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General Solution for Inverse Kinematics of Six Degrees of Freedom of a Welding Robot Arm

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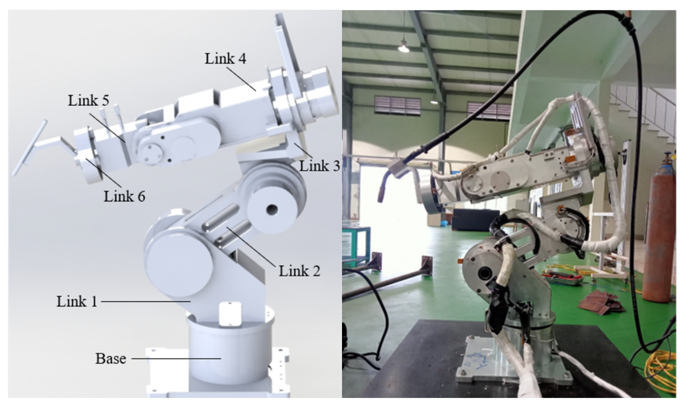
***Abstract*—In this study, we compared the efficiency of the dual quaternion and Paden-Kahan subproblems methods to the Raghavan method in solving the inverse kinematics of a six degrees of freedom robot arm. The results show that the dual quaternion method successfully overcomes the singularity problem for certain special configurations that the Raghavan method encountered in solving the inverse kinematics problem. Furthermore, using the dual quaternion method helps to simplify and intuitively visualize the representation of the forward kinematics problem. In addition to the numerical analysis, experiments were also conducted on a fabricated robot arm to validate the simulation results. Our experimental results showed that the robot was able to perform a horizontal straight-line weld with a length of approximately 380 mm.**

***Index Terms*—Raghavan method, Dual quaternion, General solution inverse kinematics, 6R robot arm, Paden-Kahan subproblems**

I. Introduction

In recent years, the use of robot arms in industrial production has become more prevalent. As a result, the field of research on robotic manipulators has piqued the interest of many people. Some works have focused on optimizing energy consumption during transport by analyzing and optimizing the applied learning response of stages in model-based transport [1]. A genetic algorithm was used to select the transport path for the manipulator steps that the robot arm must follow, such that the payoff satisfies the given optimal work condition. The consumption rate for carrying out the task must be minimized [2]. Other works have focused on modeling a robotic arm with a deformed model and designing an adaptive controller for the manipulator position and force feedback control [3], simulating and predicting a force applied to a knife when a robot manipulator automatically cuts vegetables [4], and sustainable control for an elastic robot manipulator model using a linear omnidirectional optimization design method [5]. The finite element method was used to analyze the behavior of materials in the operation stages of a 5-degree-of-freedom manipulator under specific working conditions in order to optimize the structure of the manipulator accordingly [6].

To address the above issue, it is essential to solve the kinematics problem for the manipulator. It helps in determining the position, velocity, and acceleration of any point in a certain part of the manipulator in a pre-selected fixed frame of reference. The kinematics of the robotic arm, in essence, consists of two main problems, namely forward kinematics and inverse kinematics. In this study, we focus on the inverse kinematics problem. Traditionally, the approach for this problem mainly used analyzing the geometric structure



(a) (b)

Fig. 1: Six DOFs welding robot arm: (a) CAD Model; (b) Fabricated Model.

Unlike the method of analyzing geometric structures, numerical methods apply to most manipulator structures, regardless of their complexity, and do not require much time and effort to implement algorithms. Some prominent numerical methods include the Newton-Raphson method [7] and the quasi-inverse matrix method [8]. Recently, artificial neural networks have also been used to solve inverse kinematics problems [9]. However, numerical methods have their limitations when compared to analyzing geometric structures. These limitations include longer computation time, inability to predict the occurrence of some singularities, and difficulty in finding solutions.

In the case of a robotic arm with 6 DOFs (Degrees of Freedom), when dealing with the problem of inverse kinematics, Raghavan and Roth proposed a general solution for all geometric configurations [10]. By implementing a numerical calculation engine, Manocha was able to enhance the algorithm's execution speed and achieved precise outcomes with remarkable performance [2]. Generally, these methods construct a 16-degree polynomial function that can be solved using the QR decomposition method. The majority of the time consumption is attributed to solving the 16-degree polynomial function step. Furthermore, the formula for the polynomial function has been established using Denavit-Hartenberg parameters from the forward kinematics step. Fortunately, there are better methods for solving forward and inverse kinematics problems using dual quaternions and the Paden-Kahan subproblems. The combination of these methods provides a geometric point of view for both problems and significantly reduces computation time without sacrificing accuracy [11].

Therefore, this paper will study and compare the Raghavan method and dual quaternion and Paden-Kahan subproblems method to design and analyze the kinematics for an arc welding manipulator with a reach of 600 mm, as shown in Fig. 1. Before designing the kinematics for the manipulator, the theoretical basis of manipulator kinematics and the general solution method of Raghavan and Roth are briefly presented, followed by the dual quaternion and Paden-Kahan subproblems method. The paper is structured as follows: section II briefly introduces the Raghavan method and provides a matrix formula for convenient use. In Sections III and IV, dual quaternions are introduced for forward kinematics and the step-by-step process for solving inverse kinematics using the Paden-Kahan subproblems is demonstrated in detail. Finally, Section V presents the comparison results from a simulation and experimental results from the fabricated robot arm.

II. Raghavan Method

Raghavan Method is the general analytical method that gives us the general solution for any configuration of a serial manipulation robot from 1 to 6 degrees of freedom. First of all, this one constructs general forward kinematic equations which are given by Denavit-Hatenberg Method. And then, the general solution can be established based on some linear algebra techniques. In this section, we briefly introduce a Denavit-Hatenberg method followed by the Raghavan Method to construct the general solution for the inversion kinematics problem.

A. Forward Kinematics based on Denavit-Hatenberg Method

The Denavit-Hatenberg convention is used to model the kinematics of a chain robot arm. Links are numbered from 0 to 6. Link number 0 is the fixed link, and link numbered 6 is the last link of the manipulator. We attach on the i^{th} link a coordinate system i. The coordinate system attached to link 0 is the global frame, and the coordinate system attached to the dynamic links i is a movement frame. Meanwhile, the kinematics of link i can be described indirectly by mathematical transformation from coordinate system i to $i-1$ one. Thus, the basis of coordinate system $i-1(R_{i-1})$ can be transformed into the $i(R_i)$ one as followed:

$$R_{i-1} = A_i \cdot R_i \tag{1}$$

where, A_i is square matrix called by Denavit-Hatenberg matrix in $R^{4\times4}$. The formula of A_i can be written as below:

$$A_i = \begin{bmatrix} c_i & -s_i\lambda_i & s_i\mu_i & a_ic_i \\ s_i & c_i\lambda_i & -c_i\mu_i & a_is_i \\ 0 & \mu_i & \lambda_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

where, $c_i = \cos\theta_i$, $s_i = \sin\theta_i$, $\lambda_i = \cos\alpha_i$ and $\mu_i = \sin\alpha_i$. Moreover, θ_i is the i^{th} joint angle, α_i is the i^{th} twist angle, a_i is the length of link $i+1$, and d_i the offset distance at joint i.

For convenience, we define that R_0 is the basis of the fixed coordinate system and R_6 is one of the coordinate systems attached to the end link of the robot arm. Then, the mathematical transformation between these frames can be written as

$$R_0 = A_{\text{hand}} \cdot R_6 \tag{3}$$

where,

$$A_1 \cdot A_2 \cdot A_3 \cdot A_4 \cdot A_5 \cdot A_6 = A_{\text{hand}} \tag{4}$$

B. General Inverse Kinematic Solution based on Raghavan Method

Accounting for the synthesis of robotic manipulators with 6 degrees of freedom including 6 rotations of 6R with all configurations. To facilitate the optimization of computational costs, matrix A_i can be completely decomposed into a multiplication of two matrices as follows [10]:

$$A_i = A_{iv} \cdot A_{is} \tag{5}$$

where,

$$A_{iv} = R_z(\theta_i) = \begin{bmatrix} c_i & -s_i & 0 & 0 \\ s_i & c_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

$$A_{is} = T_z(d_i)Tx(a_i)R_x(\alpha_i) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_i & -\mu_i & 0 \\ 0 & \mu_i & \lambda_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7}$$

Substituting (5) for (4) and eliminating the θ_6 yields 14 algebra equations written in matrix formula as below:

$$P \cdot \begin{bmatrix} s_4 s_5 \\ s_4 c_5 \\ c_4 s_5 \\ s_4 \\ c_4 \\ s_5 \\ c_5 \\ 1 \end{bmatrix} = Q \cdot \begin{bmatrix} s_1 s_2 \\ s_1 c_2 \\ c_1 s_2 \\ c_1 c_2 \\ s_1 \\ c_1 \\ s_2 \\ c_2 \end{bmatrix} \tag{8}$$

where, P is a matrix in $R^{14\times 9}$ whose element is a linear combination constructed from c_3, s_3. Q is a constant matrix in $R^{14\times 8}$ called by the configuration matrix.

Using Gaussian elimination on (8), we can eliminate θ_1 and θ_2. And then this action yields the result as

$$\Sigma \cdot \begin{bmatrix} s_4 s_5 \\ s_4 c_5 \\ c_4 s_5 \\ s_4 \\ c_4 \\ s_5 \\ c_5 \\ 1 \end{bmatrix} = \mathbf{0} \tag{9}$$

where, Σ is a matrix in $R^{6\times 9}$ whose element is a linear combination constructed from c_3, s_3. Substituting $c_i = \frac{1-x_i^2}{1+x_i^2}$, $s_i = \frac{2x_i}{1+x_i^2}$ such that $i \in \{3,4,5\}$ and using Sylvester elimination on equation (9) can be yields as

$$\begin{bmatrix} \Sigma' & \mathbf{0} \\ \mathbf{0} & \Sigma' \end{bmatrix} \cdot \begin{bmatrix} x_4^3 \cdot x_5^3 \\ x_4^3 \cdot x_5 \\ x_4^3 \\ x_4^2 \cdot x_5^2 \\ x_4^2 \cdot x_5 \\ x_4^2 \\ x_4 \cdot x_5^2 \\ x_4 \cdot x_5 \\ x_4 \\ x_5^2 \\ x_5 \\ 1 \end{bmatrix} = \mathbf{0} \tag{10}$$

Equation (10) represents 12 equations written in the matrix formula. The condition guarantees that (10) have the non-trivial solution is written as below:

$$\det\left(\begin{bmatrix} \Sigma' & \mathbf{0} \\ \mathbf{0} & \Sigma' \end{bmatrix}\right) = 0 \tag{11}$$

Finally, we use the R-Q decomposition method to solve x_3 numerically from (11).

C. Σ' Construction and Eigenvalues Problem

Generally, we can realize that transformation from equation (9) to one (10) can consume dramatic time because of the use of symbolic processing. First of all, we can rewrite matrix P as below:

$$P = P_1 \cdot c_3 + P_2 \cdot s_3 + P_3 \tag{12}$$

where, P_1, P_2 and P_3 are the constant matrix in $R^{14\times 9}$. And then, we define the auxiliary matrix M_1 have the form as

$$M_1 = \begin{bmatrix} Q & P_1 & P_2 & P_3 \end{bmatrix} \tag{13}$$

Using Gaussian elimination on matrix M_1 can be yields matrix M_2

$$M_2 = \begin{bmatrix} \boldsymbol{I} & P_1^* & P_2^* & P_3^* \\ \mathbf{0} & P_1' & P_2' & P_3' \end{bmatrix} \tag{14}$$

Consequently, the left-hand side of equation (11) can be rewritten as follow:

$$\det\left(\begin{bmatrix} \Sigma' & \mathbf{0} \\ \mathbf{0} & \Sigma' \end{bmatrix}\right) = \det\left(A \cdot x_3^2 + B \cdot x_3 + C\right) \tag{15}$$

where,

$$A = \begin{bmatrix} E & \mathbf{0} \\ \mathbf{0} & E \end{bmatrix} \quad \text{and} \quad E = P_3' - P_1' \tag{16}$$

$$B = \begin{bmatrix} F & \mathbf{0} \\ \mathbf{0} & F \end{bmatrix} \quad \text{and} \quad F = 2 \cdot P_2' \tag{17}$$

$$C = \begin{bmatrix} G & \mathbf{0} \\ \mathbf{0} & G \end{bmatrix} \quad \text{and} \quad G = P_3' + P_1' \tag{18}$$

By using the matrix polynomial theorem, a companion matrix M can be established as follow:

$$M = \begin{bmatrix} \mathbf{0} & \boldsymbol{I} \\ A^{-1} \cdot C & A^{-1} \cdot B \end{bmatrix} \tag{19}$$

Note that, roots of the polynomial equation (19) are eigenvalues of the matrix M.

D. Raghavan Method with special configuration

Consider a 6-degree-of-freedom robotic arm with a wrist as the last 3 links whose rotational axis always intersects at some common point, and this type of robot has special properties that $a_4 = a_5 = d_6 = 0$. Because of these special properties, this leads to some columns of Σ to be linearly dependent on each other and cause one to be singularities that the rank of Σ is less than 6. So, we cannot use equation (11) to find x_3. To overcome this consequence, we propose a trick which establishes three independent variables by grouping some dependent columns. This leads to the establishment of new variables

$$X_1 = s_4 \cdot s_5 - \lambda_4 \cdot c_4 \cdot c_5 + \frac{\lambda_5 \cdot \mu_4}{\mu_5} \cdot c_4 \tag{20}$$

$$X_2 = c_4 \cdot s_5 + \lambda_4 \cdot s_4 \cdot c_5 + \frac{\lambda_5 \cdot \mu_4}{\mu_5} \cdot s_4 \tag{21}$$

$$X_3 = c_5 \tag{22}$$

III. Dual Quaternion and Paden-Kahan Subproblems

In this section, first of all, we briefly introduce two points of view on describing the kinematic of a rigid body based on the dual quaternion and show that describing kinematics based on dual quaternion is equivalent to using a cosine matrix and translation vector. Finally, the analytical solution of Paden-Kahan Subproblems utilized frequently to solve more complex inverse kinematics problems is introduced.

A. Dual Quaternion for Forward Kinematics

Consider a rigid body moving in space that is attached to a moving coordinate system at a particular point on one, as shown in Fig 2. Let $\mathbf{R}$ be a cosine matrix that represents a rotation about unit vector $\vec{l}$ through the origin of the moving coordinate system, and translation vector $\vec{t} \in \mathbb{R}^3$. So now, an arbitrary point $\vec{v}$ belonging to the rigid body under the rotation R followed by the translation $\vec{t}$ is transformed into

$$\boldsymbol{v}' = \mathbf{R} \cdot \boldsymbol{v} + \boldsymbol{t} \tag{23}$$

In quaternion form, equation (23) can be expressed as [12]

$$\begin{bmatrix} 0 \\ \boldsymbol{v}' \end{bmatrix} = \boldsymbol{q} \begin{bmatrix} 0 \\ \boldsymbol{v} \end{bmatrix} \boldsymbol{q}^* + \begin{bmatrix} 0 \\ \boldsymbol{t} \end{bmatrix} \tag{24}$$

Davit-Hatenberg established successfully a homogeneous matrix which is comprised of the rotation matrix $\mathbf{R}$ and translation vector $\boldsymbol{t}$. These things help us to demonstrate equation (23) just by matrix multiplication. Furthermore, we can do the same thing in dual quaternion form. Indeed, let dual quaternion $\boldsymbol{\sigma}$ having formula as follows

$$\boldsymbol{\sigma} = \boldsymbol{q_1} + \varepsilon \boldsymbol{q_2} \tag{25}$$

In the right expression, $\boldsymbol{q_1}$ and $\boldsymbol{q_2}$ are both quaternions, and ε is the dual factor with the property of $\varepsilon^2 = 0$. In addition, there are identity element $\boldsymbol{\sigma_I}$ and inverse element $\boldsymbol{\sigma}^{-1}$ in dual quaternion algebra as written as below [13]

$$\boldsymbol{\sigma_I} = \boldsymbol{q_I} + \varepsilon \boldsymbol{q_o} \tag{26}$$

$$\boldsymbol{q_I} = [1,0,0,0]^T \tag{27}$$

$$\boldsymbol{q_o} = [0,0,0,0]^T \tag{28}$$

$$\boldsymbol{\sigma}^{-1} = \boldsymbol{q_1}^{-1}(\boldsymbol{q_I} - \varepsilon \boldsymbol{q_2} \boldsymbol{q_1}^{-1}) \tag{29}$$

And these definitions satisfy the property $\boldsymbol{\sigma}^{-1}\boldsymbol{\sigma} = \boldsymbol{\sigma}\boldsymbol{\sigma}^{-1} = \boldsymbol{\sigma_I}$. Originally, dual quaternions had three kinds of conjugates [11]. However, in this study, we mainly focus on the third kind of conjugate written as follows

$$\boldsymbol{\sigma}^{\diamond} = \boldsymbol{q}_1^* - \varepsilon \boldsymbol{q}_2^* \tag{30}$$

By introducing the dual quaternion in equation (30) into the left-hand side of equation (31) resulted in the right-hand side of one. Therefore, if let $\boldsymbol{q}$ be a quaternion represents rotation transformation, $\boldsymbol{t}$ be a pure quaternion represents translation, $\boldsymbol{\sigma} = \boldsymbol{q} + \frac{1}{2}\varepsilon \boldsymbol{q}\boldsymbol{t}$, the rigid body transformation [12] can be achieved by using the left-hand side of equation (31) as below:

$$\boldsymbol{\sigma} \otimes (\boldsymbol{q_I} + \varepsilon \boldsymbol{v}) \otimes \boldsymbol{\sigma}^{\diamond} = \boldsymbol{q_I} + \varepsilon\,(\boldsymbol{q}\boldsymbol{v}\boldsymbol{q}^* + \boldsymbol{t}) \tag{31}$$

Where, $\otimes$ is the dual quaternion multiplication operation. And note that, the dual vector of the right-hand side of equation (31) is identical to the right-hand side of equation (24).

In addition, the special property of dual quaternion is representing the rotation about the unit axis even without going through the origin of the moving coordinate system attached to rigid body and one describes nicely for screw theory which represents any rigid transformation according to Charles's theorem. Indeed, let vector $\vec{a}$ be the arbitrary point belong to the instantaneous rotation axis $\vec{l}$, θ be rotation angle and $d = \boldsymbol{t} \cdot \boldsymbol{l}$ called pitch of screw, as shown in Fig. 2. The formula for dual quaternion in this case can be rewritten as below:

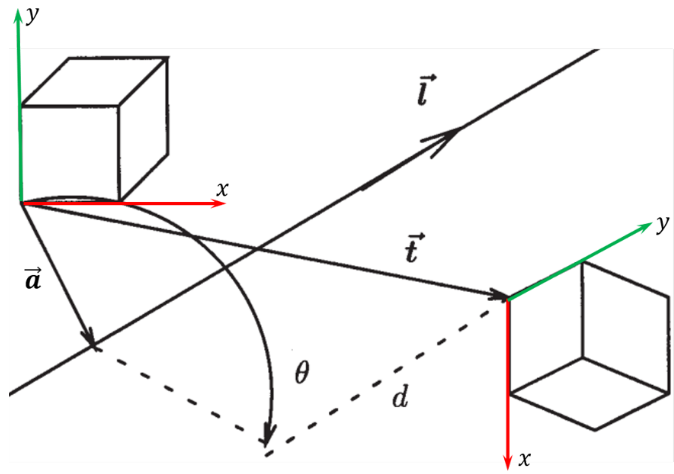


Fig. 2: Any rigid transformation can be described by screw motion [13].

$$\boldsymbol{\sigma} = \left[\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\boldsymbol{l}\right] + \varepsilon\left[\frac{d}{2}\left(-\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\boldsymbol{l}\right) + \sin\frac{\theta}{2}\boldsymbol{a} \times \boldsymbol{l}\right] \tag{32}$$

There is a difference in the point of view of Denavit-Hatenberg in forward kinematics. Generally, there are two kinds of approaches to cope with forward kinematics problems. Denavit-Hatenberg's method defines several moving frames attached to the corresponding rigid body, and then the Denavit-Hatenberg matrix describes transformation locally between adjacent moving frames, this approach is called local forward kinematics. And another approach is called global forward kinematics which defines just one frame attached to the last link (manipulation link) and then describes transformation globally from the initial configuration to desired one. Eventually, dual quaternion is a powerful tool to model forward kinematics in both ways. For instance, we can define locally moving frames O_1^l, O_2^l attached to each link as shown in Fig. 3. On the other hand, we can use dual quaternion in different ways than just using coordinate system O_0 and another attached to the end of the latest link O_2. And then, both these ones simultaneously are transformed (O_0 transform to O_1 and from O_1 to O_2) by screw transformation defined globally by screw parameters at several joint points.

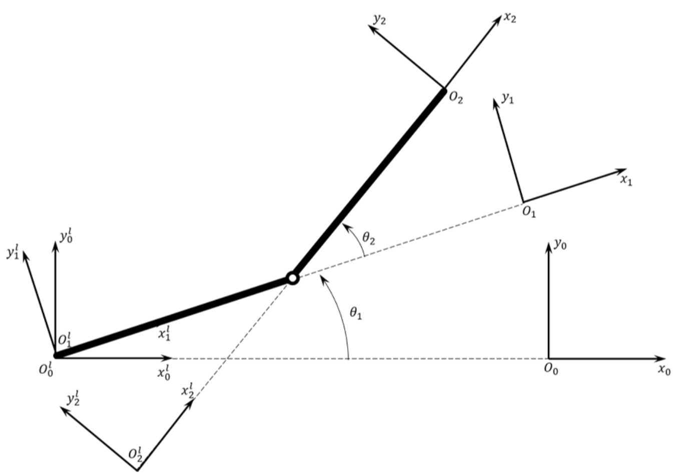


Fig. 3: Difference approach to forward kinematics.

B. Paden-Kahan Subproblems

Paden-Kahan subproblems are a set of three geometry problems that can be faced regularly when we solve inverse kinematics. Moreover, analytical solutions for three subproblems were given. Therefore, solving the inverse kinematics problem by the geometry approach is just splitting the configuration's robot arm into Paden-Kahan subproblems. For more details, the following steps can be applied to solve the inverse kinematics problem:

- Find points of intersection at which two or more axis intersect.
- Apply rigid transformation to these points (rigid transformation can not affect), and then subproblem 1 or 2 can be found.
- Subproblem 3 can be found when we subtract these points from one another.

In addition, three subproblems will be briefly introduced sequentially as follows.

Subproblem 1 [14]: Given two points $\boldsymbol{p},\boldsymbol{q} \in \mathbb{R}^3$, unit axis $\boldsymbol{\omega} \in \mathbb{R}^3$ and a point on this axis $\boldsymbol{r} \in \mathbb{R}^3$. Find θ be a angle of the rotation about unit axis $\boldsymbol{\omega}$ such that rotate vector $\boldsymbol{p}$ into $\boldsymbol{q}$ as follows:

$$e^{\boldsymbol{\omega}\theta}\boldsymbol{p} = \boldsymbol{q} \tag{33}$$

where, $e^{\boldsymbol{\omega}\theta}$ is an exponential map of the rotation. The solution θ can be written as below:

$$\theta = \tan^{-1}\left(\frac{\boldsymbol{\omega}\cdot\boldsymbol{u'}\times\boldsymbol{v'}}{\boldsymbol{u'}\cdot\boldsymbol{v'}}\right) \tag{34}$$

$$\boldsymbol{u'} = (\boldsymbol{p}-\boldsymbol{r}) - [\boldsymbol{\omega}\cdot(\boldsymbol{p}-\boldsymbol{r})]\cdot\boldsymbol{\omega} \tag{35}$$

$$\boldsymbol{v'} = (\boldsymbol{q}-\boldsymbol{r}) - [\boldsymbol{\omega}\cdot(\boldsymbol{q}-\boldsymbol{r})]\cdot\boldsymbol{\omega} \tag{36}$$

Subproblem 2 [14]: Given two points $\boldsymbol{p},\boldsymbol{q} \in \mathbb{R}^3$, unit axis $\boldsymbol{\omega_1},\boldsymbol{\omega_2} \in \mathbb{R}^3$ and a point of intersection on these axis $\boldsymbol{r} \in \mathbb{R}^3$. Find θ_1,θ_2 be angle of the rotation about unit axis $\boldsymbol{\omega_1},\boldsymbol{\omega_1}$, respectively such that rotate vector $\boldsymbol{p}$ into $\boldsymbol{q}$ as follows:

$$e^{\boldsymbol{\omega_1}\theta_1}e^{\boldsymbol{\omega_2}\theta_2}\boldsymbol{p} = \boldsymbol{q} \tag{37}$$

Equation (37) can be separated into two equations in Subproblem 1 form as below:

$$e^{\boldsymbol{\omega_2}\theta_2}\boldsymbol{p} = \boldsymbol{c} \quad \text{and} \quad e^{-\boldsymbol{\omega_1}\theta_1}\boldsymbol{p} = \boldsymbol{c} \tag{38}$$

where,

$$\boldsymbol{c} = \alpha\boldsymbol{\omega_1} + \beta\boldsymbol{\omega_2} + \gamma\boldsymbol{\omega_1}\times\boldsymbol{\omega_2} + \boldsymbol{r} \tag{39}$$

$$\alpha = \frac{(\boldsymbol{\omega_1}\boldsymbol{\omega_2})(\boldsymbol{\omega_2}[\boldsymbol{p}-\boldsymbol{r}]) - \boldsymbol{\omega_1}[\boldsymbol{q}-\boldsymbol{r}]}{(\boldsymbol{\omega_1}\boldsymbol{\omega_2})^2-1} \tag{40}$$

$$\beta = \frac{(\boldsymbol{\omega_1}\boldsymbol{\omega_2})(\boldsymbol{\omega_1}[\boldsymbol{q}-\boldsymbol{r}]) - \boldsymbol{\omega_2}[\boldsymbol{p}-\boldsymbol{r}]}{(\boldsymbol{\omega_1}\boldsymbol{\omega_2})^2-1} \tag{41}$$

$$\gamma^2 = \frac{\|\boldsymbol{p}-\boldsymbol{r}\|^2 - \alpha^2 - \beta^2 - 2\alpha\beta\boldsymbol{\omega_1}\boldsymbol{\omega_2}}{\|\boldsymbol{\omega_1}\times\boldsymbol{\omega_2}\|^2} \tag{42}$$

Subproblem 3 [14]: Given two points $\boldsymbol{p},\boldsymbol{q} \in \mathbb{R}^3$, unit axis $\boldsymbol{\omega} \in \mathbb{R}^3$, a point on this axis $\boldsymbol{r} \in \mathbb{R}^3$ and a real positive number $\delta \in \mathbb{R}$. Find θ be a angle of the rotation about unit axis $\boldsymbol{\omega}$ such that rotate vector $\boldsymbol{p}$ into $\boldsymbol{q}$ satisfying the following equation:

$$\|e^{\boldsymbol{\omega}\theta}\boldsymbol{p} - \boldsymbol{q}\| = \delta \tag{43}$$

The solution θ can be written as below:

$$\theta = \tan^{-1}\left(\frac{\boldsymbol{\omega}\cdot\boldsymbol{u'}\times\boldsymbol{v'}}{\boldsymbol{u'}\cdot\boldsymbol{v'}}\right) \pm \cos^{-1}\left(\frac{\|u'\|^2+\|v'\|^2-\delta'^2}{2\|u'\|\|v'\|}\right) \tag{44}$$

$$\boldsymbol{u'} = (\boldsymbol{p}-\boldsymbol{r}) - [\boldsymbol{\omega}\cdot(\boldsymbol{p}-\boldsymbol{r})]\cdot\boldsymbol{\omega} \tag{45}$$

$$\boldsymbol{v'} = (\boldsymbol{q}-\boldsymbol{r}) - [\boldsymbol{\omega}\cdot(\boldsymbol{q}-\boldsymbol{r})]\cdot\boldsymbol{\omega} \tag{46}$$

$$\delta'^2 = \delta^2 - (\boldsymbol{\omega}\cdot[\boldsymbol{p}-\boldsymbol{q}])^2 \tag{47}$$

IV. Analysis and Interpretation

In this section, we use a welding robot arm as shown in Fig. 4 to solve the inverse kinematics by using the methods discussed in the previous section. To provide a more detailed explanation, we will demonstrate a step-by-step approach for solving inverse kinematics using the combination of the dual quaternion and Paden-Kahan subproblems method.

TABLE I
Denavit Hatenberg parameters of welding robot arm

Joint	1	2	3	4	5	6	Unit
θ	θ_1	$\frac{\pi}{2}-\theta_2$	θ_3	θ_4	θ_5	θ_6	rad
d	0.150	0.0	0.0	0.238	0.0	0.339	m
a	0.120	0.270	0.205	0.0	0.0	0.0	m
α	$-\frac{\pi}{2}$	0	$-\frac{\pi}{2}$	$\frac{\pi}{2}$	$-\frac{\pi}{2}$	0	rad

We implemented the introduced methods on a real welding robot arm to verify the effectiveness of the inverse kinematics-solving process. It is 6 DOFs all revolute joint robot arm, with a robot configuration of SRRR (S-spherical, R-revolute), with both kinematics Denavit Hatenberg and screw parameters, as shown in Table I and Table II.

Given θ_i ($i = 1,2,...,6$), we can calculate six dual quaternions $\boldsymbol{\sigma_i}$. Hence, the forward kinematics problem can be

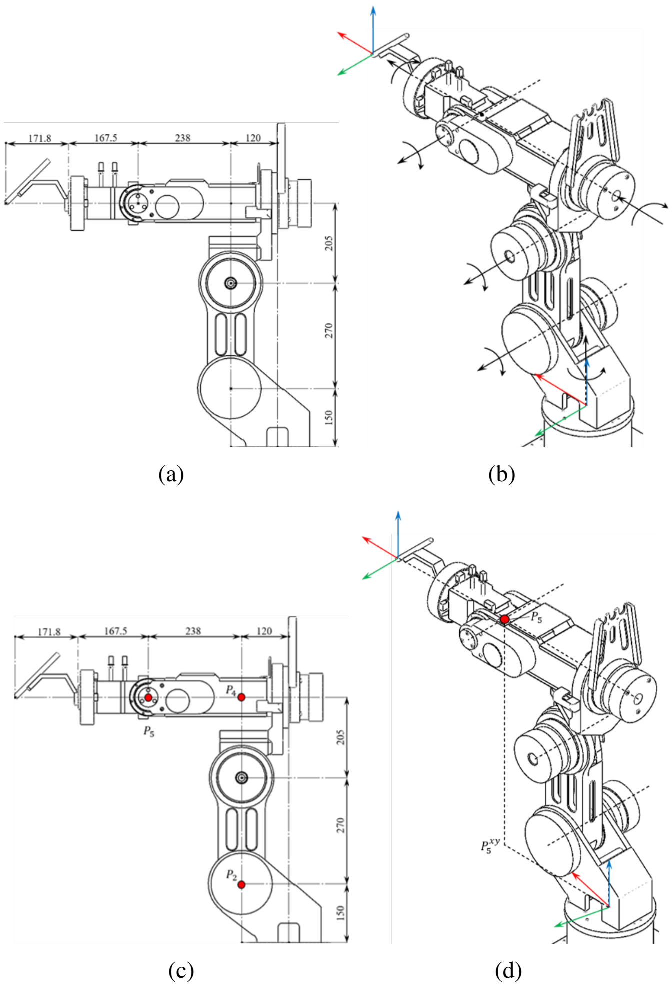


Fig. 4: Welding Robot Arm will be considered: (a) kinematic configurations; (b) moving and fixed frame; (c) intersection points; (d) special point P_5 always rely on vertical pass through fixed origin and P_5.

TABLE II
SCREW PARAMETERS OF WELDING ROBOT ARM

Joint	1	2	3	4	5	6	Unit
θ	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	rad
l_x	0	0	0	1	0	1	m
l_y	0	-1	-1	0	-1	0	m
l_z	1.000	0	0	0	0	0	m
a_x	0	0.120	0.120	0.120	0.358	0.525	m
a_y	0	0	0	0	0	0	m
a_z	0	0.150	0.420	0.625	0.625	0.625	m
d	0	0	0	0	0	0	m

solved by determining $\boldsymbol{\sigma_d}$ as shown in equation (48). Apparently, by substituting $\boldsymbol{\sigma_d}$ into the left hand side equation (31), we can obtain the position of the endpoint.

$$\boldsymbol{\sigma_d} = \prod_{i=1}^{6} \boldsymbol{\sigma_i} \tag{48}$$

Conversely, in the inverse kinematics problem, θ_i are not given, all we have is the desired position and direction of the coordinate system attached to the end of the robot arm as shown in Fig. 4b, $\boldsymbol{r} = [0, r_x, r_y, r_z]^T$ and $\boldsymbol{q} = [q_s, \boldsymbol{q_v}]^T = [q_0, q_1, q_2, q_3]^T$, respectively. We cannot construct $\boldsymbol{\sigma_d}$ using equation (48). To address this issue, we propose an alternative method in dual quaternion form for computing $\boldsymbol{\sigma_d}$ through $\boldsymbol{r}$ and $\boldsymbol{q}$, instead of using equation (48), as shown below

$$\boldsymbol{\sigma_d} = \boldsymbol{q} + \frac{1}{2}\varepsilon \boldsymbol{q}\left(\boldsymbol{r} - \boldsymbol{q}\boldsymbol{r_0}\boldsymbol{q}^*\right) \tag{49}$$

where, $\boldsymbol{r_0}$ is an initial configuration. Eventually, $\boldsymbol{\sigma_d}$ is used as an input for the inverse kinematic problem. By following from section A to section C, we can obtain the unknown rotation angle θ_i then.

A. Solution θ_1

For the first of all, consider a point P_5 that's the point of intersection at which $\boldsymbol{l_4}, \boldsymbol{l_5}, \boldsymbol{l_6}$ intersect, as shown in Fig. 4c. Because these joint $\theta_2, \theta_3, \theta_4$ are kinematics of the 3-link planar mechanism, the projection point P_5^{xy} of point P_5 onto the xy plane always lie in the plane that's perpendicular to the xy plane and the $\boldsymbol{l_4}$ axis in one, as shown in Fig. 4d. Therefore, $\boldsymbol{\sigma_4}, \boldsymbol{\sigma_5}, \boldsymbol{\sigma_6}$ can not effect point P_5. So, this implies the solution for θ_1 can be written as follows:

$$\theta_1 = \tan^{-1} \frac{P_5^y}{P_5^x} \tag{50}$$

$$P_5 = \boldsymbol{\sigma_d} \otimes P_5^0 \otimes \boldsymbol{\sigma_d^\diamond} \tag{51}$$

where, let P_5^0 be an initial position of P_5 as shown in Fig. 4a.

B. Solution $\theta_2, \theta_3, \theta_5$

To find solution for θ_3, consider a point P_2 that's point that lie on axis $\boldsymbol{l_2}$, as shown in Fig. 4a. And then, we can write the formula of Subproblem 3 in this case as follows:

$$\|\boldsymbol{\sigma_3} \cdot P_5^0 - P_2^0\| = \|\boldsymbol{\sigma_d} \cdot P_5^0 - \boldsymbol{\sigma_1} \cdot P_2^0\| \tag{52}$$

where, $\boldsymbol{\sigma}_i \cdot P_j^k$ represents the screw transformation as shown in equation (32). Consequently, we can obtain two possible solutions for θ_3. For each solution of θ_3, we establish a the formula of Subproblem 1 for finding the solution of θ_2 as below

$$\boldsymbol{\sigma_2} \cdot \left(\boldsymbol{\sigma_3} \cdot P_5^0\right) = \boldsymbol{\sigma_1}^{-1} \cdot \left(\boldsymbol{\sigma_d} \cdot P_5^0\right) \tag{53}$$

In addition, when we found $\theta_1, \theta_2, \theta_3$, Subproblem 3 solving solution θ_5 can be established as below:

$$\|\boldsymbol{\sigma_5} \cdot \boldsymbol{P_6^0} - \boldsymbol{P_4^0}\| = \|\boldsymbol{\sigma_d} \cdot \boldsymbol{P_6^0} - \boldsymbol{\sigma_1} \cdot \boldsymbol{\sigma_2} \cdot \boldsymbol{\sigma_3} \cdot \boldsymbol{P_4^0}\| \tag{54}$$

where, let P_6^0 be the initial point of the tool endpoint as shown in Fig. 4a.

C. Solution θ_4, θ_6

After that, to solve solution for θ_4, Subproblem 1 can be written as follows:

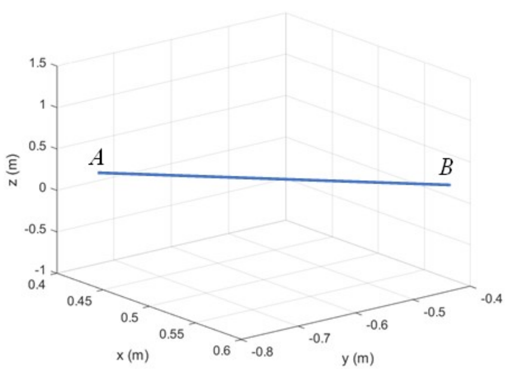
$$\boldsymbol{\sigma_4} \cdot \left(\boldsymbol{\sigma_5} \cdot \boldsymbol{P_6^0}\right) = \left(\boldsymbol{\sigma_1} \cdot \boldsymbol{\sigma_2} \cdot \boldsymbol{\sigma_3}\right)^{-1} \cdot \boldsymbol{\sigma_d} \cdot \boldsymbol{P_6^0} \tag{55}$$

Finally, we can solve solution θ_6 by choosing arbitrary point $\boldsymbol{P_6'}$ nearby $\boldsymbol{P_6^0}$ that don't lie in axis $\boldsymbol{l_6}$ to establish Subproblem 1 as below:

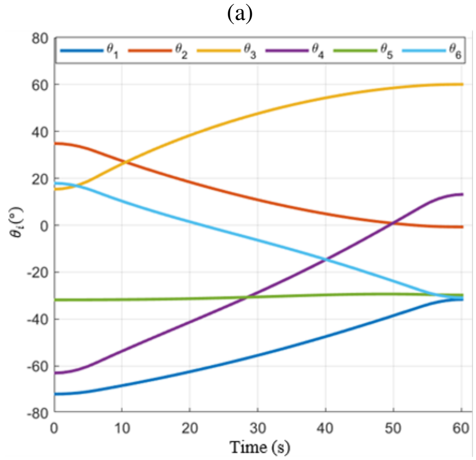
$$\boldsymbol{\sigma_6} \cdot \boldsymbol{P_6'} = \left(\boldsymbol{\sigma_1} \cdot \boldsymbol{\sigma_2} \cdot \boldsymbol{\sigma_3} \cdot \boldsymbol{\sigma_4} \cdot \boldsymbol{\sigma_5}\right)^{-1} \cdot \boldsymbol{\sigma_d} \cdot \boldsymbol{P_6'} \tag{56}$$

V. Results

To illustrate the efficiency and intuition of solving inverse kinematics using dual quaternion, the comparison with the Raghavan method for computation time and error has been considered. A case study was conducted using a line trajectory lying parallel to the $x-y$ plane, which was divided into 100 segments, as shown in Fig 5a. The inverse kinematics solution for the line trajectory, which is obtained by the dual quaternion and Paden-Kahan subproblems method, is shown in Fig. 5b. There is a maximum of eight solutions to the inverse kinematics problem that satisfies given configuration of robot arm 6 DOFs [10], then what is shown in Fig. 5b is chosen such that meet the specific criteria $\boldsymbol{\theta} = \min_{\theta_1,\ldots,\theta_6} \int \dot{\theta}^T \dot{\theta}$.



(a)



(b)

Fig. 5: (a) The line trajectory and (b) inverse kinematics solution.

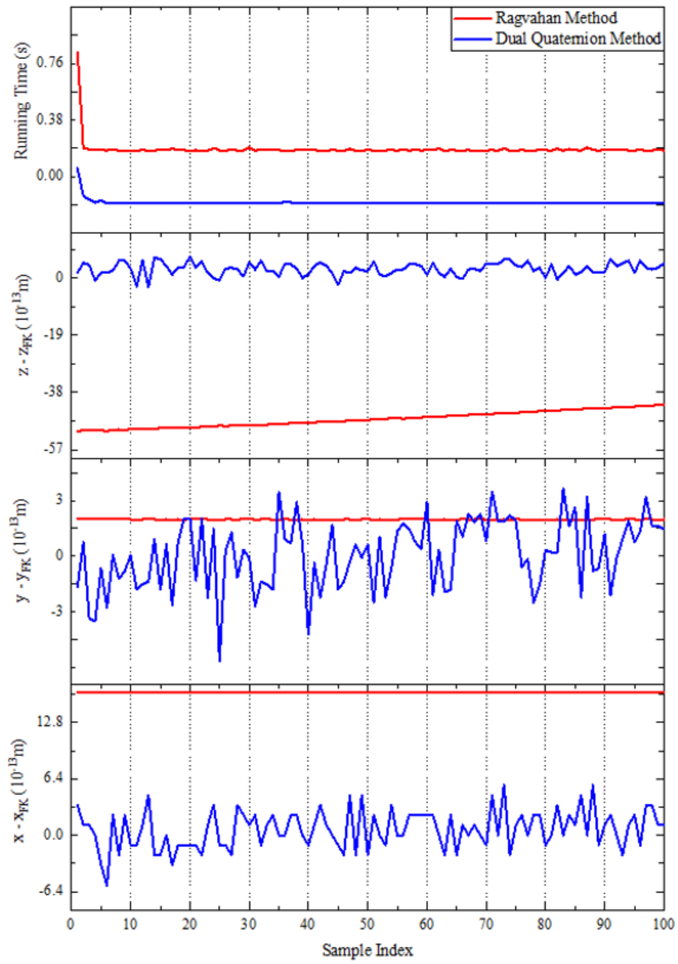
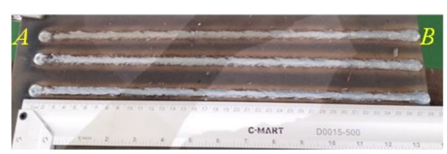
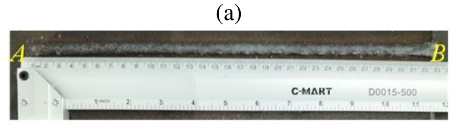


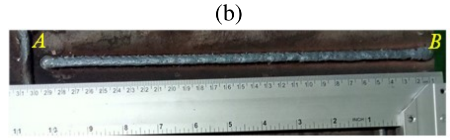
Fig. 6: Compare results for two computation methods: running time; forward kinematic errors on z, y and x axis.



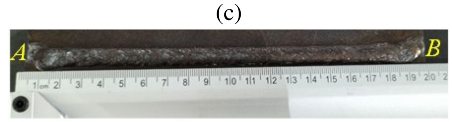
(a)



(b)



(c)



(d)

Fig. 7: (a) Triple 38 cm welding path; (b) 33 cm welding path; (c) 30 cm welding path (c); (d) 20 cm welding path.

To compare the computation time and error of these two methods, we present the simulation results in Fig. 6. Generally, the Raghavan method requires more numerical computations, specifically the RQ method to solve for θ_3 by finding the eigenvalues of matrix M in equation (19). This is why this method takes more time and is less precise than the dual quaternion method. The dual quaternion method uses special configurations to establish analytical solutions for the inverse kinematic problem.

Finally, we used the dual quaternion and Paden-Kahan subproblems method to find the inverse kinematics solution, which is used to run the fabricated welding robot arm Fig. 1. We conducted six experiments, which are shown in Fig. 7. For the fisrt three experiments, the robot arm performed three parallel welding line, each with a length of 38 cm. The result shown in Fig 7a prove that the robot arm can meet the requirements. We also perform different lengths of welding lines (Fig 7b, 7c and 7d) in different location on the x-y plane to validate the ability of the method.

VI. Conclusions

In this paper, we proposed an alternative process for solving the inverse kinematics using the Paden-Kahan subproblems method and dual quaternion algebra. We developed and implemented Matlab codes that use the dual quaternion and Paden-Kahan subproblems methods, as well as the Raghavan method, to solve both the forward and inverse kinematics problems of a 6 DOF robot arm. Our results showed that the inverse kinematic solution obtained using the dual quaternion and Paden-Kahan subproblems method was more accurate and faster to compute than the solution obtained using the Raghavan method. The dual quaternion and Paden-Kahan subproblems-based method also proved to be more robust and effective. However, due to the special configuration of the robot arm we considered, the Raghavan method faced a singular problem and we had to manually modify dependent variables, as discussed in Section II.D. In arbitrary configurations with 1 to 6 DOFs, the Raghavan method is still more general than the dual quaternion and Paden-Kahan subproblems-based method.

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