UNIVERSITY OF LEEDS

Mechanics 1

Session 13: Circular Motion – Acceleration in Radial Co-ordinates

DR BEN HANSON

1

Last Lecture

Circular Motion 1 – Velocity in Radial Co-Ordinates

We:
- Recapped everything we've done so far
- Learned that each of the concepts we have studied so far has a parallel in circular motion
- Recalled what we know of circular motion from previous studies (A-levels etc)
- Derived the fundamental kinematic equations of motion in *circular* co-ordinates

You should be able to:
- Understand that circular motion is best studied in *circular* co-ordinates
- Understand that unlike the Cartesian unit vectors $\underline{i}, \underline{j}$, circular unit vectors rotate
- Reproduce the derivation of the kinematic equations for velocity circular co-ordinates

2

This Lecture

Circular Motion 2 – Acceleration & Key Concepts

We will:
- Describe the different components of velocity and acceleration in circular coordinates
- Derive the vector form of centripetal acceleration
- Derive the vector form of velocity and acceleration with variable angular speed
- Consider what it means to have rotation in these directions

You will be able to:
- Reproduce the derivation of the *full* kinematic equations for velocity and acceleration in circular co-ordinates
- Calculate velocities, accelerations and forces in circular coordinates
- Transform from cartesian coordinates to circular coordinates

3

This Lecture

Don't be afraid…

The methods we will cover today will be very new to you all. I promise you, you are all capable of understanding this.

- You have already shown in your coursework that you understand the concepts we are about to discuss
- Don't be afraid to ask questions. If I don't see your hand, shout out!

4

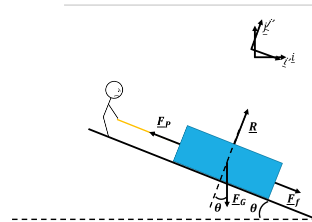
Circular Co-ordinates

A Recap

5

Circular Co-ordinates

Changing Co-ordinate Systems



Changing to "ideal" co-ordinate systems:

1. Doesn't change the underlying physics
2. Makes calculations easier
3. Can change how we measure things (positions, velocities (momentum), accelerations (forces))

6

Circular Co-ordinates

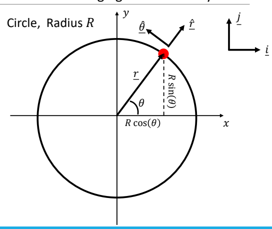
Changing Co-ordinate Systems

	Cartesian $(\underline{i}, \underline{j})$	Radial $(\hat{\underline{r}}, \hat{\underline{\theta}})$
Position, $\underline{r} =$	$R\cos(\theta)\,\underline{i} + R\sin(\theta)\,\underline{j}$	$R\hat{\underline{r}}$
Velocity, $\underline{v} =$		
Acceleration, $\underline{a} =$		

$\underline{r} = R\hat{\underline{r}} = R\cos(\theta)\,\underline{i} + R\sin(\theta)\,\underline{j}$

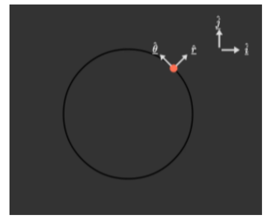
$\boxed{\hat{\underline{r}} = \cos(\theta)\,\underline{i} + \sin(\theta)\,\underline{j}}$

Circle, Radius R



Circular Co-ordinates

Changing Co-ordinate Systems



Changing to "ideal" co-ordinate systems:

1. Doesn't change the underlying physics
2. Makes calculations easier
3. Can change how we measure things (positions, velocities (momentum), accelerations (forces))

- $\underline{i}$ and $\underline{j}$ are constant unit vectors
- $\hat{\underline{r}}$ and $\hat{\underline{\theta}}$ are unit vectors that vary with θ
- θ varies with time
- $\boxed{\hat{\underline{r}} \text{ and } \hat{\underline{\theta}} \text{ vary with time!}}$

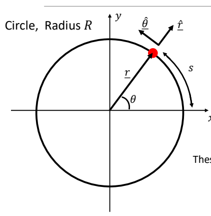
Circular Motion

Properties We've Learned So Far

Circular Motion

Properties Learned So Far

Circle, Radius R



Radial Properties $(\hat{\underline{\theta}})$	Angular	Linear
Distance	θ	$s = R\theta$
Speed	$\omega = \frac{d\theta}{dt}$	$v_\theta = \frac{ds}{dt} = R\omega$
Acceleration	$\alpha = \frac{d\omega}{dt}$	$a_\theta = \frac{dv}{dt} = R\alpha$

These properties are all in the $\hat{\underline{\theta}}$ direction:

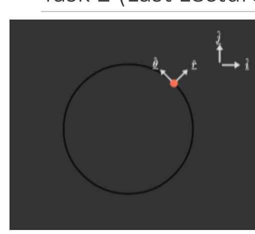
- θ is the angle moved around the circle, s is the arc length
- ω is the angular speed, v_θ is the linear speed *in the $\hat{\underline{\theta}}$ direction* i.e. the speed along the circumference itself
- α is the angular acceleration, a_θ is the linear acceleration *in the $\hat{\underline{\theta}}$ direction*

Task 2 (Last Lecture)

Around and Around the Circle We Go

Task 2 (Last Lecture)

Around and Around the Circle We Go



Scenario: A car is driving around a roundabout with $R = 12m$. Unfortunately, the driver does not know when to leave the roundabout so just keeps driving around at a constant speed.

Tasks:

1. The speedometer in the car reads $20mph$. Calculate the angular speed. (Hint: $20mph$ is about $9ms^{-1}$)
2. What is the linear velocity vector of the car? (Hint: In the UK, which way do we drive around roundabouts?)
3. Calculate the time period, the total time taken for a single revolution of the circle.
4. Calculate the frequency, the number of revolutions of the circle the car does per second.

Task 2 (Last Lecture) Around and Around the Circle We Go

Scenario: A car is driving around a roundabout with $R = 12m$. Unfortunately, the driver does not know when to leave the roundabout so just keeps driving around at a constant speed.

Tasks:
1. The speedometer in the car reads $20mph$. Calculate the angular speed. (Hint: $20mph$ is about $9ms^{-1}$)

Linear speed->Angular speed, $v_\theta = R\omega$

Rearrange, $\omega = \frac{v_\theta}{R}$

Speedometer shows linear speed, $\omega = \frac{9}{12}$

Solve, $\boxed{\omega = 0.75\ rads.s^{-1}}$

13

MECHANICS 1: CIRCULAR MOTION - ACCELERATION IN RADIAL CO-ORDINATES

Task 2 (Last Lecture) Around and Around the Circle We Go

Scenario: A car is driving around a roundabout with $R = 12m$. Unfortunately, the driver does not know when to leave the roundabout so just keeps driving around at a constant speed.

Tasks:
2. What is the linear velocity vector of the car? (Hint: In the UK, which way do we drive around roundabouts?)

Clockwise driving, $\underline{v} = 9ms^{-1}(-\underline{\hat{\theta}})$

$\boxed{\underline{v} = -9ms^{-1}\underline{\hat{\theta}}}$

DR BEN HANSON

14

MECHANICS 1: CIRCULAR MOTION - ACCELERATION IN RADIAL CO-ORDINATES

Task 2 (Last Lecture) Around and Around the Circle We Go

Scenario: A car is driving around a roundabout with $R = 12m$. Unfortunately, the driver does not know when to leave the roundabout so just keeps driving around at a constant speed.

Tasks:
3. Calculate the time period, the total time taken for a single revolution of the circle.

Method 1:

Total distance, $s = 2\pi R$

SUVAT, $a = 0$, $s = ut + \frac{1}{2}at^2$

$s = ut$

Rearrange, $t = \frac{s}{u}$

$t = \frac{2\pi R}{u}$

DR BEN HANSON

15

MECHANICS 1: CIRCULAR MOTION - ACCELERATION IN RADIAL CO-ORDINATES

Task 2 (Last Lecture) Around and Around the Circle We Go

Scenario: A car is driving around a roundabout with $R = 12m$. Unfortunately, the driver does not know when to leave the roundabout so just keeps driving around at a constant speed.

Tasks:
3. Calculate the time period, the total time taken for a single revolution of the circle.

Method 1:

Rearrange, $t = \frac{2\pi R}{u}$

Sub, $t = \frac{2\pi \times 12}{9}$

Solve, $\boxed{t \approx 8.38s}$

DR BEN HANSON

16

MECHANICS 1: CIRCULAR MOTION - ACCELERATION IN RADIAL CO-ORDINATES

Task 2 (Last Lecture) Around and Around the Circle We Go

Scenario: A car is driving around a roundabout with $R = 12m$. Unfortunately, the driver does not know when to leave the roundabout so just keeps driving around at a constant speed.

Tasks:
3. Calculate the time period, the total time taken for a single revolution of the circle.

Method 2:

Total angle, $\theta = 2\pi$

SUVAT, $\alpha = 0$, $\theta = \omega t + \frac{1}{2}\alpha t^2$

$\theta = \omega t$

Rearrange, $t = \frac{\theta}{\omega}$

$t = \frac{2\pi}{\omega}$

DR BEN HANSON

17

MECHANICS 1: CIRCULAR MOTION - ACCELERATION IN RADIAL CO-ORDINATES

Task 2 (Last Lecture) Around and Around the Circle We Go

Scenario: A car is driving around a roundabout with $R = 12m$. Unfortunately, the driver does not know when to leave the roundabout so just keeps driving around at a constant speed.

Tasks:
3. Calculate the time period, the total time taken for a single revolution of the circle.

Method 2:

Rearrange, $t = \frac{2\pi}{\omega}$

Sub, $t = \frac{2\pi}{0.75}$

Solve, $\boxed{t \approx 8.38s}$

DR BEN HANSON

18

Task 2 (Last Lecture)

Around and Around the Circle We Go

Scenario: A car is driving around a roundabout with $R = 12m$. Unfortunately, the driver does not know when to leave the roundabout so just keeps driving around at a constant speed.

Tasks:

4. Calculate the frequency, the number of revolutions of the circle the car does per second.

$f = \frac{1}{T}$, $\qquad f = \frac{1}{8.38}$

Solve, $\qquad t \approx 0.12s^{-1}$

$$\boxed{t \approx 0.12Hz}$$

19

Circular Motion

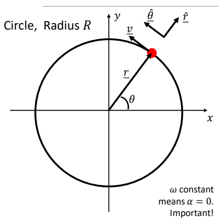
Acceleration (with constant R and ω)

20

Circular Motion

Acceleration (with constant R and ω)

Circle, Radius R



Velocity, $\quad \underline{v} = R\omega\underline{\hat{\theta}}$

Acceleration, $\quad \underline{a} = \frac{d\underline{v}}{dt}$

$$\underline{a} = \frac{d}{dt}(R\omega\underline{\hat{\theta}})$$

Again, only direction changing with time!

R and ω constant, $\quad \underline{a} = R\omega\frac{d}{dt}(\underline{\hat{\theta}})$

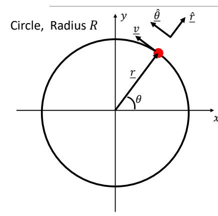
ω constant means $\alpha = 0$. Important!

21

Circular Motion

Acceleration (with constant R and ω)

Circle, Radius R



Change to Cartesian, $\quad \underline{a} = R\omega\frac{d}{dt}\left(-\sin(\theta)\,\underline{i} + \cos(\theta)\,\underline{j}\right)$

Chain Rule, $\quad \underline{a} = R\omega\frac{d\theta}{dt}\frac{d}{d\theta}\left(-\sin(\theta)\,\underline{i} + \cos(\theta)\,\underline{j}\right)$

$$\underline{a} = R\omega^2\frac{d}{d\theta}\left(-\sin(\theta)\,\underline{i} + \cos(\theta)\,\underline{j}\right)$$

Differentiate, $\quad \underline{a} = R\omega^2\left(-\cos(\theta)\,\underline{i} - \sin(\theta)\,\underline{j}\right)$

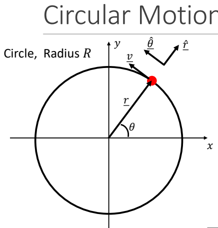
Factorise -, $\quad \underline{a} = -R\omega^2\left(\cos(\theta)\,\underline{i} + \sin(\theta)\,\underline{j}\right)$

22

Circular Motion

Acceleration (with constant R and ω)

Circle, Radius R



Factorise -, $\quad \underline{a} = -R\omega^2\left(\cos(\theta)\,\underline{i} + \sin(\theta)\,\underline{j}\right)$

Substitute, $\quad \boxed{\underline{a} = -R\omega^2\underline{\hat{r}}}$

$\omega = \frac{v_\theta}{R}$, $\quad \boxed{\underline{a} = -\frac{v_\theta^2}{R}\underline{\hat{r}}}$

The classic result from our previous studies! And it all comes simply from the fact that our unit vectors are changing direction ☺

$$\boxed{\underline{r} = R\underline{\hat{r}} \qquad \underline{v} = R\omega\underline{\hat{\theta}} \qquad \underline{a} = -R\omega^2\underline{\hat{r}}}$$

23

Circular Motion

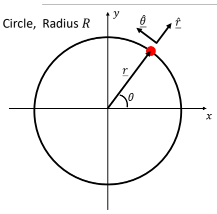
Acceleration (with constant R, variable ω)

24

Circular Motion

Acceleration (with constant R, variable ω)

Circle, Radius R



Position, $\underline{r} = R\hat{\underline{r}}$

Velocity, $\underline{v} = \frac{d\underline{r}}{dt}$

$\underline{v} = \frac{d}{dt}(R\hat{\underline{r}})$

Only direction changing with time!

R constant, $\underline{v} = R\frac{d}{dt}(\hat{\underline{r}})$

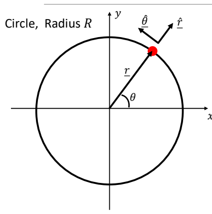
Change to Cartesian, $\underline{v} = R\frac{d}{dt}\left(\cos(\theta)\,\underline{i} + \sin(\theta)\,\underline{j}\right)$

25

Circular Motion

Acceleration (with constant R, variable ω)

Circle, Radius R



Change to Cartesian, $\underline{v} = R\frac{d}{dt}\left(\cos(\theta)\,\underline{i} + \sin(\theta)\,\underline{j}\right)$

Chain Rule, $\underline{v} = R\frac{d\theta}{dt}\frac{d}{d\theta}\left(\cos(\theta)\,\underline{i} + \sin(\theta)\,\underline{j}\right)$

$\underline{v} = R\omega\frac{d}{d\theta}\left(\cos(\theta)\,\underline{i} + \sin(\theta)\,\underline{j}\right)$

Differentiate, $\underline{v} = R\omega\left(-\sin(\theta)\,\underline{i} + \cos(\theta)\,\underline{j}\right)$

Substitute, $\boxed{\underline{v} = R\omega\hat{\underline{\theta}}}$ $\boxed{\underline{v} = v_\theta\hat{\underline{\theta}}}$

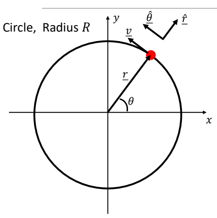
No different to before! If ω is variable, the derivation for the angular speed is still the same!

26

Circular Motion

Acceleration (with constant R, variable ω)

Circle, Radius R



Acceleration, $\underline{a} = \frac{d\underline{v}}{dt}$

$\underline{a} = \frac{d}{dt}(R\omega\hat{\underline{\theta}})$

ω now remains within the derivative.

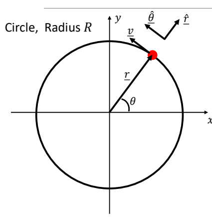
Only R constant, $\underline{a} = R\frac{d}{dt}(\omega\hat{\underline{\theta}})$

27

Circular Motion

Acceleration (with constant R, variable ω)

Circle, Radius R



Only R constant, $\underline{a} = R\frac{d}{dt}(\omega\hat{\underline{\theta}})$

Product rule, $\underline{a} = R\left(\frac{d\omega}{dt}\hat{\underline{\theta}} + \omega\frac{d\hat{\underline{\theta}}}{dt}\right)$

Simplify, $\underline{a} = R\alpha\hat{\underline{\theta}} + R\omega\frac{d\hat{\underline{\theta}}}{dt}$

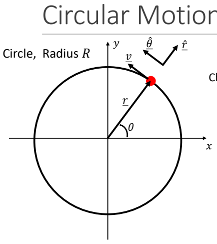
Change to Cartesian, $\underline{a} = R\alpha\hat{\underline{\theta}} + R\omega\frac{d}{dt}\left(-\sin(\theta)\,\underline{i} + \cos(\theta)\,\underline{j}\right)$

28

Circular Motion

Acceleration (with constant R, variable ω)

Circle, Radius R



Change to Cartesian, $\underline{a} = R\alpha\hat{\underline{\theta}} + R\omega\frac{d}{dt}\left(-\sin(\theta)\,\underline{i} + \cos(\theta)\,\underline{j}\right)$

Chain Rule, $\underline{a} = R\alpha\hat{\underline{\theta}} + R\omega\frac{d\theta}{dt}\frac{d}{d\theta}\left(-\sin(\theta)\,\underline{i} + \cos(\theta)\,\underline{j}\right)$

$\underline{a} = R\alpha\hat{\underline{\theta}} + R\omega^2\frac{d}{d\theta}\left(-\sin(\theta)\,\underline{i} + \cos(\theta)\,\underline{j}\right)$

Differentiate, $\underline{a} = R\alpha\hat{\underline{\theta}} + R\omega^2\left(-\cos(\theta)\,\underline{i} - \sin(\theta)\,\underline{j}\right)$

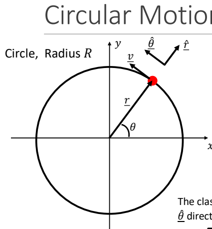
Factorise -, $\underline{a} = R\alpha\hat{\underline{\theta}} - R\omega^2\left(\cos(\theta)\,\underline{i} + \sin(\theta)\,\underline{j}\right)$

29

Circular Motion

Acceleration (with constant R, variable ω)

Circle, Radius R



Factorise -, $\underline{a} = R\alpha\hat{\underline{\theta}} - R\omega^2\left(\cos(\theta)\,\underline{i} + \sin(\theta)\,\underline{j}\right)$

Substitute, $\boxed{\underline{a} = R\alpha\hat{\underline{\theta}} - R\omega^2\hat{\underline{r}}}$

$\omega = \frac{v_\theta}{R}$, $\boxed{\underline{a} = R\alpha\hat{\underline{\theta}} - \frac{v_\theta^2}{R}\hat{\underline{r}}}$

The classic result from our previous studies, but now with an extra component in the $\hat{\underline{\theta}}$ direction! Now the vectors are changing direction, but also ω itself is changing ☺

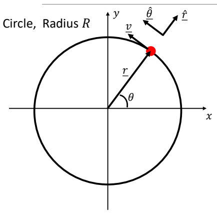
$\boxed{\underline{r} = R\hat{\underline{r}} \qquad \underline{v} = R\omega\hat{\underline{\theta}} \qquad \underline{a} = R\alpha\hat{\underline{\theta}} - R\omega^2\hat{\underline{r}}}$

30

Circular Motion

Acceleration (with constant R, variable ω)

Circle, Radius R



Centripetal acceleration. This causes the continuous change of direction of the object. <u>This is the most important component of acceleration</u>

$$\underline{a} = -R\omega^2\hat{r} + R\alpha\hat{\theta}$$

Angular acceleration. This is the change in angular speed; of the object moving around the circle. It is this component that is caused by the application of rotational forces, or "torques"

Task 1

Velocity and Acceleration in Circular Coordinates

Task 1

Velocity and Acceleration in Circular Coordinates

Scenario: An object of mass $m = 4kg$ is undergoing circular motion with an initial radius $R = 10m$. Currently, at $t = 0s$, has an acceleration vector $\underline{a} = -8ms^{-2}\hat{\underline{r}} + 5ms^{-2}\hat{\underline{\theta}}$:

Tasks:

1. Calculate the angular acceleration
2. Calculate the angular velocity
3. Calculate the force acting on this object. Identify the centripetal and angular components.
4. Is the angular velocity constant?
5. Is the magnitude of the force constant?
6. Calculate the angular speed at $t = 5s$
7. Use this new angular speed to calculate the total acceleration vector and thus, the new force on the object at $t = 5s$.
8. Imagine the centripetal force was generated by the tension in a string, like a slingshot. What would happen if I gave the object such a large angular speed, that the tension increased to more than the string could support?

Task 2

Just a quick co-ordinate change

Task 2

Just a quick co-ordinate change

Scenario: An object of mass $m = 4kg$ is undergoing circular motion with a radius $R = 10m$. At $t = 5s$, the object is at an angle of 1.15 radians to the horizontal, and has an acceleration vector $\underline{a} = -6ms^{-2}\hat{\underline{i}} - 9ms^{-2}\hat{\underline{j}}$:

Tasks:

1. By transforming into radial coordinates (writing the acceleration in terms of $\hat{\underline{r}}$ and $\hat{\underline{\theta}}$), calculate the angular and centripetal forces. *Hint: Consider the dot product. Remember our derivation from the start of the course?*