UNIVERSITY OF LEEDS

Mechanics 1

Session 7 – Variable Force, Energy & Work

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Last Lecture

Electric and Gravitational Forces

We learned:

- What the gravitational force is and how it is mathematically modelled
- Taylor series expansions (hopefully you watched the video!)
- What the electric force is and how it is mathematically modelled
- To compare and contrast electrical and gravitational forces

You should be able to:

- Calculate the (linear) motion of an object experiencing an electric or gravitational force
- Understand how vectors are used to represent forces in arbitrary directions

This Lecture

Variable Force, Energy & Work

We will:

- Consider how forces might vary over time or through space
- Understand the concept of energy
- Understand the concept of work done

You will be able to:

- Calculate the (energetic) work done by a force on a particular system with constant forces along straight lines
- Calculate the (energetic) work done by a force on a particular system with changing forces and non-linear paths

A Quick Note on Units

A Quick Note on Units

SI Units

Quantity	SI Unit
Time	Seconds (s)
Distance	Metres (m)
Speed	Metres per second (ms^{-1})
Acceleration	Metres per second-squared (ms^{-2})
Mass	Kilogram (kg)
Force	Newtons (N), $(kg.ms^{-2})$
Energy / Work	Joules (J), $(N.m)$

Variable Forces

Variable Forces

Acceleration isn't Constant Anymore ☹

Up to now, for the most part, we have mostly been considering constant forces (i.e. constant acceleration)

But many forces in our world are not constant. They continuously change from moment to moment, and throughout space.

Variable Forces

Examples

Engines:
- **Cars**
- **Planes**
- **Boats**

Air Resistance:
- **Aeroplanes**
- **Terminal Velocity**

Surface Friction:
- **Cars moving on hills**
- **Bicycles & Cars in motion**
- **Anything that rolls**

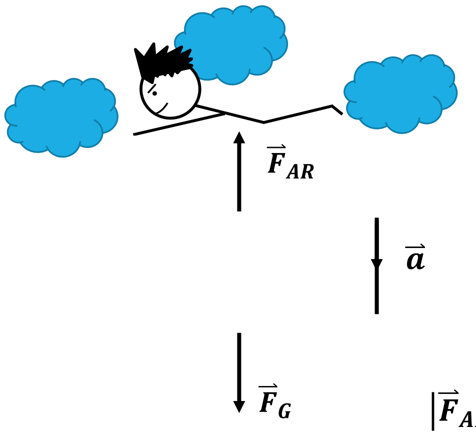
Electromagnetism:
- **Coulomb interaction**

Gravity (in general):
- **Planets in orbit**
- **Black holes**

Viscosity:
- **Swimmers**
- **Biological Objects (Organelles, proteins etc)**

Variable Forces

Going Down – Terminal Velocity



Initially,

$$\vec{F}_G - \vec{F}_{AR} = ma$$

$$mg - bv^2 = ma$$

$$a = g - \frac{b}{m}v^2$$

At equilibrium ($a = 0, v = v_T$),

$$\vec{F}_G = \vec{F}_{AR}$$

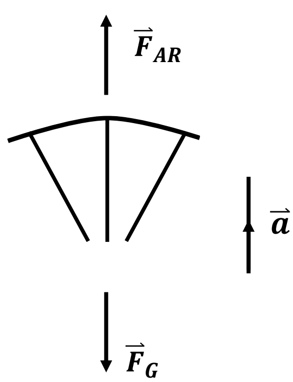
$$g - \frac{b}{m}(v_T)^2 = 0$$

$$\boxed{v_T = \sqrt{\frac{mg}{b}}}$$

$$|\vec{\boldsymbol{F}}_{\boldsymbol{AR}}| = \boldsymbol{b}|\vec{\boldsymbol{v}}|^2$$

Variable Forces

Going Down – Terminal Velocity



Initially,

$$\vec{F}_{AR} - \vec{F}_G = ma$$

$$cv^2 - mg = ma$$

$$a = \frac{c}{m}v^2 - g$$

At equilibrium ($a = 0, v = v_T$),

$$\vec{F}_G = \vec{F}_{AR}$$

$$\frac{c}{m}(v_T)^2 - g = 0$$

$$\boxed{v_T = \sqrt{\frac{mg}{c}}}$$

We can solve the equilibrium cases!

Variable Forces

Going Forward – Air Resistance

With the SUVAT equations, we know more than single points in time. We know how distance and velocity *vary* with time.

But what about when acceleration isn't constant? Can we still get an equation? Yes…sometimes

Variable Forces

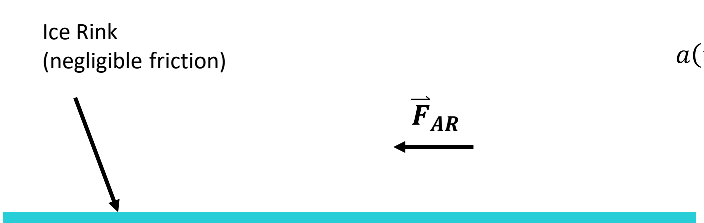
Air Resistance

Newton's 2nd Law, $F_{AR}(t) = ma(t)$

$F_{AR}(t) = -bv(t)^2$, $-bv(t)^2 = ma(t)$

Rearrange, $a(t) = -\frac{b}{m}v(t)^2$

$a(t) = \frac{dv}{dt}$, $\boxed{\frac{dv}{dt} = -\frac{b}{m}v(t)^2}$



Non-linear differential equation. Extremely complex!

Variable Forces

Air Resistance

You **do not** need to learn how to do this right now! I'm just making a point ☺

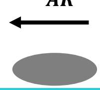
$$a(t) = \frac{dv}{dt}, \qquad \frac{dv}{dt} = -\frac{b}{m}v(t)^2$$

Chain Rule, $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt}$ $\qquad \frac{dv}{dx}\frac{dx}{dt} = -\frac{b}{m}v(t)^2$

$$v = \frac{dx}{dt}, \qquad \frac{dv}{dx} = -\frac{b}{m}v(x)$$

Rearrange, $\qquad \frac{dv}{dx} + \frac{b}{m}v(x) = 0$

$\vec{F}_{AR}$



Multiply integrating factor, $\qquad \frac{dv}{dx}e^{cx} + \frac{b}{m}v(x)e^{cx} = 0$

Variable Forces

Air Resistance

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Multiply integrating factor, $\qquad \frac{dv}{dx}e^{cx} + \frac{b}{m}v(x)e^{cx} = 0$

$$c = \frac{b}{m}, \qquad \frac{dv}{dx}e^{cx} + cv(x)e^{cx} = 0$$

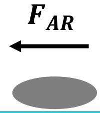
Factorise differential, $\qquad \frac{d}{dx}\left(v(x)e^{cx}\right) = 0$

Integrate, $\qquad v(x)e^{cx} = d$

$$x = 0, v(0) = 0, \qquad v(0)e^{c.0} = d$$

$$\rightarrow d = v(0)$$

$\vec{F}_{AR}$



Variable Forces

Air Resistance

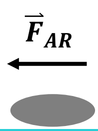
$x = 0, v(0) = 0,$ $\quad v(0)e^{c.0} = d$

$\rightarrow d = v(0)$

Solve, $\boxed{v(x) = v_0 e^{-cx}}$

You do not need to learn how to do this right now! I'm just making a point ☺

Great! We know speed as a function of position. But…we wanted position as a function of time…



Variable Forces

Air Resistance

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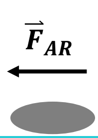
Solve, $\quad v(x) = v_0 e^{-cx}$

$v = \frac{dx}{dt},$ $\quad \frac{dx}{dt} = v_0 e^{-cx}$

Inverse, $\quad \frac{dt}{dx} = \frac{1}{v_0} e^{cx}$

Integrate, $\quad t = \frac{1}{v_0} \int e^{cx}\, dx$

$$t = \frac{1}{v_0}\left(\frac{1}{c} e^{cx} + h\right)$$



Variable Forces

Air Resistance

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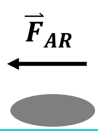
Integrate, $$t = \frac{1}{v_0}\int e^{cx}\,dx$$

$$t = \frac{1}{v_0}\left(\frac{1}{c}e^{cx} + h\right)$$

$t = 0, x = 0,$ $$\frac{1}{c}e^{c.0} + h = 0$$

$$\rightarrow h = -\frac{1}{c}$$

Substitute, $$t = \frac{1}{v_0}\left(\frac{1}{c}e^{cx} - \frac{1}{c}\right)$$



Variable Forces

Air Resistance

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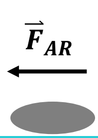
Substitute, $$t = \frac{1}{v_0}\left(\frac{1}{c}e^{cx} - \frac{1}{c}\right)$$

Make x subject, $$cv_0t + 1 = e^{cx}$$

$$\boxed{x(t) = \frac{1}{c}\ln(cv_0t + 1)}$$

That was...unpleasant. Surely there has to be a better way to analyse complex motion!



Variable Forces

Air Resistance

There is a better way! Don't worry, you don't have to solve crazy differential equations like that…until at least next year ;)

Let's learn a better way of understanding complex motion!

Energy

And a Metaphor for what it is

Energy

The Metaphor

We've all heard of the concept of energy. We've even used it. Now we're going to start thinking of it a little differently...

We're going to think about energy in terms of…money!

Energy

Money as a Metaphor for Energy

Just for now, I want you guys to think about energy as being like money. Right now we all have some amount of money that we can each spend. In principle, we can spend this money on whatever we like want! Just like money, I can spend energy on whatever I want. I can use energy to push a box against friction, lift a weight against gravity, ride a bike, push together two magnets, move a charged particle in a field. All these systems are different, with different forces and different rules, just like financial exchanges are different and have different rules. Do we not hear economists talking about "financial forces"? That's not a coincidence, the similarities between energy and money are striking. To take this metaphor even further, let's consider a chemical bond. Between two atoms there sometimes exists a bond, and it takes effort, energy, to pull them apart. Thus we can think of a chemical bond as a store of energy. Now, when we invest money (put money into) a financial asset, what do we call that? A financial bond! Exactly, same idea. So, energy (like money) is like a universally recognised quantity that can do, and exchange between, a variety of different things, no matter the specific rules and forces related to those systems.

Now, one major difference (as you'll see in your thermodynamics lectures) is that **energy cannot be created or destroyed**, whereas (as you might have noticed during Covid and heard about in the 2008 financial crash) you can actually just print money. Further, chemical bonds don't appreciate with interest and magically become stronger, but financial bonds do. So **our analogy stops at the individual scale i.e. we have some amount of energy (money) and we can use it to do stuff.**

Work

What Forces Do

Work

What is it?

So we have this idea of energy. You can have energy, but the question is: how do we get energy? More formally, how do we transfer energy between objects, between systems?

We apply forces! As well as "the cause of acceleration", another way of thinking about forces is that they act to transfer energy into and out of systems!

First, we'll find out how we give objects *kinetic* energy. To give an object kinetic energy, we must do **work** on that object.

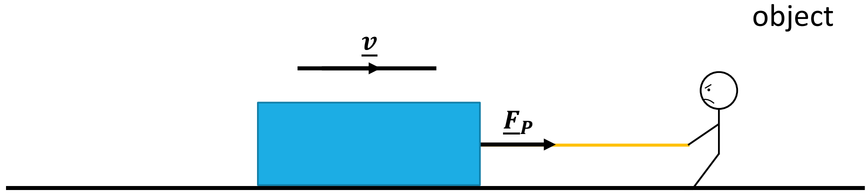
Work

Moving Objects

What is the force doing?

Force in same direction as motion! All energy from force goes into object motion

We are doing work *on* the object



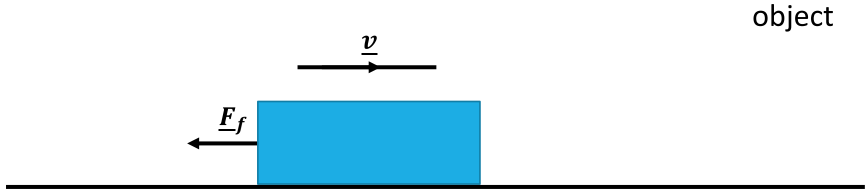
Work

A Moving Object

What is the force doing?

Force in opposite direction as motion! All energy from force takes away from object motion

We are doing work *against* the object



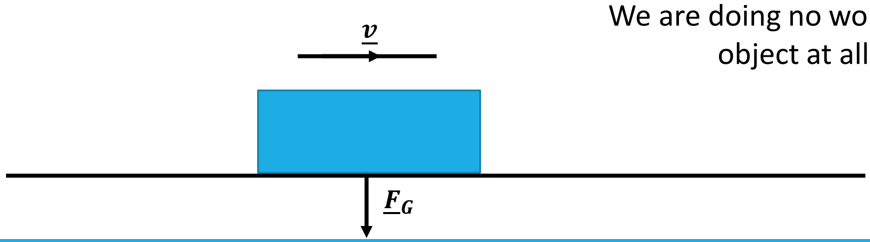
Work

Moving Objects

What is the force doing?

Force in perpendicular to the direction of motion! All energy from force does nothing to object motion

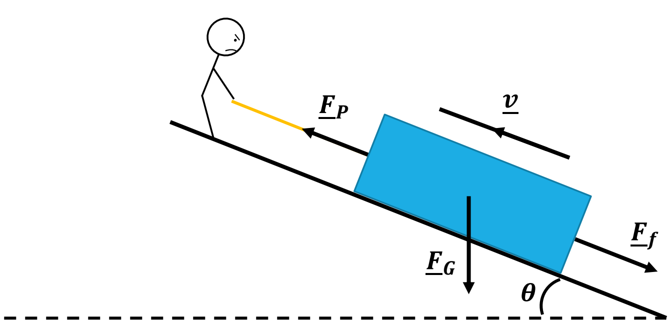
We are doing no work to the object at all



Work

Moving Objects

What are the forces doing?



- $\underline{F}_P$ is doing work *on* the object
- $\underline{F}_f$ is doing work *against* the object
- $\underline{F}_G$ is doing some work *against* the object. A large component of the force is doing no work on the object (energy wasted).

Work

The Mathematics of Work

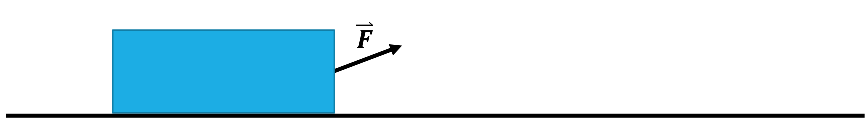
What vector operation has the properties we need?:

- If force in same direction as motion, 100% work done on
- If force in opposite direction as motion, 100% work done against (negative 100% work done on)
- If force perpendicular to direction of motion, 0% work done
- If force in intermediate direction, some *component* of the force does work.

The vector dot product!

Work

The Mathematics of Work



Work

The Mathematics of Work

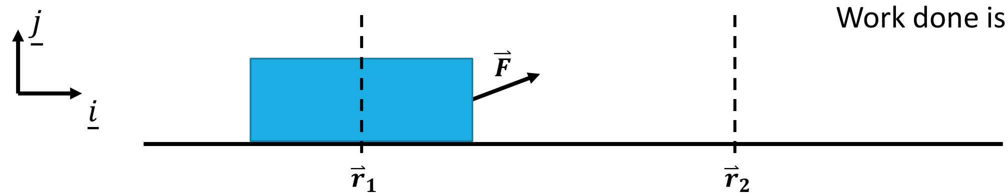
The work done, W, by a force $\vec{F}$ on an object which moves from $\vec{\boldsymbol{r}}_1$ to $\vec{\boldsymbol{r}}_2$:

$$\boxed{W = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r}}$$

$$d\vec{r} = dx\, \underline{i} + dy\, \underline{j}$$

Work done is *energy* transferred into the system

Work done is a *line integral*



Work

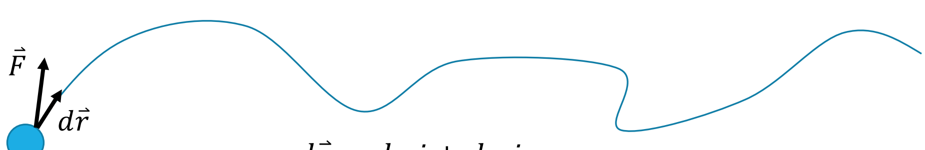
The Mathematics of Work

The work done, W, by a force $\vec{F}$ on an object which moves from $\vec{\boldsymbol{r}}_1$ to $\vec{\boldsymbol{r}}_2$:

$$\boxed{W = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r}}$$

$$dW = \vec{F} \cdot d\vec{r}$$

Work done is a *line integral*



$$d\vec{r} = dx\, \underline{i} + dy\, \underline{j}$$

Work

The Mathematics of Work

The work done, W, by a force $\vec{F}$ on an object which moves from $\vec{\boldsymbol{r}}_1$ to $\vec{\boldsymbol{r}}_2$:

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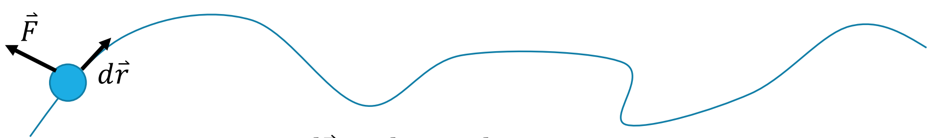
$$dW = \vec{F} \cdot d\vec{r}$$

Work done is a *line integral*



$$d\vec{r} = dx\,\underline{i} + dy\,\underline{j}$$

Work

The Mathematics of Work

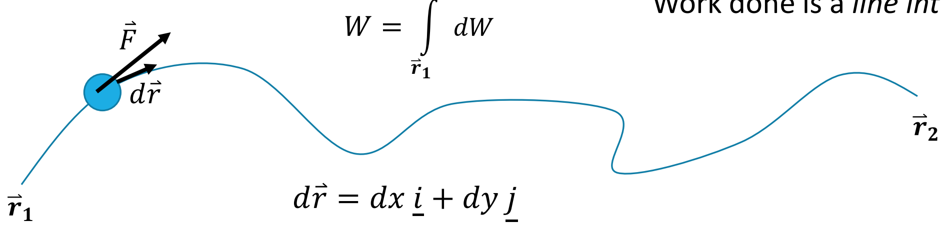
The work done, W, by a force $\vec{F}$ on an object which moves from $\vec{\boldsymbol{r}}_1$ to $\vec{\boldsymbol{r}}_2$:

$$\boxed{W = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r}}$$

$$dW = \vec{F} \cdot d\vec{r}$$

$$W = \int_{\vec{r}_1}^{\vec{r}_2} dW$$

Work done is a *line integral*



$$d\vec{r} = dx\,\underline{i} + dy\,\underline{j}$$

Work

The Mathematics of Work

$$W = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r}$$

$$d\vec{r} = dx\,\underline{i} + dy\underline{j} = \begin{pmatrix} dx \\ dy \end{pmatrix}$$

$$\vec{F} = F_x\,\underline{i} + F_y\underline{j} = \begin{pmatrix} F_x \\ F_y \end{pmatrix}$$

$$\vec{r}_1 = x_1\underline{i} + y_1\underline{j} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$$\vec{r}_2 = x_2\underline{i} + y_2\underline{j} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

$$W = \int_{\vec{r}_1}^{\vec{r}_2} \left(F_x\underline{i} + F_y\underline{j}\right) \cdot \left(dx\underline{i} + dy\underline{j}\right)$$

$$W = \int_{\vec{r}_1}^{\vec{r}_2} F_x dx\,(\underline{i}\cdot\underline{i}) + F_x dy\,\cancel{(\underline{i}\cdot\underline{j})} + F_y dx\,\cancel{(\underline{j}\cdot\underline{i})} + F_y dy\,(\underline{j}\cdot\underline{j})$$

$$\boxed{W = \int_{x_1}^{x_2} F_x dx + \int_{y_1}^{y_2} F_y dy}$$

Task 1

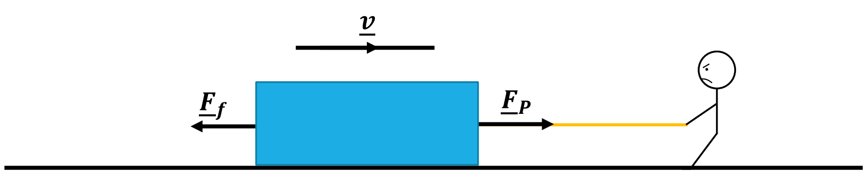
Work Done By Tension & Friction

Task 1

Work Done By Tension & Friction

Scenario: Someone is pulling a box from rest along a surface with some friction. They apply a horizontal force $|\underline{F}_P| = 50N$ to the box, which moves from $x = 5m$ to $x = 12m$. The horizontal frictional force on the box $|\underline{F}_f| = 40N$

Tasks:
1. Calculate the work done on the box by the pulling force.
2. Calculate the work done on the box by friction.
3. There is a difference in energy (work done on + work done against). What do you think this energy is? If the box has mass $m = 10kg$, what do you think the speed of the object might be?



Work – Kinetic Energy Theorem

Work – Kinetic Energy Theorem

What is it?

The net work done on an object is equal to the change in the object's kinetic energy

If we do work to an object, we give it kinetic energy. If we do work against an object (negative work on), we take kinetic energy away. If we do no work to the object, its kinetic energy does not change.

$$W_{Net} = \Delta E_k$$

$$\int_{\vec{r}_1}^{\vec{r}_2} \vec{F}_{Net} \cdot d\vec{r} = \Delta\left(\frac{1}{2}mv^2\right)$$

The velocity changes!

Task 2

Work Done By Not-Constant Tension & Friction

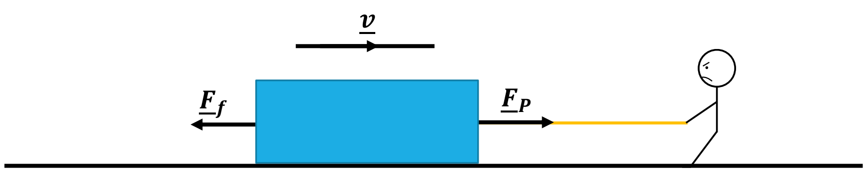
Task 2

Work Done By Not-Constant Tension & Friction

Scenario: Someone is pulling a box of mass $m = 30kg$ along a surface with some friction. It begins with a speed $v_i = 3ms^{-1}$. They apply a horizontal force $|\underline{F_P}| = (50x)N$ to the box, which moves from $x = 5m$ to $x = 12m$. The horizontal frictional force on the box $|\underline{F_f}| = 40N$.

Tasks:

1. Calculate the work done on the box by the pulling force.
2. Calculate the work done by the net force.
3. Calculate the overall change in velocity of the object



Task 3

Work Done By Tension & Friction Uphill

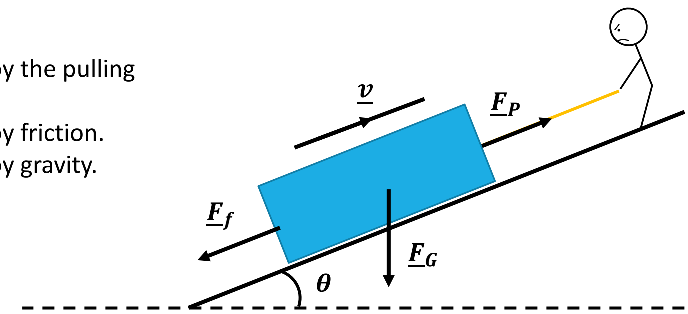
Task 3

Work Done By Tension & Friction Uphill

Scenario: Someone is pulling an $m = 10kg$ box up a $\theta = 23.2^o$ hill with some friction. They apply a force $\underline{F_P} = 300N$ to the box, which moves from $x = 5m$ to $x = 12m$, and $y = 3m$ to $y = 6m$. The frictional force on the box $\underline{F_f} = 40N$

Tasks:

1. Calculate the work done on the box by the pulling force.
2. Calculate the work done on the box by friction.
3. Calculate the work done on the box by gravity.



Task 4

Work Done By An Arbitrary Force

Task 4

Work Done By An Arbitrary Force

Scenario: An object is initially at position $\vec{r}_1 = 2\underline{i} + 8\underline{j}$ and moves to position $\vec{r}_2 = 5\underline{i} + 29\underline{j}$ along the path defined by the equation $y = x^2 + 4$. While moving along this path, it is subject to two forces: $\vec{F}_1 = x\underline{i} + y^2\underline{j}$, and $\vec{F}_2 = y\underline{i} + x^2\underline{j}$.

Tasks:

1. Calculate the work done on the object by the force $\vec{F}_1$
2. Calculate the work done on the object by the force $\vec{F}_2$

Note: Think carefully about these! Understand work done, the concepts, the derivation, what it means to do work to an object. Then, apply the equation. *<u>You can do this!</u>*

MECHANICS 1 - VARIABLE FORCE, ENERGY & WORK

Task 4

Work Done By An Arbitrary Force

Scenario: An object is initially at position $\vec{r}_1 = 2\underline{i} + 8\underline{j}$ and moves to position $\vec{r}_2 = 5\underline{i} + 29\underline{j}$ along the path defined by the equation $y = x^2 + 4$. While moving along this path, it is subject to two forces: $\vec{F}_1 = x\underline{i} + y^2\underline{j}$, and $\vec{F}_2 = y\underline{i} + x^2\underline{j}$.

Tasks:

1. Calculate the work done on the object by the force $\vec{F}_1$

Work done, $$W_1 = \int_{x_1}^{x_2} F_{1,x}\,dx + \int_{y_1}^{y_2} F_{1,y}\,dy$$

Sub forces, $$W_1 = \int_{x_1}^{x_2} x\,dx + \int_{y_1}^{y_2} y^2\,dy$$

Integrate, $$W_1 = \frac{1}{2}[x^2]_{x_1}^{x_2} + \frac{1}{3}[y^3]_{y_1}^{y_2}$$

A Quick Note on Units

A Quick Note on Units

SI Units

Quantity	SI Unit
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Distance	Metres (m)
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Acceleration	Metres per second-squared (ms^{-2})
Mass	Kilogram (kg)
Force	Newtons (N), ($kg.ms^{-2}$)
Energy / Work	Joules (J), ($N.m$)

Resources

Dr Purdy's Notes

And Examples

I've added some great notes and examples on Minerva from Rob Purdy. He proves the work-kinetic energy theorem ☺