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ON SELF-NORMALIZED SUMS AND STUDENT'S STATISTIC*

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(*Translated by the author*)

Abstract. We evaluate the accuracy of normal approximation for the distributions of some nonlinear functionals of sums of random vectors. A Berry–Esseen type inequality *with explicit constants* for the distribution of Student's statistic is established as a consequence of the main result.

Key words. self-normalized sums, Student's statistic, Berry–Esseen inequality

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1. Introduction. The problem of evaluating the accuracy of normal approximation for the distribution of a sum of independent and identically distributed (i.i.d.) random variables goes back to Liapunov [16]. The solution of Lyapunov's problem was found by Berry [5] and Esseen [12]: if $X, X_1, X_2, \dots$ are i.i.d. random variables $\mathbf{E}X = 0$ and $\mathbf{E}|X|^3 < \infty$, then

$$\sup_x \left| \mathbf{P}\left\{ \frac{\sum_{i=1}^n X_i}{\sigma\sqrt{n}} < x \right\} - \Phi(x) \right| \leqq C_* n^{-1/2} \mathbf{E}|X|^3 \sigma^{-3}$$

for some absolute constant C_*, where $\sigma^2 = \mathbf{D}X$ and Φ is the distribution function of the normal $\mathcal{N}(0;1)$ law. The constant $C_* \leqq 0.7655$ (see [24]). In most applications, however, σ^2 is not known, and one replaces σ^2 with its consistent estimator $n^{-1}\sum_{i=1}^n (X_i - \widehat{X}_n)^2$, where $\widehat{X}_n = n^{-1}\sum_{i=1}^n X_i$ is the sample mean. This leads to the problem of evaluating the accuracy of normal approximation for the distribution of Student's statistic

$$t_n = \sum_{i=1}^n X_i \left[\sum_{i=1}^n (X_i - \widehat{X}_n)^2 \right]^{-1/2}.$$

Student's statistic t_n is renowned for numerous applications. It was studied by Bentkus, Chung, Chibisov, Efron, Fisher, Giné, Götze, Hall, Mason, and other specialists.

Egorov [28] and Giné, Götze, and Mason [14] established necessary and sufficient conditions for the asymptotic normality of t_n. Slavova [23], using Chibisov's results (see [7], [8]), showed that

$$\Delta_n \equiv \sup_x \left| \mathbf{P}\{t_n < x\} - \Phi(x) \right| \leqq C n^{-1/2} \qquad (\exists\, C < \infty)\,. \tag{1}$$

Bentkus and Götze [3] proved that there exists an absolute constant $C < \infty$ such that

$$\Delta_n \leqq C\big(\mathbf{E}X^2 \mathbb{1}^{>} + n^{-1/2}\mathbf{E}|X|^3 \mathbb{1}^{<}\big), \tag{2}$$

where $\sigma = 1$, $\mathbb{1}^{<} = \mathbb{1}\{X^2 \leqq n\}$, and $\mathbb{1}^{>} = \mathbb{1}\{X^2 > n\}$. Novak [19], [20] found uniform and nonuniform bounds for $|\mathbf{P}\{t_n < x\} - \Phi(x)|$ *with explicit constants*. Estimates in [19], [20] have the rate $n^{-1/2}$ if $\mathbf{E}X^4 < \infty$, though the rate is $n^{-2/7}$ if $\mathbf{E}X^3 < \infty$.

Student's statistic t_n is closely related to the self-normalized sum

$$t_n^* = \sum_{i=1}^n X_i \left[\sum_{i=1}^n X_i^2 \right]^{-1/2}.$$

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Indeed, denote $s_n = \sum_{i=1}^n X_i$. Then for $x \in [0; \sqrt{n}\,]$ (note that $|t_n^*| \leqq \sqrt{n}$)

$$\{t_n \geqq x\} = \left\{ s_n \geqq x \sqrt{\sum_{i=1}^n (X_i - \widehat{X})^2} \right\} = \left\{ s_n^2 \geqq x^2 \left(\sum_{i=1}^n X_i^2 - \frac{s_n^2}{n} \right),\ s_n \geqq 0 \right\}$$
$$= \left\{ s_n^2 \left(1 + \frac{x^2}{n}\right) \geqq x^2 \sum_{i=1}^n X_i^2,\ s_n \geqq 0 \right\} = \left\{ t_n^* \geqq \frac{x}{\sqrt{1 + x^2/n}} \right\};$$

cf. [9] (we assume $t_n = \infty$ if $X_1 = \cdots = X_n$). Hence, the results on the distribution of t_n^* entail the corresponding ones on the distribution of t_n.

According to Shao [22], for any $\varepsilon \in (0; \frac{1}{2})$ one can find a constant $c \in (0;1)$ such that

$$e^{-(1+\varepsilon)\,x^2/2} \leqq \mathbf{P}\{t_n^* \geqq x\} \leqq e^{-(1-\varepsilon)\,x^2/2} \qquad \left(c^{-1} < x < cn^{1/2}\right) \tag{3}$$

for all large enough n. A combination of (1) and (3) yields

$$\Delta_n^*(x) \equiv \left|\mathbf{P}\{t_n^* < x\} - \Phi(x)\right| \leqq Cn^{-(1-\delta)/2} e^{-(1-\varepsilon)\,\delta x^2/2} \qquad \left(c^{-1} < x < cn^{1/2}\right)$$

for any $\delta \in (0;1)$. Theorem 2.2 and Corollary 2.3 in [26] state that there exist absolute constants $A \in (0;\infty)$ and $b \in (0;1)$ such that

$$\Delta_n^*(x) \leqq A\mathbf{E}|X|^{10/3} n^{-1/2}\, e^{-bx^2/2} \qquad (n \geqq 1,\ x \in \mathbf{R})\,. \tag{4}$$

Denote $\varphi = \Phi'$. Chibisov [8] showed that

$$\sup_x \left| \mathbf{P}\{t_n < x\} - \Phi(x) - \frac{2x^2+1}{6\sqrt{n}}\, \varphi(x)\, \mathbf{E}X^3 \right| = o(n^{-1/2}) \tag{5}$$

if $\mathbf{E}|X|^3 < \infty$ and the characteristic function of the random vector (X, X^2) obeys Cramér's condition. Hall [15] proved (5) under the assumption that the distribution of the random variable X is nonsingular. According to Bloznelis and Putter [30], relation (5) is valid if the distribution of X is nonlattice. If, in addition, $\mathbf{E}|X|^{4+\varepsilon} < \infty$ for some $\varepsilon > 0$ and Cramér's condition holds, then the right-hand side of (5) is $O(n^{-1})$ (see [4]).

The old open problem was to establish a Berry–Esseen type inequality (BETI) for the distribution of Student's statistic *with explicit constants* under correct moment restrictions (see [9]). One needs such a result, for instance, in order to construct subasymptotic confidence intervals (cf. [18], [19]).

We give a solution of this problem in section 2 below. It is a consequence of a more general result of section 3. In section 3 we deal with the problem of evaluating the accuracy of normal approximation for the distribution of a quadratic functional of sums of random vectors. The main result presents a BETI for the distribution of such a functional. The solution of Lyapunov's problem for Student's statistic is given as a consequence of Theorem 2. Proofs are left to section 4. The approach seems to be of interest on its own. Not only is a more general problem solved, but also the proof is short.

2. Self-normalized sums. In this section we deal with Student's statistic and, more generally, with self-normalized sums $S_n/\sqrt{T_n}$, where $(\xi,\eta), (\xi_1,\eta_1), (\xi_2,\eta_2), \ldots$ is a sequence of i.i.d. pairs of random variables, $S_n = \sum_{i=1}^n \xi_i$ and $T_n = \sum_{i=1}^n \eta_i$. Without loss of generality, it is sufficient to estimate $\mathbf{P}\{S_n/\sqrt{T_n} < x\} - \Phi(x)$ for nonnegative x.

Assume that $\eta \geqq 0$, $\mathbf{E}\xi = 0$, $\mathbf{E}|\xi|^3 + \mathbf{E}\eta^{3/2} < \infty$. We suppose that $\xi = 0$ as $\eta = 0$ and that $0/0$ equals 0. A bar over a random variable means that it is centered by its mathematical expectation.

Let $\sigma_\xi^2 = \mathbf{E}\xi^2$ and $m = \mathbf{E}\eta$. We introduce statistics

$$t_n^\circ = \frac{S_n/\sigma_\xi}{\sqrt{T_n/m}}, \qquad t_n^\star = \frac{S_n/\sigma_\xi}{\sqrt{T_n/m_\eta}},$$

where $m_\eta = \mathbf{E}\eta^<$, $\eta^< = \eta\, \mathbb{1}\{\eta \leqq N\}$, and N is a truncation level. Let $C_* \leqq 0.7655$ and $C_+ < 30.52$ be the constants in uniform and nonuniform Berry–Esseen inequalities for sums of random variables with nonrandom normalization (i.e., in the case $\eta \equiv 1$, see [2], [21], [24]). Denote

$$c_*^3 = C_*, \quad c_+^3 = C_+, \quad m_0 = \mathbf{E}\xi\eta^<, \quad \sigma_\eta^2 = \mathbf{E}(\eta^<)^2, \quad \psi_\xi^3 = \mathbf{E}|\xi|^3, \quad \psi_\eta^3 = \mathbf{E}|\overline{\eta}^<|^3,$$

$$\psi_* = 1 + \frac{3x(m_0 \vee 0)}{2m_\eta\sigma_\xi\sqrt{n}}, \quad \psi = \left(\psi_\xi + \frac{x\psi_\eta\sigma_\xi}{2m_\eta\sqrt{n}}\right)\psi_*.$$

We put also $\kappa = \min\{1; (\mathbf{E}|\xi|/\sigma_\xi + xn^{-1/2})\,\psi_*\}$,

$$\begin{aligned}
r &= x^2\varphi\left(\frac{x}{2}\right)\left[\frac{|m_0|/m_\eta}{\sigma_\xi\sqrt{n}} + 4|x|\,\frac{\sigma_\eta^2}{nm_\eta^2}\right],\\
r_\star &= \frac{(\psi/\sigma_\xi)^3}{\sqrt{n}}\left(1 + \frac{9}{\sqrt{2\pi}} + \sqrt{\frac{\pi}{8}}\right) + \frac{x\sigma_\eta^2\psi_*}{nm_\eta^2}\left(\sqrt{\frac{\pi}{8}} + \frac{1}{n}\right)\\
&\quad + \frac{2\kappa}{\sqrt{n}} + \frac{2^{5/3}x\psi_*}{m_\eta^2 n^{4/3}}\left(2 + \frac{(\psi/\sigma_\xi)^3}{\sqrt{n}}\right)^{1/3}\left(\mathbf{E}|\overline{\eta}^<|^{3/2}\right)^{5/3},\\
r_* &= \frac{x\psi_*^2}{m_\eta^2 n^{3/2}}\left[\frac{\psi_\xi\psi_\eta^2}{\sigma_\xi} + \frac{x\psi_\eta^3}{2m_\eta\sqrt{n}} + \frac{2^{5/3}}{n^{1/3}}\left(\frac{\mathbf{E}|\xi|\,|\overline{\eta}^<|}{\sigma_\xi} + \frac{x\sigma_\eta^2}{2m_\eta\sqrt{n}}\right)(\mathbf{E}|\overline{\eta}^<|^{3/2})^{2/3}\right],\\
R_n^* &= \frac{1}{\sqrt{n}}\left[\frac{\psi_\xi}{\sigma_\xi}\left(c_* + c_+\frac{m_0/2 \vee 0}{m_\eta\sigma_\xi\sqrt{n}}\right) + \frac{\psi_\eta/2}{m_\eta\sqrt{n}}\left(c_+ \wedge c_*\frac{m_\eta\sqrt{n}}{\sigma_\eta}\right)\right]^3.
\end{aligned}$$

THEOREM 1. *If $0 \leqq x \leqq m_\eta\sqrt{n}/3\sigma_\eta$, then*

$$-R_n^- \leqq \mathbf{P}\{t_n^\star < x\} - \Phi(x) \leqq R_n^+, \tag{6}$$

where $R_n^- = r + r_* + r_\star$ and $R_n^+ = R_n^* + r/2 + n\mathbf{P}\{\eta > N\}$.

It was pointed out in [19] that $\mathbf{P}\{t_n^\star \geqq x\} = O(n^{-2})$ if $x > m_\eta\sqrt{n}/3\sigma_\eta$.

Theorem 1 and evident estimates $\sigma_\eta^2 \leqq \mathbf{E}|\eta|^{3/2}N^{1/2}$ and $\psi_\eta^3 \leqq \mathbf{E}|\eta|^{3/2}N^{3/2}$ entail

$$\sup_x \left|\mathbf{P}\{t_n^\circ < x\} - \Phi(x)\right| \leqq Cn^{-1/2}\left(\mathbf{E}|\xi|^3 \vee \mathbf{E}\eta^{3/2}\right) \tag{7}$$

if $\mathbf{E}\xi^2 = 1$ and $\mathbf{E}\eta \geqq 1$. More precisely, there holds the following statement.

COROLLARY 1. *Uniformly in $x \geqq 0$,*

$$-An^{-1/2}(1 + o(1)) \leqq \mathbf{P}\{t_n^\circ < x\} - \Phi(x) \leqq Bn^{-1/2}\left(1 + o(1)\right) \tag{8}$$

as $n \to \infty$, where

$$\begin{aligned}
A &= \left(\frac{\psi_\xi}{\sigma_\xi}\right)^3\left(1 + \frac{9}{\sqrt{2\pi}} + \sqrt{\frac{\pi}{8}}\right) + \frac{2\mathbf{E}|\xi|}{\sigma_\xi} + \frac{8|\mathbf{E}\xi\eta|}{em\sigma_\xi\sqrt{2\pi}},\\
B &= C_*\left(\frac{\psi_\xi}{\sigma_\xi}\right)^3 + \frac{4|\mathbf{E}\xi\eta|}{m\sigma_\xi e\sqrt{2\pi}}\,.
\end{aligned}$$

Assume that $\mathbf{E}X = 0$, $\mathbf{E}X^2 = 1$, and denote

$$\begin{aligned}
A_X &= \left(1 + \frac{9}{\sqrt{2\pi}} + \sqrt{\frac{\pi}{8}} + \frac{8}{e\sqrt{2\pi}}\right)\mathbf{E}|X|^3 + 2\mathbf{E}|X|,\\
B_X &= \left(C_* + \frac{4}{e\sqrt{2\pi}}\right)\mathbf{E}|X|^3.
\end{aligned}$$

COROLLARY 2. *If* $\mathbf{E}|X|^3 < \infty$, *then*

$$\inf_{x \geqq 0} \left[\mathbf{P}\{t_n^* < x\} - \Phi(x)\right] \geqq -A_X n^{-1/2}(1 + o(1)),$$
$$\sup_{x \geqq 0} \left[\mathbf{P}\{t_n^* < x\} - \Phi(x)\right] \leqq B_X n^{-1/2}\big(1 + o(1)\big).$$

Here t_n^* *may be replaced with* t_n.

Corollary 2 yields that uniformly in $x \geqq 0$,

$$(*) \qquad -6.4\,\mathbf{E}|X|^3 - 2\mathbf{E}|X| \leqq \left[\mathbf{P}\{t_n^* < x\} - \Phi(x)\right]\sqrt{n} \leqq 1.36\,\mathbf{E}|X|^3$$

for all large enough n. Notice that Theorem 1 in the preprint version of [17] states that

$$\sup_x \left|\mathbf{P}\{t_n^* < x\} - \Phi(x)\right| < \left(43\mathbf{E}|X|^3 + 8\right) n^{-1/2}. \tag{9}$$

The key lemma, Lemma 2.7, in that preprint was not correct (corrected in the journal version).

A natural question is how small can the absolute constant C in (7) be? The answer in the case $\eta \equiv 1$ is given in [29] $C_* \geqq C_E = (3 + \sqrt{10})/6\sqrt{2\pi}$ [10]. The following lemma establishes a lower bound for C.

LEMMA 1. *If* (7) *holds for all large enough* n, *then* $C \geqq 1/\sqrt{2e} > C_E$.

Notice that in the case of sums of random variables with nonrandom normalization, Stein's method yields worse constants than other methods (see [6], [25]), while the method of characteristic functions (MCF) provides the best-known upper bound for C_* (cf. [2]). Surprisingly, Stein's method appears superior to the MCF in the case of self-normalized sums. Another key element of the proof is a passage from self-normalized sums to sums of random variables, depending on the argument (see [18], [19]).

3. Functionals of sums of random vectors.

In this section we evaluate the accuracy of normal approximation for the distribution of some nonlinear functionals of sums of random vectors.

Let $(X, Y), (X_1, Y_1), \ldots$ be i.i.d. pairs of random variables. Assume that $\mathbf{E}X = \mathbf{E}Y = 0$ and $\mathbf{E}S_{n,X}^2 = 1$, where

$$S_{n,X} = \sum_{i=1}^n X_i, \qquad S_{n,Y} = \sum_{i=1}^n Y_i.$$

We use Stein's method [25] to evaluate the accuracy of normal approximation for the distribution of the quadratic functional

$$Z_n = S_{n,X} + cS_{n,Y}^2 \qquad (c \in \mathbf{R}).$$

Recall that there exists an absolutely continuous function g such that

$$g'(y) - yg(y) = \mathbb{1}\{y < x\} - \Phi(x), \qquad \|g\| \leqq \sqrt{\frac{\pi}{8}}, \quad \|g'\| \leqq 1$$

and

$$\mathbf{P}\{Z < x\} - \Phi(x) = \mathbf{E}g'(Z) - \mathbf{E}Zg(Z) \tag{10}$$

for any random variable Z with finite first moment (see [25]). Denote

$$\Delta = \left|\mathbf{P}\{Z_n < x\} - \Phi(x)\right|, \quad r_1 = \|g\|\big(2^{-1}n\mathbf{E}|X|^3 + c\mathbf{E}S_{n,Y}^2\big),$$
$$r_2 = c\|g'\|\, n\big(\mathbf{E}|X|\,Y^2 + 2\mathbf{E}|X||Y|\,\mathbf{E}|S_{n,Y}|\big),$$
$$r_3 = \|g'\|\left(\mathbf{E}|X| + 2^{-1}n\mathbf{E}|X|^3 + c\mathbf{E}Y^2 + 2c\mathbf{E}|Y|\left(\mathbf{E}|S_{n,X}|^3\right)^{1/3}\left(\mathbf{E}|S_{n,Y}|^{3/2}\right)^{2/3}\right).$$

THEOREM 2. *If* $\mathbf{E}|X|^3 + \mathbf{E}Y^2 < \infty$, *then*

$$\Delta \leqq \frac{9n}{\sqrt{2\pi}}\, \mathbf{E}|X|^3 + 2(r_1 + r_2 + r_3). \tag{11}$$

Proof. Denote $\Delta^+ = |\mathbf{P}\{Z_n + \nu < x\} - \Phi(x)|$, where the random variable ν is independent of Z_n and has the distribution with the density

$$f_\nu(y) = n \int_y^\infty u\, \mathbf{P}\{X \in du\}. \tag{12}$$

It is easy to check that $\mathbf{E}|\nu| = n\mathbf{E}|X|^3/2$. Because of (10),

$$\Delta^+ \leqq \left|\mathbf{E}g'(Z_n + \nu) - \mathbf{E}S_{n,X}\, g(Z_n + \nu)\right| + r_1.$$

It follows from (12) that $\mathbf{E}g'(a + \nu) = n\mathbf{E}Xg(a + X)$. Hence

$$\Delta^+ \leqq \left|n\mathbf{E}Xg(Z_n + X) - \mathbf{E}S_{n,X}\, g(Z_n + \nu)\right| + r_1.$$

Obviously, $S^2_{n+1,Y} = S^2_{n,Y} + 2Y_{n+1}S_{n,Y} + Y^2_{n+1}$. Therefore,

$$n\left|\mathbf{E}Xg(Z_n + X) - \mathbf{E}X_{n+1}g(Z_{n+1})\right| \leqq c\|g'\|\, n\mathbf{E}|X_{n+1}|\big(2|Y_{n+1}||S_{n,Y}| + Y^2_{n+1}\big) \leqq r_2.$$

Notice that $n\mathbf{E}X_{n+1}g(Z_{n+1}) = \mathbf{E}S_{n,X}\,g(Z_{n+1})$. It is easy to see that

$$\left|\mathbf{E}S_{n,X}\left[g(Z_{n+1}) - g(Z_n + \nu)\right]\right| \leqq r_3.$$

Thus,

$$\Delta^+ \leqq \sum_{i=1}^{3} r_i\,. \tag{13}$$

Recall a smoothing inequality by Esseen. If η, ζ, and ν are independent random variables,

$$\Delta_\zeta = \sup_x \left|\mathbf{P}\{\zeta < x\} - \Phi(x)\right|, \quad \Delta^*_\zeta = \sup_x \left|\mathbf{P}\{\zeta + \nu < x\} - \mathbf{P}\{\eta + \nu < x\}\right|,$$

$\mathcal{L}(\eta) = \mathcal{N}(0;1)$, and $2\mathbf{P}\{|\nu| > \varepsilon\} < 1$, then

$$\Delta_\zeta \leqq \frac{\Delta^*_\zeta + \varepsilon\sqrt{2/\pi}}{1 - 2\mathbf{P}\{|\nu| > \varepsilon\}}.$$

With $\Delta^+_\zeta = \sup_x |\mathbf{P}\{\zeta + \nu < x\} - \Phi(x)|$, this implies

$$\Delta_\zeta \leqq \frac{\Delta^+_\zeta + \varepsilon\sqrt{2/\pi} + \mathbf{E}|\nu|/\sqrt{2\pi}}{1 - 2\mathbf{E}|\nu|/\varepsilon}. \tag{14}$$

Put $\zeta = Z_n$ and $\varepsilon = 2n\mathbf{E}|X|^3$ in (14). Then

$$\Delta \leqq 2\Delta^+ + \frac{9}{\sqrt{2\pi}}\, n\mathbf{E}|X|^3\,. \tag{15}$$

Combining (13) and (15), we get (11).

4. Proofs of Theorem 1 and Corollary 2. Our approach differs from those in [3], [7], [9], [15], [17], [23], [26]. It involves Stein's method and some ideas from [18] and [19].

As in [18] and [19], our first step is to reduce the problem for self-normalized sums to that for sums with nonrandom normalization but depending on the argument x. Then we apply Theorem 2. At the final step, we evaluate remainders.

Proof of Theorem 1. Denote $a = m_0/2m_\eta\sigma_\xi\sqrt{n}$, $b=\sigma_\eta/2m_\eta\sqrt{n}$. For any $x \in [0; m_\eta\sqrt{n}/3\sigma_\eta]$, we put $\Phi_c = 1-\Phi$,

$$y=\frac{x}{\sqrt{1-2ax+b^2x^2}},\qquad d=\frac{\sigma_\xi}{ay+\sqrt{1+a^2y^2-b^2y^2}},$$
$$c=\frac{y}{2},\qquad X_i=\frac{\xi_i}{d\sqrt{n}}-\frac{y\overline{\eta}_i^<}{2m_\eta n},\qquad Y_i=\frac{\overline{\eta}_i^<}{m_\eta n}.$$

Then $\mathbf{E}S_{n,X}^2=1$ (cf. [19]). Since $\sqrt{1+y}\geqq 1+y/2-y^2/2$ as $y\geqq -1$, we have

$$\mathbf{P}\{t_n^\star\geqq x\}\leqq \mathbf{P}\left\{\frac{S_n/\sigma_\xi\sqrt{n}}{\sqrt{T_n^</m_\eta n}}\geqq x\right\}\leqq \mathbf{P}\left\{\frac{S_n}{\sigma_\xi\sqrt{n}}\geqq x\left[1+\frac{\overline{T}_n^<}{2m_\eta n}-\frac{1}{2}\left(\frac{\overline{T}_n^<}{m_\eta n}\right)^2\right]\right\},$$

where $T_n^< = \sum^n \overline{\eta}_i^<$. According to (5.4) in [19], $x=yd/\sigma_\xi$. Hence

$$\mathbf{P}\{t_n^\star\geqq x\}\leqq \mathbf{P}\left\{\frac{S_n}{d\sqrt{n}}-\frac{y\overline{T}_n^<}{2m_\eta n}+\frac{y}{2}\left(\frac{\overline{T}_n^<}{m_\eta n}\right)^2\geqq y\right\}=\mathbf{P}\{S_{n,X}+cS_{n,Y}^2\geqq y\}.$$

Theorem 2 yields

$$\mathbf{P}\{S_{n,X}+cS_{n,Y}^2\geqq y\}\leqq \Phi_c(y)+2(r_1+r_2+r_3)+\frac{9n}{\sqrt{2\pi}}\,\mathbf{E}|X|^3.$$

According to Lemma 12 in [19],

$$\Phi_c(y)\leqq\Phi_c(x)+r,\quad y\leqq\frac{3x}{2},\quad \frac{\sigma_\xi}{d}\leqq 1+\frac{ym_0}{m_\eta\sigma_\xi\sqrt{n}}\leqq\psi_*.$$

Therefore, $y=x\sigma_\xi/d\leqq x\psi_*$. Note that

$$\mathbf{E}^2|X|\leqq\mathbf{E}X^2=\frac{1}{n},\qquad n^{3/2}\mathbf{E}|X|^3\leqq\left(\frac{\psi}{\sigma_\xi}\right)^3,$$
$$n^{3/2}\mathbf{E}|X||Y|\leqq\frac{\psi_*}{m_\eta}\left(\frac{\mathbf{E}|\xi|\,|\overline{\eta}^<|}{\sigma_\xi}+\frac{x\sigma_\eta^2}{2m_\eta\sqrt{n}}\right),$$
$$n^{5/2}\mathbf{E}|X|Y^2\leqq\frac{\psi_*}{m_\eta^2}\left(\frac{\psi_\xi\psi_\eta^2}{\sigma_\xi}+\frac{x\psi_\eta^3}{2m_\eta\sqrt{n}}\right).$$

From [1] we obtain $\mathbf{E}|S_{n,Y}|^{3/2}\leqq 2n\mathbf{E}|Y_i|^{3/2}$. Hence

$$\mathbf{E}|S_{n,Y}|\leqq n^{-1/3}m_\eta^{-1}\big(2\mathbf{E}|\overline{\eta}^<|^{3/2}\big)^{2/3}$$

and $2r_2\leqq r_*$. It is easy to check that $\mathbf{E}|\sum_{i=1}^n X_i|^3\leqq 2(n\mathbf{E}X^2)^{3/2}+n\mathbf{E}|X|^3$. Therefore,

$$2(r_1+r_3)+\frac{9n}{\sqrt{2\pi}}\,\mathbf{E}|X|^3\leqq n\mathbf{E}|X|^3\left(\|g\|+\|g'\|+\frac{9}{\sqrt{2\pi}}\right)+2c\big(n\|g\|+\|g'\|\big)\,\mathbf{E}Y^2$$
$$+2\|g'\|\mathbf{E}|X|+4c\|g'\|\mathbf{E}|Y|\big(2+n\mathbf{E}|X|^3\big)^{1/3}\big(2n\mathbf{E}|Y|^{3/2}\big)^{2/3}\leqq r_\star.$$

Combining these estimates, we get the lower bound in (6). The upper bound is taken from Theorem 9 in [19]. The proof is complete.

Proof of Corollary 1. We suggest using different truncation levels in lower and upper bounds. With properly chosen truncation levels, the decay rates of R_n^- and R_n^+ are $n^{-1/2}$.

First, we choose $N \asymp n$. Since $\sup_x x^2\varphi(x/2) = 8/e\sqrt{2\pi}$, it is easy to see that

$$n\mathbf{P}\{\eta > N\} = o(n^{-1/2}), \qquad r \leqq \frac{8|\mathbf{E}\xi\eta|/\mathbf{E}\eta}{e\sigma_\xi\sqrt{2\pi n}}\big(1+o(1)\big),$$

and $R_n^* \sim C_*\mathbf{E}|\xi|^3\sigma_\xi^{-3}n^{-1/2}$. Theorem 2 entails

$$\sup_{0\leqq x\leqq x_n}\big[\mathbf{P}\{t_n^\star < x\} - \Phi(x)\big] \leqq Bn^{-1/2}\big(1+o(1)\big),$$

where $x_n = m_\eta\sqrt{n}/3\sigma_\eta$. According to (3.6*) in [19],

$$\mathbf{P}\{t_n^\star \geqq x_n\} = o(n^{-1}).$$

Besides, $n^{1/2}/\sigma_\eta \gg n^{1/6}$. Therefore,

$$\sup_{x\geqq 0}\big[\mathbf{P}\{t_n^\star < x\} - \Phi(x)\big] \leqq Bn^{-1/2}\big(1+o(1)\big). \tag{16}$$

Since $m_\eta = \mathbf{E}\eta + o(n^{-1/2})$, we may replace $t_n^\star$ with t_n°.

Now we put $N \asymp n/(1+x)^2$. Then uniformly in $x \in [0; n^{1/6}]$,

$$r_* = o(n^{-1/2}), \qquad n^{1/2}r_\star = \left(\frac{\psi}{\sigma_\xi}\right)^3\left(1+\frac{9}{\sqrt{2\pi}}+\sqrt{\frac{\pi}{8}}\right) + \frac{2\mathbf{E}|\xi|}{\sigma_\xi} + o(1).$$

Theorem 2 yields

$$\inf_{0\leqq x\leqq n^{1/6}}\big[\mathbf{P}\{t_n^\star < x\} - \Phi(x)\big] \geqq -An^{-1/2}\big(1+o(1)\big).$$

Since $\mathbf{E}\eta^{3/2} < \infty$, we have $|m_\eta - \mathbf{E}\eta| = \mathbf{E}\eta 1\!\!1\{\eta > N\} = (1+x)\,n^{-1/2}v_n(x)$, where $v_n(x) \to 0$ uniformly in $x \in [0; n^{1/6}]$. Hence we may replace $t_n^\star$ with t_n°. Note that $\mathbf{P}\{t_n^\circ \geqq n^{1/6}\} = O(n^{-1})$ (cf. (3.6*) in [19]). Therefore,

$$\inf_{x\geqq 0}\big[\mathbf{P}\{t_n^\circ < x\} - \Phi(x)\big] \geqq -An^{-1/2}\big(1+o(1)\big). \tag{17}$$

The proof is complete.

Proof of Lemma 1. It suffices to construct a sequence of distributions such that $\eta = \xi^2$ and the left-hand side of (7) asymptotically equals $\mathbf{E}|\xi|^3/\sqrt{2en}$.

Let ξ be a random variable with the distribution

$$\mathbf{P}\big\{\xi = p^{1/2}(1-p)^{-1/2}\big\} = 1-p, \qquad \mathbf{P}\big\{\xi = -(1-p)^{1/2}p^{-1/2}\big\} = p, \tag{18}$$

where $p \in (0;1)$. Then $\mathbf{E}\xi = 0$, $\mathbf{E}\xi^2 = 1$, $t_n^\circ = t_n^*$, and

$$\mathbf{P}\big\{t_n^* \geqq \sqrt{n}\big\} = \mathbf{P}\big\{\xi_1 = \cdots = \xi_n = p^{1/2}(1-p)^{-1/2}\big\} = (1-p)^n.$$

If (7) holds, then

$$C\mathbf{E}|\xi|^3 \geqq \big((1-p)^n - \Phi(-\sqrt{n}\,)\big)\sqrt{n}.$$

Since $\mathbf{E}|\xi|^3 \sim 1/\sqrt{p}$ as $p \to 0$, we have

$$C \geqq (1-p)^n\sqrt{np}\,\big(1+o(1)\big) \sim \frac{1}{\sqrt{2e}}$$

if $p = 1/(2n)$. The result follows.

Example (18) highlights the disadvantage of the self-normalized sum t_n^*: it can take "large" values when $X_1, \dots, X_n$ are "small." Indeed, if $X_1 = \cdots = X_n = y > 0$, then $t_n^* = \sqrt{n}$ (in contrast, $\sum_{i=1}^n X_i/\sqrt{n} \to 0$ as $y \to 0$). This example shows also that estimate (4) fails. Moreover, no inequality of the form

$$\left|\mathbf{P}\{t_n^* < x\} - \Phi(x)\right| \leqq A\mathbf{E}|X|^3 g(x)\, n^{-1/2} \qquad (x \in \mathbf{R}), \tag{19}$$

where A is an absolute constant and a positive function $g(x) \downarrow 0$ as $x \to \infty$, may hold: take $p = 1/n$ and $x = \sqrt{n}$; then the left-hand side of (19) tends to $1 - 1/e$ while the right-hand side tends to 0.

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