinetic Energy

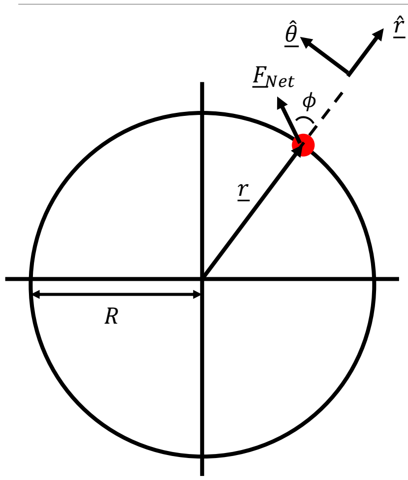
A Quick Overview

To make things rotate takes effort…takes work! So perhaps, rather than using torque (a vector) to analyse rotation, we can instead use energy!

Before we get going, have a think about what the equations for rotational kinetic energy may look like, and where they might come from ☺

Rotational Kinetic Energy

A Quick Overview



$$W = \Delta E_k = \int \underline{F}_{Net}.d\underline{l}$$

$d\underline{l}$ represents the path taken by the particle in this equation, not the $\hat{\underline{r}}$ or $\hat{\underline{\theta}}$ directions!

$$\Delta E_k = \int (F_r\hat{\underline{r}} + F_\theta\hat{\underline{\theta}}).(\cancel{dr}\,\hat{\underline{r}} + Rd\theta\hat{\underline{\theta}})$$

Extremely important! **The centripetal force does <u>no work</u> on a purely rotating object** and so does not change its kinetic energy

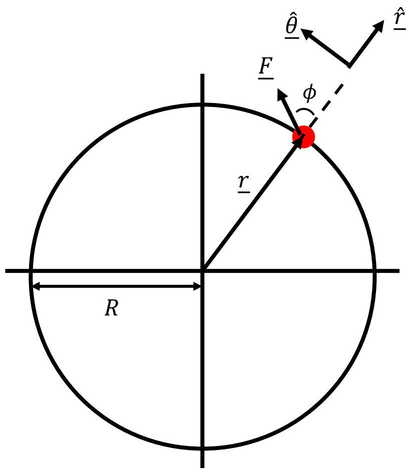
$$\Delta E_k = \int F_\theta R d\theta$$

$$\boxed{\Delta E_k = \int |\underline{\tau}| d\theta}$$

Again, same form as linear motion!

Rotational Kinetic Energy

A Quick Overview



$$\boxed{E_k = \frac{1}{2} m v_\theta^2}$$

$$E_k = \frac{1}{2} m R^2 \omega^2$$

$$\boxed{E_k = \frac{1}{2} I \omega^2}$$

Angular Momentum

A Quick Overview

Angular Momentum

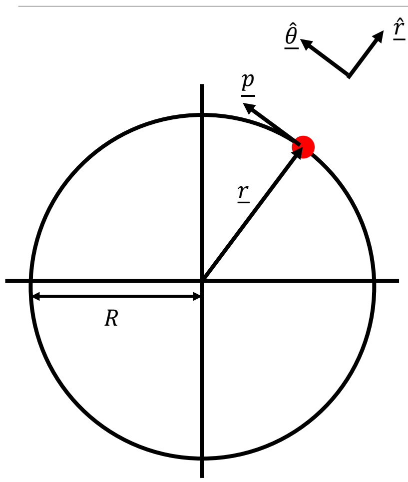
A Quick Overview

Rotating objects have velocity, and thus they also have momentum. It has the same conservation laws as with translational motion (i.e it's always conserved!)

This one is a bit more in depth than energy, so let's define it and derive some properties

Angular Momentum

A Quick Overview



$$\boxed{\underline{L} = \underline{r} \times \underline{p}}$$

Expand, $\underline{L} = |\underline{r}|\,|\underline{p}|\sin(\phi)\underline{\hat{n}}$

Simplify, $\underline{L} = Rmv_\theta\underline{\hat{n}}$

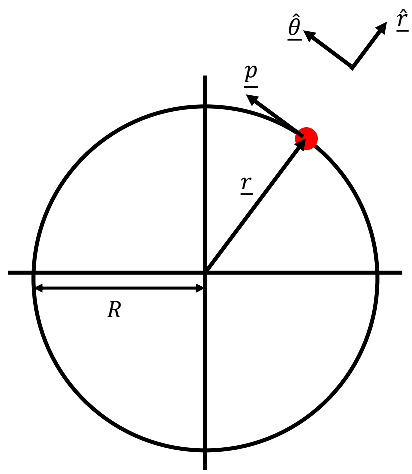
Angular speed, $\underline{L} = mR^2\omega\underline{\hat{n}}$

Replace, $\underline{L} = I\omega\underline{\hat{n}}$

Hence, $\boxed{|\underline{L}| = I\omega}$

Angular Momentum

A Quick Overview



$$\boxed{\underline{L} = \underline{r} \times \underline{p}}$$

Differentiate, $\frac{d\underline{L}}{dt} = \frac{d\underline{r}}{dt} \times \underline{p} + \underline{r} \times \frac{d\underline{p}}{dt}$

Expand, $\frac{d\underline{L}}{dt} = m\frac{d\underline{r}}{dt} \times \underline{v} + \underline{r} \times m\frac{d\underline{v}}{dt}$

Solve, $\frac{d\underline{L}}{dt} = \cancel{m\underline{v} \times \underline{v}} + \underline{r} \times m\underline{a}$

Cancel and simplify, $\frac{d\underline{L}}{dt} = \underline{r} \times \underline{F}$

Hence, $\boxed{\frac{d\underline{L}}{dt} = \underline{\tau}}$

Angular Momentum

A Quick Overview

Angular Momentum:

1. Is the perfect rotational analogue of linear momentum
   i. $|\underline{p}| = mv \rightarrow |\underline{L}| = I\omega$
   ii. $\frac{d\underline{p}}{dt} = \underline{F} \rightarrow \frac{d\underline{L}}{dt} = \underline{\tau}$
2. Is a vector that (usually) points in the same direction as torque
3. Is conserved

Angular Momentum

A Quick Overview

Angular Momentum:

3. Is conserved

Questions:

1. Describe how this is happening in terms of angular momentum.
2. Given the speed quoted, estimate the angular momentum of this ice dancer

Task 1

Energy & Angular Momentum...In Space!!

Task 1

Energy & Angular Momentum…In Space!!

Scenario: The moon is rotating around the Earth. It takes about 28 days to do this and is a distance of approximately $d = 384{,}400km$ away from the Earth. Assuming a circular motion and constant angular speed (and using the internet for the things you need):

Tasks:
1. Calculate the linear speed, angular speed, and rotational kinetic energy of the moon corresponding to its orbit around the Earth.
2. Calculate the angular momentum of the moon corresponding to its orbit around the Earth.
3. The moon also rotates about its own axis once every 28 days. Calculate the rotational kinetic energy and angular momentum of the moon corresponding to its rotation on its own axis. *Hint: You'll need the moment of inertia of a sphere.*
4. The Earth also rotates about its own axis one per day. Calculate the rotational kinetic energy and angular momentum of the Earth corresponding to its rotation on its own axis.
5. The Earth also rotates about the sun once per 365 days and is a distance of 147.42 million km away. Assuming a circular path, calculate the rotational kinetic energy and angular momentum of the of the Earth corresponding to its rotation around the sun.
6. Calculate the total angular momentum and rotational kinetic energy of the Earth-moon system.

https://moon.nasa.gov/resources/429/the-moons-orbit-and-rotation/

Two Motions

Translational & Rotational Motion Combined

Two Motions

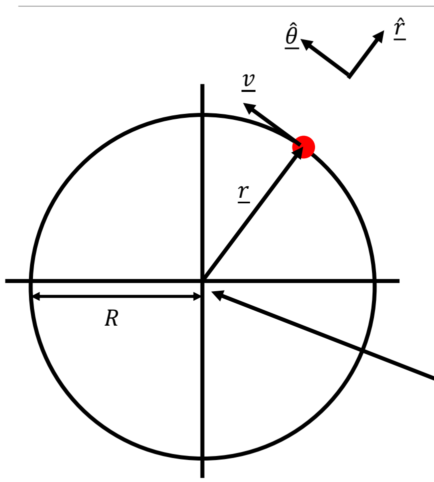
Translational & Rotational Motion Combined

Rolling objects will truly test your understanding of first-year mechanics. We're combining translational and rotational motion together, with friction as the thing that links them!

Cars, bicycles, wheelchairs, marbles, bowling balls, putting at golf. We'll have the tools to understand them all in the next two sessions!

Two Motions

Translational & Rotational Motion Combined



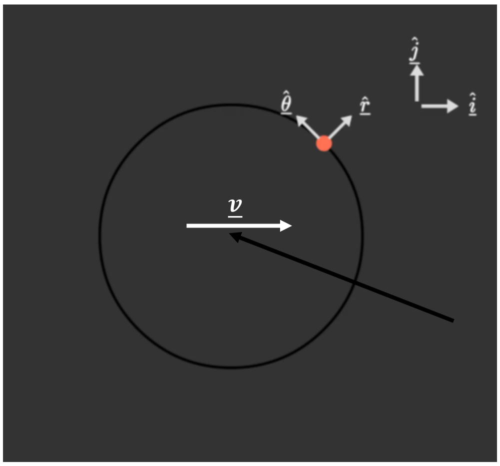
Let's treat a circular motion in its entirety.

Imagine that it's not just the rotating object, but the entire circular system that we care about.

What is the velocity of this point?

Two Motions

Translational & Rotational Motion Combined



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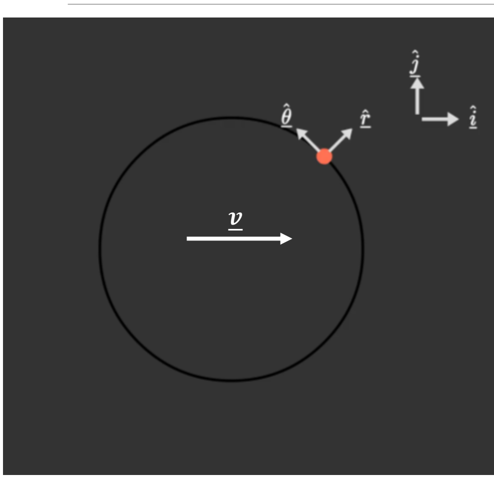
Two Motions

Translational & Rotational Motion Combined

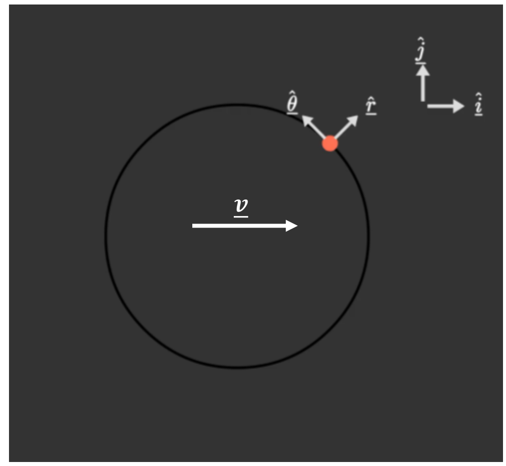


Two Motions

Translational & Rotational Motion Combined

Questions:

1. When the rotating object is at the top, what is its total linear velocity?
2. When the rotating object is at the top, what is its total linear velocity?
3. When the rotating object is at the left-hand side, what is its total linear speed?
4. When the rotating object is at the right-hand side, what is its total linear speed?
5. Can you give me some examples of real physical systems like this (rotating object who's axis of rotation itself moves)?



Two Motions

Translational & Rotational Motion Combined

In general, the rotation of an object and the motion of its axis of rotation are entirely independent. Either can occur without the other, and you can just add their effects like relative velocities.

For example, moons orbiting planets that themselves orbit a star (that also may orbit another star, or the galaxy itself!).
But what about when they are not independent?

Rolling Objects

Coupling Rotation and Translation

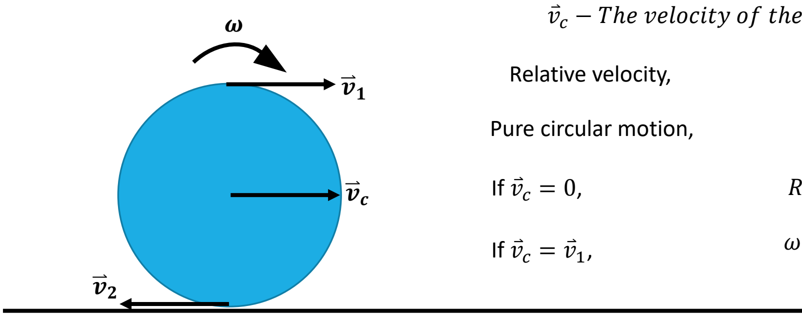
Rolling Objects

Coupling Rotation and Translation

$\vec{v}_1 - \textit{The total velocity of the top point on the object}$

$\vec{v}_2 - \textit{The total velocity of the bottom point on the object}$

$\vec{v}_c - \textit{The velocity of the centre of mass of the object}$



Relative velocity, $\vec{v}_1' = \vec{v}_1 - \vec{v}_c$

Pure circular motion, $R\omega\underline{\hat{\theta}} = \vec{v}_1 - \vec{v}_c$

If $\vec{v}_c = 0$, $R\omega\underline{\hat{\theta}} = \vec{v}_1$, *only circular motion*

If $\vec{v}_c = \vec{v}_1$, $\omega = 0$, *only translational motion (sliding)*

Rolling Objects

Coupling Rotation and Translation

$\vec{v}_1 - The\ total\ velocity\ of\ the\ top\ point\ on\ the\ object$

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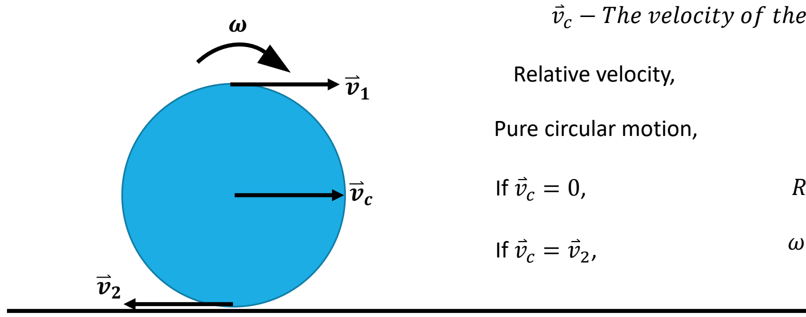
$\vec{v}_c - The\ velocity\ of\ the\ centre\ of\ mass\ of\ the\ object$



Relative velocity, $\vec{v}_2' = \vec{v}_2 - \vec{v}_c$

Pure circular motion, $R\omega\underline{\hat{\theta}} = \vec{v}_2 - \vec{v}_c$

If $\vec{v}_c = 0$, $R\omega\underline{\hat{\theta}} = \vec{v}_2$, *only circular motion*

If $\vec{v}_c = \vec{v}_2$, $\omega = 0$, *only translational motion (sliding)*

Rolling Objects

Coupling Rotation and Translation

$\vec{v}_1 - The\ total\ velocity\ of\ the\ top\ point\ on\ the\ object$

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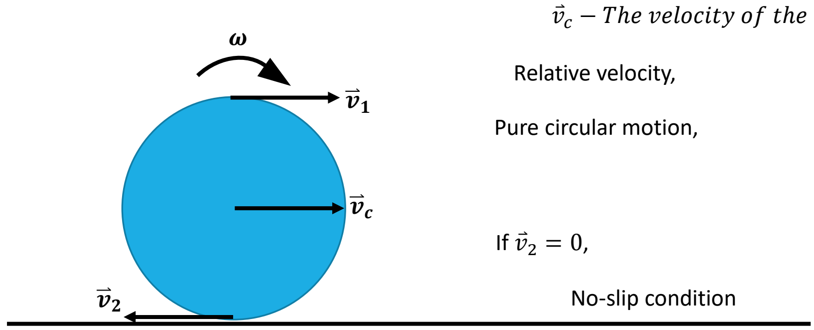


Relative velocity, $\vec{v}_2' = \vec{v}_2 - \vec{v}_c$

Pure circular motion, $R\omega\underline{\hat{\theta}} = \vec{v}_2 - \vec{v}_c$

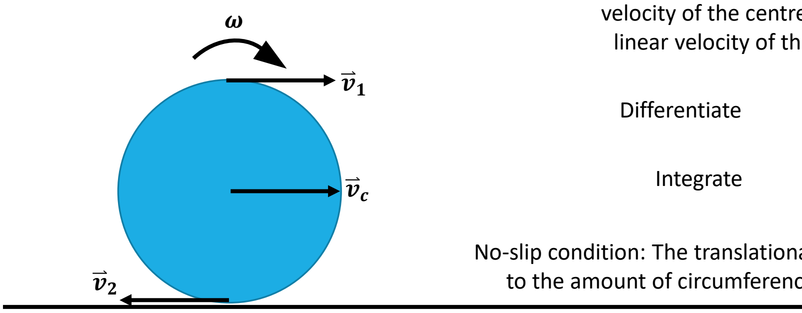
$-R\omega\underline{\hat{i}} = \vec{v}_2 - \vec{v}_c$

If $\vec{v}_2 = 0$, $-R\omega\underline{\hat{i}} = -\vec{v}_c$

No-slip condition $\boxed{|\vec{v}_c| = R\omega}$

Rolling Objects

Coupling Rotation and Translation



No-slip condition $\boxed{v_c = R\omega}$

The no slip condition occurs when the translational velocity of the centre of mass is exactly equal to the linear velocity of the surface of the rolling object.

Differentiate $\boxed{a_c = R\alpha}$

Integrate $\boxed{s_c = R\theta}$

No-slip condition: The translational distance covered is exactly equal to the amount of circumference that has touched the ground.

https://physics.info/rolling/
https://tinyurl.com/yc4hsr7f

MECHANICS 1: CIRCULAR MOTION - ROLLING

Rolling Objects

Coupling Rotation and Translation

No-slip condition

$\boxed{s_c = R\theta}$ $\boxed{v_c = R\omega}$ $\boxed{a_c = R\alpha}$

Slip condition

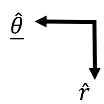
$\boxed{s_c \neq R\theta}$ $\boxed{v_c \neq R\omega}$ $\boxed{a_c \neq R\alpha}$

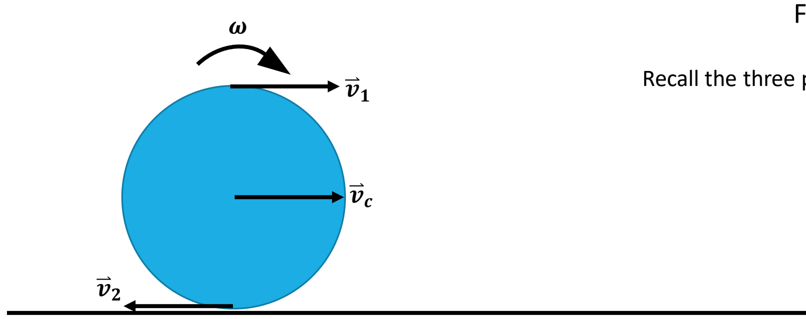
https://www.youtube.com/watch?v=gtGsU8g2WH4
https://www.youtube.com/shorts/h9HqRxUgOjs

Rolling Objects

Coupling Rotation and Translation

Axes for surface contact point





Question: What force, for rolling objects, causes the translational and rotational motion to be coupled?

Friction!

Recall the three possibilities for friction:

https://physics.info/rolling/
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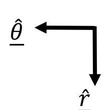
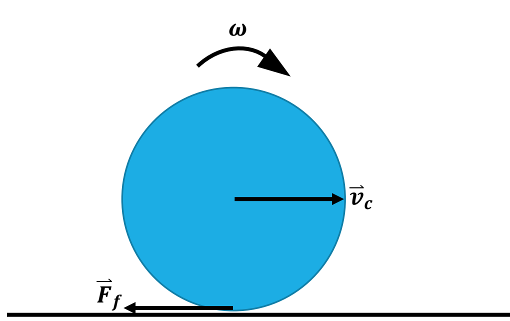
Rolling Objects

Coupling Rotation and Translation

Axes for surface contact point





Question: What force, for rolling objects, causes the translational and rotational motion to be coupled?

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Recall the three possibilities for friction:

$|\vec{\boldsymbol{v}}_c| > R\omega$

$\vec{F}_f = \mu_k N \hat{\underline{\theta}}$

Slip condition

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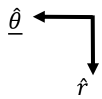
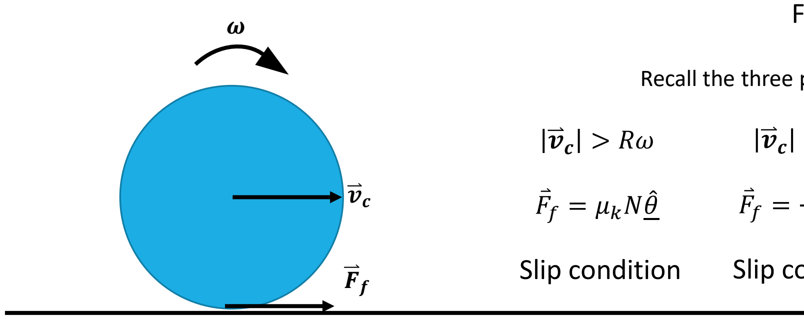
Rolling Objects

Coupling Rotation and Translation

Axes for surface contact point





Question: What force, for rolling objects, causes the translational and rotational motion to be coupled?

Friction!

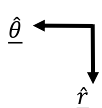
Recall the three possibilities for friction:

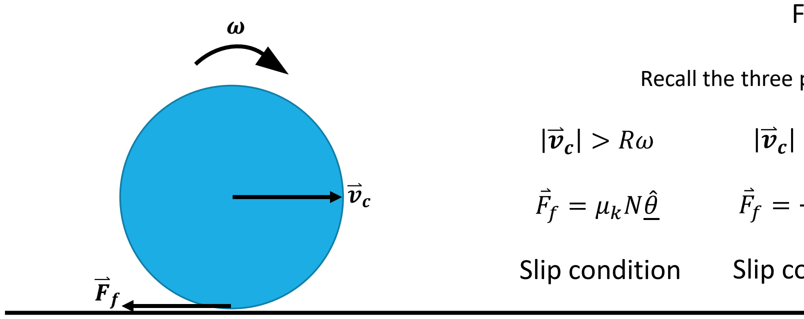
$\lvert\vec{\boldsymbol{v}}_c\rvert > R\omega$	$\lvert\vec{\boldsymbol{v}}_c\rvert < R\omega$
$\vec{F}_f = \mu_k N\underline{\hat{\theta}}$	$\vec{F}_f = -\mu_k N\underline{\hat{\theta}}$
Slip condition	Slip condition

Rolling Objects

Coupling Rotation and Translation

Axes for surface contact point





Question: What force, for rolling objects, causes the translational and rotational motion to be coupled?

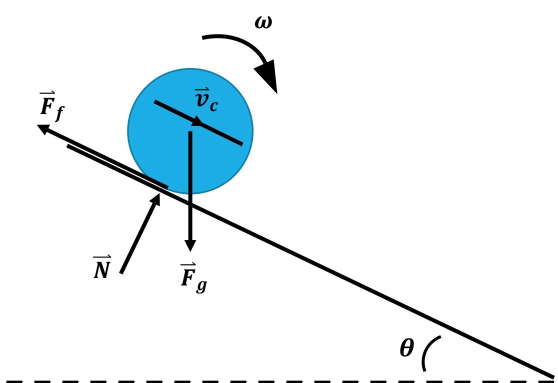
Friction!

Recall the three possibilities for friction:

$\lvert\vec{\boldsymbol{v}}_c\rvert > R\omega$	$\lvert\vec{\boldsymbol{v}}_c\rvert < R\omega$	$\lvert\vec{\boldsymbol{v}}_c\rvert = R\omega$
$\vec{F}_f = \mu_k N\underline{\hat{\theta}}$	$\vec{F}_f = -\mu_k N\underline{\hat{\theta}}$	$\vec{F}_f \leq \mu_s N\underline{\hat{\theta}}$
Slip condition	Slip condition	No-slip condition

Rolling Objects

Coupling Rotation and Translation



Step by Step:

1. Gravity acting
2. Gravity component perpendicular to hill -> Normal reaction force
3. Gravity component downhill -> sliding motion
4. Friction opposes sliding motion
5. Friction causes torque -> angular acceleration and velocity!

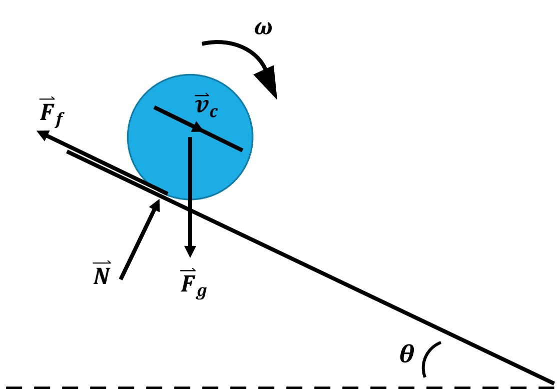
Always consider friction last. It is dependent on everything else happening in the system

https://www.youtube.com/watch?v=hxa6jAYA980
https://www.youtube.com/watch?v=CfTLS6YYPms

Rolling Objects

Coupling Rotation and Translation

No slip condition



Resolve forces perpendicular, $$|\vec{F}_g|\cos(\theta) = |\vec{N}|, \qquad (1)$$

Resolve forces parallel, $$|\vec{F}_g|\sin(\theta) - |\vec{F}_f| = ma_c, \qquad (2)$$

Resolve net torque, $$|\vec{F}_f|R = I\alpha$$

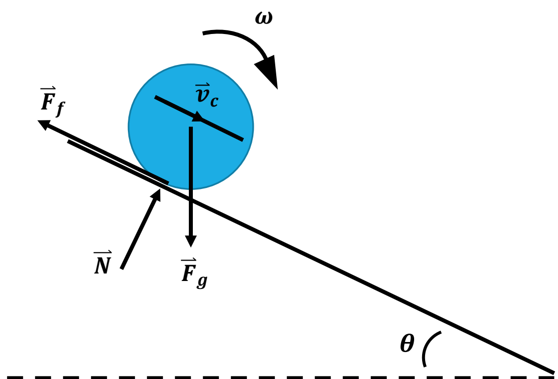
If I could calculate α and a_c independently, I could compare them to check if the object was slipping or not! For now, let's assume no-slip and see what happens

https://www.youtube.com/watch?v=hxa6jAYA980
https://www.youtube.com/watch?v=CfTLS6YYPms

Rolling Objects

Coupling Rotation and Translation

No slip condition



Couple translation and rotation via no-slip $\quad a_c = R\alpha$

Net torque, $\quad |\vec{\tau}| = |\vec{F}_f|R = I\alpha$

$$|\vec{F}_f|R = I\frac{a_c}{R}$$

Rearrange, $\quad |\vec{F}_f| = \frac{I}{R^2}a_c, \qquad (3)$

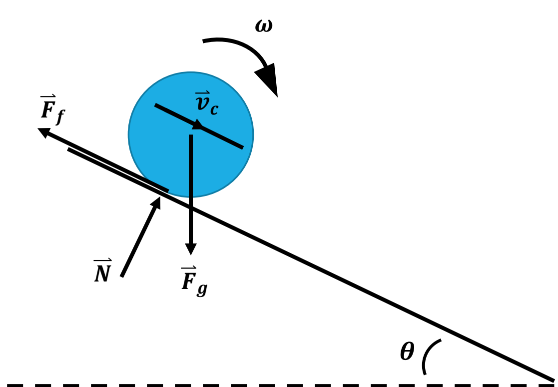
(3) Into (2), $\quad |\vec{F}_g|\sin(\theta) - \frac{I}{R^2}a_c = ma_c$

Simplify, $\quad g\sin(\theta) - \frac{I}{mR^2}a_c = a_c$

Rolling Objects

Coupling Rotation and Translation

No slip condition



Simplify, $\quad g\sin(\theta) - \frac{I}{mR^2}a_c = a_c$

Rearrange, $\quad \boxed{a_c = g\frac{\sin(\theta)}{1 + \frac{I}{mR^2}}}$

Things to note:

1. $\theta = 0, a_c = 0$
2. $I \to \infty, a_c \to 0$ (impossibly hard to turn)
3. $I \to 0, a_c \to g\sin(\theta)$ (like having zero-friction sliding)
4. a_c is constant! Therefore α is also constant! SUVAT all the way ☺

Task 2

Objects Rolling Downhill

Task 2

Objects Rolling Downhill

Scenario: An object with a circular cross-section, radius $R = 0.75m$ and mass $m = 20kg$ is rolling down a hill at an angle $\theta = 35^o$ to the horizontal. It begins from rest with the centre of mass at a height $h_1 = 10m$ above the ground and reaches the bottom at a height $h_2 = R$. For this system, the no-slip condition holds.

Tasks:

1. By considering only the energies in this system, show that the speed of the centre of mass of the object after it has moved a distance s down the hill, $v_c = \sqrt{\frac{2g\Delta h}{1+\frac{I}{mR^2}}}$, where $\Delta h = h_1 - h_2$ and I is the moment of inertia

   Hint: Static friction does no work against an object, even when rolling
2. Calculate the centre of mass speed of the object at the bottom if it is a sphere.
3. Calculate the centre of mass speed of the object at at the bottom if it is a cylinder.
4. For any object, would it reach the bottom of the hill faster or slower if it didn't rotate (i.e. it just slid down). Explain your answer
5. By considering both the net force and net torque on this object, show that $a_c = g\frac{\sin(\theta)}{1+\frac{I}{mR^2}}$
6. Using the SUVAT equations, calculate the time taken to reach the bottom for a sphere and a cylinder.