4

Markov Chain and Bayesian Filtering

In the context of embodied AI, where robots engage in real-time interactions with their environment, the intricacies of dynamic systems come to the forefront. Unlike computer vision and simple AI, embodied AI involves robots that not only process information but also move and operate in real-time scenarios. This paradigm shift necessitates a deeper exploration of the role of filters, such as the Kalman filter, within the dynamics of robotics, especially considering the Markov assumption and Markov chain.

Let's explore this perspective with a scenario involving a mobile robot exploring an environment. The robot's location is measured through sensors, and its dynamic model and measurement model are crucial components in this process. Consider a mobile robot exploring an environment, measuring its location through sensors. The robot's position is obtained either by directly measuring its distance to specific objects within the environment or by external observations using sensors. Regardless of the method, let's assume we have a measurement of the robot's position.

Given that the robot operates in real-time, continuously measuring its surroundings at the system's frame rate, the dynamic nature of its motion introduces a time-dependent parameter to the estimation problem. Unlike scenarios where a stationary robot's position or the location of a static structure is measured, the key difference lies in the fact that the quantity to be estimated continually changes over time. In the case of a mobile robot, this evolving parameter is influenced by the robot's motion model. For instance, let's assume we know the robot's position and motion model at time steps 1 to t. Intuitively, for sufficiently short time intervals, the robot's position at time $t+1$ would be determined by its position and motion model at times 1 to t. Graphically representing this relationship yields a node and edge graph. However, in the context of a real-time, continuously operating mobile robot, the accumulation of the robot's sequence over time results in a significant computational drawback. To mitigate this, we introduce the Markov assumption to the mobile robot's sequence. Simply put, in a dynamic system, the state at time t is assumed to depend only on the state at time $t-1$, making the system Markovian. This assumption implies that, for a sufficiently short time interval, knowing the robot's position and motion model at time $t-1$ allows us to estimate its position at time t. This recursive application extends to states at times $t-1$ and $t-2$ and is graphically represented as a Markov chain.

The Markov assumption, applied to the mobile robot's sequence, alleviates the computational burden, enabling more efficient state estimation. This sequential calculation can be obtained through recursive calculations, both in real-time and in a post-processing manner after acquiring all the data. This narrative sheds light on the significance of the Markov assumption and Markov chain within the embodied AI framework, emphasizing the role of filters like the Kalman filter in addressing the challenges posed by real-time, dynamic robotic systems.

4.1 Sequential Measurement and Posterior

Recall that the Posterior can be represented with the prior and the likelihood as the following:

$$\begin{aligned} p\left(A|\boldsymbol{x}\right) &= \frac{p\left(\boldsymbol{x}|A\right)p\left(A\right)}{p\left(\boldsymbol{x}\right)} \\ &= \frac{p\left(\boldsymbol{x}|A\right)p\left(A\right)}{\int p\left(\boldsymbol{x}|A\right)p\left(A\right)dA}. \end{aligned} \tag{4.1}$$

From now, we let z be the value of true location, instead of A. Since we have the series of observations $\boldsymbol{x} = \{x_0, ..., x_t\}$, we can rewrite Eqn. (4.1) as follows:

$$\begin{aligned} p\left(z|\boldsymbol{x}\right) &= p\left(z|x_0, ..., x_t\right) \\ &= \frac{p\left(x_t|z, x_0, ..., x_{t-1}\right)p\left(z|x_0, ..., x_{t-1}\right)}{\int p\left(x_t|z, x_0, ..., x_{t-1}\right)p\left(z|x_0, ..., x_{t-1}\right)dz}. \end{aligned} \tag{4.2}$$

For simplicity, from now let $x_{0:t-1} = x_0, ..., x_{t-1}$. Since the measurement x_t is generated solely from z as shown in Fig. 4.1, we have:

$$p\left(x_t|z, x_0, ..., x_{t-1}\right) = p\left(x_t|z\right). \tag{4.3}$$

Then we can rewrite Eqn. (4.2) as:

$$p\left(z|x_{0:t}\right) = \frac{p\left(x_t|z\right)p\left(z|x_{0:t-1}\right)}{\int p\left(x_t|z\right)p\left(z|x_{0:t-1}\right)dz} \tag{4.4}$$

$$\propto \underbrace{p\left(x_t|z\right)}_{likelihood}\underbrace{p\left(z|x_{0:t-1}\right)}_{prior}. \tag{4.5}$$

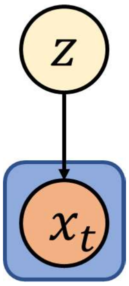

Figure 4.1: Graph of the true value z and the measurement x_t. Note that the measurement x_t is solely depend on z.

Thus, if we have *likelihood* and *prior*, then we can directly compute *posterior* by taking multiplication and integration in Eqn. (4.4). Here, the likelihood

$$p\left(x_t|z\right) \sim \mathcal{N}\left(z, \boldsymbol{R}\right) \tag{4.6}$$

denotes the measurement model in usual, and it can equivalently be represented as:

$$x_t = z + w, \quad \text{where } w \sim \mathcal{N}\left(\mathbf{0}, \boldsymbol{R}\right). \tag{4.7}$$

w denotes the measurement noise.

Note that the *prior* in Eqn. (4.5) can also be seen as the *posterior* in $(t-1)^{th}$ step:

$$p\left(z|x_{0:t-1}\right) \sim \mathcal{N}\left(\mu_{z|x_{t-1}}, \Sigma_{z|x_{t-1}}\right) \tag{4.8}$$

In other words, in Eqn. (4.5) we can use the previously computed *posterior* in $(t-1)^{th}$ time step as the *prior* in t^{th} time step. Therefore, if we obtain a new measurement (*likelihood*), then using the previous *posterior* (as the current *prior*) we can compute our desired *posterior*.

Consequently, assuming that we already know the posterior $p\left(z|x_{0:t-1}\right)$ derived from the previous step, the target posterior $p\left(z|x_{0:t}\right)$ can be directly computed from Eqn. (4.4). However, rather than computing multiplication and integration, it is way more easier to consider the joint PDF $p\left(z, x_t|x_{0:t-1}\right)$ for finding the posterior $p\left(z|x_{0:t}\right)$.

Consider the joint probability of two random variables z, x_t:

$$p\left(z, x_t|x_{0:t-1}\right) \sim \mathcal{N}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right) \tag{4.9}$$

where

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_z \\ \mu_{x_t} \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} \Sigma_{z,z} & \Sigma_{z,x_t} \\ \Sigma_{x_t,z} & \Sigma_{x_t,x_t} \end{bmatrix}. \tag{4.10}$$

The conditional PDF of this joint PDF is

$$p\left(z|x_t, x_{0:t-1}\right) = p\left(z|x_{0:t}\right), \tag{4.11}$$

which is the *posterior* that we need to find.
Recall that from the Conditional Gaussian PDF formula in Eqn. (2.34) and Eqn. (4.14). For the arbitrary random variables $\boldsymbol{Z}$ and $\boldsymbol{X}$ as an example, we have:

$$\boldsymbol{Z}|\boldsymbol{X} \sim \mathcal{N}\left(\mathbb{E}\left(\boldsymbol{Z}|\boldsymbol{X}\right), \boldsymbol{C}_{\boldsymbol{Z}|\boldsymbol{X}}\right) \tag{4.12}$$

where

$$\mathbb{E}\left(\boldsymbol{Z}|\boldsymbol{X}\right) = \mathbb{E}\left(\boldsymbol{Z}\right) + \boldsymbol{C}_{\boldsymbol{ZX}}\boldsymbol{C}_{\boldsymbol{XX}}^{-1}\left(\boldsymbol{X} - \mathbb{E}\left(\boldsymbol{X}\right)\right) \tag{4.13}$$

$$\boldsymbol{C}_{\boldsymbol{Z}|\boldsymbol{X}} = \boldsymbol{C}_{\boldsymbol{ZZ}} - \boldsymbol{C}_{\boldsymbol{ZX}}\boldsymbol{C}_{\boldsymbol{XX}}^{-1}\boldsymbol{C}_{\boldsymbol{ZX}}. \tag{4.14}$$

In our case, $\boldsymbol{X} = x_t$ and $\boldsymbol{Z} = z$. Therefore, by following those equations, the conditional PDF (=*posterior*) $p\left(z|x_t, x_{0:t-1}\right) = p\left(z|x_{0:t}\right) \sim \mathcal{N}\left(\mu_{z|x_t}, \Sigma_{z|x_t}\right)$ can be given as:

$$\mu_{z|x_t} = \mu_z + \Sigma_{z,x_t}\Sigma_{x_t,x_t}^{-1}\left(x_t - \mu_{x_t}\right) \tag{4.15}$$

$$\Sigma_{z|x_t} = \Sigma_{z,z} - \Sigma_{z,x_t}\Sigma_{x_t,x_t}^{-1}\Sigma_{x_t,z}. \tag{4.16}$$

Note that we do not need direct computation as Eqn. (4.4) no more, since we can obtain the *posterior* from Eqn. (4.15) and Eqn. (4.16), which can be computed by matrix operation.

Now, let us compute Eqn. (4.15) and Eqn. (4.16). Apparently, μ_z and $\Sigma_{z,z}$ are the parameters of the previous posterior $p\left(z|x_{0:t-1}\right)$, we simply put:

$$\mu_z = \mu_{z|x_{t-1}} \tag{4.17}$$

$$\Sigma_{z,z} = \Sigma_{z|x_{t-1}}. \tag{4.18}$$

For the other parameters, we have:

$$\begin{aligned}
\mu_{x_t} &= \mathbb{E}\left(x_t|x_{0:t-1}\right) \\
&= \mathbb{E}\left(z+w|x_{0:t-1}\right) \\
&= \mathbb{E}\left(z|x_{0:t-1}\right) = \mu_{z|x_{t-1}} && (4.19) \\
\Sigma_{x_t,x_t} &= \mathbb{E}\left[\left(x_t-\mu_{x_t}\right)^T\left(x_t-\mu_{x_t}\right)|x_{0:t-1}\right] \\
&= \mathbb{E}\left[\left(z+w-\mu_{x_t}\right)^T\left(z+w-\mu_{x_t}\right)|x_{0:t-1}\right] \\
&= \mathbb{E}\left[\left(z+w-\mu_{z|x_{t-1}}\right)^T\left(z+w-\mu_{z|x_{t-1}}\right)|x_{0:t-1}\right] \\
&= \Sigma_{z|x_{t-1}} + \boldsymbol{R} && (4.20) \\
\Sigma_{z,x_t} &= \mathbb{E}\left[\left(z-\mu_z\right)^T\left(x_t-\mu_{x_t}\right)|x_{0:t-1}\right] \\
&= \mathbb{E}\left[\left(z-\mu_z\right)^T\left(z+w-\mu_{x_t}\right)|x_{0:t-1}\right] \\
&= \mathbb{E}\left[\left(z-\mu_{z|x_{t-1}}\right)^T\left(z+w-\mu_{z|x_{t-1}}\right)|x_{0:t-1}\right] \\
&= \Sigma_{z|x_{t-1}}. && (4.21)
\end{aligned}$$

Using these parameters, we can rewrite Eqn. (4.15) and Eqn. (4.16) for the updated posterior $p\left(z|x_{0:t}\right)$ as:

$$\mu_{z|x_t} = \mu_{z|x_{t-1}} + \Sigma_{z|x_{t-1}}^T\left(\Sigma_{z|x_{t-1}} + \boldsymbol{R}\right)^{-1}\left(x_t-\mu_{z|x_{t-1}}\right) \tag{4.22}$$

$$\Sigma_{z|x_t} = \Sigma_{z|x_{t-1}} - \Sigma_{z|x_{t-1}}\left(\Sigma_{z|x_{t-1}} + \boldsymbol{R}\right)^{-1}\Sigma_{z|x_{t-1}}. \tag{4.23}$$

Consequently, when the additional measurement x_t is added, we can update the previous posterior $p\left(z|x_{0:t-1}\right)$ to $p\left(z|x_{0:t}\right)$ simply by updating the parameters.

4.2 Markov Chain and Dynamic Model

Up until now, we have only considered scenarios where the robot is in a static state, without accounting for cases involving robot mobility or considering the dynamics of the robot. The basic essential of the mobile robot is that the robot moves around, and now it is time that we should handle the dynamics of the robot as well. Recall that the posterior for the measurement model is given as the following:

$$p\left(z|x_{0:t}\right) = \frac{p\left(x_t|z\right)p\left(z|x_{0:t-1}\right)}{\int p\left(x_t|z\right)p\left(z|x_{0:t-1}\right)dz}.$$

Since we assume that the robot has no dynamics, apparently the robot‘s location z remains constant according to t. Now suppose the robot moves around and measures its location at the same time. In this case the posterior for t'th measurement can be rewritten as:

$$\begin{aligned}\underbrace{p\left(z_t|x_{0:t}\right)}_{\text{posterior}} &= \frac{p\left(x_t|z_t\right)p\left(z_t|x_{0:t-1}\right)}{\int p\left(x_t|z_t\right)p\left(z_t|x_{0:t-1}\right)dz} \\ &= \frac{1}{Z}\underbrace{p\left(x_t|z_t\right)}_{\text{likelihood}}\underbrace{p\left(z_t|x_{0:t-1}\right)}_{\text{prior}}\end{aligned}$$

where Z is a normalizing constant. Similar to the case of the robot without any dynamics, our goal is to estimate the t^{th} posterior $p\left(z_t|x_{0:t}\right)$ from $t-1^{th}$ posterior $p\left(z_{t-1}|x_{0:t-1}\right)$ and the measurement x_t. In order to achieve this, in the above equation first we need the likelihood

$$p\left(x_t|z_t\right) \sim \mathcal{N}\left(z_t, \boldsymbol{R}\right) \tag{4.24}$$

that can be rewritten as:

$$x_t = z_t + w \tag{4.25}$$

where $w \sim \mathcal{N}\left(0, \boldsymbol{R}\right)$ is the measurement noise. Similarly, the prior is expressed as a Gaussian:

$$p\left(z_t|x_{0:t-1}\right) \sim \mathcal{N}\left(\mu_{z_t|x_{t-1}}, \Sigma_{z_t|x_{t-1}}\right). \tag{4.26}$$

Fortunately, the likelihood represents the measurement model depending on the sensor noise w, thereby we assume that it does not depend on the time step t. Compared to the case in previous chapter, the only difference is the prior $p\left(z_t|x_{0:t-1}\right)$, which was previously known from the previous step. However, the only thing that we have now from the previous step is $t-1^{th}$ posterior

$$p\left(z_{t-1}|x_{0:t-1}\right) \sim \mathcal{N}\left(\mu_{z_{t-1}|x_{t-1}}, \Sigma_{z_{t-1}|x_{t-1}}\right), \tag{4.27}$$

which is different from $p\left(z_t|x_{0:t-1}\right)$, because z_t is now time-varying. Therefore, we cannot directly use the previously computed *posterior* as the current *prior*. We need more step to figure out the prior $p\left(z_t|x_{0:t-1}\right)$ from $t-1^{th}$ posterior $p\left(z_{t-1}|x_{0:t-1}\right)$.
In order to find the relation between $p\left(z_t|x_{0:t-1}\right)$ and $p\left(z_{t-1}|x_{0:t-1}\right)$, we can rewrite $p\left(z_t|x_{0:t-1}\right)$ as the following:

$$p\left(z_t|x_{0:t-1}\right) = \int p\left(z_t|z_{t-1}, x_{0:t-1}\right) \quad \underbrace{p\left(z_{t-1}|x_{0:t-1}\right)}_{\text{posterior from } t-1^{th} \text{ step}} \quad dz_{t-1}. \tag{4.28}$$

Note that $p\left(z_{t-1}|x_{0:t-1}\right)$ is the *posterior* computed from the previous step, thus we now can use our previously computed *posterior* here. Then how

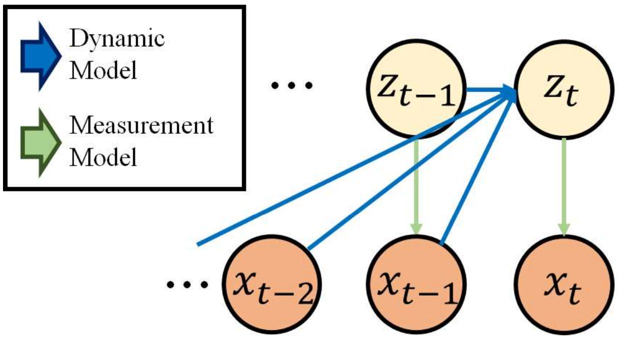

Figure 4.2: Graph of the true value z and the measurement x. Since the dynamic model involves the all measurements, the graph cannot be represented in simple structure.

about $p\left(z_t|z_{t-1}, x_{0:t-1}\right)$?

Dynamic Model

In Eqn. (4.28), $p\left(z_t|z_{t-1}, x_{0:t-1}\right)$ is called as a *dynamic model*. This PDF describes the location z_t at t^{th} time step according to the previous location z_{t-1} and the observations $x_{0:t-1}$. This relation is depicted in Fig. 4.2. Unfortunately, this relation is somewhat complex, thus it is challenging to derive the simple dynamic model. To simplify our dynamic model, we adopt *Markov assumption*. That is, the current state of robot is assumed to be determined only by its previous state. In other words, the current robot state is determined not by measurements, but by the previous robot state. The example of this Markov assumption is displayed in Fig. 4.3. Note that now the graph is represented as a chain, thus we call it as Markov Chain.
Under this Markov assumption the dynamic model $p\left(z_t|z_{t-1}, x_{0:t-1}\right)$ can be simplified to $p\left(z_t|z_{t-1}\right)$, therefore we finally have:

$$p\left(z_t|x_{0:t-1}\right) = \int \underbrace{p\left(z_t|z_{t-1}\right)}_{\text{dynamic model}} \underbrace{p\left(z_{t-1}|x_{0:t-1}\right)}_{\text{posterior from } t-1^{th} \text{ step}} dz_{t-1}. \tag{4.29}$$

Assume the dynamic model $p\left(z_t|z_{t-1}\right)$ follows Gaussian distribution, then the dynamic model can equivalently be represented as:

$$z_t = z_{t-1} + v \tag{4.30}$$

where $v \sim \mathcal{N}\left(0, \boldsymbol{Q}\right)$ is the noise of the dynamic model. Let's refer to it as the dynamic noise. This noise represents the uncertainty of our dynamic model, which apparently have gap between the real-world scenario.

The dynamic model presented above is exceedingly simplistic, representing a scenario where the robot's location z_t remains unchanged at time t,

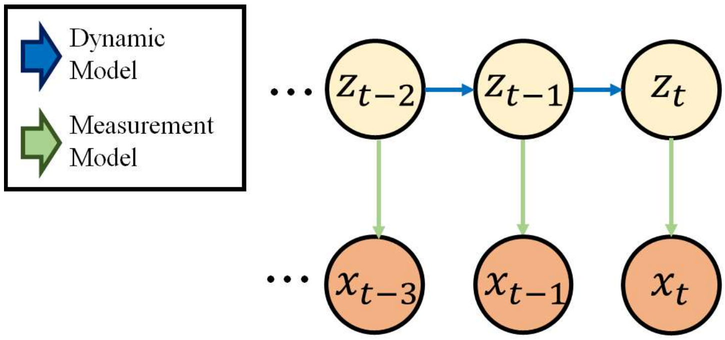

Figure 4.3: Markov chain of z and x. Since z_t is determined only by the previous robot state z_{t-1}, the dynamic model can be defined in simple form.

with only observation noise v varying over time. In other words, this dynamic model depicts a state where the robot is stationary. Now, let us explore how we can represent the dynamic model for a mobile robot with movement. Since a mobile robot moves based on the robot's control input u_{t-1} given at time $t-1$, the robot's dynamic model can be expressed as u_{t-1}. The control input for a robot can be provided through various means. For example, in a scenario where a person directly manipulates the robot, it can be defined as the manipulation input. In the case of a robot engaged in autonomous navigation, the control input is generated based on the robot's own path planning algorithm. For a mobile robot equipped with multiple sensors, in addition to sensors for environmental perception, the robot's movement can be inferred from measurements of the location obtained from robust sensors capable of recording the robot's trajectory. This type of measurement can also be utilized as odometry.

Odometry, in its literal sense, refers to a method of measuring the position of a moving object by recording the number of rotations through encoders and measuring the inclination using an device such as Inertial Measurement Unit (IMU). Incorporating this dynamic model u, the graph model of a mobile robot with a Markov chain can be expressed as follows. Let us designate both the control input and odometry as u from now on. Assume that at time $t-1$, the odometry u_{t-1} is given to the mobile robot. Then we can expect that within a very short period of time, this odometry remains approximately same during that time interval. That is, we can assume that the robot has a constant velocity model for its movement. To simplify the problem, let us assume that u_{t-1} is expressed in the same coordinate system as the one representing the robot's location. Then, for time t, our new dynamic model is expressed as follows:

$$z_t = z_{t-1} + u_{t-1} + v. \tag{4.31}$$

This dynamic model is also called as *prediction model*, since we expect the

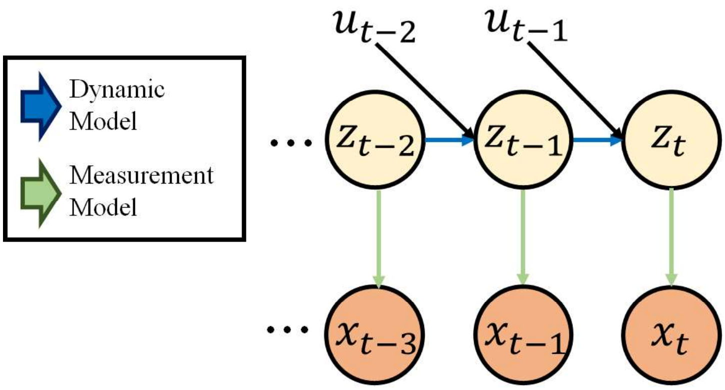

Figure 4.4: Markov chain involving the control input u. z_t is determined by the previous robot state z_{t-1} and the control input u_{t-1}.

robot location z_t is determined by the previous location z_{t-1} and our control input u_{t-1}. The graphical model including the control input u is displayed in Fig. 4.4.
Using the dynamic model, now we can find prior $p\left(z_t|x_{0:t-1}\right) \sim \mathcal{N}\left(\mu_{z_t|x_{t-1}}, \Sigma_{z_t|x_{t-1}}\right)$ by directly computing the parameters as follows:

$$\begin{aligned}
\mu_{z_t|x_{t-1}} &= \mathbb{E}\left(z_t|x_{0:t-1}\right) \\
&= \mathbb{E}\left(z_{t-1} + u_{t-1} + v|x_{0:t-1}\right) \\
&= \mu_{z_{t-1}|x_{t-1}} + u_{t-1} \qquad (4.32)\\
\Sigma_{z_t|x_{t-1}} &= \mathbb{E}\left[\left(z_t - \mu_{z_t|x_{t-1}}\right)\left(z_t - \mu_{z_t|x_{t-1}}\right)^T |x_{0:t-1}\right] \\
&= \mathbb{E}\left[\left(z_{t-1} + v - \mu_{z_{t-1}|x_{t-1}}\right)\left(z_{t-1} + v - \mu_{z_{t-1}|x_{t-1}}\right)^T |x_{0:t-1}\right] \\
&= \mathbb{E}\left[\left(z_{t-1} - \mu_{z_{t-1}|x_{t-1}} + v\right)\left(z_{t-1} - \mu_{z_{t-1}|x_{t-1}} + v\right)^T |x_{0:t-1}\right] \\
&= \Sigma_{z_{t-1}|x_{t-1}} + \boldsymbol{Q}. \qquad (4.33)
\end{aligned}$$

Note that we do not compute Eqn. (4.29), but compute the parameters of PDF, since it is much easier. The rest part remains same. By using the prior and the likelihood $p\left(x_t|z_t\right)$, we can perform the measurement update and finally have:

$$\mu_{z_t|x_t} = \mu_{z_t|x_{t-1}} + \Sigma^T_{z_t|x_{t-1}}\left(\Sigma_{z_t|x_{t-1}} + \boldsymbol{R}\right)^{-1}\left(x_t - \mu_{z_t|x_{t-1}}\right) \qquad (4.34)$$

$$\Sigma_{z_t|x_t} = \Sigma_{z_t|x_{t-1}} - \Sigma_{z_t|x_{t-1}}\left(\Sigma_{z_t|x_{t-1}} + \boldsymbol{R}\right)^{-1}\Sigma_{z_t|x_{Nt-1}}. \qquad (4.35)$$

Consequently, the sequential measurement with the dynamic model can be summarized as follows. For the robot location z_t, assume we have observations $x_{0:t}$ and the odometry $u_{0:t-1}$. our goal is to achieve the posterior of t^{th} step:

$$\begin{aligned}
\underbrace{p\left(z_t|x_{0:t}, u_{0:t-1}\right)}_{\text{posterior}} &= \frac{p\left(x_t|z_t, u_{0:t-1}\right) p\left(z_t|x_{0:t-1}, u_{0:t-1}\right)}{\int p\left(x_N|z_N, u_{0:t-1}\right) p\left(z_t|x_{0:t-1}, u_{0:t-1}\right) dz} \\
&= \frac{1}{Z} p\left(x_t|z_t, u_{0:t-1}\right) p\left(z_t|x_{0:t-1}, u_{0:t-1}\right) \\
&= \frac{1}{Z} \underbrace{p\left(x_t|z_t\right)}_{\text{likelihood (measurement model)}} \underbrace{p\left(z_t|x_{0:t-1}, u_{0:t-1}\right)}_{\text{prior}}.
\end{aligned}$$

where Z is a normalizing constant. Note that the relationships between each variable can be easily derived from the graph model. For the prior, we have the following step:

$$\underbrace{p\left(z_t|x_{0:t-1}, u_{0:t-1}\right)}_{\text{prior}} = \int \underbrace{p\left(z_t|z_{t-1}, u_{t-1}\right)}_{\text{dynamic model}} \underbrace{p\left(z_{t-1}|x_{0:t-1}, u_{0:t-2}\right)}_{\text{posterior from } t-1} dx_{t-1}.$$

For each model we have:
• Dynamic model:

$$z_t = z_{t-1} + u_{t-1} + v$$

• Measurement model:

$$x_t = z_t + w$$

where $v \sim \mathcal{N}\left(0, \boldsymbol{Q}\right)$ and $w \sim \mathcal{N}\left(0, \boldsymbol{R}\right)$ are the dynamic noise and the measurement noise. From the above models, we have
• Dynamic update:

$$\begin{aligned}
\mu_{z_t|x_{t-1}} &= \mu_{z_{t-1}|x_{t-1}} + u_{t-1} \\
\Sigma_{z_t|x_{t-1}} &= \Sigma_{z_{t-1}|x_{t-1}} + \boldsymbol{Q}
\end{aligned}$$

• Measurement update:

$$\begin{aligned}
\mu_{z_t|x_t} &= \mu_{z_t|x_{t-1}} + K_t \left(x_t - \mu_{z_t|x_{t-1}}\right) \\
\Sigma_{z_t|x_t} &= \Sigma_{z_t|x_{t-1}} - K_t \Sigma_{z_t|x_{t-1}}
\end{aligned}$$

where

$$K_t = \Sigma_{z_t|x_{t-1}} \left(\Sigma_{z_t|x_{t-1}} + \boldsymbol{R}\right)^{-1}.$$

Since we already know that the optimal solution of Bmse is equivalent to the mean value of the posterior, $\mu_{z_t|x_{t-1}}$ is directly utilized as the optimal robot location. By computing other parameters, we can sequentially estimate the optimal solution of the robot location while the robot collects measurement values.

4.3 Bayesian Linear Filter

Until now, we have assumed a simple dynamic model and measurement model. For instance, we assumed that the odometry u could be easily represented in the robot's coordinates. Unfortunately, in real-world scenarios, models are nonlinear and extremely complex. For example, let's consider that the robot's location z we aim to estimate is represented in a 3D Cartesian coordinate system. While it would be ideal for the robot's odometry u to also be expressed in Cartesian coordinates, in many cases, the robot's odometry is unfortunately represented in terms of linear velocity and angular acceleration. To represent this in Cartesian coordinates, nonlinear functions, such as trigonometric functions, need to be employed. Similarly, in the case of the measurement model, the sensor observations x for the robot's location z are often not expressed in the same Cartesian coordinates. Instead, they may be represented in terms of distance and angle. As a result, the relationships between various variables are no longer expressed as linear relationships but instead take on nonlinear forms.

Suppose the robot follows a complex dynamic model which can be represented by the function g. Similarly, assume the sensor model is given as the function h. With the Gaussian noise $v \sim \mathcal{N}(0, \boldsymbol{Q})$ and $w \sim \mathcal{N}(0, \boldsymbol{R})$, for the robot location z and the measurement x we have:

• Dynamic model:

$$z_t = g(z_{t-1}, u_{t-1}) + \boldsymbol{G}v$$

• Measurement model:

$$x_t = h(z_t) + w$$

where $v \sim \mathcal{N}(0, \boldsymbol{Q})$ and $w \sim \mathcal{N}(0, \boldsymbol{R})$ are the dynamic noise and the measurement noise.

These are equivalently represented as:

• Dynamic model:

$$p(z_t | x_{t-1}, u_{t-1}) = \mathcal{N}\left(g(z_{t-1}, u_{t-1}), \boldsymbol{G}\boldsymbol{Q}\boldsymbol{G}^T\right)$$

• Measurement model:

$$p(x_t | z_t) = \mathcal{N}(h(z_t), \boldsymbol{R}).$$

Unlike the previous simple example, in the Dynamic model even the random variable z_{t-1} is Gaussian we cannot tell z_t is also a Gaussian random variable, because g is nonlinear. Using the same logic, in the measurement model we cannot conclusively assert that x_t is also Gaussian, even if z_t is a Gaussian random variable.

To simplify our discussion, assume that we have the approximated *linear*

models represented as follows:
• Dynamic model:

$$\begin{aligned} z_t &= g\left(z_{t-1}, u_{t-1}\right) + \boldsymbol{G}v \\ &\simeq \boldsymbol{A}z_{t-1} + \boldsymbol{B}u_{t-1} + \boldsymbol{G}v \end{aligned}$$

• Measurement model:

$$\begin{aligned} x_t &= h\left(z_t\right) + w \\ &\simeq \boldsymbol{C}z_t + w \end{aligned}$$

First, for the dynamic update, we find the parameters of prior $p\left(z_t|x_{0:t-1}, u_{0:t-1}\right) \sim \mathcal{N}\left(\mu_{z_t|x_{t-1}}, \Sigma_{z_t|x_{t-1}}\right)$ as follows:

$$\begin{aligned} \mu_{z_t|x_{t-1}} &= \mathbb{E}\left(z_t|x_{0:t-1}, u_{0:t-1}\right) \\ &= \mathbb{E}\left(\boldsymbol{A}z_{t-1} + \boldsymbol{B}u_{t-1} + \boldsymbol{G}v|x_{0:t-1}, u_{0:t-1}\right) \\ &= \boldsymbol{A}\mu_{z_{t-1}|x_{t-1}} + \boldsymbol{B}u_{t-1} & (4.36) \\ \Sigma_{z_t|x_{t-1}} &= \mathbb{E}\left[\left(z_t - \mu_{z_t|x_{t-1}}\right)\left(z_t - \mu_{z_t|x_{t-1}}\right)^T |x_{0:t-1}, u_{0:t-1}\right] \\ &= \mathbb{E}\left[\left(\boldsymbol{A}z_{t-1} + \boldsymbol{G}v - \boldsymbol{A}\mu_{z_{t-1}|x_{t-1}}\right)\left(\boldsymbol{A}z_{t-1} + \boldsymbol{G}v - \boldsymbol{A}\mu_{z_{t-1}|x_{t-1}}\right)^T |\cdot\right] \\ &= \mathbb{E}\left[\left(\boldsymbol{A}z_{t-1} - \boldsymbol{A}\mu_{z_{t-1}|x_{t-1}} + \boldsymbol{G}v\right)\left(\boldsymbol{A}z_{t-1} - \boldsymbol{A}\mu_{z_{t-1}|x_{t-1}} + \boldsymbol{G}v\right)^T |\cdot\right] \\ &= \boldsymbol{A}\Sigma_{z_{t-1}|x_{t-1}}\boldsymbol{A}^T + \boldsymbol{G}\boldsymbol{Q}\boldsymbol{G}^T. & (4.37) \end{aligned}$$

For the measurement update, we compute the mean and covariance of the posterior $p\left(z_t|x_{0:t}, u_{0:t-1}\right) \sim \mathcal{N}\left(\mu_{z_t|x_t}, \Sigma_{z_t|x_t}\right)$ by using the computed prior $p\left(z_t|x_{0:t-1}, u_{0:t-1}\right)$ as follows:

$$\begin{aligned} \mu_{z_t|x_t} &= \mu_{z_t|x_{t-1}} + \Sigma_{z_t,x_t}\Sigma_{x_t,x_t}^{-1}\left(x_t - \mu_{x_t}\right) \\ \Sigma_{z_t|x_t} &= \Sigma_{z_t|x_{t-1}} - \Sigma_{z_t,x_t}\Sigma_{x_t,x_t}^{-1}\Sigma_{x_t,z_t}. & (4.38) \end{aligned}$$

Here, the parameters can be computed as follows:

$$
\begin{aligned}
\mu_{x_t} &= \mathbb{E}\left(x_t|x_{0:t-1}\right) \\
&= \mathbb{E}\left(\boldsymbol{C}z_t + w|x_{0:t-1}\right) \\
&= \mathbb{E}\left(\boldsymbol{C}z_t|x_{0:t-1}\right) = \boldsymbol{C}\mu_{z_t|x_{t-1}} \qquad (4.39)\\
\Sigma_{x_t,x_t} &= \mathbb{E}\left[\left(x_t - \mu_{x_t}\right)\left(x_t - \mu_{x_t}\right)^T|x_{0:t-1}\right] \\
&= \mathbb{E}\left[\left(\boldsymbol{C}z_t + w - \boldsymbol{C}\mu_{z_t|x_{t-1}}\right)\left(\boldsymbol{C}z_t + w - \boldsymbol{C}\mu_{z_t|x_{t-1}}\right)^T|x_{0:t-1}\right] \\
&= \boldsymbol{C}\Sigma_{z_t|x_{t-1}}\boldsymbol{C}^T + \boldsymbol{R} \qquad (4.40)\\
\Sigma_{z_t,x_t} &= \mathbb{E}\left[\left(z_t - \mu_{z|x_{t-1}}\right)\left(x_t - \mu_{x_t}\right)^T|x_{0:t-1}\right] \\
&= \mathbb{E}\left[\left(z - \mu_{z|x_{t-1}}\right)\left(\boldsymbol{C}z + w - \boldsymbol{C}\mu_{z|x_{t-1}}\right)^T|x_{0:t-1}\right] \\
&= \Sigma_{z|x_{t-1}}\boldsymbol{C}^T. \qquad (4.41)
\end{aligned}
$$

In summary, for the linear models the update process can be represented as follows:

• Dynamic update:

$$
\begin{aligned}
\mu_{z_t|x_{t-1}} &= \boldsymbol{A}\mu_{z_{t-1}|x_{t-1}} + \boldsymbol{B}u_{t-1} \\
\Sigma_{z_t|x_{t-1}} &= \boldsymbol{A}\Sigma_{z_{t-1}|x_{t-1}}\boldsymbol{A}^T + \boldsymbol{GQG}^T
\end{aligned}
$$

• Measurement update:

$$
\begin{aligned}
\mu_{z_t|x_t} &= \mu_{z_t|x_{t-1}} + K_t\left(x_t - \boldsymbol{C}\mu_{z_t|x_{t-1}}\right) \\
\Sigma_{z_t|x_t} &= \Sigma_{z_t|x_{t-1}} - K_t\boldsymbol{C}\Sigma_{z_t|x_{t-1}}
\end{aligned}
$$

where

$$
K_t = \Sigma_{z_t|x_{t-1}}\boldsymbol{C}^T\left(\boldsymbol{C}\Sigma_{z_t|x_{t-1}}\boldsymbol{C}^T + \boldsymbol{R}\right)^{-1}.
$$

The algorithm of this linear Bayesian Filter is shown in Algorithm. 1. In summary, to derive filters for dynamic and measurement models that actually follow nonlinear systems, we made several assumptions. Firstly, we assumed linear models, and secondly, we assumed Gaussian noise. Based on these two robust assumptions, the filtering technique is known as the Kalman Filter, a type of Bayesian linear filter.

Even though this filter assumes linear models, it proves useful for modeling simple robot dynamics. Let's explore how we can represent various types of dynamic models as linear models through the following examples.

Algorithm 1 Linear Kalman Filter

Require: x_t, u_{t-1}, $\mu_{z_{t-1}|x_{t-1}}$, $\Sigma_{z_{t-1}|x_{t-1}}$

[*Dynamic Update*]

1: $\mu_{z_t|x_{t-1}} = \boldsymbol{A}\mu_{z_{t-1}|x_{t-1}} + \boldsymbol{B}u_{t-1}$

2: $\Sigma_{z_t|x_{t-1}} = \boldsymbol{A}\Sigma_{z_{t-1}|x_{t-1}}\boldsymbol{A}^T + \boldsymbol{GQG}^T$

[*Measurement Update*]

3: $K_t = \Sigma_{z_t|x_{t-1}}\boldsymbol{C}^T\left(\boldsymbol{C}\Sigma_{z_t|x_{t-1}}\boldsymbol{C}^T + \boldsymbol{R}\right)^{-1}$

4: $\mu_{z_t|x_t} = \mu_{z_t|x_{t-1}} + K_t\left(x_t - \boldsymbol{C}\mu_{z_t|x_{t-1}}\right)$

5: $\Sigma_{z_t|x_t} = \Sigma_{z_t|x_{t-1}} - K_t\boldsymbol{C}\Sigma_{z_t|x_{t-1}}$

6: **return** $\mu_{z_t|x_t}$, $\Sigma_{z_t|x_t}$

Example: Constant velocity model

Let's consider a scenario where a mobile robot is navigating a 2-dimensional plane. In this case, let's denote the robot's state at time t as z_t. Up until now, we've referred to only the robot's location $\left[z^1, z^2\right]^T$ as the state. However, in reality, there are no constraints on what constitutes the state. Therefore, this time, let's include not only the robot's location but also its velocity $\left[\dot{z}^1, \dot{z}^2\right]^T$ in the state. This can be expressed as follows:

$$z_t = \left[z_t^1, \dot{z}_t^1, z_t^2, \dot{z}_t^2\right]^T .$$

Here, for the dynamic model of the robot, we can make a very simple assumption. Although the robot's velocity $\left[\dot{z}^1, \dot{z}^2\right]^T$ may vary with time t, for very short time intervals Δt, we can assume constant velocity motion. Then, for the robot's location $\left[z^1, z^2\right]^T$, it can be expressed as follows:

$$\begin{bmatrix} z_t^1 \\ z_t^2 \end{bmatrix} = \begin{bmatrix} z_{t-1}^1 \\ z_{t-1}^2 \end{bmatrix} + \begin{bmatrix} \dot{z}_{t-1}^1 \\ \dot{z}_{t-1}^2 \end{bmatrix} \Delta t.$$

Since we assume the constant velocity model, with the model noise w we have:

$$\begin{bmatrix} \dot{z}_t^1 \\ \dot{z}_t^2 \end{bmatrix} = \begin{bmatrix} \dot{z}_{t-1}^1 \\ \dot{z}_{t-1}^2 \end{bmatrix} + \begin{bmatrix} w^1 \\ w^2 \end{bmatrix} .$$

The dynamic model presented above can be expressed collectively as follows:

$$z_t = \boldsymbol{A} z_{t-1} + \boldsymbol{G} w$$

where

$$\boldsymbol{A} = \begin{bmatrix} 1 & \Delta t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \Delta t \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \boldsymbol{G} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Assume that we obtain the measurement $x = \left[x^1 x^2\right]^T$ for the robot location $\left[z^1, z^2\right]^T$, and we can write the measurement model as:

$$x_t = \boldsymbol{C} z_t + v$$

where v is a measurement noise and

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$