More on Functions

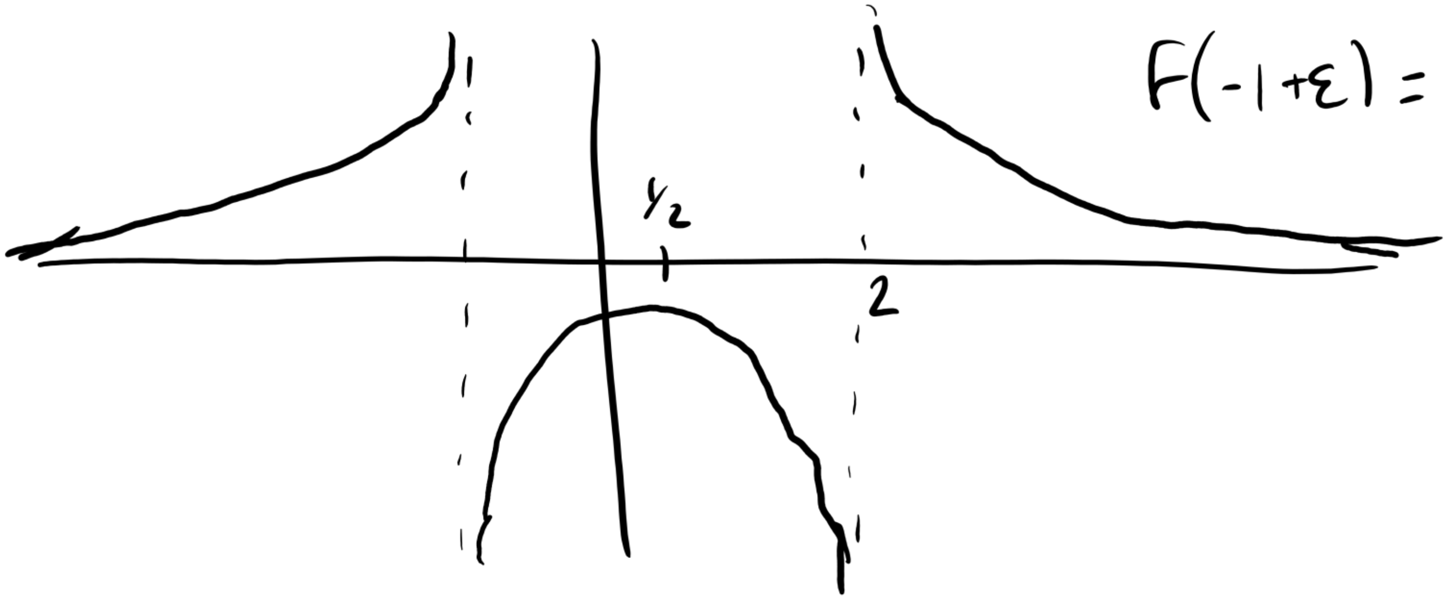
Example

- Sketch the graph of $f: \mathbb{R}\backslash\{-1,2\} \to \mathbb{R}$, $f(x) = \frac{1}{(x-2)(x+1)}$ identifying asymptotes and stationary points (there are no roots)

Vert asymptotes at $x=-1$, $x=2$

$f(2+\varepsilon) = \frac{1}{(2+\varepsilon-2)(2+\varepsilon+1)}$

$F(-1+\varepsilon) = \ldots$



$$f(x) = \frac{1}{(x-2)(x+1)} = (x^2 - x - 2)^{-1}$$

$$f'(x) = (2x-1)(-1)(x^2 - x - 2)^{-2} = \frac{-(2x-1)}{(x^2-x-2)^2}$$

s.p. @ $x = \frac{1}{2}$

Inverse Functions

- Only defined if f is **bijective**

1. for every $b \in B$ there is $a \in A$ such that $f(a) = b$ surjective
2. for any distinct $x, y \in A$, $f(x) \neq f(y)$ injective

- If f bijective, inverse is

$$f^{-1} : B \to A \qquad f^{-1} \circ f = I$$

$$I : A \to A, \quad I(x) = x$$

Inverse Functions

- Can always restrict A, B to produce bijection

Eg $\quad f(x) = x^2 \quad f: \mathbb{R} \to \mathbb{R}$

Restrict A to $\mathbb{R}^+$

Restrict B to $\mathbb{R}^+$

Finding Inverse Functions

Often can rearrange to find argument in terms of value

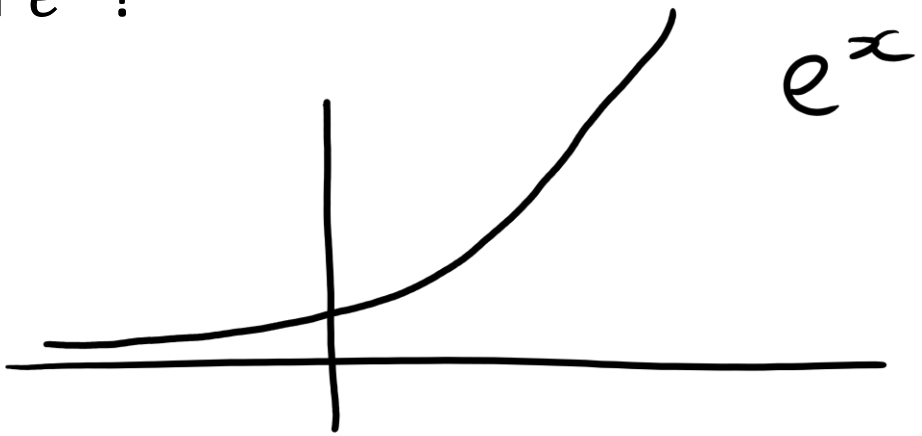
Eg. $f : \mathbb{R} \setminus \{3\} \to \mathbb{R} \setminus \{2\}$ $\quad f(x) = \dfrac{2x-5}{x-3}$

$$y = \frac{2x-5}{x-3} \implies x = \frac{3y-5}{y-2}$$

$$\implies f^{-1} : \mathbb{R} \setminus \{2\} \to \mathbb{R} \setminus \{3\} \qquad f^{-1}(x) = \frac{3x-5}{x-2}$$

Example

- What is the largest possible (real) domain on which we can define the inverse of e^x?



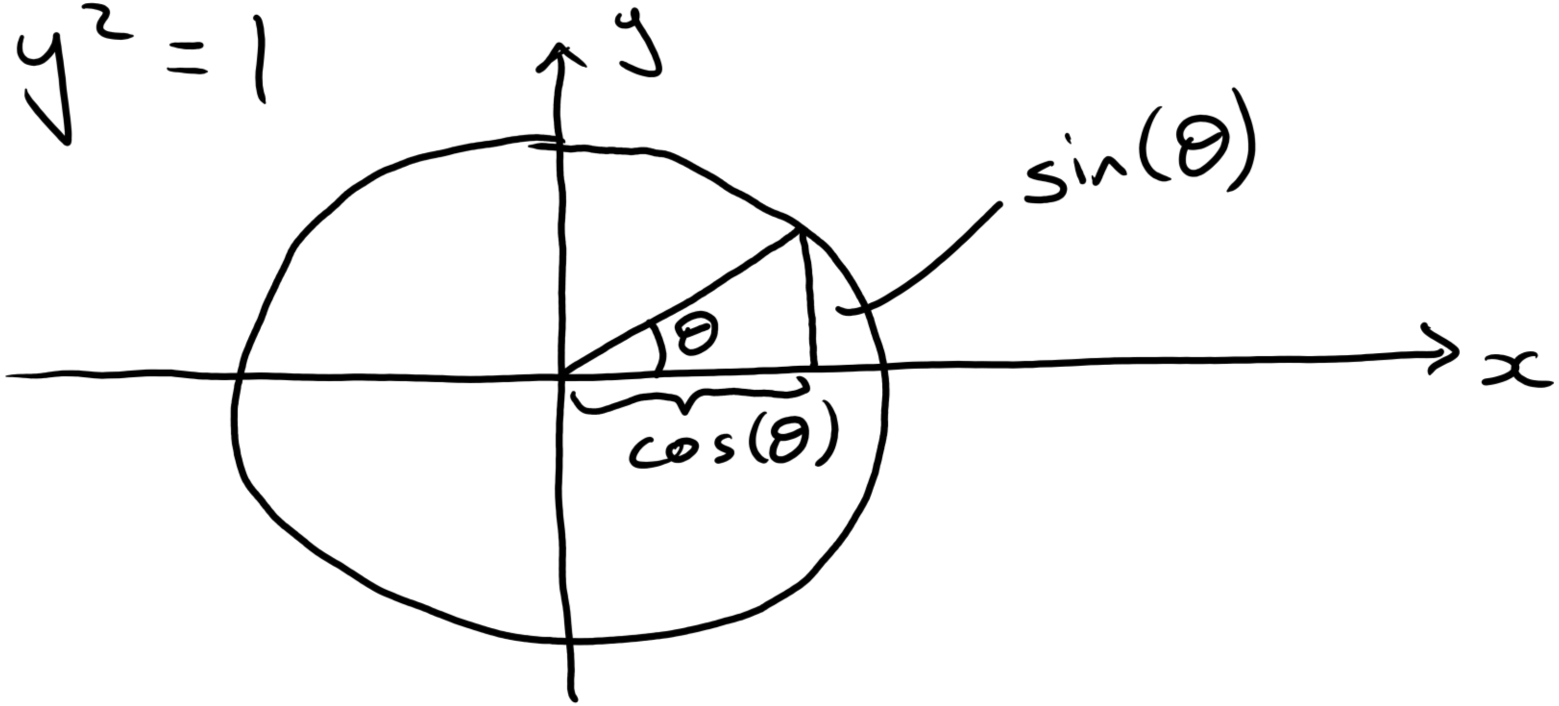
$\text{range}(e^x) = (0, \infty)$

$\text{domain}(\ln x) = (0, \infty)$

Circular Functions

- Regular trigonometric functions defined in terms of unit circle

$$x^2 + y^2 = 1$$



Circular Functions

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \qquad \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

$$\sec(\theta) = \frac{1}{\cos(\theta)} \qquad \csc(\theta) = \frac{1}{\sin(\theta)}$$

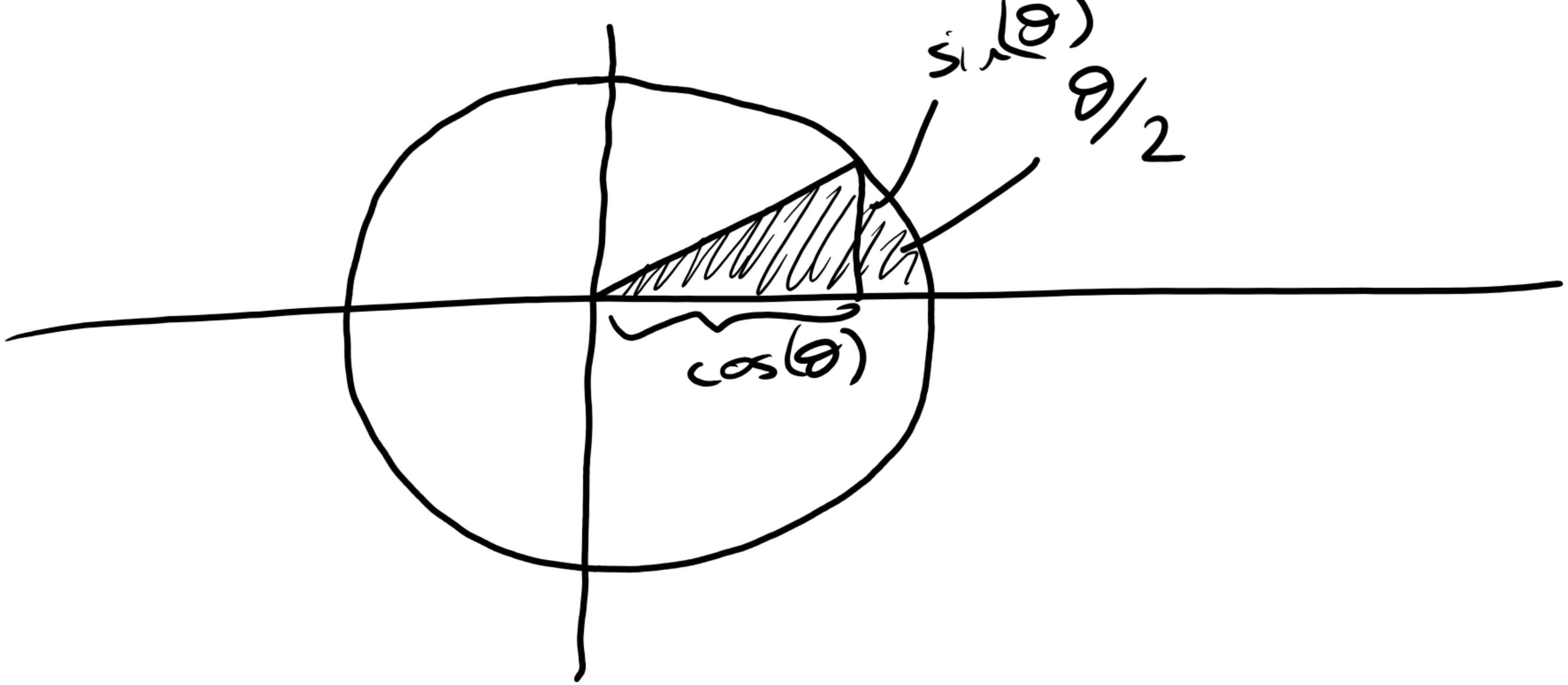
$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \cot^2\theta = \csc^2(\theta)$$

$\cdot \ \cdot \ \cdot \ \cdot$

Circular Functions

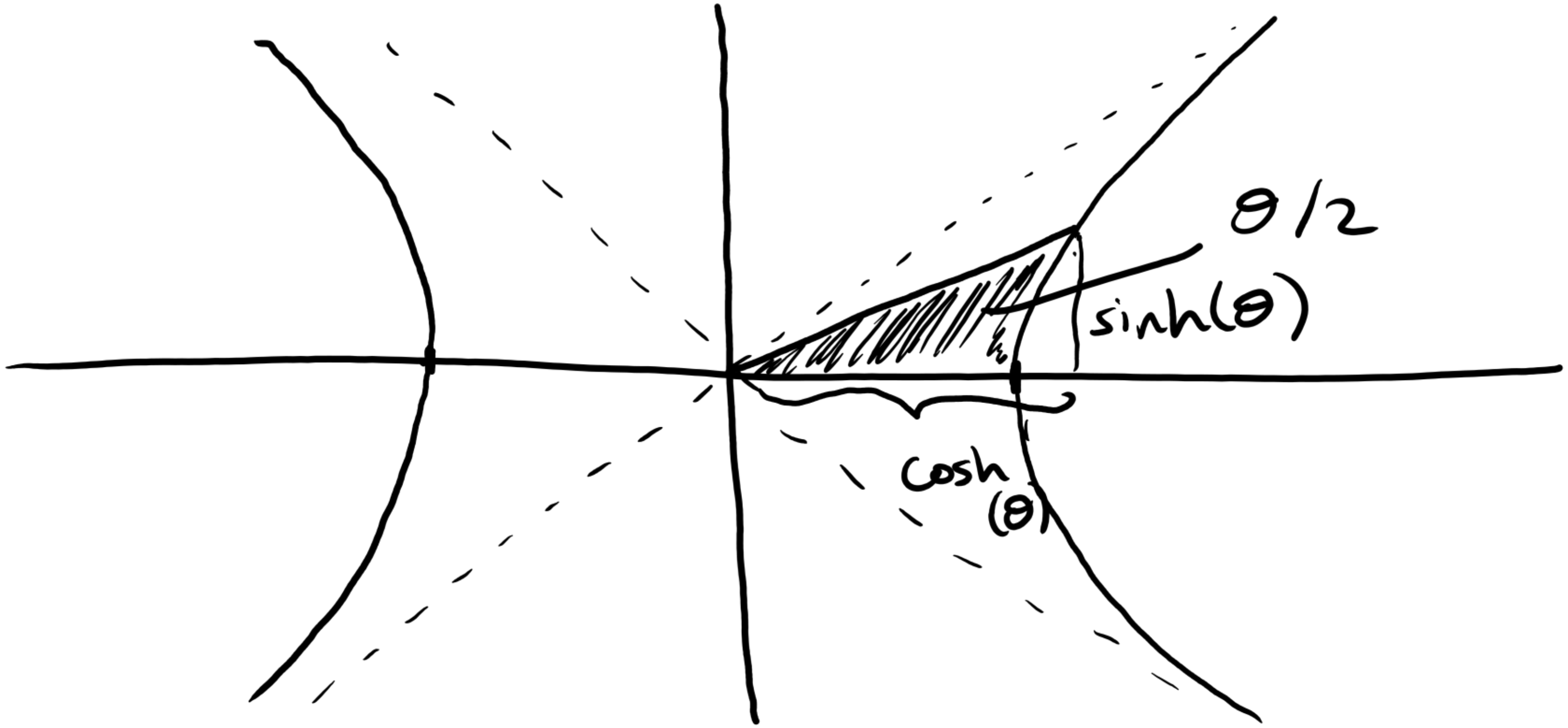
- Sector area $\frac{1}{2}r^2\theta$



Hyperbolic Functions

- Defined in terms of unit hyperbola

$x^2 - y^2 = 1$



Hyperbolic Functions

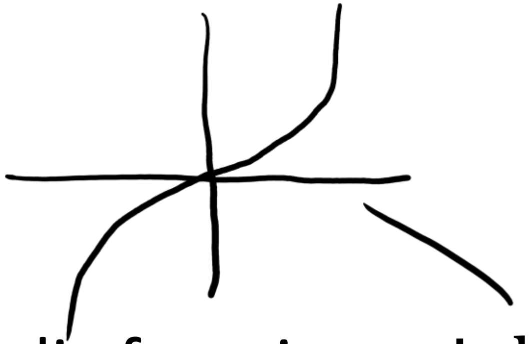
$$\tanh(\theta) = \frac{\sinh(\theta)}{\cosh(\theta)} \qquad \coth(\theta) = \frac{\cosh(\theta)}{\sinh(\theta)}$$

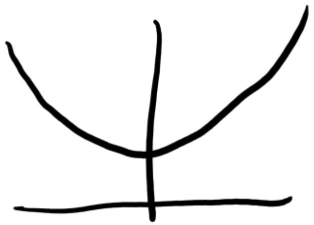
$$\operatorname{sech}(\theta) = \frac{1}{\cosh(\theta)} \qquad \operatorname{csch}(\theta) = \frac{1}{\sinh(\theta)}$$

Identity

$$\cosh(\theta) = \frac{e^{\theta} + e^{-\theta}}{2} \qquad \sinh(\theta) = \frac{e^{\theta} - e^{-\theta}}{2}$$

Exercises





- Sketch the hyperbolic functions $\sinh(x)$, $\cosh(x)$, $\tanh(x)$
- What is $\sinh(x) + \cosh(x)$? e^{x}
- What is $\cosh^2 x - \sinh^2 x$? 1
- What are

$$\frac{\mathrm{d}\sinh(x)}{\mathrm{d}x} \text{ and } \frac{\mathrm{d}\cosh(x)}{\mathrm{d}x} ?$$

$$\cosh(x) \qquad \sinh(x)$$