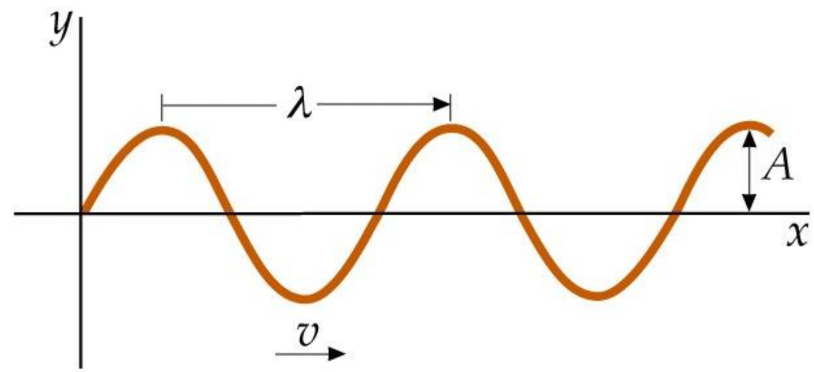
Travelling Harmonic Waves

KEY!

A harmonic wave is sinusoidal.

A special case of $y = g(x \mp vt)$ with

$$g(x) = A\sin\left(\frac{2\pi}{\lambda}x\right)$$



A is the amplitude.
λ is the wavelength.

For wave moving to the right, we have $$y = A\sin\left(\frac{2\pi}{\lambda}(x - vt)\right)$$

$2\pi\left(\frac{x}{\lambda} - \frac{vt}{\lambda}\right)$

$\frac{v}{\lambda} = f = \frac{1}{T}$

Time to travel one wavelength is the period T=1/f.

So we have $vT = \lambda$ or $\boxed{v = f\lambda}$ $\therefore$ $$y = A\sin\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right]$$

$$\therefore \quad y = A\sin\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right]$$

$\Rightarrow$ periodic in both space and time:

At any time t, y has the same value at x, x+λ, x+2λ, …
At any position x, y has the same value at times t, t+T, t+2T, …

More convenient to write $$\boxed{y = A\sin(kx - \omega t)}$$

$\omega = \frac{2\pi}{T}$ | $vT = \lambda$

with **wave-number** $k = \frac{2\pi}{\lambda}$

& **angular frequency** $\omega = \frac{2\pi}{T}$

WHY?

Hence $\boxed{v = \frac{\omega}{k}}$

$\therefore \omega = 2\pi \frac{v}{\lambda}$

$v = \frac{\omega\lambda}{2\pi}$

$\ddot{y}\ k = \frac{2\pi}{\lambda}$

$v = \omega/k$ ✓

This assumes displacement y=0 at x=0 and t=0.

For more general initial conditions, add the phase constant:

$$y = A\sin(kx - \omega t + \delta)$$

The last slide laid out clearly

wave shape $g(x) = A\sin\left(\frac{2\pi}{\lambda}x\right)$ $\rightarrow$ $\frac{1}{\lambda}$ = spatial frequency of wave (wavelengths per unit distance), or $\frac{2\pi}{\lambda}$ radians per unit distance.

if travelling wave moves distance $vt = x$
velocity time

$\rightarrow$ $y = A\sin\left(\frac{2\pi}{\lambda}(x - vt)\right)$

the period T is the time to complete 1 oscillation (cycle), but also the time for a wave (crest) to move λ, hence distance λ in time T $\rightarrow$ $v = \frac{\lambda}{T}$ or $v = f\lambda$

compare

$y = A\cos(\omega t)$

$\cos(ft)$ $\frac{\text{per}}{T}$, or cycle, or revolution

$\cos\left(\frac{1}{T}t\right)$

$\cos\left(\frac{2\pi}{T}t\right)$ convert to radians per cycle.

$\omega = \frac{2\pi}{T}$

So, $y = A\sin\left(2\pi\left(\frac{x}{\lambda} - \frac{vt}{\lambda}\right)\right)$ $v = \frac{\lambda}{T}$

$= A\sin\left(2\pi\left(\frac{x}{\lambda} - \frac{\lambda t}{T\lambda}\right)\right)$

$= A\sin\left(2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right)$ if $k = \frac{2\pi}{\lambda}$ and ω is $\frac{2\pi}{T}$

then $y = A\sin(kx - \omega t)$

where $v = \frac{\omega}{k}$

$$y = A\sin(kx - \omega t)$$

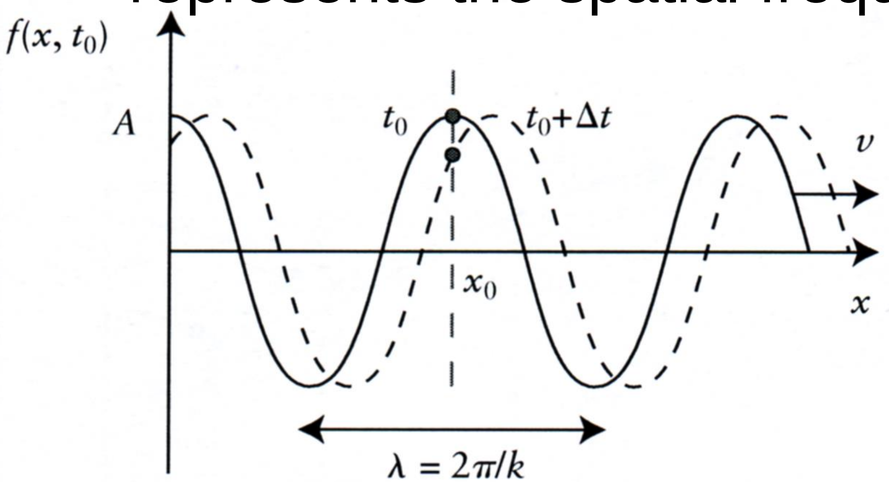
The wavenumber k is inversely proportional to the wavelength, and is defined as:

$$k = \frac{2\pi}{\lambda}$$

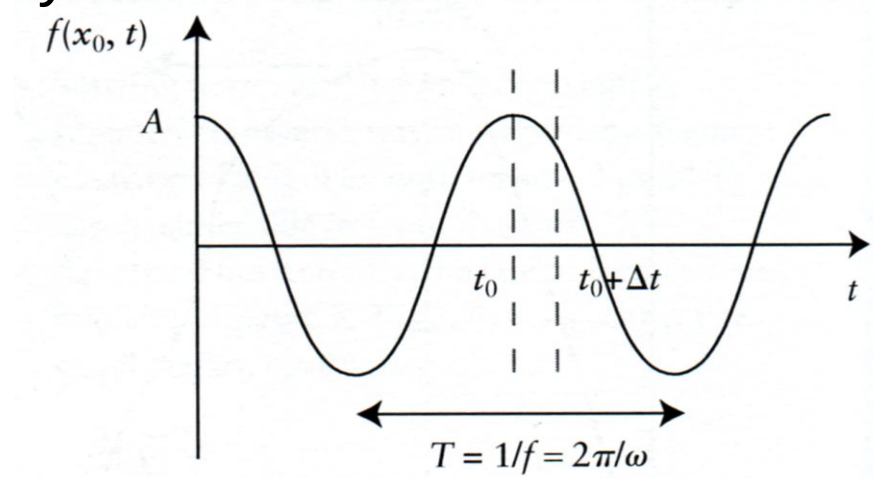
Units = rad m^{-1}

i.e. the number of wavelengths per unit distance x 2π to give number of radians per unit distance

The product kx therefore has units of radians, and represents the spatial frequency of the wave in x

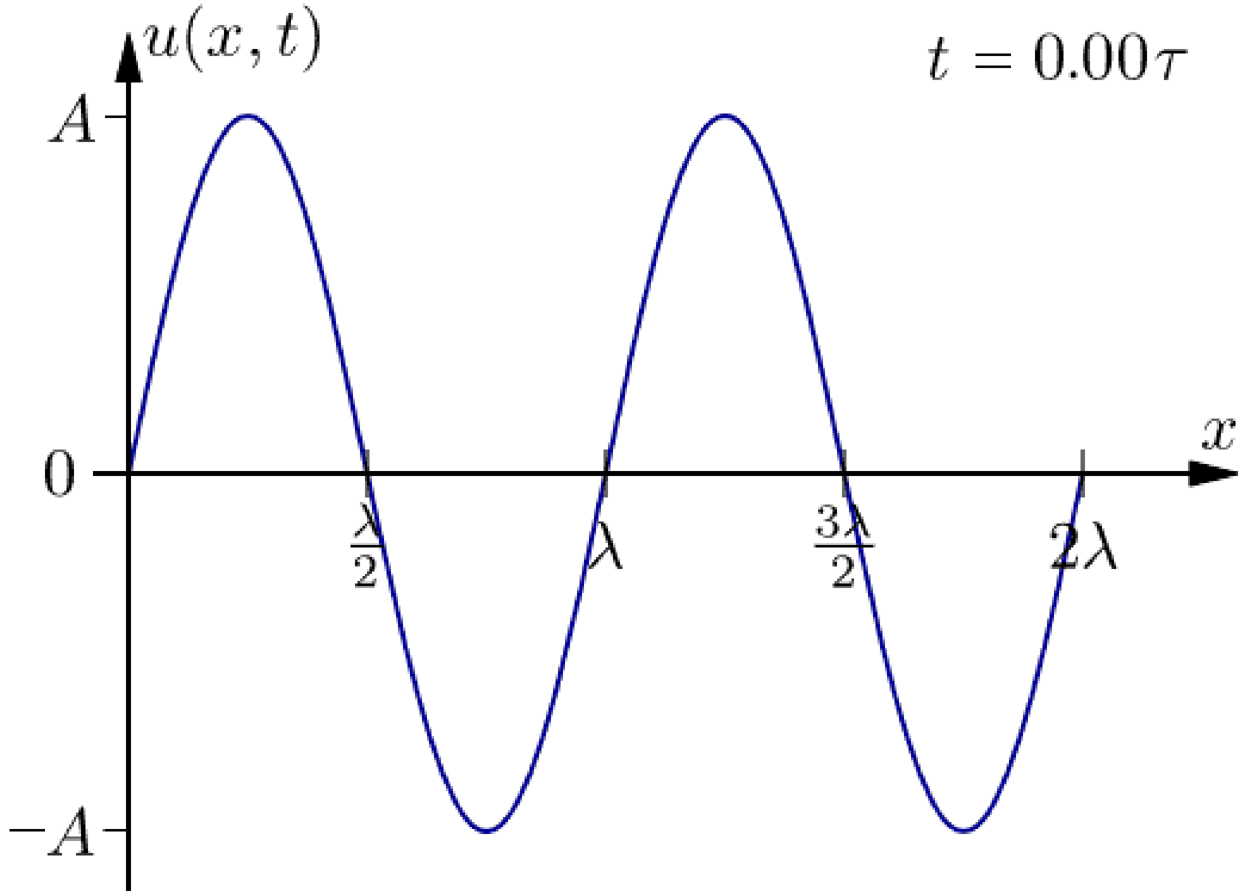


t = constant
(wave at a snapshot in time)



x = constant
(wave at one point in space)

https://www.youtube.com/watch?v=4j2bIuEBSyQ

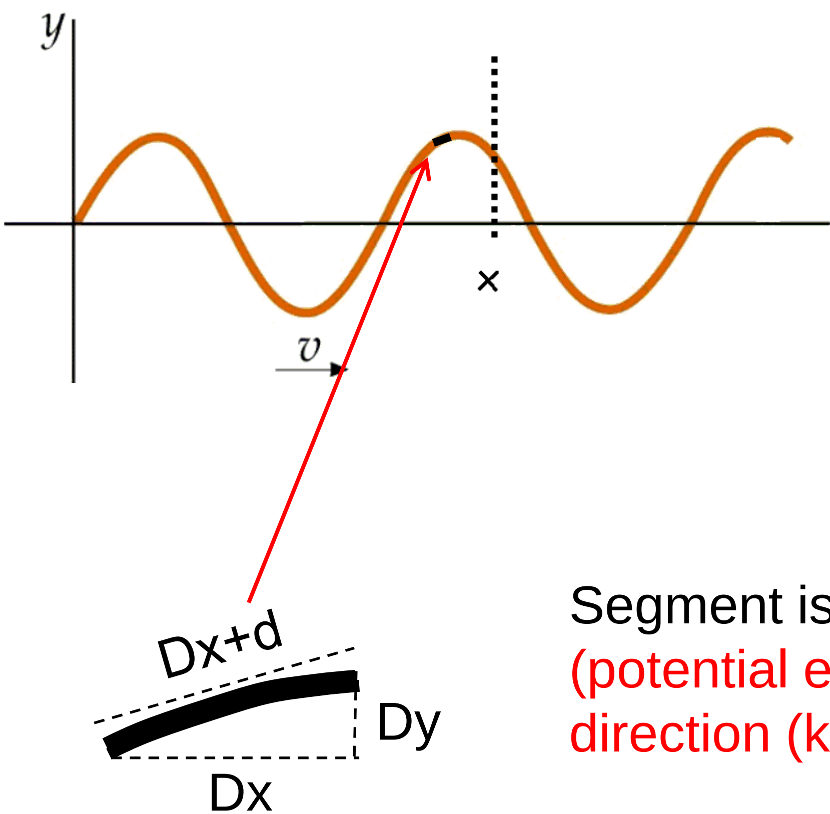


$u(x,t)$
$t = 0.00\tau$
A
0
$-A$
x
$\frac{\lambda}{2}$
λ
$\frac{3\lambda}{2}$
2λ

Energy/power of a travelling wave on a string

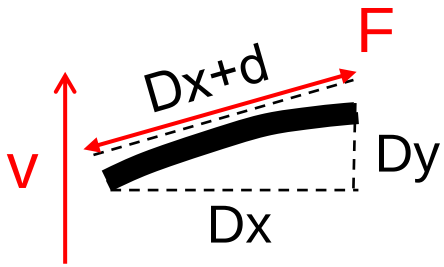
Consider position x on a string along which a harmonic wave is travelling from left to right.

String segment to the left of x is doing work on segment to the right of x, transferring energy along the string.



Segment is stretched from Dx to Dx +d (potential energy), and moves in y direction (kinetic energy)

Kinetic energy due to motion $E_{kin} = \frac{1}{2}mv^2$



velocity in y-direction:

$$v = \frac{dy}{dt} = \frac{d}{dt}\left[A\sin(kx - \omega t)\right] = -\omega A\cos(kx - \omega t)$$

$$\Rightarrow E_{kin} = \frac{1}{2}\mu\Delta x\left(\frac{dy}{dt}\right)^2 = \frac{1}{2}\mu\Delta x\omega^2 A^2\cos^2(kx - \omega t) \qquad (\mu\Delta x \text{ is mass})$$

Elastic potential energy due to stretching (as for spring): $E_{pot} = F\delta x$

$$\Delta x + \delta x = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \Delta x\left[1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right]^{1/2}$$

factor out Δx

what is δx ?

Binomial expansion:

$$\Delta x + \cancel{\delta x} = \Delta x\left[1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right]^{1/2} = \Delta x\left[1 + \frac{1}{2}\left(\frac{\Delta y}{\Delta x}\right)^2\right]$$

$$\delta x = \Delta x + \frac{\Delta x}{2}\left(\frac{\Delta y}{\Delta x}\right)^2 - \Delta x$$

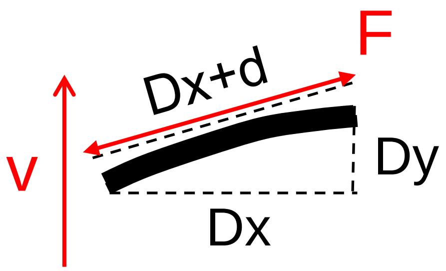
[(1+z)^n ≈ 1 + z/2 + …] and assume $(\Delta y/\Delta x) \ll 0$

$$= 1 + \left(\tfrac{1}{2}\right)x + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)x^2}{2!} + \frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{3}x^3}{3!}$$

$$= 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 +$$

$$\Rightarrow E_{pot} = F\delta x = F\left[\frac{\Delta x}{2}\left(\frac{\Delta y}{\Delta x}\right)^2\right]$$

Higher terms disappear due to small number to high power and division by higher factorial.



$$\Rightarrow E_{pot} = F\delta x = F\left[\frac{\Delta x}{2}\left(\frac{\Delta y}{\Delta x}\right)^2\right]$$

Let Dx, Dy $\rightarrow$ 0

$$\frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \frac{d}{dx}\left[A\sin(kx - \omega t)\right] = kA\cos(kx - \omega t)$$

$$\Rightarrow E_{pot} = F\delta x = \frac{F}{2}k^2A^2\cos^2(kx - \omega t)\Delta x$$

remember: $v = \frac{\lambda}{T} = \frac{\omega}{k} = \sqrt{\frac{F}{\mu}} \Rightarrow F = \mu\left(\frac{\omega}{k}\right)^2$ v is wave speed, not segment velocity!

$$\Rightarrow E_{pot} = \frac{1}{2}\mu\omega^2A^2\cos^2(kx - \omega t)\Delta x$$

from before: $E_{kin} = \frac{1}{2}\mu\omega^2A^2\cos^2(kx - \omega t)\Delta x$

Kinetic and potential energies of string segment are equal!

$$E_{kin} = E_{pot} = \frac{1}{2}\mu\omega^2 A^2 \cos^2(kx - \omega t)\Delta x \Rightarrow E_{tot} = \mu\omega^2 A^2 \cos^2(kx - \omega t)\Delta x$$

Vary with t and x, so average over one period!

satisfy yourself that this is true. Hint
$\cos 2x = \cos^2 x - \sin^2 x$
$\cos^2 x + \sin^2 x = 1$
and $\sin 2\pi = 0$

$$E_{av} = \mu\omega^2 A^2 \Delta x \frac{\int_0^T \cos^2(kx - \omega t)dt}{T} = \frac{1}{2}\mu\omega^2 A^2 \Delta x$$

or simply, the average value for $\cos^2 x = \frac{1}{2}$

length of segment = wave speed × time $\Delta x = v\Delta t$

Power = energy transmitted per unit time:

so $P_{av} = \dfrac{\frac{1}{2}\mu\omega^2 A^2 v \cancel{\Delta t}}{\cancel{\Delta t}}$

$$\boxed{P_{av} = \frac{1}{2}\mu v\omega^2 A^2} \qquad \boxed{P_{av} = \frac{1}{2}\sqrt{\mu F}\omega^2 A^2}$$

$v = \sqrt{\frac{F_T}{\mu}}$ $\quad P_{av} = \frac{1}{2}\mu F_T^{\frac{1}{2}}\mu^{-\frac{1}{2}}\omega^2 A^2$

$\sqrt{\mu F} = Z$, a quantity known as impedance

for later

Power is proportional to Amplitude^2 and Frequency^2!

ASIDE - maths from previous slide

$$\frac{\int_0^T \cos^2(kx-\omega t)\,dt}{T}$$

use $\cos^2 x + \sin^2 x = 1$ $(x = t)$

$\cos 2x = \cos^2 x - \sin^2 x$

combine

$\cos 2x = \cos^2 x - (1 - \cos^2 x)$

$\cos 2x = 2\cos^2 x - 1$

$\cos^2 x = \frac{1}{2}\cos 2x + \frac{1}{2}$ ✓

$$\int_0^T \frac{1}{2}\cos 2x + \frac{1}{2}\,dx$$

$$= \frac{1}{4}\sin 2x + \frac{x}{2} + c$$

$$T \qquad - \qquad 0$$

$$\frac{\frac{1}{4}\sin 2T + \frac{T}{2} + \cancel{c} - \cancel{\frac{1}{4}\sin(2.0)} + \cancel{\frac{0}{2}} + \cancel{c}}{T}$$

$$= \frac{\frac{1}{4}\sin(2T) + \frac{1}{2}}{T}$$

$\sin T$ (one period, 2π) $\equiv 2T$

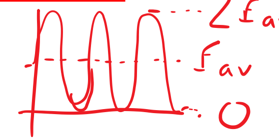
$= 0.$

$$= \frac{1}{2}$$

Energy transfer

$\boxed{P_{max} = 2 \times P_{av}}$

$E = \cos^2$ function,



Power = rate of energy transfer

Average energy ΔE_{av} flowing past a point p in time Δt is given by

$$\Delta E_{av} = \tfrac{1}{2}\mu v \omega^2 A^2 \Delta t$$

$\Delta x = v\Delta t$

And expressed per unit length Δx, we have

$$\Delta E_{av} = \tfrac{1}{2}\mu \omega^2 A^2 \Delta x$$

For A = 10 cm, F_T = 20 N, μ = 0.1 kg m^{-1}, ω = 0.6 s^{-1}

$v = \sqrt{\frac{F}{\mu}} = \sqrt{\frac{20}{0.1}} = 14\ \mathrm{ms^{-1}}$

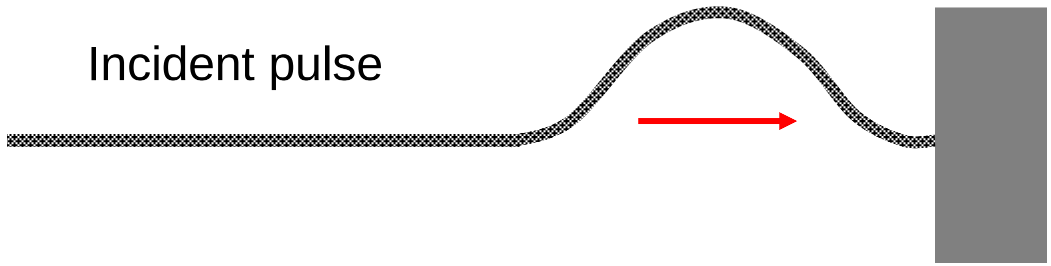
If $\Delta t = 1\,\mathrm{s}$, $\Delta E_{av} = 2.5\ \mathrm{mJ}$

If $\Delta x = 10\,\mathrm{m}$, $\Delta E_{av} = 1.8\ \mathrm{mJ}$

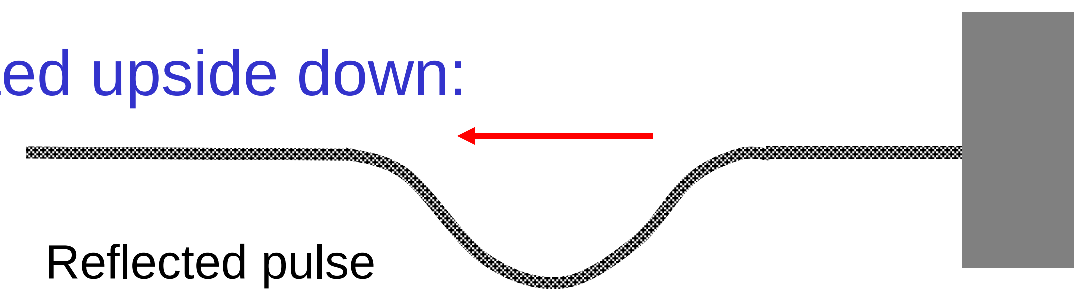
Reflection of travelling waves

Tipler 15-4 + *French Ch.8*

A pulse travelling on a string fixed at one end:



...is reflected upside down:

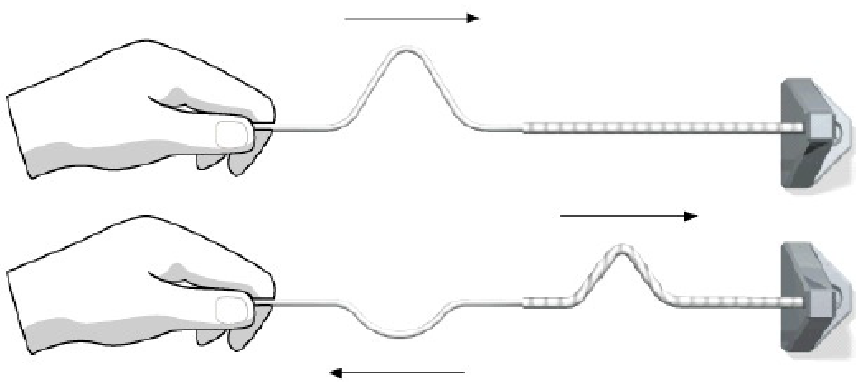


Why? - Conservation of energy !

Impedance: transmission across 'junctions'

Reflection occurs at an interface between light and heavy strings:

Reflected wave is inverted

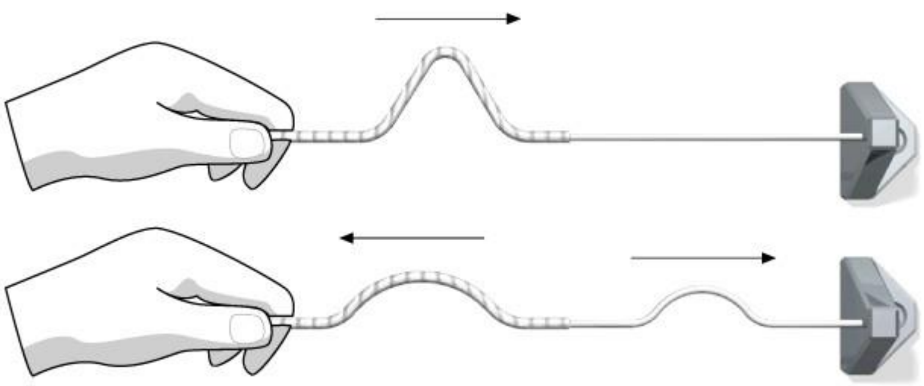


This is because they have different wave impedances, but A and ω match across the interface.

To transfer 100% of the wave power from one medium to another, the impedances must match.

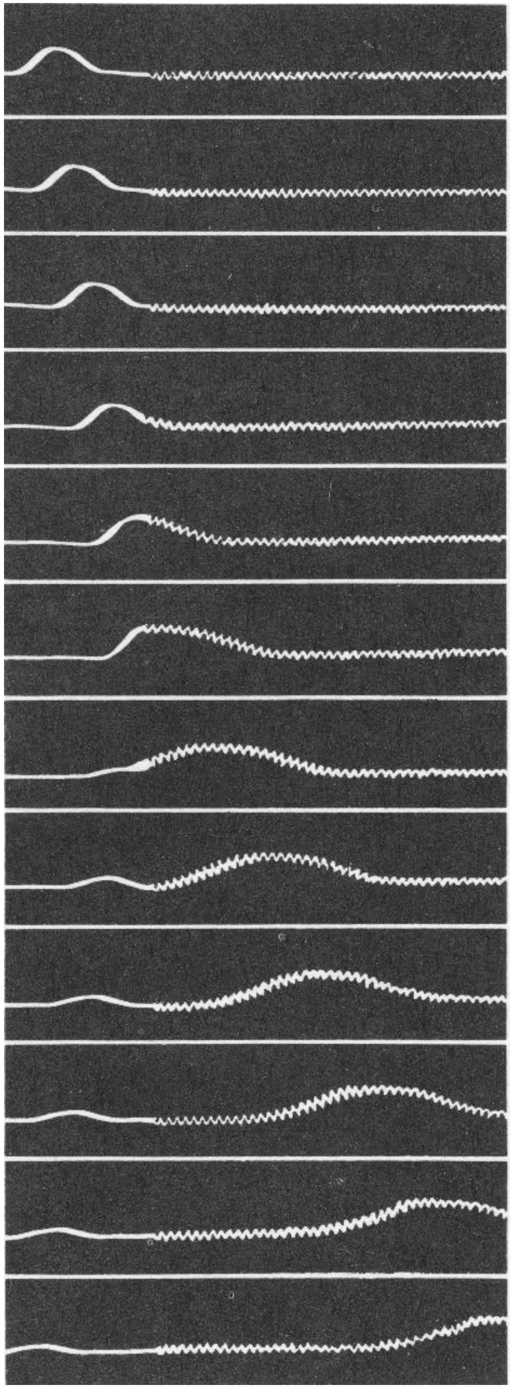
Impedance plays a similar role to refractive index for light waves travelling from one medium to another

A pulse travelling from a heavy string to a lighter string

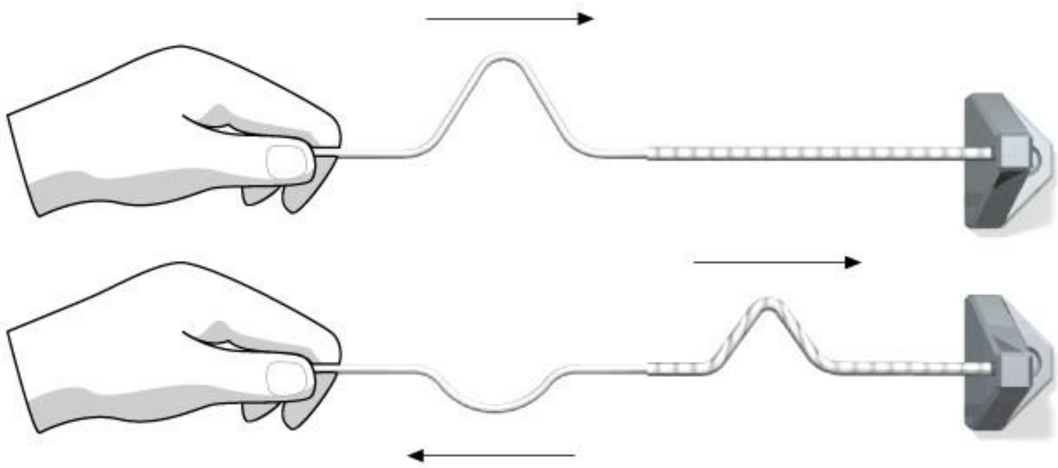


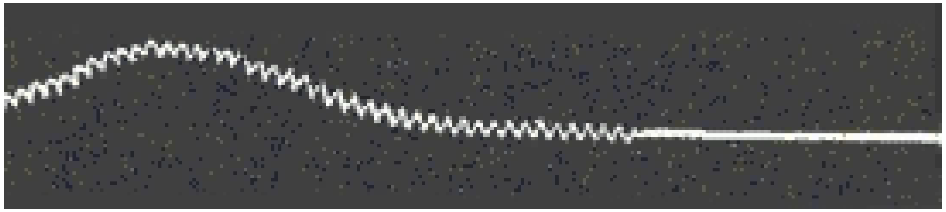
Reflected wave is not inverted in this case

Wave speed is higher on lighter string



A pulse travelling from a light string to a heavier string:





Demonstration
https://www.youtube.com/watch?v=ocLGb28_8UQ

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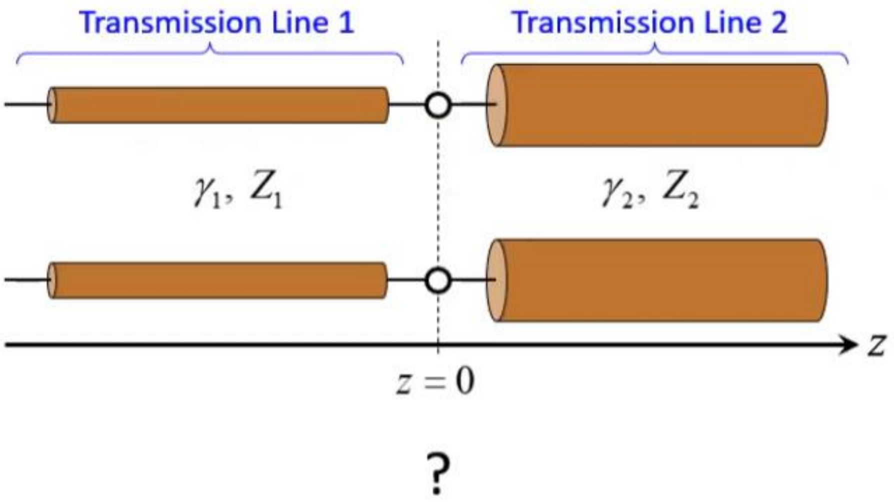
Applied Electromagnetics

Topic 4d

Scattering on a Transmission Line



signal degradation

- Scattering at an Impedance Discontinuity
- Power on a Transmission Line
- Voltage Standing Wave Ratio (VSWR)

power reflected back at junction with mismatched impedance. = BAD!