

*annotated
version*

Machine Learning Course - CS-433

Optimization

Sep 21+27, 2022

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Last updated on: September 21, 2022
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Learning / Estimation / Fitting

Given a cost function $\mathcal{L}(\mathbf{w})$, we wish to find \mathbf{w}^* which minimizes the cost:

$$\min_{\mathbf{w}} \boxed{\mathcal{L}(\mathbf{w})} \quad \text{subject to } \mathbf{w} \in \mathbb{R}^D$$

This means the learning problem is formulated as an optimization problem.

We will use an optimization algorithm to solve the problem (to find a good \mathbf{w}).

Grid Search

Grid search is one of the simplest optimization algorithms. We compute the cost over all values \mathbf{w} in a grid, and pick the best among those.

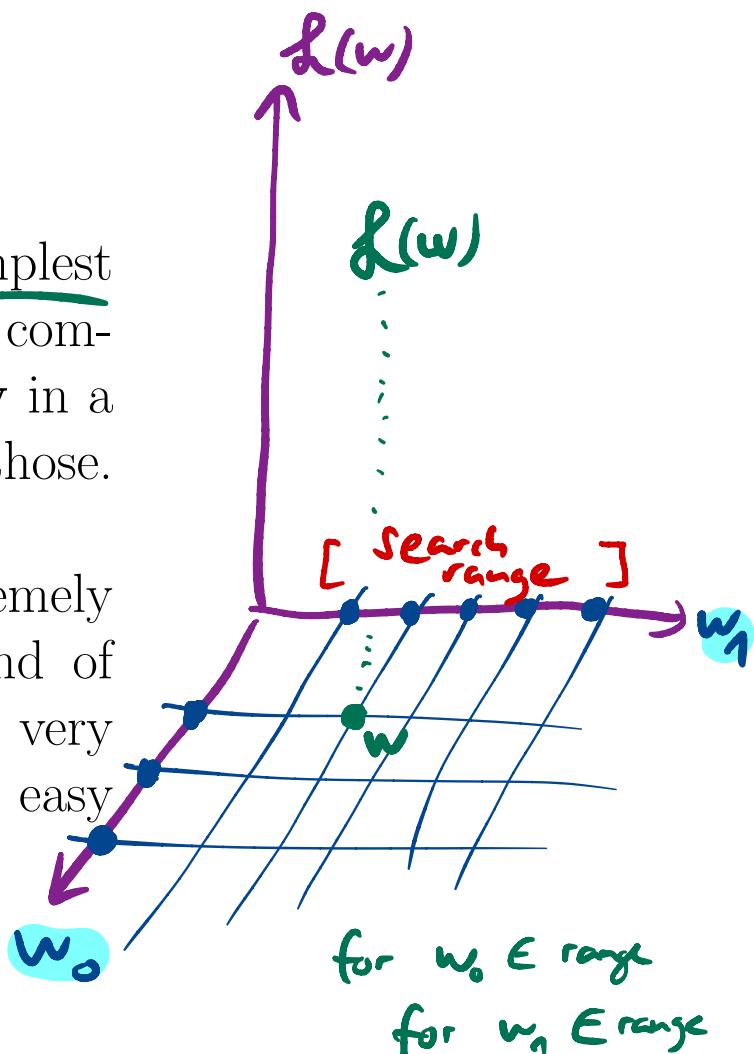
This is brute-force, but extremely simple and works for any kind of cost function when we have very few parameters and the cost is easy to compute.

cost function

$$\mathcal{L}(\mathbf{w}) = \frac{1}{N} \sum_{n=1}^N \mathcal{L}_n(\mathbf{w})$$

cost of
data point n

Example: $(y_n - f_{\mathbf{w}}(x_n))^2$



high dimensionality

① For a large number of parameters D , however, grid search has too many “for-loops”, resulting in an exponential computational complexity:

If we decide to use 10 possible values for each dimension of \mathbf{w} , then we have to check 10^D points. This is clearly impossible for most practical machine learning models, which can often have $D \approx$ millions of parameters. Choosing a good range of values for each dimension is another problem.

$$\mathbf{w} \in \mathbb{R}^D$$

search range

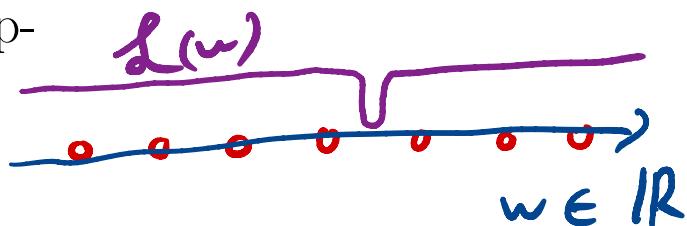
$$\mathbf{w} \in \{1, 2, \dots, 10\}^D$$

|

$$| = 10^D$$

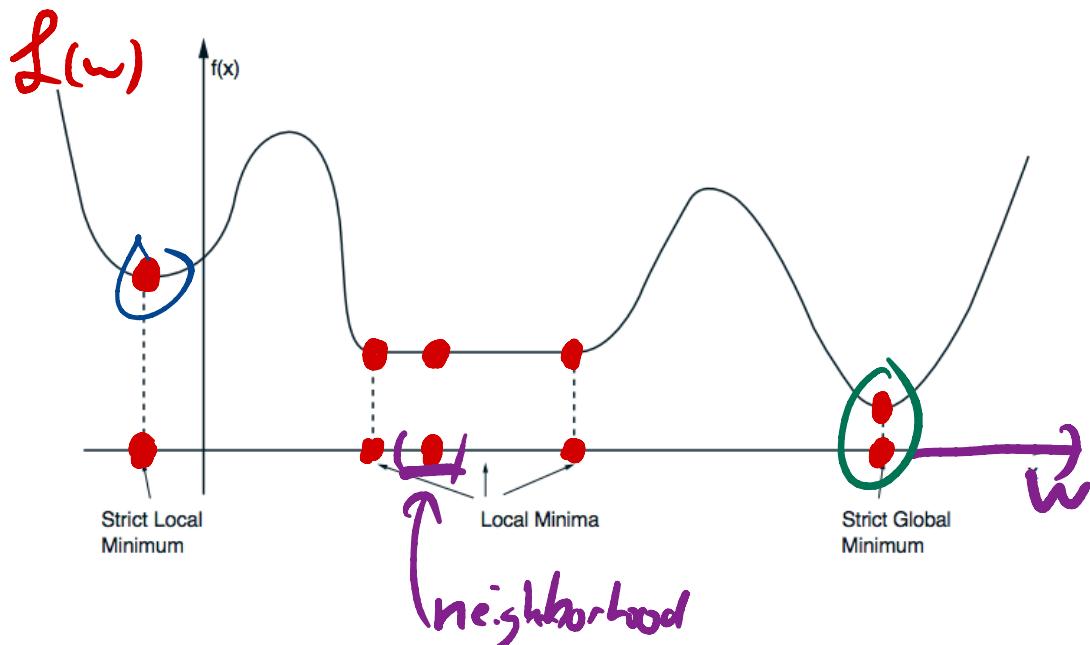
exponential cost

② Other issues: No guarantee can be given that we end up close to an optimum.



\mathcal{L} Lipschitz \Rightarrow ok

Optimization Landscapes



The above figure is taken from Bertsekas, Nonlinear programming.

A vector \mathbf{w}^* is a **local minimum** of \mathcal{L} if it is no worse than its neighbors; i.e. there exists an $\epsilon > 0$ such that,

$$\mathcal{L}(\mathbf{w}^*) \leq \mathcal{L}(\mathbf{w}), \quad \forall \mathbf{w} \text{ with } \|\mathbf{w} - \mathbf{w}^*\| < \epsilon$$

*neighborhood around \mathbf{w}^**

A vector \mathbf{w}^* is a **global minimum** of \mathcal{L} if it is no worse than all others,

$$\mathcal{L}(\mathbf{w}^*) \leq \mathcal{L}(\mathbf{w}), \quad \forall \mathbf{w} \in \mathbb{R}^D \quad \leftarrow \text{global}$$

A local or global minimum is said to be **strict** if the corresponding inequality is strict for $\mathbf{w} \neq \mathbf{w}^*$.

Smooth Optimization

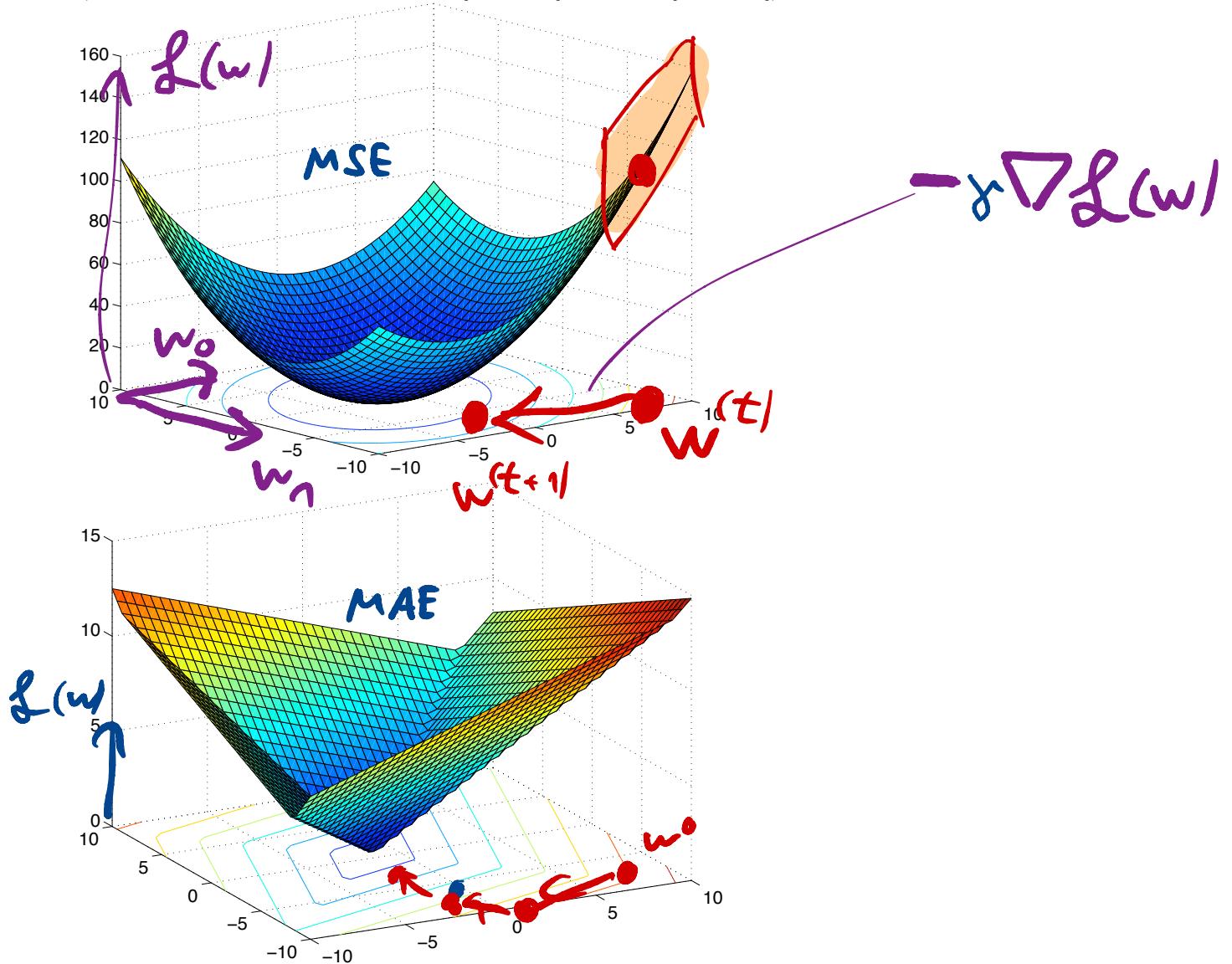
Follow the Gradient

A gradient (at a point) is the slope of the tangent to the function (at that point). It points to the direction of largest increase of the function.

Cauchy
1848

For a 2-parameter model, $\text{MSE}(\mathbf{w})$ and $\text{MAE}(\mathbf{w})$ are shown below.

(We used $\mathbf{y}_n \approx w_0 + w_1 x_{n1}$ with $\mathbf{y}^\top = [2, -1, 1.5]$ and $\mathbf{x}^\top = [-1, 1, -1]$).



Definition of the gradient:

$$\nabla \mathcal{L}(\mathbf{w}) := \left[\frac{\partial \mathcal{L}(\mathbf{w})}{\partial w_1}, \dots, \frac{\partial \mathcal{L}(\mathbf{w})}{\partial w_D} \right]^\top \quad \begin{matrix} \mathbf{w} \in \mathbb{R}^D \\ \mathbf{w} \in \mathbb{R}^D \end{matrix}$$

This is a vector, $\nabla \mathcal{L}(\mathbf{w}) \in \mathbb{R}^D$.

Gradient Descent

To minimize the function, we iteratively take a step in the (opposite) direction of the gradient

$$\mathbf{w}^{(t+1)} := \mathbf{w}^{(t)} - \gamma \nabla \mathcal{L}(\mathbf{w}^{(t)})$$

where $\gamma > 0$ is the **step-size** (or learning rate). Then repeat with the next t .

$$t=0, 1, 2, \dots$$

popular stepsize

$$\delta = \delta(\epsilon) = \frac{C}{\epsilon}$$

$$\arg \min_{\mathbf{w}} \mathcal{L}(\mathbf{w}) = \min_{\mathbf{w}} \mathcal{L}(\mathbf{w})$$

Example: Gradient descent for 1-parameter model to minimize MSE:

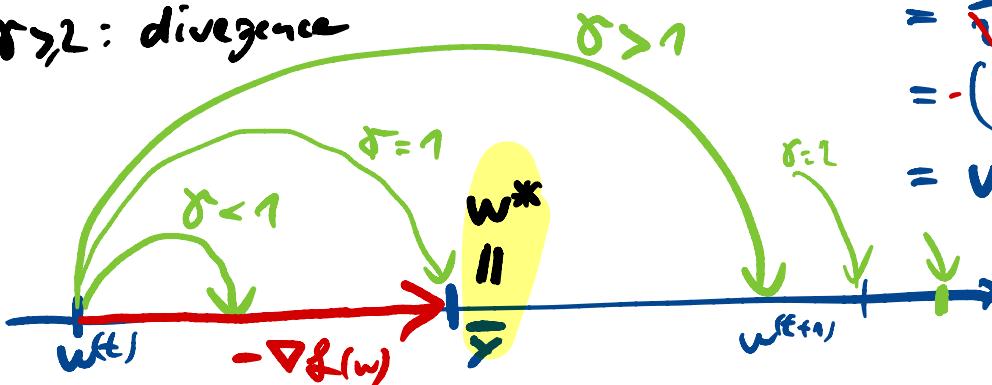
$$w_0^{(t+1)} := (1 - \gamma)w_0^{(t)} + \gamma \bar{y}$$

where $\bar{y} := \sum_n y_n / N$. When is this sequence guaranteed to converge?

$$\mathcal{L}(\mathbf{w}) = f_w(\mathbf{x}) = \frac{1}{2N} \sum_{n=1}^N (y_n - w_0)^2$$

$$\nabla \mathcal{L}(\mathbf{w}) = \frac{\partial}{\partial w_0} \mathcal{L}$$

$$\begin{aligned} &= \frac{1}{N} \sum_{n=1}^N -2(y_n - w_0) \\ &= -\left(\frac{1}{N} \sum y_n\right) + w_0 = \\ &= w_0 - \bar{y} \end{aligned}$$



Gradient Descent for Linear MSE

For linear regression

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}, \mathbf{X} = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1D} \\ x_{21} & x_{22} & \dots & x_{2D} \\ \vdots & \vdots & \ddots & \vdots \\ x_{N1} & x_{N2} & \dots & x_{ND} \end{bmatrix}$$

$\mathbf{x}_2 \in \mathbb{R}^D$

$\mathbf{w} \in \mathbb{R}^D$

We define the error vector \mathbf{e} :

$$\mathbf{e} = \mathbf{y} - \mathbf{X}\mathbf{w} \quad \leftarrow \mathbf{e} = \begin{pmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_N \end{pmatrix} = \mathbb{R}^N$$

and MSE as follows:

$$\begin{aligned} \mathcal{L}(\mathbf{w}) &:= \frac{1}{2N} \sum_{n=1}^N (y_n - \mathbf{x}_n^\top \mathbf{w})^2 \\ &= \frac{1}{2N} \mathbf{e}^\top \mathbf{e} \end{aligned}$$

$$f_{\mathbf{w}}(\mathbf{x}_n)$$

$$\mathcal{L}_n(\mathbf{w})$$

then the gradient is given by

$$\nabla \mathcal{L}(\mathbf{w}) = -\frac{1}{N} \mathbf{X}^\top \mathbf{e}$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial w_n} &= \frac{1}{N} \sum_{n=1}^N -2(y_n - \mathbf{x}_n^\top \mathbf{w}) \mathbf{x}_{n1} \\ &= -\frac{1}{N} (\mathbf{X}_{:,1})^\top \mathbf{e} \\ &\vdots \\ \frac{\partial \mathcal{L}}{\partial w_D} &= \frac{1}{N} \sum_{n=1}^N \dots \\ &= -\frac{1}{N} (\mathbf{X}_{:,D})^\top \mathbf{e} \end{aligned}$$

Computational cost. What is the complexity (# operations) of computing the gradient?

a) starting from \mathbf{w} and

① compute \mathbf{e}
cost = ~~$\Theta(N \cdot D)$~~ + ~~$\Theta(N)$~~

b) given \mathbf{e} and \mathbf{w} ?

② compute \mathbf{e} , given \mathbf{e}
cost: $\Theta(N \cdot D)$ + $\Theta(D)$
= $\Theta(N \cdot D)$

Variant with offset. Recall: Alternative trick when also incorporating an offset term for the regression:

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} \quad \tilde{\mathbf{X}} = \begin{bmatrix} 1 & x_{11} & x_{12} & \dots & x_{1D} \\ 1 & x_{21} & x_{22} & \dots & x_{2D} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{N1} & x_{N2} & \dots & x_{ND} \end{bmatrix}$$

artificial new feature

Exercise:
compute $\nabla \mathcal{L}(\mathbf{w})$

Stochastic Gradient Descent SGD

Sum Objectives. In machine learning, most cost functions are formulated as a sum over the training examples, that is

$$\mathcal{L}(\mathbf{w}) = \frac{1}{N} \sum_{n=1}^N \mathcal{L}_n(\mathbf{w}),$$

where \mathcal{L}_n is the cost contributed by the n -th training example.

Q: What are the \mathcal{L}_n for linear MSE? 

The SGD Algorithm. The stochastic gradient descent (SGD) algorithm is given by the following update rule, at step t :

- ① sample one datapoint $n \in \{1, \dots, N\}$ u.a.r.
- ② $\mathbf{w}^{(t+1)} := \mathbf{w}^{(t)} - \gamma \underbrace{\nabla \mathcal{L}_n(\mathbf{w}^{(t)})}_{\text{"stochastic gradient"}}$

Theoretical Motivation. Idea:

Cheap but unbiased estimate of the gradient!

In expectation over the random choice of n , we have

$$\begin{aligned} \mathbb{E}_{\mathbf{n}} [\nabla \mathcal{L}_n(\mathbf{w})] &= \frac{1}{N} \sum_{n=1}^N \nabla \mathcal{L}_n(\mathbf{w}) \\ &= \nabla \mathcal{L}(\mathbf{w}) \end{aligned}$$

$$\mathbb{E} [\nabla \mathcal{L}_n(\mathbf{w})] = \nabla \mathcal{L}(\mathbf{w})$$

which is the true gradient direction.
(check!)

Mini-batch SGD. There is an intermediate version, using the update direction being

$$\mathbf{g} := \frac{1}{|B|} \sum_{n \in B} \nabla \mathcal{L}_n(\mathbf{w}^{(t)})$$

average over mini-batch

again with

$$\mathbf{w}^{(t+1)} := \mathbf{w}^{(t)} - \gamma \mathbf{g} .$$

In the above gradient computation, we have randomly chosen a subset $B \subseteq [N]$ of the training examples. For each of these selected examples n , we compute the respective gradient $\nabla \mathcal{L}_n$, at the same current point $\mathbf{w}^{(t)}$.

$B \subseteq \{1, \dots, N\}$
sample $|B|$ datapoints

- $|B|=1$
→ pure SGD

- $|B|=128$
→ mini-batch

- $|B|=N$
→ full GD

The computation of \mathbf{g} can be parallelized easily. This is how current deep-learning applications utilize GPUs (by running over $|B|$ threads in parallel).

Note that in the extreme case $B := [N]$, we obtain (batch) gradient descent, i.e. $\mathbf{g} = \nabla \mathcal{L}$.

SGD for Linear MSE

See Exercise Sheet 2.

Computational cost. For linear MSE, what is the complexity (# operations) of computing the stochastic gradient?

(using only $|B| = 1$ data examples)

*N times
cheaper*

SGD

$$\begin{aligned}\hat{\mathcal{L}}_n &= \frac{1}{2}(x_n - x_n^T w)^2 \\ \nabla \hat{\mathcal{L}}_n &= -x_n^T (x_n - x_n^T w)\end{aligned}$$

$x_n \in \mathbb{R}^n$

cost

$\Theta(D)$

before:

GD

$$\mathcal{L} = \frac{1}{2n} \sum_n (\quad)^2$$

$$\nabla \mathcal{L} = -\frac{1}{n} X^T (y - Xw)$$

$y \in \mathbb{R}^n$

$\Theta(N \cdot D)$

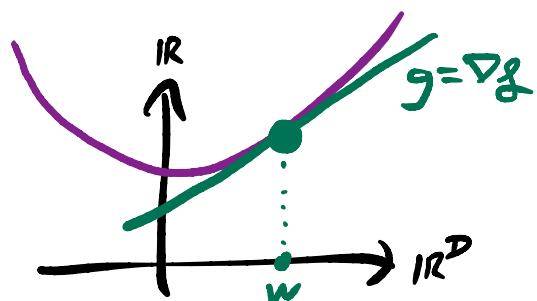
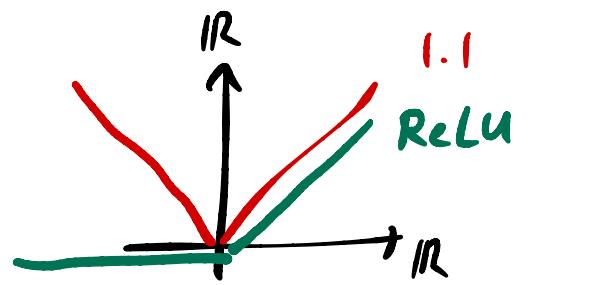
Non-Smooth Optimization

An alternative characterization of *convexity*, for differentiable functions is given by

$$\mathcal{L}(\mathbf{u}) \geq \mathcal{L}(\mathbf{w}) + \nabla \mathcal{L}(\mathbf{w})^\top (\mathbf{u} - \mathbf{w}) \quad \forall \mathbf{u}, \mathbf{w}$$

1st order Taylor

meaning that the function must always lie above its linearization.

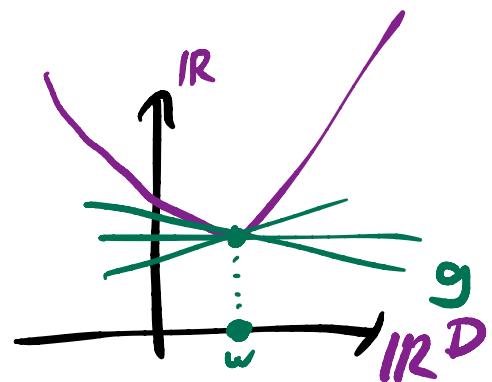


Subgradients

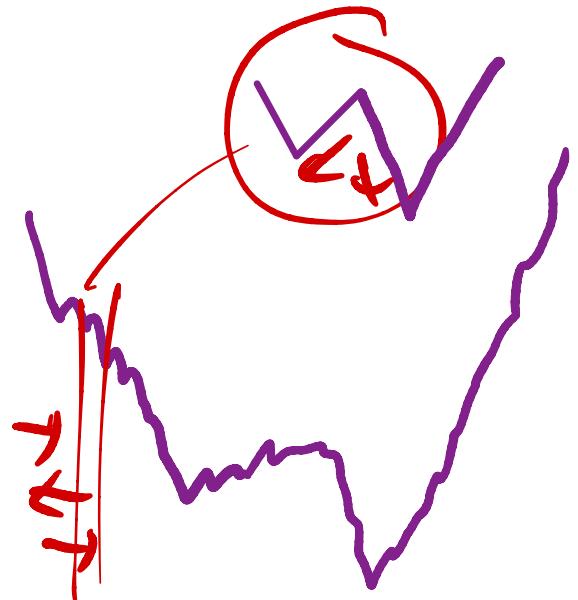
A vector $\mathbf{g} \in \mathbb{R}^D$ such that

$$\mathcal{L}(\mathbf{u}) \geq \mathcal{L}(\mathbf{w}) + \mathbf{g}^\top (\mathbf{u} - \mathbf{w}) \quad \forall \mathbf{u}$$

is called a *subgradient* to the function \mathcal{L} at \mathbf{w} .



This definition makes sense for objectives \mathcal{L} which are not necessarily differentiable (and not even necessarily convex).

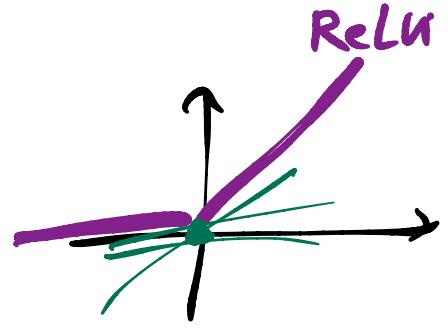


If \mathcal{L} is convex and differentiable at \mathbf{w} , then the only subgradient at \mathbf{w} is $\mathbf{g} = \nabla \mathcal{L}(\mathbf{w})$.

Subgradient Descent

Identical to the gradient descent algorithm, but using a subgradient instead of gradient. Update rule

$$\mathbf{w}^{(t+1)} := \mathbf{w}^{(t)} - \gamma \mathbf{g}$$

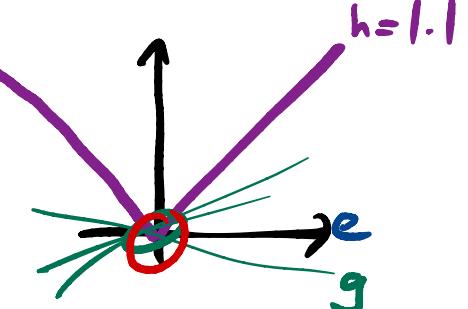


for \mathbf{g} being a subgradient to \mathcal{L} at the current iterate $\mathbf{w}^{(t)}$.

Example: Optimizing Linear MAE

1. Compute a subgradient of the absolute value function

$$h : \mathbb{R} \rightarrow \mathbb{R}, h(e) := |e|.$$



2. Recall the definition of the mean absolute error:

$$\mathcal{L}(\mathbf{w}) = \text{MAE}(\mathbf{w}) := \frac{1}{N} \sum_{n=1}^N |y_n - f_{\mathbf{w}}(\mathbf{x}_n)|$$

$$g \in \begin{cases} -1 & \text{if } e < 0 \\ [-1, 1] & \text{if } e = 0 \\ 1 & \text{if } e > 0 \end{cases}$$

For linear regression, its (sub)gradient is easy to compute using the chain rule. Compute it!

See Exercise Sheet 2.

$\mathcal{L}(\mathbf{w}) = h(q(\mathbf{w}))$
 h : non-differentiable
 q : differentiable

Subgradient of \mathcal{L} at \mathbf{w} :

$$g \in \boxed{\partial h(q(\mathbf{w}))} \quad \nabla q(\mathbf{w})$$

Notation:

∂h : set of subgradients of h

Stochastic Subgradient Descent

Stochastic SubGradient Descent
(still abbreviated SGD commonly).

Same, \mathbf{g} being a subgradient to the randomly selected \mathcal{L}_n at the current iterate $\mathbf{w}^{(t)}$.

$$\mathcal{L}(\omega) = \frac{1}{N} \sum \mathcal{L}_n(\omega)$$

↑
potentially
non-differentiable

Exercise: Compute the SGD update for linear MAE.

take a
subgradient of \mathcal{L}_n

of linear model

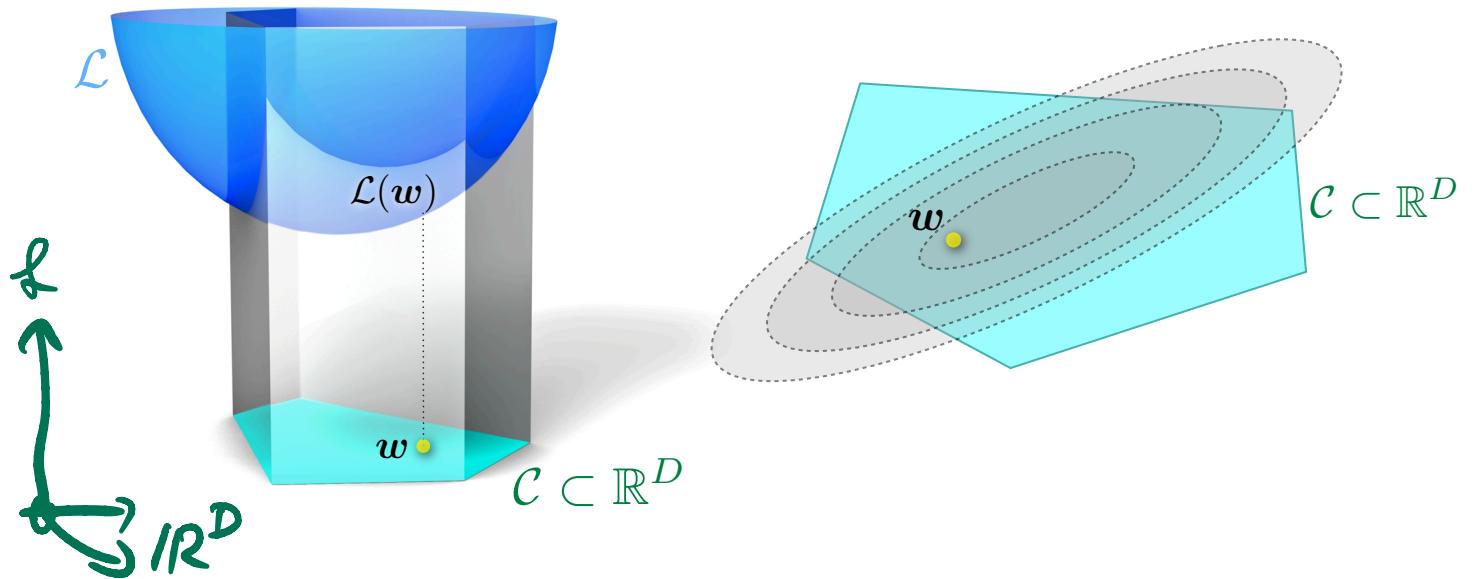
	Smooth	non-smooth
MSE	gradient $\nabla \mathcal{L}$ $\Theta(N \cdot D)$	MAE
GD	gradient $\nabla \mathcal{L}$ $\Theta(N \cdot D)$	subgradient $\mathbf{g} \in \partial \mathcal{L}$ $\Theta(N \cdot D)$
SGD	gradient $\nabla \mathcal{L}_n$ $\Theta(D)$	subgradient $\mathbf{g} \in \partial \mathcal{L}_n$ $\Theta(D)$
cost per iteration		

Constrained Optimization

Sometimes, optimization problems come posed with additional constraints:

$$\min_{\mathbf{w}} \mathcal{L}(\mathbf{w}), \quad \text{subject to } \mathbf{w} \in \mathcal{C}.$$

The set $\mathcal{C} \subset \mathbb{R}^D$ is called the constraint set.



Solving Constrained Optimization Problems

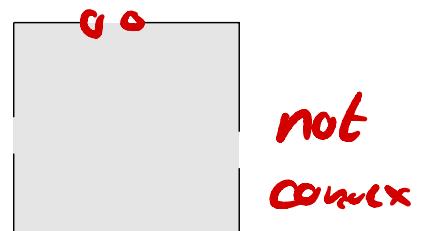
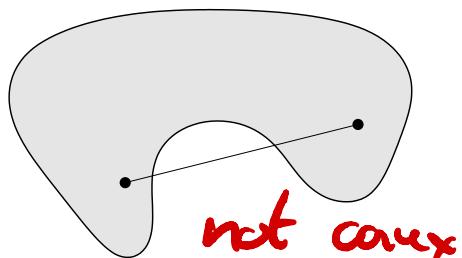
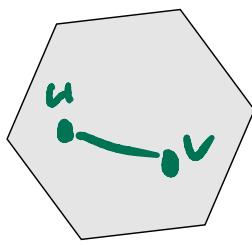
- (A) Projected Gradient Descent
- (B) Transform it into an *unconstrained* problem

Convex Sets

A set \mathcal{C} is **convex** iff

the line segment between any two points of \mathcal{C} lies in \mathcal{C} , i.e., if for any $\mathbf{u}, \mathbf{v} \in \mathcal{C}$ and any θ with $0 \leq \theta \leq 1$, we have

$$\theta\mathbf{u} + (1 - \theta)\mathbf{v} \in \mathcal{C}.$$



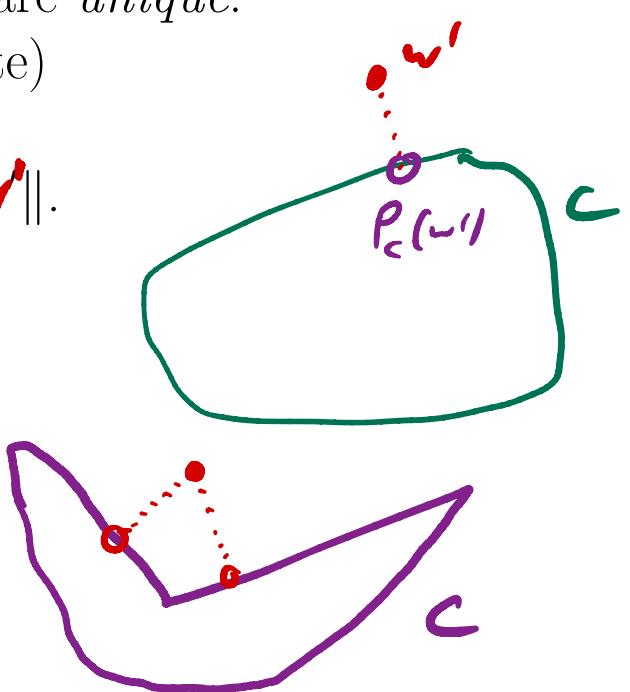
*Figure 2.2 from S. Boyd, L. Vandenberghe

Properties of Convex Sets

- Intersections of convex sets are convex
- Projections onto convex sets are *unique*.
(and often efficient to compute)

Formal definition:

$$P_{\mathcal{C}}(\mathbf{w}') := \arg \min_{\mathbf{v} \in \mathcal{C}} \|\mathbf{v} - \mathbf{w}'\|.$$



Projected Gradient Descent

Idea: add a projection onto \mathcal{C} after every step:

$$P_{\mathcal{C}}(\mathbf{w}') := \arg \min_{\mathbf{v} \in \mathcal{C}} \|\mathbf{v} - \mathbf{w}'\|.$$

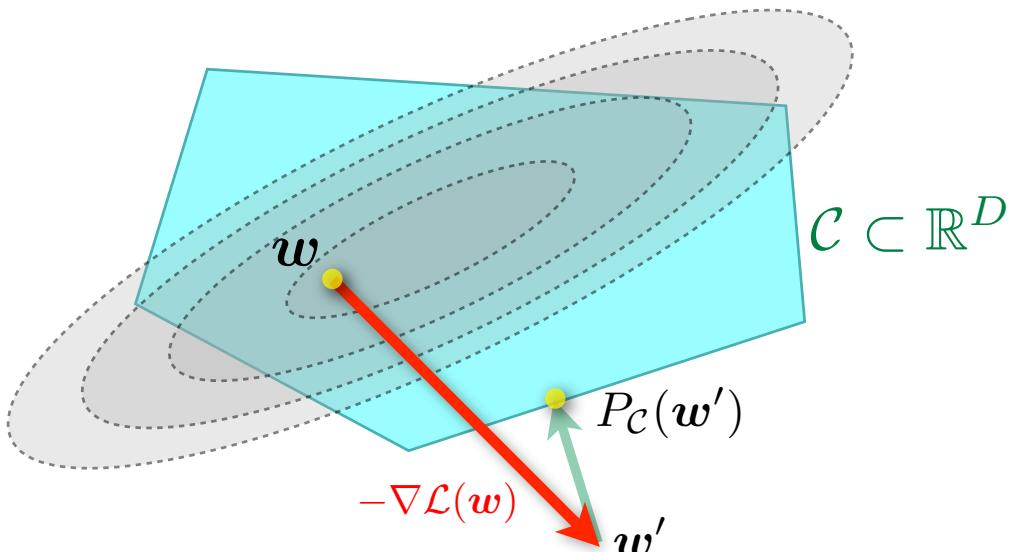
Update rule:

$$\mathbf{w}^{(t+1)} := P_{\mathcal{C}} \left[\mathbf{w}^{(t)} - \gamma \nabla \mathcal{L}(\mathbf{w}^{(t)}) \right].$$

traditional GD step

Projection

repeat



Projected SGD. Same SGD step, followed by the projection step, as above. Same convergence properties.

Computational cost of projection?
Crucial!

depends on set C

$$\min_{\mathbf{w} \in \mathcal{C}} \mathcal{L}(\mathbf{w}) \rightsquigarrow \min_{\mathbf{w}} \mathcal{L}(\mathbf{w}) + P(\mathbf{w})$$

Turning Constrained into Unconstrained Problems

(Alternatives to projected gradient methods)

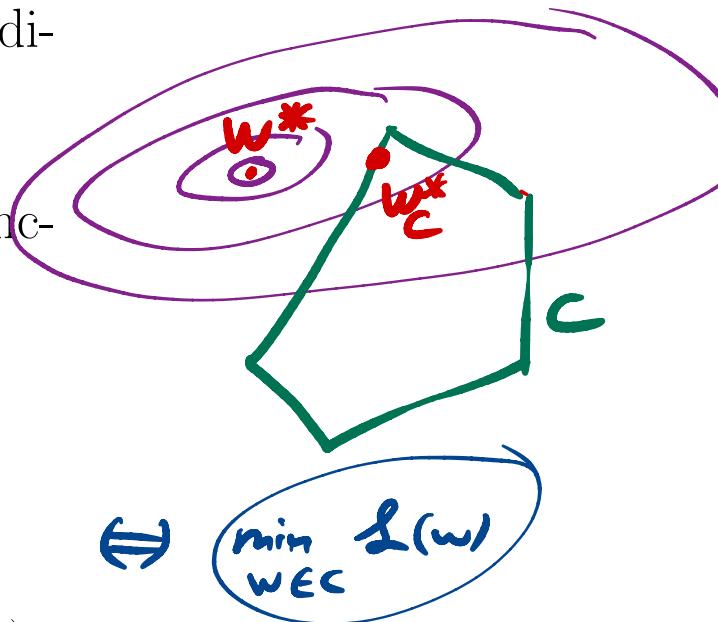
Use **penalty functions** instead of directly solving $\min_{\mathbf{w} \in \mathcal{C}} \mathcal{L}(\mathbf{w})$.

- “brick wall” (indicator function)

$$I_C(\mathbf{w}) := \begin{cases} 0 & \mathbf{w} \in \mathcal{C} \\ +\infty & \mathbf{w} \notin \mathcal{C} \end{cases}$$

$$\Rightarrow \min_{\mathbf{w} \in \mathbb{R}^D} \mathcal{L}(\mathbf{w}) + I_C(\mathbf{w})$$

(disadvantage: non-continuous objective)



- Penalize error. Example:

$$\mathcal{C} = \{\mathbf{w} \in \mathbb{R}^D \mid \underline{A}\mathbf{w} = \mathbf{b}\}$$

$$\Rightarrow \min_{\mathbf{w} \in \mathbb{R}^D} \mathcal{L}(\mathbf{w}) + \lambda \|\underline{A}\mathbf{w} - \mathbf{b}\|^2$$

- Linearized Penalty Functions
(see Lagrange Multipliers)

\hookrightarrow additional “helper” variable

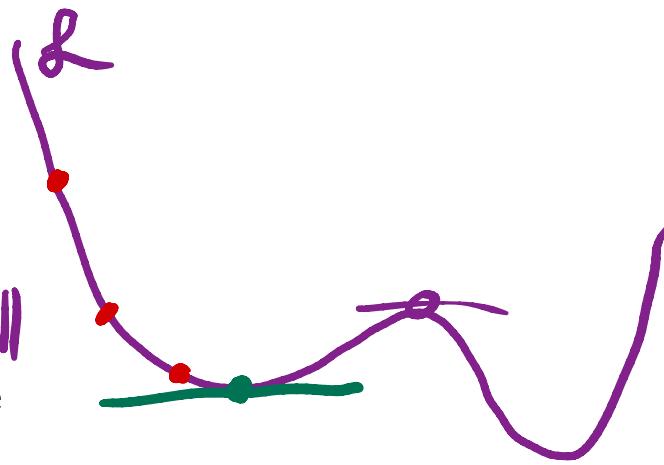
“distance” from \mathcal{C}

$$\begin{array}{c} A \\ \times \\ \mathbf{w} \\ = \\ b \end{array}$$

Implementation Issues

For gradient methods:

Stopping criteria: When $\|\nabla \mathcal{L}(\mathbf{w})\|$ is (close to) zero, we are (often) close to the optimum value.



Optimality: If the second-order derivative is positive (positive semi-definite to be precise), then it is a (possibly local) minimum. If the function is also convex, then this condition implies that we are at a global optimum. See the supplementary section on Optimality Conditions.

1st order optimality condition

if \mathcal{L} is convex
+ $\nabla \mathcal{L}(\mathbf{w}) = 0$

\Rightarrow global optimality

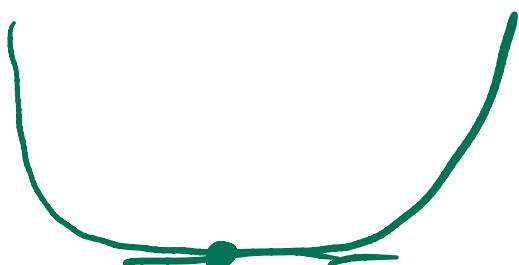
2nd-order optimality

if \mathcal{L} is potentially non-convex

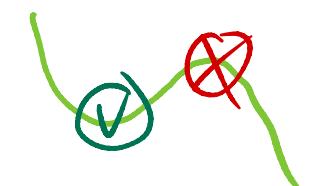
and $\nabla \mathcal{L} = 0$

$\nabla^2 \mathcal{L} > 0$
positive definite

\Rightarrow local minimum



$$\mathbf{x}^\top \nabla^2 \mathcal{L} \mathbf{x} > 0 \quad \forall \mathbf{x}$$



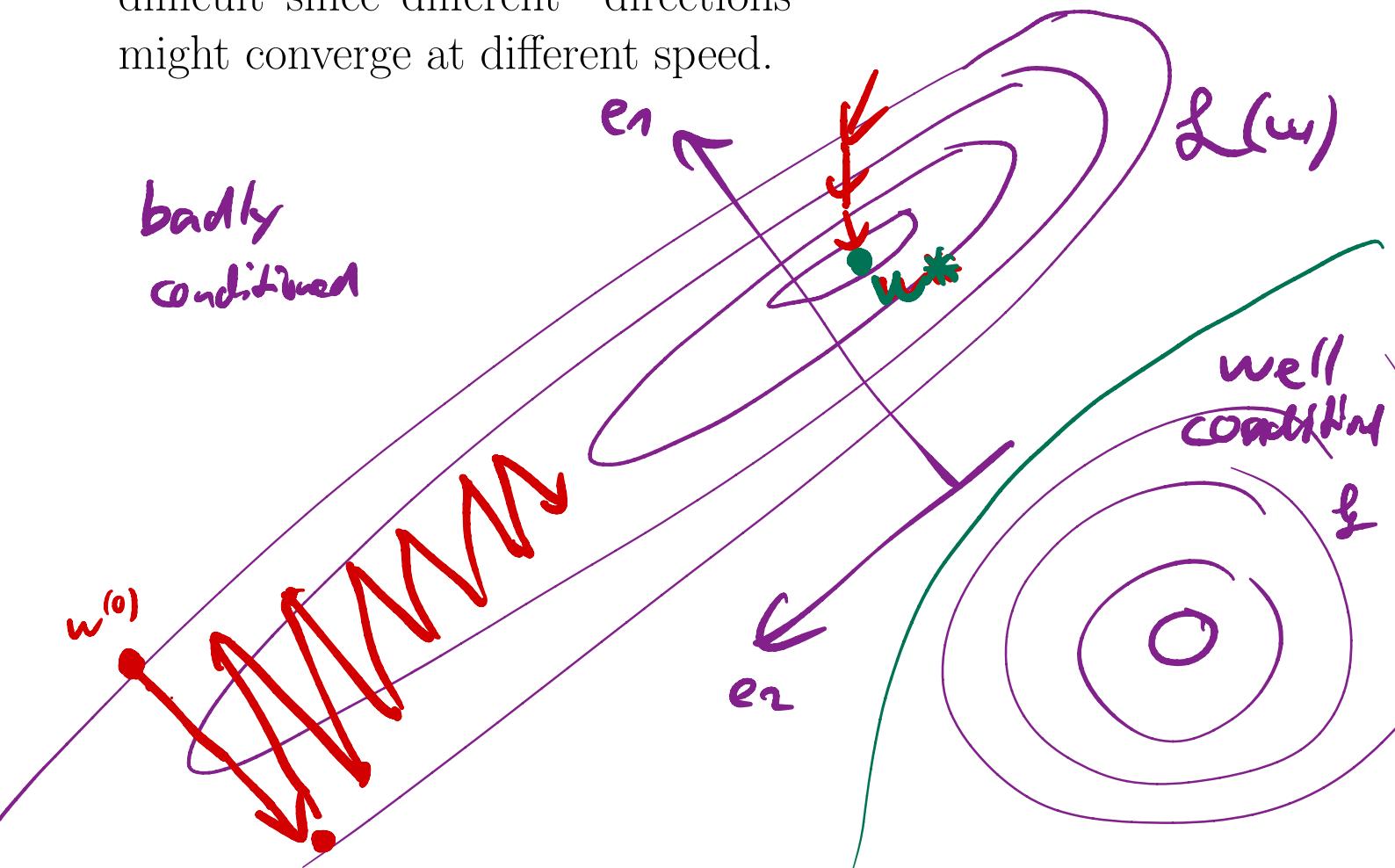
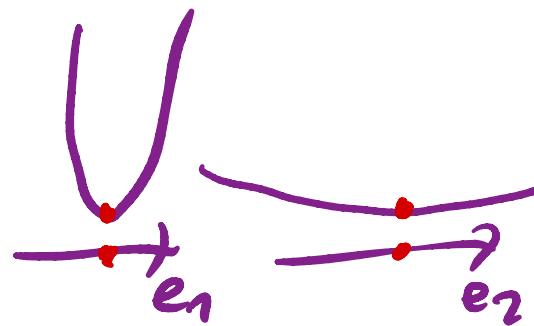
Adaptive step-sizes: Adam ...

Line-search methods: For some objectives \mathcal{L} , we can set step-size automatically using a line-search method. More details on “back-tracking” methods can be found in Chapter 1 of Bertsekas’ book on “nonlinear programming”.

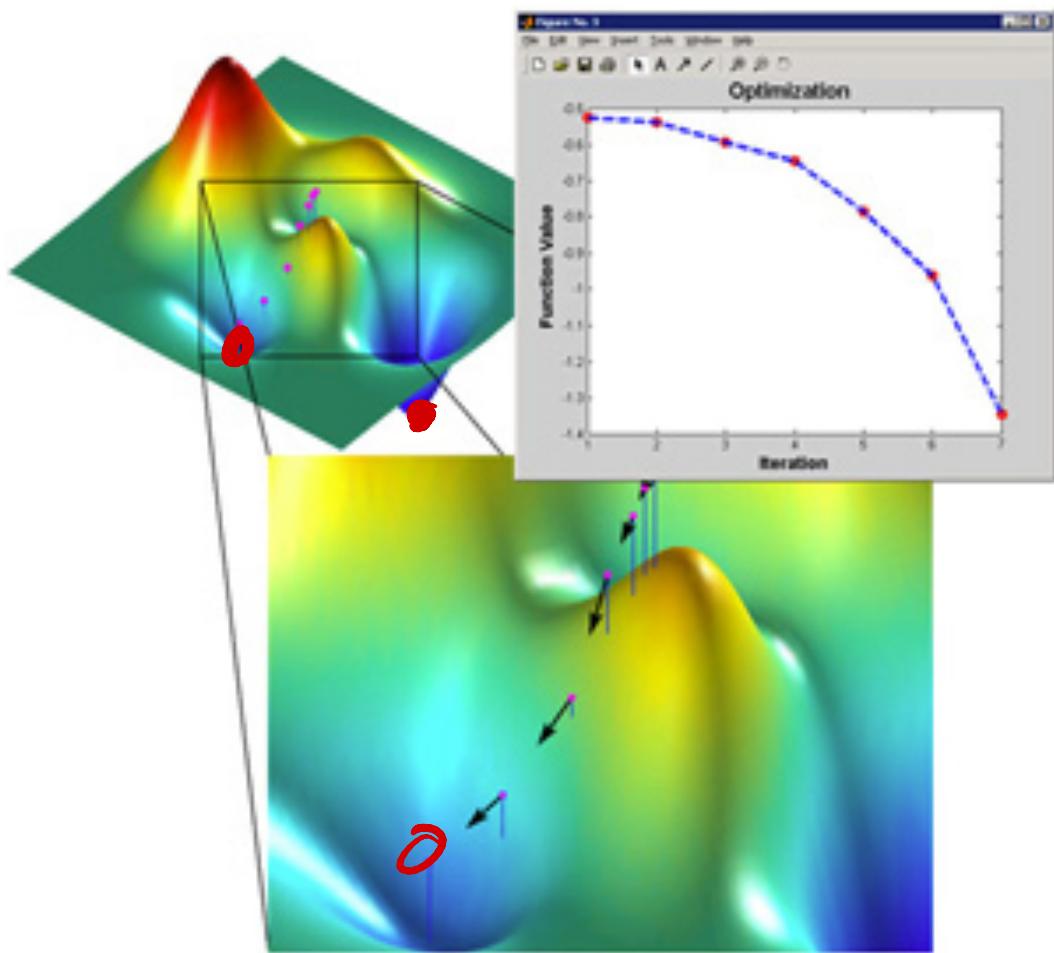
δ is different for each parameter w :

optimal δ at w

Feature normalization and pre-conditioning: Gradient descent is very sensitive to ill-conditioning. Therefore, it is typically advised to normalize your input features. In other words, we pre-condition the optimization problem. Without this, step-size selection is more difficult since different “directions” might converge at different speed.



Non-Convex Optimization



*image from mathworks.com

Real-world problems are **not convex**!

All we have learnt on algorithm design and performance of convex algorithms still helps us in the non-convex world.

Additional Notes

Grid Search and Hyper-Parameter Optimization

Read more about grid search and other methods for “hyperparameter” setting:

en.wikipedia.org/wiki/Hyperparameter_optimization#Grid_search.

Computational Complexity

The computation cost is expressed using the [big-O](#) notation. Here is a definition taken from Wikipedia. Let f and g be two functions defined on some subset of the real numbers. We write $f(x) = \mathcal{O}(g(x))$ as $x \rightarrow \infty$, if and only if there exists a positive real number c and a real number x_0 such that $|f(x)| \leq c|g(x)|$, $\forall x > x_0$.

Please read and learn more from this page in Wikipedia:

en.wikipedia.org/wiki/Computational_complexity_of_mathematical_operations#Matrix_algebra .

- What is the computational complexity of matrix multiplication?
- What is the computational complexity of matrix-vector multiplication?

Optimality Conditions

For a *convex* optimization problem, the first-order *necessary* condition says that at *an* optimum the gradient is equal to zero.

$$\nabla \mathcal{L}(\mathbf{w}^*) = \mathbf{0} \quad (1)$$

The second-order *sufficient* condition ensures that the optimum is a minimum (not a maximum or saddle-point) using the [Hessian](#) matrix,

which is the matrix of second derivatives:

$$\mathbf{H}(\mathbf{w}^*) := \frac{\partial^2 \mathcal{L}(\mathbf{w}^*)}{\partial \mathbf{w} \partial \mathbf{w}^\top} \quad \text{is positive semi-definite.} \quad (2)$$

The Hessian is also related to the convexity of a function: a twice-differentiable function is convex if and only if the Hessian is positive semi-definite at all points.

SGD Theory

As we have seen above, when N is large, choosing a random training example (\mathbf{x}_n, y_n) and taking an SGD step is advantageous:

$$\mathbf{w}^{(t+1)} := \mathbf{w}^{(t)} - \gamma^{(t)} \nabla \mathcal{L}_n(\mathbf{w}^{(t)})$$

For convergence, $\gamma^{(t)} \rightarrow 0$ “appropriately”. One such condition called the Robbins-Monroe condition suggests to take $\gamma^{(t)}$ such that:

$$\sum_{t=1}^{\infty} \gamma^{(t)} = \infty, \quad \sum_{t=1}^{\infty} (\gamma^{(t)})^2 < \infty \quad (3)$$

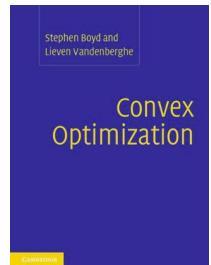
One way to obtain such sequences is $\gamma^{(t)} := 1/(t+1)^r$ where $r \in (0.5, 1)$.

More Optimization Theory

If you want, you can gain a deeper understanding of several optimization methods relevant for machine learning from this survey:

Convex Optimization: Algorithms and Complexity

- by Sébastien Bubeck



And also from the book of Boyd & Vandenberghe
(both are free online PDFs)

Exercises

1. Chain-rule



If it has been a while, familiarize yourself with it again.

2. Revise computational complexity (also see the Wikipedia link in Page 6 of lecture notes).
3. Derive the computational complexity of grid-search, gradient descent and stochastic gradient descent for linear MSE (# steps and cost per step).
4. Derive the gradients for the linear MSE and MAE cost functions.
5. Implement gradient descent and gain experience in setting the step-size.
6. Implement SGD and gain experience in setting the step-size.