QUICK ESTIMATION OF $\hat{M}$, $\hat{P}$, AND $\hat{N}$

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From before, the analytical solutions for $\hat{M}, \hat{N}|\hat{P}$ and $\hat{P}, \hat{N}|\hat{M}$ can be written as:

$$\hat{M'}^t = \left(\sum_i w_i S_i^P S_i^{P\,t}\right)^{-1} \left(\sum_i w_i S_i^P I_i^t\right) \tag{1}$$

$$\hat{P'}^t = \left(\sum_i w_i S_i^{M\,t} S_i^M\right)^{-1} \left(\sum_i w_i S_i^{M\,t} I_i\right) \tag{2}$$

where

$$\begin{aligned}
M' &= \begin{pmatrix} M & N \end{pmatrix} \\
P' &= \begin{pmatrix} P \\ N \end{pmatrix} \\
S_i^M &= \begin{pmatrix} \lambda_i M S_i & \mathrm{I_D} \end{pmatrix} \\
S_i^P &= \begin{pmatrix} \lambda_i S_i P \\ \mathrm{I_D} \end{pmatrix}
\end{aligned}$$

By breaking the matrices of each analytical solution into components, the summation over all clusters can be removed. Taking equation (1) for example:

$$\begin{aligned}
\sum_i w_i S_i^P S_i^{P\,t} &= \begin{pmatrix} \sum_i w_i \lambda_i^2 S_i P P^t S_i^t & \sum_i w_i \lambda_i S_i P \\ \sum_i w_i \lambda_i P^t S_i^t & \sum_i w_i \mathrm{I_D} \end{pmatrix} \\
&= \begin{pmatrix} \sum_i w_i \lambda_i^2 S_i P P^t S_i^t & \bar{S} P \\ P^t \bar{S}^t & \bar{w} \mathrm{I_D} \end{pmatrix} \\
\sum_i w_i S_i^P I_i^t &= \begin{pmatrix} \sum_i w_i \lambda_i S_i P I_i^t \\ \sum_i w_i I_i^t \end{pmatrix} \\
&= \begin{pmatrix} \sum_i w_i \lambda_i S_i P I_i^t \\ \bar{I}^t \end{pmatrix}
\end{aligned}$$

where $\bar{S} = \sum_i w_i \lambda_i S_i$, $\bar{w} = \sum w_i$ and $\bar{I} = \sum_i w_i I_i$. There are two sums left in the components of the equation. From Minka we have the following identity:

$$\mathrm{Vec}\,(ABC) = \left(C^t \otimes A\right) \mathrm{Vec}(B)$$

so

$$
\begin{aligned}
\mathrm{Vec}\left(\sum_i w_i \lambda_i^2 S_i P P^t S_i^t\right) &= \left[\sum_i w_i \lambda_i^2 \left(S_i \otimes S_i\right)\right] \mathrm{Vec}\left(P P^t\right) \\
&= J \mathrm{Vec}\left(P P^t\right) \quad \text{by appropriate definition of } J. \\
\mathrm{Vec}\left(\sum_i w_i \lambda_i S_i P I_i^t\right) &= \left[\sum_i w_i \lambda_i \left(I_i \otimes S_i\right)\right] \mathrm{Vec}\left(P\right) \\
&= K \mathrm{Vec}\left(P\right) \quad \text{by appropriate definition of } K.
\end{aligned}
$$

Defining the Reshape operator to be the inverse of Vec, the summation free matrices are:

$$
\begin{aligned}
\sum_i w_i S_i^P S_i^{P\,t} &= \begin{pmatrix} \mathrm{Reshape}\left(J\mathrm{Vec}(PP^t)\right) & \bar{S}P \\ P^t \bar{S}^t & \bar{w}\mathrm{I_D} \end{pmatrix} \\
\sum_i w_i S_i^P I_i^t &= \begin{pmatrix} \mathrm{Reshape}\left(K\mathrm{Vec}(P)\right) \\ \bar{I}^t \end{pmatrix}
\end{aligned}
$$

Similar solutions hold for equation (2):

$$
\begin{aligned}
\sum_i w_i S_i^{M\,t} S_i^M &= \begin{pmatrix} \mathrm{Reshape}\left(J^t\mathrm{Vec}(M^tM)\right) & \bar{S}^t M^t \\ M\bar{S} & \bar{w}\mathrm{I_D} \end{pmatrix} \\
\sum_i w_i S_i^{M\,t} I_i &= \begin{pmatrix} \mathrm{Reshape}\left(K^t\mathrm{Vec}(M^t)\right) \\ \bar{I} \end{pmatrix}
\end{aligned}
$$