UNIVERSITY OF LEEDS

Mechanics 1

Session 16: Circular Motion – The Continuous Moment of Inertia

DR BEN HANSON

Last Lecture

Circular Motion – The Moment of Inertia

We:

- Recalled that torque "causes" angular acceleration, just as force "causes" linear acceleration
- Considered the idea of "rotational mass", otherwise known as the moment of inertia

You should be able to:

- Calculate the moment of inertia of a single object undergoing circular motion
- Calculate the moment of inertia of a set of objects undergoing collective circular motion
- Calculate the moment of inertia of a single, continuous object rotating about some axis

This Lecture

Circular Motion – The Continuous Moment of Inertia

We will:

- Conceptualise continuous objects as infinitely dense collections of particles
- Understand how to determine the moment of inertia of continuous objects

You will be able to:

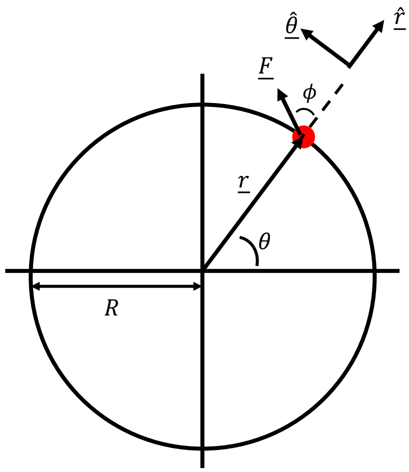
- Calculate the moment of inertia of continuous objects with regular structure
- Calculate the moment of inertia of continuous objects with variable density

Moment of Inertia

Of a Single Particle

Moment of Inertia

Of a Single Particle



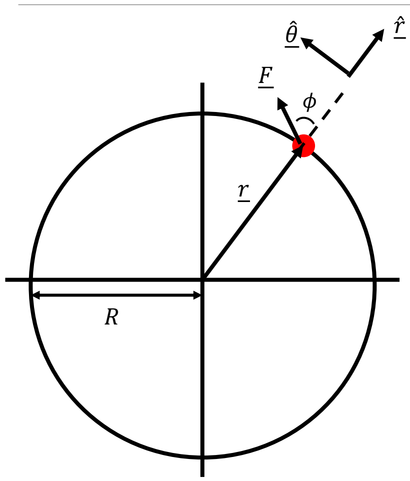
$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\vec{\tau} = |\vec{r}|\left|\vec{F}\right| \sin(\phi)\underline{\hat{n}} \qquad \boldsymbol{\phi \neq \theta}$$

$$\rightarrow |\vec{\tau}| = |\vec{r}|\left|\vec{F}\right| \sin(\phi)$$

Moment of Inertia

Of a Single Particle



$$|\vec{\tau}| = |\vec{r}|\left|\vec{F}\right| \sin(\phi)$$

$$\left|\vec{F}\right| \sin(\phi) = F_\theta, \qquad |\vec{\tau}| = |\vec{r}|F_\theta$$

$$|\vec{r}| = R, \qquad |\vec{\tau}| = RF_\theta$$

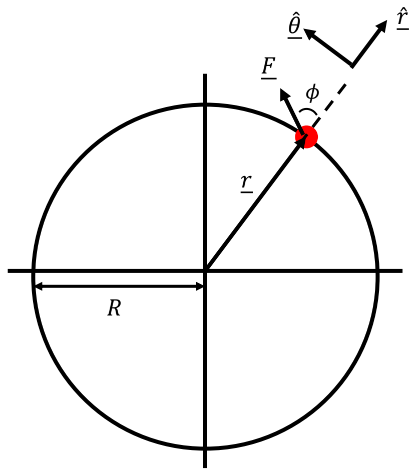
$$F_\theta = ma_\theta, \qquad |\vec{\tau}| = Rma_\theta$$

$$a_\theta = R\alpha\,, \qquad \boxed{|\vec{\tau}| = mR^2\alpha}$$

$$\boxed{|\vec{\tau}| = I\alpha}$$

Moment of Inertia

Of a Single Particle



For a single particle rotating around an axis with position vector $\hat{\underline{r}}$ and applied force $\underline{F}$:

The torque vector, $\vec{\tau} = \vec{r} \times \vec{F}$

The torque magnitude, $|\vec{\tau}| = I\alpha$

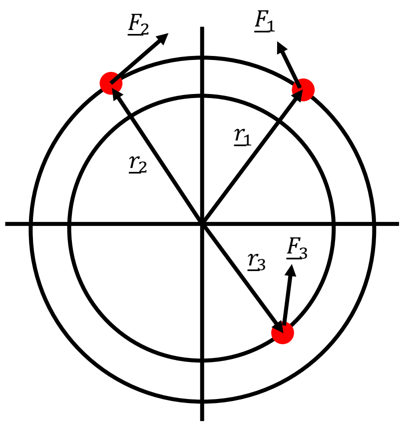
The Moment of Inertia, $I = mR^2$

Moment of Inertia

Of Multiple Particles (same axis)

Moment of Inertia

Of Multiple Particles (same axis)



$$\vec{\tau}_{Net} = \sum_{i}^{N} \vec{r}_i \times \vec{F}_i$$

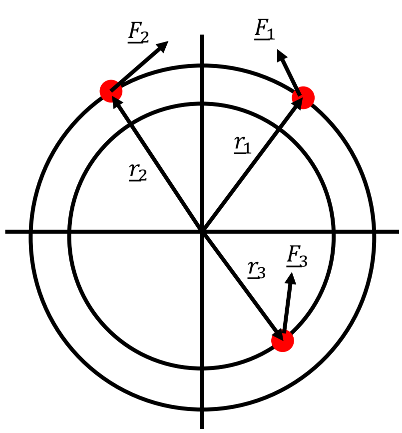
$$\vec{\tau}_{Net} = \sum_{i}^{N} |\vec{r}_i|\left|\vec{F}_i\right| \sin(\phi_i)\underline{\hat{n}}$$

All same axis!

$$\rightarrow |\vec{\tau}_{Net}| = \sum_{i}^{N} |\vec{r}_i|\left|\vec{F}_i\right| \sin(\phi_i)$$

Moment of Inertia

Of Multiple Particles (same axis)



$$|\vec{\tau}_{Net}| = \sum_{i}^{N} |\vec{r}_i|\left|\vec{F}_i\right| \sin(\phi_i)$$

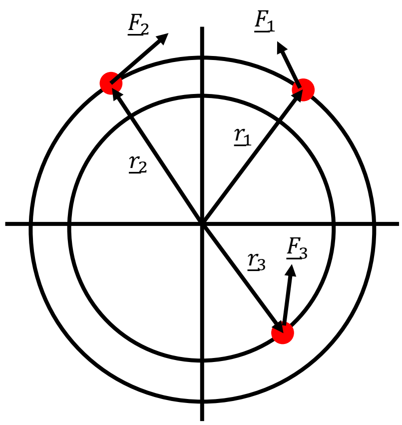
$$\left|\vec{F}_i\right| \sin(\phi_i) = F_{\theta,i}, \qquad |\vec{\tau}_{Net}| = \sum_{i}^{N} |\vec{r}_i| F_{\theta,i}$$

$$|\vec{r}_i| = R_i, \qquad |\vec{\tau}_{Net}| = \sum_{i}^{N} R_i F_{\theta,i}$$

$$F_{\theta,i} = m_i a_{\theta,i}, \qquad |\vec{\tau}_{Net}| = \sum_{i}^{N} R_i m_i a_{\theta,i}$$

Moment of Inertia

Of Multiple Particles (same axis)



$$F_{\theta,i} = m_i a_{\theta,i},$$

$$|\vec{\tau}_{Net}| = \sum_i^N R_i m_i a_{\theta,i}$$

$$a_{\theta,i} = R_i \alpha_i\ ,$$

$$|\vec{\tau}_{Net}| = \sum_i^N m_i R_i^2 \alpha_i$$

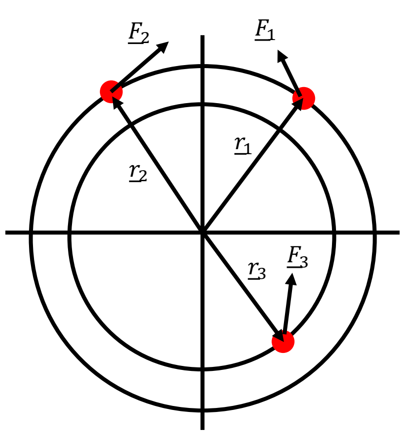
$$\boxed{|\vec{\tau}_{Net}| = \sum_i^N I_i \alpha_i}$$

Torques (about the same axis) sum like forces do! Each torque on each particle corresponds to its own moment of inertia and angular acceleration!

MECHANICS 1: THE CONTINUOUS MOMENT OF INERTIA

Moment of Inertia

Of Multiple Particles (same axis)



If, <u>and only if</u>, all $\alpha_i = \alpha$ i.e. all rotational accelerations equal,

$$|\vec{\tau}_{Net}| = \sum_i^N I_i \alpha$$

$$|\vec{\tau}_{Net}| = \alpha \sum_i^N I_i$$

Define net moment of inertia,

$$I_{Net}\alpha = \alpha \sum_i^N I_i$$

Cancel,

$$\boxed{I_{Net} = \sum_i^N I_i}$$

This is not a general result! I'm showing that all α_i must be equal to define a "net" moment of inertia

Moment of Inertia

Of Continuous Objects

Moment of Inertia

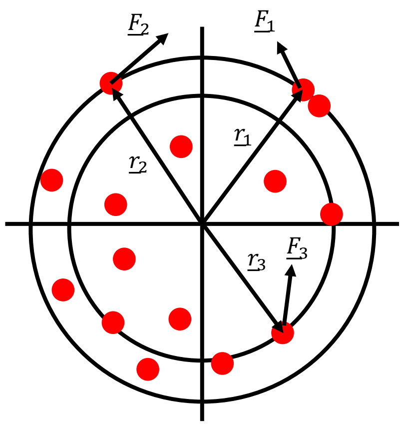
Of Continuous Objects

What if there weren't one or two individual, "discrete" particles, but instead an infinite continuum of them? Can that have a moment of inertia? Yes, it absolutely can

This will potentially be your first view of something called "continuum mechanics", but it won't be the last…so strap your calculus hats on again! ☺

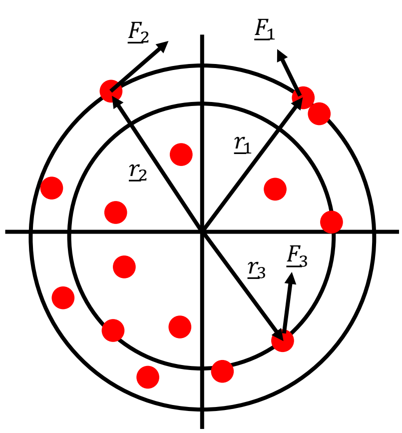
Moment of Inertia

Of Continuous Objects



Moment of Inertia

Of Continuous Objects



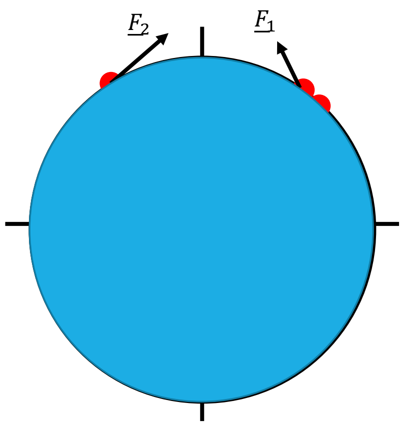
Single, collective object, hence $\alpha_i = \alpha$,

$$I_{Net} = \sum_i^N I_i$$

$$I_{Net} = \sum_i^N m_i r_i^2$$

Moment of Inertia

Of Continuous Objects



Single, collective object, hence $\alpha_i = \alpha$,

$$I_{Net} = \sum_{i}^{N} I_i$$

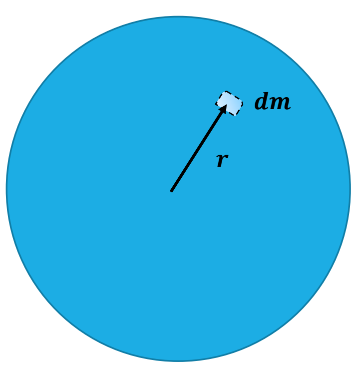
$$I_{Net} = \sum_{i}^{N} m_i r_i^2$$

Convert sum to integral,

$$\boxed{I = \int r^2\, dm}$$

Moment of Inertia

Of Continuous Objects



Convert sum to integral,

$$\boxed{I = \int r^2\, dm}$$

- dm – Small (infinitesimal) bit of mass
- r – Distance from rotation axis to the small mass

For every single (continuous) position on the object, r, we add the little bit of mass at that point, dm. That is what this integral means. Let's stop here and discuss

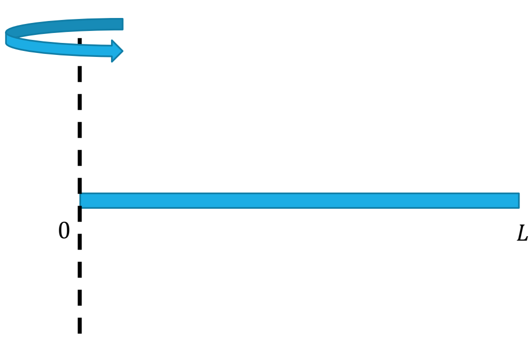
Moment of Inertia

Of a 1D Rigid Rod

MECHANICS 1: THE CONTINUOUS MOMENT OF INERTIA

Moment of Inertia

Of a 1D Rigid Rod



Continuous moment of inertia,
$$I = \int r^2 \, dm$$

Build the integral,
$$I = \int_0^L r^2 \, dm$$

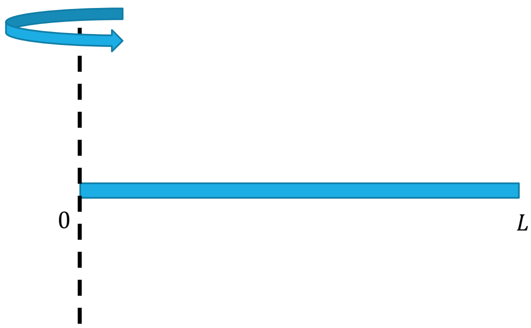
Consider how mass changes with length,
$$dm = \rho_L . dr$$

Density (mass per unit length)

Substitute,
$$I = \int_0^L r^2 \rho_L \, dr$$

Moment of Inertia

Of a 1D Rigid Rod



Substitute, $$I = \int_0^L r^2 \rho_L \, dr$$

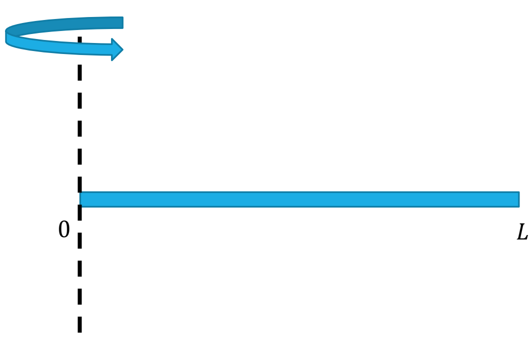
Factorise density (if constant), $$I = \rho_L \int_0^L r^2 \, dr$$

Integrate, $$I = \rho_L \left[\frac{1}{3} r^3\right]_0^L$$

Solve, $$I = \rho_L \frac{1}{3} L^3$$

Moment of Inertia

Of a 1D Rigid Rod



Solve, $$I = \rho_L \frac{1}{3} L^3$$

$\rho_L = \frac{M}{L}$, $$\boxed{I = \frac{1}{3} M L^2}$$

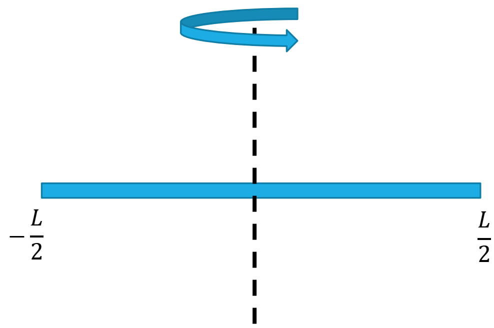
This is the moment of inertia of a 1D rigid rod rotating about one of its ends. Things to note:

1. It's proportional to L^2. All moments of inertia are
2. It's proportional to the total mass M. All moments of inertia are.
3. It has a prefactor of $\frac{1}{3}$. Every object has a different prefactor (see Wikipedia for a huge list of them!)

This is a big list of results: https://en.wikipedia.org/wiki/List_of_moments_of_inertia

Moment of Inertia

Of a 1D Rigid Rod



Continuous moment of inertia,

$$I = \int r^2 \, dm$$

Build the integral,

$$I = \int_{-\frac{L}{2}}^{\frac{L}{2}} r^2 \, dm$$

Consider how mass changes with length,

$$dm = \rho_L . dr$$

Density (mass per unit length)

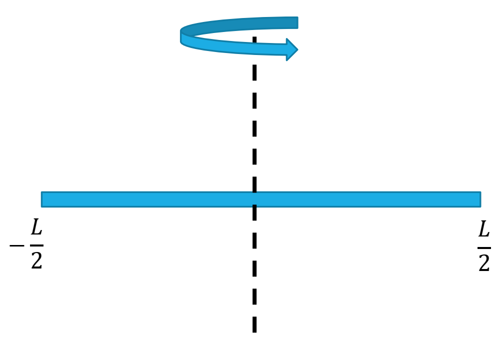
Substitute,

$$I = \int_{-\frac{L}{2}}^{\frac{L}{2}} r^2 \, \rho_L \, dr$$

MECHANICS 1: THE CONTINUOUS MOMENT OF INERTIA

Moment of Inertia

Of a 1D Rigid Rod



Substitute,

$$I = \int_{-\frac{L}{2}}^{\frac{L}{2}} r^2 \, \rho_L \, dr$$

Factorise density (if constant),

$$I = \rho_L \int_{-\frac{L}{2}}^{\frac{L}{2}} r^2 \, dr$$

Integrate,

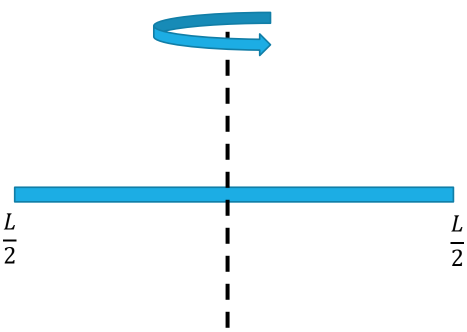
$$I = \rho_L \left[\frac{1}{3} r^3\right]_{-\frac{L}{2}}^{\frac{L}{2}}$$

Solve,

$$I = \rho_L \frac{1}{12} L^3$$

Moment of Inertia

Of a 1D Rigid Rod



Solve,

$\rho_L = \frac{M}{L}$,

$$I = \rho_L \frac{1}{12} L^3$$

$$\boxed{I = \frac{1}{12} M L^2}$$

This is the moment of inertia of a 1D rigid rod rotating about its centre. Things to note:

1. It's proportional to L^2. All moments of inertia are
2. It's proportional to the total mass M. All moments of inertia are.
3. It has a prefactor of $\frac{1}{12}$. Thus, it's 4 times "easier" to rotate a rod about its centre than its end!

This is a big list of results: https://en.wikipedia.org/wiki/List_of_moments_of_inertia

Task 1

Calculating the Moment of Inertia of a Rod

Task 1

Calculating the Moment of Inertia of a Rod

Tasks:

1. Calculate the moment of inertia of a 1D rod of length L that is rotating about an axis that is a distance x away from the edge of the rod. Express your answer algebraically (in terms of x and L)

2. Calculate the moment of inertia of a 1D rod of length L that is rotating about an axis that is a distance x away from the edge of the rod, and which has a linear density $\rho_L = ar^2$. Express your answer algebraically (in terms of x, L and a).
   Hint: You can't factor out the density this time!

Moment of Inertia

Of a 2D Rigid Beam

Moment of Inertia

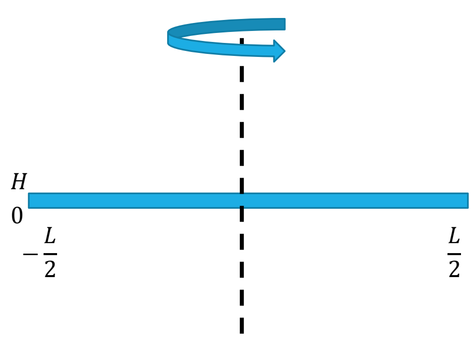
Of a 2D Rigid Beam

The 1D rod was an approximation. Let's now consider the case when the object has a height as well.

We're doing multi-dimensional integrals here…we can do it!

Moment of Inertia

Of a 2D Rigid Beam



Continuous moment of inertia,

$$I = \int r^2 \, dm$$

Build the integral,

$$I = \int_0^H \int_{-\frac{L}{2}}^{\frac{L}{2}} r^2 \, dm$$

Consider how mass changes with area,

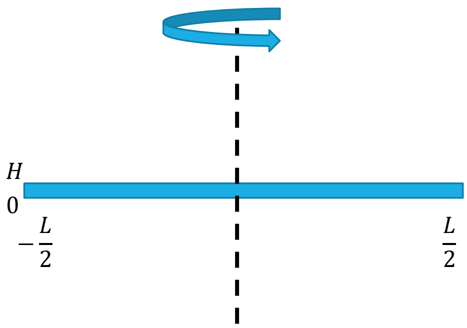
$$dm = \rho_A . dA$$

$$dm = \rho_A . dr . dh$$

Density (mass per unit area)

Moment of Inertia

Of a 2D Rigid Beam



Substitute,

$$I = \int_0^H \int_{-\frac{L}{2}}^{\frac{L}{2}} r^2 \rho_A dr.dh$$

Factorise density (if constant),

$$I = \rho_A \int_0^H \int_{-\frac{L}{2}}^{\frac{L}{2}} r^2\, dr.dh$$

First integral,

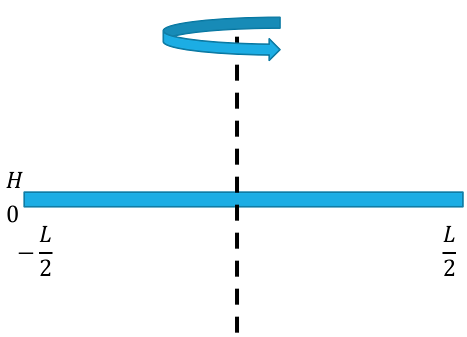
$$I = \rho_A \int_0^H \left[\frac{1}{3} r^3\right]_{-\frac{L}{2}}^{\frac{L}{2}} dh$$

$$I = \rho_A \int_0^H \frac{1}{12} L^3\, dh$$

MECHANICS 1: THE CONTINUOUS MOMENT OF INERTIA

Moment of Inertia

Of a 2D Rigid Beam



First integral,

$$I = \rho_A \int_0^H \frac{1}{12} L^3\, dh$$

Factorise,

$$I = \frac{1}{12} \rho_A L^3 \int_0^H dh$$

Second integral,

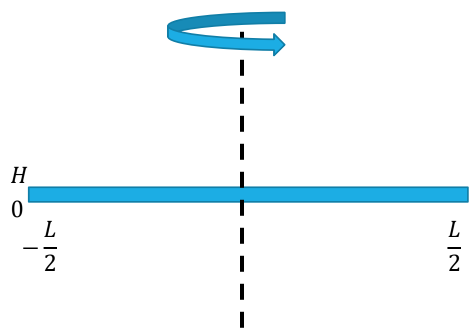
$$I = \frac{1}{12} \rho_A L^3 H$$

$\rho_A = \frac{M}{A} = \frac{M}{LH},$

$$\boxed{I = \frac{1}{12} M L^2}$$

Moment of Inertia

Of a 2D Rigid Beam



$\rho_A = \frac{M}{A} = \frac{M}{LH},$

$$\boxed{I = \frac{1}{12}ML^2}$$

This is the moment of inertia of a 2D rigid beam rotating about its centre. Things to note:

1. It's proportional to L^2. All moments of inertia are
2. It's proportional to the total mass M. All moments of inertia are.
3. It has a prefactor of $\frac{1}{12}$, same as the 1D case. This is because the distance to the axis does not change with h, and thus does not affect the integral ☺
4. It's the same as a 1D rod due to symmetry along the axis!

This is a big list of results: https://en.wikipedia.org/wiki/List_of_moments_of_inertia

Moment of Inertia

Of a 2D Circular Disk

Moment of Inertia

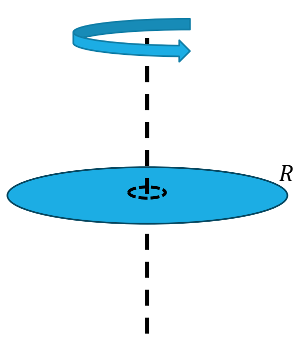
Of a 2D Circular Disk

The 2D rod was basically the same as the 1D rod due to symmetry. Let's now consider the case when the object has a shape that can't reduce to 1D.

We're doing multi-dimensional integrals here…we can do it!

Moment of Inertia

Of a 2D Circular Disk



Continuous moment of inertia,

$$I = \int r^2\, dm$$

Build the integral,

$$I = \int_0^{2\pi} \int_0^R r^2\, dm$$

Consider how mass changes with area,

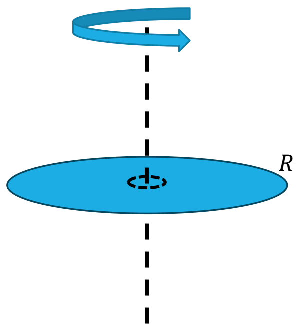
$$dm = \rho_A . dA$$

$$dm = \rho_A . r\, dr d\theta$$

Density (mass per unit area)

Moment of Inertia

Of a 2D Circular Disk



Substitute,

$$I = \int_0^{2\pi} \int_0^R r^2 \rho_A . r \, dr d\theta$$

Factorise density (if constant),

$$I = \rho_A \int_0^{2\pi} \int_0^R r^3 . dr \, d\theta$$

First integral,

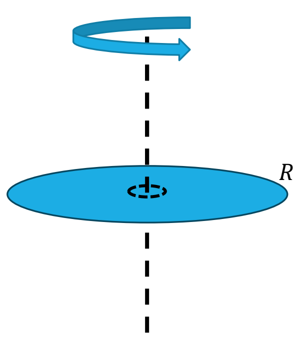
$$I = \rho_A \int_0^{2\pi} \left[\frac{1}{4} r^4\right]_0^R d\theta$$

$$I = \rho_A \int_0^{2\pi} \frac{1}{4} R^4 \, d\theta$$

MECHANICS 1: THE CONTINUOUS MOMENT OF INERTIA

Moment of Inertia

Of a 2D Circular Disk



First integral,

$$I = \rho_A \int_0^{2\pi} \frac{1}{4} R^4 \, d\theta$$

Factorise,

$$I = \frac{1}{4} \rho_A R^4 \int_0^{2\pi} d\theta$$

Second integral,

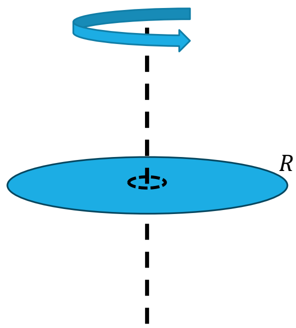
$$I = \frac{1}{2} \pi \rho_A R^4$$

$\rho_A = \frac{M}{A} = \frac{M}{\pi R^2}$,

$$\boxed{I = \frac{1}{2} M R^2}$$

Moment of Inertia

Of a 2D Circular Disk



$$\rho_A = \frac{M}{A} = \frac{M}{\pi R^2},$$

$$\boxed{I = \frac{1}{2} M R^2}$$

This is the moment of inertia of a 2D circular disk rotating about its centre. Things to note:

1. It's proportional to R^2. All moments of inertia are
2. It's proportional to the total mass M. All moments of inertia are.
3. It has a prefactor of $\frac{1}{2}$. The mass is not all at the edge (as it is with a single particle). Some mass is closer to the axis, so it is relatively easier to spin

This is a big list of results: https://en.wikipedia.org/wiki/List_of_moments_of_inertia

Task 2

Calculating the Moment of Inertia of Continuous Objects

Task 2

Calculating the Moment of Inertia of Continuous Objects

Tasks:

1. Calculate the moment of inertia of a cuboid with length L, width W and height H as it rotates around an axis straight through its centre, parallel to the length. The cube has mass M and constant density ρ_V.
   Hint 1: You might need 3 integrals here for each of the three dimensions.
   Hint 2: Remember, r is the (shortest) perpendicular distance to the axis of rotation

2. Calculate the moment of inertia of a circular ring with an inner radius R_1 and outer radius R_2, as it rotates around an axis straight through its centre, perpendicular to the circle itself. The circle has mass M and constant density ρ_A.
   Hint: Try to copy my integration from the previous example, but adjust it for the new system